Direct and Inverse Spectral Theorems for a
Class of Canonical Systems with two
Singular Endpoints. Part I: General Theory

MATTHIAS LANGER * HARALD WORACEK

Abstract: This paper deals with two-dimensional canonical systems
\[ y'(x) = zJH(x)y(x), \quad x \in (a, b), \]
whose Hamiltonian \( H \) is non-negative and locally integrable, and where Weyl’s limit point case prevails at both endpoints \( a \) and \( b \).
We investigate a class of such systems defined by growth restrictions on \( H \) towards \( a \). We develop a direct and inverse spectral theory parallel to the theory of Weyl and de Branges for systems in the limit circle case at \( a \). Our approach proceeds via — and is bound to — Pontryagin space theory. It relies on spectral theory and operator models in such spaces, and on the theory of de Branges Pontryagin spaces.
The main results concerning the direct problem are: (1) showing existence of regularized boundary values at \( a \); (2) construction of a singular Weyl coefficient and a scalar spectral measure; (3) construction of a Fourier transform and computation of its action and the action of its inverse as integral transforms. The main results for the inverse problem are: (4) characterization of the class of measures that are obtained via the above construction (positive Borel measures with power growth at \( \pm \infty \)); (5) a global uniqueness theorem (if Weyl functions or spectral measures coincide, Hamiltonians essentially coincide); (6) a local uniqueness theorem (if Weyl functions coincide up to an exponentially small error, Hamiltonians essentially coincide up to a certain point).
In Part II of the paper we shall apply these results to Sturm–Liouville equations with singular coefficients.

AMS MSC 2010: 34B05, 34L40, 34B20, 34A55, 47B50, 47B32

Keywords: canonical system, Sturm–Liouville equation, singular potential, direct and inverse spectral theorems, Pontryagin space, de Branges space

1 Introduction

By a Hamiltonian we understand a function \( H \) defined on a (possibly unbounded) interval \( (a, b) \), which takes real and non-negative \( 2 \times 2 \)-matrices as values, is locally integrable, and does not vanish on any set of positive measure.
Throughout this paper we assume that Weyl’s limit point case prevails at the endpoint \( b \); this means that for one (and hence for all) \( x_0 \in (a, b) \), we have \( \int_{x_0}^b \text{tr} H(x) \, dx = \infty \).
The canonical system associated with \( H \) is the differential equation
\[ y'(x) = zJH(x)y(x), \quad x \in (a, b), \tag{1.1} \]
where \( z \) is a complex parameter (the eigenvalue parameter), \( J \) is the signature matrix \( J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and \( y \) is a 2-vector-valued function. Canonical systems appear frequently in natural sciences, for example in Hamiltonian mechanics or
as generalizations of Sturm–Liouville problems, e.g. in the study of a vibrating string with non-homogeneous mass distribution. They provide a unifying approach to Schrödinger operators, Jacobi operators and Krein strings. Some selected references are [1], [2], [20], [58] for relevance in physics, and [4], [35], [36], [61] for the relation to scalar second order differential or difference equations.

The theory of canonical systems was developed in works of Stieltjes, Weyl, Markov, Krein, Kac and de Branges. There is a vast literature, especially on spectral theory, ranging from classical papers to very recent work. As examples we mention [31], [25], [10], [59], [34], [62], [26], [27], [64], [65], [41], [3], [47].

Our standard reference is [26], where the spectral theory of canonical systems is developed in a modern operator-theoretic language.

With a Hamiltonian \( H \) one can associate a Hilbert space \( L^2(H) \) and a (minimal) differential operator \( S(H) \); see Section 3.2. The spectral theory of \( S(H) \) changes drastically depending whether at the endpoint a Weyl’s limit circle case (LC) or Weyl’s limit point case (LP) prevails, i.e. whether for one (and hence for all) \( x_0 \in (a, b) \)

\[
(LC) : \int_a^{x_0} \text{tr} H(x) \, dx < \infty \quad \text{or} \quad (LP) : \int_a^{x_0} \text{tr} H(x) \, dx = \infty.
\]

Note that because of the non-negativity of \( H \) the Hamiltonian \( H \) is in the limit circle case if and only if all entries of \( H \) are integrable at \( a \).

**Limit circle case.**

Assume that \( H \) is in the limit circle case at its left endpoint (and, as always in this paper, in the limit point case at its right endpoint). Then the operator \( S(H) \) is symmetric with deficiency index \((1, 1)\). A complex-valued function \( q_H \), the Weyl coefficient of \( H \), can be constructed as follows. Let \( \theta(\cdot; z) \) and \( \varphi(\cdot; z) \) be the solutions of (1.1) that satisfy the initial conditions \( \theta(a; z) = (1, 0) \) and \( \varphi(a; z) = (0, 1) \), respectively; note that \( H \) is integrable at \( a \). The Weyl coefficient \( q_H \) is defined by

\[
q_H(z) := \lim_{\tau \to b} \frac{\theta_1(x; z)\tau + \Theta_2(x; z)}{\varphi_1(x; z)\tau + \varphi_2(x; z)}, \quad z \in \mathbb{C} \setminus \mathbb{R},
\]

with \( \tau \in \mathbb{R} \cup \{\infty\} \); the limit is independent of \( \tau \) since \( H \) is in the limit point case at \( b \). The function \( q_H \) belongs to the Nevanlinna class \( \mathcal{N}_0 \), i.e. it is analytic in \( \mathbb{C} \setminus \mathbb{R} \), symmetric with respect to the real line in the sense that \( q_H(z) = \overline{q_H(\overline{z})} \), \( z \in \mathbb{C} \setminus \mathbb{R} \), and maps the open upper half-plane \( \mathbb{C}^+ \) into \( \mathbb{C}^+ \cup \mathbb{R} \).

The Weyl coefficient \( q_H \) can be used to construct a spectral measure and a Fourier transform. Let \( \mu_H \) be the measure in the Herglotz integral representation of \( q_H \) (see (3.1) below) appropriately including a possible point mass at \( \infty \), and define an integral transformation \( \Theta_H \) by

\[
(\Theta_H f)(t) := \int_a^b \psi(x; t)^T H(x) f(x) \, dx, \quad f \in L^2(H), \ \sup(\text{supp} f) < b.
\]

Then a direct spectral theorem holds; more precisely, the following is true.

(1) The map \( \Theta_H \) extends to an isometric isomorphism from \( L^2(H) \) onto \( L^2(\mu_H) \), where we tacitly understand that the space \( L^2(\mu_H) \) appropriately includes a possible point mass at \( \infty \).
(2) This extension of $\Theta_H$ establishes a unitary equivalence between the self-adjoint extension of $S(H)$ that is determined by the boundary condition $y_1(a) = 0$ and the operator $M_{\mu_H}$ of multiplication by the independent variable in the space $L^2(\mu_H)$.

This direct theorem shows, in particular, that the mentioned self-adjoint extension of $S(H)$ has simple spectrum.

An inverse spectral theorem was proved by L. de Branges in [6]–[9], in particular [7, Theorem XII] and [9, Theorem VII]; see also [66] for an explicit treatment. These results include the following statements.

(1) Let a function $q$ in the Nevanlinna class $N_0$ be given. Then there exists a Hamiltonian $H$ that is in the limit circle case at its left endpoint (and in the limit point case at its right endpoint) such that $qH = q$.

(2) Let a positive scalar measure $\mu$ with $\int_{\mathbb{R}}(1 + t^2)^{-1}d\mu(t) < \infty$ be given (plus a possible point mass at $\infty$). Then there exists a Hamiltonian $H$ that is in the limit circle case at its left endpoint (and in the limit point case at its right endpoint) such that $\mu = \mu_H$ (and possible point masses at $\infty$ coincide).

(3) Let two Hamiltonians $H_1$ and $H_2$ be given, both being in the limit circle case at their left endpoints (and in the limit point case at their right endpoints). Then we have $qH_1 = qH_2$ if and only if $H_1$ and $H_2$ are reparameterizations of each other; the latter means that $H_2(x) = H_1(\gamma(x))\gamma'(x)$ with some increasing bijection $\gamma$ such that $\gamma$ and $\gamma^{-1}$ are absolutely continuous.

(4) Let two Hamiltonians $H_1$ and $H_2$ be given, both being in the limit circle case at their left endpoints (and in the limit point case at their right endpoints). Then we have $\mu_{H_1} = \mu_{H_2}$ (and possible point masses at $\infty$ coincide) if and only if there exists a real constant $\alpha$ such that the Hamiltonians $H_1$, $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} H_2 \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$ are reparameterizations of each other.

Limit point case.

If the limit point case prevails (also) at the left endpoint, much less can be said in general. The operator $S(H)$ is self-adjoint, and its spectral multiplicity cannot exceed 2. A $2 \times 2$-matrix-valued Weyl coefficient can be defined. Via the Titchmarsh–Kodaira formula, this leads to a Fourier transform onto an $L^2$-space with respect to a $2 \times 2$-matrix-valued measure; see, e.g. [26], and [44] or [24, §2] for Schrödinger equations.

For Hamiltonians being in the limit point case non-simple spectrum can appear; and this is not an exceptional case. The class of all Hamiltonians that have simple spectrum — despite being in the limit point case at both endpoints — can be characterized based on a theorem of I. S. Kac from the 1960s; see [32, Fundamental Theorem]¹. However, given the Hamiltonian $H$, the condition given in Kac’s Theorem is hardly accessible to computation. To the best of

¹A proof is given in [33] (in Russian).
our knowledge an explicit characterization of simplicity of the spectrum is not known. An easy-to-check sufficient condition for $S(H)$ having simple spectrum follows from a result of L. de Branges; see [10, Theorems 40 and 41].

In the study of limit point Hamiltonians with simple spectrum there remain some major drawbacks compared with the limit circle situation. Even in the situation of de Branges’ Theorem there is neither a canonical way to choose a scalar-valued spectral measure $\mu$ nor further information on properties of $\mu$ can be obtained. In view of this fact, naturally, there are no inverse statements asserting existence or uniqueness of a Hamiltonian which would lead to a given measure.

**The main results of the present paper.**

We specify a class $\mathbb{H}$ of Hamiltonians, which are in the limit point case at both endpoints and for which a Weyl theory analogous to the limit circle case can be developed. This class $\mathbb{H}$ is a proper subclass of the one familiar from de Branges’ theorem mentioned above, but it is still sufficiently large to cover many cases of interest.

Concerning the direct spectral problem, we show that, for each Hamiltonian $H \in \mathbb{H}$,

1. each solution of equation (1.1) attains regularized boundary values at $a$ (Theorem 4.2), and an analogue to the Weyl coefficient can be defined, which we call *singular Weyl coefficient* and which is unique up to an additive real polynomial (Theorem 4.5);

2. a Fourier transform onto an $L^2$-space generated by a scalar measure exists; one measure with this property can be constructed in a canonical way (Theorem 4.8), and the corresponding Fourier transform and its inverse can be written as integral transforms (Theorem 5.1).

Concerning the, now meaningfully posed, inverse spectral problem, we

3. characterize the class of measures occurring via the mentioned construction (Theorem 6.1);

4. establish global and local uniqueness results (Theorems 6.2 and 6.3);

5. establish a one-to-one correspondence between the growth of the Hamiltonian $H$ at $a$ and the growth of the spectral measure $\mu_H$ at infinity, measured by a positive integer $\Delta$ (Theorem 4.8).

Let us mention that during the last decade there has been a lot of interest in Sturm–Liouville equations with two singular endpoints where still a scalar Titchmarsh–Weyl coefficient can be introduced and a corresponding Fourier transform into an $L^2$ with a scalar measure can be constructed; see, in particular, [21], [23]. In Part II [57] of the paper we shall apply the results from the current paper to Sturm–Liouville equations without potential, $-(py')' = \lambda wy$, and to one-dimensional Schrödinger equations, $-y'' + Vy = \lambda y$, with coefficients that are singular at both endpoints.
Methods employed.

In order to establish our present results, we utilize the theory of indefinite inner product spaces. Our approach proceeds via Pontryagin space theory, i.e. the theory of indefinite inner product spaces with a finite-dimensional negative part. In some sense our approach reaches as far as Pontryagin space models possibly can. One key idea is to extend the Hamiltonian $H$ to the left by a so-called indivisible interval so that the original left endpoint $a$ becomes an interior point where $H$ is singular. We can then apply the theory of generalized Hamiltonians, developed in [40]–[43] and also [54], for which corresponding operator models act in Pontryagin spaces (in general, a generalized Hamiltonian can have a finite number of interior singularities).

We use operator-theoretic tools like the spectral theory of self-adjoint relations, models for generalized Nevanlinna functions and for generalized Hamiltonians, as well as the theory of de Branges Pontryagin spaces of entire functions. In particular, proofs rely heavily on the theory developed in [54] and [56] and in [40]–[43]. We would like to mention that the underlying relation in the Pontryagin space is of the most intriguing (but also most difficult to handle) kind: it is a proper relation having infinity as a singular critical point with a neutral algebraic eigenspace.

Organization of the manuscript.

The paper is divided into sections according to the following table.

<table>
<thead>
<tr>
<th>Table of contents</th>
<th>p.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2. The two basic classes</td>
<td>6</td>
</tr>
<tr>
<td>3. Preliminaries from indefinite theory</td>
<td>8</td>
</tr>
<tr>
<td>4. Construction of the spectral measure</td>
<td>18</td>
</tr>
<tr>
<td>5. The Fourier transform</td>
<td>25</td>
</tr>
<tr>
<td>6. Inverse theorems</td>
<td>46</td>
</tr>
</tbody>
</table>

In Section 2 we introduce the class $\mathbb{H}$ of Hamiltonians that is treated in our paper. The definition involves a certain growth condition of the Hamiltonian at the left endpoint $a$. We associate a positive integer, $\Delta(H)$, with each $H \in \mathbb{H}$, which measures the growth of $H$ at $a$. Further, we define a class $\mathcal{M}$ of Borel measures on $\mathbb{R}$ that satisfy a certain growth condition at infinity; this class will turn out to be the set of spectral measures of Hamiltonians from $\mathbb{H}$. In Section 3 we recall the definition and certain properties of generalized Nevanlinna functions and the operator that is connected with equation (1.1). Moreover, we recall the notion of generalized Hamiltonians, a certain subclass of generalized Hamiltonians that have only one interior singularity, and corresponding operator models. In Section 4 we show that solutions of (1.1) attain regularized boundary values at $a$ (Theorem 4.2), we construct singular Weyl coefficients (Theorem 4.5) and construct a spectral measure via a Stieltjes-type inversion formula (Theorem 4.8). The Fourier transform is constructed in Section 5 (Theorem 5.1); this shows, in particular, that the spectrum is simple. Inverse spectral theorems (existence and global and local uniqueness theorem) are proved in Section 6 (Theorems 6.1, 6.2 and 6.3). As mentioned above, we shall apply the results to Sturm–Liouville equations in Part II [57] of the paper.
2 The two basic classes

We start with the definition and a brief discussion of the two major objects of our investigation. These are a class $\mathcal{H}$ of Hamiltonians and a class $\mathcal{M}$ of measures, which will turn out to correspond to each other.

2.1 The class $\mathcal{H}$ of Hamiltonians

Let us state the definition of Hamiltonians again explicitly: by a Hamiltonian $H = (h_{ij})_{i,j=1}^2$ we understand a function defined on some (non-empty and possibly unbounded) interval $(a,b)$ whose values are non-negative $2 \times 2$-matrices, which is locally integrable and which does not vanish on any set of positive measure. In the rest of the paper we shall also write $\text{dom}(H) := (a,b)$ if $H$ is defined on $(a,b)$.

We say that two Hamiltonians $H_1$ and $H_2$ defined on intervals $(a_1,b_1)$ and $(a_2,b_2)$, respectively, are reparameterizations of each other if there exists an increasing bijection $\gamma : (a_2,b_2) \to (a_1,b_1)$ such that $\gamma$ and $\gamma^{-1}$ are both absolutely continuous and

$$H_2(x) = H_1(\gamma(x)) \cdot \gamma'(x), \quad x \in (a_2,b_2) \text{ a.e.} \quad (2.1)$$

Note that in this situation $y$ is a solution of (1.1) with $H = H_1$ if and only if $\tilde{y}$, where $\tilde{y}(x) = y(\gamma(x))$, is a solution of (1.1) with $H = H_2$.

2.1 Remark. As a rule of thumb, Hamiltonians which are reparameterizations of each other share all their essential properties. For a detailed and explicit exposition of reparameterizations in an up-to-date language, see [67] (in particular, Theorem 3.8 therein).

We also recall the notion of indivisible intervals. An interval $(\alpha,\beta) \subseteq (a,b)$ is called $H$-indivisible (or just indivisible) of type $\phi$ if

$$H(x) = h(x)\xi_{\phi}^T, \quad x \in (\alpha,\beta), \quad (2.2)$$

where $\xi_{\phi} = (\cos \phi, \sin \phi)^T$ and $h$ is a locally integrable function that is positive almost everywhere; see, e.g. [34]. An indivisible interval $(\alpha,\beta)$ is called maximal if it is not contained in any larger indivisible interval.

2.2 Definition. Let $H = (h_{ij})_{i,j=1}^2$ be a Hamiltonian defined on $(a,b)$. We say that $H$ belongs to the class $\mathcal{H}$ if $H$ is in the limit point case at both endpoints, the interval $(a,b)$ is neither one indivisible interval nor the union of two indivisible intervals, and $H$ satisfies the following conditions (I), (HS) and ($\Delta$).

(I) For one (and hence for all) $x_0 \in (a,b)$,

$$\int_a^{x_0} h_{22}(x) \, dx < \infty.$$ 

(HS) For one (and hence for all) $x_0 \in (a,b)$,

$$\int_a^{x_0} \int_a^x h_{22}(t) \, dt \, h_{11}(x) \, dx < \infty.$$ 

$\Diamond$
(Δ) Let $x_0 \in (a, b)$ and define functions $X_k : (a, x_0) \to \mathbb{C}^2$ recursively by

$$X_0(x) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x \in (a, x_0],$$

$$X_k(x) := \int_{x_0}^x JH(t)X_{k-1}(t) \, dt, \quad x \in (a, x_0], \ k \in \mathbb{N}.$$ 

There exists a number $N \in \mathbb{N}_0$ such that

$$L^2(H|_{(a, x_0)}) \cap \text{span} \{X_k : k \leq N\} \neq \{0\}. \quad (2.3)$$

If $H \in \mathbb{H}$, we denote by $\Delta(H)$ the smallest non-negative integer $N$ such that $(2.3)$ holds.

It was proved in [40, Lemma 3.12] that this definition is justified, namely that the validity of $(\Delta)$ and the number $\Delta(H)$ do not depend on the choice of $x_0$ (for (I) and (HS) this is trivial to check).

Notice that, for $H \in \mathbb{H}$, we always have $\Delta(H) > 0$. This follows since we assume limit point case at $a$. Namely, for each $x_0 \in (a, b)$, the constant function $(0, 1)^T$ belongs to $L^2(H|_{(a, x_0)})$ by (I), and hence the constant $(1, 0)^T$ cannot be in this space.

2.3 Remark. We assume that $(a, b)$ is neither one indivisible interval nor the union of two indivisible intervals since, otherwise, the corresponding space $L^2(H)$ (defined in Section 3.2) and hence also the Fourier transform would be trivial.

2.4 Remark. The conditions (I) and (HS) are, up to a normalization and exchanging upper and lower rows, precisely the conditions of de Branges’ Theorem [10, Theorem 41]. Note that under the conditions (I) and (HS) any self-adjoint realization corresponding to $H|_{(a, x_0)}$ has a Hilbert–Schmidt resolvent. The additional condition $(\Delta)$ arose only recently in the context of indefinite canonical systems; we recall more details in §3.2 below.

In general it is difficult to decide whether a given Hamiltonian satisfies $(\Delta)$. Contrasting (I) and (HS) the condition $(\Delta)$ is of recursive nature and not accessible by simple computation. An easier-to-handle (though still recursive) criterion for the validity of $(\Delta)$ is available for Hamiltonians of diagonal form, cf. [68, Theorem 3.7] and [57, Section 2]. Using this criterion, various examples can be constructed. The following two examples are taken from [68, Corollary 3.14 and Example 3.15].

2.5 Example. Let $\alpha \in \mathbb{R}$ and set

$$H_{\alpha}(x) := \begin{pmatrix} x^{-\alpha} & 0 \\ 0 & 1 \end{pmatrix}, \quad x \in (0, \infty).$$

Then $H_{\alpha}$ is in the limit point case at $\infty$ and satisfies (I) at 0. Depending on the value of $\alpha$, the following conditions hold:

<table>
<thead>
<tr>
<th>value of $\alpha$</th>
<th>(LP)/(LC) at 0</th>
<th>(HS) and $(\Delta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha &lt; 1$</td>
<td>(LC)</td>
<td>both hold (trivially)</td>
</tr>
<tr>
<td>$1 \leq \alpha &lt; 2$</td>
<td>(LP)</td>
<td>both hold</td>
</tr>
<tr>
<td>$\alpha \geq 2$</td>
<td>(LP)</td>
<td>none holds</td>
</tr>
</tbody>
</table>
Hence, we have $H_\alpha \in \mathbb{H}$ for each $\alpha \in [1, 2)$ but not for other values of $\alpha$.

The number $\Delta(H_\alpha)$ can be computed, namely,

$$\Delta(H_\alpha) = n, \quad \text{when } \alpha \in \left(2 - \frac{1}{n}, 2 - \frac{1}{n+1}\right) \text{ with } n \in \mathbb{N}.$$ 

This shows that, for a Hamiltonian of class $\mathbb{H}$, there are no a priori restrictions on the value of the number $\Delta(H)$. Computing $\Delta(H_\alpha)$ for $\alpha = 2 - \frac{1}{n}$ with $n \in \mathbb{N}$ is equally well possible, but requires more elaborate computations. These have not been carried out in [68] but will be made available elsewhere.

\[\diamondsuit\]

2.6 Example. Consider the Hamiltonian

$$H(x) := \begin{pmatrix} (x \ln x)^{-2} & 0 \\ 0 & 1 \end{pmatrix}, \quad x \in (0, 1).$$

This Hamiltonian is in the limit point case at 0 and at 1, satisfies (I) and (HS) at 0, but does not satisfy ($\Delta$).

This example shows that the presently considered class $\mathbb{H}$ is a proper subclass of the one treated in [10, Theorem 41].

\[\diamondsuit\]

2.2 The class $\mathbb{M}$ of measures

By a positive Borel measure on $\mathbb{R}$ we understand a (not necessarily finite) positive measure defined on the $\sigma$-algebra of all Borel subsets of $\mathbb{R}$ which takes finite values on compact sets.

\[\diamondsuit\]

2.7 Definition. Let $\mu$ be a positive Borel measure on $\mathbb{R}$. We say that $\mu$ belongs to the class $\mathbb{M}$ if there exists a number $N \in \mathbb{N}_0$ such that

$$\int_{\mathbb{R}} \frac{d\mu(t)}{(1+t^2)^{N+1}} < \infty. \quad (2.4)$$

If $\mu \in \mathbb{M}$, we denote by $\Delta(\mu)$ the smallest non-negative integer $N$ such that (2.4) holds.

This class of measures is known from Pontryagin space theory. A measure $\mu$ belongs to $\mathbb{M}$ if and only if it is the measure in the distributional representation of some generalized Nevanlinna function with a certain spectral behaviour, cf. [46] and [56, Theorems 2.8 and 3.9]. In the classical (positive definite) setting, this corresponds to the fact that a positive Borel measure $\mu$ satisfies $\int_{\mathbb{R}} (1+t^2)^{-1} d\mu(t)$ if and only if it is the measure in the Herglotz integral representation of some Nevanlinna function. We recall details in §3.1 below.

3 Preliminaries from indefinite theory

Our approach to direct and inverse spectral theory for Hamiltonians of class $\mathbb{H}$ is based on the theory of indefinite canonical systems and their Pontryagin space operator models as developed in [40]–[43] and further in [52], [54]. In this preliminary section we recall the relevant notions and theorems. For the theory of Pontryagin space we refer the reader, e.g. to [5].
3.1 Generalized Nevanlinna functions and the class $N_{<\infty}$

As we already mentioned in the introduction, a function $q$ is said to be a Nevanlinna function if it is analytic in $\mathbb{C} \setminus \mathbb{R}$, satisfies $q(z) = \overline{q(\overline{z})}$ for $z \in \mathbb{C} \setminus \mathbb{R}$ and $\Im q(z) \geq 0$ for $z \in \mathbb{C}^+$. We denote the set of all Nevanlinna functions by $N_0$.

In Pontryagin space theory an indefinite analogue of this notion appears and plays a significant role; see, e.g. [45], [46].

3.1 Definition. A function $q$ is called a generalized Nevanlinna function if it is meromorphic in $\mathbb{C} \setminus \mathbb{R}$ and has the following properties (i) and (ii):

(i) $q(z) = \overline{q(\overline{z})}$ for $z \in \rho(q)$, where $\rho(q)$ denotes the domain of analyticity of $q$ in $\mathbb{C} \setminus \mathbb{R}$;

(ii) the reproducing kernel

$$K_q(w, z) := \frac{q(z) - \overline{q(w)}}{z - \overline{w}}, \quad z, w \in \rho(q),$$

has a finite number of negative squares. The latter means that there exists a $\kappa \in \mathbb{N}_0$ so that for every choice of $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in \rho(q)$ the matrices $(K_q(z_i, z_j))_{i,j=1}^n$ have at most $\kappa$ negative eigenvalues.

We denote the set of all generalized Nevanlinna functions by $N_{<\infty}$. Moreover, if $q \in N_{<\infty}$, we denote the actual number of negative squares of the kernel $K_q$ (i.e. the minimal $\kappa$ in the above definition) by $\text{ind}_- q$. Further, we set $N_\kappa := \{q \in N_{<\infty} : \text{ind}_- q = \kappa\}$ for $\kappa \in \mathbb{N}_0$.

That this definition is indeed an extension of the definition of $N_0$, i.e. that the class $N_0$ in Definition 3.1 coincides with the class defined before Definition 3.1 is a classical result, which can be traced back to as far as [28] or [60].

Let $q \in N_0$. Using the representation of the positive harmonic function $\Im q$ as a Poisson integral, one easily obtains a representation of $q$ with a Cauchy-type integral.

3.2. Herglotz integral representation of $N_0$-functions. A function $q$ belongs to the class $N_0$ if and only if it can be represented in the form

$$q(z) = a + bz + \int_\mathbb{R} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu(t), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (3.1)$$

with $a \in \mathbb{R}$, $b \geq 0$ and a positive Borel measure $\mu$ satisfying $\int_\mathbb{R} (1+t^2)^{-1} d\mu(t) < \infty$.

The analogue of this integral representation in the indefinite setting is a distributional representation of a generalized Nevanlinna function. In essence this is shown in [46], where an integral representation of $q \in N_{<\infty}$ was given without using the language of distributions. The distributional viewpoint was first mentioned in [30, Introduction, p. 253], established more thoroughly in [39, Corollary 3.5] and refined in [37, Proposition 5.4]. The formulation given below is taken from our recent paper [56]. This paper contains several results which are crucial for the present discussion and is our standard reference in the context of distributional representations.

Before we can provide the actual statement, we need to introduce some notation. First, we denote by $\mathbb{R}(z)$ the set of all rational functions with real
coefficients. Second, we denote by \( \mathbb{R} \) the one-point compactification of the real line considered as a \( C^{\infty} \)-manifold in the usual way. Moreover, for each \( z \in \mathbb{C} \setminus \mathbb{R} \), let \( \beta_z : \mathbb{R} \to \mathbb{C} \) be defined by

\[
\beta_z(t) := \begin{cases} 
1 + \frac{xz}{x - z}, & x \in \mathbb{R}, \\
z, & x = \infty.
\end{cases}
\]

Third, for a function \( f \), set \( f^\#(z) := f(z) \) whenever \( z \in \mathbb{C} \) is in the domain of \( f \). Further, we denote by \( \mathcal{D}'(\mathbb{R}) \) the set of all distributional densities on \( \mathbb{R} \); see, e.g. [29] or [56]. With each \( \phi \in \mathcal{D}'(\mathbb{R}) \) one can associate a linear functional on \( C^\infty(\mathbb{R}) \), which is again denoted by \( \phi \); see [56, (2.2)]. Next, we denote by \( \mathcal{F}(\mathbb{R}) \) the set of all \( \phi \) that act as a positive measure on \( \mathbb{R} \setminus F \); for details see [56, Definitions 2.1 and 2.3]. Finally, \( \mathcal{F}_{(\infty)} \) is the set of \( \phi \in \mathcal{F}(\mathbb{R}) \) that act as a positive measure on \( \mathbb{R} \). For \( \phi \in \mathcal{F}_{(\infty)} \) we denote by \( \mu_\phi \) the unique positive Borel measure on \( \mathbb{R} \) such that

\[
\phi(f) = \int_\mathbb{R} f(x) \frac{d\mu_\phi(x)}{1 + x^2}, \quad f \in C^\infty(\mathbb{R}), \supp f \subset \mathbb{R}; \quad (3.2)
\]

see [56, Definition 2.4].

3.3 Distributional representation of \( N_{<\infty} \)-functions [37, Proposition 5.4]. Let \( \phi \in \mathcal{F}(\mathbb{R}) \) and \( r \in \mathbb{R}(z) \). Then the function

\[
q(z) := r(z) + \phi(\beta_z)
\]

belongs to \( N_{<\infty} \).

Conversely, let \( q \in N_{<\infty} \) be given. Then there exist unique \( \phi \in \mathcal{F}(\mathbb{R}) \) and \( r \in \mathbb{R}(z) \) such that

(i) the representation (3.3) holds;

(ii) \( r \) is analytic on \( \mathbb{R} \) and remains bounded for \( |z| \to \infty \).

In the present paper the following subclass of \( N_{<\infty} \) plays a central role.

3.4 Definition. We denote by \( N_\kappa(\infty) \), \( \kappa \in \mathbb{N}_0 \), the set of all functions \( q \in N_\kappa \) such that

\[
\lim_{z \to i \infty} q(z) 2^{2\kappa - 1} \in (-\infty, 0) \quad \text{or} \quad \lim_{z \to i \infty} 2^{2\kappa - 1} = \infty, \quad (3.4)
\]

where \( \lim_{z \to i \infty} \) denotes the non-tangential limit, i.e. \( z \to \infty \) inside some Stolz angle \( \{ z \in \mathbb{C} : \varepsilon \leq \arg z \leq \pi - \varepsilon \} \) for one (and hence for all) \( \varepsilon \in (0, \frac{\pi}{2}) \). Moreover, set

\[
N_{<\infty}^{(\infty)} := \bigcup_{\kappa \in \mathbb{N}_0} N_\kappa(\infty).
\]

\( \diamond \)
Let us stress that the significance of the condition in this definition is not that (3.4) holds for some $\kappa$, but that it holds exactly for $\kappa = \text{ind}_- q$. Note that $\mathcal{N}_0^{(\infty)} = \mathcal{N}_0$.

The classes $\mathcal{N}_c^{(\infty)}$ appeared often in the recent literature, where they are also denoted by $\mathcal{N}_c^{\infty}$. Let us mention, for instance, [19], [15], [22] or [48], where Sturm–Liouville equations with singular endpoints or singular perturbations of self-adjoint operators were studied, and [14] in connection with rank one perturbations at infinite coupling and [16], [17], [18] where operator models of such functions were investigated. The class $\mathcal{N}_{c<\infty}$ has an operator-theoretic interpretation, namely, the self-adjoint relation in the operator/relation representation$^2$ of $q$ has $\infty$ has its only spectral point of non-positive type; for details see also [56, §5].

For our present considerations it is essential that the distributional representation of a generalized Nevanlinna function $q$ takes a simple form if $q \in \mathcal{N}_{c<\infty}$. The following result is contained in [56, Theorem 3.9 (i), (ii)].

3.5. Distributional representation of $\mathcal{N}_{c<\infty}$-functions. A function $q$ belongs to the class $\mathcal{N}_{c<\infty}$ if and only if it can be represented as

$$ q(z) := r + \Phi(\beta z) $$

with a real constant $r$ and a distributional density $\Phi \in \mathcal{F}(\infty)$. Denote by $\mu_q$ the measure in (3.2) that is connected with the distributional density $\Phi$ in (3.5), i.e. $\mu_q := \mu_{\Phi_q}$ with notation from (3.2). Then a Stieltjes inversion formula is valid:

$$ \mu_q([a, b]) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \lim_{\delta \to 0^+} \int_{a+\epsilon}^{b+\epsilon} \text{Im} q(t+i\delta) \, dt, \quad [a, b] \subseteq \mathbb{R}; $$

see [56, Theorem 3.9 (ii)].

3.6. Operator model. With a distributional density $\Phi \in \mathcal{F}(\infty)$ one can associate a Pontryagin space $\Pi(\Phi)$, which is the completion of $C^\infty(\mathbb{R})$ with respect to the inner product

$$ [f, g]_\Phi := \Phi(f \overline{g}), \quad f, g \in C^\infty(\mathbb{R}), $$

and a self-adjoint relation $A_\Phi$ in $\Pi(\Phi)$; see [30], [39] or [56, §5]. The space $\Pi(\Phi)$ contains the following set

$$ \left\{ f \in L^2 \left( \frac{\mu_{\Phi}(x)}{1 + x^2} \right) : \text{supp } f \text{ is compact} \right\}. $$

(3.7)

Let $\mathcal{E}_{A_\Phi}(\infty)$ be the algebraic eigenspace at infinity of $A_\Phi$. By [56, Theorem 5.3] there exists an isometric, continuous, surjective map

$$ \psi(\Phi) : \mathcal{E}_{A_\Phi}(\infty)^{[1]} \to L^2 \left( \frac{\mu_{\Phi}(x)}{1 + x^2} \right), $$

which acts as the identity on functions from the set in (3.7).\hfill\Box

---

$^2$A detailed account on the operator representation of scalar-valued generalized Nevanlinna functions can be found in [46, §1]. For a slightly different viewpoint and results for operator-(or matrix-) valued functions, see [45, §3], [11], or some of the vast more recent literature on operator models.
Recall from [13] that every function \( q \in \mathcal{N}_\kappa^\infty \) can be written as \( q(z) = p(z)q_0(z) \) where \( q_0 \in \mathcal{N}_0 \) and \( p \) is a monic real polynomial of degree \( 2\kappa \). Let \( \mu_q \) be the measure associated with \( q \) as in 3.5 and let \( \mu_0 \) be the measure in the integral representation (3.1) of \( q_0 \). Then the Stieltjes inversion formula (3.6) implies that
\[
\mu_q([a,b]) = \int_{[a,b]} p(t) d\mu_0(t)
\]
for every finite interval \([a,b]\) with \( \mu_q\{a\} = \mu_q\{b\} = 0 \). This, together with [51, Corollary 3.1] immediately yields the following lemma.

**3.7 Lemma.** Let \( \kappa \in \mathbb{N} \), let \( q_n \in \mathcal{N}_\kappa^\infty \) for \( n \in \mathbb{N} \) and let \( q \in \mathcal{N}_\kappa^\infty \) such that \( q_n(z) \to q(z) \) locally uniformly on \( \mathbb{C} \setminus \mathbb{R} \). Moreover, let \( \mu_{q_n} \) and \( \mu_q \) be the measures that are associated with \( q_n \) and \( q \), respectively, as in 3.5. Then, for every interval \([a,b]\) with \( \mu_q\{a\} = \mu_q\{b\} = 0 \), we have
\[
\lim_{n \to \infty} \mu_{q_n}([a,b]) = \mu_q([a,b]).
\]

In this lemma the assumption that \( \text{ind}_\kappa q = \text{ind}_\kappa q_n \) is crucial.

### 3.2 The operator associated with a canonical system

We recall the definition of the space \( L^2(H) \) and the corresponding operator. Note that the notion of indivisible intervals and the vector \( \xi_\phi \) were defined in Section 2. The space \( L^2(H) \) is the space of measurable functions \( f \) defined on \((a,b)\) with values in \( \mathbb{C}^2 \) which satisfy \( \int_a^b f^* H f < \infty \) and have the property that \( \xi_\phi^T f \) is constant on every indivisible interval of type \( \phi \), factorized with respect to the equivalence relation \( =_H \) where
\[
f =_H g \iff H(f - g) = 0 \ a.e.
\]
In the space \( L^2(H) \) the operator \( T(H) \) is defined via its graph as
\[
T(H) := \{ (f,g) \in (L^2(H))^2 : \exists \text{ representatives } \hat{f}, \hat{g} \text{ of } f, g \text{ such that } \\
\hat{f} \text{ is locally absolutely continuous and } \hat{f}' = JH\hat{g} \text{ a.e. on } (a,b) \},
\]
Since \( H \) is in the limit point case at both endpoints, the operator \( (T) \) is self-adjoint; see, e.g. [26, §6]. If \( H \) is in the limit circle case at the left endpoint, then \( T(H) \) is the maximal operator (or relation); its adjoint, the minimal operator \( S(H) \), is a symmetric operator.

### 3.3 General Hamiltonians

The definition of the Pontryagin space analogue of a Hamiltonian and the description of its associated canonical system and the operator model is quite long and involved. Here we give only an intuitive picture; for a complete and logically sound formulation we refer to [40, §8] or [54, Definitions 2.16–2.18]. The latter paper is our standard reference in the context of general Hamiltonians.
A general Hamiltonian $\mathfrak{h}$ is a collection of data

$$\mathfrak{h} : \quad n \in \mathbb{N}_0, \quad -\infty \leq \sigma_0 < \sigma_1 < \ldots < \sigma_{n+1} \leq \infty;$$

Hamiltonians $H_i : (\sigma_i, \sigma_{i+1}) \to \mathbb{R}^{2 \times 2}, \ i = 1, \ldots, n$;

$$\tilde{\sigma}_i \in \mathbb{N}_0, \ b_{i,1}, \ldots, b_{i,\tilde{\sigma}_i+1} \in \mathbb{R}, \ d_{i,0}, \ldots, d_{i,2\Delta_i-1} \in \mathbb{R}, \ i = 1, \ldots, n;$$

$$E \subseteq \{\sigma_0, \sigma_{n+1}\} \cup \bigcup_{i=0}^n (\sigma_i, \sigma_{i+1});$$

where, among others,

- $H_0$ is in the limit circle case at $\sigma_0$; if $n \geq 1$, then $H_{i-1}$ is in the limit point case at $\sigma_i$ and $H_i$ is in the limit point case at $\sigma_i$ for $i = 1, \ldots, n$;

- the growth of the Hamiltonians $H_i$ towards the points $\sigma_1, \ldots, \sigma_n$ is restricted; the number $\Delta_i \in \mathbb{N}$ is a certain measure for this growth;

- two adjacent Hamiltonians satisfy an interface condition at their common endpoint.

The general Hamiltonian is called singular if $H_n$ is in the limit point case at $\sigma_{n+1}$; it is called regular if $H_n$ is in the limit circle case at $\sigma_{n+1}$. Moreover, let $H$ be the function defined on $\bigcup_{i=0}^n (\sigma_i, \sigma_{i+1})$ such that $H|_{(\sigma_i, \sigma_{i+1})} = H_i$ for $i = 0, \ldots, n$.

The following visualization of the canonical system associated with a general Hamiltonian may be helpful:

At $\sigma_0$ an initial condition can be prescribed. The points $\sigma_1, \ldots, \sigma_n$ are inner singularities: the Hamiltonian function is in the limit point case from both sides. On the interval $(\sigma_i, \sigma_{i+1})$ a solution of the system behaves according to the canonical differential equation $y'(x) = z J H_i(x) y(x)$. The data $\tilde{\sigma}_i, b_{ij}, d_{ij}$ describe what happens to a solution when passing through the singularity $\sigma_i$. Thereby, $\tilde{\sigma}_i, b_{ij}$ correspond to a point interaction inside the singularity $\sigma_i$, whereas $d_{ij}$ correspond to a local interaction of the Hamiltonians to the left and to the right of the singularity $\sigma_i$. The set $E$ (which we did not indicate in the picture) is used to quantitatively describe the influence of the interface conditions manifested by the data part $d_{ij}$. At the points from the set $E$ the interval $(\sigma_0, \sigma_{n+1})$ is split into smaller pieces that contain at most one singularity.

It can be proved that the spectral theory of singular general Hamiltonians defined in this way is the full Pontryagin space analogue of the theory of classical Hamiltonians (being in the limit circle at their left and in the limit point case at their right endpoint).
(1) With a singular general Hamiltonian $\mathfrak{h}$ a boundary triple $(\mathcal{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h}))$ can be associated and a Weyl coefficient $q_\mathfrak{h}$ can be constructed; see [40, Definition 8.5 and Theorem 8.7] and [42, Theorem 5.1 and Definition 5.2].

(2) The Weyl coefficient $q_\mathfrak{h}$ (see 3.8 below) belongs to the class $N_{<\infty}$ and can be interpreted as a $Q$-function of the minimal operator $S(\mathfrak{h}) := T(\mathfrak{h})^*$, which is a completely non-self-adjoint symmetry with deficiency index $(1, 1)$; see [40, Theorem 8.7] and [42, Proposition 5.19 and Corollary 6.5].

(3) An inverse spectral theorem holds, which states that each generalized Nevanlinna function is the Weyl coefficient of some singular general Hamiltonian and that the general Hamiltonian is, up to reparameterization, uniquely determined by its Weyl coefficients; see [43, Theorem 1.4] and [42, Remark 3.38].

A local uniqueness theorem holds, which states that beginning sections of Hamiltonians are uniquely determined (up to reparameterization) by the asymptotic behaviour of the Weyl function towards $i \infty$; see [53, Theorem 1.2 and Remark 1.3].

We should point out that the term “reparameterization”, which we used without further notice in item (3), actually requires some explanation. Not only the Hamiltonians $H_i$ may be reparameterized (in the sense of the classical theory, see (2.1)) but also the data $d_{ij}$ and $E$ may be changed according to certain rules; see [42, Remark 3.38].

3.8. The fundamental solution. A crucial concept in the theory of general Hamiltonians is the fundamental solution associated with a general Hamiltonian $\mathfrak{h}$. This is the indefinite analogue of the fundamental matrix solution of the system (1.1) in the positive definite case.

It is shown in [42, §5a-c] that with a general Hamiltonian $\mathfrak{h}$ a chain $\omega_\mathfrak{h}$ of entire $2 \times 2$-matrix functions is associated, namely, $\omega_\mathfrak{h} = \omega_\mathfrak{h}(x; z), x \in [\sigma_0, \sigma_{n+1}) \setminus \{\sigma_1, \ldots, \sigma_n\}, z \in \mathbb{C}$, so that, for fixed $x$, the function $\omega_\mathfrak{h}$ is entire in $z$ and, for fixed $z$, it satisfies

$$\frac{\partial}{\partial x} \omega_\mathfrak{h}(x; z)J = z\omega_\mathfrak{h}(x; z)H(x), \quad x \in (\sigma_0, \sigma_{n+1}) \setminus \{\sigma_1, \ldots, \sigma_n\}. \quad (3.9)$$

Note that the rows of $\omega_\mathfrak{h}$ satisfy the differential equation (1.1). Moreover,

$$\omega_\mathfrak{h}(x; 0) = I, \quad \det \omega_\mathfrak{h}(x; z) = 1, \quad x \in [\sigma_0, \sigma_{n+1}) \setminus \{\sigma_1, \ldots, \sigma_n\}, \quad z \in \mathbb{C} \quad (3.10)$$

and $\omega_\mathfrak{h}(\sigma_0; z) = I, \quad z \in \mathbb{C}$. The chain $\omega_\mathfrak{h}$ is used to construct the Weyl coefficient $q_\mathfrak{h}$ using a similar limiting procedure as in the positive definite case: if $\omega_\mathfrak{h}^2 = (\omega_{\mathfrak{h},ij})_{i,j=1}^2$, then

$$q_\mathfrak{h}(z) = \lim_{x \rightarrow \sigma_{n+1}} \frac{\omega_{\mathfrak{h},11}(x; z)\tau + \omega_{\mathfrak{h},12}(x; z)}{\omega_{\mathfrak{h},21}(x; z)\tau + \omega_{\mathfrak{h},22}(x; z)}$$

for $\tau \in \mathbb{R} \cup \{\infty\}$ and $z$ in the domain of holomorphy of $q_\mathfrak{h}$ (which is $\mathbb{C} \setminus \mathbb{R}$ with at most a finite number of points removed); the limit is locally uniform in $z$ and independent of $\tau$; see [39, Lemma 8.2].

In the present paper we use some specific properties of fundamental solutions; detailed references are provided at the appropriate places. Here we only would...
like to mention that two general Hamiltonians that are reparameterizations of each other give rise to the same fundamental solutions (up to reparameterization in the sense of [42, Definition 3.4]) and hence to the same Weyl coefficients; this was shown in [43, Theorem 1.6].

3.9. Splitting of general Hamiltonians. Let $h$ be a general Hamiltonian, let $s \in \bigcup_{i=0}^{n} (\sigma_i,\sigma_{i+1})$ and assume that $s$ is not inner point of an indivisible interval. Then 'restrictions' of $h$ to the intervals $(\sigma_0,s)$ and $(s,\sigma_{n+1})$ can be defined, which are denoted by $h_{\sigma_0}$ and $h_{\sigma_{n+1}}$, respectively; see [42, Definition 3.47] and also [54, §2.19].

3.4 The class $H_0$

In the present paper those general Hamiltonians are of interest whose Weyl coefficients belong to the class $N^{(\infty)}_\infty$. They can be characterized in a neat way; see [54, Theorem 3.1]. For the notion of indivisible intervals see Section 2, (2.2).

3.10 Definition. We say that a singular general Hamiltonian $h$ belongs to the class $H_0$ if

(i) $h$ has exactly one singularity, i.e. $H$ is defined on a set of the form $(\sigma_0,\sigma_1) \cup (\sigma_1,\sigma_2)$;

(ii) the interval $(\sigma_0,\sigma_1)$ is indivisible of type 0, i.e. the Hamiltonian function $H_0$ of $h$ on $(\sigma_0,\sigma_1)$ is of the form $H_0(x) = h_0(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ with some scalar function $h_0$.

It follows from the definition of a general Hamiltonian and the form of $H_0$ that the component $(H_1)_{22}$ is integrable on $(\sigma_1,x_0)$ for some (and hence all) $x_0 \in (\sigma_1,\sigma_2)$, i.e. $H_1$ satisfies condition (I) in Definition 2.2.

3.11. The relation $H_0 \leftrightarrow N^{(\infty)}_\infty$. The content of [54, Theorem 3.1] is the following: a general Hamiltonian $h$ belongs to $H_0$ if and only if its Weyl coefficient $q_h$ belongs to $N^{(\infty)}_\infty \setminus N_0$.

Thereby, the negative index of $q_h$ can be expressed in terms of the general Hamiltonian $h$:

$$\text{ind}_{-} q_h = \Delta_1 + \left\lfloor \frac{\tilde{o}_1}{2} \right\rfloor + \begin{cases} 1, & \text{if } \tilde{o}_1 \text{ odd}, \ b_{1,1} > 0, \\ 0, & \text{otherwise}. \end{cases}$$ (3.11)

This is a particular instance of a general formula shown in [43, Theorem 1.4], namely, that $\text{ind}_{-} q_h = \text{ind}_{-} h$ where $\text{ind}_{-} h$ is given by the formula [54, (2.13)].

Often is convenient to use a particular form of a general Hamiltonian from the class $H_0$, namely

\[ h: \quad \sigma_0 = -1, \ \sigma_1 = 0, \ \sigma_2 = \infty; \quad E = \{-1, x_0, \infty\}; \]

\[ H_0(x) = x^{-2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad x \in (-1, 0); \quad H_1(x), \quad x \in (0, \infty); \]

\[ \tilde{o}_1 \in \mathbb{N}_0, \quad b_{1,1}, \ldots, b_{1,\tilde{o}_1+1} \in \mathbb{R}, \quad d_{1,0}, \ldots, d_{1,2\Delta_1-1} \in \mathbb{R}, \]

where $x_0 \in (0, \infty)$; one can choose $b_{1,\tilde{o}_1+1} = 0$ if no interval of the form $(0, \varepsilon)$ with $\varepsilon > 0$ is indivisible. For every given general Hamiltonian $g \in H_0$ and given $x_0 \in (0, \infty)$ there exists an $h$ as in (3.12) which is a reparameterization of $g$. 

15
3.5 The operator model

The original definition of the boundary triple associated with a general Hamiltonian given in [40] is involved and quite abstract (using a completion procedure). The boundary triple associated with a general Hamiltonian $h$ that has only one singularity can be described isomorphically in a more concrete way; see [52, Definition 2.14 and Theorem 2.15]. Since we deal with the operator model in some depth, we recall its concrete description for a general Hamiltonian of the form (3.12). In the above mentioned reference the component $h_{11}$ is integrable around $\sigma_1$ instead of the component $h_{22}$. One has to apply a rotation isomorphism as defined in [42, Definition 2.4] and also discussed in [54, §2.4] to transform the model from [52] to the current situation.

First we need the following fact, which was shown in [40, Lemma 3.10]; also here one has to apply a rotation isomorphism.

3.12. The functions $w_k$. Let $H \in \mathbb{H}$ with $\text{dom}(H) = (a,b)$. For each $x_0 \in (a,b)$, there exists a unique sequence $(w_k)_{k \in \mathbb{N}_0}$ of absolutely continuous real 2-vector functions on $(a,b)$ such that

$$w_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$w_{l+1} = JH w_l, \quad l \geq 0,$$

$$w_l|_{(a,x_0)} \in L^2(H|_{(a,x_0)}), \quad l \geq \Delta(H),$$

$$w_l(x_0) \in \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, \quad l \geq 0.$$  

(3.13)

Let $H_1, H_2 \in \mathbb{H}$, and assume that $H_1$ and $H_2$ are reparameterizations of each other, say, $H_2(x) = H_1(\gamma(x))\gamma'(x)$, where $\gamma$ is an increasing bijection such that $\gamma$ and $\gamma^{-1}$ are absolutely continuous. Let $x_1 \in \text{dom}(H_1)$, set $x_2 := \gamma^{-1}(x_1)$ and let $w_{1,l}$ and $w_{2,l}$ be the corresponding sequences of functions for $H_1$ and $H_2$, respectively. Then, as a simple calculation shows, one has

$$w_{2,l} = w_{1,l} \circ \gamma, \quad l \geq 0.$$

$\diamond$

In the following item 3.13 we recall the above mentioned isomorphic form of the operator model. We restrict ourselves to the case that is needed in the present paper (this leads to a significant simplification of the formulae).

3.13. The boundary triple $(\mathcal{P}(h), T(h), \Gamma(h))$. Let $h \in \mathcal{H}$ be given by the data as in (3.12), and assume, in addition, that $\sigma_1 = 0$ and that no interval $(0, \varepsilon)$ with $\varepsilon > 0$ is indivisible. Due to the growth restriction imposed on the Hamiltonian functions of a general Hamiltonian in its definition (cf. [54, Definitions 2.16–2.18]) the Hamiltonian function $H_1$ of $h$ satisfies (I), (HS) and ($\Delta$), i.e. it belongs to the class $\mathbb{H}$. Let $w_i$ be the corresponding functions (3.13) and denote by $1_{(0,x_0]}$ the indicator function of the interval $(0, x_0]$.

First, we define the base space $\mathcal{P}(h)$ of the boundary triple $(\mathcal{P}(h), T(h), \Gamma(h))$. Set $\Delta := \Delta(H_1) = \Delta_1$ and

$$L^2_\Delta(H_1) := L^2(H_1) \oplus \text{span}\{ w_k 1_{(0,x_0]} : k = 0, \ldots, \Delta - 1 \}.$$
Then $\mathcal{P}(\mathfrak{h})$ is the linear space 

$$\mathcal{P}(\mathfrak{h}) := L_2^\Delta(H_1) \times \mathbb{C}^\Delta$$

endowed with an inner product as follows. Let $F = (f; \xi), G = (g; \eta) \in \mathcal{P}(\mathfrak{h})$, where $\xi = (\xi_k)_{k=0}^{\Delta-1}$, $\eta = (\eta_k)_{k=0}^{\Delta-1}$, and denote by $\lambda = (\lambda_k)_{k=0}^{\Delta-1}$ and $\mu = (\mu_k)_{k=0}^{\Delta-1}$ the unique coefficients such that

$$\tilde{f} := f - \sum_{l=0}^{\Delta-1} \lambda_l w_l 1_{\gamma x_0} \in L^2(H_1),$$

$$\tilde{g} := g - \sum_{l=0}^{\Delta-1} \mu_l w_l 1_{\gamma x_0} \in L^2(H_1).$$

Then

$$[F, G] := (\tilde{f}, \tilde{g})_{L^2(H_1)} + \sum_{k=0}^{\Delta-1} \lambda_k \overline{\eta_k} + \sum_{k=0}^{\Delta-1} \xi_k \overline{\mu_k}.$$  

Second, we define the maximal relation $T(\mathfrak{h})$. Set

$$T_{\Delta,\text{max}}(H_1) := \{(f; g) \in L_2^\Delta(H_1) \times L_2^\Delta(H_1) : \exists \tilde{f} \text{ absolutely continuous} \text{ representative of } f \text{ s.t. } \tilde{f}' = JH_1 g \}.$$ 

Then a pair $(F; G)$ of elements $F = (f; \xi), G = (g; \eta) \in \mathcal{P}(\mathfrak{h})$ belongs to $T(\mathfrak{h})$ if and only if (with $\lambda$ and $\mu$ again as in (3.14))

(i) $(f; g) \in T_{\Delta,\text{max}}(H_1)$;

(ii) for each $k \in \{0, \ldots, \Delta - 2\}$,

$$\xi_k = \eta_{k+1} + \frac{1}{2} \mu_{\Delta-1} d_{\Delta+k} + \frac{1}{2} \lambda_0 d_k - w_{k+1}(x_0) f(x_0)_2;$$

(iii) $\xi_{\Delta-1} = \int_0^{x_0} w_{\Delta} H_1 \tilde{g} + \frac{1}{2} \sum_{l=0}^{\Delta-1} \lambda_l d_l + \mu_{\Delta-1} d_{2\Delta-1} - w_{\Delta} \gamma x_0 f(x_0)_2.$

Here $w_k(x_0)_2$ denotes the lower component of the vector $w_k(x_0)$ and $f(x_0) = (f(x_0)_1, f(x_0)_2)^T$ denotes the value at $x_0$ of the unique absolutely continuous representative $\tilde{f}$ with $\tilde{f}' = JH_1 g$ (uniqueness of this representative follows since $H_1$ does not end indivisibly towards 0, cf. [26, Lemma 3.5]).

Finally, we define the boundary relation $\Gamma(\mathfrak{h})$: for $(F; G) \in T(\mathfrak{h})$, we set

$$\Gamma(\mathfrak{h})(F; G) := \begin{pmatrix} -\lambda_0 \\ \eta_0 - f(x_0)_2 + \frac{1}{2} \sum_{l=0}^{\Delta-1} \mu_l d_l \end{pmatrix}.$$  

The space $\mathcal{P}(\mathfrak{h})$ and the relation $T(\mathfrak{h})$ are related to the space $L^2(H_1)$ and the maximal relation $T_{\Delta,\text{max}}(H_1)$ therein as follows.
3.14. The map $\psi(h)$. The original definition of the map $\psi(h)$ that establishes this relation is again implicit, cf. [40, Definitions 8.5 and 4.10]. However, based on [40, (4.12)], it is easy to obtain the following description in the concrete model space $P(h)$ introduced above.

Let $h$ be a general Hamiltonian as in 3.13. Then we denote by $\psi(h) : P(h) \rightarrow L^2_\Delta(H_1)$ the projection onto the first component of $P(h)$, i.e.

$$\psi(h)(f; \xi) := f, \quad (f; \xi) \in P(h).$$

This map satisfies

$$\left(\psi(h) \times \psi(h)\right) T(h) = T_{\Delta, \text{max}}(H_1);$$

see [52, Remark 2.11]. Moreover, it is obvious that $\psi(h)$ maps $L^2(H_1) \times \mathbb{C}^\Delta$ isometrically and surjectively onto $L^2(H_1)$; note that the elements in $\mathbb{C}^\Delta$ are neutral. Using that $L^2(H_1) \times \mathbb{C}^\Delta = \left(\{0\} \times \mathbb{C}^\Delta\right)^{[\parallel]}$

we can deduce that

$$\psi(h)\left(\left(\{0\} \times \mathbb{C}^\Delta\right)^{[\parallel]}\right) = L^2(H_1),$$

$$\left(\psi(h) \times \psi(h)\right) \left(\left(T(h) \cap \left(\{0\} \times \mathbb{C}^\Delta\right)^{[\parallel]}\right)^2\right) = T_{\text{max}}(H_1);$$

(3.16)

here $[\parallel]$ denotes the orthogonal companion with respect to the inner product $[\cdot, \cdot]$, i.e. $\mathcal{M}^{[\parallel]} = \{x : [x, y] = 0\}$ for all $y \in \mathcal{M}$. ♦

3.6 The basic identification

The class $\mathcal{H}$ of Hamiltonians can be identified with the class $\mathcal{H}_0$ of general Hamiltonians up to the parameters $\tilde{o}_1, b_{1,j}, d_{1,j}$. This is nearly obvious, but is a crucial observation for our approach. Hence we point it out in this prominent way.

3.15. The relation $\mathcal{H}_0 \sim \mathcal{H}$. Let $h \in \mathcal{H}_0$ be given by the data (3.12). Then $H_1 \in \mathcal{H}$ and $\Delta(H_1) = \Delta_1$. ♦

3.16. The relation $\mathcal{H} \sim \mathcal{H}_0$. Let $H \in \mathcal{H}$ be given, assume that $H$ is defined on $(0, \infty)$ and choose $x_0 \in [0, \infty)$. Then we associate with $H$ a Hamiltonian $h \in \mathcal{H}_0$ as in (3.12) with $H_1 = H$ and $\tilde{o}_1, b_{1,j}, d_{1,j}$ arbitrary. Again one has $\Delta_1 = \Delta(H_1)$, and the negative index of $q_h$ is given by (3.11). ♦

4 Construction of the spectral measure

Let a Hamiltonian $H \in \mathcal{H}$ be given. In this section we complete the following tasks: (1) we show that each solution of the canonical system (1.1) attains regularized boundary values; (2) we construct a family of functions from the class $\mathcal{N}^{(\infty)}_{\infty}$ to which we refer as singular Weyl coefficients of $H$; and (3) we construct a positive Borel measure of class $\mathcal{M}$ to which we refer as the spectral measure of $H$. Most of these facts follow relatively easily by using the basic identification 3.16 and previous results from [54, 56].

In order to formulate the theorems, one more notation is needed.
4.1. The defect spaces \( \mathfrak{N}_z \). Let \( H \in \mathbb{H} \) and \( z \in \mathbb{C} \). We denote the set of all locally absolutely continuous solutions of the differential equation (1.1) by \( \mathfrak{N}_z \) and speak of the defect space of \( H \) at the point \( z \). Clearly, \( \mathfrak{N}_z \) is a linear space of dimension 2. Note that, for \( z = 0 \), this space is trivial in the sense that it consists of all constant functions.

Let \( H_1, H_2 \in \mathbb{H} \) and assume that \( H_1 \) and \( H_2 \) are reparameterizations of each other: \( H_2(x) = H_1(\gamma(x))\gamma'(x) \) where \( \gamma \) is an increasing bijection such that \( \gamma \) and \( \gamma^{-1} \) are absolutely continuous. Then a simple calculation shows that the mapping \( \psi \mapsto \psi \circ \gamma \) is a bijection between the corresponding defect spaces \( \mathfrak{N}_{1,z} \) and \( \mathfrak{N}_{2,z} \).

In the next theorem we show that each solution of (1.1) assumes regularized boundary values at the left endpoint. These regularized boundary values will be used later to fix a fundamental system of solutions.

4.2 Theorem (Regularized boundary values). Let \( H \in \mathbb{H} \) with \( \text{dom}(H) = (a,b) \). Then, for each fixed \( x_0 \in (a,b) \), the following statements hold (the functions \( w_k \) are as in 3.12).

(i) For each \( z \in \mathbb{C} \) and each solution \( \psi = (\psi_1, \psi_2)^T \in \mathfrak{N}_z \) the boundary value

\[
\text{rbv}_{z,1} \psi := \lim_{x \downarrow a} \psi_1(x)
\]

and the regularized boundary value

\[
\text{rbv}_{z,2} \psi := - \lim_{x \downarrow a} \left[ \sum_{l=0}^{\Delta(H)} z^l (w_l(x))^\ast J\left( \psi(x) - \lim_{t \downarrow a} \psi_1(t) \sum_{k=\Delta(H)+1}^{2\Delta(H)-l} z^k w_k(x) \right) \right]
\]

exist.

(ii) For \( z \in \mathbb{C} \) define

\[
\text{rbv}_z : \begin{cases} 
\mathfrak{N}_z &\rightarrow & \mathbb{C}^2, \\
\psi &\mapsto & (\text{rbv}_{z,1} \psi, \text{rbv}_{z,2} \psi)^T.
\end{cases}
\]

Then \( \text{rbv}_z \) is a bijection from \( \mathfrak{N}_z \) onto \( \mathbb{C}^2 \).

(iii) For each \( z \in \mathbb{C} \setminus \{0\} \) there exists an (up to scalar multiples) unique solution \( \psi = (\psi_1, \psi_2) \in \mathfrak{N}_z \setminus \{0\} \) such that \( \lim_{x \searrow a} \psi_2(x) \) exists.

This solution is characterized by the property that \( \psi|_{(a,x_0)} \in L^2(H|_{(a,x_0)}) \), and also by the property that \( \text{rbv}_{z,1} \psi = 0 \) (and \( \psi \neq 0 \)).

If \( \psi \) is such that \( \lim_{x \searrow a} \psi_2(x) \) exists, then

\[
\text{rbv}_{z,2} \psi = \lim_{x \searrow a} \psi_2(x).
\]

In contrast to \( \text{rbv}_{z,1} \psi \), the regularized boundary value \( \text{rbv}_{z,2} \psi \) depends on the choice of \( x_0 \) since the \( w_k \) depend on \( x_0 \). This dependence is controlled as follows.
(iv) Let \( x_0, \dot{x}_0 \in (a, b) \), and let \( \text{rbv}_z \) and \( \hat{\text{rbv}}_z \) be the correspondingly defined regularized boundary value mappings. Then there exists a polynomial \( p(z) \) with real coefficients which has no constant term and whose degree does not exceed \( 2\Delta(H) \) such that

\[
\text{rbv}_{z,2} \psi = \text{rbv}_{z,2} \psi + p(z) \text{rbv}_{z,1} \psi, \quad \psi \in \mathfrak{N}_z, \ z \in \mathbb{C}.
\]

4.3 Remark. For \( z = 0 \), solutions \( \psi \) of (1.1) are constant, and for such \( \psi \) the relation

\[
\text{rbv}_z \psi = \psi(x), \quad x \in (a, b),
\]

holds.

\( \diamond \)

Proof of Theorem 4.2. There is no loss of generality in assuming that \( H \) is defined on \((0, \infty)\). This follows, since the functions \( \mathfrak{m}_k \) transform naturally by composition when performing a reparameterization, cf. 3.12.

Let \( \mathfrak{h} \) be the general Hamiltonian given by the data (3.12) with \( H_1 = H \) as in the basic identification 3.16 with \( \delta_1, b_1, d_{1,j} \) all equal to 0. Items (i) and (ii) follow immediately from [54, Theorem 5.1]; we just need to match notation. Comparing the respective definitions we can deduce that

\[
\text{rbv}_{z,1} \psi \equiv \text{rbv}_z(\psi), \quad \text{rbv}_{z,2} \psi \equiv - \text{rbv}_z(\psi), \quad \text{rbv}_z \psi \equiv \text{rbv}_z(\psi),
\]

where the expressions on the right-hand sides are generalized boundary values corresponding to the general Hamiltonian \( \mathfrak{h} \) as in [54]. Item (iii) follows directly from [54, Theorem 5.2]. Only the proof of item (iv) requires an argument.

Let \( \hat{\mathfrak{h}} \) be the general Hamiltonian which is constituted by the same data as \( \mathfrak{h} \) with the exception that we take \( \hat{E} := \{-1, \dot{x}_0, \infty\} \) instead of \( E \). Then

\[
\text{rbv}_z \psi = \text{rbv}(\hat{\mathfrak{h}}, z) \psi, \quad \text{where rbv}(\hat{\mathfrak{h}}, z) \text{ denotes the regularized boundary value map defined for } \mathfrak{h} \text{ as in [54, Theorem 5.1].}
\]

By [40, Proposition 8.11] there exist numbers \( d_0, \ldots, d_{2\Delta(H) - 1} \in \mathbb{R} \) such that the general Hamiltonian \( \mathfrak{q} \) defined as in (3.12) with \( H_1 = H, \delta_1 = 0 \) and \( b_{1,1} = 0 \) is a reparameterization of \( \hat{\mathfrak{h}} \). Thereby, the increasing bijection between the domains of \( \hat{\mathfrak{h}} \) and \( \mathfrak{q} \) is the identity map. Clearly, the defect spaces \( \mathfrak{N}_z \) and the functions \( \mathfrak{m}_l \) in (3.13) built with the base point \( x_0 \) for \( \mathfrak{h} \) and for \( \mathfrak{q} \), respectively, coincide.

The fundamental solutions \( \omega_{\hat{\mathfrak{h}}} \) and \( \omega_{\mathfrak{q}} \) coincide; see 3.8. Let \( \text{rbv}(\mathfrak{q}, z) \) be the regularized boundary value map defined for \( \mathfrak{q} \) and let \( \text{rbv}_r(\mathfrak{q}, z), \text{rbv}_s(\mathfrak{q}, z) \) be its components as in [54, Theorem 5.1]. By [54, Remark 5.8], equality of fundamental solutions implies equality of regularized boundary values, i.e.

\[
\text{rbv}_r(\hat{\mathfrak{h}}, z) = \text{rbv}_r(\mathfrak{q}, z), \quad \text{rbv}_s(\hat{\mathfrak{h}}, z) = \text{rbv}_s(\mathfrak{q}, z).
\]

The first of these equalities is of course trivial; both sides, applied to a solution \( \psi = (\psi_1, \psi_2)^T \in \mathfrak{N}_z \), are equal to \( \lim_{x \searrow a} \psi_1(x) \). The second equality tells us that

\[
\text{rbv}_{z,2} \psi = - \text{rbv}_s(\mathfrak{q}, z) \psi, \quad \psi \in \mathfrak{N}_z.
\]

Comparing the definition of \( \text{rbv}_s(\mathfrak{q}, z) \) in [54, (5.3)] with the definition of \( \text{rbv}_{z,2} \) in (4.1) we obtain that

\[
\text{rbv}_s(\mathfrak{q}, z) \psi = - \text{rbv}_{z,2} \psi + \text{rbv}_{z,1} \psi \cdot \sum_{l=1}^{2\Delta(H)} z^l d_{l-1}, \quad \psi \in \mathfrak{N}_z, \ z \in \mathbb{C}.
\]
The assertion in item (iv) thus follows with the polynomial
\[ p(z) := - \sum_{l=1}^{2\Delta(H)} z^l d_{l-1}. \]

### 4.4 Remark.
In the above proof we have defined the general Hamiltonian \( \mathfrak{h} \) via the basic identification using \( \bar{\alpha}_1, b_{1,j}, d_{1,j} \) all equal to 0. This may seem artificial, and thus requires an explanation. To this end, revisit [54, (5.3)]. If we had used other values for \( \bar{\alpha}_1, b_{1,j}, d_{1,j} \), then the regularized boundary values of \( \mathfrak{h} \) as defined in [54] would have changed by the summand
\[
\left( \lim_{t \to a} \psi_1(t) \right) \left( \sum_{l=1}^{2\Delta(H)} z^l d_{l-1} - \sum_{l=0}^{\bar{\alpha}_1} z^{2\Delta(H)+l} b_{1,\bar{\alpha}_1+1-l} \right).
\]

This summand is independent of \( x \) and hence contains no information about the asymptotic behaviour of \( \psi \). We regard the inclusion of a summand (4.2) as a distracting complication from the point of our presentation and hence use the choice of vanishing \( \bar{\alpha}_1, b_{1,j}, d_{1,j} \).

Of course, notions intrinsic for \( H \) must not depend on the choice of parameters in the basic identification. Thus we shall keep track of the influence of \( \bar{\alpha}_1, b_{1,j}, d_{1,j} \).

In the next theorem a fundamental system of solutions of (1.1) is constructed. Since \( H \) is not integrable at the left endpoint, this is a non-trivial task. We fix solutions with the help of the regularized boundary values from Theorem 4.2. With this fundamental system of solutions we then construct a singular Weyl coefficient, which will be used later to obtain a spectral measure. For the definition of the class \( \mathcal{N}_h^{(\infty)} \) see Definition 3.4.

### 4.5 Theorem (Singular Weyl coefficients).
Let \( H \in \mathbb{H} \) with \( \text{dom}(H) = (a, b) \). Then, for each fixed \( x_0 \in (a, b) \), the following statements hold.

(i) For each \( z \in \mathbb{C} \) denote by \( \Theta(z) = (\Theta_1(z), \Theta_2(z)) \) and \( \varphi(z) = (\varphi_1(z), \varphi_2(z)) \) the unique elements of \( \mathcal{N}_z \) such that
\[
\text{rbv}_z \Theta(z) = (1, 0)^T, \quad \text{rbv}_z \varphi(z) = (0, 1)^T.
\]

Then, for each \( x \in (a, b) \), the functions \( \Theta(x; \cdot) \) and \( \varphi(x; \cdot) \) are entire of finite exponential type\(^3\)
\[
\int_a^x \sqrt{\det H(t)} \, dt,
\]
and they satisfy \( \Theta_1(x; z) \varphi_2(x; z) - \Theta_2(x; z) \varphi_1(x; z) = 1 \) for \( z \in \mathbb{C} \).

Moreover, let \( a_+ = \inf \{ x \in (a, b) : \int_a^x h_{22}(t) \, dt > 0 \} \). Then, for each \( z \in \mathbb{C} \), the following relations hold:
\[
\begin{align*}
\lim_{x \to a^+} \Theta_1(x; z) &= 1, & \lim_{x \to a^+} \frac{\Theta_1(x; z)}{\int_a^x h_{11}(t) \, dt} &= -z, \\
\lim_{x \to a^+} \varphi_1(x; z) &= -z, & \lim_{x \to a^+} \varphi_2(x; z) &= 1.
\end{align*}
\]

\(^3\)If the integral in (4.4) is 0, then \( \Theta(x; \cdot) \) and \( \varphi(x; \cdot) \) are either of minimal exponential type or of order less than 1.
(ii) For each \( \tau \in \mathbb{R} \cup \{ \infty \} \), the limit

\[
q_H(z) := \lim_{x \to \frac{z}{h}} \frac{\theta_1(x; z) \tau + \theta_2(x; z)}{\varphi_1(x; z) \tau + \varphi_2(x; z)}, \quad z \in \mathbb{C} \setminus \mathbb{R},
\]
exists locally uniformly on \( \mathbb{C} \setminus \mathbb{R} \), defines an analytic function in \( z \) on \( \mathbb{C} \setminus \mathbb{R} \) and does not depend on \( \tau \) (here the fraction on the right-hand side of (4.6) is interpreted as \( \frac{\theta_1(x; z)}{\varphi_1(x; z)} \) if \( \tau = \infty \)). The function \( q_H \) belongs to the class \( N^{(\infty)}_{\Delta(H)} \).

(iii) We have

\[
\theta(\cdot ; z) - q_H(z) \varphi(\cdot ; z) \in L^2(H_{(x_0, h)}), \quad z \in \mathbb{C} \setminus \mathbb{R},
\]
and this property characterizes the value \( q_H(z) \) for each \( z \in \mathbb{C} \setminus \mathbb{R} \).

The function \( q_H \) depends on the choice of \( x_0 \), which is controlled as follows.

(iv) Let \( x_0, x_0 \in (a, b) \) and let \( q_H \) and \( \tilde{q}_H \) be the correspondingly defined functions (4.6). Then there exists a polynomial \( p \) with real coefficients which has no constant term and whose degree does not exceed \( 2\Delta(H) \) such that

\[
\tilde{q}_H(z) = q_H(z) + p(z).
\]

Proof. Again we may assume without loss of generality that \( H \) is defined on \( (0, \infty) \). Let \( h \) be the general Hamiltonian (3.12) with \( H_1 = H \) and \( \delta_1, b_{1,j}, d_{1,j} \) all equal to 0, so that \( \text{rb} \psi = \text{rb} \psi \), where \( \text{rb} \psi \) is as in [54, Theorem 5.1]. It follows from [54, Corollary 5.7] that the fundamental solution \( \omega_h \) from 3.8 associated with \( h \) is given by

\[
\omega_h(x; z) = \begin{pmatrix} \theta_1(x; z) & \theta_2(x; z) \\ \varphi_1(x; z) & \varphi_2(x; z) \end{pmatrix}.
\]

The first properties of \( \theta \) and \( \varphi \) mentioned in (i) are immediate; see 3.8. The formula for the exponential type follows from [55, Theorem 4.1] if we observe the \( \det H_0(x) = 0 \) for \( x \in (-1, 0) \) with \( H_0 \) from (3.12). The limit relations in (4.5) follow from [54, Theorem 4.1 (with \( \alpha = 0 \)), Remark 4.2 (iii) and Lemma 4.14].

Corollary 5.7 in [54] also implies that the limit in (4.6) exists locally uniformly, that \( q_H \) is characterized by (4.7) and that \( q_H \) coincides with the Weyl coefficient \( q_h \) of the general Hamiltonian \( h \). In particular, this shows that \( q_H \) belongs to the class \( N^{(\infty)}_{\Delta(H)} \); see 3.11 and note that \( \Delta(H) = \Delta_1 \) and \( \delta_1 = 0 \).

For the proof of item (iv), consider again the general Hamiltonians \( \hat{h} \) and \( g \) as in the proof of Theorem 4.2 (iv). Since they are reparameterizations of each other, their Weyl coefficients coincide. An application of [54, Corollary 5.9] with \( h \) and \( g \) gives

\[
\tilde{q}_H(z) - q_H(z) = q_h(z) - q_h(z) = q_g(z) - q_h(z) = - \sum_{l=1}^{2\Delta(H)} z^l d_{l-1},
\]
which shows (iv).
Note that for \( z = 0 \) one has
\[
\theta(x;0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi(x;0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x \in (a,b),
\]
which follows from (4.3).

4.6 Remark. Let us study the influence of the parameters \( \tilde{o}_1, b_{1,j}, d_{1,j} \) in 3.16 on the above proof. If we chose other parameters than all equal to 0 and hence alter the regularized boundary values by a polynomial summand, then the same would happen to the function \( q_H \). In fact, revisiting [54, Corollary 5.9] we would pass from \( q_H \) to \( q_H + p \) where \( p \) is a polynomial with real coefficients with \( p(0) = 0 \).

Notice that, conversely, each summand \( p \in \mathbb{R}[z] \) with \( p(0) = 0 \) can be produced by a proper choice of \( \tilde{o}_1, b_{1,j}, d_{1,j} \). Moreover, changing the base point \( x_0 \) (which is the second arbitrariness in our basic identification) also manifests only in adding a polynomial summand \( p \in \mathbb{R}[z] \) with \( p(0) = 0 \).

\[\Diamond\]

In order to handle the arbitrariness in the basic identification in a structurally clean way, we introduce an equivalence relation on the set of Weyl coefficients. Namely, we set
\[
q_1 \sim q_2 \iff q_1 - q_2 \in \mathbb{R}[z], \quad (q_1 - q_2)(0) = 0.
\]
Clearly, this is an equivalence relation on \( \mathcal{N}_{<\infty}^{(\infty)} \). In this context, remember that \( q \in \mathcal{N}_{<\infty}^{(\infty)} \) implies that \( q + p \in \mathcal{N}_{<\infty}^{(\infty)} \) for all \( p \in \mathbb{R}[z] \).

4.7 Definition. Let \( H \in \mathbb{H} \) be given. Then we denote by \([q]_H\) the equivalence class modulo the relation (4.10) which contains some (and hence any) function \( q_H \) constructed in Theorem 4.5.

We speak of \([q]_H\) as the singular Weyl coefficient of \( H \). Each representative \( q_H \) of \([q]_H\) is called a (!) singular Weyl coefficient of \( H \).

By this definition we achieve that the singular Weyl coefficient \([q]_H\) of \( H \in \mathbb{H} \) is nothing but the equivalence class which consists of all Weyl coefficients of general Hamiltonians associated with \( H \) by the basic identification.

In the following theorem a measure is constructed with the help of the singular Weyl coefficient and a Stieltjes-type inversion formula.

4.8 Theorem (The spectral measure). Let \( H \in \mathbb{H} \) be given. Then there exists a unique positive Borel measure \( \mu_H \) with
\[
\mu_H([s_1,s_2]) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \lim_{\delta \to 0^+} \int_{s_1 - \varepsilon}^{s_2 + \varepsilon} \Im q_H(t + i\delta) \, dt, \quad -\infty < s_1 < s_2 < \infty,
\]
where \( q_H \in [q]_H \) is any singular Weyl coefficient of \( H \). We have \( \mu_H \in \mathcal{M} \) and \( \Delta(\mu_H) = \Delta(H) \).

Proof. Since \( q_H \in \mathcal{N}_{<\infty}^{(\infty)} \), it has a representation \( q_H(z) = r + \Phi(\beta_z) \) with \( r \in \mathbb{R} \) and a distributional density \( \Phi \in \mathcal{F}_{<\infty}^{(\infty)} \), i.e. \( \Phi \) coincides with a measure \( \mu_\Phi \) on \( \mathbb{R} \); see 3.5. It follows from (3.6) (see also [56, Theorem 3.9 (ii)]) that the measure \( \mu_\Phi \) is given by the right-hand side of (4.11) on the set of closed intervals; in
particular, the double limit exists. Moreover, [56, Theorem 2.8 (ii)] implies that \( \mu_H \in \mathcal{M} \).

The fact that \( \mu_H \) does not depend on the choice of \( q_H \in [q]_H \) is clear since a summand that is a real polynomial yields no contribution in the Stieltjes inversion formula.

Finally, we show that \( \Delta(\mu_H) = \Delta(H) \). It follows from [56, Theorem 3.9 (v)] that

\[
\Delta(\mu_H) = \min \left\{ \text{ind}_-(q_H + p) : p \in \mathbb{R}[z] \right\}
\]

holds for every \( p \in \mathbb{R}[z] \) with \( p(0) = 0 \) there exist \( \hat{\omega}_1, b_{1,j}, d_{1,j} \) such that the Weyl coefficient of the general Hamiltonian \( \eta \) in (3.12) with \( H_1 = H \) is equal to \( q_H + p \); and for each choice of \( \hat{\omega}_1, b_{1,j}, d_{1,j} \) the Weyl coefficient is of this form; see 4.6.

It follows from this and (3.11) that

\[
\Delta(\mu_H) = \Delta(\mu_H) = \min \left\{ \text{ind}_-(q_H + p) : p \in \mathbb{R}[z], p(0) = 0 \right\}.
\]

For every \( p \in \mathbb{R}[z] \) with \( p(0) = 0 \) there exist \( \hat{\omega}_1, b_{1,j}, d_{1,j} \) such that the Weyl coefficient of the general Hamiltonian \( \eta \) in (3.12) with \( H_1 = H \) and \( \hat{\omega}_1, b_{1,j}, d_{1,j} \) arbitrary

\[
\Delta(\mu_H) = \min \left\{ \text{ind}_-(q_H + p) : p \in \mathbb{R}[z], p(0) = 0 \right\}.
\]

4.9 Definition. Let \( H \in \mathbb{H} \) be given. Then we call the unique positive Borel measure \( \mu_H \) defined by (4.11) the spectral measure of \( H \).

The choice of this terminology is justified by Theorem 5.1 below where we construct a Fourier transform into the space \( L^2(\mu_H) \). Before we establish this Fourier transform, let us mention one simple observation. Namely, it is almost immediate that Hamiltonians which are reparameterizations of each other give rise to the same singular Weyl coefficients and the same spectral measures. We provide a slightly more exhaustive variant of this fact.

4.10 Proposition. Let \( H \in \mathbb{H} \) and \( \alpha \in \mathbb{R} \). Then the Hamiltonian

\[
H_\alpha := \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} H \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}
\]

belongs to \( \mathbb{H} \).

Let, in addition, \( \tilde{H} \in \mathbb{H} \) be given and assume that \( \tilde{H} \) and \( H_\alpha \) are reparametrizations of each other. Then

(i) for each pair of singular Weyl coefficients \( q_\tilde{H} \) and \( q_H \) of \( \tilde{H} \) and \( H \), respectively, the difference \( q_\tilde{H} - q_H \) is a real polynomial whose constant term equals \( \alpha \);

(ii) \( \mu_\tilde{H} = \mu_H \).

Proof. Let \( \eta \) be the general Hamiltonian as in (3.12) with \( H_1 = H \) and let \( \omega_\eta \) be its fundamental solution. It follows from [39, Lemma 10.2] that

\[
\omega_\alpha := \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \omega_\eta \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}
\]
is the fundamental solution of some general Hamiltonian \( h_\alpha \). This factorization immediately yields
\[
q_{h_\alpha} = q_b + \alpha, \tag{4.12}
\]
which implies that \( q_{h_\alpha} \in \mathcal{N}^{(\infty)} \), and hence \( h_\alpha \in \mathcal{D}_0 \). A short computation, based on \cite[Corollary 5.6]{42}, shows that the Hamiltonian function of \( h_\alpha \) on \((0, \infty)\) is equal to \( H_\alpha \), and therefore \( H_\alpha \in \mathcal{H} \). The functions \( q_{h_\alpha} \) and \( q_b \) are singular Weyl coefficients of \( H_\alpha \) and \( H \), respectively. Relation (4.12), together with Theorem 4.5 (iv) implies that \( q_{h_\alpha} - q_b \) is a real polynomial whose constant term is equal to \( \alpha \).

Assume now that \( \tilde{H} \) is a reparameterizations of \( H_\alpha \), say (2.1) holds with \( H_1 = H_\alpha \), \( H_2 = \tilde{H} \) and some \( \gamma : (0, \infty) \to (0, \infty) \). We can build general Hamiltonians \( \tilde{h} \) and \( h_\alpha \) via 3.16 with \( \ddot{o}_1 = \ddot{\tilde{o}}_1 = 0 \), \( b_{1,1} = \ddot{b}_{1,1} = 0 \), \( d_{1,j} = \ddot{d}_{1,j} = 0 \) and some \( x_0, \tilde{x}_0 \in (0, \infty) \) such that \( \gamma(\tilde{x}_0) = x_0 \). Then \( \tilde{h} \) and \( h_\alpha \) are reparameterizations of each other; see \cite[Remark 3.38]{42} and \cite[Proposition 8.13]{40}. Hence \( \tilde{h} \) and \( h_\alpha \) have the same Weyl coefficients; see \cite[Theorem 1.4]{43}. This shows that \( q_{\tilde{H}} = q_{h_\alpha} \), which in turn implies (i).

For (ii) observe that an entire summand yields no contribution in the Stieltjes inversion formula.

5 The Fourier transform

For a positive Borel measure \( \mu \) on \( \mathbb{R} \) we denote by \( M_\mu \) the operator of multiplication by the independent variable in \( L^2(\mu) \). In this section we prove that for each \( H \in \mathcal{H} \) there exists a unitary operator \( \Theta_H \) from \( L^2(H) \) onto \( L^2(\mu_H) \), the Fourier transform connected with \( H \), which establishes unitary equivalence of \( T(H) \) and \( M_\mu H \). Both \( \Theta_H \) and its inverse act as integral transformations. These results are the most involved ones in the paper. Their proofs require to go deeply into the operator model of a general Hamiltonian \( h \). The essential ingredients are the following:

1. the spectral theory of the model relation in the Pontryagin space \( \mathcal{P}(h) \), in particular, the spectral decomposition of a self-adjoint relation in a Pontryagin space,
2. \( Q \)-function theory to relate the model relation connected with \( h \) to the model relation of a distributional density,
3. the interpretation of a fundamental solution matrix as a generalized \( u \)-resolvent matrix.

5.1 Theorem (The Fourier transform). Let \( H \in \mathcal{H} \) with \( \text{dom}(H) = (a, b) \) be given, let \( T(H) \) be the self-adjoint operator as defined in Section 3.2 and let \( \mu_H \) be the spectral measure associated with \( H \) via (4.11). Moreover, let \( \varphi(\cdot ; z) = (\varphi_1(\cdot ; z), \varphi_2(\cdot ; z))^T \) be the unique element of \( \mathcal{R}_z \) with \( \text{rbv}_z \varphi(\cdot ; z) = (0, 1)^T \) as in Theorem 4.5. Then the following statements hold.

1. The map defined by
\[
(\Theta_H f)(t) := \int_a^b \varphi(x; t)^T H(x) f(x) \, dx, \quad t \in \mathbb{R}, \tag{5.1}
\]
for \( f \in L^2(H) \), \( \sup(\text{supp} f) < b \).
extends to an isometric isomorphism from $L^2(H)$ onto $L^2(\mu_H)$, which we again denote by $\Theta_H$.

(ii) The operator $\Theta_H$ establishes a unitary equivalence between $T(H)$ and $M_{\mu_H}$, i.e., we have

$$\Theta_H \circ T(H) = M_{\mu_H} \circ \Theta_H.$$ 

(iii) On the subspace of compactly supported functions, also the inverse of $\Theta_H$ acts as an integral transformation, namely,

$$\left(\Theta_H^{-1} g\right)(x) = \int g(t) \varphi(x; t) d\mu_H(t), \quad x \in (a, b),$$ 

$$g \in L^2(\mu_H), \text{ supp } g \text{ compact.} \quad (5.2)$$

5.2 Remark. Note that the integrals in (5.1) and (5.2) are always well defined. For the latter this is obvious since $\varphi(x; t)$ is continuous in $t$; for the former it follows from Theorem 4.2 (iii).

As an additional result we prove a connection between the point mass at 0 of the spectral measure and the behaviour of $H$ at $b$.

5.3 Proposition. Let $H = (h_{ij})_{i,j=1}^2 \in \mathbb{H}$, defined on $(a, b)$, and let $\mu_H$ be the spectral measure associated with $H$ via (4.11). Then $\mu_H(\{0\}) > 0$ if and only if

$$\int_a^b h_{22}(x) dx < \infty. \quad (5.3)$$

If (5.3) is satisfied, then

$$\mu_H(\{0\}) = \left[ \int_a^b h_{22}(x) dx \right]^{-1}. \quad (5.4)$$

Note that in any case,

$$\mu_H(\{0\}) = -\lim_{y \searrow 0} iy q_H(iy) \quad (5.5)$$

by [56, (3.8)].

The rest of this section is devoted to the proof of Theorem 5.1 and Proposition 5.3. We split the proof of Theorem 5.1 into three parts, which are contained in three separate subsections. First, in §5.1, we construct a Fourier transform $\Theta_H$ from $L^2(H)$ onto $L^2(\mu_H)$ in an abstract way. In §5.2 we show that this map acts as asserted in (5.1). The formula for $\Theta_H^{-1}$ is proved in §5.3. Finally, we prove Proposition 5.3 in §5.4.

Since all statements in Theorem 5.1 and Proposition 5.3 are invariant under reparameterizations, we can assume without loss of generality that $\text{dom } H = (0, \infty)$. Throughout these four subsections, keep $H \in \mathbb{H}$ fixed and let $\mathfrak{f}$ be the general Hamiltonian defined in the basic identification 3.16 with $\tilde{o}_1, b_{1,j}, d_{1,j}$ all equal to 0.
5.1 Construction of a Fourier transform

Let us first consider the case when \((0,c)\) is a maximal \(H\)-indivisible interval of type 0. Then the space \(L^2(H)\) can be identified with \(L^2(H|_{(c,\infty)})\) and \(\varphi(c; z) = (0, 1)^T\); see [54, proof of Theorem 5.1, p. 541]. Now items (i) and (ii) in Theorem 5.1 follow from [8, Theorem III]. For the rest of §5.1 and §5.2 we assume that \(H\) does not start with an indivisible interval at 0. Hence we can use the boundary triple \((\mathcal{P}(h), T(h), \Gamma(h))\) associated with the general Hamiltonian \(h\) in the form in 3.13.

The Fourier transform \(\Theta_H\) is constructed by combining several mappings. We provide a comprehensive summary in Figure 1 on page 30. The reader may find it helpful to visit this summary already while going through the construction.

It was shown in [42, Proposition 5.19] that the Weyl coefficient \(q_h\) is a \(Q\)-function of the minimal relation \(S(h) := T(h)^*\). In fact, denote by \(\pi_{l,1}\) the projection from \(C^2 \times C^2\) onto the upper entry of the first vector component, set

\[
A := \ker(\pi_{l,1} \circ \Gamma(h)),
\]

and let \(\varepsilon_z\) be the defect elements with

\[
(\pi_l \circ \Gamma(h))(\varepsilon_z; z\varepsilon_z) = \begin{pmatrix} 1 \\ -q_h(z) \end{pmatrix}.
\]

Then \(q_h\) is the \(Q\)-function of \(S(h)\) induced by \(A\) and \((\varepsilon_z)_{z \in \rho(A)}\). In particular, since \(q_h\) is analytic in \(\mathbb{C} \setminus \mathbb{R}\) and \(S(h)\) is completely non-self-adjoint, we have \(\mathbb{C} \setminus \mathbb{R} \subseteq \rho(A)\).

Let \(\psi_h\) be the distributional density in the representation (3.3) of \(q_h\), and let \(\Pi(\phi_h), A_{\phi_h}\) and \(\psi(\phi_h)\) be as in 3.6. By [39, Proposition 3.4, Proof of Corollary 3.5], there exists an isometric isomorphism (where we use the base point \(z_0 = i\))

\[
\Theta_h : \mathcal{P}(h) \to \Pi(\phi_h) \quad \text{with} \quad (\Theta_h \times \Theta_h)(A) = A_{\phi_h}.
\]

This isomorphism is determined by its action on defect elements, namely,

\[
\Theta_h(\varepsilon_z) = \hat{\varepsilon}_z, \quad z \in \rho(A),
\]

where \(\hat{\varepsilon}_z \in \Pi(\phi_h)\) is defined by

\[
\hat{\varepsilon}_z(t) := \begin{cases} 
\frac{t - i}{-iz}, & t \in \mathbb{R}, \\
1, & t = \infty.
\end{cases}
\]

Denote by \(E_{\Delta}(\infty)\) and \(E_{\phi_h}(\infty)\) the algebraic eigenspaces at infinity of the relations \(A\) and \(A_{\phi_h}\), respectively. By [54, Lemma 3.2 (d)] we have \(E_{\Delta}(\infty) = \{0\} \times \mathbb{C}^\Delta\); in particular, \(E_{\Delta}(\infty)\) is neutral. It follows from (5.8) that

\[
\Theta_h(E_{\Delta}(\infty)) = E_{\phi_h}(\infty),
\]

\(\text{Here } "l,1" \text{ stands for } "\text{left vector, first entry}". \text{ This is a generic notation: for example, } \pi_l \text{ is the projection from } C^2 \times C^2 \text{ onto the first vector component, } \pi_{r,2} \text{ onto the lower entry of the second vector component, etc. The use of } "\text{left}" \text{ and } "\text{right}" \text{ is motivated by the fact that the vectors correspond to boundary values at the left and right endpoint, respectively.}

27
By (3.15) we have
\[ \Lambda: E_A(\infty)^{[1]} / E_A(\infty) \rightarrow E_{\Phi_n}(\infty)^{[1]} / E_{\Phi_n}(\infty). \]

Then there exists an isometric isomorphism \( \Theta_b : E_A(\infty)^{[1]} / E_A(\infty) \rightarrow E_{\Phi_n}(\infty)^{[1]} / E_{\Phi_n}(\infty) \)
such that
\[ \pi_{\Phi_n} \circ \Theta_b = \Lambda_b \circ \pi_A. \]

By (3.15) we have
\[ \ker \pi_A = E_A(\infty) = \{0\} \times \mathbb{C}^2 = \ker \psi(h). \]

Since \( \psi(h)(E_A(\infty)^{[1]}) = L^2(H) \) by the first relation in (3.16), we can deduce that there exists an isometric isomorphism
\[ \Psi_A : E_A(\infty)^{[1]} / E_A(\infty) \rightarrow L^2(H) \quad \text{with} \quad \Psi_A \circ \pi_A = \psi(h)|_{E_A(\infty)^{[1]}}. \]

By [56, Theorem 5.3], \( \psi(h) \) maps \( E_{\Phi_n}(\infty)^{[1]} \) isometrically onto \( L^2\left(\frac{\mu_{\Phi_n}(x)}{1 + x^2}\right) \).

Since \( L^2\left(\frac{\mu_{\Phi_n}(x)}{1 + x^2}\right) \) is non-degenerate, we have
\[ \ker \psi(\Phi_n) = (E_{\Phi_n}(\infty)^{[1]})^\circ = E_{\Phi_n}(\infty) = \ker \pi_{\Phi_n}, \]
and we obtain an isometric isomorphism
\[ \Psi_{\Phi_n} : E_{\Phi_n}(\infty)^{[1]} / E_{\Phi_n}(\infty) \rightarrow L^2\left(\frac{\mu_{\Phi_n}(x)}{1 + x^2}\right) \]
such that
\[ \Psi_{\Phi_n} \circ \pi_{\Phi_n} = \psi(\Phi_n)|_{E_{\Phi_n}(\infty)^{[1]}}. \]

Finally, let
\[ U : L^2\left(\frac{\mu_{\Phi_n}(x)}{1 + x^2}\right) \rightarrow L^2(\mu_{\Phi_n}) \]
be the operator of multiplication by \( \frac{1}{x^2} \), which is an isomorphism from \( L^2\left(\frac{\mu_{\Phi_n}(x)}{1 + x^2}\right) \) onto \( L^2(\mu_{\Phi_n}) \).

Now we define \( \Theta_H \) as the composition of the constructed isometric isomorphisms, namely
\[ \Theta_H := U \circ \Psi_{\Phi_n} \circ \Lambda_b \circ \Psi_A^{-1}. \]

Then \( \Theta_H \) is an isometric isomorphism from \( L^2(H) \) onto \( L^2(\mu_{\Phi_n}) \).
In order to see how $\Theta_H$ is related to $T(H)$, it is enough to trace back the defining procedure\textsuperscript{5}. Using (5.8) and (5.11) we obtain
\[
(\Lambda_b \times \Lambda_b)\left(\pi_A \times \pi_A\right)\left(A \cap (E_A(\infty)^{[1]}\right)^2) = (\pi_{\Phi_b} \times \pi_{\Phi_b})\left(A_{\Phi_b} \cap (E_{\Phi_b}(\infty)^{[1]}\right)^2).
\]
By [40, Proposition 4.17 (iii)] we have
\[
(\psi(h) \times \psi(h))(T(h) \cap (E_A(\infty)^{[1]}\right)^2) = T(H),
\]
and hence
\[
(\Psi_A \times \Psi_A)\left(\pi_A \times \pi_A\right)\left(A \cap (E_A(\infty)^{[1]}\right)^2) \subseteq T(H).
\]
It follows from [56, Theorem 5.3] that
\[
(\psi(\Phi_b) \times \psi(\Phi_b))(A_{\Phi_b} \cap (E_{\Phi_b}(\infty)^{[1]}\right)^2) = M_{(1+x^2)^{-1}d\mu_{\Phi_b}(x)},
\]
and therefore
\[
(\Psi_{\Phi_b} \times \Psi_{\Phi_b})\left(\pi_{\Phi_b} \times \pi_{\Phi_b}\right)\left(A_{\Phi_b} \cap (E_{\Phi_b}(\infty)^{[1]}\right)^2) = M_{(1+x^2)^{-1}d\mu_{\Phi_b}(x)}.
\]
Putting these relations together we obtain
\[
(\Theta_H^{-1} \times \Theta_H^{-1})M_{\mu_{\Phi_b}}
\]
\[
= (\Psi_A A^{-1}\psi_{\Phi_b}^{-1} \times \Psi_A A^{-1}\psi_{\Phi_b}^{-1})M_{(1+x^2)^{-1}d\mu_{\Phi_b}(x)}
\]
\[
= (\Psi_A A^{-1} \times \Psi_A A^{-1})\left(\pi_{\Phi_b} \times \pi_{\Phi_b}\right)\left(A_{\Phi_b} \cap (E_{\Phi_b}(\infty)^{[1]}\right)^2)
\]
\[
= (\Psi_A \times \Psi_A)\left(\pi_{\Phi_b} \times \pi_{\Phi_b}\right)\left(A \cap (E_A(\infty)^{[1]}\right)^2)
\]
\[
\subseteq T(H).
\]
Strict inclusion cannot occur since both $M_{\mu_{\Phi_b}}$ and $T(H)$ are self-adjoint. We conclude that
\[
(\Theta_H \times \Theta_H)T(H) = M_{\mu_{\Phi_b}}.
\]
It remains to remember (from the proof of Theorem 4.8) that $\mu_H = \mu_{\Phi_b}$.

5.2 Computation of $\Theta_H$ as an integral transform

Since $\mathfrak{h}$ starts with an indivisible interval of type 0, [42, Theorem 6.4] is not applicable; a computation of the full Fourier transform $\Theta_{\mathfrak{h}}$ in the spirit of this result is not possible. However, we are only interested in $\Theta_H$, which is essentially a restriction of $\Theta_{\mathfrak{h}}$. And it turns out that the action of this restriction can be computed. The argument is based on a refined variant of [39, Proposition 4.6]. It requires to go into the details of the constructions made in [39] and [42].

Let us lay out the operator-theoretic setup (in five parts). Thereby we use, without much further notice, terminology and results from [38]. In particular, we ask the reader to recall definitions and usage of spaces, like $\mathcal{P}_-$, dualities $[\cdot, \cdot]_\Lambda$ and resolvent-like operators $R_{\Delta}^\pm$, as introduced and studied in [38, §3]. Moreover, we repeatedly employ terminology and results from [42]. A comprehensive summary of the involved spaces and relations can be found in Figure 2 on page 35. We advice the reader to visit this summary already on going through the construction.

\textsuperscript{5}Remember in the following that $T(H)$ is self-adjoint, and hence $T(H) = T_{\text{max}}(H) = S(H)$ in the notation of several previous papers like [40].
Let $A$ and $\varepsilon_z$ be as in the previous section, cf. (5.6), (5.7), so that $q_h$ is a $Q$-function of $S(h)$ induced by $A$ and $(\varepsilon_z)_z \in \rho(A)$. Let $u \in \mathcal{P}(h) := (S(h)^*)'$ be the element defined by

$$\left[ u, (f; g) \right]_\pm := (\pi_{l, 2} \circ \Gamma(h))(f; g), \quad (f; g) \in S(h)^*.$$  

Moreover, set

$$R_z := (A - z)^{-1}, \quad R_z^+ : \begin{cases} \mathcal{P}(h) \to S(h)^* \\ f \to (R_z f; I + zR_z f) \end{cases}$$

for $z \in \rho(A)$ and let $R_z^{-} : \mathcal{P}(h)_- \to \mathcal{P}(h)$ be the dual of $R_z^+$, i.e. the unique map with

$$[R_z^- v, f] = [v, R_z^+ f]_\pm, \quad v \in \mathcal{P}(h)_-, f \in \mathcal{P}(h).$$

Then, by [42, Theorem 4.24], we have $\varepsilon_z = R_z^- u$, $z \in \rho(A)$. 

Since $q_h \in \mathcal{N}_{<\infty}$, the only critical point of $A$ is the point $\infty$ and $A$ has no finite spectral points of non-positive type, cf. [54, Lemma 2.5] and the paragraph preceding it. This implies that for each bounded Borel set $\Delta$ the spectral projection $E(\Delta)$ of $A$ is well defined and its range is a Hilbert space. Moreover,
\( E_\Delta : \Delta' \mapsto E(\Delta \cap \Delta') \) is the spectral measure of the bounded self-adjoint operator \( A_\Delta := A \cap (\text{ran } E(\Delta) \times \text{ran } E(\Delta)) \) in the Hilbert space \( \text{ran } E(\Delta) \), and \( \sigma(A_\Delta) \subseteq \Delta \), cf. [49, Theorem II.3.1, p. 34]\(^6\).

For a bounded Borel set \( \Delta \) and elements \( f, g \in \mathcal{P}(\mathfrak{h}) \) we thus have a complex measure \( E_{\Delta,f,g} \) on \( \mathbb{R} \) defined by

\[
E_{\Delta,f,g}(\Delta') := [E(\Delta \cap \Delta')f,g], \quad \Delta' \text{ a Borel set on } \mathbb{R}.
\]

For later use let us list some simple properties of these objects.

1. We have

\[
E_{\Delta,f,g} = E_{\Delta,E(\Delta)f,g} = E_{\Delta,f,E(\Delta)g}, \quad f,g \in \mathcal{P}(\mathfrak{h}). \quad (5.12)
\]

and

\[
E_{\Delta,f,g}(\Delta') = E_{E(\Delta)f,g}(\Delta'), \quad f,g \in \mathcal{P}(\mathfrak{h}), \quad \Delta' \text{ a Borel set on } \mathbb{R}. \quad (5.13)
\]

2. Denote by \( \| \cdot \|_{\mathcal{P}(\mathfrak{h})} \) some norm which induces the Pontryagin space topology of \( \mathcal{P}(\mathfrak{h}) \), \( \| E(\Delta) \| \) the corresponding operator norm, and let \( \| E_{\Delta,f,g} \| \) denote the total variation of the measure \( E_{\Delta,f,g} \). Then

\[
\| E_{\Delta,f,g} \| \leq \| E(\Delta) \| \| f \|_{\mathcal{P}(\mathfrak{h})} \| g \|_{\mathcal{P}(\mathfrak{h})}.
\]

3. Let \( T > 0 \), let \( F : [-T,T] \to \mathbb{C} \) be continuous and let \( f,g \in \mathcal{P}(\mathfrak{h}) \). Then the map

\[
\Delta \mapsto \int_{[-T,T]} F \, dE_{\Delta,f,g}, \quad \Delta \text{ a Borel subset of } [-T,T],
\]

is a complex Borel measure on \( [-T,T] \).

4. For each bounded Borel set \( \Delta \) we have

\[
E(\Delta)R_z|_{\text{ran } E(\Delta)} = (A_\Delta - z)^{-1} = \frac{1}{\int_{\mathbb{R}} t - z} \, dE_\Delta(t), \quad z \in \rho(A).
\]

5. If \( \Delta, \Delta' \) are bounded Borel sets with \( \Delta \subseteq \Delta' \) and \( f,g \in \mathcal{P}(\mathfrak{h}) \), then

\[
E_{\Delta,f,g} \ll E_{\Delta',f,g} \quad \text{with} \quad \frac{dE_{\Delta,f,g}}{dE_{\Delta',f,g}} = 1_\Delta.
\]

\(^6\)In [49] bounded operators are treated. The extension to the case of relations is provided in [12]. In [49] results are formulated for \( \Delta \) being an interval whose endpoints are not critical points. In our case, the only critical point is \( \infty \). Moreover, with the usual measure-theoretic extension process one can define \( E(\Delta) \) for each bounded Borel set; cf. the first paragraph in the proof of Lemma 5.4 below. We tacitly use this fact and often formulate results for bounded Borel sets although in the original references only intervals were considered.
Introduce the set
\[ I_{\text{reg}} := \{ t \in (0, \infty) : t \text{ is not inner point of an indivisible interval} \}. \quad (5.14) \]

For \( t \in I_{\text{reg}} \) the boundary triple \((\mathcal{P}(h), \mathcal{T}(h), \Gamma(h))\) is isomorphic to the boundary triple that is obtained by pasting the boundary triples corresponding to \( h_{\tau} \) and \( h_{\tau'} \); for the notation of pasting boundary triples and the mentioned result see [42, Definition 3.47] and [42, Remark 3.51] (or [52, 2.2, 2.3 and Lemma 2.5]); for the definition of \( h_{\tau} \) and \( h_{\tau'} \) see 3.9. In particular, the space \( \mathcal{P}(h) \) can be decomposed as follows:
\[ \mathcal{P}(h) = \mathcal{P}(h_{\tau'}) [\dot{\top}] \mathcal{P}(h_{\tau}), \]
and we may consider \( \mathcal{P}(h_{\tau'}) \) naturally as a subspace of \( \mathcal{P}(h) \). Denote by \( P_t \), \( t \in I_{\text{reg}} \), the orthogonal projection in \( \mathcal{P}(h) \) onto \( \mathcal{P}(h_{\tau'}) \). Note that this projection acts as
\[ P_t(f; \xi) = (1_{\tau'} f; \xi), \quad (f; \xi) \in \mathcal{P}(h). \quad (5.15) \]

Let us consider the relation
\[ S_1(h_{\tau'}) := \ker [(\pi_{t,1} \times \pi_{t}) \Gamma(h_{\tau'})] \subseteq \mathcal{P}(h_{\tau'})^2. \]
This relation is symmetric and has deficiency index \((1, 1)\). Let \( \psi_t(z) \in \mathcal{P}(h_{\tau'}) \), \( z \in \mathbb{C} \), be the defect elements of \( S(h_{\tau'}) \) that satisfy
\[ (\pi_t \circ \Gamma(h_{\tau'}))(\psi_t(z); z \psi_t(z)) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad z \in \mathbb{C}; \quad (5.16) \]
then \( \psi_t(z) \) are also defect elements of \( S_1(h_{\tau'}) \), i.e.
\[ \text{ran}(S_1(h_{\tau'}) - \pi_t^\perp) = \text{span}\{ \psi_t(z) \}, \quad z \in \mathbb{C}. \]

Considering \( S_1(h_{\tau'}) \) as a linear relation in \( \mathcal{P}(h) \), we have \( S_1(h_{\tau'}) \subseteq A \) and hence \( A \subseteq S_1(h_{\tau'})^* \), where \( S_1(h_{\tau'})^* \) denotes the adjoint of \( S_1(h_{\tau'}) \) as a relation in \( \mathcal{P}(h) \). The adjoint \( S_1(h_{\tau'})^* \) of \( S_1(h_{\tau'}) \) in \( \mathcal{P}(h_{\tau'}) \) is given by \( S_1(h_{\tau'})^* = \ker(\pi'_{t,1} \circ \Gamma(h_{\tau'})) \), and it follows that
\[ S_1(h_{\tau'})^* = (P_t \times P_t)(A). \quad (5.17) \]

Let \( u_t \in \mathcal{P}(h_{\tau'})_\perp := (S_1(h_{\tau'})^*)^* \) be the unique element with
\[ [u_t, (f; g)]_t := (\pi_{1,2} \circ \Gamma(h_{\tau'}))(f; g), \quad (f; g) \in S_1(h_{\tau'})^*, \quad (5.18) \]
where \([\cdot, \cdot]_t\) denotes the duality between \( S_1(h_{\tau'})^* \) and \( \mathcal{P}(h_{\tau'})_\perp \). Since \( (P_t \times P_t)(S(h)^*) = S(h_{\tau'})^* \) and
\[ (\pi_t \circ \Gamma(h))(f; g) = (\pi_t \circ \Gamma(h_{\tau'}))(P_t f; P_t g), \quad (f; g) \in S(h)^*, \]
we have
\[ [u_t, (f; g)]_t = [u_t, (P_t f; P_t g)]_t, \quad (f; g) \in A, \quad (5.19) \]
where \([\cdot, \cdot]_t\) denotes the duality between \( S(h)^* \) and \( \mathcal{P}(h)_\perp \).
Let \( \psi_t \) be as in (5.16). For \( t \in I_{\text{reg}} \) and \( w \in \mathbb{C} \setminus \mathbb{R} \) define elements \( \psi_{t,w}(z) \in \mathcal{P}(\mathfrak{h}) \) by
\[
\psi_{t,w}(z) := (I + (z - w)R_z)\psi_t(w), \quad z \in \rho(A);
\]
remember here that \( \mathbb{C} \setminus \mathbb{R} \subseteq \rho(A) \). It follows from the resolvent identity that
\[
\psi_{t,w}(z_1) = (I + (z_1 - z_2)R_{z_1})\psi_{t,w}(z_2), \quad z_1, z_2 \in \rho(A). \quad (5.20)
\]
Moreover, since \( \psi_t(w) \perp \text{ran}(S_1(\mathfrak{h}_\nu) - \overline{m}) \) and \( A \) is a self-adjoint extension of \( S_1(\mathfrak{h}_\nu) \), we have
\[
\psi_{t,w}(z) \perp \text{ran}(S_1(\mathfrak{h}_\nu) - \overline{z}), \quad z \in \rho(A). \quad (5.21)
\]
Hence there exists a scalar function \( \lambda_{t,w} \), which is analytic on \( \rho(A) \), such that
\[
P_t\psi_{t,w}(z) = \lambda_{t,w}(z) \cdot \psi_t(z), \quad z \in \rho(A). \quad (5.22)
\]
Clearly, \( \lambda_{t,w}(w) = 1 \), and the zeros of \( \lambda_{t,w} \) form a discrete subset of \( \rho(A) \). Set
\[
S_{t,w} := \{(f;g) \in A : g - zf \perp \psi_{t,w}(z), \ z \in \rho(A)\} \subseteq \mathcal{P}(\mathfrak{h})^2.
\]
Relation (5.21) implies that \( S_{t,w} \) is a symmetric extension of \( S_1(\mathfrak{h}_\nu) \subseteq \mathcal{P}(\mathfrak{h}_\nu)^2 \), which acts in the larger Pontryagin space \( \mathcal{P}(\mathfrak{h}) \) and has deficiency index \((1,1)\).

It follows that \( (P_t \times P_t)((S_{t,w})^*) \subseteq S_1(\mathfrak{h}_\nu)^* \). In fact, due to (5.17), equality must hold.

Set \( \mathcal{P}(\mathfrak{h})_-^w := (S_{t,w}^*)_r \) and let \( P_{t,w} : \mathcal{P}(\mathfrak{h}_\nu)_- \rightarrow \mathcal{P}(\mathfrak{h})_-^w \) be the adjoint of the map \( P_t \times P_t : S_{t,w}^* \rightarrow S_1(\mathfrak{h}_\nu)^* \), i.e.
\[
[P_{t,w}, (f;g)]_{\pm t,w} = [v, (P_t f; P_t g)]_{\pm t}, \quad v \in \mathcal{P}(\mathfrak{h}_\nu)_-, \ (f;g) \in (S_{t,w})^*;
\]
where \([\cdot, \cdot]_{\pm t,w}\) is the duality between \( S_{t,w}^* \) and \( \mathcal{P}(\mathfrak{h})_-^w \). Set \( u_{t,w} := P_{t,w} u_t \), so that
\[
[u_{t,w}, (f;g)]_{\pm t,w} = [u_t, (P_t f; P_t g)]_{\pm t}, \quad (f;g) \in S_{t,w}^*.
\]
Remembering (5.19) we have, in particular,
\[
[u_{t,w}, (f;g)]_{\pm t,w} = [u, (f;g)]_{\pm}, \quad (f;g) \in A. \quad (5.24)
\]
Note here that \( A \) is contained in both \( S(\mathfrak{h})^* \) and \( S_{t,w}^* \).

Denote by \( R_{t,w;z} \) the dual of \( R_{\mathfrak{h}}^+ : \mathcal{P}(\mathfrak{h}) \rightarrow A \subseteq S_{t,w}^* \) corresponding to the duality \([\cdot, \cdot]_{\pm t,w} \). Then, by (5.24), we have
\[
[R_{t,w;z} u_{t,w}, f] = [u_{t,w}, R_{\mathfrak{h}}^+ f]_{\pm t,w} = [u, R_{\mathfrak{h}}^+ f]_{\pm} = [R_{\mathfrak{h}}^- u, f], \quad f \in \mathcal{P}(\mathfrak{h}),
\]
and hence
\[
R_{t,w;z} u_{t,w} = R_{\mathfrak{h}}^- u = \varepsilon_z, \quad z \in \rho(A). \quad (5.25)
\]
For \( t \in I_{\text{reg}} \) let \( \phi_t(z) \), \( z \in \mathbb{C} \), be the defect elements of \( S(h_{\eta t}) \) with

\[
\left( \pi_t \circ \Gamma(h_{\eta t}) \right) (\phi_t(z); z \phi_t(z)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad z \in \mathbb{C}.
\] (5.26)

By [42, Theorem 4.19], the map \( \Xi_t \), which is defined by

\[
(\Xi_t f)(z) = \begin{bmatrix} [f, \phi_t(\overline{z})] \\ [f, \psi_t(\overline{z})] \end{bmatrix}, \quad f \in \mathcal{P}(h_{\eta t}),
\] (5.27)

is an isomorphism from \( \mathcal{P}(h_{\eta t}) \) onto the reproducing kernel space \( K(\omega_{h(t)}) \) with the kernel (here \( \omega_h \) is the fundamental solution of \( h \))

\[
H_{\omega_h(t)}(w, z) = \frac{\omega_h(t; z)J_{\omega_h(t)}(w; z^*) - J_{\omega_h(t)}(w; z^*) \ast J_{\omega_h(t)}(w; z)}{z - \overline{w}}.
\]

By [42, Proposition 4.4] the kernel \( H_{\omega_h(t)} \) can be written as

\[
H_{\omega_h(t)}(w, z) = \begin{bmatrix} [\phi_t(z), \phi_t(w)] & [\phi_t(z), \psi_t(w)] \\ [\psi_t(z), \phi_t(w)] & [\psi_t(z), \psi_t(w)] \end{bmatrix},
\]

On the dense subspace span \( \{\phi_t(z) : z \in \mathbb{C}\} \cup \{\psi_t(z) : z \in \mathbb{C}\} \) of \( \mathcal{P}(h_{\eta t}) \), the action of \( \Xi_t \) is determined by linearity and the formula

\[
\Xi_t (\lambda \phi_t(z) + \mu \psi_t(z)) = H_{\omega_h(t)}(\overline{\zeta}, \cdot) \begin{pmatrix} \lambda \\ \mu \end{pmatrix}, \quad \lambda, \mu \in \mathbb{C}, \; z \in \mathbb{C}.
\] (5.28)

This relation is seen as follows (\( \zeta \in \mathbb{C} \)):

\[
\Xi_t (\lambda \phi_t(z) + \mu \psi_t(z))(\zeta) = \lambda \begin{bmatrix} [\phi_t(z), \phi_t(\overline{\zeta})] \\ [\phi_t(z), \psi_t(\overline{\zeta})] \end{bmatrix} + \mu \begin{bmatrix} [\psi_t(z), \phi_t(\overline{\zeta})] \\ [\psi_t(z), \psi_t(\overline{\zeta})] \end{bmatrix}
\]

\[
= \left( H_{\omega_h(t)}(\zeta, \cdot) \right)^T \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \left( H_{\omega_h(t)}(\zeta, \overline{\zeta}) \right)^* \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = H_{\omega_h(t)}(\overline{\zeta}, \zeta) \begin{pmatrix} \lambda \\ \mu \end{pmatrix}.
\]

In order to prove (5.1), we proceed in several steps and compute various inner products and the action of several maps. The desired result will then follow by putting these formulae together. A comprehensive overview of the involved maps is provided in the diagram on page 43.

For \( t \in I_{\text{reg}} \) denote by \( \Theta_t \) the map \( \Xi_t \) followed by projection onto the second component, i.e.

\[
(\Theta_t f)(z) := [f, \psi_t(\overline{z})], \quad f \in \mathcal{P}(h_{\eta t}).
\] (5.29)
Figure 2: Computation of $\Theta_H$ in §5.2

\[
\begin{align*}
\mathcal{P}(h) & = \mathcal{P}(h_\gamma)[\pi_\gamma] \mathcal{P}(h_\gamma^*) \\
\mathcal{P}(h) & \xrightarrow{P} \mathcal{P}(h_\gamma) \\
\pi_\gamma \circ \Gamma(h) & = \pi_\gamma \circ \Gamma(h_\gamma) \\
[u_\gamma,]_2 & := \pi_{1,2} \circ \Gamma(h), \quad [u_\gamma,]_2^\prime := \pi_{1,2} \circ \Gamma(h_\gamma) \\
[u_\gamma,]_2^{t,w} & := \pi_{1,2} \circ \Gamma(h_\gamma) \circ (P_t \times P_t) \\

S(h) & := \ker [\pi_\gamma \circ \Gamma(h)] \\
S(h_\gamma) & := \ker [(\pi_\gamma \times \pi_\tau) \circ \Gamma(h_\gamma)] \\
A & := \ker [\pi_{1,1} \circ \Gamma(h)] \\
S_1(h_\gamma) & := \ker [(\pi_{1,1} \times \pi_\tau) \circ \Gamma(h_\gamma)] \\
S_{t,w} & \text{ defect elements } \psi_{t,w}(z), \\
(P_t \psi_{t,w}(z) = \lambda_{t,w}(z) \psi_{t}(z))
\end{align*}
\]

\[
\begin{align*}
S(h) & \xrightarrow{\mathcal{P}(h)^2} S^*_t \xrightarrow{P_t \times P_t} S_1(h_\gamma) & \quad & \text{in } \mathcal{P}(h_\gamma)^2 \\
S^*_t \xrightarrow{P_t \times P_t} S_1(h_\gamma) & \xrightarrow{\mathcal{P}(h_\gamma)^*} S(h_\gamma)^* & \quad & \text{in } \mathcal{P}(h_\gamma)^2 \\
S(h) & \xrightarrow{\mathcal{P}(h)^2} S^*_t \xrightarrow{P_t \times P_t} S_1(h_\gamma) & \quad & \text{in } \mathcal{P}(h_\gamma)^2 \\
S(h) & \xrightarrow{\mathcal{P}(h)^2} S^*_t \xrightarrow{P_t \times P_t} S_1(h_\gamma) & \quad & \text{in } \mathcal{P}(h_\gamma)^2
\end{align*}
\]
Recall that the entries of $\omega_h(t; z)$ are by definition (see [42, Definitions 5.3, 4.3]) just the right-hand boundary values of the elements $\phi_t(z)$ and $\psi_t(z)$:

$$
(\pi_r \circ \Gamma(h_{\gamma t}))(\phi_t(z); z\phi_t(z)) = \omega_h(t; z)^T \begin{pmatrix} 1 \\ 0 \end{pmatrix},
$$

$$
(\pi_r \circ \Gamma(h_{\gamma t}))(\psi_t(z); z\psi_t(z)) = \omega_h(t; z)^T \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
$$

With the same method that was used in the proof of [39, Proposition 4.1] we obtain the following statement.

**5.4 Lemma.** Let $\Delta$ be a bounded Borel set, $z_0 \in \mathbb{C} \setminus \mathbb{R}$ and $t \in I_{\text{reg}}$. Then

$$
[E(\Delta) f, \psi_{t,w}(\zeta)] = \int_{-\infty}^{\infty} (\Theta_t f)(x) \cdot (x - z_0) \, dE_{\Delta, z_0, \psi_{t,w}(\zeta)}(x),
$$

where $f \in \mathcal{P}(h_{\gamma t})$, $\zeta, w \in \mathbb{C} \setminus \mathbb{R}$. 

**Proof.** The algebra of all bounded Borel sets is the union $\bigcup_{T > 0} \mathcal{B}_{(-T, T)}$ of the $\sigma$-algebras $\mathcal{B}_{(-T, T)}$ of all Borel subsets of $[-T, T]$. Since $\mathcal{B}_{(-T, T)}$, as a $\sigma$-algebra, is generated by the set of all closed intervals whose endpoints do not carry a point mass of $E$, it is enough to establish (5.30) for such intervals.

Throughout the proof fix $f \in \mathcal{P}(h_{\gamma t})$ and an interval $[a_-, a_+]$ with $E([a_-]) = E([a_+]) = 0$. Moreover, choose a bounded open interval $\Delta_0$ which contains $[a_-, a_+]$.

**Step 1: some computations.** To start with, we compute (indicating which equations are used)

$$
[u_{t,w}, (\psi_{t,w}(\zeta); \bar{\pi} \psi_{t,w}(\bar{\zeta}))]_{\pm t} \overset{(5.21)}{=} [u_{t, \Theta_t \psi_{t,w}(\bar{\zeta})}; \bar{\pi} \psi_{t,w}(\bar{\zeta})]_{\pm t},
$$

$$
\overset{(5.22)}{=} \Lambda_{t,\bar{\zeta}}(\bar{\pi}) [u_{t, (\psi_{t,w}(\bar{\zeta}); \bar{\pi} \psi_{t,w}(\bar{\zeta}))}]_{\pm t}
$$

which yields

$$
[f, \psi_{t,w}(\zeta)] \overset{(5.22)}{=} \Lambda_{t,\bar{\zeta}}(\bar{\pi}) [f, \psi_{t,w}(\bar{\zeta})]
$$

$$
= (\Theta_t f)(z) \cdot [u_{t,w}, (\psi_{t,w}(\bar{\zeta}); \bar{\pi} \psi_{t,w}(\bar{\zeta}))]_{\pm t},
$$

(5.31)

Since

$$(\bar{\pi} - z_0) R_{\pi_0} \psi_{t,w}(\bar{\zeta}) \overset{(5.20)}{=} \psi_{t,w}(\bar{\zeta}) - \psi_{t,w}(\bar{z}_0),$$

we have

$$(z - z_0)[\bar{\pi} \psi_{t,w}(\bar{\zeta})] \overset{(5.25)}{=} (z - z_0)[R_{\bar{\pi} \psi_{t,w}(\bar{\zeta})}]_{\pm t}
$$

$$
= [u_{t,w}, (\bar{\pi} - z_0) R_{\bar{\pi} \psi_{t,w}(\bar{\zeta})}]_{\pm t}
$$

$$
= [u_{t,w}, (\bar{\pi} - z_0) R_{\bar{\pi}_0} \psi_{t,w}(\bar{\zeta}); (\bar{\pi} - z_0) (I + z_0 R_{\bar{\pi}_0}) \psi_{t,w}(\bar{\zeta})]_{\pm t}.
$$

36
For each \(\gamma\), second, by analyticity we may apply Cauchy’s theorem to replace the path of integration on the right-hand side into (5.31) we obtain

\[
[f, \psi_{t,w}(\tau)] = (\Theta_t f)(z) \cdot \left( (z - \tau_0) [\varepsilon_{z_0}, \psi_{t,w}(\tau)] \\
+ [u_{t,w}, (\psi_{t,w}(\tau); \varphi_0\psi_{t,w}(\tau))] \right)_{\pm t,w},
\]

Bringing the very last summand to the left-hand side and substituting the first term on the right-hand side into (5.31) we obtain

\[
[f, \psi_{t,w}(\tau)] = (\Theta_t f)(z) \cdot \left( (z - \tau_0) [\varepsilon_{z_0}, \psi_{t,w}(\tau)] \\
+ [u_{t,w}, (\psi_{t,w}(\tau); \varphi_0\psi_{t,w}(\tau))] \right)_{\pm t,w}.
\]

For each \(z, \zeta \in \mathbb{C} \setminus \mathbb{R}\), we have

\[
[(I + (z - \zeta)R_z)f, \psi_{t,w}(\zeta)] = [f, (I + (\bar{\zeta} - \zeta)R_{\bar{\zeta}})\psi_{t,w}(\zeta)] \quad (5.32)
\]

and

\[
[\varepsilon_{z_0}, \psi_{t,w}(\zeta)] \equiv [\varepsilon_{z_0}, (I + (\bar{\zeta} - \zeta)R_{\bar{\zeta}})\psi_{t,w}(\zeta)] = \left[ (I + (z - \zeta)R_z)\varepsilon_{z_0}, \psi_{t,w}(\zeta) \right]
\]

\[
= [\varepsilon_{z_0}, \psi_{t,w}(\zeta)] + (z - \zeta) [R_z E(\Delta_0)\varepsilon_{z_0}, \psi_{t,w}(\zeta)] + (z - \zeta) [R_z E(\Delta_0^\ast)\varepsilon_{z_0}, \psi_{t,w}(\zeta)].
\]

If \(z \neq \zeta\), then

\[
\frac{1}{z - \zeta} [f, \psi_{t,w}(\zeta)] + [R_z f, \psi_{t,w}(\zeta)] \quad (5.33) \equiv \frac{1}{z - \zeta} [f, \psi_{t,w}(\tau)]
\]

\[
= \frac{1}{z - \zeta} (\Theta_t f)(z) \cdot \left( (z - \tau_0) [\varepsilon_{z_0}, \psi_{t,w}(\tau)] \\
+ [u_{t,w}, (\psi_{t,w}(\tau); \varphi_0\psi_{t,w}(\tau))] \right)_{\pm t,w}
\]

\[
\equiv (\Theta_t f)(z) \cdot \left( \frac{z - \tau_0}{z - \zeta} [\varepsilon_{z_0}, \psi_{t,w}(\zeta)] \\
+ (z - \zeta) [R_z E(\Delta_0)\varepsilon_{z_0}, \psi_{t,w}(\zeta)] + (z - \zeta) [R_z E(\Delta_0^\ast)\varepsilon_{z_0}, \psi_{t,w}(\zeta)] \right) \quad (5.34)
\]

\[
+ \frac{1}{z - \zeta} [u_{t,w}, (\psi_{t,w}(\tau); \varphi_0\psi_{t,w}(\tau))] \right)_{\pm t,w}.
\]

**Step 2: use of the Stieltjes–Liščik inversion formula.** We shall apply the Stieltjes–Liščik inversion formula as stated in [49, p. 24, Corollary II.2 (second formula)] with two minor modifications. First, since we assume that the endpoints \(a_+\) and \(a_-\) carry no point mass of \(E\), the limit “\(\varepsilon \searrow 0\)” is not needed. Second, by analyticity we may apply Cauchy’s theorem to replace the path of integration used in [49] by the path \(\gamma_3\) composed of the two oriented line segments connecting the points \(a_- - i\delta, a_+ - i\delta\), and \(a_+ + i\delta, a_- + i\delta\), respectively. Then,
for each function \( g \) that is analytic in some open neighbourhood of \([a_-, a_+]\) and \( u, v \in \text{ran } E(\Delta_0)\),

\[
\lim_{\delta \to 0} \frac{-1}{2\pi i} \int_{\gamma_\delta} g(z)[R_z u, v] dz = \int_{[a_-, a_+]} g(x) \frac{dE_{\Delta_0; v, w}}{dx}(x). \quad (5.39)
\]

Since \( \Theta_t f \) is entire and \( \zeta \notin \mathbb{R} \), the first term on the left-hand side of (5.35), the terms in (5.36) and (5.38) and the second term in (5.37) are holomorphic in a neighbourhood of \([a_-, a_+]\). Hence

\[
\lim_{\delta \to 0} \frac{-1}{2\pi i} \int_{\gamma_\delta} [R_z f, \psi_{t, w}(\zeta)] dz = \lim_{\delta \to 0} \frac{-1}{2\pi i} \int_{\gamma_\delta} (\Theta_t f)(z) \cdot (z - z_0) [R_z E(\Delta_0) \epsilon_{z_0}, \psi_{t, w}(\zeta)] dz. \quad (5.40)
\]

Applying (5.39) twice, namely with the entire functions \( g(z) = 1 \) and \( g(z) = (\Theta_t f)(z) \cdot (z - z_0) \), we obtain

\[
\begin{align*}
\left[ E([a_-, a_+]) f, \psi_{t, w}(\zeta) \right] \\
= \left[ E([a_-, a_+]) E(\Delta_0) f, E(\Delta_0) \psi_{t, w}(\zeta) \right] = \int \frac{dE_{\Delta_0; E(\Delta_0) f, E(\Delta_0) \psi_{t, w}(\zeta)}}{dx}(x) [a_-, a_+] \\
= \lim_{\delta \to 0} \frac{-1}{2\pi i} \int_{\gamma_\delta} [R_z E(\Delta_0) f, E(\Delta_0) \psi_{t, w}(\zeta)] dz \\
= \lim_{\delta \to 0} \frac{-1}{2\pi i} \int_{\gamma_\delta} (\Theta_t f)(z) \cdot (z - z_0) [R_z E(\Delta_0) \epsilon_{z_0}, E(\Delta_0) \psi_{t, w}(\zeta)] dz \\
= \lim_{\delta \to 0} \frac{-1}{2\pi i} \int_{\gamma_\delta} (\Theta_t f)(z) \cdot (z - z_0) dE_{\Delta_0; E(\Delta_0) \epsilon_{z_0}, E(\Delta_0) \psi_{t, w}(\zeta)}(x) [a_-, a_+] \\
= \int_{[a_-, a_+]} (\Theta_t f)(x) \cdot (x - z_0) dE_{\Delta_0; \epsilon_{z_0}, \psi_{t, w}(\zeta)}(x).
\end{align*}
\]

It remains to remember that \( \frac{dE_{\Delta_0; \epsilon_{z_0}, \psi_{t, w}(\zeta)}}{dx}(x) = 1_{[a_-, a_+]} \).

The next statement is the key lemma.

**5.5 Lemma.** Assume that \( H \) does not end with an indivisible interval towards \( \infty \). Then, for each \( s_0 \in (0, \infty) \), we have

\[
c.l.s. \left\{ \psi_{t, w}(z) : t \in I_{\text{reg}}, t \geq s_0, z, w \in \mathbb{C} \setminus \mathbb{R} \right\} = \mathcal{P}(\mathfrak{h}) \left[ - \right] \mathcal{E}_A(\infty), \quad (5.41)
\]

where c.l.s. stands for “closed linear span” and \( I_{\text{reg}} \) is defined in (5.14).
Proof.

Step 1. Fix a point \( s \in I_{\text{reg}} \) and let \( M \subseteq \mathbb{C} \) be a set with a finite accumulation point. Let \( \delta_j \in \mathcal{P}(\mathfrak{h}_{\gamma s}), \ j = 0, \ldots, \Delta(H) - 1 \), be the elements \( \delta_j := (0, (\delta_{jk})_{k=0}^{\Delta(H)-1}) \) where \( \delta_{jk} \) is the Kronecker delta. Then \((0; \delta_0), (\delta_0; \delta_1), \ldots, (\delta_{\Delta(H)-2}; \delta_{\Delta(H)-1}) \) \( \in T(\mathfrak{h}_{\gamma s}) \), and the boundary values of these elements vanish. Repeatedly applying the abstract Green identity \([40, (2.6) \text{ and Proposition 5.2}] \) we obtain, for \( k = 0, \ldots, \Delta(H) - 1 \) and \( z \in C \),

\[
[\delta_k, \psi_s(z)] = [\delta_{k-1}, z\psi_s(z)] = \ldots = [\delta_0, z^k\psi_s(z)] = [0, z^{k+1}\psi_s(z)] = 0.
\]

In particular, we have

\[
\mathcal{P}(\mathfrak{h}_{\gamma s}) \cdot \{ \psi_s(z) : z \in M \} \supseteq \text{span}\{\delta_0, \ldots, \delta_{\Delta(H)-1}\}. \tag{5.42}
\]

Applying the isomorphism \( \Xi_s : \mathcal{P}(\mathfrak{h}_{\gamma s}) \rightarrow \mathcal{H}(\omega_b(s)) \) from (5.27) and using (5.28) we can deduce that

\[
\Xi_s\left(\mathcal{P}(\mathfrak{h}_{\gamma s}) \cdot \{ \psi_s(z) : z \in M \} \right) = \left( \text{c.l.s.}\left\{ H_{\omega_b(s)}(\Xi, \cdot) \left( \begin{array}{c} 0 \\ 1 \end{array} \right) : z \in M \right\} \right)^{\perp}.
\]

By analyticity, the space on the right-hand side equals \( \ker \pi_- \), where \( \pi_- \) denotes the projection onto the second component in \( \mathcal{H}(\omega_b(s)) \). As shown in the proof of \([69, \text{Lemma 6.3 (subcase 3b)}] \), we have \( \dim \ker \pi_- = \Delta(H) \). Thus equality must hold in (5.42).

Step 2. First note that \( P_s\delta_k = \delta_k \) where \( P_s \) is as in (5.15), and hence

\[
[\psi_{s,w}(z), \delta_k] = [P_s\psi_{s,w}(z), \delta_k] = \lambda_{s,w}(z)[\psi_s(z), \delta_k] = 0,
\]

\( k = 0, \ldots, \Delta(H) - 1, \ z, w \in \rho(A) \).

Together with the fact that \( \mathcal{E}_A(\infty) = \text{span}\{\delta_0, \ldots, \delta_{\Delta(H)-1}\} \) (see \([54, \text{Lemma 3.2 (d)}] \)) this gives

\[
\mathcal{E}_A(\infty) \subseteq \left\{ \psi_{t,w}(z) : t \in I_{\text{reg}}, t \geq s_0, z, w \in \mathbb{C} \setminus \mathbb{R} \right\}^{\perp},
\]

i.e. the inclusion ‘\( \subseteq \)’ in (5.41). To show the reverse inclusion, let \( f \in \mathcal{P}(\mathfrak{h}) \) be given and assume that \( f \left[ \mathcal{H}_{\omega_b(s)}(\Xi, \cdot) \right] \psi_{s,w}(z), s \in I_{\text{reg}}, s \geq s_0, z, w \in \mathbb{C} \setminus \mathbb{R} \). Then, for each fixed \( s \geq s_0 \) and \( z \in \mathbb{C} \setminus \mathbb{R} \),

\[
[P_s f, \psi_s(z)] = [f, \psi_{s,z}(z)] = 0, \quad z \in \mathbb{C} \setminus \mathbb{R}.
\]

By Step 1 we therefore have \( P_s f \in \text{span}\{\delta_0, \ldots, \delta_{\Delta(H)-1}\} \). This tells us that \( \psi(\mathfrak{h}_{\gamma s})P_s f = 0 \). Since

\[
\psi(\mathfrak{h})f|_{(-1,0) \cup (0,s)} = \psi(\mathfrak{h}_{\gamma s})P_s f,
\]

and \( s \) may be chosen arbitrarily large by our hypothesis that \( H \) does not end indivisibly, it follows that \( \psi(\mathfrak{h})f = 0 \). Hence

\[
f \in \text{span}\{\delta_0, \ldots, \delta_{\Delta(H)-1}\} = \mathcal{E}_A(\infty),
\]

which proves the reverse inclusion. \( \square \)
If \( s_1, s_2 \in I_{\text{reg}}, s_1 < s_2 \), then clearly \( P_s \psi_{s_2}(z) = \psi_{s_1}(z) \), and hence
\[
\Theta_{s_1} f = \Theta_{s_2} f, \quad f \in \mathcal{P}(\mathfrak{h}_{s_1}),
\]
where \( P_s \) and \( \Theta_s \) are defined in (5.15) and (5.29). Thus a map \( \Theta_\infty \) on \( \bigcup_{s \in I_{\text{reg}}} \mathcal{P}(\mathfrak{h}_s) \) is well defined by setting
\[
\Theta_\infty f := \Theta_{s_f} f \quad \text{with} \quad s_f \quad \text{sufficiently large so that} \quad f \in \mathcal{P}(\mathfrak{h}_{s_f}). \tag{5.43}
\]
Using Lemma 5.5 we can extend Lemma 5.4.

5.6 Proposition. Let \( \Delta \) be a bounded Borel set and let \( z_0 \in \mathbb{C} \setminus \mathbb{R} \). Then
\[
[E(\Delta)f, g] = \int_{-\infty}^{\infty} (\Theta_\infty f)(x) \cdot (x - z_0) \, dE_{\Delta;x_{z_0},g}(x),
\]
\[
f \in \bigcup_{s \in I_{\text{reg}}} \mathcal{P}(\mathfrak{h}_s), \quad g \in \mathcal{P}(\mathfrak{h}).
\]  \tag{5.44}

Proof. Fix \( f \in \bigcup_{s \in I_{\text{reg}}} \mathcal{P}(\mathfrak{h}_s) \) and let \( s_f \) be so large that \( f \in \mathcal{P}(\mathfrak{h}_{s_f}) \). By Lemma 5.4 the asserted relation holds for all \( g \in \text{span}\{\psi_{s,w}(z) : s \in I_{\text{reg}}, s \geq s_f, \ z, w \in \mathbb{C} \setminus \mathbb{R}\} \).

Both sides of (5.44) depend continuously on \( g \). For the left-hand side this is obvious, for the right-hand side remember that \( E_\Delta \) is compactly supported, and hence, for each continuous function \( F \) on \( \mathbb{R} \), the integral \( \int_{-\infty}^{\infty} F \, dE_\Delta \) exists in the strong operator topology. We obtain from Lemma 5.5 that (5.44) holds for all \( g \in \mathcal{E}_A(\infty)^{[1]} \).

Finally, let an arbitrary element \( g \in \mathcal{P}(\mathfrak{h}) \) be given. Since \( E(\Delta)g \in \mathcal{E}_A(\infty)^{[2]} \), we may apply what we have already shown and obtain
\[
[E(\Delta)f, g] = [E(\Delta)f, E(\Delta)g] = \int_{-\infty}^{\infty} (\Theta_\infty f)(x) \cdot (x - z_0) \, dE_{\Delta;x_{z_0},E(\Delta)g}(x).
\]

5.7 Lemma. Let \( \Delta \) be a bounded Borel set and \( g \in \mathcal{P}(\mathfrak{h}) \). Then
\[
(x - z_0) \, dE_{\Delta;x_{z_0},g} = (x - z_1) \, dE_{\Delta;x_{z_1},g}, \quad z_0, z_1 \in \mathbb{C} \setminus \mathbb{R}. \tag{5.45}
\]

Proof. To see this, note that
\[
I + (z_0 - z_1)(A - z_0)^{-1} = \int_{\mathbb{R}} \frac{x - z_1}{x - z_0} \, dE_\Delta, \quad z_0, z_1 \in \mathbb{C} \setminus \mathbb{R}.
\]

From the identity [38, (3.2)] we obtain
\[
(I + (z_0 - z_1)(A - z_0)^{-1})E(\Delta)R_{z_1}^- = R_{z_0}^-.
\]

Hence, for each Borel set \( \Delta' \) and \( g \in \mathcal{P}(\mathfrak{h}) \),
\[
E_{\Delta;x_{z_0},g}(\Delta') \overset{(5.25)}{=} E(\Delta \cap \Delta')R_{z_0}^-u, g \overset{\quad (5.25)}{=} [E(\Delta')E(\Delta)R_{z_0}^-u, E(\Delta)g]
\]
\[
= [E(\Delta')E(\Delta)(I + (z_0 - z_1)(A - z_0)^{-1})R_{z_1}^-u, E(\Delta)g]
\]
Proof. In particular, acting as in (5.9) the Weyl coefficient \( w \) and the \( \infty \) Lemma. Let \( \Delta \) and \( \delta \) be the distributional density in the representation (3.5) of the Weyl coefficient \( q_h \) of \( h \), let \( \psi(\phi_h) \) be as in 3.6, let \( \Theta_h \) be the isomorphism acting as in (5.9) and let \( \Theta_{\infty} \) be as in (5.43). Then

\[
(\Theta_{\infty} f)(x) = \frac{1}{x-i} \left( \psi(\phi_h) \circ \Theta_h \right)(x) \quad \mu_H - a.e.,
\]

\[ f \in \bigcup_{s \in \mathcal{L}_{reg}} \mathcal{P}(\eta_h_s) \cap \mathcal{E}_A(\infty)^{[1,1]}.
\]

In particular, \( \Theta_{\infty} \) maps \( \bigcup_{s \in \mathcal{L}_{reg}} \mathcal{P}(\eta_h_s) \) into \( L^2(\mu_H) \).

Proof. Let \( f \in \bigcup_{s \in \mathcal{L}_{reg}} \mathcal{P}(\eta_h_s) \cap \mathcal{E}_A(\infty)^{[1,1]} \) be given. For a bounded open interval \( \Delta \) and \( w \in \mathbb{C} \setminus \mathbb{R} \), we compute the inner product \( [E(\Delta)f, \varepsilon_w] \) in two ways.

On one hand, we have

\[
[E(\Delta)f, \varepsilon_w] \overset{(5.44)}{=} \int_{\mathbb{R}} (\Theta_{\infty} f)(x) (x-i) \, dE_{\varepsilon, \varepsilon_w}(x) \overset{(5.13, 5.45)}{=} \int_{\mathbb{R}} (\Theta_{\infty} f)(x) (x+i) \, dE_{\varepsilon, \varepsilon_w}(x).
\]

Using a standard argument we now relate \( E_{\varepsilon, \varepsilon_w} \) to \( \mu_H \). Since \( q_H \) is the \( \mathbb{Q} \)-function induced by \( A \) and the family \((\varepsilon)_{\varepsilon \in \mathbb{R}(A)}\), we have the representation

\[
q_h(z) = \overline{q_h(i)} + (z+i)[\varepsilon_i, \varepsilon_i] + (z^2+1)[R_\varepsilon \varepsilon_i, \varepsilon_i] = \overline{q_h(i)} + (z+i)[\varepsilon_i, \varepsilon_i] + (z^2+1)\left[ R_\varepsilon E(\Delta)^c \varepsilon_i, E(\Delta)^c \varepsilon_i \right] + (z^2+1) \left[ (A_\varepsilon - z)^{-1} E(\Delta) \varepsilon_i, E(\Delta) \varepsilon_i \right].
\]

Let \([a_-, a_+] \subseteq \Delta \) be such that \( \mu_H \{a_- \} = \mu_H \{a_+ \} = 0 \) and \( E(\{a_- \}) = E(\{a_+ \}) = 0 \). Observing that all summands on the right-hand side of (5.47) apart from the last one are analytic across \( \Delta \) we compute (where \( \gamma_3 \) is as in the
Proof of Lemma 5.4)

\[ \mu_H([a_-, a_+]) = \frac{1}{12} \lim_{\delta \searrow 0} \int_{[a_-, a_+]} \text{Im} q_H(x + i\delta) \, dx \]

\[ q_H(r \pm i\delta) = \frac{1}{\pi} \lim_{\delta \searrow 0} -\frac{1}{2i} \int_{\gamma_{\delta}} q_H(z) \, dz \]

\[ \lim_{\delta \searrow 0} \frac{1}{2\pi i} \int_{\gamma_{\delta}} \left( z^2 + 1 \right) \left[ (A_\Delta - z)^{-1} E(\Delta) \xi_i, E(\Delta) \xi_i \right] \, dz \]

\[ \int_{[a_-, a_+]} \left( x^2 + 1 \right) dE_{\Delta, E(\Delta) \xi_i, E(\Delta) \xi_i} \]

From this it follows that \( 1_\Delta(x) \, d\mu_H(x) \ll dE_{\Delta, \xi_i} \) and

\[ \frac{1_\Delta(x) \, d\mu_H(x)}{dE_{\Delta, \xi_i}} = x^2 + 1. \]

Hence, we can further rewrite the last integral in (5.46), and obtain

\[ \int_\mathbb{R} (\Theta_{\infty} f)(x)(x - i) \frac{x + i}{x - \frac{i}{w}} \frac{1_\Delta(x) \, d\mu_H(x)}{1 + x^2}. \]

(5.48)

On the other hand, let \( E_{\Phi_\theta}(\infty)^{[1]} \) be the algebraic eigenspace at infinity of \( A_{\Phi_\theta} \), let \( E_{\Phi_\theta} \) be the spectral measure of \( A_{\Phi_\theta} \) and let \( \xi_w \) be as in (5.10). Since \( \Theta_\theta \) is isometric, \( \text{ran} E_{\Phi_\theta}(\Delta) \subseteq E_{\Phi_\theta}(\infty)^{[1]} \) and \( \psi(\Phi_\theta) \) is isometric on \( E_{\Phi_\theta}(\infty)^{[1]} \) (by [56, Theorem 5.3]), we have

\[ [E(\Delta)f, \xi_w] = [E(\Delta)f, E(\Delta)\xi_w] \]

\[ = [\Theta_\theta E(\Delta)f, \Theta_\theta E(\Delta)\xi_w]_{\Pi(\Phi_\theta)} \]

\[ = [E_{\Phi_\theta}(\Delta) \Theta_\theta f, E_{\Phi_\theta}(\Delta) \Theta_\theta \xi_w]_{\Pi(\Phi_\theta)} \]

\[ = \int_\mathbb{R} \left( \psi(\Phi_\theta) E_{\Phi_\theta}(\Delta) \Theta_\theta f \right)(x) \left( \psi(\Phi_\theta) E_{\Phi_\theta}(\Delta) \xi_w \right)(x) \frac{d\mu_H(x)}{1 + x^2}. \]

(5.49)

The mappings \( \psi(\Phi_\theta) \circ E_{\Phi_\theta}(\Delta) \xi_{\Phi_\theta}(\infty)^{[1]} \) and \( (-1) \circ \psi(\Phi_\theta) \) are continuous on \( E_{\Phi_\theta}(\infty)^{[1]} \). By the definition of \( \psi(\Phi_\theta) \) and [39, Proposition 3.1], they coincide on all compactly supported functions of \( B_2(\Phi_\theta) \) (for this notation see [56, §5]). Their continuity implies that they coincide on \( E_{\Phi_\theta}(\infty)^{[1]} \). The assumption \( f \in \mathcal{E}_A(\infty)^{[1]} \) and the relation (5.11) imply that \( \Theta_\theta f \in \mathcal{E}_{\Phi_\theta}(\infty)^{[1]} \). Hence

\[ \left( \psi(\Phi_\theta) E_{\Phi_\theta}(\Delta) \Theta_\theta f \right)(x) = 1_\Delta(x) \left( \psi(\Phi_\theta) \Theta_\theta f \right)(x) \quad x \in \mathbb{R} \quad \mu_H\text{-a.e.} \]

Using [39, Proposition 3.1] and the fact that \( \psi(\Phi_\theta) \) acts as the identity on compactly supported functions we obtain

\[ \left( \psi(\Phi_\theta) E_{\Phi_\theta}(\Delta) \xi_w \right)(x) = \left( \psi(\Phi_\theta) (1_\Delta \xi_w) \right)(x) = 1_\Delta(x) \cdot \frac{x - i}{x - w}. \]
The integral on the right-hand side of (5.49) thus equals
\[
\int \psi(\phi_h)\Theta_h f(x) \cdot \frac{x+i}{\sqrt{x-w}} \Delta(x) \frac{d\mu_H(x)}{1+x^2}.
\] (5.50)

Since the linear span of the functions \(x \mapsto x + i x - w, w \in \rho(A)\), is dense in \(L^2(\frac{1}{1+x^2}\mu_H(x))\), we conclude from (5.48), (5.49) and (5.50) that
\[
(\Theta_\infty f)(x)(x+i) = (\psi(\phi_h)\Theta_h f)(x), \quad x \in \Delta \mu_H\text{-a.e.}
\]

Since \(\Delta\) was an arbitrary bounded open interval, the assertion follows.

To finish the proof of (5.1), let \([a, b] \subseteq (0, \infty)\) and denote by \(\Omega\) the map defined on \(L^2(H|_{[a,b]})\) as
\[
(\Omega f)(z) := \left( \int_{[a,b]} \left[ \psi(\phi_h)\phi_h(z) \right]^* H(t)f(t) dt \right),
\]
where \(\phi_h\) and \(\psi_h\) are as in (5.26) and (5.16), respectively. The function \([\psi(\phi_h)\psi_h(z)](t)\) is a solution of (1.1) with \(z\) replaced by \(\Xi\), which assumes boundary values at 0, namely \((0, 1)^T\). The function \(\phi(t; z)\) shares these properties, and hence we have \([\psi(\phi_h)\psi_h(z)](t) = \phi(t; z)\). Thus the second component of \(\Omega\) can be rewritten as
\[
\int_{[a,b]} \phi(t; z)^* H(t)f(t) dt.
\]

The asserted formula (5.1) for the action of \(\Theta_H\) is now obtained by putting together the so far collected knowledge. Consider the following diagram (here \(\pi_\infty\) denotes the projection onto the second component, and references between \# are for the proof of the commutativity of the corresponding part of the diagram):
We see that (5.1) holds for each \( f \in L^2(H) \) supported on \([a, b]\). For each fixed \( T > 0 \) both sides of (5.1) depend continuously on \( f \) when \( f \) varies in the set of all elements of \( L^2(H) \) whose support is bounded above by \( T \). Remember here that \( \varphi(\cdot; z) \in L^2(0, T) \). Hence, (5.1) holds for all \( f \in L^2(H) \) whose support is bounded above by \( T \).

5.3 Computation of \( \Theta_H^{-1} \) as an integral transform

The final task in the proof of Theorem 5.1 is to establish the formula (5.2) for the action of \( \Theta_H^{-1} \).

Denote by \( E \) the spectral family associated with \( T(H) \). Let \( f \in L^2(H) \) with \( \text{sup} \text{supp} f < b \), let \( h \in L^2(H) \) be bounded and with \( \text{sup} \text{supp} h < b \), and let \( \Delta \subseteq \mathbb{R} \) be a finite interval. Using (i), (ii) of Theorem 5.1 and Fubini’s theorem we obtain

\[
(E(\Delta)f, h)_{L^2(H)} = \int_{\mathbb{R}} \mathbb{1}_\Delta(t)(\Theta_H f)(t)(\Theta_H h)(t) \, d\mu_H(t)
\]

\[
= \int_{\Delta} (\Theta_H f)(t) \int_a^b h(x)^* H(x) \varphi(x; t) \, dx \, d\mu_H(t)
\]

\[
= \int_a^b h(x)^* H(x) \int_{\Delta} (\Theta_H f)(t) \varphi(x; t) \, d\mu_H(t) \, dx.
\]

Since the set \( \{ h \in L^2(H) : \text{sup} \text{supp} h < b, h \text{ bounded} \} \) is dense in \( L^2(H) \), it follows that

\[
(E(\Delta)f)(x) = \int_{\Delta} (\Theta_H f)(t) \varphi(x; t) \, d\mu_H(t) \quad \text{H-a.e.} \quad (5.51)
\]

Both sides of this equality depend continuously on \( f \) and therefore this relation is valid for arbitrary \( f \in L^2(H) \).

To complete the proof, let \( g \in L^2(\mu_H) \) with compact support be given. Choose a finite interval \( \Delta \) which contains \( \text{supp} g \). Since \( \Theta_H \) intertwines \( T(H) \) with the multiplication operator in \( L^2(\mu_H) \), we have

\[
E(\Delta) \circ \Theta_H^{-1} = \Theta_H^{-1} \circ (\cdot \mathbb{1}_\Delta).
\]

Thus

\[
(\Theta_H^{-1} g)(x) = (\Theta_H^{-1}(\mathbb{1}_\Delta g))(x)^{(5.52)} = (E(\Delta)(\Theta_H^{-1} g))(x)
\]

\[
(5.51)
\]

\[
\int_{\Delta} g(t)\varphi(x; t) \, d\mu_H(t) = \int_{\mathbb{R}} g(t)\varphi(x; t) \, d\mu_H(t) \quad \text{H-a.e.}
\]

This finishes the proof of Theorem 5.1.

5.4 The connection between the point mass at 0 and the behaviour of \( H \)

Before we prove Proposition 5.3, we need a lemma.
5.9 Lemma. Let $H \in \mathbb{H}$ with $\text{dom} \ H = (0, \infty)$ and let $\omega$ be the general Hamiltonian as in 3.15. Moreover, let $\omega_{h}$ be the chain of matrices as in 3.8. Then, for each $x \in (0, \infty)$,

$$\left[ \frac{\partial}{\partial z} \omega_{h}(x; z) \right]_{z=0} = - \int_{0}^{x} h_{22}(t)dt.$$  \hspace{1cm} (5.53)

Proof. Let $x_{1} \in (0, x)$. Integrating (3.9) we obtain

$$\omega_{h}(x; z) - \omega_{h}(x_{1}; z) = z \int_{x_{1}}^{x} \omega_{h}(t; z)H(t)Jdt$$

for $z \in \mathbb{C}$. If we differentiate both sides with respect to $z$, set $z = 0$ and use (3.10), we arrive at

$$\left[ \frac{\partial}{\partial z} \omega_{h}(x; z) \right]_{z=0} = \int_{x_{1}}^{x} \omega_{h}(t; 0)H(t)Jdt.$$  \hspace{1cm} (5.54)

It follows from [54, Theorem 4.1] that

$$\lim_{x_{1} \searrow 0} \left[ \frac{\partial}{\partial z} \omega_{h}(x_{1}; z) \right]_{z=0} = 0,$$

which, together with (5.54), implies (5.53).

Proof of Proposition 5.3. Assume first that $\mu_{H}(\{0\}) > 0$. Define $g \in L^{2}(\mu_{H})$ by

$$g(t) = \begin{cases} 1, & t = 0, \\ 0, & t \neq 0. \end{cases}$$

Then

$$\Theta_{H}^{-1}g)(x) = \int_{-\infty}^{\infty} g(t)\phi(x; t)d\mu_{H}(t) = \mu_{H}(\{0\})\phi(x; 0) = \mu_{H}(\{0\}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ 

Since $\Theta_{H}$ is an isometric isomorphism, we obtain

$$\mu_{H}(\{0\}) = \|g\|_{L^{2}(\mu_{H})}^{2} = \|\Theta_{H}^{-1}g\|_{L^{2}(H)}^{2} = [\mu_{H}(\{0\})]^{2} \int_{0}^{\infty} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\ast} H(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} dx$$

$$= \mu_{H}(\{0\})^{2} \int_{0}^{\infty} h_{22}(x)dx,$$

which implies (5.3) and (5.4).

Now assume that (5.3) is satisfied. Let $c > 0$ be large enough such that $(0, c)$ is not an indivisible interval and introduce the Hamiltonian function

$$H_{c}(x) := \begin{cases} H(x), & x \in (0, c], \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & x \in (c, \infty), \end{cases}$$

45
which belongs to \(\mathbb{H}\), satisfies \(\Delta(H_c) = \Delta(H)\) and is in the limit point case at infinity. Moreover, let \(h_c\) be the corresponding singular general Hamiltonian as in 3.15 and let \(\omega_{h_c}\) be the chain of matrices as in 3.8. The singular Weyl coefficient \(q_{H_c}\) is given by

\[
q_{H_c}(z) = q_{h_c}(z) = \frac{\omega_{h_c,11}(c;z)}{\omega_{h_c,21}(c;z)}, \quad z \in \mathbb{C} \setminus \mathbb{R};
\]

it belongs to \(N^{(\infty)}_{\Delta(H)}\) and is meromorphic in \(\mathbb{C}\). Let \(\mu_{H_c}\) be the spectral measure associated with \(H_c\) via (4.11). Then Lemma 5.9 and (3.10) imply that, for every \(\varepsilon > 0\),

\[
\mu_{H_c}((\varepsilon, \varepsilon)) \geq \mu_{H_c}(\{0\}) = \text{Res}(q_{H_c}, 0) = \frac{\omega_{h_c,11}(c;0)}{\sqrt{2} \omega_{h_c,21}(c;0)} = M.
\]

By Theorem 4.5 (ii) we have \(q_{H}(z) = \lim_{c \to \infty} q_{H_c}(z)\) locally uniformly in \(\mathbb{C} \setminus \mathbb{R}\). Since \(\text{ind}_{-} q_{H_c} = \Delta(H_c) = \Delta(H) = \text{ind}_{-} q_{H}\), we can apply Lemma 3.7, which implies that, for every \(\varepsilon > 0\) such that \(\mu_{H}((\varepsilon, \varepsilon)) = 0\),

\[
\mu_{H}((\varepsilon, \varepsilon)) = \lim_{c \to \infty} \mu_{H_c}((\varepsilon, \varepsilon)) \geq M.
\]

Hence \(\mu_{H}(\{0\}) \geq M > 0\).

6 Inverse theorems

By the procedure described in Theorem 4.8 a map from \(\mathbb{H}\) to \(\mathbb{M}\), namely \(H \mapsto \mu_{H}\), is well defined. Hence it is meaningful to pose inverse problems. Concisely formulated, we face the task to determine the range and kernel of the mapping \(H \mapsto \mu_{H}\).

In this section we complete this task. In fact, we provide somewhat more detailed results. They include singular Weyl functions and a local version of the uniqueness theorem. Proofs are again relatively simple; they are carried out in the same manner as in §4, using the basic identifications 3.15, 3.16, and some results taken from the literature. Recall the notation from Definition 4.7. Theorems 6.1 and 6.2 are the analogues for our class \(\mathbb{H}\) of Hamiltonians with two singular endpoints to de Branges’ celebrated inverse spectral theorem.

6.1 Theorem (Existence Theorem).

The following statements hold.

(i) Let \(q \in N^{(\infty)}\) with \(\text{ind}_{-} q > 0\). Then there exists a Hamiltonian \(H \in \mathbb{H}\) with \(q \in [q]_H\).

(ii) Let \(\mu \in \mathbb{M}\) with \(\Delta(\mu) > 0\). Then there exists a Hamiltonian \(H \in \mathbb{H}\) with \(\mu_{H} = \mu\).

Proof. To show (i) let \(q \in N^{(\infty)}\) be given. According to [43, Theorem 1.4] there exists a general Hamiltonian \(h_0\) with \(q_{h_0} = q\). Moreover, [54, Theorem 3.1] implies that \(h_0 \in \mathcal{S}_0\). Applying an appropriate reparameterization we may assume that...
- $h_0$ is defined on $(-1, 0) \cup (0, \infty)$;
- the Hamiltonian function of $h$ equals $x^{-2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ for $x \in (-1, 0)$;
- $b_{\alpha+1} = 0$;
- $E = \{-1, 1, \infty\}$.

Let $H$ be the Hamiltonian function of $h_0$ on the interval $(0, \infty)$. Then, by our basic identification 3.15, we have $H \in \mathcal{H}$.

Let $h$ be the general Hamiltonian built from $H$ in the basic identification 3.16 with $\tilde{a}_1, b_{1,j}, d_{1,j}$ all equal to 0. Then $h$ and $h_0$ differ only in the data part $\tilde{a}_1, b_{1,j}, d_{1,j}$. From [54, Corollary 5.9] we obtain that

$$q_H(z) = q_h = q_{h_0} - \sum_{l=1}^{2\Delta(H)} z^l d_{1,l-1} + \sum_{l=1}^{2\Delta(H)+1} z^l b_{1,\tilde{a}_1+1-l},$$

i.e. $q_H$ and $q_h$ differ only by a polynomial with real coefficients and vanishing constant term. Hence $q \in \mathcal{Q}_{\mathbb{R}}(\mathbb{H})$.

For the proof of (ii) let $\mu \in \mathbb{M}$ be given. Choose $q \in \Lambda^\infty_{\leq \infty}$ with $\mu_q = \mu$: this is possible by [56, Theorem 3.9 (v)]\(^7\). An application of the already proved item (i) provides us with a Hamiltonian $H \in \mathbb{H}$ such that $q_H - q$ is a real polynomial. By the Stieltjes inversion formula (3.6) and the definition of $\mu_H$, it follows that $\mu_H = \mu_{q_H} = \mu$. \(\square\)

As we have seen in Proposition 4.10, Hamiltonians which are — essentially — reparameterizations of each other have — essentially — the same singular Weyl coefficients and have the same spectral measures. The converse of this fact is an important result.

### 6.2 Theorem (Global Uniqueness Theorem).

Let $H_1, H_2 \in \mathbb{H}$ be given.

1. Assume that there exist singular Weyl coefficients $q_{H_1}$ and $q_{H_2}$ of $H_1$ and $H_2$, respectively, such that $q_{H_1} - q_{H_2}$ is a real polynomial, and set $\alpha := (q_{H_1} - q_{H_2})(0)$. Then the Hamiltonians

$$H_1 \quad \text{and} \quad \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} H_2 \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \quad (6.1)$$

are reparameterizations of each other.

In particular, if $[q]_{H_1} = [q]_{H_2}$, then $H_1$ and $H_2$ are reparameterizations of each other.

2. If $\mu_{H_1} = \mu_{H_2}$, then there exists a real constant $\alpha$ such that the Hamiltonians in (6.1) are reparameterizations of each other.

Proof. Let $H_1, H_2 \in \mathbb{H}$ be given. Assume that both Hamiltonians are defined on $(0, \infty)$. This is no loss in generality since it can always be achieved by a reparameterization, and reparameterizations change neither singular Weyl coefficients nor spectral measures; see Proposition 4.10.

\(^7\)The set on the right-hand side of [56, Theorem 3.9 (v)] is certainly non-empty.
First we consider the case when \([q]H_1 = [q]H_2\). Let \(h_1\) and \(h_2\) be the general Hamiltonians defined for \(H_1\) and \(H_2\) by the basic identification 3.16 with \(\delta_1, b_{1,j}, d_{1,j}\) all equal to 0 with some base points \(x_1\) and \(x_2\). Then
\[ q_{h_1} \in [q]H_1 \quad \text{and} \quad q_{h_2} \in [q]H_2. \]

Applying a reparameterization to \(h_2\), we can achieve that \(x_1 = x_2\). By [54, Corollary 5.9] there exist numbers \(d_{1,0}, \ldots, d_{1,2\Delta(H_2)} - 1 \in \mathbb{R}\) and \(\delta_1 \in \mathbb{N}_0, b_{1,1}, \ldots, b_{1,\delta_1} \in \mathbb{R}\) such that the Weyl coefficient of the general Hamiltonian \(h_2\) constituted by the same data as \(h_2\) with exception of \(d_{1,j}, \delta_1, b_{1,j}\) is equal to \(q_{h_1}\). By the uniqueness part in [43, Theorem 1.4], thus, \(h_1\) and \(h_2\) are reparameterizations of each other. In particular, their Hamiltonian functions on \((0, \infty)\) are reparameterizations of each other. However, the Hamiltonian function of \(h_1\) on this interval is \(H_1\) and the one of \(h_2\) is \(H_2\).

Now assume that some singular Weyl coefficients \(q_{H_1}\) and \(q_{H_2}\) differ by a real constant, say \(\alpha\). Consider the Hamiltonian \(H_0 := \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}H_2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\). We know from Proposition 4.10 (and its proof) that \(H_0 \in \mathcal{H}\) and that \(q_{H_0} - q_{H_2} = \alpha\) when we choose the same base point in the construction of \(q_{H_1}\) and \(q_{H_2}\), respectively. We thus have \(q_{H_1} = q_{H_0}\). By what we proved in the previous paragraph, this implies that \(H_1\) and \(H_0\) are reparameterizations of each other; hence (i) is shown.

For the proof of (ii) assume that \(\mu_{H_1} = \mu_{H_2}\). Choose \(x_0 \in (0, \infty)\), and let \(q_{H_1}\) and \(q_{H_2}\) be singular Weyl coefficients of \(H_1\) and \(H_2\), respectively, built with the base point \(x_0\). By [56, Theorem 3.9 (iv)] the difference \(q_{H_1} - q_{H_2}\) is a real polynomial. Now the already proved item (i) yields (6.1).

Our last result in this section is a refined version of the above uniqueness theorem. It asserts that certain beginning sections of two Hamiltonians \(H_1, H_2 \in \mathcal{H}\) coincide if (and only if) some of their singular Weyl coefficients are exponentially close. Local uniqueness theorems for one-dimensional Schrödinger operators were first proved by B. Simon in [63, Theorem 1.2]. For canonical systems with a regular left endpoint a local uniqueness theorem was proved in [53, Theorem 1.2]; see also [50, Section 4] for a formulation in terms of transfer functions.

### 6.3 Theorem (Local Uniqueness Theorem)

Let \(H_1, H_2 \in \mathcal{H}\) with \(\text{dom}(H_i) = (a_i, b_i), i = 1, 2\), be given and set
\[ s_i(\tau) := \sup \left\{ x \in (a_i, b_i) : \int_{a_i}^x \sqrt{\det H_i(\xi)} \, d\xi < \tau \right\}, \quad \tau > 0, \ i = 1, 2. \quad (6.2) \]

Then, for each \(\tau > 0\), the following statements are equivalent.

(i) The Hamiltonian \(H_1|_{(a_1,s_1(\tau))}\) is a reparameterization of \(H_2|_{(a_2,s_2(\tau))}\).

(ii) There exist singular Weyl coefficients \(q_{H_1}\) and \(q_{H_2}\) of \(H_1\) and \(H_2\), respectively, and there exists a \(\beta \in (0, \pi)\) such that for each \(\varepsilon > 0\),
\[ q_{H_1}(re^{i\beta}) - q_{H_2}(re^{i\beta}) = O(e^{(-2\tau + \varepsilon)\sin \beta}), \ r \to \infty. \]

(iii) There exist singular Weyl coefficients \(q_{H_1}\) and \(q_{H_2}\) of \(H_1\) and \(H_2\), respectively, and there exists a \(k \geq 0\) such that for each \(\delta \in (0, \frac{\pi}{2})\),
\[ q_{H_1}(z) - q_{H_2}(z) = O((\text{Im} z)^ke^{-2\tau \text{Im} z}), \ |z| \to \infty, \ z \in \Gamma_\delta, \quad (6.3) \]
where \(\Gamma_\delta\) is the Stolz angle \(\Gamma_\delta := \{ z \in \mathbb{C} : \delta \leq \text{arg} z \leq \pi - \delta \}\).

48
Note that the integral in (6.2) is always finite. This is a consequence of [55, Theorem 4.1], which also implies that this integral is equal to the exponential type of each entry of \( \Theta(x; \cdot) \) and \( \varphi(x; \cdot) \).

**Proof of Theorem 6.3.** This theorem is a consequence of the indefinite version of [53, Theorem 1.2] indicated in [53, Remark 1.3]. Let \( H_1, H_2 \in \mathbb{H} \) be given, and assume w.l.o.g. that both are defined on \((0, \infty)\). The proof again proceeds via considering general Hamiltonians \( h_1 \) and \( h_2 \) built in our basic identification 3.16 from \( H_1 \) and \( H_2 \), respectively.

Assuming (i) we choose \( \bar{a}_1, b_{1,j}, d_{1,j} \) all equal to 0 and the same base point \( x_0 \) in the definition of \( h_1 \) and \( h_2 \). Then, by [53, Theorem 1.2 (indefinite variant)], it follows that \( q_{h_1} \) and \( q_{h_2} \) are exponentially close in the sense of (ii) and (iii).

Conversely, assume (ii) or (iii), and choose \( \bar{a}_1^{(1)}, b_{1,j}^{(1)}, d_{1,j}^{(1)} \) and \( x_1 \) and \( \bar{a}_1^{(2)}, b_{1,j}^{(2)}, d_{1,j}^{(2)} \) and \( x_2 \) in the definition of \( h_1 \) and \( h_2 \) so that \( q_{H_1} = q_{h_1} \). This is possible; cf. Remark 4.6. Then, again by [53, Theorem 1.2 (indefinite variant)], \( h_1, \varphi_{s_1(\tau)} \) and \( h_2, \varphi_{s_2(\tau)} \) are reparameterizations of each other. In particular, (i) holds. \( \square \)

**6.4 Remark.** If one (and hence all) of the equivalent conditions of Theorem 6.3 hold, then (6.3) holds with

\[
k := 8 \max \{ \Delta(H_1), \Delta(H_2) \} + 3. \tag{6.4}
\]

This can be seen by tracing the proof in [53, Theorem 1.2 (indefinite version)] as indicated in the footnotes in this paper.

The value (6.4) of the constant \( k \) in (6.3) is probably not the best possible. However, it is noteworthy that (6.4) depends only on \( \Delta(H_1) \) and \( \Delta(H_2) \).

**6.5 Remark.** The dependence of these results on the choices of \( q_{H_1} \) and \( q_{H_2} \) is not essential. If (ii/iii) holds with some pair \((q_{H_1}, q_{H_2}) \in [q|H_1] \times [q|H_2] \), then for each \( q_1 \in [q|H_1] \) there exists a unique element \( q_2 \in [q|H_2] \) such that (ii/iii) holds for \((q_1, q_2) \). This corresponding function \( q_2 \) can be determined by starting with some \( q \in [q|H_2] \), computing the polynomial asymptotics of \( \varphi = q_{H_2} \) at \( \infty \), and subtracting this polynomial from \( q \).

Viewing the above remark from a slightly different perspective leads to the following more effective test for (ii/iii) to hold, which removes the dependence on the choice of \( q_{H_1} \) and \( q_{H_2} \).

**6.6 Corollary.** Assume that we are in the situation of Theorem 6.3. Pick some singular Weyl coefficients \( q_1 \in [q|H_1] \) and \( q_2 \in [q|H_2] \) and let

\[
q_2(iy) - q_1(iy) = \alpha_0 y^n + \alpha_{n-1} y^{n-1} + \ldots + \alpha_1 y + o(y), \quad y \rightarrow \infty.
\]

Then (ii/iii) of Theorem 6.3 hold if and only if the conditions stated in (ii/iii) hold with \( q_{H_1} = q_1 \) and

\[
q_{H_2}(z) = q_2(z) - \sum_{l=1}^{n} \alpha_l (-i)^l z^l.
\]

The following corollary of Theorem 6.3 is also worth mentioning. It says that under the a priori hypothesis of finite exponential type, the global uniqueness result Theorem 6.2 can be strengthened; and it may be much easier to establish exponential closeness of singular Weyl coefficients than their actual equality.
6.7 Corollary. Let $H_1, H_2 \in \mathbb{H}$ with $\text{dom}(H_i) = (a_i, b_i)$, $i = 1, 2$, be given and assume that
\[
\int_{a_i}^{b_i} \sqrt{\det H_i(y)} \, dy < \infty, \quad i = 1, 2.
\]
If there exist singular Weyl coefficients $q_{H_1}$ and $q_{H_2}$ of $H_1$ and $H_2$, respectively, and there exist $\beta \in (0, \pi)$ and $\tau > \max\left\{\int_{a_i}^{b_i} \sqrt{\det H_i(y)} \, dy : i = 1, 2\right\}$ such that
\[
q_{H_1}(re^{i\beta}) - q_{H_2}(re^{i\beta}) = O(e^{-2\tau r \sin \beta}), \quad r \to \infty,
\]
then $H_1$ and $H_2$ are reparameterizations of each other.

Acknowledgements.
The first author gratefully acknowledges the support of the Nuffield Foundation, grant no. NAL/01159/G, and the Engineering and Physical Sciences Research Council (EPSRC), grant no. EP/E037844/1. The second author was supported by a joint project of the Austrian Science Fund (FWF, I1536–N25) and the Russian Foundation for Basic Research (RFBR, 13-01-91002-ANF).

References


52


Liouville operators, canonical systems and strings. *Integral Equations Op-

[51] H. Langer, A. Luger and V. Matsaev. Convergence of generalized Nevan-

[52] M. Langer and H. Woracek. A function space model for canonical systems

[53] M. Langer and H. Woracek. A local inverse spectral theorem for Hamilto-

[54] M. Langer and H. Woracek. Indefinite Hamiltonian systems whose
Titchmarsh–Weyl coefficients have no finite generalized poles of non-

[55] M. Langer and H. Woracek. The exponential type of the fundamental solu-


[57] M. Langer and H. Woracek. Direct and inverse spectral theorems for a class
of canonical systems with two singular endpoints. Part II: Sturm-Liouville
Equations. *submitted*.


[59] B.C. Orcutt. *Canonical Differential Equations*. ProQuest LLC, Ann Arbor,

[60] G. Pick. Über die Beschränkungen analytischer Funktionen, welche durch
vorgegebene Funktionswerte bewirkt werden [German]. *Math. Ann.* **77**
(1915), 7–23.


Method of Operator Identities*. Translated from the Russian manuscript

[63] B. Simon. A new approach to inverse spectral theory. I. Fundamental form-

[64] H. de Snoo and H. Winkler. Canonical systems of differential equations


M. Langer
Department of Mathematics and Statistics
University of Strathclyde
26 Richmond Street
Glasgow G1 1XH
UNITED KINGDOM
email: m.langer@strath.ac.uk

H. Woracek
Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße, 8-10/101
1040 Wien
AUSTRIA
email: harald.woracek@tuwien.ac.at