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Canonical systems with discrete spectrum

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ABSTRACT

We study spectral properties of two-dimensional canonical systems $y'(t) = zJH(t)y(t)$, $t \in [a, b)$, where the Hamiltonian H is locally integrable on $[a, b)$, positive semidefinite, and Weyl's limit point case takes place at b . We answer the following questions explicitly in terms of H :

Is the spectrum of the associated selfadjoint operator discrete?

If it is discrete, what is its asymptotic distribution?

Here asymptotic distribution means summability and limit superior conditions relative to comparison functions growing sufficiently fast. Making an analogy with complex analysis, this corresponds to convergence class and type w.r.t. proximate orders having order larger than 1. It is a surprising fact that these properties depend only on the diagonal entries of H . In 1968 L.de Branges posed the following question as a fundamental problem:

Which Hamiltonians are the structure Hamiltonian of some de Branges space?

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We give a complete and explicit answer.

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1. Introduction

We study the spectrum of the selfadjoint model operator associated with a two-dimensional canonical system

$$y'(t) = zJH(t)y(t), \quad t \in (a, b). \quad (1.1)$$

Here H is the Hamiltonian of the system, $-\infty < a < b \leq \infty$, J is the symplectic matrix $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $z \in \mathbb{C}$ is the eigenvalue parameter. We assume throughout that H satisfies

- ▷ $H \in L^1([a, c], \mathbb{R}^{2 \times 2})$, $c \in (a, b)$, and $\{t \in [a, b) : H(t) = 0\}$ has measure 0,
- ▷ $H(t) \geq 0$, $t \in [a, b)$ a.e. and $\int_a^b \operatorname{tr} H(t) dt = \infty$.

Differential equations of this form originate from Hamiltonian mechanics, and appear frequently in theory and applications. Various kinds of equations can be rewritten to the form (1.1), and several problems of classical analysis can be treated with the help of canonical systems. For example we mention Schrödinger operators [33], Dirac systems [41], or the extrapolation problem of stationary Gaussian processes via Bochner's theorem [22]. Other instances can be found, e.g., in [19,18], [27], [2], or [1].

The direct and inverse spectral theory of the equation (1.1) was developed in [11,7]. Recent references are [34,35].

With a Hamiltonian H a Hilbert space $L^2(H)$ is associated, and in $L^2(H)$ a selfadjoint operator $A_{[H]}$ is given by the differential expression (1.1) and by prescribing the boundary condition $(1, 0)y(a) = 0$ (in one exceptional situation $A_{[H]}$ is a multivalued operator, but this is only a technical difficulty). This operator model behind (1.1) was given its final form in [14,15]. A more accessible reference is [12], and the relation with de Branges' work on Hilbert spaces of entire functions was made explicit in [43,44].

In the present paper we answer the following questions:

Is the spectrum of $A_{[H]}$ discrete?

If it is discrete, what is its asymptotic distribution?

The question about asymptotic distribution is understood as the problem of finding the convergence exponent and the upper density of eigenvalues in terms of the Hamiltonian.

Discreteness of the spectrum. In our first theorem we characterise discreteness of the spectrum of $A_{[H]}$.

1.1 Theorem. Let $H = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}$ be a Hamiltonian on $[a, b]$ and assume that $\int_a^b h_1(s) ds < \infty$. Then the spectrum of $A_{[H]}$ is discrete if and only if

$$\lim_{t \nearrow b} \left(\int_t^b h_1(s) ds \cdot \int_a^t h_2(s) ds \right) = 0. \tag{1.2}$$

1.2 Remark. The assumption that $\int_a^b h_1(s) ds < \infty$ in Theorem 1.1 is made for normalisation and is no loss in generality. First, a necessary condition that $0 \notin \sigma_{\text{ess}}(A_{[H]})$ is that there exists some angle $\phi \in \mathbb{R}$ such that $\int_a^b (\cos \phi, \sin \phi)H(s)(\cos \phi, \sin \phi)^* ds < \infty$. Second, applying rotation isomorphism always allows to reduce to the case that $\phi = 0$. We will give details in Section 5.2. \diamond

Let us remark that Theorem 1.1 yields a new proof of the discreteness criterion for strings given by I.S. Kac and M.G. Krein in [21, Theorema 4,5], of [37, Theorem 1.4], and of [17, Theorem 1] (see Appendix B).

Structure Hamiltonians of de Branges spaces. Recall that a de Branges space $\mathcal{H}(E)$ is a reproducing kernel Hilbert space of entire functions with certain additional properties, whose kernel is generated by a Hermite-Biehler function E . For each de Branges space there exists a unique maximal chain of de Branges subspaces $\mathcal{H}(E_t)$, $t \leq 0$, contained isometrically (on exceptional intervals only contractively) in $\mathcal{H}(E)$. The generating Hermite-Biehler functions E_t satisfy a canonical system on the interval $(-\infty, 0]$ with some Hamiltonian H , and this Hamiltonian is called the structure Hamiltonian of $\mathcal{H}(E)$.

L. de Branges identified in [6, Theorem IV] (see also [7, Theorem 41]) a particular class of Hamiltonians which are structure Hamiltonians of de Branges spaces. Namely those corresponding to functions E of Polya class. A mild generalisation of de Branges’ theorem can be found in [29, Theorem 4.11], and a further class of structure Hamiltonians is identified (in a different language) by the already mentioned work of I.S. Kac and M.G. Krein [21] and its generalisation in [37]. These classes do by far not exhaust the set of all structure Hamiltonians. In 1968, after having finalised his theory of Hilbert spaces of entire functions, de Branges posed the following question as a fundamental problem, cf. [7, p.140]:

Which Hamiltonians H are the structure Hamiltonian of some de Branges space $\mathcal{H}(E)$?

In the following decades there was no significant progress towards a solution of this question. One result was claimed by I.S. Kac in 1995; proofs have never been published.

He states a sufficient condition and a (different) necessary condition for H to be a structure Hamiltonian. Unfortunately, his conditions are difficult to handle.

The connection with Theorem 1.1 is the following: a Hamiltonian is the structure Hamiltonian of some de Branges space $\mathcal{H}(E)$, if and only if the operator $A_{[\tilde{H}]}$ associated with the reversed Hamiltonian

$$\tilde{H}(t) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} H(-t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t \in [0, \infty),$$

has discrete spectrum. This can be seen by a simple “juggling with fundamental solutions”–argument. A proof based on a different argument was published in [20], see also [29, Theorem 2.3].

Hence we obtain from Theorem 1.1 a complete and explicit answer to de Branges’ question.

Summability properties. We turn to discussing the asymptotic distribution of $\sigma(A_{[H]})$. Consider a Hamiltonian H with discrete spectrum. Then its spectrum is a (finite or infinite) sequence of simple eigenvalues without a finite accumulation point. If $\sigma(A_{[H]})$ is finite, any questions about the asymptotic behaviour of the eigenvalues are obsolete. Moreover, under the normalisation that $\int_a^b h_1(s) ds < \infty$, the point 0 is not an eigenvalue of $A_{[H]}$. Hence, we can think of $\sigma(A_{[H]})$ as a sequence $(\lambda_n)_{n=1}^\infty$ of pairwise different real numbers arranged such that

$$0 < |\lambda_1| \leq |\lambda_2| \leq |\lambda_3| \leq \dots \tag{1.3}$$

In our second theorem we characterise summability of the sequence $(\lambda_n^{-1})_{n=1}^\infty$ relative to suitable comparison functions. In particular, this answers the question whether $(\lambda_n^{-1})_{n=1}^\infty \in \ell^p$ when $p > 1$. The only known result in this direction is [25, Theorem 2.4], which settles the case $p = 2$; we reobtain this theorem.

As comparison functions we use regularly varying functions, i.e., measurable functions $g: [0, \infty) \rightarrow (0, \infty)$ such that there exists $\rho_g \in \mathbb{R}$ with

$$\lim_{x \rightarrow \infty} \frac{\psi(kx)}{\psi(x)} = k^{\rho_g}, \quad \forall k > 0.$$

The number ρ_g is called the order (or index) of g . Regularly varying functions form a comparison scale which is finer than the scale of powers r^ρ . The history of working with growth scales other than powers probably starts with the paper [28], where E. Lindelöf compared the growth of an entire function with functions of the form

$$g(r) = r^\rho \cdot (\log r)^{\beta_1} \cdot (\log \log r)^{\beta_2} \cdot \dots \cdot \underbrace{(\log \dots \log r)^{\beta_m}}_{m^{\text{th-iterate}}} \quad \text{for } r \text{ large.}$$

A function of this form is regularly varying with order ρ . In what follows the reader may think of $g(r)$ for simplicity as a concrete function of this form, or simply as a power r^ρ .

1.3 Theorem. Let $H = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}$ be a Hamiltonian on an interval $[a, b]$ such that $\int_a^b h_1(s) ds < \infty$ and that $A_{[H]}$ has discrete spectrum. Moreover, assume that h_1 does not vanish a.e. on any interval (c, b) with $c \in (a, b)$. Let g be a regularly varying function with order $\rho_g > 1$. Then

$$\sum_{n=1}^{\infty} \frac{1}{g(|\lambda_n|)} < \infty \Leftrightarrow \int_a^b \left[g \left(\left(\int_t^b h_1(s) ds \cdot \int_a^t h_2(s) ds \right)^{-\frac{1}{2}} \right) \right]^{-1} \cdot \frac{h_1(t) dt}{\int_t^b h_1(s) ds} < \infty.$$

1.4 Remark. For the same reasons as explained in Remark 1.2, the assumption that $\int_a^b h_1(s) ds < \infty$ is just a normalisation and no loss in generality. Also the assumption that h_1 cannot vanish a.e. on any interval (c, b) is no loss of generality. The reason being that, if h_1 does vanish on an interval of this form, then the Krein-de Branges formula, cf. [23, p.369 (english translation)], [6, Theorem X], says that

$$\lim_{n \rightarrow \infty} \frac{n}{\lambda_n^+} = \lim_{n \rightarrow \infty} \frac{n}{\lambda_n^-} = \frac{1}{\pi} \int_a^b \sqrt{\det H(s)} ds, \tag{1.4}$$

where λ_n^\pm denote the sequences of positive and negative, respectively, eigenvalues arranged according to increasing modulus. In particular, the series $\sum_{n=1}^{\infty} \frac{1}{g(|\lambda_n|)}$ converges whenever $\rho_g > 1$. ◊

Theorem 1.3 yields new proofs for the conditions for square summability given in [25] and in [21, p.139f] in the string case. It also gives a new approach to the results on the convergence exponent of the spectrum of a string given in [13, Theorema 1,2] and in [16] for the case of orders between $\frac{1}{2}$ and 1.

Limit superior properties. In our third theorem, we characterise lim sup-properties of the sequence $(\lambda_n)_{n=1}^\infty$, again relative to regularly varying functions g with $\rho_g > 1$. While the characterisations in Theorems 1.1 and 1.3 are perfectly explicit in terms of H , the conditions occurring in this context are somewhat more complicated. The reason for this is intrinsic, and manifests itself in the necessity to pass to the nonincreasing rearrangement of a certain sequence.

1.5 Theorem. Let $H = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}$ be a Hamiltonian on an interval $[a, b]$ such that $\int_a^b h_1(s) ds < \infty$ and that $A_{[H]}$ has discrete spectrum. Moreover, assume that h_1 does not vanish a.e. on any interval (c, b) with $c \in (a, b)$. Let g be a regularly varying function with order $\rho_g > 1$.

Choose a right inverse χ of the nonincreasing surjection

$$\begin{cases} [a, b] \rightarrow [0, 1] \\ t \mapsto \left(\int_a^b h_1(s) ds \right)^{-1} \left(\int_t^b h_1(s) ds \right) \end{cases}$$

and let $(\omega_n^*)_{n \in \mathbb{N}}$ be the nonincreasing rearrangement of the sequence $(\omega_n)_{n \in \mathbb{N}}$ defined as

$$\omega_n := 2^{-\frac{n}{2}} \left(\int_{\chi(2^{1-n})}^{\chi(2^{-n})} h_2(s) ds \right)^{\frac{1}{2}}, \quad n \in \mathbb{N}.$$

Then

- (i) $\limsup_{n \rightarrow \infty} \frac{n}{\mathfrak{g}(|\lambda_n|)} < \infty \Leftrightarrow \limsup_{n \rightarrow \infty} \frac{n}{\mathfrak{g}((\omega_n^*)^{-1})} < \infty,$
- (ii) $\lim_{n \rightarrow \infty} \frac{n}{\mathfrak{g}(|\lambda_n|)} = 0 \Leftrightarrow \lim_{n \rightarrow \infty} \frac{n}{\mathfrak{g}((\omega_n^*)^{-1})} = 0.$

Remember here Remark 1.4.

Outline of the proofs. The proof of Theorems 1.1, 1.3, and 1.5 proceeds through four stages.

- ① The first stage is to pass from eigenvalue distribution to operator theoretic properties. This is done in a standard way using symmetrically normed operator ideals: discreteness of the spectrum of $A_{[H]}$ is equivalent to $(A_{[H]} - z)^{-1}$ being compact, summability properties of $\sigma(A_{[H]})$ are equivalent to $(A_{[H]} - z)^{-1}$ belonging to Orlicz ideals, and \limsup -properties of $\sigma(A_{[H]})$ are equivalent to $(A_{[H]} - z)^{-1}$ belonging to Lorentz spaces.
- ② Simple considerations (apparently known in the folklore) resumed in Theorem 3.4 show that the eigenvalues of the canonical system with Hamiltonian H are estimated by those of the system with Hamiltonian $\text{diag } H$, where $\text{diag } H$ is obtained from H by replacing its off-diagonal entries by 0. In fact, it holds that $|\lambda_n(A_{[H]})|^{-1} \leq 2|\lambda_n(A_{[\text{diag } H]})|^{-1}$, and therefore $(A_{[\text{diag } H]} - z)^{-1} \in \mathfrak{I}$ implies $(A_{[H]} - z)^{-1} \in \mathfrak{I}$ for any operator ideal $\mathfrak{I} \subseteq \mathfrak{S}^\infty$. The second stage is to prove the probably surprising fact that for a wide class of ideals the converse is also true. If a weak variant of Matsaev’s Theorem on real and imaginary parts of Volterra operators holds in \mathfrak{I} , then membership of resolvents $(A_{[H]} - z)^{-1}$ in \mathfrak{I} is independent of h_3 . This is the reason why the conditions in our theorems do not involve the off-diagonal entry h_3 of the Hamiltonian H .
- ③ In a work of A.B. Aleksandrov, S. Janson, V.V. Peller, and R. Rochberg, membership in Schatten classes of integral operators whose kernel has a particular form is characterised using a dyadic discretisation method. The third stage is to realise that

a minor generalisation of one of their results suffices to prove the mentioned weak Matsaev Theorem in the ideal \mathfrak{S}_∞ of all compact operators. For Orlicz- and Lorentz ideals, it is known that (the full) Matsaev Theorem holds. Thus the Independence Theorem stated in ② will apply to all ideals occurring in ①.

- ④ The final stage is to characterise membership in the mentioned ideals for a diagonal Hamiltonian (meaning that $h_3 = 0$). This again rests on the discretisation method from [3], which yields characterisations of a sequential form (as the one stated in Theorem 1.5) for all ideals occurring in ①. For the cases of \mathfrak{S}_∞ and Orlicz ideals, sequential characterisations can be rewritten to a continuous form (as stated in Theorems 1.1 and 1.3). This is nearly obvious for \mathfrak{S}_∞ , while for Orlicz ideals a little more effort and passing to dual spaces is needed.

The threshold $\rho_{\mathbf{g}} = 1$. It is not difficult to find Hamiltonians $H = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}$ whose spectrum is discrete and satisfies $\lim_{n \rightarrow \infty} \frac{n}{|\lambda_n|^\rho} \in (0, \infty)$ for some $\rho < 1$, and for which $h_1 h_2$ does not vanish a.e., see Example 1.7. Together with Example 1.6, this shows that for every Schatten–von Neumann class \mathfrak{S}_p with $\rho < p \leq 1$, the Independence Theorem mentioned in ② fails. This shows that our method necessarily must break down at (and below) trace class, i.e., growth of speed $\mathbf{g}(r) := r$.

On a less concrete level, growth of order 1 is a threshold because of (at least) four reasons.

- ▷ Orders larger than 1, meaning eigenvalue distribution more dense than integers, can occur only from the behaviour of tails of H at its singular endpoint b . In fact, for $\rho > 1$, the spectrum of $A_{[H]}$ is discrete with convergence exponent ρ if and only if for some $c \in (a, b)$ the spectrum of $A_{[H]_{[c,b]}}$ is discrete with convergence exponent ρ . Contrasting this, orders less than 1 will in general accumulate over the whole interval (a, b) . In fact, it may happen that $\sigma(A_{[H]})$ has convergence exponent 1 while for every $c \in (a, b)$ the spectrum of the tail $H|_{[c,b]}$ has convergence exponent 0.
- ▷ Entire functions of bounded type have very specific properties related to exponential type. In complex analysis, orders larger than 1 are usually considered as more stable than smaller orders.
- ▷ The theory of symmetrically normed operator ideals is significantly more complicated for ideals close to trace class than for ideals containing some Schatten–von Neumann class \mathfrak{S}_p with $p > 1$. When going even below trace class, a lot of the theory breaks down completely.
- ▷ Rewriting asymptotic conditions on the spectral distribution to conditions on membership in Orlicz- and Lorentz ideals is not anymore possible when coming close to trace class.

Let us now give two examples which illustrate our results. They are simple, and given by Hamiltonians related to a string, but, as we hope, still illustrative. At this point we

only state their spectral properties; the proof is given in Section 5.3, where we in fact treat a more general example.

1.6 Example. Given $\alpha > 1$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, we consider the Hamiltonian (to avoid bulky notation, we skip indices $\alpha, \alpha_1, \alpha_2$ at h_2)

$$H_{\alpha; \alpha_1, \alpha_2}(t) := \begin{pmatrix} 1 & 0 \\ 0 & h_2(t) \end{pmatrix}, \quad t \in [0, 1),$$

where

$$h_2(t) := \left(\frac{1}{1-t}\right)^\alpha \left(1 + \log \frac{1}{1-t}\right)^{-\alpha_1} \left(1 + \log^+ \log \frac{1}{1-t}\right)^{-\alpha_2}, \quad t \in [0, 1). \quad (1.5)$$

Since $\alpha > 1$, we have $\int_0^1 h_2(t) dt = \infty$.

If $\alpha > 2$, then 0 belongs to the essential spectrum of $A_{[H_{\alpha; \alpha_1, \alpha_2}]}$, and if $\alpha \in (1, 2)$, then the spectrum is discrete with convergence exponent 1 but $\liminf_{n \rightarrow \infty} \frac{n}{|\lambda_n|} > 0$, in particular, $\sum_{n=1}^\infty \frac{1}{|\lambda_n|} = \infty$.

A behaviour between those extreme situations occurs when $\alpha = 2$. First, the spectrum of $A_{[H_{2; \alpha_1, \alpha_2}]}$ is discrete, if and only if

$$(\alpha_1 > 0) \vee (\alpha_1 = 0, \alpha_2 > 0).$$

For such parameter values, the convergence exponent of the spectrum is

$$\text{conv.exp. of } \sigma(A_{[H_{2; \alpha_1, \alpha_2}]}) = \begin{cases} \infty & , \quad \alpha_1 = 0, \\ \frac{2}{\alpha_1} & , \quad \alpha_1 \in (0, 2), \\ 1 & , \quad \alpha_1 \geq 2. \end{cases} \quad (1.6)$$

For $\alpha_1 \in (0, 2)$, we have a more refined lim sup-property relative to a comparison function which is not a power:

$$0 < \limsup_{n \rightarrow \infty} \frac{n}{|\lambda_n|^{\frac{2}{\alpha_1}} (\log |\lambda_n|)^{-\frac{\alpha_2}{\alpha_1}}} < \infty. \quad \diamond$$

1.7 Example. Given $\alpha_1 > 0$ and $\alpha_2 \in \mathbb{R}$ consider the Hamiltonian (again indices at h_2 are skipped)

$$\mathring{H}_{\alpha_1, \alpha_2} := \begin{pmatrix} 1 & -\sqrt{h_2(t)} \\ -\sqrt{h_2(t)} & h_2(t) \end{pmatrix}, \quad t \in [0, 1),$$

where h_2 is as in (1.5) with $\alpha = 2$. Then the spectrum of $A_{[\mathring{H}_{\alpha_1, \alpha_2}]}$ is discrete, and its convergence exponent is

$$\text{conv.exp. of } \sigma(A_{[\dot{H}_{\alpha_1, \alpha_2}]}) = \begin{cases} \frac{2}{\alpha_1} & , \quad \alpha_1 \in (0, 4), \\ \frac{1}{2} & , \quad \alpha_1 \geq 4. \end{cases} \tag{1.7}$$

The diagonalisation of $\dot{H}_{\alpha_1, \alpha_2}$, i.e., the Hamiltonian obtained by skipping its off-diagonal entries, is $H_{2; \alpha_1, \alpha_2}$. Comparing the convergence exponents computed in (1.6) and (1.7), illustrates validity of the Independence Theorem from ② as long as the convergence exponent is not less than 1, and its failure for other values. \diamond

Organisation of the manuscript. Section 2 is of preliminary character. We recall some facts about operator ideals which are crucial for the present investigations. In particular, we recall a theorem of G.I. Russu characterising a class of ideals for which Matsaev’s Theorem about real and imaginary parts of Volterra operators holds. A standard reference about symmetrically normed ideals is [10]; another classical reference is [42]. A standard reference about Volterra operators is [11].

Sections 3 and 4 deal with the general operator theoretic aspect. We prove the central Independence Theorem mentioned in ②, and the AJPR-type Theorem mentioned in ③ and ④. The latter is only a minor generalisation of [3], and is established by just the same method. For the convenience of the reader, we give a self-contained proof.

Connecting the general theory with spectral asymptotics is done in Section 5. There we prove Theorems 1.1 and 1.5, a slightly more general variant of Theorem 1.3 (Theorem 5.6), and a characterisation of bounded invertibility (Theorem 5.2).

In Section 6 we round off the presentation by discussing the normalisation condition $\int_a^b h_1(s) ds < \infty$, giving details for a somewhat more general variant of Examples 1.6 and 1.7, and showing that the class of comparison functions introduced to measure the growth of eigenvalues is a natural one.

The paper closes with two appendices. In Appendix A we provide detailed proof for some technical facts used in the text, and in Appendix B we make the connection of our present work with the results of I.S. Kac.

2. Some facts about operator ideals

This section is of preliminary nature. We collect some basic definitions and facts about operator ideals which are essential for the present investigation.

Let \mathcal{H} be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators on \mathcal{H} . For an operator $T \in \mathcal{B}(\mathcal{H})$ we denote by $a_n(T)$ the n -th approximation number of T , i.e.,

$$a_n(T) := \inf \{ \|T - A\| : A \in \mathcal{B}(\mathcal{H}), \dim \text{ran } A < n \}, \quad n \in \mathbb{N}.$$

The Calkin correspondence [5] is the map assigning to each $T \in \mathcal{B}(\mathcal{H})$ the sequence $(a_n(T))_{n=1}^\infty$ of its approximation numbers.

An operator ideal \mathfrak{J} in \mathcal{H} is a two-sided ideal of the algebra $\mathcal{B}(\mathcal{H})$. Every nonzero operator ideal \mathfrak{J} contains the ideal of all finite rank operators. Provided \mathcal{H} is separable,

every operator ideal $\mathfrak{I} \neq \mathcal{B}(\mathcal{H})$ is contained in the ideal \mathfrak{S}_∞ of all compact operators. Moreover, every operator ideal contains with an operator T also its adjoint T^* .

Via the Calkin correspondence, operator ideals can be identified with certain sequence spaces.

2.1 Remark. A linear subspace \mathcal{S} of ℓ^∞ ($\subseteq \mathbb{R}^{\mathbb{N}}$) is called solid, if

$$(\alpha_n)_{n=1}^\infty \in \mathcal{S} \wedge |\beta_n| \leq |\alpha_n|, n \in \mathbb{N} \Rightarrow (\beta_n)_{n=1}^\infty \in \mathcal{S},$$

and it is called symmetric, if

$$(\alpha_n)_{n=1}^\infty \in \mathcal{S}, \sigma \text{ permutation of } \mathbb{N} \Rightarrow (\alpha_{\sigma(n)})_{n=1}^\infty \in \mathcal{S}.$$

It is shown in [8, Theorem 1] that for every separable Hilbert space \mathcal{H} the Calkin correspondence induces a bijection Seq of the set of all operator ideals of \mathcal{H} onto the set of all solid symmetric sequence spaces. The action of this bijection and its inverse Idl is as follows:

- ▷ If \mathcal{S} is a solid symmetric subspace of ℓ^∞ , then $\text{Idl}(\mathcal{S})$ is the operator ideal $\{T \in \mathcal{B}(\mathcal{H}) : (a_n(T))_{n=1}^\infty \in \mathcal{S}\}$.
- ▷ If \mathfrak{I} is an operator ideal, then $\text{Seq}(\mathfrak{I})$ is the linear subspace of ℓ^∞ generated by the convex cone $\{(a_n(T))_{n=1}^\infty : T \in \mathfrak{I}\}$.

In other words, we have for all $T \in \mathcal{B}(\mathcal{H})$

$$T \in \mathfrak{I} \Leftrightarrow (a_n(T))_{n=1}^\infty \in \text{Seq}(\mathfrak{I}). \quad \diamond$$

For example, the ideal \mathfrak{S}_∞ of all compact operators corresponds to c_0 , the trivial ideal $\mathcal{B}(\mathcal{H})$ to ℓ^∞ , and the Schatten–von Neumann classes \mathfrak{S}_p to ℓ^p .

Taking the viewpoint of sequence spaces is natural in (at least) two respects.

- ▷ It allows to compare ideals in $\mathcal{B}(\mathcal{H})$ for different base spaces \mathcal{H} . A solid symmetric sequence space \mathcal{S} invokes the family of “same-sized” operator ideals

$$\{T \in \mathcal{B}(\mathcal{H}) : (a_n(T))_{n=1}^\infty \in \mathcal{S}\}, \quad \mathcal{H} \text{ Hilbert space.}$$

- ▷ Virtually all examples of operator ideals \mathfrak{I} which “appear in nature” are defined by a specifying their sequence space $\text{Seq}(\mathfrak{I})$.

Let \mathcal{H} be a separable Hilbert space. A symmetrically normed ideal \mathfrak{I} in \mathcal{H} is a proper (meaning $\mathfrak{I} \neq \{0\}, \mathcal{B}(\mathcal{H})$) operator ideal which is endowed with a norm $\|\cdot\|_{\mathfrak{I}}$, such that

- (i) $(\mathfrak{I}, \|\cdot\|_{\mathfrak{I}})$ is complete,

- (ii) $\|ATB\|_{\mathfrak{J}} \leq \|A\| \cdot \|T\|_{\mathfrak{J}} \cdot \|B\|, \quad T \in \mathfrak{J}, \quad A, B \in \mathcal{B}(\mathcal{H}),$
- (iii) $\|T\|_{\mathfrak{J}} = \|T\|$ for T with $\dim \operatorname{ran} T = 1$.

Basic examples of symmetrically normed ideals are the Schatten–von Neumann classes $\mathfrak{S}_p, 1 \leq p \leq \infty$.

While the relation between operator ideals and sequence spaces goes back to the early 1940’s, and also the theory of symmetrically normed ideals is a classical topic, the appropriate correspondent of symmetrically normed ideals on the sequence space side has been identified only recently in [24].

2.2 Remark. A symmetric Banach sequence space is a solid and symmetric linear subspace \mathcal{S} of c_0 which is endowed with a norm $\|\cdot\|_{\mathcal{S}}$ such that

- (i) $(\mathcal{S}, \|\cdot\|_{\mathcal{S}})$ is complete,
- (ii) it holds that

$$\begin{aligned}
 (\alpha_n)_{n=1}^{\infty} \in \mathcal{S} \wedge |\beta_n| \leq |\alpha_n|, n \in \mathbb{N} &\Rightarrow \|(\beta_n)_{n=1}^{\infty}\|_{\mathcal{S}} \leq \|(\alpha_n)_{n=1}^{\infty}\|_{\mathcal{S}}, \\
 (\alpha_n)_{n=1}^{\infty} \in \mathcal{S}, \sigma \text{ permutation of } \mathbb{N} &\Rightarrow \|(\alpha_{\sigma(n)})_{n=1}^{\infty}\|_{\mathcal{S}} = \|(\alpha_n)_{n=1}^{\infty}\|_{\mathcal{S}},
 \end{aligned}$$

- (iii) $\|(1, 0, 0, \dots)\|_{\mathcal{S}} = 1$.

The mutually inverse bijections Seq and Idl exhibited above induce mutually inverse bijections between the set of all symmetrically normed ideals and the set of all symmetric Banach sequence spaces. The action of these maps is as follows:

- ▷ If $(\mathcal{S}, \|\cdot\|_{\mathcal{S}})$ is a symmetric Banach sequence function space, then $\operatorname{Idl}(\mathcal{S})$ endowed with $\|T\|_{\operatorname{Idl}(\mathcal{S})} := \|(a_n(T))_{n=1}^{\infty}\|_{\mathcal{S}}$ is a symmetrically normed ideal.
- ▷ Let \mathcal{H} be a separable Hilbert space, and choose an orthonormal basis $\{\phi_n: n \in \mathbb{N}\}$ of \mathcal{H} . Let $(\mathfrak{J}, \|\cdot\|_{\mathfrak{J}})$ be a symmetrically normed ideal in \mathcal{H} . Then $\operatorname{Seq}(\mathfrak{J})$ endowed with $\|(a_n)_{n=1}^{\infty}\|_{\operatorname{Seq}(\mathfrak{J})} := \|\sum_{n=1}^{\infty} a_n(\cdot, \phi_n)\phi_n\|_{\mathfrak{J}}$ is a symmetric Banach sequence space. \diamond

Convention: From now on we do not anymore distinguish explicitly between sequence spaces and operator ideals, and tacitly apply the Calkin correspondence if needed. \diamond

Every nonzero operator ideal contains the space $c_{00} (\subseteq \mathbb{R}^{\mathbb{N}})$ of all finitely supported sequences. For a symmetrically normed ideal \mathfrak{J} , we denote by \mathfrak{J}° the closure of c_{00} in \mathfrak{J} . The space \mathfrak{J}° is again a symmetrically normed ideal.

Many symmetrically normed ideals can be generated using so-called symmetric norming functions.

2.3 Remark. A symmetric norming function is a function $\Phi: c_{00} \rightarrow [0, \infty)$, such that

- (i) Φ is a norm,

- (ii) $(\alpha_n)_{n=1}^\infty \in c_{00}, \sigma$ permutation of $\mathbb{N} \Rightarrow \Phi((|\alpha_{\sigma(n)}|)_{n=1}^\infty) = \Phi((\alpha_n)_{n=1}^\infty)$,
- (iii) $\Phi((1, 0, 0, \dots)) = 1$.

A symmetric norming function Φ has a natural extension, again denoted as Φ , to a larger subspace. Namely, by $(\mathbb{1}_N$ denotes the indicator function of the set $\{1, \dots, N\}$)

$$c_\Phi := \{(\alpha_n)_{n=1}^\infty \in c_0 : \sup_{N \in \mathbb{N}} \Phi((\mathbb{1}_N(n)\alpha_n)_{n=1}^\infty) < \infty\},$$

$$\Phi((\alpha_n)_{n=1}^\infty) := \lim_{N \rightarrow \infty} \Phi((\mathbb{1}_N(n)\alpha_n)_{n=1}^\infty) \text{ for } (\alpha_n)_{n=1}^\infty \in c_\Phi.$$

Then (c_Φ, Φ) is a symmetrically normed ideal.

Conversely, if $(\mathfrak{J}, \|\cdot\|_{\mathfrak{J}})$ is a symmetrically normed ideal, then $\Phi := \|\cdot\|_{\mathfrak{J}}|_{c_{00}}$ is a symmetric norming function. We have $\mathfrak{J}^\circ \subseteq c_\Phi$, and $\|(\alpha_n)_{n=1}^\infty\|_{\mathfrak{J}} = \Phi((\alpha_n)_{n=1}^\infty)$ for all $(\alpha_n)_{n=1}^\infty \in \mathfrak{J}^\circ$. However, it may happen that $\|(\alpha_n)_{n=1}^\infty\|_{\mathfrak{J}} \neq \Phi((\alpha_n)_{n=1}^\infty)$ for some $(\alpha_n)_{n=1}^\infty \in (\mathfrak{J} \setminus \mathfrak{J}^\circ) \cap c_\Phi$. If $\mathfrak{J} \subseteq c_\Phi$ and $\|\cdot\|_{\mathfrak{J}} = \Phi$ on all of \mathfrak{J} , we say that $\|\cdot\|_{\mathfrak{J}}$ is induced by a symmetric norming function. \diamond

For an operator ideal \mathfrak{J} , we denote by \mathfrak{J}^+ the convex cone of all nondecreasing nonnegative elements of \mathfrak{J} . A symmetrically normed ideal \mathfrak{J} is called fully symmetric, if for $(\alpha_n)_{n=1}^\infty \in \mathfrak{J}^+$ and $(\beta_n)_{n=1}^\infty \in (\ell^\infty)^+$ with

$$\forall n \in \mathbb{N}. \sum_{k=1}^n \beta_k \leq \sum_{k=1}^n \alpha_k,$$

it holds that

$$(\beta_n)_{n=1}^\infty \in \mathfrak{J}, \quad \|(\beta_n)_{n=1}^\infty\|_{\mathfrak{J}} \leq \|(\alpha_n)_{n=1}^\infty\|_{\mathfrak{J}}.$$

If Φ is a symmetric norming function, then both symmetrically normed ideals c_Φ and c_Φ° are fully symmetric.

Next, we recall an operator theoretic notion.

2.4 Definition. Let \mathfrak{J} be a symmetrically normed ideal which is properly contained in \mathfrak{S}_∞ . We say \mathfrak{J} has the Matsaev property, if the following statement is true.

- \triangleright Let \mathcal{H} be a Hilbert space, and let T be a Volterra operator in \mathcal{H} . Then $\text{Re}T \in \mathfrak{J}$ implies $T \in \mathfrak{J}$. \diamond

A result of G.I. Russu stated in [39, Theorem 1]³ gives a characterisation of the Matsaev property for a certain class of ideals. In Theorem 2.5 below we formulate the part of

³ Proofs are given in [40]; a translation to english is not available.

Russu’s Theorem used in the sequel. To this end, we need one more notation. Let \mathfrak{J} be an operator ideal, and $\mathcal{T}: \mathfrak{J}^+ \rightarrow \mathfrak{J}^+$ a convex and positively homogeneous map. Then we denote

$$\|\mathcal{T}\| := \sup \{ \|\mathcal{T}(\alpha_n)_{n=1}^\infty\|_{\mathfrak{J}} : (\alpha_n)_{n=1}^\infty \in \mathfrak{J}^+, \|(\alpha_n)_{n=1}^\infty\|_{\mathfrak{J}} = 1 \}.$$

2.5 Theorem ([39,40]). *Let \mathfrak{J} a symmetrically normed ideal whose norm is induced by a symmetric norming function. For each $n \in \mathbb{N}$ define maps \mathcal{T}_n and $\mathcal{T}_{1/n}$ acting in the cone \mathfrak{J}^+ as*

$$\mathcal{T}_n(\alpha_k)_{k=1}^\infty := (\underbrace{\alpha_1, \dots, \alpha_1}_{n \text{ times}}, \underbrace{\alpha_2, \dots, \alpha_2}_{n \text{ times}}, \dots), \tag{2.1}$$

$$\mathcal{T}_{1/n}(\alpha_k)_{k=1}^\infty := \left(\frac{1}{n} \sum_{k=1}^n \alpha_k, \frac{1}{n} \sum_{k=1}^n \alpha_{n+k}, \frac{1}{n} \sum_{k=1}^n \alpha_{2n+k}, \dots \right). \tag{2.2}$$

Then \mathfrak{J} has the Matsaev property, if and only if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \|\mathcal{T}_n\| = 0, \quad \lim_{n \rightarrow \infty} \|\mathcal{T}_{1/n}\| = 0. \tag{2.3}$$

Let us show that for a separable ideal the second condition in (2.3) automatically holds.

2.6 Lemma. *Let \mathfrak{J} be a separable symmetrically normed ideal. Then $\lim_{n \rightarrow \infty} \|\mathcal{T}_{\frac{1}{n}}\| = 0$.*

Proof. Set $\Phi := \|\cdot\|_{\mathfrak{J}}$, then $\mathfrak{J} = c_\Phi^0$. For $(\alpha_n)_{n=1}^\infty \in \mathfrak{J}^+$ we have $\lim_{N \rightarrow \infty} \Phi((\alpha_{N+n})_{n=1}^\infty) = 0$. Now let $(\alpha_n)_{n=1}^\infty \in \mathfrak{J}^+$ with $\|(\alpha_n)_{n=1}^\infty\|_{\mathfrak{J}} \leq 1$ be given. Then, for every $N \in \mathbb{N}$,

$$\Phi((\alpha_{N+n})_{n=1}^\infty) \leq \Phi((\alpha_n)_{n=1}^\infty) \leq 1.$$

Let $\varepsilon > 0$ and choose $N_0 \in \mathbb{N}$ such that $\Phi((\alpha_{N+n})_{n=1}^\infty) \leq \varepsilon$, $N \geq N_0$. For $n \in \mathbb{N}$ and $j \in \{1, \dots, n\}$ set

$$(\alpha_k^{(n,j)})_{k=1}^\infty := (\alpha_{j+n(k-1)})_{k=1}^\infty.$$

Then $\mathcal{T}_{\frac{1}{n}}((\alpha_n)_{n=1}^\infty) = \frac{1}{n} \sum_{j=1}^n (\alpha_k^{(n,j)})_{k=1}^\infty$. We have

$$\|(\alpha_k^{(n,j)})_{k=1}^\infty\|_{\mathfrak{J}} \leq \Phi((\alpha_j, \alpha_{j+1}, \alpha_{j+2}, \dots)) \leq \begin{cases} 1 & , \quad j \leq N_0, \\ \varepsilon & , \quad j > N_0. \end{cases}$$

Now choose $N_1 \in \mathbb{N}$ such that $\frac{1}{N_1} \leq \varepsilon$. Then, for $n \geq N_0 N_1$, we have

$$\|J_{\frac{1}{n}}((\alpha_k)_{k=1}^\infty)\|_{\mathfrak{J}} \leq \frac{1}{n} \sum_{j=1}^{N_0} \|(\alpha_k^{(n,j)})_{k=1}^\infty\|_{\mathfrak{J}} + \frac{1}{n} \sum_{j=N_0+1}^n \|(\alpha_k^{(n,j)})_{k=1}^\infty\|_{\mathfrak{J}} \leq 2\varepsilon. \quad \square$$

3. The independence theorem

To start with, let us recall some facts about the model operator $A_{[H]}$ associated with a Hamiltonian H . The first lemma is folklore; one possible reference is [25] where it appears implicitly. Recall that the model space $L^2(H)$ associated with a Hamiltonian H is a closed subspace of the L^2 -space of 2-vector valued functions on (a, b) with respect to the matrix measure $H(t)dt$. Namely, each indivisible interval of H contributes only a one-dimensional space.⁴

3.1 Lemma. *Under the assumption that $\int_a^b h_1(s) ds < \infty$, the operator $A_{[H]}$ is injective and its inverse $B_{[H]} := A_{[H]}^{-1}$ acts as*

$$(B_{[H]}f)(t) = -\lim_{c \nearrow b} \int_a^c \begin{pmatrix} 0 & \mathbb{1}_{s < t}(t, s) \\ \mathbb{1}_{s > t}(t, s) & 0 \end{pmatrix} H(s)f(s) ds, \tag{3.1}$$

on the domain

$$\text{dom } B_{[H]} = \left\{ f \in L^2(H) : \begin{array}{l} \lim_{c \nearrow b} (0, 1) \int_a^c JH(s)f(s) ds \text{ exists,} \\ \text{r.h.s. of (3.1) belongs to } L^2(H) \end{array} \right\}. \tag{3.2}$$

Denote by $L^2(Idx)$ the L^2 -space of 2-vector valued functions on (a, b) with respect to the matrix measure $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} dt$. The function

$$\Phi : f(t) \mapsto H(t)^{\frac{1}{2}} f(t)$$

maps the model space $L^2(H)$ isometrically onto some closed subspace of $L^2(Idx)$.

Let $C_{[H]}$ be the (closed, but possibly unbounded) integral operator on $L^2(Idx)$ with kernel

$$C_{[H]} : -H(t)^{\frac{1}{2}} \begin{pmatrix} 0 & \mathbb{1}_{s < t}(t, s) \\ \mathbb{1}_{s > t}(t, s) & 0 \end{pmatrix} H(s)^{\frac{1}{2}} \tag{3.3}$$

and the natural maximal domain.

The next lemma says that the operator $B_{[H]}$ can be transformed into $C_{[H]}$, and was shown in [25, Proof of Lemma 2.2].

3.2 Lemma. *Assume that $\int_a^b h_1(s) ds < \infty$, and denote by P the orthogonal projection of $L^2(Idx)$ onto $\text{ran } \Phi$. Then*

$$B_{[H]} = \Phi^{-1}PC_{[H]}\Phi \quad \text{and} \quad C_{[H]} = \Phi B_{[H]}\Phi^{-1}P.$$

⁴ In [12] and [25], this space is called $L^2_s(H)$. However, for simplicity of notation, we prefer to use $L^2(H)$.

As a consequence of Lemma 3.2, the operators $B_{[H]}$ and $C_{[H]}$ are together bounded or unbounded, and if they are bounded their approximation numbers coincide. Thus, for every operator ideal \mathfrak{J} , we have

$$B_{[H]} \in \mathfrak{J} \Leftrightarrow C_{[H]} \in \mathfrak{J}.$$

Next we introduce integral operators whose kernel has a very special form. Let $-\infty < a < b \leq \infty$, and $\kappa, \varphi: (a, b) \rightarrow \mathbb{C}$ be measurable functions such that $\kappa \in L^2(a, b)$ and $\mathbb{1}_{(a,c)}\varphi \in L^2(a, b)$ for every $c \in (a, b)$. Then we consider the (closed, but possibly unbounded) integral operator T in $L^2(a, b)$ with kernel

$$T : \mathbb{1}_{t < s}(t, s)\varphi(t)\overline{\kappa(s)} \tag{3.4}$$

Explicitly, this is the operator acting as

$$(Tf)(t) := \varphi(t) \int_t^b f(s)\overline{\kappa(s)} ds, \quad t \in (a, b),$$

on its natural maximal domain

$$\text{dom } T := \left\{ f \in L^2(a, b) : \left(t \mapsto \varphi(t) \int_t^b f(s)\overline{\kappa(s)} ds \right) \in L^2(a, b) \right\}.$$

Note that $\text{dom } T$ always contains the dense linear subspace

$$L_{00} := \{ f \in L^2(a, b) : \text{supp } f < b \}.$$

The adjoint of T is the integral operator with kernel

$$T^* : \mathbb{1}_{s < t}(t, s)\kappa(t)\overline{\varphi(s)}$$

explicitly,

$$(T^*f)(t) := \kappa(t) \int_a^t f(s)\overline{\varphi(s)} ds, \quad t \in (a, b).$$

Since $\kappa \in L^2(a, b)$, we again have $L_{00} \subseteq \text{dom } T^*$. The operator $\frac{1}{2}(T + T^*)$ is densely defined and symmetric; we denote its closure as $\text{Re } T$.

We are going to compare a Hamiltonian H with its diagonal part.

3.3 Definition. Let H be a Hamiltonian and write $H = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}$. Then we denote the corresponding diagonal Hamiltonian as

$$\text{diag } H := \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}. \quad \diamond$$

Concerning operator theoretic properties, the diagonalised Hamiltonian always dominates the original one.

3.4 Theorem (Diagonal Dominance). *Let H be a Hamiltonian defined on an interval $[a, b]$, and assume that $\int_a^b h_1(s) ds < \infty$. If $C_{[\text{diag } H]}$ is bounded, then also $C_{[H]}$ is bounded and there exists an operator λ in $L^2(\text{Idt})$ with $\|\lambda\| \leq \sqrt{2}$ such that*

$$C_{[H]} = \lambda \circ C_{[\text{diag } H]} \circ \lambda^*.$$

In particular, $a_n(C_{[H]}) \leq 2a_n(C_{[\text{diag } H]})$, $n \in \mathbb{N}$, and $C_{[\text{diag } H]} \in \mathfrak{J}$ implies $C_{[H]} \in \mathfrak{J}$ for every operator ideal \mathfrak{J} .

Proof. For any nonnegative 2×2 -matrix G it holds that

$$2 \text{diag } G - G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} G \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \geq 0.$$

Hence we find a matrix V with $\|V\| \leq \sqrt{2}$, such that

$$G^{\frac{1}{2}} = V(\text{diag } G)^{\frac{1}{2}} = (\text{diag } G)^{\frac{1}{2}} V^*.$$

If $\det(\text{diag } G) \neq 0$, then clearly $V = G^{\frac{1}{2}}(\text{diag } G)^{-\frac{1}{2}}$. Otherwise, we can choose $V = I$.

It follows that there exists a measurable matrix function $V(t)$, $t \in (a, b)$, with $\|V(t)\| \leq \sqrt{2}$, $t \in (a, b)$ a.e., and

$$H(t)^{\frac{1}{2}} = V(t)[\text{diag } H(t)]^{\frac{1}{2}} = [\text{diag } H(t)]^{\frac{1}{2}} V(t)^*, \quad t \in (a, b) \text{ a.e.}$$

On substituting this into the definition (3.3) of $C_{[H]}$, we obtain the required assertion where λ is the multiplication operator with $V(t)$. \square

Let us remark that Theorem 3.4 implies the first inequality [36, Theorem 1.1] (personal communication by C. Remling).

The central result in this section is that under a certain assumption on the ideal \mathfrak{J} also the converse holds, i.e., membership of $B_{[H]}$ in the ideal does not depend on the off-diagonal entry of H .

3.5 Definition. Let \mathfrak{J} be an operator ideal. We say that the weak Matsaev Theorem holds in \mathfrak{J} , if the following statement is true.

- ▷ Let $-\infty \leq a < b \leq \infty$, let $\kappa, \varphi: (a, b) \rightarrow \mathbb{C}$ be measurable functions such that $\kappa \in L^2(a, b)$ and $\mathbb{1}_{(a,c)}\varphi \in L^2(a, b)$ for every $c \in (a, b)$, and let T be the integral operator with kernel (3.4). Then $\text{Re } T \in \mathfrak{J}$ implies $T \in \mathfrak{J}$. \diamond

This property is not very strong. Intuitively speaking, it fails only close to trace class, close (but away from) the ideal of all compact operators, and for some ideals which hardly ever appear in nature.

3.6 Theorem (Independence Theorem). *Let H be a Hamiltonian defined on an interval $[a, b]$, and assume that $\int_a^b h_1(s) ds < \infty$. Let \mathfrak{I} be an operator ideal, and assume that the weak Matsaev Theorem holds in \mathfrak{I} . If $B_{[H]} \in \mathfrak{I}$, then $B_{[\text{diag } H]} \in \mathfrak{I}$.*

The following simple computation is a key step to the proof of Theorem 3.6.

3.7 Lemma. *Let H be a Hamiltonian on $[a, b]$ with $\int_a^b h_1(s) ds < \infty$. Denote*

$$H(t)^{\frac{1}{2}} = \begin{pmatrix} v_1(t) & v_3(t) \\ v_3(t) & v_2(t) \end{pmatrix}, \quad t \in [a, b],$$

and let T_{ij} , $(i, j) \in \{2, 3\} \times \{1, 3\}$, be the integral operators in $L^2(a, b)$ with kernel

$$T_{ij} : \quad \mathbf{1}_{t < s}(t, s)v_i(t)v_j(s)$$

Then

$$C_{[H]}f = - \begin{pmatrix} T_{31} + T_{31}^* & T_{21}^* + T_{33} \\ T_{21} + T_{33}^* & T_{23} + T_{23}^* \end{pmatrix} f, \quad f \in L_{00} \times L_{00}.$$

Proof. Multiplying out the kernel (3.3) of the integral operator $C_{[H]}$ gives

$$\begin{pmatrix} \mathbf{1}_{t < s}(t, s)v_3(t)v_1(s) & \mathbf{1}_{t < s}(t, s)v_3(t)v_3(s) \\ \quad + \mathbf{1}_{t > s}(t, s)v_1(t)v_3(s) & \quad + \mathbf{1}_{t > s}(t, s)v_1(t)v_2(s) \\ \mathbf{1}_{t < s}(t, s)v_2(t)v_1(s) & \mathbf{1}_{t < s}(t, s)v_2(t)v_3(s) \\ \quad + \mathbf{1}_{t > s}(t, s)v_3(t)v_3(s) & \quad + \mathbf{1}_{t > s}(t, s)v_3(t)v_2(s) \end{pmatrix}$$

The adjoint T_{ij}^* is the integral operator with kernel

$$T_{ij}^* : \quad \mathbf{1}_{t > s}(t, s)v_j(t)v_i(s)$$

and the assertion follows. \square

3.8 Corollary. *Let $H = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$ be a diagonal Hamiltonian, and let S_{21} be the integral operator in $L^2(a, b)$ with kernel*

$$S_{21} : \quad \mathbf{1}_{t < s}(t, s)\sqrt{h_2(t)}\sqrt{h_1(s)}$$

Then for every operator ideal \mathfrak{I} we have

$$B_{[H]} \in \mathfrak{I} \Leftrightarrow S_{21} \in \mathfrak{I}.$$

Proof. Lemma 3.7 gives

$$C_{[H]}f = - \begin{pmatrix} 0 & S_{21}^* \\ S_{21} & 0 \end{pmatrix} f, \quad f \in L_{00} \times L_{00}. \quad \square \tag{3.5}$$

For a bounded function ψ , we denote by M_ψ the multiplication operator with ψ on $L^2(a, b)$:

$$(M_\psi f)(t) := \psi(t)f(t), \quad \|M_\psi\| = \|\psi\|_\infty.$$

Proof of Theorem 3.6. It holds that $h_1 = v_1^2 + v_3^2$ and $h_2 = v_2^2 + v_3^2$, and hence

$$\sqrt{h_1} \leq v_1 + |v_3|, \quad \sqrt{h_2} \leq v_2 + |v_3|.$$

Thus the functions (quotients are understood as 0 if their denominator vanishes)

$$\psi_1 := \frac{\sqrt{h_1}}{v_1 + |v_3|}, \quad \psi_2 := \frac{\sqrt{h_2}}{v_2 + |v_3|},$$

are bounded. Set $\vartheta_j := \psi_j \cdot \operatorname{sgn} v_3$, then for all $f \in L_{00}$

$$S_{21}f = M_{\vartheta_2}T_{33}M_{\vartheta_1}f + M_{\vartheta_2}T_{31}M_{\psi_1}f + M_{\vartheta_2}T_{23}M_{\vartheta_1}f + M_{\vartheta_2}T_{21}M_{\psi_1}f.$$

We see that $S_{21} \in \mathfrak{J}$ if $T_{21}, T_{23}, T_{31}, T_{33} \in \mathfrak{J}$.

Assume that the weak Matsaev Theorem holds for \mathfrak{J} . If $B_{[H]} \in \mathfrak{J}$, then

$$\operatorname{Re} T_{31}, \operatorname{Re} T_{23}, \operatorname{Re} T_{21} + \operatorname{Re} T_{33} \in \mathfrak{J}.$$

The operator $\operatorname{Re} T_{33}$ is one-dimensional, hence certainly belongs to \mathfrak{J} , and it follows that also $\operatorname{Re} T_{21} \in \mathfrak{J}$. We conclude that T_{23}, T_{31}, T_{21} , and T_{33} , all belong to \mathfrak{J} . From this $S_{21} \in \mathfrak{J}$, and in turn $B_{[\operatorname{diag} H]} \in \mathfrak{J}$. \square

4. Invoking the AJPR-method

The Independence Theorem together with Corollary 3.8 leaves us with two tasks:

- ▷ Determine for which operator ideals the weak Matsaev Theorem holds.
- ▷ Characterise membership in an operator ideal for integral operators of the form (3.4).

In Theorem 4.1 below we complete these tasks to an extent sufficient for our present purposes. This rests on a method developed in a paper by A.B. Aleksandrov, S. Janson, V.V. Peller, and R. Rochberg. In order to formulate the result, we still need to introduce some notation.

Let $-\infty \leq a < b \leq \infty$, let $\kappa, \varphi: (a, b) \rightarrow \mathbb{C}$ be measurable functions such that $\kappa \in L^2(a, b)$ and $\mathbf{1}_{(a,c)}\varphi \in L^2(a, b)$ for every $c \in (a, b)$. Then the function $t \mapsto \|\mathbf{1}_{(t,b)}\kappa\|^2$ is a nonincreasing surjection of $[a, b]$ onto $[0, \|\kappa\|^2]$. Hence, we can choose an increasing sequence $c_0 := a < c_1 < c_2 < \dots < b$ such that $\|\mathbf{1}_{(c_n,b)}\kappa\|^2 = 2^{-n}\|\kappa\|^2$, $n \in \mathbb{N}$. Note that this requirement is equivalent to

$$\|\mathbf{1}_{(c_{n-1},c_n)}\kappa\|^2 = \left(\frac{1}{2}\right)^n \|\kappa\|^2, \quad n \in \mathbb{N}. \tag{4.1}$$

Having chosen c_n , we denote

$$J_n := (c_{n-1}, c_n), \quad \omega_n := \|\mathbf{1}_{J_n}\kappa\| \cdot \|\mathbf{1}_{J_n}\varphi\|, \quad n \in \mathbb{N}. \tag{4.2}$$

Explicitly, by (4.1),

$$\omega_n = \|\kappa\| \cdot 2^{-\frac{n}{2}} \left(\int_{c_{n-1}}^{c_n} |\varphi(s)|^2 ds \right)^{\frac{1}{2}}, \quad n \in \mathbb{N}.$$

The following theorem is a (minor) extension of [3, Theorems 3.1,3.2,3.3].

4.1 Theorem (AJPR-type Theorem). *Let \mathfrak{J} be an operator ideal which is either ℓ^∞ or c_0 , or a symmetrically normed ideal which is fully symmetric and has the Matsaev property. Moreover, let $-\infty \leq a < b \leq \infty$, let $\kappa, \varphi: (a, b) \rightarrow \mathbb{C}$ be measurable functions with $\kappa \in L^2(a, b)$ and $\mathbf{1}_{(a,c)}\varphi \in L^2(a, b)$, $c \in (a, b)$, and consider the integral operator T on $L^2(a, b)$ with kernel given by (3.4).*

Then the following statements are equivalent.

- (i) $T \in \mathfrak{J}$;
- (ii) $\text{Re } T \in \mathfrak{J}$;
- (iii) $(\omega_n)_{n=1}^\infty \in \mathfrak{J}$, where ω_n are as in (4.2).

4.2 Remark. The implication “(i) \Rightarrow (ii)” holds of course for every operator ideal. Our proof will show that “(ii) \Rightarrow (iii)” holds for every fully symmetric operator ideal, and “(iii) \Rightarrow (i)” holds for every symmetrically normed ideal with the Matsaev property. \diamond

The argument needed to establish Theorem 4.1 is nearly verbatim the same as in [3]. For the convenience of the reader we provide a self-contained proof.

Proof of “ $\text{Re } T \in \mathfrak{J} \Rightarrow (\omega_n)_{n=1}^\infty \in \mathfrak{J}$ ”. Let \mathfrak{J} be ℓ^∞ , c_0 , or a fully symmetric operator ideal, and assume that $\text{Re } T \in \mathfrak{J}$. Moreover, denote by P_n the orthogonal projection $P_n f := \mathbf{1}_{J_n} f$ of $L^2(a, b)$ onto its subspace $L^2(J_n)$.

Since \mathfrak{J} is of one of the stated forms, we obtain that $\sum_{n=1}^\infty P_n(\operatorname{Re}T)P_{n+1} \in \mathfrak{J}$. For $\mathfrak{J} = \ell^\infty$ or $\mathfrak{J} = c_0$, this is obvious. For \mathfrak{J} being fully symmetric, it is a consequence of [10, Theorem II.5.1].⁵

Clearly, $P_nTP_{n+1} = (\sqcup, \mathbb{1}_{J_{n+1}}\kappa)\mathbb{1}_{J_n}\varphi$. The adjoint of T is an integral operator whose kernel vanishes for $s > t$, and hence $P_nT^*P_{n+1} = 0$. Together,

$$\sum_{n=1}^\infty P_n(\operatorname{Re}T)P_{n+1} = \frac{1}{2} \sum_{n=1}^\infty 2^{-1/2}\omega_n \cdot \left(\sqcup, \frac{\mathbb{1}_{J_{n+1}}\kappa}{\|\mathbb{1}_{J_{n+1}}\kappa\|} \right) \frac{\mathbb{1}_{J_n}\varphi}{\|\mathbb{1}_{J_n}\varphi\|},$$

and hence $a_n(\sum_{n=1}^\infty P_n(\operatorname{Re}T)P_{n+1}) = 2^{-\frac{1}{2}}\omega_n^*$. We see that $(\omega_n)_{n=1}^\infty \in \mathfrak{J}$. \square

Proof of “ $(\omega_n)_{n=1}^\infty \in \mathfrak{J} \Rightarrow T \in \mathfrak{J}$ ”. Let \mathfrak{J} be ℓ^∞ , c_0 , or a symmetrically normed ideal with the Matsaev property, and assume that $(\omega_n)_{n=1}^\infty \in \mathfrak{J}$. Note that in every case $(\omega_n)_{n=1}^\infty$ is bounded.

① The crucial point is to handle the diagonal cell sum $\sum_{n=1}^\infty P_nTP_n$. Our aim is to show that this series converges to an operator in \mathfrak{J} .

The summand P_nTP_n is the integral operator in $L^2(a, b)$ with kernel

$$P_nTP_n : \quad \mathbb{1}_{t < s}(t, s)\mathbb{1}_{J_n}(t)\mathbb{1}_{J_n}(s)\varphi(t)\overline{\kappa(s)}$$

Since $\mathbb{1}_{J_n}\kappa, \mathbb{1}_{J_n}\varphi \in L^2(a, b)$, it is compact and

$$\begin{aligned} \|P_nTP_n\| &\leq \left(\int_a^b \int_a^b |\mathbb{1}_{t < s}(t, s)\mathbb{1}_{J_n}(t)\mathbb{1}_{J_n}(s)\varphi(t)\overline{\kappa(s)}|^2 ds dt \right)^{\frac{1}{2}} \\ &\leq \|\mathbb{1}_{J_n}\kappa\| \|\mathbb{1}_{J_n}\varphi\| = \omega_n. \end{aligned}$$

The sequence $(\omega_n)_{n=1}^\infty$ is bounded, and hence the series $\sum_{n=1}^\infty P_nTP_n$ converges strongly, and its sum is a bounded operator with

⁵ This is actually a variant of [10, Theorem II.5.1] which is easy to obtain in the present situation since all spaces $L^2(J_n)$ have the same Hilbert space dimension. Choose unitary operators $U_n : L^2(J_n) \rightarrow L^2(J_{n+1})$, let $S : L^2(a, b) \rightarrow L^2(a, b)$ be the block shift

$$S := \begin{pmatrix} 0 & & & & \\ U_1 & 0 & & & \\ & U_2 & 0 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \end{pmatrix} : \begin{pmatrix} L^2(J_1) \\ \oplus \\ L^2(J_2) \\ \oplus \\ \vdots \end{pmatrix} \rightarrow \begin{pmatrix} L^2(J_1) \\ \oplus \\ L^2(J_2) \\ \oplus \\ \vdots \end{pmatrix},$$

and apply [10, Theorem II.5.1] to the operator $(\operatorname{Re}T)S$.

$$\left\| \sum_{n=1}^{\infty} P_n T P_n \right\| \leq \|(\omega_n)_{n=1}^{\infty}\|_{\infty}.$$

This settles the case that $\mathfrak{J} = \ell^{\infty}$. If $\lim_{n \rightarrow \infty} \omega_n = 0$, the series converges w.r.t. the operator norm and hence its sum is a compact operator. This settles the case that $\mathfrak{J} = c_0$.

Consider the remaining case. Then, in particular, $\lim_{n \rightarrow \infty} \omega_n = 0$. Let Q_0 be the compact operator given by the Schmidt-series

$$Q_0 := \sum_{n=1}^{\infty} \omega_n \cdot \left(\sqcup, \frac{\mathbb{1}_{J_n} \kappa}{\|\mathbb{1}_{J_n} \kappa\|} \right) \frac{\mathbb{1}_{J_n} \varphi}{\|\mathbb{1}_{J_n} \varphi\|}.$$

Then $a_n(Q_0) = \omega_n^*$, and hence $Q_0 \in \mathfrak{J}$. By the Matsaev property the triangular truncation transformer \mathcal{C} , cf. [11], is defined on all of \mathfrak{J} and maps \mathfrak{J} boundedly into itself. Thus $\mathcal{C}Q_0 \in \mathfrak{J}$, and

$$\mathcal{C}Q_0 = \sum_{n=1}^{\infty} \mathcal{C}((\sqcup, \mathbb{1}_{J_n} \kappa) \mathbb{1}_{J_n} \varphi).$$

However, $\mathcal{C}((\sqcup, \mathbb{1}_{J_n} \kappa) \mathbb{1}_{J_n} \varphi) = P_n T P_n$. Thus we have $\sum_{n=1}^{\infty} P_n T P_n \in \mathfrak{J}$.

② The rest of the proof merely uses completeness. For $l \in \mathbb{N}$ let Q_l be the compact operator given by the Schmidt-series

$$Q_l := \sum_{n=1}^{\infty} 2^{-\frac{l}{2}} \omega_n \cdot \left(\sqcup, \frac{\mathbb{1}_{J_{n+l}} \kappa}{\|\mathbb{1}_{J_{n+l}} \kappa\|} \right) \frac{\mathbb{1}_{J_n} \varphi}{\|\mathbb{1}_{J_n} \varphi\|}.$$

Then $Q_l \in \mathfrak{J}$ and $\|Q_l\|_{\mathfrak{J}} = 2^{-l/2} \|(\omega_n)_{n=1}^{\infty}\|_{\mathfrak{J}}$. Hence, the series $\sum_{l=1}^{\infty} Q_l$ converges w.r.t. $\|\cdot\|_{\mathfrak{J}}$ and its sum belongs to \mathfrak{J} .

A short computation shows that

$$Tf = \left(\sum_{n=1}^{\infty} P_n T P_n \right) f + \left(\sum_{l=1}^{\infty} Q_l \right) f, \quad f \in L^2(a, b), \sup \text{supp } f < b.$$

Since T is closed, it follows that $T = \sum_{n=1}^{\infty} P_n T P_n + \sum_{l=1}^{\infty} Q_l$, and we conclude that $T \in \mathfrak{J}$. \square

The AJPR-type Theorem has the following obvious consequence (which was our motivation for the choice of terminology). Note that we do not assume \mathfrak{J} to be fully symmetric in this statement.

4.3 Corollary. *Let \mathfrak{J} be an operator ideal which is either ℓ^{∞} or c_0 , or a symmetrically normed ideal with the Matsaev property. Then the weak Matsaev Theorem holds in \mathfrak{J} .*

Proof. If $\mathfrak{J} = \ell^\infty$ or $\mathfrak{J} = c_0$, this is immediate from the AJPR-type Theorem.

Assume now that \mathfrak{J} is a symmetrically normed ideal with the Matsaev property. An integral operator T whose kernel has the form (3.4) obviously has no nonzero eigenvalues. If $\operatorname{Re} T \in \mathfrak{J}$, then in particular $\operatorname{Re} T \in c_0$. Thus $T \in c_0$ by the c_0 -case of the AJPR-type Theorem. This means that T is a Volterra operator, and now the Matsaev property implies that $T \in \mathfrak{J}$. \square

5. Connecting with spectral properties

We instantiate the general results from the previous sections to characterise properties of the spectrum of $A_{[H]}$.

5.1. The ideals c_0 and ℓ^∞

The characterisation of discreteness of the spectrum stated in Theorem 1.1 is obtained from the general results applied with the ideal $\mathfrak{J} := c_0$. The proof follows a very structured scheme, which will repeat in later theorems.

Proof of Theorem 1.1. Let $H = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}$ be a Hamiltonian on $[a, b]$ and assume that $\int_a^b h_1(s) ds < \infty$.

- ① The spectrum of $A_{[H]}$ is discrete if and only if $B_{[H]}$ is compact.
- ② We invoke the results of Sections 3 and 4.
 - ▷ By the AJPR-type Theorem the weak Matsaev Theorem holds in c_0 , cf. Corollary 4.3.
 - ▷ The Independence Theorem is applicable, and yields that $B_{[H]}$ is compact if and only if the integral operator S_{21} with kernel (3.4) where

$$\kappa := \sqrt{h_1}, \quad \varphi := \sqrt{h_2},$$

is compact.

- ▷ The AJPR-type Theorem says that compactness of S_{21} is equivalent to the sequential condition $\lim_{n \rightarrow \infty} \omega_n = 0$.
- ③ A simple elementary argument (explicit proof is provided in Appendix A) shows that the sequential condition $\lim_{n \rightarrow \infty} \omega_n = 0$ is equivalent to the continuous condition

$$\lim_{t \nearrow b} \left(\int_t^b h_1(s) ds \cdot \int_a^t h_2(s) ds \right) = 0. \quad \square$$

5.1 Remark. Using the connection between Krein strings and diagonal canonical systems elaborated in [27], the present Theorem 1.1 yields a new proof of the classical criterion [21, Theorema 4,5] for a string to have discrete spectrum. \diamond

Using $\mathfrak{J} := \ell^\infty$ we obtain a condition for the model operator $A_{[H]}$ to be boundedly invertible.

5.2 Theorem. *Let $H = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}$ be a Hamiltonian on $[a, b]$ and assume that $\int_a^b h_1(s) ds < \infty$. Then $0 \in \rho(A_{[H]})$ if and only if*

$$\sup_{t \in [a, b)} \left(\int_t^b h_1(s) ds \cdot \int_a^t h_2(s) ds \right) < \infty.$$

Proof. We execute the same scheme as in the proof of Theorem 1.1.

- ① We have $0 \in \rho(A_{[H]})$ if and only if $B_{[H]}$ is bounded.
- ② Invoking the general results in exactly the same way as above yields that $B_{[H]}$ is bounded if and only if

$$\sup_{n \in \mathbb{N}} \omega_n < \infty.$$

- ③ This sequential condition is easily seen to be equivalent to the stated continuous condition (for an explicit proof see again Appendix A). \square

We remark that Theorem 5.2 implies [37, Theorem 1.5].

5.2. Ideals with the Matsaev property

Executing our generic proof scheme for ideals with the Matsaev property leads to the following theorem. Remember here Russu’s characterisation of the Matsaev property, cf. Theorem 2.5.

5.3 Theorem. *Let $H = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}$ be a Hamiltonian on $[a, b]$ and assume that $\int_a^b h_1(s) ds < \infty$. Moreover, assume that h_1 does not vanish a.e. on any interval (c, b) with $c \in (a, b)$. Let \mathfrak{J} a symmetrically normed ideal whose norm is induced by a symmetric norming function, and assume that*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \|\mathcal{T}_n\| = 0, \quad \lim_{n \rightarrow \infty} \|\mathcal{T}_{1/n}\| = 0,$$

where \mathcal{T}_n and $\mathcal{T}_{1/n}$ are the maps (2.1) and (2.2). Then the following statements are equivalent.

- (i) *The spectrum of $A_{[H]}$ is discrete and $(\lambda_n^{-1})_{n=1}^\infty \in \mathfrak{J}$, where λ_n are the eigenvalues of $A_{[H]}$ arranged according to increasing modulus (and if necessary extended to an infinite sequence by setting $\lambda_n := \infty$ for $n > \dim \text{ran } A_{[H]}^{-1}$).*

(ii) $(\omega_n)_{n=1}^\infty \in \mathfrak{J}$, where ω_n is as in (4.2) with $\kappa := \sqrt{h_1}, \varphi := \sqrt{h_2}$.

Note that we do not claim a continuous characterisation similar as in Theorem 1.1 or Theorem 5.2. Most probably, such a continuous form does not exist in the general setting.

Proof of Theorem 5.3. The property (i) just means that $B_{[H]} \in \mathfrak{J}$. The general results are applicable since by Russu’s Theorem and Corollary 4.3 the weak Matsaev Theorem holds in \mathfrak{J} . Thus $B_{[H]} \in \mathfrak{J}$ if and only if $(\omega_n)_{n=1}^\infty \in \mathfrak{J}$. \square

5.3. Spectral asymptotics I. Convergence class conditions

As usual let $H = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}$ be a Hamiltonian on $[a, b)$ and assume that $\int_a^b h_1(s) ds < \infty$. Further assume that the spectrum of $A_{[H]}$ is discrete.

We are interested in measuring the density of the point set $\sigma(A_{[H]})$ by means of convergence class w.r.t. a suitable comparison function M . To be precise, we wish to characterise whether

$$\sum_n M\left(\frac{1}{|\lambda_n|}\right) < \infty,$$

where λ_n denote the spectral points of $A_{[H]}$. Since the inverses of absolute values of eigenvalues are the approximation numbers of $B_{[H]}$, we may say equivalently that the task is to decide whether

$$B_{[H]} \in \mathfrak{S}_{[M]} := \left\{ (\alpha_n)_{n=1}^\infty \in c_0 : \sum_{n=1}^\infty M(|\alpha_n|) < \infty \right\}$$

via the Calkin correspondence.

5.4 Example. If $M(x) := x^p$ with some $p > 0$, then $\mathfrak{S}_{[M]}$ is the usual Schatten–von Neumann class \mathfrak{S}_p . Completing the above task for this class of comparison functions would mean to determine the convergence exponent (including convergence class) of the spectrum of $A_{[H]}$. \diamond

Since we work in the regime of symmetrically normed ideals and the Matsaev property plays a decisive role, this example already suggests that our method has a natural border (namely, a slight bit above trace class). This corresponds in some sense – but not fully – to the threshold discussed in the introduction where the Independence Theorem is known to fail.

Let us now rigorously define what we understand by a “suitable comparison function”.

5.5 Definition. Let $M : [0, \infty) \rightarrow [0, \infty)$. Then we say that M is a suitable comparison function, if

- (i) M is increasing and $M(0) = 0$,
- (ii) M is continuous, satisfies $\lim_{t \rightarrow \infty} \frac{M(t)}{t} = \infty$, and $M(1) = 1$,
- (iii) the Δ_2^0 -condition holds: $\limsup_{x \downarrow 0} \frac{M(2x)}{M(x)} < \infty$,
- (iv) M is convex,
- (v) $\alpha_M^0 := \lim_{t \downarrow 0} \left(\frac{1}{\log t} \cdot \log \left[\limsup_{u \downarrow 0} \frac{M(tu)}{M(u)} \right] \right) > 1$. \diamond

We will discuss these conditions in more detail in Section 5.3, and see that they form the natural range of comparison functions for our method.

The theorem we are going to prove now reads as follows.

5.6 Theorem. *Let $H = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}$ be a Hamiltonian on an interval $[a, b]$ such that $\int_a^b h_1(s) ds < \infty$ and that $A_{[H]}$ has discrete spectrum. Moreover, assume that h_1 does not vanish a.e. on any interval (c, b) with $c \in (a, b)$. Let $m : [0, \infty) \rightarrow [0, \infty)$ be equivalent at 0 to some suitable comparison function M in the sense that*

$$\exists c_1, c_2 > 0, x_0 > 0 \forall x \in [0, x_0). \quad c_1 M(x) \leq m(x) \leq c_2 M(x).$$

Then

$$\sum_{n=1}^{\infty} m(a_n(B_{[H]})) < \infty \Leftrightarrow \int_a^b m \left(\left(\int_t^b h_1(s) ds \cdot \int_a^t h_2(s) ds \right)^{\frac{1}{2}} \right) \cdot \frac{h_1(t) dt}{\int_t^b h_1(s) ds} < \infty.$$

This theorem includes Theorem 1.3 as a particular case. Based on [4, Theorem 1.3.3], for every regularly varying function q with order $\rho_q > 1$ the function

$$m(x) := q(x^{-1})^{-1} \tag{5.1}$$

is equivalent at 0 to a suitable comparison function.

The proof runs along the same lines as in the discrete spectrum case. However, significantly heavier machinery is needed to establish that we actually may proceed along these lines.

- ① Neither the left side nor the right side in the asserted equivalence in Theorem 5.6 change their truth value when passing from a function m to another one which is equivalent to m at 0. For the left side this is obvious, while for the right side we should remember Theorem 1.1. Hence, we may assume without loss of generality that $m = M$ is a suitable comparison function.

- ② The space $\mathfrak{S}_{[M]}$ is known as the Orlicz class associated with M . Based on the conditions (i)–(iv) in Definition 5.5, it becomes a separable Banach space when endowed with the Luxemburg norm

$$\|(\alpha_n)_{n=1}^\infty\|_{\mathfrak{S}_{[M]}} := \inf \left\{ \beta > 0 : \sum_{n=1}^\infty M\left(\frac{|\alpha_n|}{\beta}\right) \leq 1 \right\}.$$

This is a classical fact, and shown e.g. in [31, Section 4.a].

Obviously $\mathfrak{S}_{[M]}$ is a symmetric Banach sequence space, i.e., via the Calkin correspondence a symmetrically normed ideal.

- ③ We show that the condition Definition 5.5(v) ensures the Matsaev property for $\mathfrak{S}_{[M]}$.

5.7 Lemma. *Let M be a suitable comparison function. Then $\mathfrak{S}_{[M]}$ has the Matsaev property.*

Proof. To show this, we use Russu’s Theorem. Since $\mathfrak{S}_{[M]}$ is separable, the norm $\|\cdot\|_{\mathfrak{S}_{[M]}}$ is induced by a symmetric norming function. Moreover, by Lemma 2.6, we have $\lim_{n \rightarrow \infty} \|\mathcal{T}_{1/n}\| = 0$.

Consider an element $(\alpha_n)_{n=1}^\infty \in \mathfrak{S}_{[M]}$ with $\|(\alpha_n)_{n=1}^\infty\|_{\mathfrak{S}_{[M]}} \leq 1$. This means that $\sum_{n=1}^\infty M(\alpha_n) \leq 1$, and it follows in particular that $\alpha_n \leq 1, n \in \mathbb{N}$. From Definition 5.5(v) and [32, Theorem 11.13] we find $\delta > 1$ and $C > 0$ such that

$$M\left(\frac{t}{\beta}\right) \leq C\left(\frac{1}{\beta}\right)^\delta M(t), \quad t \leq 1, \beta \geq 1.$$

Let $n \geq \frac{1}{C}$, then $\beta_0 := (Cn)^{\frac{1}{\delta}} \geq 1$, and we obtain

$$\sum_{k=1}^\infty M\left(\frac{|\alpha_k|}{\beta_0}\right) \leq \underbrace{C\left(\frac{1}{\beta_0}\right)^\delta}_{=\frac{1}{n}} \cdot \underbrace{\sum_{k=1}^\infty M(|\alpha_k|)}_{\leq 1} \leq \frac{1}{n},$$

and now we can estimate

$$\|\mathcal{T}_n((\alpha_k)_{k=1}^\infty)\|_{\mathfrak{S}_{[M]}} = \inf \left\{ \beta > 0 : n \sum_{n=1}^\infty M\left(\frac{|\alpha_n|}{\beta}\right) \leq 1 \right\} \leq (Cn)^{\frac{1}{\delta}}.$$

This shows that $\|\mathcal{T}_n\| \leq (Cn)^{\frac{1}{\delta}}$. \square

- ④ The left side of the equivalence asserted in Theorem 5.6 means that $B_{[H]} \in \mathfrak{S}_{[M]}$. By Theorem 5.3 this is equivalent to $(\omega_n)_{n=1}^\infty \in \mathfrak{S}_{[M]}$.
- ⑤ It remains to rewrite the sequential condition “ $(\omega_n)_{n=1}^\infty \in \mathfrak{S}_{[M]}$ ” to the continuous condition stated in Theorem 5.6.

5.8 Lemma. Let $-\infty \leq a < b \leq \infty$, let $\kappa, \varphi: (a, b) \rightarrow \mathbb{C}$ be measurable functions with $\kappa \in L^2(a, b)$ and $\mathbb{1}_{(a,c)}\varphi \in L^2(a, b)$, $c \in (a, b)$. Moreover, let M be a suitable comparison function. Then

$$\sum_{n=1}^{\infty} M(\omega_n) < \infty \Leftrightarrow \int_a^b M(\|\mathbb{1}_{(a,t)}\varphi\| \|\mathbb{1}_{(t,b)}\kappa\|) \cdot \frac{|\kappa(t)|^2 dt}{\|\mathbb{1}_{(t,b)}\kappa\|^2} < \infty.$$

The proof of this lemma requires some technical arguments about Orlicz spaces and is carried out in Appendix A.

5.4. Finite- or minimal-type conditions

Again assuming that the spectrum of $A_{[H]}$ is discrete, we now investigate the density of the point set $\sigma(A_{[H]})$ in terms of “big-O or small-o” conditions on its counting function w.r.t. a regularly varying function g . To be precise, we wish to characterise whether (n denotes the counting function)

$$\limsup_{r \rightarrow \infty} \frac{n((\lambda_n)_{n=1}^{\infty}, r)}{g(r)} < \infty \quad \text{or} \quad \lim_{r \rightarrow \infty} \frac{n((\lambda_n)_{n=1}^{\infty}, r)}{g(r)} = 0. \tag{5.2}$$

The threshold discussed in the introduction says that we have to assume that the order ρ_g is not less than 1. It will turn out that the limit of our method is actually a slight bit higher.

Obviously, the properties in (5.2) do not change their truth value when passing from a function g to another one g_1 which is equivalent to g at ∞ in the sense that

$$\exists c_1, c_2 > 0, r_0 > 0 \quad \forall r \geq r_0. \quad c_1 g_1(r) \leq g(r) \leq c_2 g_1(r).$$

Hence, based on [4, Theorem 1.3.3], we may always assume that g is smooth, strictly monotone, and normalised by $g(1) = 1$.

Passing to the language of approximation numbers of $B_{[H]}$, our task thus is to decide whether

$$a_n(B_{[H]}) = O(\pi_n) \quad \text{or} \quad a_n(B_{[H]}) = o(\pi_n), \tag{5.3}$$

where we have set (g^{-1} denotes the inverse function of g)

$$\pi_n := \frac{1}{g^{-1}(n)}, \quad n \in \mathbb{N}.$$

Assume now that $\rho_g > 1$. Then [10, Theorem III.14.2] applies, and the conditions (5.3) can further be reformulated as

$$B_{[H]} \in \mathfrak{S}_{[\pi]} \quad \text{or} \quad B_{[H]} \in \mathfrak{S}_{[\pi]}^\circ,$$

where $\mathfrak{S}_{[\pi]}$ is the Lorentz ideal

$$\mathfrak{S}_{[\pi]} := \left\{ (\alpha_n)_{n=1}^\infty \in c_0 : \sup_{n \in \mathbb{N}} \left(\sum_{k=1}^n \alpha_k^* / \sum_{k=1}^n \pi_k \right) < \infty \right\},$$

$$\|(\alpha_n)_{n=1}^\infty\|_{\mathfrak{S}_{[\pi]}} := \sup_{n \in \mathbb{N}} \left(\sum_{k=1}^n \alpha_k^* / \sum_{k=1}^n \pi_k \right),$$

and $\mathfrak{S}_{[\pi]}^\circ$ is its separable part. For this type of sequence spaces we refer to [10, Theorem III.14.1] or [30, Example 1.2.7].

Again using that $\rho_q > 1$, we can apply [11, Theorem III.9.1] (or Russu’s Theorem) and conclude that $\mathfrak{S}_{[\pi]}$ and $\mathfrak{S}_{[\pi]}^\circ$ have the Matsaev property.

Now we easily obtain the proof of Theorem 1.5.

Proof of Theorem 1.5. The properties on eigenvalues of $A_{[H]}$ on the left sides of the asserted equivalences just mean that $B_{[H]} \in \mathfrak{S}_{[\pi]}$ or $B_{[H]} \in \mathfrak{S}_{[\pi]}^\circ$, respectively. Theorem 5.3 tells that this is equivalent to $(\omega_n)_{n=1}^\infty \in \mathfrak{S}_{[[M]]}$. By a property of regularly varying functions, it holds that

$$\limsup_{n \rightarrow \infty} \omega_n^* g^{-1}(n) < \infty \Leftrightarrow \limsup_{n \rightarrow \infty} \frac{n}{g((\omega_n^*)^{-1})} < \infty,$$

$$\lim_{n \rightarrow \infty} \omega_n^* g^{-1}(n) = 0 \Leftrightarrow \lim_{n \rightarrow \infty} \frac{n}{g((\omega_n^*)^{-1})} = 0,$$

and the proof is complete. \square

6. Normalisation, examples, and comparison functions

6.1. The normalisation $\int_a^b h_1(s) ds < \infty$

In this section we provide the arguments announced in Remark 1.2. Denote by $T_{\min}(H)$ and $T_{\max}(H)$ the minimal and maximal operators induced by the equation (1.1), cf. [12, Section 3]. First observe that, in the cases of present interest, the space $L^2(H)$ always contains some constant.

6.1 Lemma. *Assume that $0 \notin \sigma_{\text{ess}}(A_{[H]})$. Then there exists $\phi \in \mathbb{R}$ such that*

$$\begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix} \in L^2(H). \tag{6.1}$$

This follows from [34, Theorem 3.8(b)] (use $t = 0$); for the convenience of the reader we recall the argument.

Proof of Lemma 6.1. Since $0 \notin \sigma_{\text{ess}}(A_{[H]})$, 0 is a point of regular type for $T_{\min}(H)$. Thus there exists a selfadjoint extension \tilde{A} of $T_{\min}(H)$ such that $0 \in \sigma_p(\tilde{A})$ (see, e.g., [9, Propositions 3.3 and 3.5]), and it follows that $\ker T_{\max}(H) \neq \{0\}$. This kernel, however, consists of all constant functions in $L^2(H)$. \square

To achieve the normalisation $\int_a^b h_1(s) ds < \infty$, equivalently, $\phi = 0$ in (6.1), one uses rotation isomorphisms.

6.2 Definition. Let $\alpha \in \mathbb{R}$, and denote

$$N_\alpha := \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}.$$

(i) For a Hamiltonian H defined on some interval $[a, b]$, we set

$$(\circlearrowleft_\alpha H)(t) := N_\alpha H(t) N_\alpha^{-1}, \quad t \in [a, b].$$

(ii) For a 2-vector valued function defined on some interval $[a, b]$, we set

$$(\omega_\alpha f)(t) := N_\alpha f(t), \quad t \in [a, b]. \quad \diamond$$

6.3 Remark. The following facts hold (see, e.g., [26, p.263]):

- ▷ $\circlearrowleft_\alpha H$ is a Hamiltonian,
- ▷ ω_α induces an isometric isomorphism of $L^2(H)$ onto $L^2(\circlearrowleft_\alpha H)$,
- ▷ $T_{\min}(\circlearrowleft_\alpha H) \circ \omega_\alpha = \omega_\alpha \circ T_{\min}(H)$.

Consequently, the Hamiltonians H and $\circlearrowleft_\alpha H$ will share all operator theoretic properties. \diamond

6.2. Discussion of Examples 1.6 and 1.7

In this section we consider Hamiltonians of a particularly simple form for which the conditions given in Theorems 1.1, 1.3, and 1.5 can be evaluated. Namely, we assume that H defined on the interval $(0, 1)$, where $h_1(t) = 1$ a.e., and where $h_2(t)$ varies regularly at the singular endpoint 1 in the following sense.

6.4 Definition. We call a function $\varphi: [0, 1) \rightarrow (0, \infty)$ regularly varying at 1 with index $\rho \in \mathbb{R}$, if the function

$$\psi(x) := \varphi\left(\frac{x-1}{x}\right): [1, \infty) \rightarrow (0, \infty)$$

is regularly varying with index ρ . \diamond

The Hamiltonians considered in Examples 1.6 and 1.7 obviously are of this form. Let us point out that the reason for the behaviour exhibited in Examples 1.6 and 1.7 is not the concrete form of the functions studied there, but their property to be regularly varying.

6.5 Lemma. *Let $\varphi: [0, 1) \rightarrow (0, \infty)$ be continuous and regularly varying at 1 with index $\rho > 0$, and set $\kappa(t) := 1, t \in [0, 1)$. Then the numbers ω_n constructed in (4.2) satisfy⁶*

$$\omega_n \asymp 2^{-n} \varphi(1 - 2^{-n}).$$

Proof. We have $\|\mathbb{1}_{(t,1)}\kappa\|^2 = 1 - t$, and hence the sequence $(c_n)_{n=0}^\infty$ is given as

$$c_n = 1 - 2^{-n}, \quad n = 0, 1, 2, \dots$$

Since φ is continuous, we find $t_n \in J_n$ with

$$\omega_n = 2^{-\frac{n}{2}} \left(\int_{J_n} |\varphi(s)|^2 ds \right)^{\frac{1}{2}} = 2^{-\frac{n}{2}} (c_n - c_{n-1})^{\frac{1}{2}} \varphi(t_n) = 2^{-n} \varphi(t_n).$$

Set

$$\psi(x) := \varphi\left(\frac{x-1}{x}\right), \quad x_n := \frac{1}{1-c_n}, \quad y_n := \frac{1}{1-t_n}.$$

Since $\frac{x_n-1}{x_n} = \frac{1}{2}$, we have $y_n = k_n x_n$ with $k_n \in [\frac{1}{2}, 1]$. From the Uniform Convergence Theorem, see e.g. [4, Theorem 1.5.2], we obtain that

$$\left(\frac{1}{4}\right)^\rho \leq \frac{\psi(y_n)}{\psi(x_n)} \leq \left(\frac{5}{4}\right)^\rho, \quad n \text{ sufficiently large.}$$

Passing back to φ, c_n, t_n , this yields $\varphi(t_n) \asymp \varphi(c_n)$. \square

Proof of Example 1.6. We apply Lemma 6.5 with the function $\varphi(t) := \sqrt{h_2(t)}$. This is justified, since the corresponding function ψ is

$$\psi(x) = x^{\frac{\alpha}{2}} (1 + \log x)^{-\frac{\alpha-1}{2}} (1 + \log^+ \log x)^{-\frac{\alpha-2}{2}},$$

and hence is regularly varying with index $\frac{\alpha}{2}$. Therefore the numbers ω_n which decide about the behaviour of the operator S_{21} satisfy

$$\begin{aligned} \omega_n &\asymp 2^{-n} \varphi(1 - 2^{-n}) \\ &= 2^{-n} \cdot 2^{n \frac{\alpha}{2}} (1 + \log 2^n)^{-\frac{\alpha-1}{2}} (1 + \log^+ \log 2^n)^{-\frac{\alpha-2}{2}} \\ &\asymp 2^{n(\frac{\alpha}{2}-1)} n^{-\frac{\alpha-1}{2}} (\log n)^{-\frac{\alpha-2}{2}}. \end{aligned}$$

⁶ We write $\alpha_n \asymp \beta_n$, if there exist $c_1, c_2 > 0$ such that $c_1 \alpha_n \leq \beta_n \leq c_2 \alpha_n, n \in \mathbb{N}$.

Using this relation, the stated spectral properties of $H_{\alpha;\alpha_1,\alpha_2}$ follow immediately from the known sequential characterisations. Let us go through the cases.

▷ First of all the Krein-de Branges formula implies

$$\liminf_{n \rightarrow \infty} \frac{n}{|\lambda_n|} \geq \int_0^1 \sqrt{h_2(s)} ds > 0. \tag{6.2}$$

- ▷ If $(\alpha > 2)$ or $(\alpha = 2, \alpha_1 < 0)$ or $(\alpha = 2, \alpha_1 = 0, \alpha_2 < 0)$, then $\lim_{n \rightarrow \infty} \omega_n = \infty$, and hence 0 belongs to the essential spectrum.
- ▷ If $(\alpha = 2, \alpha_1 = \alpha_2 = 0)$, then $\omega_n \asymp 1$, and hence the spectrum is not discrete, but bounded invertibility takes place.
- ▷ If $(\alpha < 2)$ or $(\alpha = 2, \alpha_1 > 0)$ or $(\alpha = 2, \alpha_1 = 0, \alpha_2 > 0)$, then $\lim_{n \rightarrow \infty} \omega_n = 0$, and hence the spectrum is discrete.
- ▷ If $(\alpha = 2, \alpha_1 > 0)$, then the convergence exponent of $(\omega_n)_{n=1}^\infty$ equals $\frac{2}{\alpha_1}$, while in the case $(\alpha = 2, \alpha_1 = 0, \alpha_2 > 0)$, the convergence exponent of $(\omega_n)_{n=1}^\infty$ is infinite. From this and (6.2) it follows that (for $\alpha = 2$)

$$\text{conv.exp. of } (|\lambda_n|)_{n=1}^\infty = \begin{cases} \infty & , \quad \alpha_1 = 0, \alpha_2 > 0, \\ \frac{2}{\alpha_1} & , \quad \alpha_1 \in (0, 2), \alpha_2 \in \mathbb{R}, \\ 1 & , \quad \alpha_1 \geq 2, \alpha_2 \in \mathbb{R}. \end{cases}$$

▷ If $(\alpha = 2, \alpha_1 \in (0, 2), \alpha_2 \in \mathbb{R})$ and $g(r) := r^{\frac{2}{\alpha_1}} (\log r)^\gamma$, then

$$\begin{aligned} g\left(\frac{1}{\omega_n}\right) &\asymp g\left(n^{\frac{\alpha_1}{2}} (\log n)^{\frac{\alpha_2}{2}}\right) \\ &= \left[n^{\frac{\alpha_1}{2}} (\log n)^{\frac{\alpha_2}{2}}\right]^{\frac{2}{\alpha_1}} \left[\log\left(n^{\frac{\alpha_1}{2}} (\log n)^{\frac{\alpha_2}{2}}\right)\right]^\gamma \\ &\asymp n(\log n)^{\frac{\alpha_2}{\alpha_1} + \gamma}. \end{aligned}$$

This shows that for $\gamma = -\frac{\alpha_2}{\alpha_1}$ we have $n \cdot g\left(\frac{1}{\omega_n}\right)^{-1} \asymp 1$. Since the sequence $(\omega_n)_{n=1}^\infty$ is comparable to a monotone sequence, it follows that

$$0 < \limsup_{n \rightarrow \infty} \frac{n}{g((\omega_n^*)^{-1})} < \infty. \quad \square$$

Proof of Example 1.7. With a simple trick properties of $\mathring{H}_{\alpha_1,\alpha_2}$ can be obtained from Example 1.6. To explain this, we start in the reverse direction. Consider the Hamiltonian $H_{2;\alpha_1,\alpha_2}$, and set

$$\mathring{H}_{2;\alpha_1,\alpha_2}(t) := \begin{pmatrix} 1 & -m(t) \\ -m(t) & m(t)^2 \end{pmatrix}, \quad t \in [0, 1),$$

where

$$m(t) := \int_0^t h_2(s) ds.$$

Moreover, let q be the Weyl-coefficient of $H_{2;\alpha_1,\alpha_2}$ and \hat{q} the one of $\hat{H}_{2;\alpha_1,\alpha_2}$. Then, by [27, Theorem 4.2], we have

$$q(z) = \frac{1}{z} \hat{q}(z^2).$$

Thus the spectra of $A_{[\hat{H}_{2;\alpha_1,\alpha_2}]}$ and $A_{[H_{2;\alpha_1,\alpha_2}]}$ are together discrete or not. If these spectra are discrete, then the convergence exponent of $\sigma(A_{[H_{2;\alpha_1,\alpha_2}]})$ is twice the convergence exponent of $\sigma(A_{[\hat{H}_{2;\alpha_1,\alpha_2}]})$. This yields

$$\text{conv.exp. of } \sigma(A_{[\hat{H}_{2;\alpha_1,\alpha_2}]}) = \begin{cases} \frac{1}{\alpha_1} & , \alpha_1 \in (0, 2), \\ \frac{1}{2} & , \alpha_1 \geq 2. \end{cases}$$

Integrating by parts gives

$$\lim_{t \nearrow 1} \frac{m(t)}{h_2(t)(1-t)} = 1.$$

The function $(h_2(t)(1-t))^2$ is again of the form (1.5) with $\alpha = 2$, but with the parameters $2\alpha_1$ and $2\alpha_2$ instead of α_1 and α_2 . Thus the spectrum of $A_{[\hat{H}_{\alpha_1,\alpha_2}]}$ has the same asymptotic behaviour as the spectrum of $A_{[\hat{H}_{2;\frac{\alpha_1}{2},\frac{\alpha_2}{2}}]}$. \square

6.3. On the notion of suitable comparison functions

Let us discuss the conditions (i)–(v) from Definition 5.5. The basic need to apply the general theorems about operator ideals is that the Orlicz class $\mathfrak{S}_{[M]}$ is a symmetrically normed ideal with the Matsaev property.

- ▷ Monotonicity of M is necessary and sufficient in order that $\mathfrak{S}_{[M]}$ contains with an operator T also all operators \tilde{T} having smaller approximation numbers and $\|\tilde{T}\|_{\mathfrak{S}_{[M]}} \leq \|T\|_{\mathfrak{S}_{[M]}}$. The condition $M(0) = 0$ is necessary and sufficient that $\mathfrak{S}_{[M]}$ is not empty.
- ▷ Continuity is a very mild regularity assumption. The normalisation $M(1) = 1$ and the requirement that M grows sufficiently fast towards ∞ is no loss in generality, since $\mathfrak{S}_{[M]}$ does not change when passing to another function equivalent at 0 to M .
- ▷ The Δ_2^0 -condition is necessary and sufficient that $\mathfrak{S}_{[M]}$ is a linear space. This follows from [32, §3.Corollary c)].

- ▷ $\mathfrak{S}_{[M]}$ is always naturally topologised, and in fact is a Frechet-space. Convexity of M implies that $\mathfrak{S}_{[M]}$ is a Banach space (with the Luxemburg norm). On the other hand, if $\mathfrak{S}_{[M]}$ is locally convex then M is equivalent to a convex function, cf. [32, Theorem 5.3].
- ▷ The quantity α_M^0 in (v) is in the literature known as one of the Matuszewska-Orlicz indices associated with M . We saw in Lemma 5.7 that $\alpha_M^0 > 1$ is sufficient for $\mathfrak{S}_{[M]}$ having the Matsaev property.

Consider the Matuszewska-Orlicz index

$$\beta_M^0 := \lim_{t \rightarrow \infty} \left(\frac{1}{\log t} \cdot \log \left[\limsup_{u \downarrow 0} \frac{M(tu)}{M(u)} \right] \right).$$

Then $\mathfrak{S}_{[M]}$ having the Matsaev property implies that $\beta_M^0 > 1$.

Recall at this point that always $\alpha_M^0 \leq \beta_M^0$, cf. [32, p.84, Remark 2]. If M corresponds to a regularly varying function g via (5.1), then

$$\alpha_M^0 = \beta_M^0 = \rho_g.$$

Let us provide explicit proof for the necessity part in the last item.

Proof of necessity. Assume that M is subject to the conditions Definition 5.5(i)–(iv), and assume that $\mathfrak{S}_{[M]}$ has the Matsaev property. Then, by Russu’s Theorem, $\lim_{n \rightarrow \infty} \frac{1}{n} \|\mathcal{T}_n\| = 0$. A simple argument shows that we can find $\rho \in [0, 1), C > 0$ such that $\|\mathcal{T}_n\| \leq Cn^\rho$, cf. [38, Theorema 1].

For each $n \in \mathbb{N}$ consider the sequence

$$\xi^{(n)} := \left(\underbrace{M^{-1}\left(\frac{1}{n}\right), \dots, M^{-1}\left(\frac{1}{n}\right)}_{n \text{ times}}, 0, 0, \dots \right).$$

Then $\sum_{k=1}^\infty M(\xi_k^{(n)}) = 1$, and hence $\|\xi^{(n)}\|_{\mathfrak{S}_{[M]}} = 1$ (remember here that we assume M to be increasing). We have

$$\mathcal{T}_n \xi^{(n)} := \left(\underbrace{M^{-1}\left(\frac{1}{n}\right), \dots, M^{-1}\left(\frac{1}{n}\right)}_{n^2 \text{ times}}, 0, 0, \dots \right),$$

and hence $\sum_{k=1}^\infty M([\mathcal{T}_n \xi^{(n)}]_k) = n$. By our choice of ρ and C , it holds that $\|\mathcal{T}_n \xi^{(n)}\|_{\mathfrak{S}_{[M]}} \leq Cn^\rho$, and hence for each $C' > C$

$$n^2 M\left(\frac{M^{-1}\left(\frac{1}{n}\right)}{C'n^\rho}\right) = \sum_{k=1}^\infty M\left(\frac{[\mathcal{T}_n \xi^{(n)}]_k}{C'n^\rho}\right) \leq 1.$$

This implies that

$$\frac{M^{-1}(\frac{1}{n})}{C'n^\rho} \leq M^{-1}(\frac{1}{n^2}),$$

and hence, for each $\rho' \in (\rho, 1)$,

$$\frac{n^{\rho'-\rho}}{C'} \leq \frac{M^{-1}(\frac{1}{n} \cdot \frac{1}{n})}{(\frac{1}{n})^{\rho'} \cdot M^{-1}(\frac{1}{n})}.$$

Letting n tend to ∞ , we see that the Matuszewska-Orlicz index $\alpha_{M^{-1}}^0$ is not larger than ρ . By [32, Theorem 11.5], it follows that

$$\beta_M^0 \geq \frac{1}{\rho} > 1. \quad \square$$

Appendix A. Rewriting sequential to continuous conditions

In this section we give details on how to rewrite the sequential conditions obtained from the AJPR-type Theorem to the continuous conditions stated in our theorems.

Recall the relevant notation. We are given a finite or infinite interval (a, b) , and measurable functions $\kappa, \varphi: (a, b) \rightarrow \mathbb{C}$ with $\kappa \in L^2(a, b)$ and $\mathbb{1}_{(a,c)}\varphi \in L^2(a, b)$, $c \in (a, b)$. Further, $c_0 := a < c_1 < c_2 < \dots < b$ is a sequence with

$$\|\mathbb{1}_{(c_n,b)}\kappa\|^2 = \left(\frac{1}{2}\right)^n \|\kappa\|^2, \text{ equivalently, } \|\mathbb{1}_{(c_{n-1},c_n)}\kappa\|^2 = \left(\frac{1}{2}\right)^n \|\kappa\|^2,$$

and $J_n := (c_{n-1}, c_n)$ and $\omega_n := \|\mathbb{1}_{J_n}\kappa\| \cdot \|\mathbb{1}_{J_n}\varphi\|$. Moreover, denote

$$\Omega(t) := \|\mathbb{1}_{(t,b)}\kappa\| \cdot \|\mathbb{1}_{(a,t)}\varphi\|, \quad t \in (a, b).$$

The proof of what is needed in Theorem 1.1 and Theorem 5.2 is simple.

A.1 Lemma. *Letting notation be as in Theorem 4.1, we have*

$$\lim_{n \rightarrow \infty} \omega_n = 0 \Leftrightarrow \lim_{t \nearrow b} \|\mathbb{1}_{(a,t)}\varphi\| \|\mathbb{1}_{(t,b)}\kappa\| = 0,$$

and

$$\sup_{n \rightarrow \infty} \omega_n < \infty \Leftrightarrow \limsup_{t \nearrow b} \|\mathbb{1}_{(a,t)}\varphi\| \|\mathbb{1}_{(t,b)}\kappa\| < \infty.$$

Proof. A sequence $(\alpha_n)_{n=1}^\infty$ of nonnegative numbers is bounded (tends to 0), if and only if the sequence $(2^{-n} \sum_{k=1}^n 2^k \alpha_k)_{n=1}^\infty$ is bounded (tends to 0, respectively). Applying this with

$$\alpha_n := 2^{-n} \int_{c_{n-1}}^{c_n} |\varphi(s)|^2 ds, \quad n \in \mathbb{N},$$

yields the assertion. \square

The proof of Lemma 5.8 is based on dualising and requires some technique for Orlicz ideals. From our assumption Definition 5.5(v) and [32, Theorem 11.13], we find $c > 0, p > 1$ such that

$$M(tu) \leq CM(u)t^p, \quad u, t \in (0, 1].$$

Using $u = 1$ and letting t tend to 0 shows that $\lim_{t \downarrow 0} \frac{M(t)}{t} = 0$. In the language of [32, Chapter 8, p.47] this means that M belongs to the class N (remember here Definition 5.5(ii)). Thus the duality theory for Orlicz spaces is available.

We start with a preparatory lemma.

A.2 Lemma. *Set $I := \mathbb{N}$ and let $q \in (0, 1)$.*

- (i) *Set $J := \{(n, k) \in I \times I : k \leq n\}$ and let $\mathfrak{S}_{[M]}$ denote the Orlicz space of sequences indexed by I or by J depending on the context. For a sequence $(\alpha_n)_{n \in I}$ define sequences $(\beta_{(n,k)})_{(n,k) \in J}$ and $(\beta'_{(n,k)})_{(n,k) \in J}$ as*

$$\beta_{n,k} := \alpha_n q^{n-k}, \quad \beta'_{(n,k)} := \alpha_k q^{n-k}, \quad (n, k) \in J.$$

If $(\alpha_n)_{n \in I} \in \mathfrak{S}_{[M]}$, then $(\beta_{(n,k)})_{(n,k) \in J}, (\beta'_{(n,k)})_{(n,k) \in J} \in \mathfrak{S}_{[M]}$. There exists a constant $C_1 > 0$ such that

$$\begin{aligned} & \max \{ \|(\beta_{(n,k)})_{(n,k) \in J}\|_{\mathfrak{S}_{[M]}}, \|(\beta'_{(n,k)})_{(n,k) \in J}\|_{\mathfrak{S}_{[M]}} \} \\ & \leq C_1 \|(\alpha_n)_{n \in I}\|_{\mathfrak{S}_{[M]}}, \quad (\alpha_n)_{n \in I} \in \mathfrak{S}_{[M]}. \end{aligned}$$

- (ii) *Consider a sequence $(\alpha_n)_{n \in I}$ with $(q^n \alpha_n)_{n \in I} \in \mathfrak{S}_{[M]}$, and define a sequence $(\beta_n)_{n \in I}$ as*

$$\beta_n := \sum_{\substack{k \in I \\ k \leq n}} \alpha_k, \quad n \in I.$$

Then $(q^n \beta_n)_{n \in I} \in \mathfrak{S}_{[M]}$. There exists a constant $C_2 > 0$ such that

$$\|(q^n \beta_n)_{n \in I}\|_{\mathfrak{S}_{[M]}} \leq C_2 \|(q^n \alpha_n)_{n \in I}\|_{\mathfrak{S}_{[M]}}, \quad (q^n \alpha_n)_{n \in I} \in \mathfrak{S}_{[M]}.$$

Proof. For the proof of item (i) let $(\alpha_n)_{n \in I} \in \mathfrak{S}_{[M]}$ with $\|(\alpha_n)_{n \in I}\|_{\mathfrak{S}_{[M]}} \leq 1$ be given. Then, in particular, $|\alpha_n| \leq 1, n \in I$. By our assumption Definition 5.5(v) and [32, Theorem 11.13], we have

$$C := \sup_{0 < \gamma, t \leq 1} \frac{M(\gamma t)}{\gamma M(t)} < \infty.$$

Note that $C \geq 1$. Thus we can estimate

$$\begin{aligned} \sum_{(n,k) \in J} M(|\beta_{n,k}|) &= \sum_{(n,k) \in J} M(|\alpha_n|q^{n-k}) \leq \sum_{(n,k) \in J} Cq^{n-k}M(|\alpha_n|) \\ &= C \sum_{n \in I} \left(\underbrace{\sum_{\substack{k \in I \\ k \leq n}} q^{n-k}}_{\leq \frac{1}{1-q}} \right) M(|\alpha_n|) \leq \frac{C}{1-q} \sum_{n \in I} M(|\alpha_n|) \leq \frac{C}{1-q}. \end{aligned}$$

Since $|\beta_{n,k}| \leq |\alpha_n| \leq 1$ and $\frac{C^2}{1-q} \geq 1$, it follows that

$$\sum_{(n,k) \in J} M\left(\frac{|\beta_{n,k}|}{C^2/(1-q)}\right) \leq \sum_{(n,k) \in J} C \frac{1-q}{C^2} M(|\beta_{n,k}|) \leq 1.$$

This shows that $\|(\beta_{(n,k)})_{(n,k) \in J}\|_{\mathfrak{S}_{[M]}} \leq \frac{C^2}{1-q}$.

The sequence $(\beta'_{(n,k)})_{(n,k) \in J}$ is handled in the same way. Namely

$$\begin{aligned} \sum_{(n,k) \in J} M(\beta'_{n,k}) &= \sum_{(n,k) \in J} M(|\alpha_k|q^{n-k}) \leq \sum_{(n,k) \in J} Cq^{n-k}M(|\alpha_k|) \\ &= C \sum_{k \in I} \left(\underbrace{\sum_{\substack{n \in I \\ n \geq k}} q^{n-k}}_{= \frac{1}{1-q}} \right) M(|\alpha_k|) = \frac{C}{1-q} \sum_{k \in I} M(|\alpha_k|) \leq \frac{C}{1-q}, \end{aligned}$$

from which we again obtain that $\|(\beta'_{(n,k)})_{(n,k) \in J}\|_{\mathfrak{S}_{[M]}} \leq \frac{C^2}{1-q}$.

The proof of (ii) is based on dualising. Let M^* be the Orlicz function complementary to M , cf. [32, Chapter 8,p.48]. The Matuszewska-Orlicz indices of M and M^* are related as

$$\frac{1}{\alpha_M^0} + \frac{1}{\beta_{M^*}^0} = \frac{1}{\alpha_{M^*}^0} + \frac{1}{\beta_M^0} = 1,$$

cf. [32, Corollary 11.6]. By Definition 5.5(iii) and [32, Theorem 11.7] we have $\beta_M^0 < \infty$, and Definition 5.5(v) is $\alpha_M^0 > 1$. It follows that also $\beta_{M^*}^0 < \infty$ and $\alpha_{M^*}^0 > 1$. From [32, Theorem 11.13] we obtain

$$C^* := \sup_{0 < \gamma, t \leq 1} \frac{M^*(\gamma t)}{\gamma M^*(t)} < \infty.$$

Now let $(\sigma_n)_{n \in I} \in \mathfrak{S}_{[M^*]}$. Then we can use (i), the Hölder inequality [32, Chapter 8, Corollary 3], and the relation [32, Theorem 1.1] between Amemiya- and Luxemburg norms, to estimate

$$\begin{aligned} \left| \sum_{n \in I} \sigma_n \cdot q^n \beta_n \right| &\leq \sum_{n \in I} |\sigma_n| q^n \sum_{\substack{k \in I \\ k \leq n}} |\alpha_k| = \sum_{(n,k) \in J} |\sigma_n| (\sqrt{q})^{n-k} \cdot (q^k |\alpha_k|) (\sqrt{q})^{n-k} \\ &\leq 2 \left\| (\sigma_n \cdot (\sqrt{q})^{n-k})_{(n,k) \in J} \right\|_{\mathfrak{S}_{[M^*]}} \left\| (q^k \alpha_k \cdot (\sqrt{q})^{n-k})_{(n,k) \in J} \right\|_{\mathfrak{S}_{[M]}} \\ &\leq \frac{2C^2(C^*)^2}{(1 - \sqrt{q})^2} \left\| (\sigma_n)_{n \in I} \right\|_{\mathfrak{S}_{[M^*]}} \left\| (q^n \alpha_n)_{n \in I} \right\|_{\mathfrak{S}_{[M]}}. \end{aligned}$$

By [32, Theorem 8.6] it follows that $(q^n \beta_n)_{n \in I} \in \mathfrak{S}_{[M]}$ and

$$\left\| (q^n \beta_n)_{n \in I} \right\|_{\mathfrak{S}_{[M]}} \leq \frac{2C^2(C^*)^2}{(1 - \sqrt{q})^2} \left\| (q^n \alpha_n)_{n \in I} \right\|_{\mathfrak{S}_{[M]}}. \quad \square$$

Proof of Lemma 5.8. For $t \in J_n$ it holds that

$$\Omega(t) \geq \|\mathbb{1}_{(a, c_{n-1})} \varphi\| \|\mathbb{1}_{(c_n, b)} \kappa\| \geq \|\mathbb{1}_{J_{n-1}} \varphi\| \|\mathbb{1}_{J_{n+1}} \kappa\| = \frac{\omega_{n-1}}{2},$$

and we can estimate

$$\begin{aligned} \sum_{n=1}^{\infty} M\left(\frac{\omega_n}{2}\right) &= \sum_{n=1}^{\infty} \left[M\left(\frac{\omega_n}{2}\right) \cdot \frac{1}{\log 2} \int_{J_{n+1}} \frac{|\kappa(t)|^2 dt}{\|\mathbb{1}_{(t,b)} \kappa\|^2} \right] \\ &\leq \sum_{n=1}^{\infty} \left[\frac{1}{\log 2} \int_{J_{n+1}} M(\Omega(t)) \cdot \frac{|\kappa(t)|^2 dt}{\|\mathbb{1}_{(t,b)} \kappa\|^2} \right] \\ &\leq \frac{1}{\log 2} \int_a^b M(\Omega(t)) \cdot \frac{|\kappa(t)|^2 dt}{\|\mathbb{1}_{(t,b)} \kappa\|^2} < \infty. \end{aligned}$$

This shows that the implication “ \Leftarrow ” holds.

Conversely, we have for $t \in J_n$

$$\Omega(t) \leq \|\mathbb{1}_{(a, c_n)} \varphi\| \|\mathbb{1}_{(c_{n-1}, b)} \kappa\| \leq \left(\sum_{k=1}^n \|\mathbb{1}_{J_k} \varphi\| \right) \cdot \left(\frac{1}{\sqrt{2}} \right)^{n-1} \|\kappa\|.$$

Assume that $(\omega_n)_{n=1}^{\infty} \in \mathfrak{S}_{[M]}$. Since $\omega_n = \|\kappa\| \left(\frac{1}{\sqrt{2}}\right)^n \cdot \|\mathbb{1}_{J_n} \varphi\|$, we can apply Lemma A.2(ii) with the sequence $\alpha_n := \|\mathbb{1}_{J_n} \varphi\|$, $n \in \mathbb{N}$. This shows that

$$\left(\left(\frac{1}{\sqrt{2}} \right)^n \sum_{k=1}^n \|\mathbb{1}_{J_k} \varphi\| \right)_{n=1}^{\infty} \in \mathfrak{S}_{[M]},$$

and we obtain

$$\int_a^b M(\Omega(t)) \cdot \frac{|\kappa(t)|^2 dt}{\|\mathbb{1}_{(t,b)}\kappa\|^2} = \sum_{n=1}^{\infty} \int_{J_{n+1}} M(\Omega(t)) \cdot \frac{|\kappa(t)|^2 dt}{\|\mathbb{1}_{(t,b)}\kappa\|^2}$$

$$\leq \log 2 \sum_{n=1}^{\infty} M\left(\|\kappa\|\left(\frac{1}{\sqrt{2}}\right)^{n-1} \sum_{k=1}^n \|\mathbb{1}_{J_k}\varphi\|\right) < \infty. \quad \square$$

Appendix B. I.S. Kac’s compactness theorem

As mentioned in the introduction, the only result on the discreteness of the spectrum in the nondiagonal case is a theorem announced by I.S. Kac in [17, Theorem 1]; we reformulate it below. We were unable to find a proof of it in published papers or other sources. In this appendix we derive the meaningful part of Kac’ result from our discreteness criterion Theorem 1.1.

Let us elaborate on his theorem first. It consists of one necessary condition for the discreteness of the spectrum and one sufficient condition, the latter having two cases. We shall show that the necessary condition and one of the cases of the sufficient condition follow from Theorem 1.1. As for the second case in the sufficiency condition, we shall show that it is either empty, that is, no Hamiltonian satisfies it, or it is wrong.

We start by formulating Kac’ theorem. Let $H = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}$ be a Hamiltonian on $[0, \infty)$ such that $\text{tr } H(t) = 1$ a.e.⁷ For $K \geq 0$ we define

$$m_j(t) := \int_0^t h_j(t) dt,$$

$$A_K := \left\{ \lambda \in \mathbb{R} \setminus \{0\} : \limsup_{t \rightarrow \infty} \left(\int_t^\infty h_1(s) e^{\lambda m_3(s)} ds \cdot \int_0^t h_2(s) e^{-\lambda m_3(s)} ds \right) \leq \frac{K}{\lambda^2} \right\},$$

$$B_K := \left\{ \lambda \in \mathbb{R} \setminus \{0\} : \limsup_{t \rightarrow \infty} \left(\int_t^\infty h_2(s) e^{-\lambda m_3(s)} ds \cdot \int_0^t h_1(s) e^{\lambda m_3(s)} ds \right) \leq \frac{K}{\lambda^2} \right\}.$$

B.1 Theorem ([17]). *The implications “(i)⇒(ii)⇒(iii)” hold, where*

- (i) $\sup \left(\bigcup_{K < 1} A_K \cup \bigcup_{K < 1} B_K \right) = +\infty, \inf \left(\bigcup_{K < 1} A_K \cup \bigcup_{K < 1} B_K \right) = -\infty.$
- (ii) *The operator $A_{[H]}$ has discrete spectrum, with⁸ either $h_1 \in L^1(0, \infty)$, or $h_2 \in L^1(0, \infty)$.*
- (iii) $A_1 \cup B_1 = \mathbb{R} \setminus \{0\}.$

⁷ Recall that this trace normalisation can always be achieved by means of a change of scale in the independent variable, which does not change any operator theoretic properties.

⁸ This normalisation does not appear in [17]. However, without it the statement is false.

We are going to establish the following theorem.

B.2 Theorem. *Using the above notation, the following statements are equivalent.*

- (i) *There exists a $K > 0$ such that the set $\bigcup_{K' < K} A_{K'}$ accumulates at both $-\infty$ and $+\infty$.*
- (ii) *The operator $A_{[H]}$ has discrete spectrum and $h_1 \in L^1(0, \infty)$.*
- (iii) *We have*

$$\lim_{t \rightarrow \infty} \left(\int_t^\infty h_1(s) e^{\lambda m_3(s)} ds \cdot \int_0^t h_2(s) e^{-\lambda m_3(s)} ds \right) = 0, \quad \lambda \in \mathbb{R} \setminus \{0\}. \tag{B.1}$$

The analogous statement holds when A_K is replaced by B_K in (i), and h_1 and h_2 are interchanged in (ii) and (iii).

Proof. The assertion (iii) means simply that $A_0 = \mathbb{R} \setminus \{0\}$, hence the implication “(iii) \Rightarrow (i)” holds trivially.

We come to the proof of “(i) \Rightarrow (ii)”. Fix $c \in (0, \infty)$ such that the function h_2 does not vanish almost everywhere on $[0, c]$, and consider the function $F: [c, \infty) \times \mathbb{R} \rightarrow [0, \infty]$ defined as

$$F(t, \lambda) := \int_t^\infty h_1(s) e^{\lambda m_3(s)} ds \cdot \int_0^t h_2(s) e^{-\lambda m_3(s)} ds.$$

Consider an arbitrary interval $[\mu_1, \mu_2] \subset \mathbb{R}$ and a number $\lambda = v\mu_1 + (1 - v)\mu_2$, $v \in (0, 1)$, from this interval. The Hölder inequality gives

$$\begin{aligned} \int_t^\infty h_1(s) e^{2\lambda m_3(s)} ds &= \int_t^\infty \left(h_1(s) e^{\mu_1 m_3(s)} \right)^v \cdot \left(h_1(s) e^{\mu_2 m_3(s)} \right)^{1-v} ds \\ &\leq \left(\int_t^\infty h_1(s) e^{\mu_1 m_3(s)} ds \right)^v \cdot \left(\int_t^\infty h_1(s) e^{\mu_2 m_3(s)} ds \right)^{1-v}, \\ \int_0^t h_2(s) e^{-\lambda m_3(s)} ds &\leq \left(\int_0^t h_2(s) e^{-\mu_1 m_3(s)} ds \right)^v \cdot \left(\int_0^t h_2(s) e^{-\mu_2 m_3(s)} ds \right)^{1-v}. \end{aligned}$$

Multiplying these inequalities we obtain,

$$F(t, \lambda) \leq (F(t, \mu_1))^v (F(t, \mu_2))^{1-v}.$$

Let $\mu_1 \in A_{K_1} \cap (-\infty, 0)$, $\mu_2 \in A_{K_2} \cap (0, \infty)$, for some $K_1, K_2 < K$. Then

$$\limsup_{t \rightarrow \infty} F(t, \lambda) \leq \left(\frac{K}{\mu_1^2}\right)^v \left(\frac{K}{\mu_2^2}\right)^{1-v} = \frac{K}{\mu_1^{2v} \mu_2^{2(1-v)}}.$$

Consider now $\lambda = 0$ and let μ_2 tend to ∞ in the set $\bigcup_{K' < K} A_{K'}$. Then $v \rightarrow 1$, and therefore

$$\limsup_{t \rightarrow \infty} F(t, 0) \leq \frac{K}{\mu_1^2}.$$

Since $\inf \bigcup_{K' < K} A_{K'} = -\infty$ this implies that

$$\lim_{t \rightarrow \infty} \left(\int_t^\infty h_1(s) ds \cdot \int_0^t h_2(s) ds \right) = \lim_{t \rightarrow \infty} F(t, 0) = 0.$$

In particular, $h_1 \in L^1$, and we infer from Theorem 1.1 that the spectrum of $A_{[H]}$ is discrete.

The proof of “(ii) \Rightarrow (iii)” is carried out in three steps.

- ① Assume that the spectrum of $A_{[H]}$ is discrete and $h_1 \in L^1(0, \infty)$. By Theorem 1.1 we have

$$\lim_{t \rightarrow \infty} \left(\int_t^\infty h_1(s) ds \cdot \int_0^t h_2(s) ds \right) = 0. \tag{B.2}$$

Since we have the normalisation $\text{tr } H(t) = 1$, it holds that

$$(B.2) \Leftrightarrow \lim_{t \rightarrow \infty} t \int_t^\infty h_1(s) ds = 0 \Leftrightarrow \lim_{t \rightarrow \infty} \sup_{x \geq t} \underbrace{\left(x \int_x^\infty h_1(s) ds \right)^{\frac{1}{2}}}_{=: p(t)} = 0$$

Note that p is nonincreasing. Moreover, again from trace normalisation,

$$\max\{h_1(t), h_2(t)\} \leq 1, \quad h_3(t)^2 \leq h_1(t)h_2(t) \leq \min\{h_1(t), h_2(t)\}. \tag{B.3}$$

- ② In this step we show that

$$|m_3(y) - m_3(x)| \leq p(x) \left(1 + \log \frac{y}{x} \right), \quad 0 < x < y. \tag{B.4}$$

We have

$$|m_3(y) - m_3(x)| = \left| \int_x^y h_3(s) ds \right| \leq \left(\int_x^y \frac{1}{s} ds \right)^{\frac{1}{2}} \left(\int_x^y s h_3(s)^2 ds \right)^{\frac{1}{2}}$$

$$= \left(\log \frac{y}{x}\right)^{\frac{1}{2}} \left(\int_x^y sh_3(s)^2 ds\right)^{\frac{1}{2}}.$$

Using (B.3) and integrating by parts gives

$$\begin{aligned} \int_x^y sh_3(s)^2 ds &\leq \int_x^y sh_1(s) ds = x \int_x^y h_1(s) ds + \int_x^y \left(\int_s^y h_1(t) dt\right) ds \\ &\leq p(x)^2 + \int_x^y \frac{1}{s} \cdot p(s)^2 ds \leq p(x)^2 \left(1 + \log \frac{y}{x}\right). \end{aligned}$$

Combined these imply the estimate (B.4).

- ③ Fix $t_0 > 0$ such that $p(t_0) < \frac{1}{|\lambda|}$. This is possible, since $p(t) \rightarrow 0$ as $t \rightarrow \infty$. We are going to show that for all $t \geq t_0$

$$\int_t^\infty h_1(s)e^{\lambda m_3(s)} ds \cdot \int_{t_0}^t h_2(s)e^{-\lambda m_3(s)} ds \leq p(t)^2 \cdot \frac{e^2}{(1 - |\lambda|p(t_0))^2}. \tag{B.5}$$

Notice that the left hand side in (B.5) equals to

$$\int_t^\infty h_1(s)e^{\lambda(m_3(s)-m_3(t))} ds \cdot \int_0^t h_2(s)e^{-\lambda(m_3(s)-m_3(t))} ds.$$

We estimate the integrals in this product separately.

$$\begin{aligned} \int_t^\infty h_1(s) e^{\lambda(m_3(s)-m_3(t))} ds &\leq \int_t^\infty h_1(s) \exp\left(|\lambda|p(t)\left(1 + \log \frac{s}{t}\right)\right) ds \\ &\leq \frac{e}{t^{|\lambda|p(t)}} \int_t^\infty h_1(s) s^{|\lambda|p(t)} ds = \frac{e}{t^{|\lambda|p(t)}} \left[t^{|\lambda|p(t)} \underbrace{\int_t^\infty h_1(s) ds}_{\leq \frac{1}{t}p(t)^2} \right. \\ &\quad \left. + \int_t^\infty |\lambda|p(t) s^{|\lambda|p(t)-1} \cdot \left(\underbrace{\int_s^\infty h_1(x) dx}_{\leq \frac{1}{s}p(s)^2 \leq \frac{1}{s}p(t)^2}\right) ds \right] \\ &\leq \frac{e}{t} p(t)^2 + \frac{e}{t^{|\lambda|p(t)}} |\lambda|p(t)^3 \frac{t^{|\lambda|p(t)-1}}{1 - |\lambda|p(t)} \leq \frac{p(t)^2}{t} \cdot \frac{e}{1 - |\lambda|p(t_0)}; \end{aligned}$$

$$\begin{aligned} \int_{t_0}^t h_2(s) e^{-\lambda(m_3(s)-m_3(t))} ds &\leq \int_{t_0}^t \underbrace{h_2(s)}_{\leq 1} \exp\left(|\lambda| \underbrace{p(s)}_{\leq p(t_0)} \left(1 + \log \frac{t}{s}\right)\right) ds \\ &\leq e^{|\lambda|p(t_0)} \frac{t^{1-|\lambda|p(t_0)}}{1-|\lambda|p(t_0)} = t \cdot \frac{e}{1-|\lambda|p(t_0)}. \end{aligned}$$

Combined, these estimates imply (B.5). The required limit (B.1) follows from it since $p(t) \rightarrow 0$ as $t \rightarrow \infty$. \square

The theorem just proved immediately establishes the implication “(ii) \Rightarrow (iii)” in Kac’ theorem (even in a stronger form). Consider now the implication “(i) \Rightarrow (ii)” in the latter.

B.3 Lemma. *We have $A_K \cap B_{K'} = \emptyset$ for all $K, K' > 0$.*

Proof. Arguing by contradiction, assume that $\lambda \in A_K \cap B_{K'}$. Then $h_1 e^{\lambda m_3}, h_2 e^{-\lambda m_3} \in L^1(\mathbb{R}_+)$, which implies that $h_3 \in L^1(\mathbb{R}_+)$ in view of the estimate

$$\begin{aligned} \int_{-\infty}^{\infty} |h_3(x)| dx &\leq \int_{-\infty}^{\infty} \sqrt{h_1(x)h_2(x)} dx \\ &\leq \left(\int_{-\infty}^{\infty} h_1(x) e^{\lambda m_3(x)} dx \cdot \int_{-\infty}^{\infty} h_2(x) e^{-\lambda m_3(x)} dx \right)^{\frac{1}{2}}. \end{aligned}$$

Thus, the function m_3 is bounded, and $h_1, h_2 \in L^1(\mathbb{R}_+)$, which obviously contradicts the normalization $\text{tr } H(x) = 1$. \square

Proceeding, notice that it follows from the proof of the implication “(i) \Rightarrow (ii)” in Theorem B.2 that the sets $\bigcup_{K>0} A_K$ and $\bigcup_{K>0} B_K$ are convex. Together with Lemma B.3 this shows that there are just four possible situations in Theorem B.1(i):

- $\triangleright \sup \left(\bigcup_{K<1} A_K \right) = \infty$ and $\inf \left(\bigcup_{K<1} A_K \right) = -\infty$,
- $\triangleright \sup \left(\bigcup_{K<1} A_K \right) = \infty$ and $\inf \left(\bigcup_{K<1} B_K \right) = -\infty$,
- \triangleright each of the above obtained by switching A and B .

In the first case the assertion of the Kac theorem is immediately contained in Theorem B.2. Assume now that a Hamiltonian H is in the second case, and Theorem B.1 holds. By the implication “(ii) \Rightarrow (iii)” in Theorem B.2 there exists a point $\lambda \in A_K \cap B_{K'}$ on the negative semiaxis. However, by Lemma B.3, the sets A_K and $B_{K'}$ are disjoint and we have reached a contradiction. Hence, either no Hamiltonian belongs to the second case, or the assertion of Kac’ theorem in this case is false.

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