Indefinite Canonical Systems. Theory and Examples

Harald Woracek

joint work with Michael Kaltenbäck, Matthias Langer and Henrik Winkler
CONTENTS

- Review of the positive definite theory
- Some examples of canonical systems
- Indefinite analogue of canonical systems
- A short account on the literature
REVIEW OF THE POSITIVE DEFINITE THEORY
A Hamiltonian is a function

- \( H : [\sigma_0, \sigma_1) \rightarrow \mathbb{R}^{2 \times 2} \) defined a.e., measurable;
- \( H(t) \geq 0, H \in L^1_{loc}(\sigma_0, \sigma_1); \)
- \( \int_{\sigma_0}^{\sigma_0+\epsilon} \text{tr} \ H(t) \, dt < \infty \) (initial value problem);
- \( H \) does not vanish on any set of positive measure.
A Hamiltonian is a function

- \( H : [\sigma_0, \sigma_1) \rightarrow \mathbb{R}^{2 \times 2} \) defined a.e., measurable;
- \( H(t) \geq 0, H \in L^1_{loc}((\sigma_0, \sigma_1)) \);
- \( \int_{\sigma_0}^{\sigma_0+\epsilon} \text{tr } H(t) \, dt < \infty \) (initial value problem);
- \( H \) does not vanish on any set of positive measure.

The canonical system with Hamiltonian \( H \) is the differential equation

\[
y'(x) = z \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} H(x)y(x), \ x \in [\sigma_0, \sigma_1). \]
A Hamiltonian $H$ is said to be in the

- **limit circle case** if $\int_{\sigma_1 - \epsilon}^{\sigma_1} \text{tr} H(t) \, dt < +\infty$;

- **limit point case** if $\int_{\sigma_1 - \epsilon}^{\sigma_1} \text{tr} H(t) \, dt = +\infty$. 
Hamiltonian

\[ H(t) \]
**Summary**

- **Hamiltonian**: $H(t)$
- **Matrix chain**: $(W_t)$
- **Model space**: $L^2(H)$
Summary

Hamiltonian

\[ H(t) \]

Matrix chain

\( W_t \)

Model space

\( L^2(H) \)
Summary (limit point case)

Hamiltonian

\[ H(t) \]

Matrix chain

\( (W_t) \)

Model space

\[ L^2(H) \]

Weyl coefficient

\[ q_H(z) \]
Summary (limit point case)

Hamiltonian

$H(t)$

Model space

$L^2(H)$

Matrix chain

$(W_t)$

Weyl coefficient

$q_H(z)$
Summary (limit point case)

Hamiltonian

\[ H(t) \]

Matrix chain

\[ (W_t) \]

Solution

\[ \downarrow \downarrow \]

Construction

Matrix chain

\[ (W_t) \]

Defect elements

\[ \text{rks} \cong L^2(H|_{[0,t)}) \]

Weyl coefficient

\[ q_H(z) \]

Model space

\[ L^2(H) \]

Limit

\[ \lim W_t \star \tau \]

Resolvent matrix

\[ L^2(\sigma) \cong L^2(H) \]
The Inverse Spectral Theorem

The assignment

\[ H(t) \mapsto q_H(z) \]

yields a bijection between the set of all Hamiltonians (up to reparameterization) and the Nevanlinna class \( \mathcal{N}_0 \).
SOME EXAMPLES OF CANONICAL SYSTEMS
Let $a \in (0, \infty)$. A function $f : (-2a, 2a) \to \mathbb{C}$ is called positive definite, if $f(-t) = f(t)$ and if the kernel

$$K_f(s, t) = f(t - s), \quad s, t \in (-a, a),$$

is positive definite. The set of all continuous positive definite functions on the interval $(-2a, 2a)$ is denoted by $P_{0,a}$. 
Positive definite functions

Continuation problem: Let $f \in \mathcal{P}_{0,a}$. Do there exist continuations $\tilde{f} \in \mathcal{P}_{0,\infty}$?
Continuation problem: Let $f \in \mathcal{P}_{0,a}$. Do there exist continuations $\tilde{f} \in \mathcal{P}_{0,\infty}$?

Solution: There exists either a unique continuation or infinitely many continuations. In the second case the set of all continuations is parameterized by

$$i \int_{0}^{\infty} e^{itz} \tilde{f}(t) \, dt = W_f(z) \ast \tau(z)$$

where $W_f$ is a certain entire $2 \times 2$-matrix function and the parameter $\tau$ runs through the Nevanlinna class $\mathcal{N}_0$. 
Let \( f \in \mathcal{P}_{0,\infty} \). Assume that the set

\[
I_f := \{ a > 0 : f|_{(-2a,2a)} \text{ has infinitely many continuations} \}
\]

is nonempty.
Then the family

\[
W_t(z) := \begin{cases} 
  \left( \begin{array}{cc} 1 & 0 \\ -(f(0)^{-1} + t)z & 1 \end{array} \right), & t \in [-f(0)^{-1}, 0) \\
  W_f|_{(-2t,2t)}(z), & t \in I_f
\end{cases}
\]

is the matrix chain of a certain Hamiltonian \( H_f \).
The Bessel equation

The Bessel equation is the eigenvalue problem with singular endpoint 0

\[-u''(x) + \frac{\nu^2 - \frac{1}{4}}{x^2}u(x) = \lambda u(x), \ x > 0\]

Here \(\nu\) is a parameter \(\nu > \frac{1}{2}\) and \(\lambda\) is the eigenvalue parameter.
Rewriting this equation as a first-order-system, making a substitution in the independent variable, and setting $\alpha := 2\nu - 1$, $\lambda = z^2$, yields an equation of the form of a canonical system with

$$H_\alpha(x) = \begin{pmatrix} x^\alpha & 0 \\ 0 & x^{-\alpha} \end{pmatrix}$$

In order that $H_\alpha$ is integrable at 0, we need that $\alpha < 1$, i.e. $\nu < 1$. In this case the matrix chain $(W_{\alpha,t})_{t \in [0,\infty)}$ and the Weyl coefficient $q_{H_\alpha}$ can be computed explicitly:
Let $\alpha \in (0, 1)$. Then

$$W_{\alpha, t}(z) =$$

$$= \begin{pmatrix}
2^{\nu_1} \Gamma(\nu) z^{-\nu_1} t^{-\nu_1} J_{\nu_1}(t z) & 2^{\nu_1} \Gamma(\nu) z^{-\nu_1} t^\nu J_{\nu}(t z) \\
-2^{-\nu} \Gamma(-\nu_1) z^\nu t^{-\nu_1} J_{-\nu_1}(t z) & 2^{-\nu} \Gamma(-\nu_1) z^\nu t^\nu J_{-\nu}(t z)
\end{pmatrix}$$

with $\nu_1 := \frac{\alpha - 1}{2} = \nu - 1$, and

$$q H_{\alpha}(z) = c_\alpha z^{-\alpha}$$

with $c_\alpha := \frac{2^\alpha}{\pi} \Gamma(\nu)^2 \sin \nu e^{i\nu \pi}$. 
INDEFINITE ANALOGUE OF CANONICAL SYSTEMS
# Indefinite can.systems / Motivation

<table>
<thead>
<tr>
<th>CONCEPTS</th>
<th>POSITIVE DEFINITE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hamiltonian</td>
<td></td>
</tr>
<tr>
<td>Matrix chain</td>
<td></td>
</tr>
<tr>
<td>Boundary triplet</td>
<td></td>
</tr>
<tr>
<td>Nevanlinna class</td>
<td>$\mathcal{N}_0$</td>
</tr>
</tbody>
</table>
### Indefinite can.systems / Motivation

<table>
<thead>
<tr>
<th>I.S.T. CONCEPTS</th>
<th>POSITIVE DEFINITE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hamiltonian</td>
<td></td>
</tr>
<tr>
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<td></td>
</tr>
<tr>
<td>Boundary triplet</td>
<td></td>
</tr>
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</tr>
</tbody>
</table>

\[
\{\text{Hamiltonians}\} \xrightarrow{\text{WC}} \mathcal{N}_0
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**Indefinite can.systems / Motivation**

<table>
<thead>
<tr>
<th>EXAMPLES</th>
<th>I.S.T.</th>
<th>CONCEPTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>POSITIVE DEFINITE</td>
<td></td>
<td></td>
</tr>
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<td></td>
</tr>
</tbody>
</table>

{Hamiltonians} $\leftrightarrow \mathcal{N}_0$

Positive def.functions
Bessel equation $\alpha < 1$
<table>
<thead>
<tr>
<th><strong>POSITIVE DEFINITE</strong></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Hamiltonian</td>
<td></td>
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<tr>
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<td></td>
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<td></td>
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<th><strong>CONCEPTS</strong></th>
</tr>
</thead>
<tbody>
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</tr>
</tbody>
</table>

- Positive def. functions
- Bessel equation $\alpha < 1$
- Moment problems
- Strings
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<table>
<thead>
<tr>
<th><strong>POSITIVE DEFINITE</strong></th>
<th><strong>INDEFINITE</strong></th>
</tr>
</thead>
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<td></td>
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<td></td>
</tr>
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<td></td>
</tr>
</tbody>
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$I.S.T. \{ \text{Hamiltonians} \} \leftrightarrow \mathcal{N}_0$

<table>
<thead>
<tr>
<th><strong>EXAMPLES</strong></th>
<th><strong>I.S.T.</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive def.functions</td>
<td></td>
</tr>
<tr>
<td>Bessel equation $\alpha &lt; 1$</td>
<td></td>
</tr>
<tr>
<td>Moment problems</td>
<td></td>
</tr>
<tr>
<td>Strings</td>
<td></td>
</tr>
</tbody>
</table>
### Indefinite canonical systems / Motivation

<table>
<thead>
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<th>INDEFINITE</th>
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<td></td>
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<tr>
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<td>$\mathcal{N}_\kappa$ (generalized Nevanlinna class)</td>
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</tbody>
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**Examples**

- Positive def. functions
- Bessel equation $\alpha < 1$
- Moment problems
- Strings
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<table>
<thead>
<tr>
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<th>POSITIVE DEFINITE</th>
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<td></td>
<td>B. triplet in Pontryagin space</td>
</tr>
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<th>I.S.T.</th>
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<td>Positive def.functions</td>
<td></td>
<td></td>
</tr>
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<td></td>
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<tr>
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<td></td>
<td></td>
</tr>
<tr>
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</tr>
</tbody>
</table>
# Indefinite can.systems / Motivation

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<td></td>
</tr>
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<td></td>
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</tr>
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<td></td>
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</tr>
<tr>
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<td></td>
</tr>
<tr>
<td>Positive def.functions</td>
<td></td>
<td>Hermitian indef.functions</td>
</tr>
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<td>Bessel equation $\alpha &lt; 1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Moment problems</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Strings</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
# Indefinite can.systems / Motivation

## POSITIVE DEFINITE
- Hamiltonian
- Matrix chain
- Boundary triplet
- Nevanlinna class \( \mathcal{N}_0 \)

## INDEFINITE
- B.triplet in Pontryagin space
- Generalized Nev.class \( \mathcal{N}_{\kappa} \)

## EXAMPLES | I.S.T.

<table>
<thead>
<tr>
<th>Concepts</th>
<th>POSITIVE DEFINITE</th>
<th>INDEFINITE</th>
</tr>
</thead>
<tbody>
<tr>
<td>{Hamiltonians}</td>
<td>( \leftrightarrow \mathcal{N}_0 )</td>
<td>( \leftrightarrow \mathcal{N}_{\kappa} )</td>
</tr>
<tr>
<td>Positive def.functions</td>
<td></td>
<td>Hermitian indef.functions</td>
</tr>
<tr>
<td>Bessel equation ( \alpha &lt; 1 )</td>
<td></td>
<td>Bessel equation ( \alpha \in \mathbb{R}^+ )</td>
</tr>
<tr>
<td>Moment problems</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Strings</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Indefinite can. systems / Motivation

<table>
<thead>
<tr>
<th>Concepts</th>
<th>Positive Definite</th>
<th>Indefinite</th>
</tr>
</thead>
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<tr>
<td>Hamiltonian</td>
<td></td>
<td></td>
</tr>
<tr>
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<td></td>
<td></td>
</tr>
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<td></td>
<td>B. triplet in Pontryagin space</td>
</tr>
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<td></td>
<td>generalized Nev. class $\mathcal{N}_\kappa$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Examples</th>
<th>I.S.T.</th>
<th>Indefinite Canonical Systems – p.15/51</th>
</tr>
</thead>
<tbody>
<tr>
<td>${\text{Hamiltonians}} \leftrightarrow \mathcal{N}_0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Positive def. functions</td>
<td></td>
<td>Hermitian indef. functions</td>
</tr>
<tr>
<td>Bessel equation $\alpha &lt; 1$</td>
<td></td>
<td>Bessel equation $\alpha \in \mathbb{R}^+$</td>
</tr>
<tr>
<td>Moment problems</td>
<td></td>
<td>indefinite moment problems</td>
</tr>
<tr>
<td>Strings</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Indefinite can.systems / Motivation

<table>
<thead>
<tr>
<th>Concepts</th>
<th>Positive Definite</th>
<th>Indefinite</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hamiltonian</td>
<td></td>
<td></td>
</tr>
<tr>
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<td></td>
<td></td>
</tr>
<tr>
<td>Boundary triplet</td>
<td></td>
<td>B. triplet in Pontryagin space</td>
</tr>
<tr>
<td>Nevanlinna class $N_0$</td>
<td></td>
<td>generalized $N_{\kappa}$</td>
</tr>
</tbody>
</table>

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<table>
<thead>
<tr>
<th>I.S.T.</th>
<th>Positive Definite</th>
<th>Indefinite</th>
</tr>
</thead>
<tbody>
<tr>
<td>${\text{Hamiltonians}}$ $\leftrightarrow N_0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Positive def. functions</td>
<td>Hermitian indef. functions</td>
<td></td>
</tr>
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<td></td>
</tr>
<tr>
<td>Moment problems</td>
<td>indefinite moment problems</td>
<td></td>
</tr>
<tr>
<td>Strings</td>
<td>generalized strings</td>
<td></td>
</tr>
</tbody>
</table>
### Indefinite canonical systems / Motivation

<table>
<thead>
<tr>
<th>Concepts</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>POSITIVE DEFINITE</strong></td>
<td><strong>INDEFINITE</strong></td>
</tr>
<tr>
<td>Hamiltonian</td>
<td>general Hamiltonian</td>
</tr>
<tr>
<td>Matrix chain</td>
<td>maximal chain of matrices</td>
</tr>
<tr>
<td>Boundary triplet</td>
<td>B.triplet in Pontryagin space</td>
</tr>
<tr>
<td>Nevanlinna class ( \mathcal{N}_0 )</td>
<td>generalized Nev.class ( \mathcal{N}_\kappa )</td>
</tr>
<tr>
<td>{Hamiltonians} [\leftrightarrow] ( \mathcal{N}_0 )</td>
<td></td>
</tr>
<tr>
<td>Positive def.functions</td>
<td>Hermitian indef.functions</td>
</tr>
<tr>
<td>Bessel equation ( \alpha &lt; 1 )</td>
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</tr>
<tr>
<td>Moment problems</td>
<td>indefinite moment problems</td>
</tr>
<tr>
<td>Strings</td>
<td>generalized strings</td>
</tr>
</tbody>
</table>
### Indefinite can. systems / Motivation

#### Concepts

<table>
<thead>
<tr>
<th><strong>POSITIVE DEFINITE</strong></th>
<th><strong>INDEFINITE</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Hamiltonian</td>
<td>general Hamiltonian</td>
</tr>
<tr>
<td>Matrix chain</td>
<td>maximal chain of matrices</td>
</tr>
<tr>
<td>Boundary triplet</td>
<td>B. triplet in Pontryagin space</td>
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<td>generalized Nev. class $\mathcal{N}_\kappa$</td>
</tr>
</tbody>
</table>

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<table>
<thead>
<tr>
<th><strong>I.S.T.</strong></th>
<th><strong>CONCEPTS</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>{Hamiltonians} $\leftrightarrow \mathcal{N}_0$</td>
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</tr>
<tr>
<td>Positive def. functions</td>
<td>Hermitian indef. functions</td>
</tr>
<tr>
<td>Bessel equation $\alpha &lt; 1$</td>
<td>Bessel equation $\alpha \in \mathbb{R}^+$</td>
</tr>
<tr>
<td>Moment problems</td>
<td>indefinite moment problems</td>
</tr>
<tr>
<td>Strings</td>
<td>generalized strings</td>
</tr>
</tbody>
</table>
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<table>
<thead>
<tr>
<th>Concepts</th>
<th>POSITIVE DEFINITE</th>
<th>INDEFINITE</th>
</tr>
</thead>
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<tr>
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<td>general Hamiltonian</td>
<td></td>
</tr>
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<td></td>
</tr>
<tr>
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<td>B.triplet in Pontryagin space</td>
<td></td>
</tr>
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<td></td>
</tr>
</tbody>
</table>

### I.S.T. | \{Hamiltonians\} $\leftrightarrow \mathcal{N}_0$ | \{general Hamiltonians\} $\leftrightarrow \bigcup_{\kappa \geq 0} \mathcal{N}_\kappa$ |

### Examples

- Positive def.functions
- Bessel equation $\alpha < 1$
- Moment problems
- Strings

- Hermitian indef.functions
- Bessel equation $\alpha \in \mathbb{R}^+$
- indefinite moment problems
- generalized strings

Indefinite Canonical Systems – p.15/51
A general Hamiltonian consists of the data

\[ \sigma_0, \ldots, \sigma_{n+1} \in \mathbb{R} \cup \{ \pm \infty \}, \quad \sigma_0 < \sigma_1 < \ldots < \sigma_{n+1}, \]

\[ H_i : (\sigma_i, \sigma_{i+1}) \rightarrow \mathbb{R}^{2 \times 2}, \quad i = 0, \ldots, n, \]

\[ E \subseteq \bigcup_{i=0}^{n} (\sigma_i, \sigma_{i+1}) \cup \{ \sigma_0, \sigma_{n+1} \} \text{ finite} \]

\[ d_{i,0}, \ldots, d_{i,2 \Delta_i - 1} \in \mathbb{R}, \quad \ddot{\sigma}_i \in \mathbb{N}_0, b_{i,1}, \ldots, b_{i,\ddot{\sigma}_i+1} \in \mathbb{R} \]

subject to certain conditions.
General Hamiltonians

- $\sigma_0 = \text{starting point}$
- $\sigma_{n+1} = \text{endpoint}$
- $\sigma_1, \ldots, \sigma_n = \text{singularities}$
General Hamiltonians

- $H_0, \ldots, H_n = \text{Hamiltonians, not integrable at } \sigma_1, \ldots, \sigma_n (\sigma_1, \ldots, \sigma_n \text{ singularities})$
- $H_0$ integrable at 0 (initial value problem)
- $H_n$ integrable/not at $\sigma_{n+1}$ (limit circle/point case)
- Growth of $H_i$ towards singularity is restricted
General Hamiltonians

- $d_{i,j}$ = interface conditions at a singularity
- $E$ quantitative measurement of ‘local at a singularity’
General Hamiltonians

\[ H_0 \quad e_0 \quad e_1 \quad H_1 \quad e_2 \quad H_n \quad e_{m-1} \quad e_m \]

\[ \sigma_0 \quad \sigma_1 \quad \sigma_2 \quad \sigma_n \quad \sigma_{n+1} \]

\[ \ddot{o}_i, b_{i,j} = \text{contribution concentrated in the singularity} \]
Maximal chains of matrices

Axiomization of ‘fundamental solution’

- \( W_0 = I \) and \( W_t \in \bigcup_{\kappa \geq 0} \mathcal{M}_\kappa \) for \( t \in [\sigma_0, \sigma_1) \cup (\sigma_1, \sigma_2) \cup \ldots \cup (\sigma_n, \sigma_{n+1}) \)

- \( W_s^{-1}W_t \in \bigcup_{\kappa \geq 0} \mathcal{M}_\kappa \) and
  \[ \ind_- W_t = \ind_- W_s + \ind_- W_s^{-1}W_t \]

- If \( W \in \bigcup_{\kappa \geq 0} \mathcal{M}_\kappa \), \( W^{-1}W_t \in \bigcup_{\kappa \geq 0} \mathcal{M}_\kappa \) and
  \[ \ind_- W_t = \ind_- W + \ind_- W^{-1}W_t \]
  then \( W = W_t \) for some \( t \)

- some technical conditions
Theory of indefinite can.sys.ystems

General Hamiltonian $\hbar$

Maximal chain $(W_t)$

Model space $\mathcal{P}(\hbar)$

Weyl coefficient $q_\hbar(z)$
Theory of indefinite can. systems

Maximal chain \( (W_t) \)

\( \lim \) \( W_t \star \tau \)

Weyl coefficient \( q_H(z) \)

General Hamiltonian \( \mathcal{H} \)

Solution \( \rightarrow \)

Construction \( \rightarrow \)

Defect elements \( \rightarrow \)

Resolvent matrix \( \rightarrow \)

Maximal chain \( (W_t) \) \( \rightarrow \) Model space \( \mathcal{P}(\mathcal{H}) \)

\( \text{rks} \cong L^2(H_{[0,t]}) \)

\( L^2(\sigma) \cong L^2(H) \)
Theory of indefinite can. systems

Maximal chain \( (W_t) \)

general Hamiltonian \( h \)

Maximal chain \( (W_t) \)

Model space \( \mathcal{V}(h) \)

\text{solution off singularities}

\text{construction}

\text{defect elements}

\text{resolvent matrix}

\text{defect elements}

\text{resolvent matrix}

\text{lim} \; W_t \ast \tau

\Pi(\phi) \cong \mathcal{V}(h)

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\Pi(\phi) \cong \mathcal{V}(h)
The assignment \( h \mapsto q_h(z) \)
yields a bijection between the set of all general Hamiltonians (up to reparameterization) and \( \bigcup_{\kappa \geq 0} N_\kappa \).
In our examples we had obtained generalized Nevanlinna function which seemed to be the ‘Weyl coefficient of the underlying indefinite canonical system’, namely:

- \( q_f(z) = \frac{i}{z^2} - \frac{1}{z} \in \mathcal{N}_1 \) from the hermitian indefinite function \( f(t) = 1 - |t| \).

- \( q_{\alpha}(z) = c_{\alpha} z^{-\alpha} \in \mathcal{N}_{\kappa(\alpha)} \) where \( \kappa(\alpha) := \left[ \frac{\alpha+1}{2} \right] \), from the Bessel equation with parameter \( \alpha \in \mathbb{R}^+ \setminus \{1, 3, 5, \ldots\} \).
In our examples we had obtained generalized Nevanlinna function which seemed to be the ‘Weyl coefficient of the underlying indefinite canonical system’, namely:

- \( q_f(z) = \frac{i}{z^2} - \frac{1}{z} \in \mathcal{N}_1 \) from the hermitian indefinite function \( f(t) = 1 - |t| \).

- \( q_\alpha(z) = c_\alpha z^{-\alpha} \in \mathcal{N}_{\kappa(\alpha)} \) where \( \kappa(\alpha) := \left[ \frac{\alpha+1}{2} \right] \), from the Bessel equation with parameter \( \alpha \in \mathbb{R}^+ \setminus \{1, 3, 5, \ldots\} \).

In order to fit these examples, we have to find the general Hamiltonian whose Weyl coefficient is \( q_f \) or \( q_\alpha \), and see how it is related to the ‘Hamiltonians’ \( H_f \) and \( H_\alpha \).
A Short Account on the Literature
References (positive definite)

This lecture was based on:

- **M. KALTEMBÄCK, H. WORACEK**: *Pontryagin spaces of entire functions V*, manuscript in preparation.
References (indefinite)

- M.G.Krein, H.Langer: *On some extension problems which are closely connected with the theory of hermitian operators in a space \( \Pi_\kappa \). III. Indefinite analogues of the Hamburger and Stieltjes moment problems (Part 1,2)*, Beiträge zur Analysis 14 (1979), 25–40, Beiträge zur Analysis 15 (1981), 27–45.


• M. G. Krein, H. Langer: *Continuation of Hermitian positive definite functions and related questions*, unpublished manuscript.

References (Bessel equation)

- M. Langer, H. Woracek: *A Pontryagin space model for a canonical system connected with the Bessel equation*, manuscript in preparation.


• M.Kaltenbäck, H.Winkler, H.Woracek: *Diagonal general Hamiltonians. A full indefinite analogue of the theory of strings*, manuscript in preparation.

THE END
The matrix chain \((W_t)\)

Let \(H\) be a Hamiltonian defined on \([\sigma_0, \sigma_1]\). Then \(W_t, t \in [\sigma_0, \sigma_1]\), denotes the unique solution of the initial value problem

\[
\frac{d}{dt}W_t(z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = zW_t(z)H(t), \quad x \in [\sigma_0, \sigma_1],
\]

\[
W_0(z) = I.
\]
The Weyl coefficient $q_H(z)$

For $W = (w_{ij})_{i,j=1}^{2} \in \mathbb{C}^{2 \times 2}$ and $\tau \in \mathbb{C}$ denote

$$W \star \tau := \frac{w_{11}\tau + w_{12}}{w_{21}\tau + w_{22}}$$

The assignment $\tau \mapsto W \star \tau$ maps the upper half plane to some (general) disk:
The Weyl coefficient \( q_H(z) \)

Let \( (W_t)_{t \in [\sigma_0, \sigma_1]} \) be the matrix chain associated with the Hamiltonian \( H \). The assignments \( \tau \mapsto W_t \star \tau \) map \( \mathbb{C}^+ \) to a nested sequence of disks contained in \( \mathbb{C}^+ \). The disk \( W_t \star \mathbb{C}^+ \) is contained in the upper half plane and its radius is \( \left[ \int_{\sigma_0}^{t} \text{tr} H(x) \, dx \right]^{-1} \).
The Weyl coefficient \( q_H(z) \)

Let \( (W_t)_{t \in [\sigma_0, \sigma_1]} \) be the matrix chain associated with the Hamiltonian \( H \). The assignments \( \tau \mapsto W_t \ast \tau \) map \( \mathbb{C}^+ \) to a nested sequence of disks contained in \( \mathbb{C}^+ \). The disk \( W_t \ast \mathbb{C}^+ \) is contained in the upper half plane and its radius is \( \left[ \int_{\sigma_0}^{\sigma_1} \text{tr} \, H(x) \, dx \right]^{-1} \).

Thus the limit \( q_H(z) := \lim_{t \uparrow \sigma_1} W_t(z) \ast \tau \) exists, does not depend on \( \tau \in \mathbb{C}^+ \), and belongs to \( \mathbb{N}_0 \).
The model space $L^2(H)$

Supressing some technicalities which arise from ‘indivisible intervals’, we have

$$L^2(H) := \left\{ f : (\sigma_0, \sigma_1) \to \mathbb{C}^2 : \int_{\sigma_0}^{\sigma_1} f(t)^T H(t) f(t) \, dt < \infty \right\}$$

$$T_{\text{max}}(H) := \left\{ (f; g) \in L^2(H)^2 : f \text{ absolutely continuous}, \right. $$

$$\left. f(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} H(t) g(t), \text{ a.e.} \right\}$$

$$\Gamma(H)(f; g) := f(\sigma_0), (f; g) \in T_{\text{max}}(H)$$
Let \( y_1(z, x) = (y_1(z, x)_2, y_1(z, x)_2)^T \) and \( y_2(z, x) = (y_2(z, x)_2, y_2(z, x)_2)^T \) be the elements of \( \ker(T_{max}(H|_{(s_0,t)} - z)) \), such that \( y_1(z, s_0) = (1, 0)^T \) and \( y_2(z, s_0) = (0, 1)^T \). Then

\[
W_t(z) = \begin{pmatrix}
y_1(z, t)_1 & y_1(z, t)_2 \\
y_2(z, t)_1 & y_2(z, t)_2
\end{pmatrix}
\]
Consider

\[ S_1 := \left\{ (x; y) \in T_{max}(H|_{(\sigma_0,t)}) : \pi_{l,1} \Gamma(H|_{(\sigma_0,t)})(x; y) = 0, \pi_{r} \Gamma(H|_{(\sigma_0,t)})(x; y) = 0 \right\} \]

\[ u : (x; y) \mapsto \pi_{l,2} \Gamma(H|_{(\sigma_0,t)})(x; y), (x; y) \in T_{max}(H|_{(\sigma_0,t)}) \]

Then \( S_1 \) is symmetric with defect 1 and \( u|_{S_1^*} \) is continuous. The matrix function \( W_t \) is a \( u \)-resolvent matrix of \( S_1 \).
The reproducing kernel space of $W_t$

The kernel

$$K_{W_t}(w, z) := \frac{W_t(z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} W_t(w)^* - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}{z - \overline{w}}$$

is positive definite, thus generates a reproducing kernel Hilbert space $\mathcal{H}(W_t)$. The elements of $\mathcal{H}(W_t)$ are entire 2-vector-functions.

The operator $S(W_t)$ of multiplication by $z$ is a symmetry with defect 2. The map $\Gamma(W_t) : f \mapsto f(0)$ is a boundary map for $S(W_t)$. 
The reproducing kernel space of $W_t$

The boundary triplet
\[ \langle L^2(H|_{(\sigma_0,t)}), T_{\text{min}}(H|_{(\sigma_0,t)}), \Gamma(H|_{(\sigma_0,t)}) \rangle \]
is isomorphic to \[ \langle \mathcal{K}(W_t), S(W_t), \Gamma(W_t) \rangle \]. The isomorphism of
\[ L^2(H|_{(\sigma_0,t)}) \]
to \[ \mathcal{K}(W_t) \] is given by
\[ f(x) \mapsto \int_{\sigma_0}^{t} W_x(z) H(x) f(x) \, dx . \]
The Fourier transform

The Weyl coefficient admits an integral representation of the form

\[ q_H(z) = a + b z + \int_{\mathbb{R}} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\sigma(t). \]

Assuming \( b = 0 \), the map

\[ f(x) \mapsto \int_{\sigma_0}^{\sigma_1} (0, 1) W_x(z) H(x) f(x) \, dx \]

is an isomorphism of \( L^2(H) \) onto \( L^2(\sigma) \).
The class $\mathcal{N}_0$

Denote by $\mathcal{N}_0$ the set of all functions $\tau$, which are analytic in $\mathbb{C} \setminus \mathbb{R}$, satisfy $\tau(\overline{z}) = \overline{\tau(z)}$, and are such that the Nevanlinna kernel

$$Q_{\tau}(w, z) := \frac{\tau(z) - \overline{\tau(w)}}{z - \overline{w}}$$

is nonnegative definite. This means that each of the quadratic forms

$$q_{\tau}(\xi_1, \ldots, \xi_m) := \sum_{i,j=1}^{m} Q_f(z_j, z_i) \xi_i \overline{\xi_j}$$

is nonnegative definite.
The class $\mathcal{N}_0$

A more classical approach to this class of functions is given by the following result:

A function $\tau$ which is analytic in $\mathbb{C} \setminus \mathbb{R}$ and satisfies $\tau(\bar{z}) = \overline{\tau(z)}$ belongs to the class $\mathcal{N}_0$, if and only if it maps the open upper half plane into the closed upper half plane.
Hermitian indefinite functions

Let $a \in (0, \infty)$. A function $f : (-2a, 2a) \to \mathbb{C}$ is called hermitian indefinite, if $f(-t) = \overline{f(t)}$ and if the kernel

$$K_f(s, t) = f(t - s), \ s, t \in (-a, a),$$

has a finite number of negative squares. The set of all continuous hermitian indefinite functions with $\kappa$ negative squares on the interval $(-2a, 2a)$ is denoted by $\mathcal{P}_{\kappa,a}$. 
Continuation problem: Let $f \in \mathcal{P}_{\kappa_0,a}$. Do there exist continuations $\tilde{f} \in \mathcal{P}_{\kappa,\infty}$?

Clearly, for the existence of a continuation $\tilde{f} \in \mathcal{P}_{\kappa,\infty}$, it is necessary that $\kappa \geq \kappa_0$. 
Hermitian indefinite functions

Solution: There exists a number $\Delta(f) \in \mathbb{N} \cup \{0, \infty\}$:

- If $\Delta(f) = 0$, then $f$ has infinitely many continuations in each of the classes $\mathcal{P}_{\kappa,\infty}$, $\kappa \geq \kappa_0$.

- If $0 < \Delta(f) < \infty$, then $f$ has a unique continuation in $\mathcal{P}_{\kappa_0,\infty}$, no continuations in $\mathcal{P}_{\kappa,\infty}$ with $\kappa_0 < \kappa < \kappa_0 + \Delta(f)$, and infinitely many continuations in each of the classes $\mathcal{P}_{\kappa,\infty}$, $\kappa \geq \kappa_0 + \Delta(f)$.

- If $\Delta(f) = \infty$, then $f$ has a unique continuation in $\mathcal{P}_{\kappa_0,\infty}$, and no continuations in any of the classes $\mathcal{P}_{\kappa,\infty}$, $\kappa > \kappa_0$. 
Assume that \( \Delta(f) < \infty \). Then there exists an entire \( 2 \times 2 \)-matrix function \( W_f \) such that the formula

\[
i \int_0^\infty e^{itz} \tilde{f}(t) \, dt = W_f(z) \ast \tau(z)
\]

parameterizes the continuations of \( f \) in \( \bigcup_{\kappa \geq \kappa_0} \mathcal{P}_{\kappa,\infty} \). Thereby continuations \( \tilde{f} \in \mathcal{P}_{\kappa,\infty} \) correspond to parameters \( \tau \) in the class \( \mathcal{K}_{\kappa_0}^{\Delta(f)} \). If \( \Delta(f) > 0 \), the unique solution in \( \mathcal{P}_{\kappa_0,\infty} \) is given by the parameter \( \tau = \infty \).
An example: The function $f(t) := 1 - |t|$ belongs to $\mathcal{P}_{1,\infty}$. Again consider the restrictions $f|_{(-2t,2t)}$. Then

$$\Delta(f|_{(-2t,2t)}) = \begin{cases} 
0 & , \ 0 < t < 1 \text{ or } t > 1 \\
1 & , \ t = 1 
\end{cases}$$

$f|_{(-2t,2t)} \in \mathcal{P}_{0,t}, 0 < t < 1, f|_{(-2t,2t)} \in \mathcal{P}_{1,t}, t > 1$, and

$$W_{f|_{(-2t,2t)}}(z) = \begin{pmatrix} 
\frac{\sin tz - z \cos tz}{(t-1)z} & \left(\frac{1}{z^2} - (t - 1)\right) \sin t z - \frac{t \cos tz}{z} \\
\frac{z \cos tz}{t-1} & (t - 1)(z \sin t z + \cos tz)
\end{pmatrix}$$
The family

\[ W_t(z) := \begin{cases} 
\begin{pmatrix} 1 & 0 \\ -(1 + t)z & 1 \end{pmatrix}, & t \in [-1, 0] \\
W_{f|(-2t,2t)}(z), & t \in (0, 1) \cup (1, \infty) 
\end{cases} \]

satisfies a differential equation of the form of a canonical system with

\[ H_f(t) = \begin{cases} 
\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & t \in (-1, 0) \\
\begin{pmatrix} (t - 1)^2 & 0 \\ 0 & \frac{1}{(t-1)^2} \end{pmatrix}, & t \in (0, 1) \cup (1, \infty) 
\end{cases} \]
The function $H_f$ is locally integrable on $[-1, 1) \cup (1, \infty)$, but NOT at the point 1. Moreover, 
\[
\int_T^\infty \operatorname{tr} H_f(x) \, dx = +\infty \text{ for } T > 1, \text{ i.e. the ‘limit point case’ prevails at infinity.}
\]
If we formally carry out the construction of the Weyl coefficient, we obtain

\[
q_{H_f}(z) = \frac{i}{z^2} - \frac{1}{z}, \quad z \in \mathbb{C}^+
\]

This function belongs to $\mathcal{N}_1$. 
Bessel equation \((\alpha \geq 1)\)

The function \(H_\alpha(t)\) and the matrices \(W_{\alpha,t}(z)\) are well-defined for \(\alpha \in \mathbb{R}^+ \setminus \{1, 3, 5, \ldots \}\). Moreover, \(W_{\alpha,t}\) satisfies a differential equation of the form of a canonical system with \(H_\alpha\). The function \(H_\alpha\) is locally integrable on \((0, \infty)\), but NOT at the point 0. Moreover,

\[
\int_T^\infty \text{tr } H_\alpha(x) \, dx = +\infty \text{ for } T > 0, \text{ i.e. the ‘limit point case’ prevails at infinity.}
\]

If we formally carry out the construction of the Weyl coefficient, we obtain

\[
q_{H_\alpha}(z) = c_\alpha z^{-\alpha}, \quad z \in \mathbb{C}^+
\]

This function belongs to \(\mathcal{N}_{\kappa(\alpha)}\) with \(\kappa(\alpha) := \left[\frac{\alpha + 1}{2}\right]\).
The class $\mathcal{K}_\nu^\Delta$

For $\nu, \Delta \in \mathbb{N}_0$, denote by $\mathcal{K}_{\nu}^{\Delta}$ the set of all functions $\tau$, which are meromorphic in $\mathbb{C} \setminus \mathbb{R}$, satisfy $\tau(\overline{z}) = \overline{\tau(z)}$, and are such that the maximal number of negative squares of quadratic forms

$$q_\tau\left(\xi_1, \ldots, \xi_m; \eta_0, \ldots, \eta_{\Delta-1}\right) := \sum_{i,j=1}^{m} \frac{\tau(z_i) - \tau(z_j)}{z_i - \overline{z_j}} \xi_i \overline{\xi_j} + \sum_{k=0}^{\Delta-1} \sum_{i=1}^{m} \text{Re}(z_i^k \xi_i \overline{\eta_k})$$

is $\nu$. Note that $\mathcal{K}_{\nu}^{\Delta} = \mathcal{N}_{\nu}$. 
The class $\mathcal{N}_\kappa$

For $\kappa \in \mathbb{N}_0$, denote by $\mathcal{N}_\kappa$ the set of all functions $\tau$, which are meromorphic in $\mathbb{C} \setminus \mathbb{R}$, satisfy $\tau(\bar{z}) = \tau(z)$, and are such that the Nevanlinna kernel

$$Q_f(w, z) := \frac{\tau(z) - \tau(w)}{z - \bar{w}}$$

has $\kappa$ negative squares. This means that the maximal number of negative squares of quadratic forms

$$q_\tau(\xi_1, \ldots, \xi_m) = \sum_{i, j=1}^{m} Q_f(z_j, z_i) \xi_i \overline{\xi_j}$$

is equal to $\kappa$. 
The Weyl coefficient \( q_{\hbar} (z) \)

The limit

\[
q_{\hbar} (z) := \lim_{t \to \sigma_n} W_t(z) \ast \tau
\]

exists as a meromorphic function locally uniformly on \( \mathbb{C} \setminus \mathbb{R} \) and does not depend on \( \tau \in \mathbb{C}^+ \).
The maximal chain \((W_t)\)

The matrix function \(W_t, t \in \bigcup_{i=1}^{n} (\sigma_{i-1}, \sigma_i)\), is for every \(i = 1, \ldots, n\) a solution of the differential equation

\[
\frac{d}{dt} W_t(z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = z W_t(z) H_i(t), \ x \in (\sigma_{i-1}, \sigma_i)
\]

On the interval \([\sigma_0, \sigma_1]\) it is uniquely determined by its initial value \(W_{\sigma_0} = I\).
The reproducing kernel space of $W_t$

The kernel

$$K_{W_t}(w, z) := \frac{W_t(z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} W_t(w)^* - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}{z - \overline{w}}$$

has a finite number of negative squares, thus generates a reproducing kernel Pontryagin space $\mathcal{K}(W_t)$. The elements of $\mathcal{K}(W_t)$ are entire 2-vector-functions.

The operator $S(W_t)$ of multiplication by $z$ is a symmetry with defect 2. The map $\Gamma(W_t) : f \mapsto f(0)$ is a boundary map for $S(W_t)$. 
The reproducing kernel space of $W_t$

There exists an isomorphism $\Phi_t$ of the boundary triplets $\langle \mathcal{K}(W_t), \mathcal{S}(W_t), \Gamma(W_t) \rangle$ and $\langle \mathcal{P}^2(h|_{(\sigma_0,t)}), \mathcal{S}(h|_{(\sigma_0,t)}), \Gamma(h|_{(\sigma_0,t)}) \rangle$

If $J := [s_-, s_+] \subseteq (\sigma_{i-1}, \sigma_i)$, then the map

$$\lambda_J : f(x) \mapsto \int_{s_-}^{s_+} W_x(z) H(x) f(x) \, dx$$

is an isomorphism of $L^2(H_i|[s_-,s_+])$ onto $\mathcal{K}(W_{s_+})[\vdash] \mathcal{K}(W_{s_-})$. We have

$$L^2(H_i|J) \xrightarrow{\iota_J} \mathcal{P}(h|_{(\sigma_0,s_+)})$$

$$\lambda_J \downarrow \Phi_{s_+}$$

$$\mathcal{K}(W_{s_+})[\vdash] \mathcal{K}(W_{s_-})$$
The Weyl coefficient admits a representation of the form

\[ q_{\eta}(z) = \phi \left( \frac{1}{t - z} \right) \]

with some distribution on \( \mathbb{R} \). This distribution generates a Pontryagin space \( \Pi(\phi) \). There exists an isomorphism of \( \mathcal{P}(\mathfrak{h}) \) onto \( \Pi(\phi) \).
For $t \in I$ consider

$$S_1 := \{(x; y) \in T(h|_{(\sigma_0,t)}) : \quad \pi_{l,1} \Gamma(h|_{(\sigma_0,t)})(x; y) = 0, \pi_{r} \Gamma(h|_{(\sigma_0,t)})(x; y) = 0\}$$

$$u : (x; y) \mapsto \pi_{l,2} \Gamma(h|_{(\sigma_0,t)})(x; y), \quad (x; y) \in T(h|_{(\sigma_0,t)})$$

Then $S_1$ is symmetric with defect 1 and $u|_{S_1^*}$ is continuous. The matrix function $W_t$ is a $u$-resolvent matrix of $S_1$. 
$W_t$ from defect elements

Let $\phi_z, \psi_z \in \ker(T(h|_{(\sigma_0,t)}))$ be such that

$$
\pi_l \Gamma(h|_{(\sigma_0,t)})(\phi_z; z\phi_z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \pi_l \Gamma(h|_{(\sigma_0,t)})(\psi_z; z\psi_z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
$$

Then

$$
W_t(z) = \begin{pmatrix} 
\pi_r \Gamma(h|_{(\sigma_0,t)})(\phi_z; z\phi_z)^T \\
\pi_r \Gamma(h|_{(\sigma_0,t)})(\psi_z; z\psi_z)^T 
\end{pmatrix}
$$
The model space $\mathcal{H}(\mathfrak{h})$

Given a general Hamiltonian $\mathfrak{h}$ we construct an operator model, which is a Pontryagin space boundary triplet

$$\langle \mathcal{H}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h}) \rangle$$

The actual construction is quite involved and too complicated to be elaborated here.
The model space $\mathcal{P}(\mathfrak{h})$

If $J = [s_-, s_+] \subseteq (\sigma_i, \sigma_{i+1})$, there exists an isometric and homeomorphic embedding

$$\iota_J : L^2(H_i | J) \rightarrow \mathcal{P}(\mathfrak{h})$$

If $J \subseteq J'$, then

$$L^2(H_i | J) \xrightarrow{\iota_J} \mathcal{P}(\mathfrak{h})$$

$$L^2(H_i | J) \xleftarrow{\iota_{J'}} \mathcal{P}(\mathfrak{h})$$
Hamiltonian for $q_f$

The general Hamiltonian made up of the data

$$\sigma_0 = -1, \sigma_1 = 1, \sigma_2 = +\infty, \quad E = \{-1, 0, 2, +\infty\}$$

$$H_0(t) = \begin{cases} 
\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & t \in (-1, 0) \\
\begin{pmatrix} (t-1)^2 & 0 \\ 0 & (t-1)^{-2} \end{pmatrix}, & t \in (0, 1) 
\end{cases}$$

$$H_1(t) = \begin{pmatrix} (t-1)^2 & 0 \\ 0 & (t-1)^{-2} \end{pmatrix}$$

$$\ddot{\sigma}_1 = 1, b_{1,1} = 2, b_{1,2} = 0, \quad d_0 = -2, d_1 = 0$$

has Weyl coefficient $q_f$. 

Indefinite Canonical Systems – p.49/51
Hamiltonian for $q_\alpha$

The general Hamiltonian made up of the data

$$\sigma_0 = -1, \sigma_1 = 0, \sigma_2 = +\infty, \quad E = \{-1, 1, +\infty\}$$

$$H_0(t) = \frac{1}{t^2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad H_1(t) = \begin{pmatrix} t^{\alpha} & 0 \\ 0 & t^{-\alpha} \end{pmatrix}$$

$$\ddot{o}_1 = 0, \quad d_0 = \frac{1}{\alpha - 1}, \quad d_1 = 0$$

has Weyl coefficient $q_\alpha$. 
The class $\mathcal{M}_\kappa$

$W \in \mathcal{M}_\kappa$ if

- $W$ is entire $2 \times 2$-matrix function
- $W(0) = I$
- The kernel

$$K_W(w, z) := \frac{W(z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} W(w)^* - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}{z - \overline{w}}$$

has $\kappa$ negative squares