Direct and Inverse Spectral Theorems for a Class of Canonical Systems with two Singular Endpoints. Part II: Applications to Sturm–Liouville Equations

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Abstract: In Part I of this paper we established direct and inverse spectral theorems for a certain class of canonical systems with two singular endpoints.

In this second part we apply the results of Part I to Sturm–Liouville equations with singular coefficients. We investigate classes of equations without potential (in particular, equations in impedance form) and one-dimensional Schrödinger equations, where coefficients are assumed to be singular at both endpoints but subject to a growth restriction at the left endpoint. In the case of Schrödinger equations our class of potentials includes Bessel-type potentials and also highly oscillatory potentials. For both classes of equations we construct singular Titchmarsh–Weyl coefficients, spectral measures and Fourier transforms. Moreover, we prove global and local inverse spectral theorems.

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1 Introduction

The Titchmarsh–Weyl coefficient is a central object in the theory of Sturm–Liouville equations with a regular left endpoint. Its construction goes back to H. Weyl, and it can be used to obtain a spectral measure and a generalized Fourier transform. Moreover, many inverse spectral theorems have been proved, in particular the Borg–Marchenko uniqueness result and local versions of it.

During the last decade, Sturm–Liouville equations, and in particular one-dimensional Schrödinger equations, where both endpoints are singular have attracted a lot of attention. In general, one cannot define a scalar Titchmarsh–Weyl coefficient, and the spectral multiplicity may be two. C. Fulton [14] (and later C. Fulton and H. Langer [15]) studied Schrödinger equations where the potential admits a Laurent series expansion around the singular left endpoint, which is a pole of order 2. Using the Frobenius method he could fix two linearly independent solutions in order to define a singular Titchmarsh–Weyl coefficient. The latter can be used to obtain a spectral measure and construct a Fourier transform. F. Gesztesy and M. Zinchenko [18] considered a large class of one-dimensional Schrödinger equations for which they proved that the spectrum is simple, and they constructed a spectral measure via a generalized Titchmarsh–Weyl coefficient, which is not unique, however. Their assumption was that there exists a solution that depends analytically on the spectral parameter and which is in $L^2$ at the left endpoint. Many other papers appeared in the last decade where various classes of Sturm–Liouville equations were studied; let us just mention the following papers: [16], [23], [26], [25], [24], [27], [43].
The aim of the current paper is to consider large classes of Sturm–Liouville equations where a singular Titchmarsh–Weyl coefficient can be defined, which can be used to obtain a spectral measure. The Titchmarsh–Weyl coefficient is unique up to an additive real polynomial, and hence the spectral measure we obtain with our method is unique — in contrast to many other papers. This allows us to prove inverse spectral uniqueness theorems in terms of the Titchmarsh–Weyl coefficient or the spectral measure. Moreover, we can relate the growth of the spectral measure at infinity to the growth of the coefficients at the left endpoint.

We consider two classes of Sturm–Liouville equations in detail: first, equations without potential, i.e.

$$-(p y')' = \lambda w y$$

(1.1)

with $p(x), w(x) > 0$ a.e., $1/p, w$ locally integrable and either $1/p$ or $w$ integrable at $a$. Such equations, which are treated in Sections 2 and 3, have many applications (see, e.g. [7] and [40]) and include equations in impedance form, i.e. equations where $p = w$; see, e.g. [1]. Our classes of potentials include functions with an asymptotic behaviour $p(x) \asymp (x - a)^\alpha$, $w(x) \asymp (x - a)^\beta$ as $x \searrow a$ with $\alpha \geq 1$ or $\alpha \leq -1$. As a second class of equations we consider one-dimensional Schrödinger equations, i.e.

$$-y'' + Vy = \lambda y$$

(1.2)

with locally integrable $V$. The class of potentials includes Bessel-type potentials of the form

$$V(x) = \frac{l(l + 1)}{x^2} + V_0(x)$$

with $l > -\frac{1}{2}$ and $V_0$ such that $xV_0(x)$ is integrable at 0, which appear as radial parts of Schrödinger equations in $\mathbb{R}^3$ with spherically symmetric potentials. However, we allow also potentials that have a stronger singularity and are highly oscillatory; see Example 4.5.

We establish the following main results for the two classes of Sturm–Liouville equations mentioned above: we construct singular Titchmarsh–Weyl coefficients (Theorems 2.7 and 4.7) and spectral measures (Theorem 2.11 and 4.8) using regularized boundary values at the left endpoint; we show that the spectral multiplicity is one, and we construct a Fourier transform (Theorems 2.13 and 4.10); we prove global and local inverse spectral theorems (Theorems 2.15, 2.17, 3.3, 4.12 and 4.13).

For the proofs of these results we use the theory of $2 \times 2$ canonical systems of the form $y'(x) = zJH(x)y(x)$ that was developed in the first part [37] of the paper, together with the theory of generalized Nevanlinna functions, in particular, that developed in [36].

**Organization of the manuscript.**

The paper is divided into sections according to the following table.

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First we consider equations of the form (1.1). The case when $1/p$ is not integrable at $a$ is considered in Section 2; the case when $w$ is not integrable at $a$ is studied in Section 3. Finally, Schrödinger equations of the form (1.2) are investigated in Section 4.

2 Sturm–Liouville equations without potential: singular $1/p$

In this section we consider Sturm–Liouville equations of the form

$$-(py)' = \lambda wy$$

(2.1)

on an interval $(a, b)$ with $-\infty \leq a < b \leq \infty$ where $\lambda \in \mathbb{C}$ and the functions $p$ and $w$ satisfy the conditions

$$p(x) > 0, \ w(x) > 0 \ \text{a.e.,} \quad \frac{1}{p}, w \in L^1_{\text{loc}}(a, b).$$

(2.2)

In the following we write $\text{dom}(p; w) := (a, b)$. Moreover, let $L^2(w)$ be the weighted $L^2$-space with inner product $(f, g) = \int_a^b fgw$.

We consider the following class of coefficients.

2.1 Definition. We say that $(p; w) \in \mathbb{K}_{\text{SL}}$ if $p$ and $w$ are defined on some interval $(a, b)$ and they satisfy (2.2) and the following conditions.

(i) For one (and hence for all) $x_0 \in (a, b)$,

$$\int_a^{x_0} \frac{1}{p(x)} \, dx = \infty \quad \text{and} \quad \int_a^{x_0} w(x) \, dx < \infty.$$  

(2.3)

(ii) For one (and hence for all) $x_0 \in (a, b)$,

$$\int_a^{x_0} \int_a^x w(t) \, dt \, \frac{1}{p(x)} \, dx < \infty.$$  

(2.4)

(iii) Let $x_0 \in (a, b)$ and define functions $w_l$, $l = 0, 1, \ldots$, recursively by

$$w_0(x) = 1,$$

$$w_l(x) = \begin{cases} \int_x^{x_0} \frac{1}{p(t)} w_{l-1}(t) \, dt & \text{if } l \text{ is odd,} \\ \int_a^x w(t) w_{l-1}(t) \, dt & \text{if } l \text{ is even.} \end{cases}$$  

(2.5)

There exists an $n \in \mathbb{N}_0$ such that

$$w_n |_{(a, x_0)} \in \begin{cases} L^2(w |_{(a, x_0)}) & \text{if } n \text{ is even,} \\ L^2(w |_{(a, x_0)}) & \text{if } n \text{ is odd.} \end{cases}$$  

(2.6)
Equation (2.1) is in the limit point case at \( b \), i.e. for \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), equation (2.1) has (up to a scalar multiple) only one solution in \( L^2(w|(a,b)) \) for \( x_0 \in (a,b) \).

If \( (p;w) \in \mathcal{K}_{\text{SL}} \), we denote by \( \Delta_{\text{SL}}(p,w) \) the minimal \( n \in \mathbb{N}_0 \) such that (2.6) holds.

2.2 Remark.

(i) Under the assumption of (2.3), condition (2.4) is equivalent to

\[
\int_a^{x_0} \int_x^{x_0} \frac{1}{p(t)} \, dt \, w(x) \, dx < \infty;
\]

see, e.g. [35, Lemma 4.3].

(ii) Assume that (2.3) holds. Then (2.4) and (2.6) with \( n = 1 \) are satisfied if and only if equation (2.1) is in the limit circle case at \( a \); this is true because the solutions of (2.1) with \( \lambda = 0 \) are \( y(x) = c_1 w_1(x) + c_2 \) with \( c_1, c_2 \in \mathbb{C} \) and the limit circle case prevails at \( a \) if and only if all these solutions are in \( L^2(w|_{(a,x_0)}) \).

(iii) The functions \( w_0 \) and \( w_1 \) are solutions of (2.1) with \( \lambda = 0 \). Since \( w_1(x) \to \infty \) as \( x \searrow a \), the function \( w_0 \) is a principal solution and \( w_1 \) is a non-principal solution, i.e. \( w_0(x) = o(w_1(x)) \) as \( x \searrow 0 \); for the notions of principal and non-principal solutions see, e.g. [41]. Moreover, one can easily verify that

\[
-w(pw_1')' = w \quad \text{when} \ l \in \mathbb{N} \text{ is odd.}
\]

For many proofs we use results on canonical systems from the first part, [37], of the paper; these canonical systems are of the form

\[
y'(x) = JH(x)y(x), \quad x \in (a,b),
\]

where \( y \) is a 2-vector-valued function, \( z \in \mathbb{C}, \ J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), and the Hamiltonian \( H \) is a function whose values are real non-negative \( 2 \times 2 \)-matrices.

For given \( p \) and \( w \) satisfying (2.2) define the Hamiltonian

\[
H(x) := \begin{pmatrix} \frac{1}{p(x)} & 0 \\ 0 & w(x) \end{pmatrix}, \quad x \in (a,b).
\]

If \( \psi = (\psi_1,\psi_2)^T \) is a solution of equation (2.7) with \( H \) as in (2.8), then

\[
\psi'_1 = -zw\psi_2, \quad \psi'_2 = z\frac{1}{p}\psi_1,
\]
and hence $\psi_2$ is a solutions of (2.1) with $\lambda = z^2$. Conversely, if $\psi$ is a solution of (2.1) and $z \in \mathbb{C}$ is such that $z^2 = \lambda$, then

$$
\psi(x) = \begin{pmatrix} p(x)\psi'(x) \\ z\psi(x) \end{pmatrix} 
$$

satisfies (2.7) with $H$ as in (2.8).

In the following assume that $(p; w) \in \mathcal{K}_{SL}$. The first relation in (2.3) implies that $H$ is in the limit point case at $a$. Since (2.1) is in the limit point case at $b$, the Hamiltonian $H$ is also in the limit point case at $b$ because $\psi \in L^2(H|_{(x_0,b)})$ with $\psi$ as in (2.9), $z \neq 0$ and $x_0 \in (a,b)$ implies that $\psi \in L^2(w|_{(x_0,b)})$. Therefore the operator $T(H)$, which is defined in [37, (3.8)] and acts in the space $L^2(H) = L^2(\frac{1}{p}) \oplus L^2(w)$, is self-adjoint. Since $H(x)$ is invertible for a.e. $x \in (a,b)$, the operator $T(H)$ can be written as

$$
T(H)f = H^{-1}J^{-1}f' = \begin{pmatrix} pf_2' \\ -\frac{1}{w}f_1' \end{pmatrix}
$$

with maximal domain

$$
\text{dom}(T(H)) = \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : f_1, f_2 \text{ abs. cont.}, \frac{1}{p}f_2' \in L^2(\frac{1}{p}), \frac{1}{w}f_1' \in L^2(w) \right\}
$$

Hence $(T(H))^2$ acts as follows

$$
(T(H))^2 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -pf_2' \\ -\frac{1}{w}(pf_2')' \end{pmatrix}. 
$$

(2.10)

With the mappings

$$
\iota_2 : \begin{cases} L^2(w) \rightarrow L^2(H) \\ g \mapsto \begin{pmatrix} 0 \\ g \end{pmatrix}, \end{cases} \quad P_2 : \begin{cases} L^2(H) \rightarrow L^2(w) \\ \begin{pmatrix} f \\ g \end{pmatrix} \mapsto g, \end{cases}
$$

(2.11)

we define the self-adjoint operator

$$
A_{p,w} := P_2(T(H))^2\iota_2. 
$$

(2.12)

This operator acts like

$$
A_{p,w}y = -\frac{1}{w}(py')' 
$$

$$
\text{dom}(A_{p,w}) = \left\{ y \in L^2(w) : y, py' \text{ locally absolutely continuous}, \int_a^b p(x)|y'(x)|^2dx < \infty, \frac{1}{w}(py')' \in L^2(w) \right\}
$$

(2.13)

and is the Friedrichs extension of the minimal operator associated with (2.1) since all functions in $\text{dom}(A_{p,w})$ are in the form domain; note that $A_{p,w}$ is
non-negative. If \((2.1)\) is also in the limit point case at \(a\) (that is, when
\(\Delta_{SL}(p, w) \geq 2\)), then \(A_{p, w}\) coincides with the maximal operator, i.e. the condition
\(\int_a^b p|y'|^2 < \infty\) is automatically satisfied. If \((2.1)\) is in the limit circle case
at \(a\), one can replace the condition \(\int_a^b p|y'|^2 < \infty\) in \((2.13)\) by any of the two
boundary conditions
\[
\lim_{x \searrow a} \frac{y(x)}{w_1(x)} = 0, \quad \lim_{x \searrow a} p(x)y'(x) = 0; \tag{2.14}
\]
see, e.g. \([41, \text{Theorem 4.3}]\).

2.3 Remark. One can also treat the situation when \((2.1)\) is either regular or in
the limit circle case at \(b\). In the former case one extends \(H\) by an indivisible
interval of infinite length; in the latter case \(H\) is in the limit point case. In both
cases elements in the domain of \(A_{p, w}\) defined via \((2.12)\) satisfy some boundary
condition of \(b\).

\[
\begin{align*}
\text{Assume that } (p; w) &\in \mathbb{K}_{SL} \text{ and let } H \text{ be as in } (2.8). \text{ It follows from } [47, \text{Theorem 3.7}] \text{ that } \Delta_{SL}(p, w) = \Delta(H). \tag{2.16} \\
\text{Therefore we can apply the results from } [37] \text{ to the Hamiltonian } H. \text{ Using } \\
\text{the connection between } (2.1) \text{ and } (2.7) \text{ we can show that regularized boundary } \\
\text{values of solutions of } (2.1) \text{ exist at } a. \text{ Denote by } \mathcal{N}_{SL}^\lambda \text{ the set of all solutions of } \\
\text{the Sturm–Liouville equation } (2.1). \\
\end{align*}
\]

2.4 Theorem (Regularized boundary values). Let \((p; w) \in \mathbb{K}_{SL}\) with
\(\text{dom}(p; w) = (a, b)\) and set \(\Delta := \Delta_{SL}(p, w)\). Then, for \(x_0 \in (a, b)\), the following
statements hold.

(i) For each \(\lambda \in \mathbb{C}\) and each solution \(\psi \in \mathcal{N}_{SL}^\lambda\) the boundary value
\[
\text{rbv}_{\lambda, 1}^\psi := \lim_{x \searrow a} p(x)\psi'(x) \tag{2.17}
\]
and the regularized boundary value
\[
\text{rbv}_{\lambda, 2}^\psi := \lim_{x \searrow a} \left[ \lambda^{\frac{\Delta - 1}{2}} \sum_{k=0}^{\Delta - 1} \lambda^k \left( w_{2k}(x)\psi(x) + w_{2k+1}(x)p(x)\psi'(x) \right) \right. \\
+ \left\{ \begin{array}{ll} \lambda^\frac{\Delta}{2} w_{\Delta}(x)\psi(x) & \text{if } \Delta \text{ is even} \\
0 & \text{if } \Delta \text{ is odd} \end{array} \right\} \\
+ \left( \lim_{x \searrow a} p(t)\psi'(t) \right) \sum_{k=0}^{2\Delta - \Delta} \sum_{l=0}^{2k-\Delta} (-1)^l \lambda^k w_l(t) w_{2k-l+1}(t) \right] \\
\text{exist.}
\]
(ii) For each \( \lambda \in \mathbb{C} \) we define
\[
\text{rbv}_{\lambda}^{SL} : \{ N_{\lambda}^{SL} \rightarrow \mathbb{C}^2 \psi \mapsto (\text{rbv}_{\lambda,1}^{SL} \psi, \text{rbv}_{\lambda,2}^{SL} \psi)^T.}
\]
Then \( \text{rbv}_{\lambda}^{SL} \) is a bijection from \( N_{\lambda}^{SL} \) onto \( \mathbb{C}^2 \).

(iii) For each \( \lambda \in \mathbb{C} \) there exists an (up to scalar multiples) unique solution \( \psi \in N_{\lambda}^{SL} \setminus \{0\} \) such that \( \lim_{x \searrow a} \psi(x) \) exists.

This solution is characterized by the property that \( \int_a^{x_0} p|\psi'|^2 < \infty \) and also by the property that \( \text{rbv}_{\lambda,1}^{SL} \psi = 0 \) (and \( \psi \not\equiv 0 \)).

If \( \psi \) is a solution such that \( \lim_{x \searrow a} \psi(x) \) exists, then \( \text{rbv}_{\lambda,2}^{SL} \psi = \lim_{x \searrow a} \psi(x) \).

The regularized boundary value \( \text{rbv}_{\lambda,2}^{SL} \) depends on the choice of \( x_0 \) in the following way.

(iv) Let \( x_0, \hat{x}_0 \in (a, b) \), and let \( \text{rbv}_{\lambda}^{SL} \) and \( \text{\hat{r}bv}_{\lambda}^{SL} \) be the correspondingly defined regularized boundary value mappings. Then there exists a polynomial \( p_{x_0, \hat{x}_0} \) with real coefficients whose degree does not exceed \( \Delta - 1 \) such that
\[
\text{\hat{r}bv}_{\lambda,2}^{SL} \psi = \text{rbv}_{\lambda,2}^{SL} \psi + p_{x_0, \hat{x}_0}(\lambda) \text{rbv}_{\lambda,1}^{SL} \psi, \quad \psi \in N_{\lambda}^{SL}, \ \lambda \in \mathbb{C}.
\]

Moreover, clearly, \( \text{\hat{r}bv}_{\lambda,1}^{SL} = \text{rbv}_{\lambda,1}^{SL} \).

2.5 Remark.

(i) Let
\[
W_p(y_1, y_2)(x) := p(x)(y_1(x)y_2'(x) - y_1'(x)y_2(x))
\]
be the weighted Wronskian with weight \( p \). Using (2.5) we can rewrite the expression that appears within the round brackets in (2.18) as follows:
\[
w_{2k}\psi + w_{2k+1}\psi' = W_p(w_{2k+1}, \psi).
\]

(ii) When \( \Delta_{SL}(p, w) = 1 \), then \( \text{rbv}_{\lambda,2}^{SL} \psi \) does not depend on \( \lambda \) explicitly, and it takes the form
\[
\text{rbv}_{\lambda,2}^{SL} \psi = \lim_{x \searrow a} \left( \psi(x) + p(x)w_1(x)\psi'(x) \right) = \lim_{x \searrow a} W_p(w_1, \psi)(x).
\]

Note also that \( \text{rbv}_{\lambda,1}^{SL} \psi = \lim_{x \searrow a} W_p(1, \psi)(x) \).

(iii) Instead of the functions \( w_l \) one can use functions \( \hat{w}_l \) that are defined by the recurrence relation \( \hat{w}_0 \equiv 1 \) and
\[
\hat{w}_l(x) = \begin{cases} 
\int_{x_0}^{x} \frac{1}{p(t)} \hat{w}_{l-1}(t) dt + c_l & \text{if } l \text{ is odd}, \\
\int_{x_0}^{x} w(t) \hat{w}_{l-1}(t) dt & \text{if } l \text{ is even},
\end{cases}
\]
with arbitrary real numbers $c_l$ for odd $l$. To add the extra constants $c_l$ is useful for practical calculations, in particular, when $w_l$ has an asymptotic expansion (for $x \searrow a$) in which a constant term can be removed by adjusting $c_l$. One can show that the corresponding regularized boundary value $\text{rbv}_{\lambda,2}^{SL}$ satisfies

$$\text{rbv}_{\lambda,2}^{SL} \psi = \text{rbv}_{\lambda,2}^{SL} \psi + \hat{p}(\lambda) \text{rbv}_{\lambda,1}^{SL} \psi$$

with

$$\hat{p}(\lambda) = \sum_{k=0}^{\Delta-1} \lambda^k \sum_{i=0}^{k} c_{2k+1-2i} \lim_{t \searrow a} v_{2i}(t)$$

(2.20)

where $v_0 \equiv 1$ and

$$v_l(x) = \int_{x}^{x_0} \int_{a}^{t} w(t)v_{l-2}(t)dt \frac{1}{p(x)} dx, \quad l \text{ even}.$$ 

The limit $\lim_{t \searrow a} v_{2i}(t)$ exists because of condition (2.4).

Proof. Let $H$ be as in (2.8) and set $\Delta := \Delta(H)$. The relations in (2.22) are clear from the considerations around equation (2.9) and the fact that (2.1) does

Before we prove Theorem 2.4, we show the following lemma, where $\mathfrak{N}_z$ denotes the set of all solutions of (2.7); see [37, §4.1].

2.6 Lemma. Let $z \in \mathbb{C}$ and let $\psi$ be a solution of (2.1) with $\lambda := z^2$. Moreover, set

$$\psi(x) := \begin{pmatrix} p(x)\psi'(x) \\ z\psi(x) \end{pmatrix}, \quad \hat{\psi}(x) := \begin{pmatrix} p(x)\psi'(x) \\ -z\psi(x) \end{pmatrix}.$$ 

(2.21)

Then

$$\psi \in \mathfrak{N}_z, \quad \hat{\psi} \in \mathfrak{N}_{-z}, \quad \text{rbv}_{z,1} \psi = \text{rbv}_{-z,1} \hat{\psi}$$

(2.22)

and

$$\text{rbv}_{z,2} \psi = -\text{rbv}_{-z,2} \hat{\psi}$$

$$= z \lim_{x \searrow a} \left[ \sum_{l=0}^{\Delta-1} z^l \left( w_l(x)\psi(x) + w_{l+1}(x)p(x)\psi'(x) \right) + \begin{cases} z^\Delta w_\Delta(x)\psi(x) & \text{if } \Delta \text{ is even} \\
0 & \text{if } \Delta \text{ is odd} \end{cases} \right] + \begin{pmatrix} \lim_{t \searrow a} p(t)\psi'(t) \\ 0 \end{pmatrix} \sum_{k=\frac{\Delta+1}{2}}^{\Delta-1} \sum_{l=0}^{2k-\Delta} (-1)^l z^{2k} w_l(x)w_{2k-l+1}(x) \right].$$

Proof. Let $H$ be as in (2.8) and set $\Delta := \Delta(H)$. The relations in (2.22) are clear from the considerations around equation (2.9) and the fact that (2.1) does
not change when we replace $z$ by $-z$. From [37, (4.1)] and (2.15) we obtain

$$rbv_{z,2} \psi = - \lim_{x \searrow a} \left[ \sum_{l=0}^{\Delta} z^l \left( \psi(x) - (rbv_{z,1} \psi) \sum_{k=\Delta+1}^{2\Delta-l} z^k w_k(x) \right) \right]$$

$$= \lim_{x \searrow a} \left[ \sum_{l: 0 \leq l \leq \Delta \text{ even}} z^l w_l(x) \left( \psi(x) - (rbv_{z,1} \psi) \sum_{k=\Delta+1}^{2\Delta-l} z^k w_k(x) \right) \right]$$

$$+ \sum_{l: 1 \leq l \leq \Delta \text{ odd}} z^l w_l(x) \left( \psi(x) - (rbv_{z,1} \psi) \sum_{k=\Delta+1}^{2\Delta-l} z^k w_k(x) \right)$$

$$= \lim_{x \searrow a} \left[ \sum_{l: 0 \leq l \leq \Delta \text{ even}} z^l w_l(x) \left( \psi(x) + \lim_{t \searrow a} p(t) \psi'(t) \sum_{k=\Delta+1}^{\Delta+1} z^k w_k(x) \right) \right]$$

$$+ \sum_{l: 1 \leq l \leq \Delta \text{ odd}} z^l w_l(x) \left( \psi(x) + \lim_{t \searrow a} p(t) \psi'(t) \sum_{k=\Delta+1}^{\Delta+1} z^k w_k(x) \right)$$

$$= \lim_{x \searrow a} \left[ \sum_{l: 0 \leq l \leq \Delta \text{ even}} z^l w_l(x) \psi(x) + \sum_{l: 0 \leq l \leq \Delta-1} z^l w_{l+1}(x) p(x) \psi'(x) \right]$$

$$+ \left( \lim_{t \searrow a} p(t) \psi'(t) \right) \left[ \sum_{l: 0 \leq l \leq \Delta \text{ even}} \sum_{k=\Delta+1}^{\Delta+2-\Delta-l} z^{l+k-1} w_l(x) w_k(x) \right]$$

$$- \sum_{l: 1 \leq l \leq \Delta \text{ odd}} \sum_{k=\Delta+1}^{\Delta+2-\Delta-l} z^{l+k-1} w_l(x) w_k(x)$$

$$= \lim_{x \searrow a} \left[ \sum_{l: 0 \leq l \leq \Delta \text{ even}} z^l w_l(x) \psi(x) + \sum_{l: 0 \leq l \leq \Delta-1} z^l w_{l+1}(x) p(x) \psi'(x) \right]$$

$$+ \left( \lim_{t \searrow a} p(t) \psi'(t) \right) \sum_{l=0}^{\Delta} \sum_{k=\Delta+1}^{\Delta+2-\Delta-\Delta-l} \left( -1 \right)^l z^{l+k-1} w_l(x) w_k(x) \right]$$
Since the conditions in (iii) are all equivalent to \( \lim_{p \psi} (2.21) \). The existence of the limit in (2.17) and the equality 
\[ \lim_{t \to a} \left( \int_{x_0}^x \frac{dt}{p(t)} \right) \]

which proves the statement for \( \text{rbv}_{z,2} \psi \). Inside the limit only even powers of \( z \) appear, and hence, as \( \text{rbv}_{z,2} \hat{\psi} \) is obtained from \( \text{rbv}_{z,2} \psi \) by replacing \( z \) by \(-z\), the equality \( \text{rbv}_{z,2} \psi = -\text{rbv}_{z,2} \hat{\psi} \) follows.

**Proof of Theorem 2.4.** First we settle the case \( \lambda = 0 \). The solutions of (2.1) with \( \lambda = 0 \) are of the form

\[ \psi(x) = c_1 \int_{x_0}^x \frac{dt}{p(t)} + c_2 \]

with \( c_1, c_2 \in \mathbb{C} \). For such a solution the limits in (2.17) and (2.18) exist and

\[ \text{rbv}_{0,2} \psi = \lim_{x \to a} \left( w_0(x)\psi(x) + w_1(x)p(x)\psi'(x) \right) \]

is obtained from \( \text{rbv}_{0,2} \hat{\psi} \) by replacing \( z \) by \(-z\), the equality \( \text{rbv}_{0,2} \psi = -\text{rbv}_{0,2} \hat{\psi} \)

This shows that \( \text{rbv}_{0,2} \psi : \Omega_{0,2}^P \to \mathbb{C}^2 \) is a bijective mapping. Moreover, the conditions in (iii) are all equivalent to \( c_1 = 0 \) since \( 1/p \) is not integrable at \( a \).

For the rest of the proof assume that \( \lambda \neq 0 \).

(i) Let \( \psi \) be a solution of (2.1), let \( z \in \mathbb{C} \) with \( z^2 = \lambda \) and define \( \hat{\psi} \) as in (2.21). The existence of the limit in (2.17) and the equality

\[ \text{rbv}_{\lambda,1} \psi = \text{rbv}_{z,1} \hat{\psi} \]

are immediate. The existence of the limit in (2.18) and the relation

\[ \text{rbv}_{\lambda,2} \hat{\psi} = \frac{1}{z} \text{rbv}_{z,2} \psi \]

follows from Lemma 2.6 by observing that \( z^2 = \lambda \).

(ii) Theorem 4.2 (ii) in [37] and the relations in (2.23) and (2.24) show that the mapping \( \text{rbv}_{\lambda,2} : \Omega_{\lambda,2}^P \to \mathbb{C}^2 \) is bijective.

(iii) The first and the last assertion follow immediately from [37, Theorem 4.2 (iii)]. For the second statement note that there is (up to a scalar multiple) a unique solution \( \Psi_{\text{reg}} \) such that \( \Psi_{\text{reg}((a,x_0))} \in L^2(H|_{(a,x_0)}) \). Hence \( p\Psi_{\text{reg}((a,x_0))} = \Psi_{\text{reg}1((a,x_0))} \in L^2(L^P|_{(a,x_0)}) \). Any other solution \( \psi \) is such that \( \lim_{x \to a} p(x)\psi'(x) \neq 0 \) according to the already proved third statement of (iii). Since \( \frac{1}{p} \) is not integrable at \( a \) by assumption, such a \( \psi \) satisfies \( p\psi'|_{(a,x_0)} \notin L^P|_{(a,x_0)} \).
\( L^2\left( \frac{1}{p^2(x)} \right) \). Now the claim follows because \( p\psi'\big|_{(a,x_0)} \in L^2\left( \frac{1}{p^2(x)} \right) \) if and only if \( \int_{x_0}^{a} p|\psi'|^2 < \infty \).

(iv) It follows from [37, Theorem 4.2 (iv)] that there exists a polynomial \( \hat{p} \) of degree at most \( 2\Delta \) with real coefficients and no constant term such that

\[
rbv_{z,2} \psi = rbv_{z,2} \psi + \hat{p}(z) rbv_{z,1} \psi
\]

(2.25)

for all \( \psi \in \mathcal{R}_z \). If we choose \( \psi \) as in (2.21) for \( \psi \in \mathcal{R}_{2L}^\text{SL} \), then, by (2.23) and (2.24), we have

\[
rbv_{\hat{z},2} \psi = \frac{1}{z} \rbv_{z,2} \psi = \frac{1}{z} \left( \rbv_{z,2} \psi + \hat{p}(z) \rbv_{z,1} \psi \right)
\]

(2.26)

Since this relation must be true for all \( z \in \mathbb{C} \setminus \{0\} \) and all \( \psi \in \mathcal{R}_{2L}^\text{SL} \), it follows by replacing \( z \) by \(-z\) that \( \hat{p} \) is an odd polynomial. Hence one can define a polynomial \( p_{x_0,x_0}(z) \) by the relation \( p_{x_0,x_0}(z^2) = \frac{p(z)}{z^2} \), which is a real polynomial of degree at most \( \Delta - 1 \). Now the assertion follows from (2.26).

In the next theorem we establish the existence of a singular Titchmarsh–Weyl coefficient, which is used in Theorem 2.11 below to obtain a spectral measure. Recall from [37, Definition 3.1] that \( N_\kappa, \kappa \in \mathbb{N}_0 \), is the set of all generalized Nevanlinna functions with \( \kappa \) negative squares and that \( N_{<\infty} = \bigcup_{\kappa \in \mathbb{N}_0} N_\kappa \). Further, denote by \( N_{\kappa}^{(\infty)} \), \( \kappa \in \mathbb{N}_0 \), the set of functions from \( N_\kappa \) whose only generalized pole of non-positive type is infinity and set \( N_{\kappa}^{(\infty)} = \bigcup_{\kappa \in \mathbb{N}_0} N_{\kappa}^{(\infty)} \); see [37, Definition 3.4]. Note that a function \( q \), defined on \( \mathbb{C} \setminus \mathbb{R} \), belongs to \( N_{\kappa}^{(\infty)} \) if and only if

\[
q(z) = p_{2\kappa}(z)q_0(z)
\]

where \( p_{2\kappa} \) is a monic real polynomial of degree \( 2\kappa \) and \( q_0 \in N_0 \), i.e. \( q_0(\overline{z}) = \overline{q_0(z)} \) and \( \text{Im} q_0(z) \geq 0 \) for \( z \in \mathbb{C}^+ = \{ z \in \mathbb{C} : \text{Im} z > 0 \} \).

2.7 Theorem (Singular Titchmarsh–Weyl coefficients). Let \( (p;w) \in \mathbb{B}_{\text{SL}} \) with \( \text{dom}(p;w) = (a,b) \) be given. Then, for each fixed \( x_0 \in (a,b) \), the following statements hold.

(i) For \( \lambda \in \mathbb{C} \) let \( \theta(\cdot;\lambda) \) and \( \varphi(\cdot;\lambda) \) be the unique solutions of (2.1) such that

\[
rbv_{\lambda}^\text{SL} \theta(\cdot;\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad rbv_{\lambda}^\text{SL} \varphi(\cdot;\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

(2.27)

Then, for each \( x \in (a, b) \), the functions \( \theta(x;\cdot) \) and \( \varphi(x;\cdot) \) are entire of order \( \frac{1}{2} \) and finite (positive) type

\[
\int_{a}^{x} \sqrt{\frac{w(t)}{p(t)}} \, dt.
\]
Moreover, for each \( \lambda \in \mathbb{C} \), one has \( W_p(\varphi(\cdot; \lambda), \theta(\cdot; \lambda)) \equiv 1 \) where the weighted Wronskian \( W_p \) is as in (2.19), and the following relations hold:

\[
\begin{align*}
\lim_{x \searrow a} \varphi(x; \lambda) &= 1, & \lim_{x \searrow a} \frac{p(x)\varphi'(x; \lambda)}{\int_a^x w(t)dt} &= -\lambda, \\
\lim_{x \searrow a} \theta(x; \lambda) &= -1, & \lim_{x \searrow a} p(x)\theta'(x; \lambda) &= 1.
\end{align*}
\]

(2.28)

(ii) The limit

\[
m_{p,w}(\lambda) := \lim_{x \nearrow b} \frac{\theta(x; \lambda)}{\varphi(x; \lambda)}, \quad \lambda \in \mathbb{C} \setminus [0, \infty),
\]

exists locally uniformly on \( \mathbb{C} \setminus [0, \infty) \) and defines an analytic function in \( \lambda \). The function \( m_{p,w} \) belongs to the class \( \mathcal{N}(\infty) \) with \( \kappa = \lfloor \Delta_{\text{SL}}(p,w) \rfloor \).

(iii) We have

\[
\theta(\cdot; \lambda) - m_{p,w}(\lambda)\varphi(\cdot; \lambda) \in L^2(w|_{(x_0,b)}), \quad \lambda \in \mathbb{C} \setminus [0, \infty),
\]

and this property characterizes the value \( m_{p,w}(\lambda) \) for each \( \lambda \in \mathbb{C} \setminus [0, \infty) \).

(iv) For \( \lambda \in \mathbb{C} \setminus [0, \infty) \) let \( \psi \) be any non-trivial solution of (2.1) such that \( \psi|_{(x_0,b)} \in L^2(w|_{(x_0,b)}) \). Then

\[
m_{p,w}(\lambda) = -\frac{\text{rbv}_{\lambda,2} \psi}{\text{rbv}_{\lambda,1} \psi}.
\]

The function \( m_{p,w} \) depends on the choice of \( x_0 \). This dependence is controlled as follows.

(v) Let \( \tilde{x}_0 \in (a,b) \), and let \( \tilde{m}_{p,w} \) be the correspondingly defined singular Titchmarsh–Weyl coefficient. Then there exists a polynomial \( p_{x_0,\tilde{x}_0} \) with real coefficients whose degree does not exceed \( \Delta_{\text{SL}}(p) - 1 \) such that

\[
\tilde{m}_{p,w}(\lambda) = m_{p,w}(\lambda) - p_{x_0,\tilde{x}_0}(\lambda).
\]

(2.30)

Before we prove Theorem 2.7, let us introduce some notation.

2.8 Definition. We refer to each function \( m_{p,w} \) constructed as in Theorem 2.7 as a singular Titchmarsh–Weyl coefficient associated with the Sturm–Liouville equation (2.1) when \( (p; w) \in \mathcal{K}_{\text{SL}} \). We denote by \([m]_{p,w}\) the equivalence class of \( \mathcal{N}(\infty) \)-functions modulo the relation

\[
m_1 \sim m_2 :\iff m_1 - m_2 \in \mathbb{R}[z]
\]

which contains some (and hence any) function \( m_{p,w} \) in Theorem 2.7; we call \([m]_{p,w}\) the singular Titchmarsh–Weyl coefficient.
2.9 Remark.

(i) If $\Delta_{SL}(p, w) = 1$, then $m_{p,w} \in \mathcal{N}_0$ by Theorem 2.7 (ii), which is in accordance with the classical theory since (2.1) is in the limit circle case at $\alpha$; see, e.g. [5, Corollary 8.1] and [25, Corollary A.9].

(ii) According to Theorem 2.4 (iii), $\varphi(\cdot; \lambda)$ is the — up to a multiplicative scalar — unique solution of (2.1) that satisfies $\int_{a}^{x_0} |p(x)||\varphi'(x; \lambda)|^2dx < \infty$ for some $x_0$. If (2.1) is in the limit circle case at $\alpha$ (i.e. if $\Delta_{SL}(p, w) = 1$), then $\varphi(\cdot; \lambda)$ is the only solution of (2.1) that satisfies the boundary condition (2.14); if (2.1) is in the limit point case, then $\varphi(\cdot; \lambda)$ is the only solution in $L^2(w\{(a, x_0)\})$.

\[\theta(x; z) = \left(\begin{array}{c} p(x) \theta'(x; z^2) \\ z\theta(x; z^2) \end{array}\right), \quad \varphi(x; z) = \left(\begin{array}{c} \frac{1}{z} p(x) \varphi'(x; z^2) \\ \varphi(x; z^2) \end{array}\right), \tag{2.32}\]

where $'$ denotes the derivative with respect to $x$; cf. (2.23) and (2.24). It follows from Lemma 2.6 that

\[\theta(x; -z) = \left(\begin{array}{c} p(x) \theta'(x; z^2) \\ -z\theta(x; z^2) \end{array}\right), \quad \varphi(x; -z) = \left(\begin{array}{c} -\frac{1}{z} p(x) \varphi'(x; z^2) \\ \varphi(x; z^2) \end{array}\right). \tag{2.33}\]

Moreover, let $q_H$ be a singular Weyl coefficient of $H$ as in [37, (4.6)].

Proof of Theorem 2.7. Let $\theta$ and $\varphi$ be as in (2.32).

Item (i) follows directly from [37, Theorem 4.5 (i)]. Note that $a_+ = a$ in [37, Theorem 4.5] since $w(x) > 0$, $x \in (a, b)$ a.e.

(ii) It follows from [37, Theorem 4.5 (ii)] that, for $z \in \mathbb{C} \setminus \mathbb{R}$,

\[\frac{\theta(x; z^2)}{\varphi(x; z^2)} = \frac{\theta_2(x; z)}{z\varphi_2(x; z)} \rightarrow \frac{1}{z} q_H(z) \text{ as } x \nearrow b.\]

The function $q_H$ is an odd function as the following calculation shows (where we use (2.32) and (2.33)):

\[q_H(-z) = \lim_{x \nearrow b} \frac{\theta_2(x; -z)}{\varphi_2(x; -z)} = \lim_{x \nearrow b} - \frac{\theta_2(x; z)}{\varphi_2(x; z)} = -q_H(z). \tag{2.34}\]

Hence

\[\lim_{x \nearrow b} \frac{\theta(x; \lambda)}{\varphi(x; \lambda)} = \lim_{x \nearrow b} \frac{\theta(x; z)}{z\varphi(x; z)} = \frac{1}{z} q_H(z) \tag{2.35}\]

is independent of the choice of $z$ such that $z^2 = \lambda$, and therefore $m_{p,w}$ is well defined as a function of $\lambda$ and analytic on $\mathbb{C} \setminus [0, \infty)$. It follows from [36, Theorem 4.4] that $m_{p,w} \in \mathcal{N}_{<\infty}^{(\infty)}$. Set $m_+(\lambda) := \lambda m_{p,w}(\lambda)$; then $m_+ \in \mathcal{N}_{<\infty}^{(\infty)}$
again by [36, Theorem 4.4]. Now [21, Proposition 4.8] implies that \( \text{ind}_{-} m_{p,w} + \text{ind}_{-} m_{+} = \text{ind}_{-} q_{H} \). Moreover, \( m_{p,w} \) and \( m_{+} \) have infinity as only generalized pole of non-positive type and therefore, by the definition of \( m_{+} \) and by [37, (3.4)], we have \( \text{ind}_{-} m_{+} - 1 \leq \text{ind}_{-} m_{p,w} \leq \text{ind}_{-} m_{+} \). This, together with the fact that \( \text{ind}_{-} q_{H} = \Delta(H) = \Delta_{\text{SL}}(p, w) \) by [37, Theorem 4.5 (ii)] and (2.16), yields \( \text{ind}_{-} m_{p,w} = \left\lfloor \Delta_{\text{SL}}(p, w) \right\rfloor \).

(iii) We can write

\[
\theta(\cdot; z^2) - m_{p,w}(z^2)\varphi(\cdot; z^2) = \frac{1}{z}\left(\theta_{2}(\cdot; z) - q_{H}(z)\varphi_{2}(\cdot; z)\right).
\]

By [37, Theorem 4.5 (iii)] the right-hand side of this equality is in \( L^2(w|_{(x_0,b)}) \). Since (up to a scalar multiple) only one solution is in \( L^2(w) \) at \( b \) (because of the limit point assumption at \( b \)), the value of \( m_{p,w}(z^2) \) is uniquely determined by the \( L^2 \) property.

(iv) The formula follows from item (iii) since any such \( \psi \) is a multiple of \( \psi_0 := \theta(\cdot; \lambda) - m_{p,w}(\lambda)\varphi(\cdot; \lambda) \) and \( \text{rvb}_{\lambda}^{\text{SL}} \psi_0 = 1 \), \( \text{rvb}_{\lambda}^{\text{SL}} \psi_0 = -m_{p,w}(\lambda) \).

(v) Let \( \psi \in \mathcal{M}_{\lambda} \) be such that \( \psi|_{(x_0,b)} \in L^2(w|_{(x_0,b)}) \). It follows from item (iv) and Theorem 2.4 (iv) with the notation used there that

\[
\hat{m}_{p,w}(\lambda) = -\frac{\text{rvb}_{\lambda}^{\text{SL}} \psi}{\text{rvb}_{\lambda}^{\text{SL}} \psi} - \frac{\text{rvb}_{\lambda}^{\text{SL}} \psi + p_{x_0,\hat{x}_0}(\lambda) \text{rvb}_{\lambda}^{\text{SL}} \psi}{\text{rvb}_{\lambda}^{\text{SL}} \psi} = m_{p,w}(\lambda) - p_{x_0,\hat{x}_0}(\lambda).
\]

Since \( p_{x_0,\hat{x}_0} \) is the polynomial from Theorem 2.4 (iv), it has the properties stated there. \( \Box \)

From (2.35) we obtain the following relation between \( m_{p,w} \) and \( q_{H} \) if the same base point \( x_0 \) is chosen:

\[
m_{p,w}(z^2) = \frac{1}{z} q_{H}(z), \quad z \in \mathbb{C} \setminus \mathbb{R}.
\]

Next we construct a measure with the help of the Stieltjes inversion formula and the singular Titchmarsh–Weyl coefficient. Before we formulate the theorem, we introduce the following class of measures.

2.10 Definition. Let \( \nu \) be a Borel measure on \( \mathbb{R} \). We say that \( \nu \in \mathbb{M}_- \) if there exists an \( n \in \mathbb{N}_0 \) such that

\[
\nu\left(\left(-\infty, 0\right)\right) = 0 \quad \text{and} \quad \int_{[0, \infty)} \frac{d\nu(t)}{(1 + t)^{n+1}} < \infty.
\]

If \( \nu \in \mathbb{M}_- \), we denote by \( \Delta^-(\nu) \) the minimal \( n \in \mathbb{N}_0 \) such that (2.37) holds. \( \diamond \)

The reason for the use of the minus sign in the notation will become clearer in the next section; see, in particular, item 3 in §3.2.

In the next theorem a measure is constructed, which will turn out to be a spectral measure for the Sturm–Liouville equation (2.1).
Let \((p, w) \in \mathbb{K}_{\text{SL}}\) with \(\text{dom}(p; w) = (a, b)\) be given. Then there exists a unique Borel measure \(\mu_{p, w}\) that satisfies
\[
\mu_{p, w}(\{s_1, s_2\}) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \int_{s_1 - \varepsilon}^{s_2 + \varepsilon} \text{Im} m_{p, w}(t + i\delta) \, dt, \quad -\infty < s_1 < s_2 < \infty,
\]
where \(m_{p, w} \in [m]_{p, w}\) is any singular Titchmarsh–Weyl coefficient associated with (2.1). We have \(\mu_{p, w} \in M^-\) and \(\Delta^- (\mu_{p, w}) = \Delta_{\text{SL}}(p, w)\).

Moreover, \(\mu_{p, w}(\{0\}) > 0\) if and only if
\[
\int_a^b w(x) \, dx < \infty.
\]
If (2.39) is satisfied, then
\[
\mu_{p, w}(\{0\}) = \left[\int_a^b w(x) \, dx\right]^{-1}.
\]

We refer to the measure \(\mu_{p, w}\) given by (2.38) as the spectral measure associated with the Sturm–Liouville equation (2.1). This is justified by Theorem 2.13 below.

2.12 Remark.

(i) The number \(\Delta^- (\mu_{p, w})\), which describes the behaviour of the spectral measure \(\mu_{p, w}\) at infinity, gives a finer measure of the growth of the coefficients \(p\) and \(w\) at the endpoint \(a\) than \(\text{ind}_{-} m_{p, w}\), the negative index of the singular Titchmarsh–Weyl coefficient.

(ii) If \(\Delta_{\text{SL}}(p, w) = 1\), then \(\int_0^\infty \frac{\text{d} \mu_{p, w}(t)}{t^{1/2}} < \infty\), which is the classical growth condition for a spectral measure corresponding to a Nevanlinna function. This is in accordance with the fact that in this case (2.1) is in the limit circle case and therefore classical Hilbert space theory is sufficient to obtain a scalar spectral measure; see, e.g. [5, Theorem 11.1].

(iii) If one uses the more general functions \(\tilde{w}_l\) instead of \(w_l\) as in Remark 2.5(iii), then — similarly to (2.30) — the singular Titchmarsh–Weyl coefficient changes only by the real polynomial \(p\) from (2.20), which is of degree at most \(\Delta - 1\), and hence the spectral measure is unchanged.

\[\diamond\]

Proof of Theorem 2.11. Let us first note that [37, (3.6) and (4.11)] and (2.38) imply that
\[
\mu_H = \mu_{q_H} \quad \text{and} \quad \mu_{p, w} = \mu_{m_{p, w}}
\]
with the notation from [37, §3.5]. Define \(\tau(s) := s^2, \ s \in \mathbb{R}\), and denote by \(\mu^\tau\) the corresponding push-forward measure for a Borel measure \(\mu\), i.e. \(\mu^\tau(B) = \)
\( \mu(x^{-1}(B)) \) for Borel sets \( B \subseteq \mathbb{R} \). It follows from [36, Theorem 4.4] and (2.36) that
\[
\mu_{p,w} \ll \mu_T \quad \text{and} \quad \frac{d\mu_{p,w}}{d\mu_T}(t) = \mathbb{I}_{[0,\infty)}(t),
\] (2.41)
where \( \mathbb{I}_{[0,\infty)} \) denotes the characteristic function of the interval \([0, \infty)\).

For \( m \in \mathbb{N}_0 \), we obtain from (2.41) that
\[
\int_{\mathbb{R}} \frac{d\mu_T(t)}{(1 + t^2)^{m+1}} = \mu_H(\{0\}) + \int_{(0,\infty)} \frac{d\mu_H^T(s)}{(1 + s)^{m+1}},
\]
which shows that \( \mu_{p,w} \in \mathcal{M}^- \) and \( \Delta^-(\mu_{p,w}) = \Delta(\mu_T) = \Delta(H) = \Delta_{\text{SL}}(p,w) \), where the last equality follows from (2.16). The fact that a real polynomial makes no contribution in the Stieltjes inversion formula implies that the measure \( \mu_{p,w} \) does not depend on the choice of the representative from \([m]_{p,w}\).

Since \( \mu_{p,w}(\{0\}) = \mu_H(\{0\}) \) by (2.41), the equivalence of \( \mu_{p,w}(\{0\}) > 0 \) and (2.39) and the relation (2.40) follow from [37, Proposition 5.3].

In the following theorem a Fourier transform is constructed, which yields the unitary equivalence of the operator \( A_{p,w} \) and the multiplication operator in the space \( L^2(\mu_{p,w}) \). In particular, this implies that the spectrum of \( A_{p,w} \) is simple. Note that the function \( \varphi \) that appears in the Fourier transform is the (up to a scalar multiple) unique solution of (2.1) that is “regular” at \( a \); cf. Remark 2.9 (ii).

2.13 Theorem (The Fourier transform).
Let \( (p,w) \in \mathcal{K}_{\text{SL}} \) with \( \text{dom}(p,w) = (a,b) \) be given, and let \( \mu_{p,w} \) be the spectral measure associated with (2.1) via (2.38). Then the following statements hold.

(i) The map defined by
\[
(\Theta_{p,w} f)(t) := \int_a^b \varphi(x; t) f(x) w(x) \, dx, \quad t \in \mathbb{R},
\]
(2.42)
extends to an isometric isomorphism from \( L^2(w) \) onto \( L^2(\mu_{p,w}) \).

(ii) The operator \( \Theta_{p,w} \) establishes a unitary equivalence between \( A_{p,w} \) and the operator \( M_{\mu_{p,w}} \) of multiplication by the independent variable in \( L^2(\mu_{p,w}) \), i.e. we have
\[
\Theta_{p,w} A_{p,w} = M_{\mu_{p,w}} \Theta_{p,w}.
\]

(iii) For compactly supported functions, the inverse of \( \Theta_{p,w} \) acts as an integral transformation, namely,
\[
(\Theta_{p,w}^{-1} g)(x) = \int_0^\infty \varphi(x; t) g(t) \, d\mu_{p,w}(t), \quad x \in (a,b),
\]
where \( g \in L^2(\mu_{p,w}) \), \( \text{supp} \ g \) compact.
First we need a lemma.

**2.14 Lemma.** Set

\[ L_{2\text{od}}(\mu_H) := \{ g \in L^2(\mu_H) : g \text{ is odd} \}, \]
\[ L_{2\text{even}}(\mu_H) := \{ g \in L^2(\mu_H) : g \text{ is even} \}. \]

The Fourier transform \( \Theta_H \) from [37, Theorem 5.1] maps

\[ \left\{ \begin{array}{l} (f_1, f_2) \in L^2(H) : f_2 \equiv 0 \\ \Theta_H(f_1, f_2) \end{array} \right\} \text{ bijectively onto } L_{2\text{od}}(\mu_H) \]

and

\[ \left\{ \begin{array}{l} (f_1, f_2) \in L^2(H) : f_1 \equiv 0 \\ \Theta_H(f_1, f_2) \end{array} \right\} \text{ bijectively onto } L_{2\text{even}}(\mu_H). \]

**Proof.** Since \( \varphi_1(x; -z) = -\varphi_1(x; z) \) and \( \varphi_2(x; -z) = \varphi_2(x; z) \) by (2.33), we have, for \( f_1 \in L^2(\frac{1}{b^2}) \) with \( \text{supp}(f_1) < b \), that

\[
\Theta_H \begin{pmatrix} f_1 \\ 0 \end{pmatrix} (-t) = \int_a^b \varphi(x; -t) H(x) \begin{pmatrix} f_1(x) \\ 0 \end{pmatrix} dx = \int_a^b \varphi(x; -t) \frac{1}{p(x)} f_1(x) dx
\]

and similarly,

\[
\Theta_H \begin{pmatrix} 0 \\ f_2 \end{pmatrix} (-t) = \Theta_H \begin{pmatrix} 0 \\ f_2 \end{pmatrix}(t)
\]

for \( f_2 \in L^2(w) \). Now the result follows from the bijectivity of \( \Theta_H \). \( \square \)

**Proof of Theorem 2.13.**

(i) The operator

\[
U : \left\{ \begin{array}{l} L_{2\text{even}}(\mu_H) \to L^2(\mu_p, w) \\ f \mapsto g \text{ with } g(s) = f(\sqrt{s}), \ s \in [0, \infty) \end{array} \right. \]

is well defined since \( \mu_p((\infty, 0)) = 0 \). It is isometric because (2.41) implies that, for \( f \in L_{2\text{even}}(\mu_H) \),

\[
\|Uf\|_{L^2(\mu_p, w)}^2 = \int_{[0, \infty]} |f(\sqrt{s})|^2 d\mu_{p, w}(s) = \int_{[0, \infty]} |f(\sqrt{s})|^2 d\mu_H(s)
\]

\[
= \int_\mathbb{R} |f(t)|^2 d\mu_H(t) = \|f\|_{L^2(\mu_H)}^2.
\]

Moreover, \( U \) is surjective and the inverse is given by \( (U^{-1}g)(t) = g(t^2) \).

Let \( \nu_2 \) and \( P_2 \) be as (2.11). By Lemma 2.14 the operator \( U\Theta_{H^2} \) is well defined and isometric from \( L^2(w) \) onto \( L^2(\mu_p, w) \). Moreover,

\[
(U\Theta_{H^2}f)(t) = U \left[ \int_a^b \varphi_2(x; t) w(x) f(x) dx \right](t)
\]

\[
= \int_a^b \varphi_2(x; \sqrt{2}t) w(x) f(x) dx = (\Theta_{p, w} f)(t)
\]
by (2.32), which shows that

\[ \Theta_{p,w} = U\Theta_{H^2}. \tag{2.44} \]

(ii) Let \( M_{\mu_H} \) be the multiplication operator by the independent variable in \( L^2(\mu_H) \) as in [37, Theorem 5.1 (ii)]. With [37, Theorem 5.1 (ii)], the definition of \( A_{p,w} \) and (2.44) we obtain

\[ \Theta_{p,w} A_{p,w} = U\Theta_{H^2} = U\Theta_H \Theta_{H^2} = U\Theta_{H^2} = U\Theta_{p,w} = \Theta_{p,w} \Theta_{H^2}. \tag{2.45} \]

(iii) It follows from (2.44) that \( \Theta_{-1,p,w} = P_2 \Theta_{H}^{-1} U^{-1} \). If \( g \in L^2(\mu_{p,w}) \) with compact support, then, by (2.32) and (2.41), we have

\[ (\Theta_{-1,p,w} g)(x) = P_2 \int_{\mathbb{R}} g(t^2) \phi(x,t) \, d\mu_H(t) = \int_{\mathbb{R}} g(t^2) \phi(x,t^2) \, d\mu_H(t) \]

\[ = \int_{[0,\infty)} g(s) \phi(x,s) \, d\mu_B(s) = \int_{[0,\infty)} g(s) \phi(x,s) \, d\mu_{p,w}(s), \]

which shows the desired representation for \( \Theta_{-1,p,w} \).

From [37, Theorem 6.2] we obtain the following uniqueness result, which says that equality of singular Titchmarsh–Weyl coefficients or spectral measures implies equality of the coefficients up to a reparameterization of the independent variable. We cannot prove an existence result since we cannot characterize those spectral measures that lead to diagonal Hamiltonians with non-vanishing determinant. If we considered strings and the corresponding Krein–Feller operators, we would also obtain an existence result: namely every measure from the class \( \mathbb{M}^- \) is the spectral measure of a certain string with two singular endpoints.

2.15 Theorem (Global Uniqueness Theorem).

Let \( (p_1;w_1), (p_2;w_2) \in \mathbb{K}_{\text{SI}} \) be given with \( \text{dom}(p_i;w_i) = (a_i,b_i), i = 1,2 \). Then the following statements are equivalent:

(i) there exists an increasing bijection \( \gamma : (a_2,b_2) \to (a_1,b_1) \) such that \( \gamma \) and \( \gamma^{-1} \) are locally absolutely continuous and

\[ p_2(x) = \frac{1}{\gamma'(x)} p_1(\gamma(x)), \quad w_2(x) = \gamma'(x) w_1(\gamma(x)) \tag{2.46} \]

for \( x \in (a_2,b_2) \) a.e.;

(ii) \( [m]_{p_1,w_1} = [m]_{p_2,w_2} \);

(iii) \( \mu_{p_1,w_1} = \mu_{p_2,w_2} \).

Proof. The theorem follows from [37, Theorem 6.2 and Proposition 4.10] if we recall [37, (2.1)] and observe that

\[ q_{n_2}(z) - q_{n_1}(z) = z \left( m_{p_1,w_1}(z^2) - m_{p_2,w_2}(z^2) \right), \]

which vanishes at 0 and that hence \( \alpha = 0 \) in [37, Theorem 6.2].
2.16 Remark. If one considers Sturm–Liouville equations in impedance form, i.e. when $p = w$ and assumes that the left endpoints coincide, then equality of spectral measures is equivalent to the equality of coefficients a.e. because in this case one has $\gamma' = 1$ if (2.46) is satisfied. We refer to [1] and the references therein for other types of inverse spectral theorems for Sturm–Liouville equations in impedance form.

We also obtain a local version of the uniqueness result, which is an extension of [34, Theorem 1.5] to the case of two singular endpoints. See also [33, §4.4] for a result on strings with regular left endpoint. The next theorem follows immediately from [37, Theorem 6.3].

2.17 Theorem (Local Inverse Spectral Theorem).
Let $(p_1; w_1), (p_2; w_2) \in \mathbb{K}_{SL}$ be given with $\text{dom}(p_i; w_i) = (a_i, b_i), i = 1, 2$, and let $\tau > 0$. Moreover, for $i = 1, 2$, let $s_i(\tau)$ be the unique value $s_i$ such that

$$\int_{a_i}^{s_i} \frac{w_i(\xi)}{p_i(\xi)} d\xi = \tau$$

if $\int_{a_i}^{b_i} \frac{w_i(\xi)}{p_i(\xi)} d\xi > \tau$ and set $s_i(\tau) := b_i$ otherwise. Then the following statements are equivalent.

(i) There exists an increasing bijection $\gamma : (a_2, s_2(\tau)) \to (a_1, s_1(\tau))$ such that $\gamma$ and $\gamma^{-1}$ are locally absolutely continuous and (2.46) holds for $x \in (a_2, s_2(\tau))$ a.e.

(ii) There exist singular Titchmarsh–Weyl coefficients $m_{p_1, w_1}$ and $m_{p_2, w_2}$ and there exists a $\beta \in (0, 2\pi)$ such that, for each $\epsilon > 0$,

$$m_{p_1, w_1}(re^{i\beta}) - m_{p_2, w_2}(re^{i\beta}) = O\left(e^{(-2\tau + \epsilon)\sqrt{\pi} \sin \frac{\beta}{2}}\right), \quad r \to \infty.$$

(iii) There exist singular Titchmarsh–Weyl coefficients $m_{p_1, w_1}$ and $m_{p_2, w_2}$ and there exists a $k \geq 0$ such that, for each $\delta \in (0, \pi),$

$$m_{p_1, w_1}(\lambda) - m_{p_2, w_2}(\lambda) = O\left(|\lambda|^k e^{-2\tau \Im \sqrt{\lambda}}\right),$$

$$|\lambda| \to \infty, \quad \lambda \in \{z \in \mathbb{C} : \delta \leq \arg z \leq 2\pi - \delta\},$$

where $\sqrt{\lambda}$ is chosen so that $\Im \sqrt{\lambda} > 0$.

In the next proposition we provide a sufficient condition for $(p; w) \in \mathbb{K}_{SL}$. This result is also used in Section 4 below. Let us recall the following notation: we write $f(x) \asymp g(x)$ as $x \searrow 0$ if there exist $c, C > 0$ and $x_0 > 0$ such that $cg(x) \leq f(x) \leq Cg(x)$ for all $x \in (0, x_0)$.

2.18 Proposition. Let $\alpha \geq 1$ and assume that $p$ and $w$, defined on $(0, b)$ with $b > 0$ or $b = \infty$ satisfy (2.2) with $a = 0$ and

$$p(x) \asymp x^\alpha, \quad w(x) \asymp x^\alpha \quad \text{as} \quad x \searrow 0. \quad (2.47)$$

If (2.1) is in the limit point case at $b$, then $(p; w) \in \mathbb{K}_{SL}$ and $\Delta_{SL}(p, w) = \lfloor \frac{a+1}{2} \rfloor$. 

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Before we prove Proposition 2.18, we need a lemma.

2.19 Lemma. Let \( \alpha > 1 \) and assume that \( p \) and \( w \), defined on \((0, b)\) with \( b > 0 \) or \( b = \infty \) satisfy (2.47). Moreover, choose \( x_0 \in (0, b) \). Then

\[
    w_l(x) \propto \begin{cases} 
        x^l & \text{if } l \text{ is even and } l < \alpha + 1, \\
        x^{-\alpha + l} & \text{if } l \text{ is odd and } l < \alpha, 
    \end{cases} \tag{2.48}
\]

as \( x \downarrow 0 \).

Proof. We prove the lemma by induction. For \( l = 0 \) the statement is clear from the definition of \( w_0 \). Now assume that (2.48) is true for \( l \in \mathbb{N}_0 \). If \( l \) is even and \( l + 1 < \alpha + 1 \), then

\[
    w_{l+1}(x) \propto \int_x^{x_0} t^{-\alpha} t^l dt = \frac{1}{\alpha - l - 1} (x^{-\alpha + l + 1} - x_0^{-\alpha + l + 1}) \propto x^{-\alpha + l + 1}.
\]

If \( l \) is odd and \( l + 1 < \alpha + 1 \), then

\[
    w_{l+1}(x) \propto \int_0^x t^{\alpha - \alpha + l} dt = \frac{1}{l + 1} x^{l+1} \propto x^{l+1}.
\]

In both cases it follows that (2.48) is true for \( l + 1 \) instead of \( l \). Hence the statement follows by induction.

Proof of Proposition 2.18. The conditions (i) and (ii) in Definition 2.1 are easy to check. For (iii) let us first consider the case \( \alpha = 1 \). Then \( w_1(x) \propto -\ln x \) and hence \( w_1 \in L^2(w|_{(0,x_0)}) \), which shows that \( \Delta_{SL}(p, w) = 1 = \left\lfloor \frac{\alpha + 1}{2} \right\rfloor \).

Now let \( \alpha > 1 \). If \( l \) is even and \( l < \alpha + 1 \), then

\[
    w_l \in L^2\left(\frac{1}{p}|(0,x_0)\right) \iff \int_0^{x_0} x^{-\alpha} x^{2l} dx < \infty \iff l > \frac{\alpha - 1}{2}. \tag{2.49}
\]

If \( l \) is odd and \( l < \alpha \), then

\[
    w_l \in L^2(w|_{(0,x_0)}) \iff \int_0^{x_0} x^{\alpha-\alpha+l} dx < \infty \iff l > \frac{\alpha - 1}{2}. \tag{2.50}
\]

The minimal integer \( l \) that satisfies \( l > \frac{\alpha - 1}{2} \) is \( \left\lfloor \frac{\alpha + 1}{2} \right\rfloor \). Since \( \left\lfloor \frac{\alpha + 1}{2} \right\rfloor < \alpha \) for \( \alpha > 1 \), the asymptotic relations (2.48) are valid for \( w_l, l \leq \left\lfloor \frac{\alpha + 1}{2} \right\rfloor \). \( \Box \)

2.20 Example. Equations of the form

\[-a_2 y'' - a_1 y' = \lambda y,\]

where \( a_1 \) and \( a_2 \) are continuous functions on \((a,b)\) and \( a_2(x) > 0 \) for \( x \in (a,b) \), can be written in the form (2.1) with

\[
p(x) = \exp\left(\int_{x_0}^x \frac{a_1(t)}{a_2(t)} dt\right), \quad w(x) = \frac{1}{a_2(x)} \exp\left(\int_{x_0}^x \frac{a_1(t)}{a_2(t)} dt\right)
\]

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with some \( x_0 \in [a, b] \). As an example we consider the associated Laguerre equation

\[-xy''(x) - (1 + \alpha - x)y'(x) = \lambda y(x), \quad x \in (0, \infty),\]

with \( \alpha \geq 0 \). For \( p \) and \( w \) one obtains

\[ p(x) = x^{\alpha+1}e^{-x}, \quad w(x) = x^{\alpha}e^{-x}. \]

It can be shown in a similar way as in Proposition 2.18 that \((p; w) \in \mathbb{K}_{SL}\) with \( \Delta = [\alpha + 1] \). Hence the singular Titchmarsh–Weyl coefficient belongs to \( \mathcal{N}^p_{\kappa}(\infty) \) with \( \kappa = \lfloor \alpha + 1 \rfloor / 2 \). This is in agreement with \([10]\) where a model for this singular Titchmarsh–Weyl coefficient was constructed. For \( \alpha < -1 \) the associated Laguerre equation was studied with the help of Pontryagin spaces in \([28]\), \([29]\) and \([9]\); in this case the results of the next subsection can be applied.

3 Sturm–Liouville equations without potential: singular \( w \)

In this section we consider the case when \( w \) is not integrable at \( a \) but \( \frac{1}{p} \) is. In Definition 2.1 and most theorems one just has to swap the roles of \( \frac{1}{p} \) and \( w \). Let us state the definition of the class of coefficients explicitly.

3.1 Definition. We say that \((p; w) \in \mathbb{K}_{SL}^+\) if \( p \) and \( w \) are defined on some interval \((a, b)\) and they satisfy (2.2) and the following conditions.

(i) For one (and hence for all) \( x_0 \in (a, b)\),

\[ \int_a^{x_0} w(x)dx = \infty \quad \text{and} \quad \int_a^{x_0} \frac{1}{p(x)} dx < \infty. \]

(ii) For one (and hence for all) \( x_0 \in (a, b)\),

\[ \int_a^{x_0} \int_a^x \frac{1}{p(t)} dt w(x)dx < \infty. \]

(iii) Let \( x_0 \in (a, b) \) and define functions \( w_l, l = 0, 1, \ldots, \) recursively by

\[ w_0(x) = 1, \]

\[ w_l(x) = \begin{cases} \int_x^{x_0} w(t)w_{l-1}(t)dt & \text{if } l \text{ is odd}, \\ \int_a^x \frac{1}{p(t)} w_{l-1}(t)dt & \text{if } l \text{ is even}. \end{cases} \]
There exists an \( n \in \mathbb{N}_0 \) such that
\[
 w_n_{|_{(a,x_0)}} \in \begin{cases} 
 L^2(w|_{(a,x_0)}) & \text{if } n \text{ is even}, \\
 L^2\left( \tfrac{1}{p} |_{(a,x_0)} \right) & \text{if } n \text{ is odd.}
\end{cases}
\] (3.1)

(iv) Equation (2.1) is in the limit point case at \( b \), i.e. for \( \lambda \in \mathbb{C} \\setminus \mathbb{R} \), equation (2.1) has (up to a scalar multiple) only one solution in \( L^2(w|_{(x_0,b)}) \) for \( x_0 \in (a,b) \).

If \( (p;w) \in \mathbb{K}_{SL}^+ \), we denote by \( \Delta_{SL}^+ (p, w) \) the minimal \( n \in \mathbb{N}_0 \) such that (3.1) holds.

\[ \square \]

3.2. Differences between the classes \( \mathbb{K}_{SL} \) and \( \mathbb{K}_{SL}^+ \). In the following list we mention the major differences that occur in theorems and other important statements corresponding to coefficients in \( \mathbb{K}_{SL} \) and \( \mathbb{K}_{SL}^+ \), respectively.

1. The analogue of Remark 2.2 (ii) is not true; the equation (2.1) is always in the limit point case at \( a \) if \( \Delta_{SL}^+ (p, w) \geq 1 \). This follows from the fact that \( 1 \in L^2(w|_{(a,x_0)}) \).

2. The regularized boundary values have the following form (with \( \Delta = \Delta_{SL}^+(p, w) \)):
\[
\begin{align*}
\text{rbv}_{\lambda,1}^\text{SL} & = \lim_{x \downarrow a} \psi(x), \\
\text{rbv}_{\lambda,2}^\text{SL} & = \lim_{x \downarrow a} \left[ p(x) \psi'(x) + \sum_{k=1}^{\lfloor \frac{\Delta}{2} \rfloor} \lambda^k \left( w_{2k}(x) p(x) \psi'(x) - w_{2k-1}(x) \psi(x) \right) \\
& - \left\{ \begin{array}{ll}
\lambda^\Delta w_{\Delta} (x) \psi (x) & \text{if } \Delta \text{ is odd} \\
0 & \text{if } \Delta \text{ is even}
\end{array} \right. \\
& + \lim_{t \uparrow a} \psi(t) \left( \sum_{k=1}^{\lfloor \frac{\Delta+3}{2} \rfloor} \sum_{l=0}^{2k-\Delta-2} (-1)^{l+1} \lambda^k w_l(x) w_{2k-l-1}(x) \right) \right].
\end{align*}
\]

3. The singular Titchmarsh–Weyl coefficient \( m_{p,w}^+ \), which is defined as in (2.29), is connected with the singular Weyl coefficient of the corresponding canonical system via
\[
m_{p,w}^+(z^2) = z q_H(z).
\] (3.2)

This relation explains the use of the notation with \( + \) as this was used, e.g. in [21] and [36]. The singular Titchmarsh–Weyl coefficient \( m_{p,w}^+ \) belongs to \( \Lambda^\infty_{\kappa} \) where \( \kappa = \lfloor \Delta_{sl}^+(p, w) + 1 \rfloor \). The equivalence classes \( [m]_{p,w}^+ \) are defined not with respect to the equivalence relation \( \sim \) defined in (2.31) but with the equivalence relation \( \sim \) defined in [37, (4.10)], which is
\[
m_1 \sim m_2 :\iff m_1 - m_2 \in \mathbb{R}[z], \ (m_1 - m_2)(0) = 0.
\]

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4. The spectral measure $\mu_{p,w}^+$ belongs to the class $\mathbb{M}^+$, which is the set of Borel measures on $\mathbb{R}$ such that

$$\nu((\infty, 0]) = 0 \quad \text{and} \quad \int_{(0, \infty)} \frac{d\nu(t)}{t(1 + t)^{n+1}} < \infty. \quad (3.3)$$

If $\nu \in \mathbb{M}^+$, we denote by $\Delta^+(\nu)$ the minimal $n \in \mathbb{N}_0$ such that (3.3) holds. Then $\Delta^+(\mu_{p,w}^+) = \Delta^+_S(p, w)$.

5. Instead of (2.40) one has

$$- \lim_{\lambda \searrow 0} m^+_{p,w}(\lambda) = \left[ \int_a^b \frac{1}{p(x)} \, dx \right]^{-1}, \quad (3.4)$$

and that the left-hand side is zero if and only if the integral on the right-hand side is infinite. Relation (3.4) follows from \cite[(5.4) and (5.5)]{37} and (3.2).

The global uniqueness result is different from the one in the previous subsection since adding a constant to the singular Titchmarsh–Weyl coefficient corresponds to a more complicated transformation; cf. also \cite[Corollary 3.6]{12} for the case when the equations are in impedance form.

### 3.3 Theorem (Global Uniqueness Theorem)

(i) Let $(p_1; w_1), (p_2; w_2) \in K^+_S$ be given with $\text{dom}(p_i; w_i) = (a_i, b_i)$, $i = 1, 2$. Assume that there exist singular Titchmarsh–Weyl coefficients $m^+_{p_i,w_i}$ corresponding to $(p_i; w_i)$ for $i = 1, 2$ such that

$$m^+_{p_1,w_1}(\lambda) - m^+_{p_2,w_2}(\lambda) = c_n \lambda^n + \ldots + c_1 \lambda + c_0 \quad (3.5)$$

with $c_0, \ldots, c_n \in \mathbb{R}$. Then there exists an increasing bijection $\gamma : (a_2, b_2) \to (a_1, b_1)$ such that $\gamma$ and $\gamma^{-1}$ are locally absolutely continuous and

$$p_2(x) = \frac{1}{\gamma'(x)} \left( 1 + c_0 \int_{a_1}^{\gamma(x)} \frac{1}{p_1(t)} \, dt \right)^2 p_1(\gamma(x)), \quad (3.6)$$

$$w_2(x) = \gamma'(x) \left( 1 + c_0 \int_{a_1}^{\gamma(x)} \frac{1}{p_1(t)} \, dt \right)^2 w_1(\gamma(x))$$

for $x \in (a_2, b_2)$; for all $x \in (a_2, b_2)$ one has

$$1 + c_0 \int_{a_1}^{\gamma(x)} \frac{1}{p_1(t)} \, dt > 0. \quad (3.7)$$

Moreover, $\Delta^+_S(p_1, w_1) = \Delta^+_S(p_2, w_2)$.

(ii) Let $(p_1; w_1) \in K^+_S$ be given with $\text{dom}(p_1; w_1) = (a_1, b_1)$. Let $(a_2, b_2) \subseteq \mathbb{R}$ be an open interval, $\gamma : (a_2, b_2) \to (a_1, b_1)$ an increasing bijection such that $\gamma$ and $\gamma^{-1}$ are locally absolutely continuous, and let $c_0 \in \mathbb{R}$ such that

$$1 + c_0 \int_{a_1}^{b_1} \frac{1}{p_1(t)} \, dt \geq 0. \quad (3.8)$$
Define functions $p_2, w_2$ by (3.6). Then $(p_2; w_2) \in K_+^{\text{SL}}$ with $\Delta_{\text{SL}}^+(p_1, w_1) = \Delta_{\text{SL}}^+(p_2, w_2)$, and there exist singular Titchmarsh–Weyl coefficients $m_{p_i, w_i}^+$, $i = 1, 2$, such that

$$m_{p_1, w_1}^+(\lambda) - m_{p_2, w_2}^+(\lambda) = c_0.$$

(iii) Let $(p_1; w_1), (p_2; w_2) \in K_+^{\text{SL}}$ be given with $\text{dom}(p_i; w_i) = (a_i, b_i)$, $i = 1, 2$. Then $\mu_{p_1, w_1} = \mu_{p_2, w_2}^+$ if and only if there exists $\gamma$ as above and $c_0 \in \mathbb{R}$ such that (3.6) and (3.8) hold.

Before we can prove the theorem we need a lemma about a transformation of diagonal Hamiltonians.

3.4 Lemma. Let $H \in \mathbb{H}$ be a diagonal Hamiltonian with $\text{dom} H = (a, b)$ of the form

$$H(x) = \begin{pmatrix} h_{11}(x) & 0 \\ 0 & h_{22}(x) \end{pmatrix}.$$  

Assume that $h_{22}(x) > 0$ for almost all $x \in (a, b)$ and let $q_H$ be a singular Weyl coefficient and $\mu_H$ the corresponding spectral measure. Moreover, let $c \in \mathbb{R}$ and define the functions

$$\alpha(x) := 1 + c \int_a^x h_{22}(t)dt, \quad x \in (a, b), \quad (3.9)$$

$$\tilde{q}(z) := q_H(z) - \frac{c}{z}. \quad (3.10)$$

Then the following statements are equivalent:

(i) $\alpha(x) > 0$ for all $x \in (a, b)$;

(ii) $c + \mu_H([0]) \geq 0$;

(iii) $\tilde{q} \in \mathcal{N}^\infty_c$.

If these conditions are satisfied, then

$$\tilde{H}(x) := \begin{pmatrix} (\alpha(x))^2 h_{11}(x) & 0 \\ 0 & h_{22}(x) / (\alpha(x))^2 \end{pmatrix}, \quad x \in (a, b), \quad (3.11)$$

belongs to $\mathbb{H}$ and $\tilde{q}$ is a singular Weyl coefficient for $\tilde{H}$.

Proof. First note that the integral in (3.9) is positive for all $x \in (a, b)$ and strictly increasing in $x$. Hence (i) is equivalent to

$$c + \left[ \int_a^b h_{22}(t)dt \right]^{-1} \geq 0,$$

where we use $1/\infty = 0$ in the case when the integral is infinite. Since $\mu_H([0]) = \left[ \int_a^b h_{22}(t)dt \right]^{-1}$ by [37, Proposition 5.3], the equivalence of (i) and (ii) follows.
The function $\tilde{q}$ is in $\mathcal{N}_{<\infty}$. The only possible finite generalized pole of non-positive type of $\tilde{q}$ is 0. Hence $\tilde{q} \in \mathcal{N}_{<\infty}^{(\infty)}$ if and only if
\[
\lim_{\epsilon \searrow 0} i\epsilon \tilde{q}(i\epsilon) \leq 0;
\]
see, e.g. [32, Theorem 3.1]. It follows from [36, Theorem 3.9 (ii)] that
\[
c + \mu_H(\{0\}) = c - \lim_{\epsilon \searrow 0} i\epsilon q_H(i\epsilon) = -\lim_{\epsilon \searrow 0} i\epsilon \tilde{q}(i\epsilon),
\]
which implies the equivalence of (ii) and (iii).

For the rest of the proof assume that $\alpha(x) > 0$ for all $x \in (a, b)$. One can easily show that the assertions of the lemma are unaffected by reparameterizations. So we can assume that $H$ is defined on $(0, \infty)$. Let $h$ be the indefinite Hamiltonian associated with $H$ as in [37, §3.16] with Weyl coefficient $q_h$ such that $q_h = q_H$, and let $\omega_h$ be the corresponding maximal chain of matrices as in [37, §3.8]. It follows from [37, Lemma 5.9] that
\[
\tilde{\alpha}(x) := 1 - c \partial \frac{\partial}{\partial z} \omega_{h, 21}(x; z) \bigg|_{z=0} = \begin{cases} 1, & x \in [-1, 0), \\ \alpha(x), & x \in (0, \infty). \end{cases}
\]
The transformation $\Sigma_c$ from [22, Definition 4.1] applied to $\omega_h$ yields a maximal chain of matrices $\tilde{W}$, where
\[
\tilde{W}(x; z) := (\Sigma_c \omega_h)(x; z) = \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} \omega_h(x; z) \begin{pmatrix} 1 & \frac{c}{z} \\ 0 & \tilde{\alpha}(x) \end{pmatrix},
\]
so that $\tilde{W}$ corresponds to an indefinite Hamiltonian $\tilde{h}$ whose Weyl coefficient is
\[
q_h(z) = q_h(z) - \frac{c}{z};
\]
see [22, Theorem 4.4]. Differentiating $\tilde{W}$ with respect to $x$ and using the differential equation [37, (3.9)], i.e.
\[
\frac{\partial}{\partial x} \tilde{W}(x; z) J = z \tilde{W}(x; z) \tilde{H}(x),
\]
one can easily show that the Hamiltonian function $\tilde{H}$ that corresponds to $\tilde{h}$ is given by (3.11); cf. [46, Rule 4].

**Proof of Theorem 3.3.** Throughout the proof let $H_1$ and $H_2$ be the corresponding Hamiltonians
\[
H_i(x) = \begin{pmatrix} w_i(x) & 0 \\ 0 & 1/p_i(x) \end{pmatrix}, \quad i = 1, 2,
\]
and let $q_{H_1}, q_{H_2}$ be the singular Weyl coefficients such that $m_{p_i, w_i}(z^2) = z q_{H_i}(z)$, $i = 1, 2$ as in (3.2).
(i) Suppose that (3.5) holds. Define \( \tilde{q} \) as in (3.10) with \( H = H_1 \) and \( c = c_0 \), i.e.
\[
\tilde{q}(z) = q_{H_1}(z) - \frac{c_0}{z} = q_{H_2}(z) + c_n z^{2n-1} + c_{n-1} z^{2n-3} + \ldots + c_1 z.
\] (3.12)

The equality of the first and the last expression in (3.12) implies that \( \tilde{q} \in \mathcal{N}_<^{(\infty)} \), i.e. condition (iii) in Lemma 3.4 is satisfied. Hence we can apply Lemma 3.4, which yields a Hamiltonian \( \tilde{H} \) with corresponding singular Weyl coefficient \( \tilde{q} \).

(ii) Condition (3.8) implies that (i) in Lemma 3.4 is satisfied with \( H = H_1 \) and \( c = c_0 \). Hence we can apply Lemma 3.4, which yields \( \tilde{H} \). The assertion follows since \( H_2 \) is a reparameterization of \( \tilde{H} \).

(iii) It follows from [36, Theorem 3.9 (iv)] that \( \mu_{p_1,w_1}^+ = \mu_{p_2,w_2}^+ \) if and only if \( m_{p_1,w_1}^+ \) and \( m_{p_2,w_2}^+ \) differ by a real polynomial. Now the claim follows from (i) and (ii).

Let us finally point out that Proposition 2.18 remains valid for the situation in this section if (2.47) is replaced by
\[
p(x) \asymp x^{-\alpha}, \quad w(x) \asymp x^{-\alpha} \quad \text{as } x \searrow 0.
\]

4 Schrödinger equations

Let \( V \in L^1_{\text{loc}}(0,b) \) with \( b > 0 \) or \( b = \infty \) and consider the one-dimensional Schrödinger equation
\[
-u''(x) + V(x)u(x) = \lambda u(x).
\] (4.1)

In this section the left endpoint needs to be finite, which without loss of generality we assume to be 0. In the following we write \( \text{dom}(V) := (0,b) \).

Assume that, for \( \lambda = 0 \), equation (4.1) has a solution \( \phi \) (i.e. \( V = \frac{\phi''}{\phi} \)) in \( W^{2,1}_{\text{loc}}(0,b) \) that satisfies
\[
\phi(x) > 0 \quad \text{for all } x \in (0,b),
\]
\[
\phi|_{(0,x_0)} \in L^2(0,x_0), \quad \frac{1}{\phi}|_{(0,x_0)} \notin L^2(0,x_0) \quad \text{for some } x_0 \in (0,b).
\] (4.2)

A similar approach, namely to assume the existence of a particular solution instead of explicit conditions on the coefficients, was used in [8].

Note that \( \phi \) with the above properties is determined only up to a multiplicative positive constant; see Remark 4.11 below for a further discussion of this non-uniqueness.
Let $x_0 \in (0, b)$ and define functions $\tilde{w}_l$, $l = 0, 1, \ldots$, recursively by

\[
\tilde{w}_0(x) = \frac{1}{\phi(x)}, \\
\tilde{w}_k(x) = \begin{cases} 
\phi(x) \int_{x_0}^x \frac{1}{\phi(t)} \tilde{w}_{k-1}(t) dt & \text{if } k \text{ is odd}, \\
\frac{1}{\phi(x)} \int_{0}^x \phi(t) \tilde{w}_{k-1}(t) dt & \text{if } k \text{ is even}.
\end{cases}
\] (4.3)

4.1 Remark. It follows from the last condition in (4.2) and [41, Theorem 2.2] that $\phi$ is a principal solution of (4.1) with $\lambda = 0$. The function $\tilde{w}_1$ is a non-principal solution of (4.1) with $\lambda = 0$, and one has

\[-\tilde{w}_k'' + V \tilde{w}_k = \tilde{w}_1 \quad \text{when } k \in \mathbb{N} \text{ is odd}.\] (4.4)

In [31] a sequence of functions $g_k$ was used which satisfy the relations

\[-g_{k+1}'' + V g_{k+1} - \mu_k g_{k+1} = g_k\]

with pairwise distinct numbers $\mu_k$. ♦

4.2 Remark. Instead of $\tilde{w}_k$ one can use more general functions $\check{w}_k$ that are defined as in (4.3) but with the relation

\[
\check{w}_k(x) = \phi(x) \left[ \frac{1}{\phi(x)} \int_{x_0}^x \phi(t) \tilde{w}_{k-1}(t) dt + c_k \right], \quad k \text{ odd},
\]

with arbitrary constants $c_k \in \mathbb{R}$; cf. Remarks 2.5 (iii) and 2.12 (iii). The spectral measure that is constructed below remains the same. ♦

In this section we consider the following class of potentials.

4.3 Definition. We say that $V \in K_{\text{Schr}}$ if $V \in L^1_{\text{loc}}(0, b)$, there exists a $\phi$ so that $\phi$ is a solution of (4.1) with $\lambda = 0$, that (4.2) holds and that the following conditions are satisfied.

(i) For one (and hence for all) $x_0 \in (0, b)$,

\[
\int_{0}^{x_0} \phi(x) \tilde{w}_1(x) dx < \infty.
\]

(ii) There exists an $n \in \mathbb{N}$ such that

\[
\tilde{w}_n \big|_{(0, x_0)} \in L^2(0, x_0).\] (4.5)

(iii) Equation (4.1) is in the limit point case at $b$.

If $V \in K_{\text{Schr}}$, we denote by $\Delta_{\text{Schr}}(V)$ the minimal $n \in \mathbb{N}$ such that (4.5) holds. ♦

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4.4 Remark.

(i) Since the function \( \tilde{w}_1 \) is a non-principal solution of (4.1) with \( \lambda = 0 \), we have \( \Delta(V) = 1 \) if and only if equation (4.1) is regular or in the limit circle case at the left endpoint.

(ii) One can also consider the case when (4.1) is regular or in the limit circle case at the right endpoint \( b \). In this case one has to impose a fixed self-adjoint boundary condition at \( b \); cf. Remark 2.3. This situation was considered, e.g. in [44] in connection with \( n \)-entire operators.

\[ \diamond \]

In order to apply the results from Section 2, we set

\[ p(x) = w(x) := (\phi(x))^2, \quad x \in (0, b). \]  \hspace{1cm} (4.6)

With \( w_k \) defined as in (2.5) we have

\[ \tilde{w}_k(x) = \begin{cases} \frac{w_k(x)}{\phi(x)} & \text{if } k \text{ is even}, \\ \phi(x)w_k(x) & \text{if } k \text{ is odd}. \end{cases} \]  \hspace{1cm} (4.7)

It is easy to see that \( V \in \mathbb{K}_{Schr} \) if and only if \( (p, w) \in \mathbb{K}_{SL} \) with \( p, w \) from (4.6); for the equivalence of the limit point property at \( b \) see (4.15) below. Moreover, if \( V \in \mathbb{K}_{Schr} \), then \( \Delta_{Schr}(V) = \Delta_{SL}(p, w) \).

4.5 Example.

(i) A large subclass of \( \mathbb{K}_{Schr} \) is the following. Let \( b > 0 \) or \( b = \infty \), let \( V_0 \in L^1_{\text{loc}}(0, b) \) and let \( l \in [ -\frac{1}{2}, \infty ) \). Moreover, set

\[ V(x) = \frac{l(l+1)}{x^2} + V_0(x) \]  \hspace{1cm} (4.8)

and assume that

\[ xV_0(x)|_{(0, x_0)} \in L^1(0, x_0) \]  \hspace{1cm} if \( l > -\frac{1}{2} \),

\[ (\ln x)xV_0(x)|_{(0, x_0)} \in L^1(0, x_0) \]  \hspace{1cm} if \( l = -\frac{1}{2} \)  \hspace{1cm} (4.9)

with some \( x_0 \in (0, b) \). Moreover, suppose that the minimal operator associated with (4.2) is bounded from below and that (4.1) is in the limit point case at \( b \). Under the assumption that (4.9) is valid, it follows from [26, Lemma 3.2] that there exists a solution \( \phi \) of (4.1) with \( \lambda = 0 \) such that

\[ \phi(x) = x^{l+1}(1 + o(x)), \quad x \downarrow 0. \]  \hspace{1cm} (4.10)

Assume that \( \phi(x) > 0 \) for \( x \in (0, b) \), which is satisfied, e.g. if the minimal operator is uniformly positive, which can be achieved by a shift of the spectral parameter. Now it follows from Proposition 2.18 that \( (p, w) \in \mathbb{K}_{SL} \), and hence \( V \in \mathbb{K}_{Schr} \) and \( \Delta_{Schr}(V) = \Delta_{SL}(p, w) = \left[ l + \frac{3}{2} \right] \).
Since \( l = 0 \) is allowed in (4.8), the class \( K_{\text{Schr}} \) contains potentials where \( 0 \) is a regular endpoint. If \( l \in \left( -\frac{1}{2}, \frac{1}{2} \right) \setminus \{0\} \), then (4.1) is in the limit circle case at \( 0 \) and \( \Delta_{\text{Schr}}(V) = 1 \).

Potentials of the form (4.8) have been studied in many papers; see, e.g. [14], [15], [20], [23], [26], [31], [16], [2], [24], [27], [44], [11], [38].

(ii) The class \( K_{\text{Schr}} \) contains also potentials that have a stronger singularity at the left endpoint than those considered in (i). If \( V(x) = \phi''(x) \phi(x) \) where \( \phi(x) \approx x^\beta \), \( x \searrow 0 \) (4.11) with \( \beta \geq \frac{1}{2} \), \( \phi(x) > 0 \) for \( x \in (0, b) \) and (4.1) is in the limit point case at \( b \), then \( V \in K_{\text{Schr}} \) with \( \Delta_{\text{Schr}}(V) = \left\lfloor \beta + \frac{1}{2} \right\rfloor \); cf. Proposition 2.18. For instance, functions of the form \( \phi(x) = x^\beta \phi(x) \) with \( \beta > 0 \) lead to oscillatory potentials that do not satisfy (4.9), namely, \( V(x) = -\frac{1}{x^4} \cdot \frac{1}{2} + O\left( \frac{1}{x^3} \right) \), \( x \searrow 0 \).

It follows from Lemma 2.19 and (4.7) that if \( V \) is of the form in (4.11), then
\[
\tilde{w}_k(x) \approx x^{-\beta+k}, \quad x \searrow 0, \quad k \in \mathbb{N}_0, \quad k < 2\beta. \tag{4.12}
\]
In particular, the relation in (4.12) is valid for \( k = 0, 1, \ldots, 2\Delta - 1 \) if \( \beta \) is not an odd integer multiple of \( \frac{1}{2} \), and it is valid for \( k = 0, 1, \ldots, 2\Delta - 2 \), otherwise.

(iii) The function \( V(x) = \frac{1}{x^2} \) does not belong to \( K_{\text{Schr}} \). It can easily be checked that the only possible choice for \( \phi \) (up to scalar multiples) is \( \phi(x) = xe^{-1/x} \).

Moreover, one can show that
\[
\tilde{w}_n(x) \sim C_n x^{\alpha_n} e^{x^{1/2}}, \quad x \searrow 0,
\]
with some \( C_n > 0 \), \( n \in \mathbb{N} \), and \( \alpha_n = \frac{3n-1}{2} \) when \( n \) is odd. Hence, condition (i) in Definition 4.3 is satisfied, but there is no \( n \in \mathbb{N} \) such that (4.5) holds. This potential was also studied in [39], where it was shown that the approach with super-singular perturbations, as developed in [31] and [38], cannot be applied to this potential.

(iv) Potentials from the class \( H_{\text{loc}}^{-1}(0, b) \) could also be treated by our method if we relaxed the assumption \( V \in L_{\text{loc}}^2(0, b) \). In this case, one would only have \( \phi \in H_{\text{loc}}^1(0, b) \). Operators with such potentials were considered, e.g. in [19], [42] and [12]. Note that the class \( H_{\text{loc}}^{-1}(0, b) \) includes measure coefficients.

Let us introduce the unitary operator
\[
U : \begin{cases} 
L^2(0, b) & \to L^2(w), \\
h \to \frac{h}{\phi} 
\end{cases} \quad \tag{4.13}
\]
and define the self-adjoint operator

\[ A_V := U^{-1}A_{p,w}U \]  

(4.14)

with \( p \) and \( w \) as in (4.6) and \( A_{p,w} \) from (2.12). For \( u \in W^{2,1}(0,b) \) with compact support we have

\[
A_V u = \frac{1}{w} \left( p \left( \frac{u}{\phi} \right) \right)' = \frac{1}{\phi} \left( \phi^2 \frac{\phi u' - \phi' u}{\phi^2} \right)'
= -\frac{\phi u'' - \phi' u}{\phi} = -u'' + Vu.
\]

(4.15)

Therefore \( A_V \) is the Friedrichs extension of the minimal operator connected with the equation (4.1); cf. the discussion below (2.13). In particular, if \( \Delta_{\text{Schr}}(V) = 1 \), then a possible boundary condition at \( 0 \) to characterize \( A_V \) is

\[
\lim_{x \to 0} u(x) = 0;
\]

see [41, Theorem 4.3]. As mentioned above, if \( \Delta_{\text{Schr}}(V) \geq 2 \), then (4.1) is in the limit point case at \( 0 \) and hence no boundary condition is needed there.

We can apply all theorems from Section 2. In order to rewrite these results in a more intrinsic form, we define regularized boundary values by

\[ rbv_{\lambda,1}^\text{Schr} u := rbv_{\lambda}^\text{SL} u \]  

for \( \lambda \in \mathbb{C} \) and \( u \) a solution of (4.1). Then Theorem 2.4, together with a straightforward calculation, yields the following theorem.

4.6 Theorem (Regularized boundary values). Let \( V \in \mathbb{K}_{\text{Schr}} \) with \( \text{dom}(V) = (0,b) \), set \( \Delta := \Delta_{\text{Schr}}(V) \) and let \( \mathcal{N}_\lambda^\text{Schr} \) be the set of all solutions of (4.1). Then, for \( x_0 \in (0,b) \), the following statements hold.

(i) For each \( \lambda \in \mathbb{C} \) and each solution \( u \in \mathcal{N}_\lambda^\text{Schr} \) the boundary value

\[
rbv_{\lambda,1}^\text{Schr} u = \lim_{x \searrow x_0} \left( \phi(x) u'(x) - \phi'(x) u(x) \right),
\]

and the regularized boundary value

\[
rbv_{\lambda,2}^\text{Schr} u = \lim_{x \searrow x_0} \left[ \sum_{k=0}^{\Delta \div 2} \lambda^k \left( \tilde{w}_{2k+1}(x) u'(x) - \tilde{w}'_{2k+1}(x) u(x) \right) \right]
+ \begin{cases} 
\lambda^{\Delta \over 2} \tilde{w}_\Delta(x) u(x) & \text{if } \Delta \text{ is even} \\
0 & \text{if } \Delta \text{ is odd} 
\end{cases}
+ \left( rbv_{\lambda,1}^\text{Schr} u \right) \left( \sum_{k=1}^{\Delta - 1} \sum_{t=0}^{2k-\Delta} (-1)^t \lambda^k \tilde{w}_t(x) \tilde{w}_{2k-t+1}(x) \right).
\]

exist.

(ii) For each \( \lambda \in \mathbb{C} \) we define

\[ rbv_{\lambda}^\text{Schr} : \mathcal{N}_\lambda^\text{Schr} \to \mathbb{C}^2 \]

\[ u \mapsto (rbv_{\lambda,1}^\text{Schr} u, rbv_{\lambda,2}^\text{Schr} u)^T. \]

Then \( rbv_{\lambda}^\text{Schr} \) is a bijection from \( \mathcal{N}_\lambda^\text{Schr} \) onto \( \mathbb{C}^2 \).
(iii) For each \( \lambda \in \mathbb{C} \) there exists an (up to scalar multiples) unique solution \( u \in \mathcal{N}_{\lambda}^{\text{Schr}} \setminus \{0\} \) such that \( \lim_{x \searrow 0} \frac{u(x)}{\phi(x)} \) exists. This solution is characterized by the property that \( \int_0^{\infty} |\phi'|^2 \left( \frac{u}{\phi} \right)^2 < \infty \) and also by the property that \( \text{rbv}_{\lambda,1}^{\text{Schr}} u = 0 \) (and \( u \not\equiv 0 \)).

If \( u \) is a solution such that \( \lim_{x \searrow 0} \frac{u(x)}{\phi(x)} \) exists, then \( \text{rbv}_{\lambda,2}^{\text{Schr}} u = \lim_{x \searrow 0} \frac{u(x)}{\phi(x)}. \)

The regularized boundary value \( \text{rbv}_{\lambda,2}^{\text{Schr}} \) depends on the choice of \( x_0 \) in the following way.

(iv) Let \( x_0, \hat{x}_0 \in (0,b) \), and let \( \text{rbv}_{\lambda}^{\text{Schr}} \) and \( \text{rbv}_{\lambda}^{\text{Schr}}(x_0) \) be the correspondingly defined regularized boundary value mappings. Then there exists a polynomial \( p_{x_0,\hat{x}_0}(z) \) with real coefficients whose degree does not exceed \( \Delta - 1 \) such that

\[
\text{rbv}_{\lambda,2}^{\text{Schr}} u = \text{rbv}_{\lambda,2}^{\text{Schr}}(x_0) u + p_{x_0,\hat{x}_0}(\lambda) \text{rbv}_{\lambda,1}^{\text{Schr}} u, \quad u \in \mathcal{N}_{\lambda}^{\text{Schr}}, \ \lambda \in \mathbb{C}.
\]

Moreover, clearly, \( \text{rbv}_{\lambda,1}^{\text{Schr}} = \text{rbv}_{\lambda,1}^{\text{Schr}}(x_0) \).

The next theorem about a fundamental system of solutions of (4.1) and the existence of a singular Titchmarsh–Weyl coefficient follows from Theorem 2.7 (i) with the help of the unitary operator \( U \) from (4.13). For the formulation recall the definition of the classes \( \mathcal{N}_{\kappa}^{(\infty)} \) in the paragraph before Theorem 2.7; see also [37, Definition 3.4].

4.7 Theorem (Singular Titchmarsh–Weyl coefficients). Let \( V \in \mathcal{K}_{\text{Schr}} \) with \( \text{dom}(V) = (0,b) \) be given. Then, for each fixed \( x_0 \in (0,b) \), the following statements hold.

(i) For \( \lambda \in \mathbb{C} \) let \( \tilde{\theta}(\cdot;\lambda) \) and \( \tilde{\varphi}(\cdot;\lambda) \) be the unique solutions of (4.1) such that

\[
\text{rbv}_{\lambda}^{\text{Schr}} \tilde{\theta}(\cdot;\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{rbv}_{\lambda}^{\text{Schr}} \tilde{\varphi}(\cdot;\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Then, for each \( x \in (0,b) \), the functions \( \tilde{\theta}(x;\cdot) \) and \( \tilde{\varphi}(x;\cdot) \) are entire of order \( \frac{1}{x} \) and finite type \( x \). Moreover, for each \( \lambda \in \mathbb{C} \), one has \( W(\tilde{\varphi}(\cdot;\lambda), \tilde{\theta}(\cdot;\lambda)) \equiv 1 \) where \( W := W_1 \) denotes the Wronskian as in (2.19) with \( p \equiv 1 \), and the following relations hold:

\[
\lim_{x \searrow 0} \frac{\tilde{\varphi}(x;\lambda)}{\phi(x)} = 1, \quad \lim_{x \searrow 0} \frac{\phi(x)\tilde{\varphi}'(x;\lambda) - \phi'(x)\tilde{\varphi}(x;\lambda)}{\int_0^x (\phi(t))^2 dt} = -\lambda,
\]

\[
\lim_{x \searrow 0} \frac{\tilde{\theta}(x;\lambda)}{\tilde{w}_1(x)} = -1, \quad \lim_{x \searrow 0} (\phi(x)\tilde{\theta}'(x;\lambda) - \phi'(x)\tilde{\theta}(x;\lambda)) = 1.
\]

Further, one has \( \tilde{\varphi}(x;0) = \phi(x) \) and \( \tilde{\theta}(x;0) = -\tilde{w}_1(x) \), \( x \in (0,b) \).

(ii) The limit

\[
\hat{m}_V(\lambda) := \lim_{x \searrow 0} \frac{\tilde{\theta}(x;\lambda)}{\tilde{\varphi}(x;\lambda)}, \quad \lambda \in \mathbb{C} \setminus [0,\infty),
\]

exists locally uniformly on \( \mathbb{C} \setminus [0,\infty) \) and defines an analytic function in \( \lambda \). The function \( \hat{m}_V \) belongs to the class \( \mathcal{N}_{\kappa}^{(\infty)} \) with \( \kappa = \left\lfloor \frac{\Delta_{\text{ome}}(V)}{2} \right\rfloor \).
(iii) We have
\[ \tilde{\theta}(\cdot; \lambda) - \tilde{m}_V(\lambda) \tilde{\varphi}(\cdot; \lambda) \in L^2(x_0, b), \quad \lambda \in \mathbb{C} \setminus [0, \infty), \]
and this property characterizes the value \( \tilde{m}_V(\lambda) \) for each \( \lambda \in \mathbb{C} \setminus [0, \infty) \).

(iv) For \( \lambda \in \mathbb{C} \setminus [0, \infty) \) let \( u \) be any non-trivial solution of (4.1) such that \( u|_{(x_0, b)} \in L^2(x_0, b) \). Then
\[ \tilde{m}_V(\lambda) = -\frac{rbv_{\lambda, 2} u}{rbv_{\lambda, 1} u}. \]
The function \( \tilde{m}_V \) depends on the choice of \( x_0 \). This dependence is controlled as follows.

(v) Let \( \hat{x}_0 \in (a, b) \), and let \( \hat{\tilde{m}}_V \) be the correspondingly defined singular Titchmarsh–Weyl coefficient. Then there exists a polynomial \( p_{x_0, \hat{x}_0} \) with real coefficients whose degree does not exceed \( \Delta_{\text{Schr}}(V) - 1 \) such that
\[ \hat{\tilde{m}}_V(\lambda) = \tilde{m}_V(\lambda) - p_{x_0, \hat{x}_0}(\lambda). \]

The functions \( \tilde{\theta} \) and \( \tilde{\varphi} \) are related to the functions \( \theta \) and \( \varphi \) corresponding to (2.1) with \( p \) and \( w \) as in (4.6) as follows:
\[ \tilde{\theta}(x; \lambda) = \phi(x) \theta(x; \lambda), \quad \tilde{\varphi}(x; \lambda) = \phi(x) \varphi(x; \lambda), \quad x \in (0, b). \quad (4.16) \]
The function \( \tilde{m}_V \) is called singular Titchmarsh–Weyl coefficient. It follows from (2.29) and (4.16) that \( \tilde{m}_V = m_{p, w} \). As in Section 2 one defines equivalence classes \([\tilde{m}]_V\) with respect to the equivalence relation \( \sim \) defined in (2.31).

The next theorem about the existence of a spectral measure follows from Theorem 2.11. For the definition of the class \( M^- \) see Definition 2.10.

4.8 Theorem (The spectral measure).
Let \( V \in k_{\text{Schr}} \) with \( \text{dom}(V) = (0, b) \) be given. Then there exists a unique Borel measure \( \tilde{\mu}_V \) that satisfies
\[ \tilde{\mu}_V([s_1, s_2]) = \frac{1}{\pi} \lim_{\epsilon \to 0} \lim_{\delta \to 0} \int_{s_1 - \epsilon}^{s_2 + \epsilon} \text{Im} \tilde{m}_V(t + i\delta) \, dt, \quad -\infty < s_1 < s_2 < \infty, \]
where \( \tilde{m}_V \in [\tilde{m}]_V \) is any singular Titchmarsh–Weyl coefficient associated with (4.1). We have \( \tilde{\mu}_V \in M^- \) and \( \Delta^-(\tilde{\mu}_V) = \Delta_{\text{Schr}}(V) \).
Moreover, \( \tilde{\mu}_V(\{0\}) > 0 \) if and only if
\[ \int_0^b (\phi(x))^2 \, dx < \infty. \quad (4.17) \]
If (4.17) is satisfied, then
\[ \tilde{\mu}_V(\{0\}) = \left[ \int_0^b (\phi(x))^2 \, dx \right]^{-1}. \]
Clearly, we have $\tilde{\mu} V = \mu_{p,w}$ where $p$ and $w$ are as in (4.6).

4.9 Example. Consider $V$ as in Example 4.5 (i). Since $\Delta_{\text{Schr}}(V) = \lfloor l + \frac{3}{2} \rfloor$, we obtain from Theorem 4.7 that $\tilde{m}_V \in N_\kappa^{(\infty)}$ with $\kappa = \lfloor \frac{l}{2} + \frac{3}{4} \rfloor$. Moreover, Theorem 4.8 yields that $\tilde{\mu}_V \in M^-$ with $\Delta^-(\tilde{\mu}_V) = \Delta_{\text{Schr}}(V) = \lfloor l + \frac{3}{2} \rfloor$. ♦

In the next theorem we consider the corresponding Fourier transform and its inverse. This theorem follows from Theorem 2.13.

4.10 Theorem (The Fourier transform).

Let $V \in \mathcal{K}_{\text{Schr}}$ with $\text{dom}(V) = (0, b)$ be given, and let $\tilde{\mu}_V$ be the spectral measure associated with (4.1) as in Theorem 4.8. Then the following statements hold.

(i) The map defined by

$$\left( \tilde{\Theta}_V f \right)(t) := \int_0^b \tilde{\varphi}(x; t) f(x) \, dx, \quad t \in \mathbb{R},$$

for $f \in L^2(0, b)$, $\text{supp}(\text{supp } f) < b$, (4.18)

extends to an isometric isomorphism from $L^2(0, b)$ onto $L^2(\tilde{\mu}_V)$.

(ii) The operator $\tilde{\Theta}_V$ establishes a unitary equivalence between $A_V$ and the operator $M_{\tilde{\mu}_V}$ of multiplication by the independent variable in $L^2(\tilde{\mu}_V)$, i.e. we have

$$\tilde{\Theta}_V A_V = M_{\tilde{\mu}_V} \tilde{\Theta}_V.$$

(iii) For compactly supported functions, the inverse of $\tilde{\Theta}_V$ acts as an integral transformation, namely,

$$\left( \tilde{\Theta}_V^{-1} g \right)(x) = \int_0^\infty \tilde{\varphi}(x; t) g(t) \, d\tilde{\mu}_V(t), \quad x \in (a, b),$$

$$g \in L^2(\tilde{\mu}_V), \quad \text{supp } g \text{ compact}.$$

The existence of a Fourier transform into a scalar $L^2$-space shows in particular that the spectrum of $A_V$ is simple.

4.11 Remark. Recall that the solution $\varphi$ is not unique. If $\varphi$ is multiplied by a positive constant $r$, then $\tilde{w}_l$ and $\tilde{\theta}$ are divided by $r$, $\tilde{\varphi}$ is multiplied by $r$, and $\tilde{m}_V$ and $\tilde{\mu}_V$ are divided by $r^2$. However, in the situation of Example 4.5 (i) one can normalize $\varphi$ such that (4.10) holds. ♦

Finally, let us state global and local uniqueness theorems. For the case of Bessel-type potentials as in Example 4.5 (i) see, e.g. [11, Theorem 5.1].

4.12 Theorem (Global Uniqueness Theorem).

Let $V_1, V_2 \in \mathcal{K}_{\text{Schr}}$ be given with $\text{dom}(V_i) = (0, b_i)$, $i = 1, 2$. Then the following statements are equivalent:

(i) $b_1 = b_2$ and $V_1(x) = V_2(x)$, $x \in (0, b_1)$ a.e.;

(ii) there exists a $c > 0$ such that $[\tilde{m}]_{V_1} = c[\tilde{m}]_{V_2};$

(iii) there exists a $c > 0$ such that $\tilde{\mu}_V = c\tilde{\mu}_V$. 33
Proof. For the implication (i) $\Rightarrow$ (ii) see Remark 4.11. The equivalence of (ii) and (iii) is clear from the definition of $\tilde{\mu}_V$. Now suppose that (ii) holds and let $p_1 = w_1 = \phi_1^2$ be as in (4.6). By rescaling $\phi_2$ we may assume that $c = 1$. Then we have $[m]_{p_1,w_1} = [m]_{p_2,w_2}$. It follows from Theorem 2.15 that there exists $\gamma : (0,b_2) \to (0,b_1)$ such that (2.46) holds. However, this implies that $\gamma'(x) = 1$ a.e., and hence $b_1 = b_2$ and $\phi_1 = \phi_2$. This shows that $V_1 = V_2$, i.e. (i) is satisfied.

Local uniqueness theorems for Schrödinger equations have attracted a lot of attention recently. For the case of a regular left endpoint B. Simon proved the first version of such a theorem in [45, Theorem 1.2]; alternative proofs were given in [17], [3] and [33]. For Bessel-type operators with potentials as in Example 4.5 (i) a local uniqueness theorem was proved in [27, Theorem 4.1].

4.13 Theorem (Local Uniqueness Theorem).
Let $V_1, V_2 \in \mathcal{K}_{\text{Schr}}$ be given with $\text{dom}(V_i) = (0,b_i)$, $i = 1, 2$. Then, for $\tau > 0$, the following statements are equivalent:

(i) one has $V_1(x) = V_2(x)$, $x \in (0, \min\{\tau, b_1, b_2\})$ a.e.;

(ii) there exist singular Titchmarsh–Weyl coefficients $\tilde{m}_{V_1}$ and $\tilde{m}_{V_2}$ and there exist $c > 0$ and $\beta \in (0, 2\pi)$ such that, for each $\varepsilon > 0$,

$$\tilde{m}_{V_1}(re^{i\beta}) - c\tilde{m}_{V_2}(re^{i\beta}) = O(e^{(-2\varepsilon + \varepsilon)\sqrt{\tau}\sin \frac{\pi}{\theta}}), \quad r \to \infty;$$

(iii) there exist singular Titchmarsh–Weyl coefficients $\tilde{m}_{V_1}$ and $\tilde{m}_{V_2}$ and there exist $c > 0$ and $k \geq 0$ such that, for each $\delta \in (0, \pi)$,

$$\tilde{m}_{V_1}(\lambda) - c\tilde{m}_{V_2}(\lambda) = O(|\lambda|^k e^{-2\tau \text{Im} \sqrt{\lambda}}),$$

where $|\lambda| \to \infty$, $\lambda \in \{z \in \mathbb{C} : \delta \leq \arg z \leq 2\pi - \delta\}$,

where $\sqrt{\lambda}$ is chosen so that $\text{Im} \sqrt{\lambda} > 0$.

Proof. This theorem follows from Theorem 2.17; we only have to observe that $s_i(\tau) = \min\{\tau, b_i\}$ and that the validity of (2.46) with $p_i = w_i$ implies that $\gamma(x) = x$ for $x \in (0, \tau)$.

Let us conclude this section with two remarks about possible extensions.

4.14 Remark. With a similar method one can also treat general Sturm–Liouville equations of the form

$$-(Py')' + Qy = \lambda Wy$$

(4.19)

where $1/P$, $Q$ and $W$ are locally integrable. If a positive solution $\phi$ of (4.19) with $\lambda = 0$ exists such that $\phi \in L^2(W|_{a,z_0})$, then one can use the mapping $u \mapsto \frac{u}{z}$ to transform (4.19) to an equation of the form (2.1) with

$$p := P\phi^2, \quad w := W\phi^2;$$

cf. [41, Lemma 3.2]. Using Theorem 2.15 one can show that if the spectral measures corresponding two equations of the form (4.19) coincide, then the coefficients are related via a Liouville transform; see [12, Theorem 3.4] for a related result and [4, Theorem 4.2] for the case when $W \equiv 1$ and the left endpoint is regular.

\[\Diamond\]
4.15 Remark. One can also apply the results of the first part [37] of the paper to Dirac systems of the form
\[-Ju' + Vu = zu\] (4.20)
on an interval \((a, b)\), where \(V\) is a real-valued, symmetric and locally integrable \(2 \times 2\)-matrix function, \(z \in \mathbb{C}\) is the spectral parameter and \(u\) is a 2-vector function. Assume that there exists a solution \(\phi\) of (4.20) with \(z = 0\) (i.e. \(J\phi' = V\phi\)) which is in \((L^2(a, x_0))^2\) for some \(x_0 \in (a, b)\). Under this assumption we can transform (4.20) into a canonical system (2.7) as it was done in [30, Section 4.1, pp. 336, 337]. Let \(\Phi\) be a \(2 \times 2\)-matrix solution of \(J\Phi' = V\Phi\) (i.e. columns of \(\Phi\) are solutions of (4.20) with \(z = 0\)) such that
\[
\begin{pmatrix}
\Phi_{12} \\
\Phi_{22}
\end{pmatrix} = \phi \quad \text{and} \quad \det \Phi(x_0) = 1.
\]
From the second relation it follows that \(\Phi(x_0)^T J \Phi(x_0) = J\). Since \(\frac{d}{dx} (\Phi^T J \Phi) = 0\), we have
\[
\Phi^T J \Phi = J \quad \text{on} \quad (a, b),
\]
and hence \(\det \Phi(x) = 1\), \(x \in (a, b)\). Set
\[
H := \Phi^T \Phi,
\]
which is clearly symmetric and non-negative and does not vanish on any set of positive measure. It is easy to see that \(y\) is a solution of (2.7) if and only if \(u := \Phi y\) is a solution of (4.20). Since \(h_{22} = \Phi_{12}^2 + \Phi_{22}^2 = \phi_1^2 + \phi_2^2\), condition (I) in [37, Definition 2.2] is satisfied. If \(H \in \mathbb{H}\), i.e. also (HS) and (\(\Delta\)) are fulfilled, then one can apply the results from [37, Sections 4–6].

In order to write the results in a more intrinsic form, one can use the unitary transformation
\[
U : \begin{cases} L^2(H) \to (L^2(a, b))^2, \\
y \mapsto \Phi u,
\end{cases}
\]
whose inverse acts like \(U^{-1} u = \Phi^{-1} u = -J \Phi^T J u\). For instance, one can define regularized boundary values by \(rbv^\text{Dir}_z u := rbv^\text{Dir}_z U^{-1} u\) as in Section 4. Details are left to the reader. See also, e.g. [6], [13] for different approaches to Dirac operators.

\[\diamondsuit\]

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