Perturbation of chains of de Branges spaces

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Abstract:

We investigate the structure of the set of de Branges spaces of entire functions which are contained in a space $L^2(\mu)$. Thereby, we follow a perturbation approach. The main result is a growth dependent stability theorem: Assume that two measures μ_1 and μ_2 are close to each other in a sense quantified relative to a proximate order. Consider the sections of corresponding chains of de Branges spaces $\mathcal{C}_1$ and $\mathcal{C}_2$ which consist of those spaces whose elements have finite (possibly zero) type w.r.t. the given proximate order. Then these sections coincide, or one is smaller than the other but its complement consists only of a (finite or infinite) sequence of spaces.

Among others we apply – and refine – this general theorem in two important particular situations. (1) the given measures μ_1 and μ_2 differ in essence only on a compact set; then stability of whole chains rather than sections can be shown. (2) the linear space of all polynomials is dense in $L^2(\mu_1)$; then conditions for density of polynomials in the space $L^2(\mu_2)$ are obtained.

In the proof of the main result we employ a method used by P.Yuditskii in the context of density of polynomials. Another vital tool is the notion of the index of a chain, which is a generalisation of the index of determinacy of a measure having all power moments. We undertake a systematic study of this index, which is also of interest on its own right.

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1 Introduction

In this paper, a de Branges space is a reproducing kernel Hilbert space of entire functions with certain additional properties. Formulated in an operator theoretic way, one may say that the operator of multiplication by the independent variable should be symmetric with defect index $(1,1)$ and real w.r.t. a natural involution. Spaces of this kind were introduced in the late 1950's by L. de Branges as a generalisation of Fourier analysis, cf. [Bra59b], [Bra59a]. They receive a lot of attention up to the present day. Besides their intrinsic beauty, a reason for this continuous interest is that de Branges' Hilbert spaces of entire functions appear in many places of functional analysis and complex analysis. To mention some: spectral theory of canonical systems, cf. [Bra68] (for a more explicit treatment see also [Win95]), and of Schrödinger operators, cf. [Dym70; Rem02], the prediction theory of stationary Gaussian processes, cf. [DM70; Pit72], bases, interpolation and sampling, cf. [OS02; Bar06], or Beurling–Malliavin type theorems, cf. [HM03a; HM03b; BBH07].

In theory and applications of de Branges spaces the notion of isometric embeddings into spaces $L^2(\mu)$, where μ is a positive Borel measure on the real line, plays a crucial role. For a measure μ, we denote by $\mathsf{Sub}[\mu]$ the set of

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all linear spaces of entire functions whose elements are square integrable w.r.t. μ along the real line, and which become a de Branges space when endowed with the $L^2(\mu)$-inner product. The structure theory of $\mathsf{Sub}[\mu]$ lies at the core of de Branges' theory. The basic information, formulated informally, is that $\mathsf{Sub}[\mu]$ is the union of disjoint chains (subsets which are totally ordered w.r.t. set-theoretic inclusion), and that each of these chains enjoys certain completeness properties; we recall details in 2.16 below.

In this paper we follow a perturbation approach to investigate $\mathsf{Sub}[\mu]$. Namely: *Given two measures μ_1 and μ_2 which are small perturbations of each other, can one relate chains from $\mathsf{Sub}[\mu_1]$ and $\mathsf{Sub}[\mu_2]$?*

Our main result is the growth dependent stability theorem Theorem 3.9. The size of the perturbation is measured relative to a growth function (or proximate order; we shall recall details in §2 below). Again speaking informally the theorem states the following: *Let λ be a growth function, assume that μ_1 and μ_2 are close in a sense quantified via λ, and consider chains $\mathcal{C}_i \in \mathsf{Sub}[\mu_i]$ (which fit each other in a certain weak sense). Then their beginning sections consisting of those spaces whose elements are entire functions having growth limited by λ satisfy an ordering property (meaning, one of them is contained in the other). The situation that these sections are not equal may occur. However, if this is the case, then the complement of the smaller section in the larger one consists of a (finite or infinite) sequence of spaces.*

Perturbation approaches are classical and widely used in spectral theory. The probably most prominent example is the Gelfand–Levitan approach to the inverse spectral problem for Schrödinger operators, cf. [GL51]. In [Rem02], the connection with the theory of de Branges spaces was elaborated. It turns out that the decisive property for a measure to be the spectral measure for some potential is that $\mathsf{Sub}[\mu]$ contains a very specific chain (consisting of spaces whose elements are given by cosine transforms of square integrable functions). When viewing spectral problems from the point of de Branges' theory, we may say that the stability of this chain under perturbations of the spectral measure is the crucial point. In this context, it is worth to notice that to some extent the Gelfand–Levitan method has been pushed further to the case of Krein strings in [DK78a; DK78b], and canonical systems in [Win00].

Comparatively recently a stability result was shown in the context of the Hamburger power moment problem. Namely, in [Yud00] it is proved that density of polynomials in a space $L^2(\mu)$ with infinite index of determinacy (we recall details in §6.4) is preserved under sufficiently small perturbations of μ.

Our present work is inspired by P.Yuditskii's paper (and contains his result as a particular case). The essential idea – here, as well as in [Yud00] – is to exploit a compactness property. Though we obtain more general results, the core of our proof is the same. In [Yud00] it is shown via an argument based on orthogonal polynomials that a certain selfadjoint operator is of trace-class. However, it is only needed that it is a compact perturbation of a positive operator strictly smaller than the identity; and this is not difficult to verify (also in the presently considered general situation). Moreover, using a well-known perturbation result for quadratic forms, the explicit determinant-based argument elaborated in [Yud00] can be shortened. The major novelties in the present paper are

(1) The permitted perturbation may possibly be of much larger size than in

[Yud00]; it can be adjusted to a priori knowledge on growth (slow growing functions allow large perturbations of measures).

(2) Moving portions of a measure within its (measure theoretic) support is admitted up to a fixed ratio.

(3) The results hold for arbitrary chains of de Branges spaces including the situation that the spaces are not invariant under difference quotients.

A different notion of smallness of perturbation of a measure was introduced in [BS11] for studying the type problem from a perturbation viewpoint. The methods in [BS11] are very different from our present methods. In fact, the authors also state a stability result which includes Yuditskii's Theorem (see §3.1 in [BS11]), and raise the question whether Yuditskii's approach can be used to treat their kind of perturbations. Our present investigations suggest that this is only partially the case. Roughly speaking, the perturbations considered in [BS11] are composed of two contributions. One, the measure of a set may increase, two, the support of the measure may be shifted. In our present results increase of the measure (possibly even on much larger scale) and limited redistribution of mass within the support is permitted, but shifting large parts of the support is ruled out.

Let us explain organisation and content of the paper. Section 2 is of preliminary nature. There we set up our notation and recall facts about growth functions and de Branges spaces up to the extent necessary. In Section 3 we establish a first perturbation result which is simple but essential. Namely, in Theorem 3.5 we prove an ordering property for certain parts of chains of de Branges spaces for arbitrary measures, i.e., without assuming closeness of measures. Thereby, the situation that one measure majorizes the other is of particular importance. After having settled the general situation, we formulate and illustrate our main theorem; the growth dependend stability result Theorem 3.9. In Section 4 we define the index of a chain, and undertake a systematic study of this notion. The index of a chain is a generalisation of the index of determinacy of a measure having all power moments, cf. [BD95]. The case of inifinite index appeared not only in connection with moment problems, but also in the above mentioned context of the type problem for measures of polynomial growth (there the authors use the term "stable density"). Our main result in this section is the de Branges space theoretic interpretation of the index of a chain given in Theorem 4.10. This result is of independent interest, and includes as particular cases several results for the index of determinacy of a measure having all power moments. Section 5 is devoted to the proof of Theorem 3.9. Finally, in Section 6, we apply this theorem in particular situations where essential stability of whole chains (rather than beginning sections) can be shown. We deal with: (1) instances when necessary a priori growth hypothesis are automatically fullfilled, (2) perturbations which are small outside a compact set, (3) consequences of majorization of one measure by the other, (4) the chain of polynomials. As corollaries we reobtain a certain part of the stability result for type from [BS11, Corollary 1.5] (in Corollary 6.6), an inclusion result from [Win00] (in Corollary 6.8), the stability result [Yud00, Theorem] (in Corollary 6.12), and the classical sufficient determinacy condition [Fre69, Satz 5.2] (in Remark 6.14).

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2 Growth functions and de Branges spaces

Growth functions

Let us recall the notion of a growth function.

2.1 Definition. A function $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *growth function* if it satisfies the following axioms.

(gf1) The function λ is differentiable, strictly increasing, and $\lambda(0) = 1$.

(gf2) $\lim\limits_{r\to\infty} \lambda(r) = \infty$.

(gf3) The limit $\rho_\lambda := \lim\limits_{r\to\infty} \frac{\ln \lambda(r)}{\ln r}$ exists and is finite and non-negative.

(gf4) $\lim\limits_{r\to\infty} \Big(r\frac{\lambda'(r)}{\lambda(r)} \Big/ \frac{\ln \lambda(r)}{\ln r} \Big) = 1$.

$\Diamond$

Growth functions are used to measure growth on a scale which is more refined than the usual scale of order and type. Typical examples are functions of the form

$$\lambda(r) = r^a \cdot \big(\ln_{(m_1)} r\big)^{b_1} \cdot \ldots \cdot \big(\ln_{(m_n)} r\big)^{b_n}$$

for large enough r, where $a > 0$, $m_i \in \mathbb{N}$, $m_1 < \ldots < m_n$, $b_1, \ldots, b_n \in \mathbb{R}$, and $\ln_{(n)}$ is defined by

$$\ln_{(1)} r := \ln r, \qquad \ln_{(k+1)} r := \ln\big(\ln_{(k)} r\big), \quad k \in \mathbb{N},$$

for sufficiently large r. Comparing the growth of an entire function with this kind of functions goes back as far as to work of E.Lindelöf and G.Valiron.

The condition (gf1) could be replaced by the weaker one

(gf1') For all sufficiently large values of r the function λ is differentiable.

However, since the whole importance of a growth function lies in its behaviour at infinity, this yields no gain in generality. Note here that (gf4) implies that $\lambda'(r) > 0$ for r sufficiently large.

Standard references for the theory of growth functions and their use in complex analysis are [Lev80; LG86; Rub96]. In the literature one sometimes rather works with the function $\rho(r) := \frac{\ln \lambda(r)}{\ln r}$ instead of $\lambda(r)$, and speaks of $\rho(r)$ as a *proximate order*. The axioms (gf2)–(gf4) translate as follows:

– $\lim\limits_{r\to\infty} \lambda(r) = \infty \;\Leftrightarrow\; \lim\limits_{r\to\infty} \rho(r) \ln r = \infty$;

– $\lim_{r\to\infty} \dfrac{\ln \lambda(r)}{\ln r} = \rho \;\Leftrightarrow\; \lim_{r\to\infty} \rho(r) = \rho$;

– $\lim_{r\to\infty} \left(r\frac{\lambda'(r)}{\lambda(r)} \Big/ \frac{\ln\lambda(r)}{\ln r} \right) = 1 \;\Leftrightarrow\; \lim_{r\to\infty} \dfrac{r\rho'(r)\ln r}{\rho(r)} = 0.$

Often the condition (gf4) is substituted by

(gf4') $\lim_{r\to\infty} r\dfrac{\lambda'(r)}{\lambda(r)} = \rho_\lambda.$

Clearly, for $\rho_\lambda > 0$ this is equivalent to (gf4). However, if $\rho_\lambda = 0$, it is weaker and in some contexts is not enough to yield the desired properties. In any case, in terms of proximate orders, (gf4') translates as follows:

– Assuming (gf3), we have $\lim_{r\to\infty} r\dfrac{\lambda'(r)}{\lambda(r)} = \rho_\lambda \;\Leftrightarrow\; \lim_{r\to\infty} r\rho'(r)\ln r = 0.$

2.2 Definition. Let λ be a growth function. An entire function f is said to have finite λ-type, if

$$\limsup_{|z|\to\infty} \frac{\ln|f(z)|}{\lambda(|z|)} < \infty.$$

For a function of finite λ-type its indicator w.r.t. λ is

$$h(f,\lambda;\theta) := \limsup_{r\to\infty} \frac{\ln|f(re^{i\theta})|}{\lambda(r)}, \quad \theta\in[0,2\pi).$$

$\Diamond$

2.3 Remark. We use the following classical fact: *Assume that f is an entire function of finite λ-type. Then also f' has this property, and $h(f,\lambda;\theta) = h(f',\lambda;\theta)$, $\theta\in[0,2\pi)$.*

For growth functions λ with $\rho_\lambda > 0$, this is well-known and can be deduced from results in standard textbooks, e.g., from [Lev80, Ch.1,Theorems 27,28]. The case that $\rho_\lambda = 0$ seems to be less widely known. If $\rho_\lambda = 0$, the above statement follows using that the indicator $h(f,\lambda;\theta)$ is constant. This fact in turn goes back to [Gol62]. A more recent reference, which contains a nice proof due to W.Hayman, is [BP07, Appendix]. $\Diamond$

Positive Borel measures

We denote by $\mathbb{M}_+(\mathbb{R})$ the set of all positive Borel measures on $\mathbb{R}$. Thereby, we agree that the term "Borel measure" includes the requirement that compact sets have finite measure.

We will deal with differences of positive measures which are (by the finiteness of measure of compact sets) σ-finite, but not necessarily finite. To make it explicit, let us state the following conventions.

2.4. Notational convention: Expressions like $\mu-\nu$ or $|\mu-\nu|$, where $\mu,\nu\in\mathbb{M}_+(\mathbb{R})$, are understood as set functions defined on the collection of all bounded Borel subsets of the real line. Correspondingly, inequalities between linear combinations of measures are understood to hold for all bounded Borel sets. $\Diamond$

Notice that inequalities between linear combinations of measures which hold for their restrictions to bounded sets certainly imply that the corresponding inequalities hold for integrals of nonnegative functions.

We will deal a lot with inclusions of spaces of analytic functions into spaces $L^2(\mu)$. In this context it is sometimes necessary, or at least helpful, to be very precise. (1) If μ is a discrete measure, such inclusions are not necessarily injective, (2) different inclusions may give rise to different (non-equivalent) norms. Let us introduce some notation to take care of these facts. Let $\mu \in \mathbb{M}_+(\mathbb{R})$ be given. If f and g are complex valued functions on the real line, we write $f \sim_\mu g$ if $f(x) = g(x)$ for μ-a.a. $x \in \mathbb{R}$. We denote the equivalence class modulo $\sim_\mu$ of f as $[f]_\mu$. Next, we denote by $(.,.)_\mu$ the $L^2(\mu)$-inner product, by $\|.\|_\mu$ the corresponding norm, and

$$\rho_\mu : f \mapsto [f|_\mathbb{R}]_{\sim_\mu}. \tag{2.1}$$

Zero divisors

We use the common formal way to locate the zeroes of an entire function.

2.5 Definition. Let f be an entire function which does not vanish identically. Then, for each $w \in \mathbb{C}$, we denote by $\mathfrak{d}_f(w)$ the multiplicity of w as a zero of f. The function $\mathfrak{d}_f : \mathbb{C} \to \mathbb{N}_0$ is called the *zero divisor* of f.

If $\mathcal{L}$ is a set of entire functions which contains at least one element that does not vanish identically, we set

$$\mathfrak{d}_\mathcal{L}(w) := \min\big\{\mathfrak{d}_f(w) : f \in \mathcal{L} \setminus \{0\}\big\}.$$

$\Diamond$

Embeddings of de Branges spaces

2.6 Definition. Let $\mathcal{H}$ be a linear space whose elements are entire functions[1]. We call $\mathcal{H}$ an *algebraic de Branges space*, if it contains a function which does not vanish identically and satisfies the following axioms.

(adB1) If $f \in \mathcal{H}$, $w \in \mathbb{C} \setminus \mathbb{R}$, and $f(w) = 0$, then also the function $\frac{f(z)}{z-w}$ belongs to $\mathcal{H}$.

(adB2) If $f \in \mathcal{H}$, then also the function $f^\#(z) := \overline{f(\overline{z})}$ belongs to $\mathcal{H}$.

$\Diamond$

Algebraic de Branges spaces appear in several contexts.

2.7 Example.

(i) The space $\mathbb{C}[z]$ of all polynomials with complex coefficients is an algebraic de Branges space. This space occurs in the context of power moment problems: Let $\mu \in \mathbb{M}_+(\mathbb{R})$. Then $\rho_\mu(\mathbb{C}[z]) \subseteq L^2(\mu)$ holds if and only if μ has power moments of arbitrary order, and[2] $\operatorname{Clos}_{L^2(\mu)} \rho_\mu(\mathbb{C}[z]) = L^2(\mu)$ if and only if μ is in addition uniquely determined by its power moments.

[1]Here, and always, we tacitly assume that linear operations are defined by pointwise addition and scalar multiplication.

[2]We denote by "Clos M" the closure of the set M (it will always be clear from the context within which topological space this closure has to be understood).

(*ii*) For each $a > 0$, let $\mathcal{E}(a)$ be the image under the Fourier transform of the space $C_{00}^{\infty}(-a, a)$ of all infinitely differentiable functions compactly supported in $(-a, a)$. Then $\mathcal{E}(a)$ is an algebraic de Branges space (note here that the Fourier transform of a compactly supported function is entire). These spaces occur in the type problem for a measure, that is, the problem of determining the minimal width of frequencies needed to approximate any function in $L^2(\mu)$: Let $\mu \in \mathbb{M}_+(\mathbb{R})$ be a (for simplicity) finite measure, then $\rho_\mu(\mathcal{E}(a)) \subseteq L^2(\mu)$. To have $\operatorname{Clos}_{L^2(\mu)} \rho_\mu(\mathcal{E}(a)) = L^2(\mu)$, means that the type of μ does not exceed a.

$\Diamond$

Often algebraic de Branges space occur by means of the following fact: *The union of an increasing chain of algebraic de Branges spaces is itself an algebraic de Branges space.* For example, for each $n \in \mathbb{N}$, the space

$$\mathbb{C}[z]_n := \{p \in \mathbb{C}[z] : \deg p < n\}$$

is an algebraic de Branges space, and $\mathbb{C}[z]$ is the union of the chain $\{\mathbb{C}[z]_n : n \in \mathbb{N}\}$.

We call a space $\langle \mathcal{H}, (.,.)_{\mathcal{H}} \rangle$ a *reproducing kernel Hilbert space of entire functions*, if the elements of $\mathcal{H}$ are entire functions and for each $w \in \mathbb{C}$ the *point-evaluation functional* $\chi_w : f \mapsto f(w)$, $f \in \mathcal{H}$, is continuous.

2.8 Definition. Let $\mathcal{H}$ be a linear space whose elements are entire functions, $\mathcal{H} \neq \{0\}$, and let $(.,.)_{\mathcal{H}}$ be a positive definite inner product on $\mathcal{H}$. We call $\langle \mathcal{H}, (.,.)_{\mathcal{H}} \rangle$ a *de Branges space*, if $\mathcal{H}$ is an algebraic de Branges space and the following axioms are fullfilled[3].

(dB1) $\displaystyle \Big\| \frac{z - \overline{w}}{z - w} f(z) \Big\|_{\mathcal{H}} = \|f\|_{\mathcal{H}}, \quad f \in \mathcal{H}, w \in \mathbb{C} \setminus \mathbb{R}, f(w) = 0.$

(dB2) $\|f^{\#}\|_{\mathcal{H}} = \|f\|_{\mathcal{H}}, \quad f \in \mathcal{H}.$

(dB3) $\langle \mathcal{H}, (.,.)_{\mathcal{H}} \rangle$ is a reproducing kernel Hilbert space of entire functions.

$\Diamond$

Note that a linear space of functions may carry at most one Banach-space topology such that all point evaluations are continuous (as is seen by applying the Closed Graph Theorem with the identity map). Hence, if $\langle \mathcal{H}, (.,.)_1 \rangle$ and $\langle \mathcal{H}, (.,.)_2 \rangle$ are de Branges spaces, necessarily $\|.\|_1$ and $\|.\|_2$ are equivalent.

2.9. de Branges spaces via Hermite-Biehler functions: In the above definition we used an axiomatic way to introduce de Branges spaces. A more concrete, equivalent, approach proceeds via a certain class of entire functions, cf. [Bra68, §19]: We call an entire function a *Hermite-Biehler function*, if it satisfies

$$|E(\overline{z})| < |E(z)|, \quad z \in \mathbb{C}^+.$$

Given a Hermite-Biehler function E, consider the function K_E defined as

$$K_E(w, z) = \frac{i}{2\pi} \frac{E(z)E^{\#}(\overline{w}) - E^{\#}(z)E(\overline{w})}{z - \overline{w}}, \quad z, w \in \mathbb{C}, \tag{2.2}$$

[3] Here, and always, we denote by $\|.\|_{\mathcal{H}}$ the norm induced by $(.,.)_{\mathcal{H}}$. Corresponding notation is applied also with other inner products.

where the formula has to be interpreted as a derivative if $z = \overline{w}$. Then K_E is a positive semidefinite kernel, and the reproducing kernel Hilbert space $\mathcal{H}(E)$ generated by K_E is a de Branges space. Conversely, for each de Branges space $\langle \mathcal{H}, (.,.)_{\mathcal{H}} \rangle$ there exist Hermite-Biehler functions E, such that the reproducing kernel $k_{\mathcal{H}}$ of the space $\langle \mathcal{H}, (.,.)_{\mathcal{H}} \rangle$ coincides with K_E. Given a Hermite-Biehler function E, we denote the de Branges space it generates as $\mathcal{H}(E)$. $\Diamond$

2.10 Example. The probably most classical infinite dimensional de Branges spaces are the *Paley-Wiener spaces.* These spaces arise from the Fourier transform. Namely, for $a > 0$ let $\mathcal{P}W_a$ be the image under the Fourier transform of the space $L^2(-a, a)$ endowed with scalar product $(f, g)_{\mathcal{P}W_a} := \int_{\mathbb{R}} f(t)\overline{g(t)}\, dt$. Then $\mathcal{P}W_a$ is a de Branges space. It is generated by the Hermite-Biehler function given by $E_a(z) := e^{-iaz}$, $z \in \mathbb{C}$. By a theorem of Paley and Wiener, the space $\mathcal{P}W_a$ is equal to the set of all entire functions which are of bounded type in the upper- and lower half planes, whose exponential type does not exceed a, and which are square integrable along the real line. $\Diamond$

The norm of point-evaluation functionals χ_w in a normed (not necessarily complete) space of functions, plays a decisive role. For a normed space $\langle \mathcal{X}, \|.\|_{\mathcal{X}} \rangle$ of functions we denote

$$\nabla_{\langle \mathcal{X}, \|.\|_{\mathcal{X}} \rangle}(w) := \sup \big\{ |f(w)| : f \in \mathcal{H}, \|f\|_{\mathcal{X}} \leq 1 \big\},$$

where the supremum is understood as an element of $[0, \infty]$. To shorten notation, the dependency of this definition on the norm $\|.\|_{\mathcal{X}}$ is suppressed when no confusion is possible.

If the space under consideration is a reproducing kernel Hilbert space $\langle \mathcal{H}, (.,.)_{\mathcal{H}} \rangle$, and $k_{\mathcal{H}}$ denotes its reproducing kernel, then the norm of point evaluation functionals is given as

$$\nabla_{\mathcal{H}}(w) = k_{\mathcal{H}}(w, w)^{\frac{1}{2}}.$$

2.11. Inner products defined by integration: Let $\mathcal{H}$ be an algebraic de Branges space, let $\mu \in \mathbb{M}_+(\mathbb{R})$, and assume that $\int_{\mathbb{R}} |f|^2\, d\mu < \infty$, $f \in \mathcal{H}$. Then we may consider the inner product space $\langle \mathcal{H}, (.,.)_\mu \rangle$ where (in order to avoid cumbersome notation, we slightly overload useage of the symbol $(.,.)_\mu$)

$$(f, g)_\mu := (\rho_\mu f, \rho_\mu g)_\mu = \int_{\mathbb{R}} f\overline{g}\, d\mu, \quad f, g \in \mathcal{H}.$$

As usual, denote $\|f\|_\mu := (f, f)_\mu^{\frac{1}{2}}$, $f \in \mathcal{H}$. Note, however, that $\|.\|_\mu$ is in general only a seminorm on $\mathcal{H}$.

Since integration takes place along the real line, the isometry conditions in (dB1) and (dB2) are automatically satisfied. Hence, if $\mathcal{H}$ is an algebraic de Branges space, then $\langle \mathcal{H}, (.,.)_\mu \rangle$ is a de Branges space if and only if

(1) $(.,.)_\mu$ is nondegenerate on $\mathcal{H}$;

(2) $\mathcal{H}$ is complete w.r.t. the norm $\|.\|_\mu$;

(3) for each $w \in \mathbb{C}$, the point evaluation functional $\chi_w : f \mapsto f(w)$, $f \in \mathcal{H}$, is continuous w.r.t. $\|.\|_\mu$.

Equivalently, we may say that

(1′) $\rho_\mu|_{\mathcal{H}}$ is injective;

(2′) $\rho_\mu(\mathcal{H})$ is closed;

(3′) for each $w \in \mathbb{C}$, we have $\nabla_{\mathcal{H}}(w) < \infty$.

$\Diamond$

2.12 Lemma. *Let $\langle \mathcal{H}, (.,.)_{\mathcal{H}} \rangle$ be a de Branges space, $\mu \in \mathbb{M}_+(\mathbb{R})$, and assume that $\int_{\mathbb{R}} |f|^2 \, d\mu < \infty$, $f \in \mathcal{H}$. Then the map $\rho_\mu|_{\mathcal{H}}$ is continuous. It is bicontinuous, if and only if $\langle \mathcal{H}, (.,.)_\mu \rangle$ is a de Branges space.*

Proof. Assume that $f_n \to f$ in $\mathcal{H}$ and $\rho_\mu(f_n) \to g$ in $L^2(\mu)$. Choose a subsequence $(f_{n_k})_{k=1}^\infty$, such that $\rho_\mu(f_{n_k}) \to g$ pointwise almost everywhere. Since $\mathcal{H}$ is a reproducing kernel space, we have $f_{n_k}(x) \to f(x)$, $x \in \mathbb{R}$. Thus $\rho_\mu(f) = g$ almost everywhere. The closed graph theorem implies that ρ_μ is continuous.

Clearly, if ρ_μ is bicontinuous, then ρ_μ is in particular injective and $\langle \mathcal{H}, \|.\|_\mu \rangle$ is complete. Conversely, if $\langle \mathcal{H}, (.,.)_\mu \rangle$ is a de Branges space, then ρ_μ is a continuous bijection onto $\rho_\mu(\mathcal{H})$ and we may apply the open mapping theorem to conclude that ρ_μ is bicontinuous. ❑

2.13. A word of caution concerning notation: In the present paper inclusions are understood *solely* in their set-theoretic sense. Writing "$\mathcal{H}_1 \subseteq \mathcal{H}_2$" means that

$$\forall f: \ \big(f \in \mathcal{H}_1 \Rightarrow f \in \mathcal{H}_2\big). \tag{2.3}$$

By writing "$\rho_\mu(\mathcal{H}) \subseteq L^2(\mu)$" we mean that

$$\forall f: \ \Big(f \in \mathcal{H} \Rightarrow \int_{\mathbb{R}} |f|^2 \, d\mu < \infty\Big). \tag{2.4}$$

If the norms coincide and we wish to emphasise this, we say that $\mathcal{H}_1 \subseteq \mathcal{H}_2$ *isometrically* (if (2.3) and $\|f\|_{\mathcal{H}_1} = \|f\|_{\mathcal{H}_2}$, $f \in \mathcal{H}_1$) and $\rho_\mu(\mathcal{H}) \subseteq L^2(\mu)$ *isometrically* (if (2.4) and $\|f\|_{\mathcal{H}} = \|\rho_\mu f\|_\mu$, $f \in \mathcal{H}$). $\Diamond$

Chains of de Branges spaces

2.14 Definition. Let $\mu \in \mathbb{M}_+(\mathbb{R})$.

(1) We denote by $\mathsf{Sub}[\mu]$ the set of all algebraic de Branges spaces $\mathcal{H}$ with the properties that $\rho_\mu(\mathcal{H}) \subseteq L^2(\mu)$ and that $\langle \mathcal{H}, (.,.)_\mu \rangle$ is a de Branges space.

(2) We call a subset $\mathcal{C}$ of $\mathsf{Sub}[\mu]$ a *partial chain for* μ, if

  (*a*) $\mathcal{C} \neq \emptyset$;

  (*b*) $\mathcal{C}$ is totally ordered w.r.t. inclusion;

  (*c*) $\mathfrak{d}_{\mathcal{H}_1} = \mathfrak{d}_{\mathcal{H}_2}$ for every pair of spaces $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{C}$ and that each quotient $\frac{f_1}{f_2}$ with $f_i \in \mathcal{H}_i \setminus \{0\}$, $i = 1, 2$, is a meromorphic function of bounded characteristic in the open upper half-plane $\mathbb{C}^+$.

(3) We call $\mathcal{C}$ a *chain for* μ, if $\mathcal{C}$ is a maximal element in the set of all partial chains for μ. The set of all chains for μ is denoted as $\mathsf{Chains}[\mu]$.

$\Diamond$

2.15 Example. We inspect the set $\mathsf{Sub}[\mu]$ when μ is the Lebesgue measure μ on $\mathbb{R}$. We already have seen one family of de Branges spaces in $L^2(\mu)$, namely the Paley-Wiener spaces $\mathcal{P}W_a$, $a > 0$. The set

$$\mathcal{C} := \{\mathcal{P}W_a : a > 0\}$$

clearly is a partial chain. It has the properties

$$\begin{aligned}
&\bigcap_{a>0} \mathcal{P}W_a = \{0\}, \quad \operatorname{Clos}_{L^2(\mu)} \bigcup_{a>0} \mathcal{P}W_a = L^2(\mu), \\
&\bigcap_{a>a_0} \mathcal{P}W_a = \mathcal{P}W_{a_0}, \quad \operatorname{Clos}_{L^2(\mu)} \bigcup_{a_0>a>0} \mathcal{P}W_a = \mathcal{P}W_{a_0}, \qquad a_0 \in (0,\infty).
\end{aligned} \tag{2.5}$$

Hence, $\mathcal{C}$ is maximal, i.e., $\mathcal{C} \in \mathsf{Chains}[\mu]$.

There are many other chains contained in $L^2(\mu)$; let us exhibit one of them. Denote by H_n the n-th Hermite polynomial, i.e.,

$$H_n(z) := (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}, \quad z \in \mathbb{C}, n \in \mathbb{N}_0.$$

Each of the spaces

$$\mathcal{H}_n := e^{-\frac{z^2}{2}} \mathbb{C}[z]_n, \quad n \in \mathbb{N},$$

is contained in $L^2(\mu)$. The family $\{e^{-\frac{z^2}{2}} H_k(z) : k = 0, \dots, n-1\}$ forms an orthogonal basis of $\langle \mathcal{H}_n, (.,.)_{L^2(\mu)} \rangle$. The family

$$\mathcal{C}' := \{\mathcal{H}_n : n \in \mathbb{N}\}$$

is a partial chain, and

$$\bigcap_{n>m} \mathcal{H}_n = \mathcal{H}_{m+1}, \qquad \bigcup_{0<n<m} \mathcal{H}_n = \mathcal{H}_{m-1}, \quad m \in \mathbb{N}.$$

Since $\{e^{-\frac{z^2}{2}} H_k(z) : k = 0, 1, 2, \dots\}$ is an orthogonal basis of $L^2(\mu)$, we obtain properties similar to (2.5):

$$\begin{aligned}
&\bigcap_{n\in\mathbb{N}} \mathcal{H}_n = \operatorname{span}\{1\}, \quad \operatorname{Clos}_{L^2(\mu)} \bigcup_{n\in\mathbb{N}} \mathcal{H}_n = L^2(\mu), \\
&\dim\Big(\bigcap_{n>m} \mathcal{H}_n / \mathcal{H}_m\Big) = 1, \quad \dim\Big(\mathcal{H}_m / \bigcup_{0<n<m} \mathcal{H}_n\Big) = 1, \qquad m \in \mathbb{N}.
\end{aligned}$$

It follows that $\mathcal{C}' \in \mathsf{Chains}[\mu]$. Clearly, $\mathcal{C} \cap \mathcal{C}' = \emptyset$. $\Diamond$

Using the above notation a portion of the core results of de Branges' theory (as presented in [Bra68]) can be summarised as follows.

2.16. *Structure of* $\mathsf{Sub}[\mu]$*:* *Let $\mu \in \mathbb{M}_+(\mathbb{R})$.*

(1) $\mathsf{Sub}[\mu] = \bigcup_{\mathcal{C}\in\mathsf{Chains}[\mu]} \mathcal{C}$, *and for each* $\mathcal{C} \in \mathsf{Chains}[\mu]$ *we have* $\operatorname{Clos}\rho_\mu\big(\bigcup_{\mathcal{H}\in\mathcal{C}} \mathcal{H}\big) = L^2(\mu)$.

(2) *Let $\mathcal{C}_1, \mathcal{C}_2 \in \mathsf{Chains}[\mu]$. Then the following statements are equivalent.*

(a) $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$.

(b) $\mathcal{C}_1 = \mathcal{C}_2$.

(c) *There exist* $\mathcal{H}_i \in \mathcal{C}_i$, $i = 1, 2$, *and* $f_i \in \mathcal{H}_i \setminus \{0\}$, $i = 1, 2$, *such that* $\mathfrak{d}_{\mathcal{H}_1} = \mathfrak{d}_{\mathcal{H}_2}$ *and* $\frac{f_1}{f_2}$ *is a meromorphic function of bounded characteristic in* $\mathbb{C}^+$.

The properties in (c) *are the crucial hypothesis in de Branges' Ordering Theorem, cf. [Bra68, Theorem 35].*

(3) *Let* $\mathcal{C} \in \mathsf{Chains}[\mu]$ *and* $\mathcal{H} \in \mathcal{C}$. *Then*

$$\Big(\operatorname{Clos} \bigcup_{\substack{\mathcal{L} \in \mathcal{C} \cup \{\{0\}\} \\ \mathcal{L} \subsetneq \mathcal{H}}} \mathcal{L}\Big) \in \mathcal{C} \cup \{\{0\}\} \ \textit{and} \ \dim \Big(\mathcal{H} \Big/ \operatorname{Clos} \bigcup_{\substack{\mathcal{L} \in \mathcal{C} \cup \{\{0\}\} \\ \mathcal{L} \subsetneq \mathcal{H}}} \mathcal{L}\Big) \leq 1,$$

and

$$\bigcap_{\mathcal{L} \in \mathcal{C}, \mathcal{L} \supsetneq \mathcal{H}} \mathcal{L} \in \mathcal{C} \ \textit{and} \ \dim \Big(\bigcap_{\mathcal{L} \in \mathcal{C}, \mathcal{L} \supsetneq \mathcal{H}} \mathcal{L} \Big/ \mathcal{H}\Big) \leq 1,$$

if $\mathcal{H}$ *is not maximal element of* $\mathcal{C}$.

(4) *Let* $\mathcal{C} \in \mathsf{Chains}[\mu]$. *Then the following statements are equivalent.*

(a) $\mathcal{C}$ *contains a maximal element.*

(b) $\exists \mathcal{H} \in \mathcal{C} : \ \rho_\mu(\mathcal{H}) = L^2(\mu)$.

(c) *In each of the upper and lower half-planes* $\mathbb{C}^+$ *and* $\mathbb{C}^-$ *there exist points* w *with* $\sup\{\nabla_{\mathcal{H}}(w) : \mathcal{H} \in \mathcal{C}\} < \infty$.

(d) $\sup\{\nabla_{\mathcal{H}}(w) : \mathcal{H} \in \mathcal{C}\} < \infty$ *for every point* $w \in \mathbb{C}$.

(5) *Let* $\mathcal{C} \in \mathsf{Chains}[\mu]$. *Then the following statements are equivalent.*

(a) $\mathcal{C}$ *contains a minimal element.*

(b) $\mathcal{C}$ *contains a one-dimensional element.*

(c) *There exists a point* $w \in \mathbb{C}$ *with* $\inf\{\nabla_{\mathcal{H}}(w) : \mathcal{H} \in \mathcal{C}\} > 0$.

The following fact can be regarded as common knowledge.

2.17 Remark. Let $\mu \in \mathbb{M}_+(\mathbb{R})$, let $\Omega \subseteq \mathbb{C}$ be open and connected with $\mathbb{R} \subseteq \Omega$, and let $p \in [1, \infty]$. Moreover, let $\mathcal{L}$ be a linear space of functions analytic in Ω. If $L^p(\mu) \subseteq \rho_\mu(\mathcal{L})$, then μ is discrete (this is seen by an application of the Identity Theorem). $\Diamond$

In particular, we have the following.

2.18 Lemma. *Let* $\mu \in \mathbb{M}_+(\mathbb{R})$, $\mathcal{C} \in \mathsf{Chains}[\mu]$, *and assume that* $\mathcal{C}$ *has a maximal element. Then* μ *is discrete.*

Proof. Let $\mathcal{H}$ be the maximal element of $\mathcal{C}$. Then $\rho_\mu(\mathcal{H}) = L^2(\mu)$. ❑

In our present studies chains which contain elements with finite codimension in the space $L^2(\mu)$ appear frequently. Let us briefly comment on this situation.

2.19 Remark. Let $\mu \in \mathbb{M}_+(\mathbb{R})$ and $\mathcal{C} \in \mathsf{Chains}[\mu]$. Assume that $\mathcal{C}$ contains a space $\mathcal{H}_0$ with

$$0 < \underbrace{\dim \Big(L^2(\mu) \Big/ \rho_\mu(\mathcal{H}_0) \Big)}_{=:d} < \infty.$$

Then there exist spaces $\mathcal{H}_1, \dots, \mathcal{H}_d$ with $\mathcal{H}_0 \subsetneq \mathcal{H}_1 \subsetneq \dots \mathcal{H}_d$ and $\dim \mathcal{H}_i/\mathcal{H}_{i-1} = 1$, $i = 1, \dots, d$, such that[4]

$$\mathcal{C} = \{\mathcal{H} \in \mathcal{C} : \mathcal{H} \subseteq \mathcal{H}_0\} \dot\cup \{\mathcal{H}_1, \dots, \mathcal{H}_d\}.$$

In particular, $\mathcal{C}$ has a maximal element and μ is discrete. $\Diamond$

In many, but not all, applications of canonical systems a chain of de Branges spaces with a particular property plays a decisive role.

2.20 Remark. Let $\mu \in \mathbb{M}_+(\mathbb{R})$ and assume that

$$\int_{\mathbb{R}} \frac{d\mu(x)}{1 + x^2} < \infty.$$

Then there exists a unique chain $\mathcal{C} \in \mathsf{Chains}[\mu]$ with the property that each de Branges space $\mathcal{H} \in \mathcal{C}$ is invariant under difference quotients. This means that

$$\forall F \in \mathcal{H}, w \in \mathbb{C} : \quad G(z) := \frac{F(z) - F(w)}{z - w} \in \mathcal{H}$$

Another, equivalent, way to express this property is to say that

$$\forall \mathcal{H} \in \mathcal{C} : \quad 1 \in \mathcal{H} + z\mathcal{H} \tag{2.6}$$

It is a nontrivial fact that (2.6) already follows when it is known that one single space $\mathcal{H} \in \mathcal{C}$ has the property that $1 \in \mathcal{H} + z\mathcal{H}$, cf. [Bra61, Lemma 12] and [Bra68, Problem 72]. $\Diamond$

In de Branges' original work this situation was exhibited as an important particular case, however, his work includes all other situations also. Curiously, still most of the literature dealing in one or the other way with chains of de Branges spaces focuses on considering chains $\mathcal{C}$ with (2.6).

3 Comparing chains for different measures

Our aim is to compare chains for different measures. Let $\mu_1, \mu_2 \in \mathbb{M}_+(\mathbb{R})$ and let $\mathcal{C}_1 \in \mathsf{Chains}[\mu_1]$ and $\mathcal{C}_2 \in \mathsf{Chains}[\mu_2]$. An obvious necessary condition for a space $\mathcal{H}$ to be an element of both chains is that those properties which appeared in Definition 2.14, (2/c), and in 2.16, (2/c), hold, and that the elements of $\mathcal{H}$ are square integrable w.r.t. both measures μ_1 and μ_2.

In the following two definitions we notationally single out these properties.

3.1 Definition. Let $\mu_1, \mu_2 \in \mathbb{M}_+(\mathbb{R})$, and let $\mathcal{C}_i \in \mathsf{Chains}[\mu_i]$, $i = 1, 2$. We say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are *admissible for comparison*, if

(1) $\exists\, \mathcal{H}_1 \in \mathcal{C}_1, \mathcal{H}_2 \in \mathcal{C}_2 : \ \mathfrak{d}_{\mathcal{H}_1} = \mathfrak{d}_{\mathcal{H}_2}$,

[4] We use the symbol $\dot\cup$ to emphasise that the union is a union of disjoint sets.

(2) $\exists\, \mathcal{H}_1 \in \mathcal{C}_1, \mathcal{H}_2 \in \mathcal{C}_2,\ f_1 \in \mathcal{H}_1 \setminus \{0\}, f_2 \in \mathcal{H}_2 \setminus \{0\} :\ \frac{f_1}{f_2}$ is a meromorphic function of bounded characteristic in $\mathbb{C}^+$.

$\Diamond$

If two chains have nonempty intersection, then they are admissible for comparison. The converse is not true; see, e.g., Example 3.2 below. Saying that two chains are admissible for comparison means that they have a *chance* to have common parts. Since we are interested in showing that two chains actually do have common parts, we may always restrict our attention to chains which are admissible for comparison.

Also notice that replacing in this definition (all or some) existential quantifiers with universal quantifiers leads to equivalent conditions. In view of 2.16, $(2/c)$, the conditions (1) and (2) are equivalent (e.g.) to the following statements $(1')$ and $(2')$:

$(1')$ $\forall\, \mathcal{H}_1 \in \mathcal{C}_1, \mathcal{H}_2 \in \mathcal{C}_2 :\ \mathfrak{d}_{\mathcal{H}_1} = \mathfrak{d}_{\mathcal{H}_2}$,

$(2')$ $\forall\, \mathcal{H}_1 \in \mathcal{C}_1, \mathcal{H}_2 \in \mathcal{C}_2,\ f_1 \in \mathcal{H}_1 \setminus \{0\}, f_2 \in \mathcal{H}_2 \setminus \{0\} :\ \frac{f_1}{f_2}$ is a meromorphic function of bounded characteristic in $\mathbb{C}^+$.

Let us show by an example that being admissible for comparison is only necessary but not sufficient that two chains have nonempty intersection.

3.2 Example. Let μ be the Lebesgue measure on $\mathbb{R}$, and let $\mathcal{C}$ and $\mathcal{C}'$ be the chains exhibited in Example 2.15, i.e., $\mathcal{C}$ is the chain of Paley-Wiener spaces, and $\mathcal{C}'$ is the chain originating from the Hermite-polynomials multiplied by $e^{-\frac{x^2}{2}}$ as orthonormal basis.

As example for another measure, we consider a space which occurs in number theory (rather than harmonic analysis). Let Ξ be the (upper case) Riemann Ξ-function, and set $E(z) := \Xi(\sqrt{iz}),\ z \in \mathbb{C}$. Then E is a Hermite-Biehler function, and the de Branges space $\mathcal{H}(E)$ generated by E contains $\mathbb{C}[z]$ as a dense subspace, cf. [KW05b, Theorem 3.1, Example 3.2]. Let ν be the measure given by $d\nu(x) := \frac{1}{|E(x)|^2}dx$. It is a consequence of [KW05a, Theorem 2.7] that the set

$$\mathcal{D} := \big\{\mathbb{C}[z]_n : n \in \mathbb{N}\big\} \cup \big\{\mathcal{H}(E(z)e^{-iaz}) : a \geq 0\big\} \tag{3.1}$$

is ordered as $(0 < a < b)$

$$\mathbb{C}[z]_1 \subsetneq \cdots \subsetneq \mathbb{C}[z]_n \subsetneq \mathbb{C}[z]_{n+1} \subsetneq \cdots \subsetneq \mathcal{H}(E(z)) \subsetneq \cdots$$
$$\cdots \subsetneq \mathcal{H}(E(z)e^{-iaz}) \subsetneq \cdots \subsetneq \mathcal{H}(E(z)e^{-ibz}) \subsetneq \cdots$$

and belongs to $\mathsf{Chains}[\nu]$. The chains $\mathcal{C}'$ and $\mathcal{D}$ are not admissible for comparison. The chains $\mathcal{C}$ and $\mathcal{D}$ are admissible for comparison, since all functions $\frac{\sin z}{z-k\pi}$, $k \in \mathbb{Z}$, belong to $\mathcal{P}W_1$, and the function 1 belongs to some (actually every) space in $\mathcal{D}$. However, they are still disjoint since the function 1 is not square integrable along the real axis and hence cannot belong to any Paley-Wiener space. $\Diamond$

3.3 Definition. Let $\nu \in \mathbb{M}_+(\mathbb{R})$ and $\mathcal{C}$ be a chain of de Branges spaces. Then we denote

$$\mathsf{P}[\mathcal{C}, \nu] := \big\{\mathcal{H} \in \mathcal{C} : \rho_\nu(\mathcal{H}) \subseteq L^2(\nu)\big\}.$$

$\Diamond$

Clearly, $\mathsf{P}[\mathcal{C},\nu]$ is a beginning section of $\mathcal{C}$, i.e., if it contains one element of $\mathcal{C}$, then it contains also all smaller elements of $\mathcal{C}$.

Let us inspect some examples which indicate that chains for different measures may be related in various different ways. We focus on the case that one measure is larger than the other.

3.4 Example. Let ν again be the measure $d\nu(x) := \frac{1}{|E(x)|^2}dx$, $E(z) := \Xi(\sqrt{iz})$, which we studied in Example 3.2, and let $\mathcal{D}$ be the chain (3.1). We construct another measure μ in a similar way, namely as

$$d\mu(x) := \frac{1}{|F(x)|^2}dx, \quad F(z) := \prod_{n=1}^{\infty}\Big(1+\frac{z}{in^3}\Big).$$

Then (the ordering is analogous to the ordering in (3.1))

$$\mathcal{C} := \big\{\mathbb{C}[z]_n : n\in\mathbb{N}\big\} \cup \big\{\mathcal{H}(F(z)e^{-iaz}) : a\geq 0\big\} \in \mathsf{Chains}[\mu].$$

We have, by classical function theory,

$$\lim_{x\to\pm\infty}\frac{\ln|E(x)|}{\sqrt{|x|}\ln|x|} = \frac{1}{4\sqrt{2}}, \quad \lim_{x\to\pm\infty}\frac{\ln|F(x)|}{\sqrt[3]{|x|}} = \pi. \tag{3.2}$$

Hence, $\nu\leq\gamma\mu$ for some appropriate constant $\gamma>0$. Moreover, $E(x)(1+|x|)^{-1}\notin L^2(\mu)$, and hence $\mathcal{H}(E)\nsubseteq L^2(\mu)$. We see that $\mathsf{P}[\mathcal{D},\mu] = \{\mathbb{C}[z]_n : n\in\mathbb{N}\}$, and hence

$$\mathcal{D}\supsetneq\mathsf{P}[\mathcal{D},\mu]\subsetneq\mathcal{C} \qquad \big(\mathcal{D}\in\mathsf{Chains}[\nu],\mathcal{C}\in\mathsf{Chains}[\mu],\nu\leq\gamma\mu\big).$$

Let σ be the measure defined as $d\sigma(x) := e^{-|x|}dx$. By the limit relation (3.2) we have $\sigma\leq\gamma\nu$ for some appropriate constant $\gamma>0$. From the classical determinacy condition [Fre69, Satz 5.2] we have $\mathcal{C} := \{\mathbb{C}[z]_n : n\in\mathbb{N}\}\in\mathsf{Chains}[\sigma]$. We see that

$$\mathcal{C} = \mathsf{P}[\mathcal{C},\nu]\subsetneq\mathcal{D} \qquad \big(\mathcal{C}\in\mathsf{Chains}[\sigma],\mathcal{D}\in\mathsf{Chains}[\nu],\sigma\leq\gamma\nu\big).$$

Finally, for an example of two measures having equal chains, it is enough to take any two measures subject to the determinacy condition [Fre69, Satz 5.2]. $\Diamond$

The following result is easy to show, but is a basic fact when it comes to comparing chains. Its core, which is the statement in item (2), says in essence that the situations encountered in Example 3.4 represent all possibilities.

3.5 Theorem. *Let $\mu_1,\mu_2\in\mathbb{M}_+(\mathbb{R})$ be given.*

(1) *Let $\mathcal{C}_1\in\mathsf{Chains}[\mu_1]$, $\mathcal{C}_2\in\mathsf{Chains}[\mu_2]$, and assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ are admissible for comparison. Then either $\mathsf{P}[\mathcal{C}_1;\mu_2]$ is a beginning section of $\mathsf{P}[\mathcal{C}_2;\mu_1]$, or $\mathsf{P}[\mathcal{C}_2;\mu_1]$ is a beginning section of $\mathsf{P}[\mathcal{C}_1;\mu_2]$.*

Assume that there exists $\gamma>0$ such that $\mu_1\leq\gamma\mu_2$, let $\mathcal{C}_1\in\mathsf{Chains}[\mu_1]$, and assume that $\mathsf{P}[\mathcal{C}_1,\mu_2]\neq\emptyset$.

(2) *There exists a unique chain $\mathcal{C}_2\in\mathsf{Chains}[\mu_2]$ which is admissible for comparison with $\mathcal{C}_1$, and $\mathsf{P}[\mathcal{C}_1;\mu_2]$ is a beginning section of $\mathcal{C}_2$.*

(3) *Let* $\mathcal{C}_2 \in \mathsf{Chains}[\mu_2]$ *be admissible for comparison with* $\mathcal{C}_1$*. Then*

$$\sup_{\mathcal{H}\in\mathsf{P}[\mathcal{C}_1;\mu_2]} \nabla_{\langle\mathcal{H},\|.\|_{\mu_2}\rangle}(w) < \infty, \quad w \in \mathbb{C},$$

if and only if either $\mathcal{C}_2 \neq \mathsf{P}[\mathcal{C}_1;\mu_2]$*, or* $\mathcal{C}_2 = \mathsf{P}[\mathcal{C}_1;\mu_2]$ *and* $\mathcal{C}_2$ *contains a maximal element.*

If $\mathcal{C}_2 = \mathsf{P}[\mathcal{C}_1;\mu_2]$ *and* $\mathcal{C}_2$ *does not contain a maximal element, then* $\mathcal{C}_1 = \mathcal{C}_2$*.*

Proof. We start with showing property (2). Consider a space $\mathcal{H} \in \mathsf{P}[\mathcal{C}_1;\mu_2]$. Then, by Lemma 2.12, $\rho_{\mu_2}|_{\mathcal{H}}$ is continuous, i.e., $\|f\|_{\mu_2} \leq \gamma'\|f\|_{\mu_1}$, $f \in \mathcal{H}$, for some $\gamma' > 0$. However, by our assumption $\|f\|_{\mu_1} \leq \gamma\|f\|_{\mu_2}$, $f \in \mathcal{H}$, and hence $\|.\|_{\mu_1}$ and $\|.\|_{\mu_2}$ are equivalent on $\mathcal{H}$. Thus $\mathcal{H} \in \mathsf{Sub}[\mu_2]$ and

$$\{\mathcal{K} \in \mathsf{Sub}[\mu_1] : \mathcal{K} \subseteq \mathcal{H}\} = \{\mathcal{K} \in \mathsf{Sub}[\mu_2] : \mathcal{K} \subseteq \mathcal{H}\}. \tag{3.3}$$

Since $\mathsf{P}[\mathcal{C}_1;\mu_2]$ is a nonempty partial chain, there exists a unique chain $\mathcal{C}_2 \in \mathsf{Chains}[\mu_2]$ with $\mathcal{C}_2 \supseteq \mathsf{P}[\mathcal{C}_1;\mu_2]$. Since (3.3) holds for every space $\mathcal{H} \in \mathsf{P}[\mathcal{C}_1;\mu_2]$, we see that $\mathsf{P}[\mathcal{C}_1;\mu_2]$ is a beginning section of $\mathcal{C}_2$.

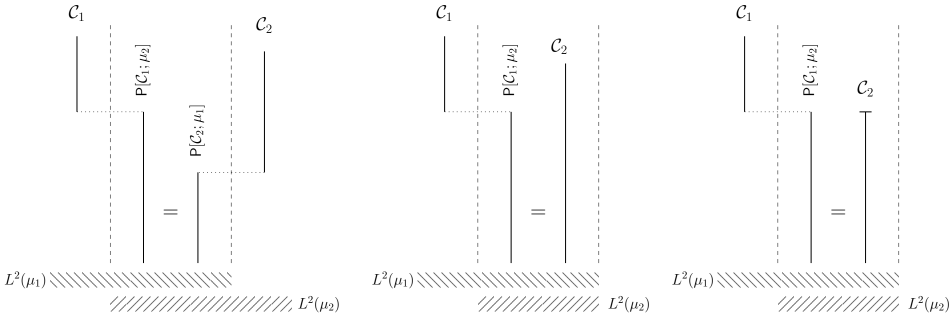
We come to the proof of (1). Introduce the measure $\mu := \mu_1 + \mu_2$. If $\mathsf{P}[\mathcal{C}_1;\mu_2] = \emptyset$ or $\mathsf{P}[\mathcal{C}_2;\mu_1] = \emptyset$, there is nothing to prove. Hence, assume that both sets are nonempty. Clearly, $\mathsf{P}[\mathcal{C}_1,\mu_2] = \mathsf{P}[\mathcal{C}_1,\mu]$ and $\mathsf{P}[\mathcal{C}_2,\mu_1] = \mathsf{P}[\mathcal{C}_2,\mu]$. Choose a space $\mathcal{H}_0 \in \mathsf{P}[\mathcal{C}_1;\mu_2]$. Then, by the already proved item (2), $\mathcal{H}_0 \in \mathsf{Sub}[\mu]$. Let $\mathcal{C} \subseteq \mathsf{Sub}[\mu]$ be the chain which contains $\mathcal{H}_0$. Then $\mathcal{C}_1$ and $\mathcal{C}$ are admissible for comparison. Hence, also $\mathcal{C}_2$ and $\mathcal{C}$ have this property. Item (2) yields that both, $\mathsf{P}[\mathcal{C}_1;\mu_2]$ and $\mathsf{P}[\mathcal{C}_2;\mu_1]$ are beginning sections of $\mathcal{C}$. From this (1) follows.

Finally, let us turn to the proof of (3). If $\mathcal{C}_2 \neq \mathsf{P}[\mathcal{C}_1;\mu_2]$, or $\mathcal{C}_2 = \mathsf{P}[\mathcal{C}_1;\mu_2]$ and $\mathcal{C}_2$ contains a maximal element, then trivially $\sup_{\mathcal{H}\in\mathsf{P}[\mathcal{C}_1;\mu_2]} \nabla_{\langle\mathcal{H},\|.\|_{\mu_2}\rangle}(w) < \infty$. Assume that $\mathcal{C}_2 = \mathsf{P}[\mathcal{C}_1;\mu_2]$ and $\mathcal{C}_2$ does not contain a maximal element. Since the unit ball of a space $\mathcal{H}$ w.r.t. the norm $\|.\|_{\mu_1}$ contains the $\|.\|_{\mu_2}$-ball centered at 0 with radius $\frac{1}{\gamma}$, we have

$$\begin{aligned}\sup_{\mathcal{H}\in\mathsf{P}[\mathcal{C}_1;\mu_2]} \nabla_{\langle\mathcal{H},\|.\|_{\mu_1}\rangle}(w) &\geq \frac{1}{\gamma}\sup_{\mathcal{H}\in\mathsf{P}[\mathcal{C}_1;\mu_2]} \nabla_{\langle\mathcal{H},\|.\|_{\mu_2}\rangle}(w)\\ &= \frac{1}{\gamma}\sup_{\mathcal{H}\in\mathcal{C}_2} \nabla_{\langle\mathcal{H},\|.\|_{\mu_2}\rangle}(w) = \infty.\end{aligned}$$

The asserted equivalence follows. Moreover, since $\mathsf{P}[\mathcal{C}_1;\mu_2]$ is a beginning section of $\mathcal{C}_1$, it follows that $\mathsf{P}[\mathcal{C}_1;\mu_2] = \mathcal{C}_1$. ❑

We may picture the possible situations described in Theorem 3.5 as follows (as examples show all these cases can occur).



Situation in general. | Case $\mu_1 \leq \gamma\mu_2$ | Case $\mu_1 \leq \gamma\mu_2$ (possible only if $\mathcal{C}_2$ has a maximal element)

The case that μ_1 and μ_2 differ only by a compactly supported part deserves particular attention. Let us comment on this situation.

3.6 Remark. Let $\mu_1, \mu_2 \in \mathbb{M}_+(\mathbb{R})$ and assume that $\mu_1 - \mu_2$ is compactly supported. Let $\mathcal{C}_i \in \mathsf{Chains}[\mu_i]$, $i = 1, 2$, be admissible for comparison. Then

$$\mathsf{P}[\mathcal{C}_1, \mu_2] = \mathcal{C}_1 \quad \text{and} \quad \mathsf{P}[\mathcal{C}_2, \mu_1] = \mathcal{C}_2.$$

Theorem 3.5 implies that either $\mathcal{C}_1$ is a beginning section of $\mathcal{C}_2$, or $\mathcal{C}_2$ is one of $\mathcal{C}_1$. If $\mu_1 \leq \gamma\mu_2$ with some $\gamma > 0$, then the first of these cases occurs. ◊

Formulation of the Main Theorem

Let us now state our main theorem, the below Theorem 3.9. In this result we show stability of the section of a chain of de Branges spaces which is defined by a restriction speed of growth of functions when the measure containing the chain is perturbed where the maximal admissible size of the perturbation corresponds to the maximal speed of growth. The proof of Theorem 3.9 needs some more machinery (to be built up in §4) and will be carried out in §5.

To make the meaning of "growth restrictions" precise, we need to introduce an appropriate notation for growth classes, and an appropriate quantification of smallness of a measure.

3.7 Definition. Let λ be a growth function and let $c \in \mathbb{R} \cup \{\infty\}$. We denote by $\mathbb{G}(\lambda, c)$ the set of all algebraic de Branges spaces $\mathcal{H}$ with the property that

$$\limsup_{|z|\to\infty} \frac{\ln|f(z)|}{\lambda(|z|)} < \infty, \ \limsup_{x\to\pm\infty} \frac{\ln|f(x)|}{\lambda(|x|)} \leq c, \quad f \in \mathcal{H} \setminus \{0\}.$$

Observe that for $c = \infty$ the second of these conditions is void. ◊

Note that $\mathbb{G}(\lambda, \infty) = \bigcup_{c\in\mathbb{R}} \mathbb{G}(\lambda, c)$.

3.8 Definition. Let λ be a growth function and let $\mu \in \mathbb{M}_+(\mathbb{R})$. Then we set

$$p(\lambda, \mu) := \sup\Big\{\beta \in \mathbb{R} : \int_{\mathbb{R}} e^{\beta\lambda(|x|)}\, d\mu(x) < \infty\Big\},$$

where the supremum is understood as an element of $\mathbb{R} \cup \{\pm\infty\}$. ◊

One may think of $p(\lambda, \mu)$ as a convergence exponent of μ w.r.t. λ; the number $p(\lambda, \mu)$ being large means that μ decays fast relative to λ.

3.9 Theorem. *Let $\mu_1, \mu_2 \in \mathbb{M}_+(\mathbb{R})$, and let $\mathcal{C}_i \in \mathsf{Chains}[\mu_i]$, $i = 1, 2$, be admissible for comparison. Let λ be a growth function, $c \in \mathbb{R} \cup \{\infty\}$, $\nu \in \mathbb{M}_+(\mathbb{R})$, and assume that*

$$c < \frac{1}{2} p(\lambda, \nu) \text{ or } p(\lambda, \nu) = \infty, \tag{3.4}$$

and that

$$\exists \epsilon \in (0,1) : \ |\mu_1 - \mu_2| \le (1-\epsilon)(\mu_1 + \mu_2) + \nu. \tag{3.5}$$

Then:

(1) *The inclusions $\mathcal{C}_1 \cap \mathbb{G}(\lambda, c) \subseteq \mathsf{P}[\mathcal{C}_1; \mu_2]$ and $\mathcal{C}_2 \cap \mathbb{G}(\lambda, c) \subseteq \mathsf{P}[\mathcal{C}_2; \mu_1]$ hold.*

(2) *Either $\mathcal{C}_1 \cap \mathbb{G}(\lambda, c)$ is a beginning section of $\mathcal{C}_2 \cap \mathbb{G}(\lambda, c)$, or $\mathcal{C}_2 \cap \mathbb{G}(\lambda, c)$ is a beginning section of $\mathcal{C}_1 \cap \mathbb{G}(\lambda, c)$.*

(3) *Moreover, if $\mathcal{C}_1 \cap \mathbb{G}(\lambda, c) \subsetneq \mathcal{C}_2 \cap \mathbb{G}(\lambda, c)$, then:*

   (*a*) $\mathcal{C}_1 = \mathsf{P}[\mathcal{C}_1; \mu_2] = \mathcal{C}_1 \cap \mathbb{G}(\lambda, c)$.

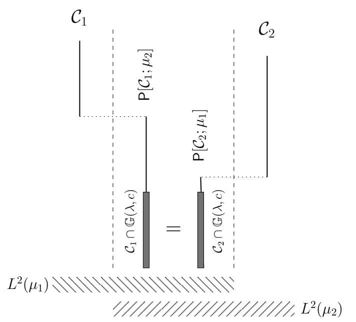
   (*b*) *There exist $N \in \mathbb{N} \cup \{\infty\}$, and de Branges spaces $\mathcal{H}_n$, $n \in \mathbb{N}_0$, $n < N$, with $\mathcal{H}_{n-1} \subsetneq \mathcal{H}_n$ and $\dim \mathcal{H}_n / \mathcal{H}_{n-1} = 1$, $n \in \mathbb{N}$, $n < N$, such that*

$$\mathcal{C}_2 \cap \mathbb{G}(\lambda, c) = \mathcal{C}_1 \dot{\cup} \big\{ \mathcal{H}_n : n \in \mathbb{N}_0, n < N \big\}. \tag{3.6}$$

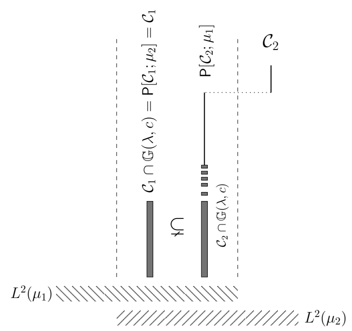
   (*c*) *There exists a finitely supported positive Borel measure on the real line with*

$$\sup_{\mathcal{H} \in \mathcal{C}} \nabla_{\langle \mathcal{H}, \|\cdot\|_{\mu + \mu_0} \rangle}(w) < \infty, \quad w \in \mathbb{C}. \tag{3.7}$$

The assertions analogous to $(3, a\text{-}c)$ hold when $\mathcal{C}_2 \cap \mathbb{G}(\lambda, c) \subsetneq \mathcal{C}_1 \cap \mathbb{G}(\lambda, c)$.



Equality of section.



Proper inclusion.

Remember in this place that two chains with nonempty intersection are certainly admissible for comparison.

Although the assumptions and conclusions of Theorem 3.9 are fully symmetric in μ_1 and μ_2, one may sometimes take a perturbation viewpoint, and consider one of the given measures as the unperturbed one, and the other as the perturbed one.

Let us demonstrate Theorem 3.9 with a concrete example.

3.10 Example. Let μ_1 be the measure $d\mu_1(x) := \frac{1}{|\Xi(\sqrt{ix})|^2}dx$ investigated in Examples 3.2 and 3.4, and let $\mathcal{C}_1$ be the chain (3.1). Let ν be the measure defined by

$$d\nu(x) := e^{-\sqrt{|x|}\ln^+|x|}dx,$$

let μ_2 be any measure with

$$|\mu_1 - \mu_2| \leq \nu,$$

and let $\mathcal{C}_2 \in \mathsf{Chains}[\mu_2]$ be the chain with $1 \in \mathcal{H} + z\mathcal{H}$, $\mathcal{H} \in \mathcal{C}_2$. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are admissible for comparison.

We use a growth function λ which is (for sufficiently large values of r) equal to $\sqrt{r}\ln r$, and $c := \frac{1}{4}$. This data obviously satisfies (3.4), in fact, $p(\lambda, \nu) = 1$.

The chain $\mathcal{C}_1$ can be analysed further. Due to the asymptotics of the function Ξ, cf. (3.2), we have $\mathcal{H}(\Xi(\sqrt{iz})) \in \mathbb{G}(\lambda, c)$. By [KW05a, Theorem 2.7], the space $\mathcal{H}(\Xi(\sqrt{iz})e^{-iaz})$, $a > 0$, can be written as

$$\mathcal{H}\big(\Xi(\sqrt{iz})e^{-iaz}\big) = \Xi(\sqrt{iz})\mathcal{P}W_a \oplus e^{iaz}\mathcal{H}(\Xi(\sqrt{iz})).$$

Thus, each such space contains functions of positive exponential type. Moreover, for each $f \in \mathcal{H}(\Xi(\sqrt{iz})e^{-iaz})$, we have $\int_{-\infty}^{\infty} |f(x)|^2 \cdot e^{-\sqrt{|x|}\ln^+|x|}\,dx < \infty$. Together, this leads to

$$\mathcal{C}_1 \cap \mathbb{G}(\lambda, c) = \big\{\mathbb{C}[z]_n : n \in \mathbb{N}\big\} \cup \big\{\mathcal{H}(\Xi(\sqrt{iz}))\big\} \subsetneq \mathsf{P}[\mathcal{C}_1; \mu_2] = \mathcal{C}_1.$$

Since both measures μ_1 and μ_2 are not discrete, there cannot exist a finitely supported measure with (3.7), see Corollary 4.7 below. Theorem 3.9 now yields that

$$\mathcal{C}_1 \cap \mathbb{G}(\lambda, c) = \mathcal{C}_2 \cap \mathbb{G}(\lambda, c).$$

$\Diamond$

4 The index of a chain

Let $\mu \in \mathbb{M}_+(\mathbb{R})$ and $\mathcal{C} \in \mathsf{Chains}[\mu]$. Then $\rho_\mu\big(\bigcup_{\mathcal{H}\in\mathcal{C}} \mathcal{H}\big)$ is dense in $L^2(\mu)$. This density property may or may not be stable under finite-dimensional perturbations of μ. For spaces of polynomials this is known from the theory of moment problems under the name of infinite- or finite index of determinacy, cf. [BD95], and for the spaces $\mathcal{E}(a)$ (from Example 2.7) related with the type of a measure under the name of stable- or unstable density, cf. [BS11]. However, it is a general phenomenon. Since it plays an important role in the subsequent considerations (as well as in other contexts), we make the effort to undertake a systematic study.

For illustration, let us first provide two examples (based on Paley-Wiener spaces).

4.1 Example. Let μ be the Lebesgue measure, and let $\mathcal{C} \in \mathsf{Chains}[\mu]$ be the chain of Paley-Wiener spaces. Let μ_0 be a positive and finitely supported measure, and let $\mathcal{C}_0 \in \mathsf{Chains}[\mu + \mu_0]$ be the chain which contains $\mathcal{C}$, cf. Remark 3.6. Since $\bigcup_{a>0}\mathcal{P}W_a$ contains functions of arbitrary large exponential type, we must have $\mathcal{C}_0 \setminus \mathcal{C} = \emptyset$. Hence, $\bigcup_{a>0}\mathcal{P}W_a$ is dense in $L^2(\mu + \mu_0)$. $\Diamond$

4.2 Example. Denote by δ_x the unit point-mass concentrated at the point x, and consider the measure $\mu := \sum_{n=-\infty}^{\infty} \delta_n$. In the Paley-Wiener space $\mathcal{P}W_\pi$ we have the complete orthogonal system

$$\Big\{\frac{\sin(\pi(z-n))}{z-n} : n \in \mathbb{Z}\Big\}.$$

The Parseval-identity yields that $\rho_\mu : \mathcal{P}W_\pi \to L^2(\mu)$ is an isometric isomorphism. Thus, we have the chain

$$\mathcal{C} := \big\{\mathcal{P}W_a : 0 < a \leq \pi\big\} \in \mathsf{Chains}[\mu].$$

Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, and consider the measure $\mu + \delta_\alpha$. Let g be the function

$$g(x) := \begin{cases} \frac{\sin(\pi(x-\alpha))}{\pi(x-\alpha)} \;, & x \in \mathbb{Z} \\ -1 \qquad\quad , & x = \alpha \end{cases}$$

Then g belongs to $L^2(\mu+\delta_\alpha)$ and is orthogonal to all functions in $\rho_{\mu+\delta_\alpha}(\mathcal{P}W_\pi)$. Thus $\bigcup_{0<a\leq\pi} \mathcal{P}W_a$ fails to be dense in $L^2(\mu+\delta_\alpha)$. $\Diamond$

We come to the general definition of the index of a chain. Denote by $\mathbb{M}_+^{\mathrm{fin}}(\mathbb{R})$ the set of all finitely supported positive Borel measures on $\mathbb{R}$.

4.3 Definition. Let $\mu \in \mathbb{M}_+(\mathbb{R})$ and $\mathcal{C} \in \mathsf{Chains}[\mu]$. Then we denote by $\mathcal{M}(\mu, \mathcal{C})$ the (possibly empty) set of all measures $\mu_0 \in \mathbb{M}_+^{\mathrm{fin}}(\mathbb{R})$ with the property that

$$\nabla_{\mathcal{C},\mu_0}(w) := \sup_{\mathcal{H}\in\mathcal{C}} \nabla_{\langle \mathcal{H}, \|\cdot\|_{\mu+\mu_0}\rangle}(w) < \infty, \quad w \in \mathbb{C}. \tag{4.1}$$

We define the *index of the chain* $\mathcal{C}$ as

$$\operatorname{ind}\mathcal{C} := \inf\{|\operatorname{supp}\mu_0| : \mu_0 \in \mathcal{M}(\mu,\mathcal{C})\} \in [0,\infty].$$

Moreover, for $\mu_0 \in \mathcal{M}(\mu, \mathcal{C})$, we denote

$$\delta(\mu_0) := \big|\{x \in \operatorname{supp}\mu_0 : \mu(\{x\}) = 0\}\big|.$$

$\Diamond$

Observe that

$$\nabla_{\mathcal{C},\mu_0} = \nabla_{\langle \bigcup_{\mathcal{H}\in\mathcal{C}} \mathcal{H}, \|\cdot\|_{\mu+\mu_0}\rangle}.$$

Hence, finiteness of $\nabla_{\mathcal{C},\mu_0}$ means that the closure of the algebraic de Branges space $\bigcup_{\mathcal{H}\in\mathcal{C}} \mathcal{H}$ inside $L^2(\mu+\mu_0)$ is a de Branges space.

By 2.16, (4), a chain $\mathcal{C}$ has index 0 if and only if it contains a maximal element. This suggests that chains with finite index may be viewed as a generalisation of chains having a maximal element. And indeed, it is the case that the property to have finite index has in many respects quite similar consequences as having a maximal element.

4.4 Remark. Let $\mu_0 \in \mathbb{M}_+(\mathbb{R})$ and set $\Delta := \{x \in \mathbb{R} : \mu(\{x\}) = 0\}$. If $\mu_0 \in \mathbb{M}_+^{\mathrm{fin}}(\mathbb{R})$, then we find a constant $\gamma > 0$ such that (here $\mathbb{1}_\Delta$ denotes the indicator function of the set Δ)

$$\mu + \mathbb{1}_\Delta\mu_0 \leq \mu + \mu_0 \leq \gamma(\mu + \mathbb{1}_\Delta\mu_0).$$

In particular, the measure μ_0 satisfies (4.1) if and only if $\mathbb{1}_\Delta \mu_0$ does. Moreover, $\delta(\mu_0) = \delta(\mathbb{1}_\Delta \mu_0) = |\operatorname{supp} \mathbb{1}_\Delta \mu_0|$. We conclude that

$$\operatorname{ind} \mathcal{C} = \inf\{\delta(\mu_0) : \mu_0 \in \mathcal{M}(\mu, \mathcal{C})\} = \inf\{|\operatorname{supp} \mu_0| : \mu_0 \in \mathcal{M}(\mu, \mathcal{C}), \mu_0 \perp \mu\}.$$

$\Diamond$

With a chain $\mathcal{C} \in \mathsf{Chains}[\mu]$ there is associated a family of (possibly) larger chains. Theorem 3.5 and Remark 3.6 justify the following definition.

4.5 Definition. Let $\mu \in \mathbb{M}_+(\mathbb{R})$, $\mathcal{C} \in \mathsf{Chains}[\mu]$, and let $\mu_0 \in \mathbb{M}_+^{\text{fin}}(\mathbb{R})$. Then we denote by $\mathcal{C}_{\mu_0}$ the unique element of $\mathsf{Chains}[\mu + \mu_0]$ with $\mathcal{C} \subseteq \mathcal{C}_{\mu_0}$. $\Diamond$

Appealing to 2.16, (4), we may say that $\mu_0 \in \mathcal{M}(\mu, \mathcal{C})$ if and only if $\mathcal{C}_{\mu_0}$ has a maximal element.

Our first step towards understanding the structure of the set $\mathcal{M}(\mu, \mathcal{C})$ is the following lemma.

4.6 Lemma. *Let $\mu \in \mathbb{M}_+(\mathbb{R})$, $\mathcal{C} \in \mathsf{Chains}[\mu]$, and let $\mu_0 \in \mathcal{M}(\mu, \mathcal{C})$. Denote $\mathcal{L} := \bigcup_{\mathcal{H} \in \mathcal{C}} \mathcal{H}$. Then*

$$\dim \Big(L^2(\mu + \mu_0) \Big/ \operatorname{Clos} \rho_{\mu + \mu_0}(\mathcal{L}) \Big) \leq \delta(\mu_0).$$

Proof. Let P_0 be the map

$$P_0 : \begin{cases} L^2(\mu + \mu_0) & \to & L^2(\mu) \\ [f]_{\mu + \mu_0} & \mapsto & [f]_\mu \end{cases} \tag{4.2}$$

Then P_0 is contractive, surjective, and

$$\ker P_0 = \operatorname{span} \big\{ [\mathbb{1}_{\{x\}}]_{\mu + \mu_0} : x \in \operatorname{supp} \mu_0, \mu(\{x\}) = 0 \big\}.$$

In particular, $\dim \ker P_0 = \delta(\mu_0) < \infty$. Clearly, $\rho_\mu = P_0 \circ \rho_{\mu + \mu_0}$.

Since $\mathcal{C} \in \mathsf{Chains}[\mu]$, the set $\rho_\mu(\mathcal{L})$ is dense in $L^2(\mu)$. Hence, $P_0^{-1}(\rho_\mu(\mathcal{L}))$ is dense in $L^2(\mu + \mu_0)$. However, we have

$$P_0^{-1}(\rho_\mu(\mathcal{L})) = P_0^{-1}\big(P_0(\rho_{\mu + \mu_0}(\mathcal{L}))\big) = \rho_{\mu + \mu_0}(\mathcal{L}) + \ker P_0,$$

and using that $\dim \ker P_0 < \infty$, we thus obtain

$$L^2(\mu + \mu_0) = \operatorname{Clos} \big[P_0^{-1}(\rho_\mu(\mathcal{L})) \big] = \operatorname{Clos} \big[\rho_{\mu + \mu_0}(\mathcal{L}) \big] + \ker P_0. \tag{4.3}$$

❑

4.7 Corollary. *Let $\mu \in \mathbb{M}_+(\mathbb{R})$ and $\mathcal{C} \in \mathsf{Chains}[\mu]$. If $\operatorname{ind} \mathcal{C} < \infty$, then μ is discrete.*

Proof. The set $\mathcal{M}(\mu, \mathcal{C})$ is nonempty. By the above lemma, the chain $\mathcal{C}_{\mu_0}$ contains an element with finite codimension in $L^2(\mu + \mu_0)$. Therefore it must have a maximal element. Lemma 2.18 implies that $\mu + \mu_0$ is discrete, and hence μ also is. ❑

For $\mu_0 \in \mathcal{M}(\mu, \mathcal{C})$ we set

$$d(\mu_0) := \delta(\mu_0) - \dim \Big(L^2(\mu + \mu_0) \Big/ \operatorname{Clos} \rho_{\mu + \mu_0}(\mathcal{L}) \Big).$$

By Lemma 4.6, the number $d(\mu_0)$ is a well-defined nonnegative integer.

4.8 Proposition. *Let $\mu \in \mathbb{M}_+(\mathbb{R})$ and $\mathcal{C} \in \mathsf{Chains}[\mu]$. Then the number $d(\mu_0)$ is independent of $\mu_0 \in \mathcal{M}(\mu, \mathcal{C})$.*

Proof. If $\mathcal{M}(\mu, \mathcal{C}) = \emptyset$, there is nothing to prove. Hence, assume that $\mathcal{M}(\mu, \mathcal{C})$ is nonempty. Then, in particular, μ is discrete.

Let $\mu_1, \mu_2 \in \mathcal{M}(\mu, \mathcal{C})$ and let P_1, P_2 be the respective maps (4.2). Moreover, to shorten notation, set $\rho_i := \rho_{\mu+\mu_i}$, $i = 1, 2$. Note that, clearly,

$$P_1 \circ \rho_1 = P_2 \circ \rho_2.$$

Denote again $\mathcal{L} := \bigcup_{\mathcal{H} \in \mathcal{C}} \mathcal{H}$. Then, for each $f \in \mathcal{L}$,

$$\begin{aligned} \|f\|^2_{\mu+\mu_2} &= \int_{\mathbb{R}} |f|^2 \, d\mu + \sum_{x \in \operatorname{supp} \mu_2} |f(x)|^2 \mu_2(\{x\}) \\ &\leq \Big(1 + \sum_{x \in \operatorname{supp} \mu_2} \nabla_{\mathcal{C},\mu_1}(x)^2 \mu_2(\{x\})\Big) \|f\|^2_{\mu+\mu_1}, \end{aligned}$$

$$\begin{aligned} \|f\|^2_{\mu+\mu_1} &= \int_{\mathbb{R}} |f|^2 \, d\mu + \sum_{x \in \operatorname{supp} \mu_1} |f(x)|^2 \mu_1(\{x\}) \\ &\leq \Big(1 + \sum_{x \in \operatorname{supp} \mu_1} \nabla_{\mathcal{C},\mu_2}(x)^2 \mu_1(\{x\})\Big) \|f\|^2_{\mu+\mu_2}. \end{aligned}$$

These relations imply that the set

$$\big\{(\rho_1 f, \rho_2 f) \in L^2(\mu + \mu_1) \times L^2(\mu + \mu_2) : f \in \mathcal{L}\big\} \tag{4.4}$$

is the graph of a bicontinuous bijection ψ_0 of $\rho_1(\mathcal{L})$ onto $\rho_2(\mathcal{L})$.

Let ψ be the extension by continuity of ψ_0. Then ψ is a bicontinuous bijection of $\operatorname{Clos} \rho_1(\mathcal{L})$ onto $\operatorname{Clos} \rho_2(\mathcal{L})$. Since the graph of ψ contains (4.4), we have

$$\rho_2 = \psi \circ \rho_1.$$

It follows that $P_1 \circ \rho_1 = P_2 \circ \rho_2 = P_2 \circ \psi \circ \rho_1$, and by continuity that

$$P_1|_{\operatorname{Clos} \rho_1(\mathcal{L})} = P_2 \circ \psi.$$

In the same way, we obtain $P_2|_{\operatorname{Clos} \rho_2(\mathcal{L})} = P_1 \circ \psi^{-1}$. Together these relations yield

$$\psi\big([\operatorname{Clos} \rho_1(\mathcal{L})] \cap \ker P_1\big) = [\operatorname{Clos} \rho_2(\mathcal{L})] \cap \ker P_2.$$

Since ψ is injective, in particular,

$$\dim \big([\operatorname{Clos} \rho_1(\mathcal{L})] \cap \ker P_1\big) = \dim \big([\operatorname{Clos} \rho_2(\mathcal{L})] \cap \ker P_2\big). \tag{4.5}$$

As we saw in (4.3),

$$[\operatorname{Clos} \rho_i(\mathcal{L})] + \ker P_i = L^2(\mu + \mu_i), \quad i = 1, 2,$$

and we conclude that

$$\dim \Big(L^2(\mu + \mu_i) \Big/ \operatorname{Clos} \rho_i(\mathcal{L})\Big) = \underbrace{\dim \ker P_i}_{=\delta(\mu_i)} - \dim \big([\operatorname{Clos} \rho_i(\mathcal{L})] \cap \ker P_i\big).$$

Now (4.5) implies that $d(\mu_1) = d(\mu_2)$. ❑

Our next aim is to investigate the chains $\mathcal{C}_{\mu_0}$ for $\mu_0 \in \mathcal{M}(\mu, \mathcal{C})$ more closely. A crucial result in this context is the following general fact.

4.9 Proposition. *Let $\mu \in \mathbb{M}_+(\mathbb{R})$ be a discrete measure, and let Δ be a finite subset of* $\operatorname{supp}\mu$. *If $\mathcal{C} \in \mathsf{Chains}[\mu]$, then*

$$\Big\{\mathcal{H} \in \mathcal{C} : \dim\Big(L^2(\mu)\Big/\operatorname{Clos}\rho_\mu(\mathcal{H})\Big) \geq |\Delta|\Big\} \in \mathsf{Chains}[\mathbb{1}_{\Delta^c}\mu].$$

Proof. We shall settle the case of a 1-element subset Δ. The general assertion then follows by repeated application of this case.

First, we make a preliminary notice. Let $x \in \operatorname{supp}\mu$, then the space $\{[f]_\mu \in L^2(\mu) : f(x) = 0\}$ is a closed and proper subspace of $L^2(\mu)$. Since $\bigcup_{\mathcal{H}\in\mathcal{C}} \mathcal{H}$ is dense in $L^2(\mu)$, there must exist $\mathcal{H}_x \in \mathcal{C}$ and $f_x \in \mathcal{H}_x$ with $f_x(x) \neq 0$. This shows that $\mathfrak{d}_{\mathcal{H}_x}(x) = 0$. Hence, for all $\mathcal{H} \in \mathcal{C}$ we have $\mathfrak{d}_{\mathcal{H}}(x) = 0$. The point x was arbitrary in $\operatorname{supp}\mu$, and we infer that

$$\mathfrak{d}_{\mathcal{H}}(x) = 0, \quad \mathcal{H} \in \mathcal{C}, x \in \operatorname{supp}\mu.$$

Assume now that we are given a point $x_0 \in \operatorname{supp}\mu$ and consider the subset $\Delta := \{x_0\}$. Let P be the map

$$P : \begin{cases} L^2(\mu) & \to & L^2(\mathbb{1}_{\Delta^c}\mu) \\ [f]_\mu & \mapsto & [f]_{\mathbb{1}_{\Delta^c}\mu} \end{cases}$$

then P is a contractive surjection and

$$\ker P = \big\{[f]_\mu : f(x) = 0, x \in [\operatorname{supp}\mu] \setminus \{x_0\}\big\} = \operatorname{span}\{[\mathbb{1}_\Delta]_\mu\}.$$

As a continuous surjection with finite-dimensional kernel, P maps closed subspaces to closed subspaces. Moreover, clearly, $P \circ \rho_\mu = \rho_{\mathbb{1}_{\Delta^c}\mu}$.

The next step in the proof is to show that for each space $\mathcal{H} \in \mathcal{C}$ with $\rho_\mu(\mathcal{H}) \cap \ker P \neq \{0\}$ it holds that $\rho_\mu(\mathcal{H}) = L^2(\mu)$. Assume that $\mathcal{H} \in \mathcal{C}$ and that $\mathcal{H}$ contains a function g_{x_0} with $\rho_\mu g_{x_0} = [\mathbb{1}_\Delta]_\mu$. This means that

$$g_{x_0}(x) = \begin{cases} 0 \ , & x \in [\operatorname{supp}\mu] \setminus \{x_0\} \\ 1 \ , & x = x_0 \end{cases}$$

For $x \in [\operatorname{supp}\mu] \setminus \{x_0\}$, set $d_x := \mathfrak{d}_{g_{x_0}}(x)$ and consider the function

$$g_x(z) := \frac{d_x!}{g_{x_0}^{(d_x)}(x)}(z - x_0)\frac{g_{x_0}(z)}{(z-x)^{d_x}}.$$

Then $g_x \in \mathcal{H}$ and $\rho_\mu g_x = [\mathbb{1}_{\{x\}}]_\mu$. We conclude that $\rho_\mu(\mathcal{H})$ contains all finitely supported functions in $L^2(\mu)$, and hence is dense in $L^2(\mu)$. However, $\rho_\mu(\mathcal{H})$ is closed in $L^2(\mu)$, and thus indeed $\rho_\mu(\mathcal{H}) = L^2(\mu)$.

Now let $\mathcal{H} \in \mathcal{C}$ with $\dim(L^2(\mu)/\rho_\mu(\mathcal{H})) \geq 1$ be given. By what we showed above, $\rho_\mu(\mathcal{H}) \cap \ker P = \{0\}$, and hence $P|_{\rho_\mu(\mathcal{H})}$ maps $\rho_\mu(\mathcal{H})$ bijectively onto some closed subspace of $L^2(\mathbb{1}_{\Delta^c}\mu)$. By the Open Mapping Theorem, $P|_{\rho_\mu(\mathcal{H})}$ is bicontinuous. This yields that $\rho_{\mathbb{1}_{\Delta^c}\mu}$ maps $\mathcal{H}$ injectively onto a closed subspace of $L^2(\mathbb{1}_{\Delta^c}\mu)$, and that

$$\nabla_{\langle\mathcal{H},\|.\|_{\mathbb{1}_{\Delta^c}\mu}\rangle}(w) \leq \|(P|_{\mathcal{H}})^{-1}\|\nabla_{\langle\mathcal{H},\|.\|_\mu\rangle}(w) < \infty, \quad w \in \mathbb{C}.$$

Thus $\mathcal{H} \in \mathsf{Sub}[\mathbb{1}_{\Delta^c}\mu]$. We see that $\mathcal{C}_1 := \{\mathcal{H} \in \mathcal{C} : \dim(L^2(\mu)/\rho_\mu(\mathcal{H})) \geq 1\}$ is a partial chain for $\mathbb{1}_{\Delta^c}\mu$.

Let $\mathcal{C}_0 \in \mathsf{Chains}[\mathbb{1}_{\Delta^c}\mu]$ be the chain with $\mathcal{C}_0 \supseteq \mathcal{C}_1$. By Remark 3.6, we have $\mathcal{C}_0 \subseteq \mathcal{C}$. The set $\mathcal{C} \setminus \mathcal{C}_1$ contains at most one element: namely, $\mathcal{C} \setminus \mathcal{C}_1$ contains the maximal element of $\mathcal{C}$ if it exists, and is empty otherwise. In the second case, we already have $\mathcal{C}_0 = \mathcal{C}_1$. Assume $\mathcal{C}$ has a maximal element, say $\mathcal{H}$. Then $\rho_\mu(\mathcal{H}) = L^2(\mu)$, and hence $\rho_{\mathbb{1}_{\Delta^c}\mu}|_{\mathcal{H}}$ is not injective. Hence, $\mathcal{H} \notin \mathsf{Sub}[\mathbb{1}_{\Delta^c}\mu]$, and we conclude again that $\mathcal{C}_0 = \mathcal{C}_1$. ❑

Using Proposition 4.8 and Proposition 4.9 we can establish the main result of this section.

4.10 Theorem. *Let $\mu \in \mathbb{M}_+(\mathbb{R})$ and $\mathcal{C} \in \mathsf{Chains}[\mu]$. Then the following statements hold.*

(1) $\mathcal{M}(\mu, \mathcal{C}) = \{\mu_0 \in \mathbb{M}_+^{\text{fin}}(\mathbb{R}) : \delta(\mu_0) \geq \operatorname{ind}\mathcal{C}\}$.

(2) *For each $\mu_0 \in \mathcal{M}(\mu, \mathcal{C})$ we have*

$$\dim\Big(L^2(\mu+\mu_0)\Big/\operatorname{Clos}\rho_{\mu+\mu_0}\Big(\bigcup_{\mathcal{H}\in\mathcal{C}}\mathcal{H}\Big)\Big) = \delta(\mu_0) - \operatorname{ind}\mathcal{C}.$$

(3) *Assume that $\operatorname{ind}\mathcal{C} = 0$, and let $\mu_0 \in \mathcal{M}_+^{\text{fin}}(\mathbb{R})$. Denote the maximal element of $\mathcal{C}$ by $\mathcal{H}_0$. Then there exist de Branges spaces $\mathcal{H}_1, \ldots, \mathcal{H}_{\delta(\mu_0)}$ with $\mathcal{H}_{i-1} \subsetneq \mathcal{H}_i$, $\dim \mathcal{H}_i/\mathcal{H}_{i-1} = 1$, $i = 1, \ldots, \delta(\mu_0)$, such that*

$$\mathcal{C}_{\mu_0} = \mathcal{C}\dot\cup\big\{\mathcal{H}_1, \ldots, \mathcal{H}_{\delta(\mu_0)}\big\}.$$

(4) *Assume that $0 < \operatorname{ind}\mathcal{C} < \infty$, and let $\mu_0 \in \mathcal{M}(\mu, \mathcal{C})$. Then there exist de Branges spaces $\mathcal{H}_0, \ldots, \mathcal{H}_{\delta(\mu_0)-\operatorname{ind}\mathcal{C}}$ with $\mathcal{H}_{i-1} \subsetneq \mathcal{H}_i$, $\dim \mathcal{H}_i/\mathcal{H}_{i-1} = 1$, $i = 1, \ldots, \delta(\mu_0) - \operatorname{ind}\mathcal{C}$, such that*

$$\mathcal{C}_{\mu_0} = \mathcal{C}\dot\cup\big\{\mathcal{H}_0, \ldots, \mathcal{H}_{\delta(\mu_0)-\operatorname{ind}\mathcal{C}}\big\},$$

$$\mathcal{H}_0 = \operatorname{Clos}\bigcup_{\mathcal{H}\in\mathcal{C}}\mathcal{H}, \qquad \dim \mathcal{H}_0/\mathcal{H} = \infty,\ \mathcal{H} \in \mathcal{C}.$$

(5) *Let $\mu_0 \in \mathbb{M}_+^{\text{fin}}(\mathbb{R}) \setminus \mathcal{M}(\mu, \mathcal{C})$. Then we have $\mathcal{C}_{\mu_0} = \mathcal{C}$, in particular,*

$$\operatorname{Clos}\rho_{\mu+\mu_0}\Big(\bigcup_{\mathcal{H}\in\mathcal{C}}\mathcal{H}\Big) = L^2(\mu+\mu_0).$$

Item (5) is easy to see.

Proof (of Theorem 4.10, (5)). If $\mathcal{C}_{\mu_0} \supsetneq \mathcal{C}$, then obviously (4.1) holds for μ_0. Thus, if $\mu_0 \in \mathbb{M}_+^{\text{fin}}(\mathbb{R}) \setminus \mathcal{M}(\mu, \mathcal{C})$, we must have $\mathcal{C}_{\mu_0} = \mathcal{C}$. In particular, $\bigcup_{\mathcal{H}\in\mathcal{C}}\mathcal{H} = \bigcup_{\mathcal{H}\in\mathcal{C}_{\mu_0}}\mathcal{H}$, and hence $\rho_{\mu+\mu_0}\big(\bigcup_{\mathcal{H}\in\mathcal{C}}\mathcal{H}\big)$ is dense in $L^2(\mu+\mu_0)$. ❑

Next, we settle the case that $\operatorname{ind}\mathcal{C}$ equals zero.

Proof (of Theorem 4.10*, Case* $\operatorname{ind}\mathcal{C} = 0$*).* Assume that $\mathcal{C}$ has index zero. Then $\mathcal{C}$ has a maximal element; denote this space as $\mathcal{H}_0$. Since $\nabla_{\mathcal{C},\mu_0}(w) \geq \nabla_{\mathcal{C},\mu_1}(w)$ whenever $\mu_0 \leq \mu_1$, it follows that $\mathcal{M}(\mu, \mathcal{C}) = \mathbb{M}_+^{\mathrm{fin}}(\mathbb{R})$. This is (1). For the zero measure we have

$$d(0) = \underbrace{\delta(0)}_{=0} - \dim\Big(L^2(\mu)\Big/\operatorname{Clos}\rho_\mu\Big(\underbrace{\bigcup_{\mathcal{H}\in\mathcal{C}}\mathcal{H}}_{=\mathcal{H}_0}\Big)\Big) = 0.$$

By Proposition 4.8, thus

$$\delta(\mu_0) - \dim\Big(L^2(\mu+\mu_0)\Big/\operatorname{Clos}\rho_{\mu+\mu_0}\mathcal{H}_0\Big) = d(\mu_0) = d(0) = 0, \quad \mu_0 \in \mathbb{M}_+^{\mathrm{fin}}(\mathbb{R}).$$

This is (2). Item (3) follows immediately, remember Remark 2.19. ❑

The case that $\operatorname{ind}\mathcal{C}$ is a finite positive number is the most involved one.

Proof (of Theorem 4.10*, Case* $0 < \operatorname{ind}\mathcal{C} < \infty$*).* Let us fix a measure $\lambda \in \mathcal{M}(\mu, \mathcal{C})$ with $\lambda \perp \mu$ and $|\operatorname{supp}\lambda| = \operatorname{ind}\mathcal{C}$. Moreover, set again $\mathcal{L} := \bigcup_{\mathcal{H}\in\mathcal{C}}\mathcal{H}$.

In the first part of this proof we investigate the chain $\mathcal{C}_\lambda$. This chain has a maximal element; denote it as $\mathcal{H}_0$. Our aim is to show that $\rho_{\mu+\lambda}(\mathcal{L})$ is dense in $L^2(\mu+\lambda)$. Assume the contrary, then by Lemma 4.6

$$0 < \dim\Big(L^2(\mu+\lambda)\Big/\operatorname{Clos}\rho_{\mu+\lambda}(\mathcal{L})\Big) \leq |\operatorname{supp}\lambda| < \infty.$$

Hence, the chain $\mathcal{C}_\lambda$ contains an element $\mathcal{H}_1$ with $\dim \mathcal{H}_0/\mathcal{H}_1 = 1$. Choose $x_0 \in \operatorname{supp}\lambda$, then by Proposition 4.9 the space $\mathcal{H}_1$ is the maximal element of the chain $\mathcal{C}_{\mathbb{1}_{\{x_0\}^c}\lambda}$. This implies that $\mathbb{1}_{\{x_0\}^c}\lambda \in \mathcal{M}(\mu, \mathcal{C})$, and we have reached a contradiction.

Since $\operatorname{ind}\mathcal{C} > 0$, the chain $\mathcal{C}$ has no maximal element, and hence $\dim(L^2(\mu)/\rho_\mu(\mathcal{H})) = \infty$ for all $\mathcal{H} \in \mathcal{C}$. Since $\rho_{\mu+\lambda}(\mathcal{H}_0) = L^2(\mu+\lambda)$, also $\rho_\mu(\mathcal{H}_0) = L^2(\mu)$, and it follows that

$$\dim(\mathcal{H}_0/\mathcal{H}) \geq \dim(L^2(\mu)/\rho_\mu(\mathcal{H})) = \infty, \quad \mathcal{H} \in \mathcal{C}.$$

In the second part of the proof we establish (1). By Remark 4.4 it is enough to show that

$$\big\{\mu_0 \in \mathbb{M}_+^{\mathrm{fin}}(\mathbb{R}) : \mu_0 \perp \mu, |\operatorname{supp}\mu_0| \geq \operatorname{ind}\mathcal{C}\big\} \subseteq \mathcal{M}(\mu, \mathcal{C}),$$

and that (2) and (4) hold for measures $\mu_0 \in \mathcal{M}(\mu, \mathcal{C})$ with $\mu_0 \perp \mu$.

Let $\mu_0 \in \mathbb{M}_+^{\mathrm{fin}}(\mathbb{R})$ with $\mu_0 \perp \mu$ and $|\operatorname{supp}\mu_0| \geq \operatorname{ind}\mathcal{C}$ be given, and set

$$\nu := \mathbb{1}_{(\operatorname{supp}\lambda)^c}\mu_0.$$

Consider the chain $\mathcal{C}_\lambda \in \mathsf{Chains}[\mu+\lambda]$. Then $\operatorname{ind}\mathcal{C}_\lambda = 0$, and we may apply what we already proved for this case. Clearly, $\nu \perp (\mu+\lambda)$, and it follows that (note that $\bigcup_{\mathcal{H}\in\mathcal{C}_\lambda}\mathcal{H} = \mathcal{H}_0 \in \mathcal{C}_{\lambda+\nu}$)

$$\dim\Big(L^2((\mu+\lambda)+\nu)\Big/\rho_{(\mu+\lambda)+\nu}(\mathcal{H}_0)\Big) = |\operatorname{supp}\nu|.$$

Set $\Delta := \operatorname{supp}\lambda \setminus \operatorname{supp}\mu_0$. We have

$$\begin{aligned}|\operatorname{supp}\nu| =&|\operatorname{supp}\mu_0 \setminus \operatorname{supp}\lambda| = |\operatorname{supp}\mu_0| - |\operatorname{supp}\mu_0 \cap \operatorname{supp}\lambda| \\ &\geq \operatorname{ind}\mathcal{C} - |\operatorname{supp}\mu_0 \cap \operatorname{supp}\lambda| = |\operatorname{supp}\lambda| - |\operatorname{supp}\mu_0 \cap \operatorname{supp}\lambda| \\ &=|\operatorname{supp}\lambda \setminus \operatorname{supp}\mu_0| = |\Delta|.\end{aligned}$$

Since $\mu(\Delta) = 0$, we have $\mathbb{1}_{\Delta^c}((\mu+\lambda)+\nu) = \mu + \mathbb{1}_{\Delta^c}(\lambda+\nu)$, and Proposition 4.9 yields that $\mathcal{H}_0 \in \mathcal{C}_{\mathbb{1}_{\Delta^c}(\lambda+\nu)}$ and that

$$\dim\Big(L^2(\mu + \mathbb{1}_{\Delta^c}(\lambda+\nu))\Big/\rho_{\mu+\mathbb{1}_{\Delta^c}(\lambda+\nu)}(\mathcal{H}_0)\Big) = |\operatorname{supp}\nu| - |\Delta| < \infty. \tag{4.6}$$

This implies that $\mathcal{C}_{\mathbb{1}_{\Delta^c}(\lambda+\nu)}$ has a maximal element, and hence that $\mathbb{1}_{\Delta^c}(\lambda+\nu) \in \mathcal{M}(\mu,\mathcal{C})$. However, $\operatorname{supp}\mathbb{1}_{\Delta^c}(\lambda+\nu) = \operatorname{supp}\mu_0$, and hence we find constants $\gamma, \gamma' > 0$ with

$$\gamma\mu_0 \leq \mathbb{1}_{\Delta^c}(\lambda+\nu) \leq \gamma'\mu_0. \tag{4.7}$$

It follows that $\mu_0 \in \mathcal{M}(\mu,\mathcal{C})$, and the proof of (1) is complete.

Finally, for the proof of (2) and (4), we compute

$$\begin{aligned}|\operatorname{supp}\mu_0|-&\operatorname{ind}\mathcal{C} = |\operatorname{supp}\mu_0| - |\operatorname{supp}\lambda| \\ &=\big(|\operatorname{supp}\mu_0 \setminus \operatorname{supp}\lambda| + |\operatorname{supp}\mu_0 \cap \operatorname{supp}\lambda|\big) \\ &-\big(|\operatorname{supp}\lambda \setminus \operatorname{supp}\mu_0| + |\operatorname{supp}\mu_0 \cap \operatorname{supp}\lambda|\big) = |\operatorname{supp}\nu| - |\Delta|.\end{aligned}$$

Since $\mathcal{H}_0 \in \mathcal{C}_{\mathbb{1}_{\Delta^c}(\lambda+\nu)}$, $\mathcal{C}_\lambda$ is a beginning section of $\mathcal{C}_{\mathbb{1}_{\Delta^c}(\lambda+\nu)}$. Hence, $\mathcal{H}_0 = \operatorname{Clos}\bigcup_{\mathcal{H}\in\mathcal{C}}\mathcal{H}$. We have the two-sided estimate (4.7), and thus the relation (4.6) gives the formula required in (2). Item (4) is an immediate consequence of (2) and Remark 2.19. ❑

5 Proof of Theorem 3.9

There are three essential ingredients in the proof of Theorem 3.9. The first one (Lemma 5.1) is to show a compactness property under an assumption on the norm of point-evaluation functionals and relies on a geometric perturbation argument. The second one (Lemma 5.2) is an analytic argument which shows that the assumption of Lemma 5.1 can be verified from growth restrictions. Last, but not least, the theory of the index of a chain developed in the previous section enters in a crucial way.

5.1 Lemma. *Let $\Omega \subseteq \mathbb{C}$ be a domain which contains the real line, and let $\langle\mathcal{H}, \|.\|_{\mathcal{H}}\rangle$ be a reproducing kernel space of functions analytic in Ω. Moreover, let $\nu \in \mathbb{M}_+(\mathbb{R})$, and assume that*

$$\nabla_{\mathcal{H}}|_{\mathbb{R}} \in L^2(\nu). \tag{5.1}$$

Then:

(1) *$\rho_\nu(\mathcal{H}) \subseteq L^2(\nu)$ and $\rho_\nu|_{\mathcal{H}} : \langle\mathcal{H}, \|.\|_{\mathcal{H}}\rangle \to L^2(\nu)$ is compact.*

(2) *There exist $\gamma_0, \gamma_1 > 0$ and $\mu_0 \in \mathbb{M}_+^{\mathrm{fin}}(\mathbb{R})$, such that*

$$\gamma_0\|f\|_{\mathcal{H}}^2 \leq \|f\|_{\mathcal{H}}^2 - \|f\|_\nu^2 + \|f\|_{\mu_0}^2 \leq \gamma_1\|f\|_{\mathcal{H}}^2, \quad f \in \mathcal{H}.$$

In other words, (5.1) says that the reproducing kernel $k_{\mathcal{H}}$ of the Hilbert space $\langle\mathcal{H},(.,.)_{\mathcal{H}}\rangle$ satisfies $\int_{\mathbb{R}} k_{\mathcal{H}}(x,x)\,d\nu<\infty$.

Proof. We have $|f(x)|\leq\|f\|_{\mathcal{H}}\cdot\nabla_{\mathcal{H}}(x)$, $f\in\mathcal{H}$, $x\in\mathbb{R}$, and hence (5.1) immediately implies that ρ_ν maps $\mathcal{H}$ boundedly into $L^2(\nu)$. To show compactness, consider a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ which converges weakly to some element $f\in\mathcal{H}$. Then $f_n(x)-f(x)\to 0$ for $x\in\mathbb{R}$ pointwise, and $\beta:=\sup_{n\in\mathbb{N}}\|f_n\|_{\mathcal{H}}<\infty$. The function $(2\beta\nabla_{\mathcal{H}}|_{\mathbb{R}})^2$ is a ν-integrable majorant for $|f_n-f|^2$, and by bounded convergence

$$\|f_n-f\|_\nu^2=\int_{\mathbb{R}}|f_n-f|^2\,d\nu\to 0.$$

Thus $\rho_\nu|_{\mathcal{H}}$ is compact.

The sesquilinear form

$$[f,g]:=(f,g)_{\mathcal{H}}-\int_{\mathbb{R}} f\overline{g}\,d\nu,\quad f,g\in\mathcal{H},$$

is bounded w.r.t. $\|.\|_{\mathcal{H}}$, and its Gram operator G w.r.t. $(.,.)_{\mathcal{H}}$ is given as

$$G:=I-(\rho_\nu|_{\mathcal{H}})^*(\rho_\nu|_{\mathcal{H}}).$$

Let $\mathcal{L}$ be the spectral subspace of G corresponding to the set $[\frac{1}{2},\infty)$. Then, by compactness of $\rho_\nu|_{\mathcal{H}}$, we have $\dim\mathcal{H}/\mathcal{L}<\infty$. Clearly, $[.,.]|_{\mathcal{L}\times\mathcal{L}}$ is positive definite on $\mathcal{L}$ and induces a norm equivalent to $(\|.\|_{\mathcal{H}})|_{\mathcal{L}}$.

The family of point-evaluation functionals $\{\chi_x:x\in\mathbb{R}\}$ is point separating on $\mathcal{H}$. Hence, by a standard perturbation argument (for an explicit proof see, e.g., [Wor14, Proposition A.9]), we find $x_1,\ldots,x_n\in\mathbb{R}$ and $\alpha_1,\ldots,\alpha_n>0$, such that the sesquilinear form

$$(f,g):=[f,g]+\sum_{i=1}^n\alpha_i f(x_i)\overline{g(x_i)},\quad f,g\in\mathcal{H},$$

is positive definite on $\mathcal{H}$ and induces a norm equivalent to $\|.\|_{\mathcal{H}}$. Denote by δ_x the unit point-mass concentrated at the point x. Then the assertion of the lemma follows with

$$\mu_0:=\sum_{i=1}^n\alpha_i\delta_{x_i}.$$

❑

5.2 Lemma. *Let λ be a growth function, $c\in\mathbb{R}\cup\{\infty\}$, and let $\langle\mathcal{H},(.,.)_{\mathcal{H}}\rangle$ be a de Branges space which is contained in $\mathbb{G}(\lambda,c)$. Then the following statements hold.*

(1) *We have*[5]

$$\limsup_{|z|\to\infty}\frac{\ln\nabla_{\mathcal{H}}(z)}{\lambda(|z|)}<\infty\quad\text{and}\quad\limsup_{x\to\pm\infty}\frac{\ln\nabla_{\mathcal{H}}(x)}{\lambda(|x|)}\leq c.\tag{5.2}$$

(2) *If $\nu\in\mathbb{M}_+(\mathbb{R})$ with $c<\frac{1}{2}p(\lambda,\nu)$ or $p(\lambda,\nu)=\infty$, then* (5.1) *holds.*

[5] Again, the second condition is void if $c=\infty$.

Proof. Choose a function E of Hermite-Biehler class which generates the de Branges space $\langle\mathcal{H},(.,.)_{\mathcal{H}}\rangle$. The representation (2.2) of the reproducing kernel $k_{\mathcal{H}}$ of $\mathcal{H}$ can be rewritten in terms of the functions $A:=\frac{1}{2}(E+E^{\#})$ and $B:=\frac{i}{2}(E-E^{\#})$. Then it reads as (again this formula has to be interpreted as a derivative if $z=\overline{w}$)

$$k_{\mathcal{H}}(w,z)=\frac{1}{\pi}\frac{B(z)A(\overline{w})-A(z)B(\overline{w})}{z-\overline{w}},\quad z,w\in\mathbb{C}.$$

In particular, we obtain that

$$\nabla_{\mathcal{H}}(x)^2=k_{\mathcal{H}}(x,x)=\frac{1}{\pi}\big[B'(x)A(x)-A'(x)B(x)\big],\quad x\in\mathbb{R}.$$

Since $A\in\mathcal{H}+z\mathcal{H}$, we have

$$\limsup_{|z|\to\infty}\frac{\ln|A(z)|}{\lambda(|z|)}<\infty,\quad \limsup_{x\to\pm\infty}\frac{\ln|A(x)|}{\lambda(|x|)}\leq c, \tag{5.3}$$

and the same for B. Remembering Remark 2.3, we see that (5.2) holds.

In view of the definition of $p(\lambda,c)$, these relations immediately imply (2). ❑

Proof (of Theorem 3.9). Let $K\subseteq\mathbb{R}$ be compact, then (3.5) yields

$$\mu_2(K)\leq\mu_1(K)+\big|\mu_2(K)-\mu_1(K)\big|\leq\mu_1(K)+(1-\epsilon)\big(\mu_1(K)+\mu_2(K)\big)+\nu(K).$$

This implies that

$$\epsilon\mu_2(K)\leq(2-\epsilon)\mu_1(K)+\nu(K). \tag{5.4}$$

Exchanging the roles of μ_1 and μ_2, yields

$$\epsilon\mu_1(K)\leq(2-\epsilon)\mu_2(K)+\nu(K). \tag{5.5}$$

Let $\mathcal{H}\in\mathcal{C}_1\cap\mathbb{G}(\lambda,c)\cup\mathcal{C}_2\cap\mathbb{G}(\lambda,c)$. Lemma 5.2, (2), shows that (5.1) holds and hence that $\rho_\nu(\mathcal{H})\subseteq L^2(\nu)$. We conclude that

$$\mathcal{C}_1\cap\mathbb{G}(\lambda,c)\subseteq\mathsf{P}[\mathcal{C}_1;\mu_2]\quad\text{and}\quad\mathcal{C}_2\cap\mathbb{G}(\lambda,c)\subseteq\mathsf{P}[\mathcal{C}_2;\mu_1], \tag{5.6}$$

and this is (1). It follows that $\mathcal{C}_1\cap\mathbb{G}(\lambda,c)$ and $\mathcal{C}_2\cap\mathbb{G}(\lambda,c)$ are both beginning sections of the larger of $\mathsf{P}[\mathcal{C}_1;\mu_2]$ and $\mathsf{P}[\mathcal{C}_2;\mu_1]$, remember Theorem 3.5. This already implies that (2) holds.

If $\mathcal{C}_1\cap\mathbb{G}(\lambda,c)$ and $\mathcal{C}_2\cap\mathbb{G}(\lambda,c)$ coincide, there is nothing more to prove. We consider the case that $\mathcal{C}_1\cap\mathbb{G}(\lambda,c)\subsetneq\mathcal{C}_2\cap\mathbb{G}(\lambda,c)$; the case $\mathcal{C}_2\cap\mathbb{G}(\lambda,c)\subsetneq\mathcal{C}_1\cap\mathbb{G}(\lambda,c)$ is treated completely analogously.

Choose $\mathcal{K}\in\mathcal{C}_2\cap\mathbb{G}(\lambda,c)\setminus\mathcal{C}_1\cap\mathbb{G}(\lambda,c)$. From (5.4) and (5.5) we conclude that for each compact set $K\subseteq\mathbb{R}$

$$\frac{\epsilon}{2-\epsilon}\mu_2(K)\leq\mu_1(K)+\frac{1}{2-\epsilon}\nu(K)\leq\frac{2-\epsilon}{\epsilon}\mu_2(K)+\Big(\frac{1}{\epsilon}+\frac{1}{2-\epsilon}\Big)\nu(K). \tag{5.7}$$

Denote by $\|\rho_\nu|_{\mathcal{K}}\|$ the norm of $\rho_\nu|_{\mathcal{K}}$ as an operator from $\langle\mathcal{K},\|.\|_{\mu_2}\rangle$ to $L^2(\nu)$. Then

$$\begin{aligned}\frac{\epsilon}{2-\epsilon}\int_{\mathbb{R}}|f|^2\,d\mu_2&\leq\int_{\mathbb{R}}|f|^2\,d\mu_1+\int_{\mathbb{R}}|f|^2\frac{d\nu}{2-\epsilon}\\&\leq\frac{2-\epsilon}{\epsilon}\int_{\mathbb{R}}|f|^2\,d\mu_2+\Big(\frac{1}{\epsilon}+\frac{1}{2-\epsilon}\Big)\int_{\mathbb{R}}|f|^2\,d\nu\\&\leq\Big(\frac{2-\epsilon}{\epsilon}+\Big(\frac{1}{\epsilon}+\frac{1}{2-\epsilon}\Big)\|\rho_\nu|_{\mathcal{K}}\|^2\Big)\int_{\mathbb{R}}|f|^2\,d\mu_2,\quad f\in\mathcal{K}.\end{aligned}$$

Hence, as the norm $\|.\|_{\mu_2}$ is equivalent to the norm based on $\mu_1 + \frac{\nu}{2-\epsilon}$, the space $\mathcal{K}$ becomes a de Branges space when endowed with the norm $\|f\|_{\mathcal{K}}^2 := \|f\|_{\mu_1}^2 + \|f\|_{\frac{d\nu}{2-\epsilon}}^2$, $f \in \mathcal{K}$.

We apply Lemma 5.1, (2), with the space $\langle \mathcal{K}, \|.\|_{\mathcal{K}} \rangle$ and the measure $\frac{d\nu}{2-\epsilon}$; the required hypothesis (5.1) holds by Lemma 5.2 since $\mathcal{K} \in \mathbb{G}(\lambda, c)$. This provides us with $\mu_0 \in \mathbb{M}_+^{\mathrm{fin}}(\mathbb{R})$ and $\gamma_0, \gamma_1 > 0$, such that

$$\gamma_0 \|f\|_{\mathcal{K}}^2 \leq \underbrace{\|f\|_{\mu_1}^2 + \|f\|_{\mu_0}^2}_{=\|f\|_{\mu_1+\mu_0}^2} \leq \gamma_1 \|f\|_{\mathcal{K}}^2, \quad f \in \mathcal{K}.$$

It follows that $\mathcal{K} \in \mathsf{Sub}[\mu_1 + \mu_0]$.

Consider the chain $\mathcal{C}_{1,\mu_0}$, i.e., the unique element of $\mathsf{Chains}[\mu_1 + \mu_0]$ which contains $\mathcal{C}_1$. By Theorem 4.10, we have

$$\mathcal{C}_{1,\mu_0} = \mathcal{C}_1 \dot{\cup} \big\{ \mathcal{K}_m : m \in \mathbb{N}, m \leq M \big\}$$

with some $M \in \mathbb{N}_0$ and de Branges spaces $\mathcal{K}_m$ satisfying

$$\mathcal{K}_{i-1} \subsetneq \mathcal{K}_i, \dim \mathcal{K}_i / \mathcal{K}_{i-1} = 1,\ i = 2, \dots, M, \qquad \bigcup_{\mathcal{H} \in \mathcal{C}_1} \mathcal{H} \subseteq \mathcal{K}_1 \text{ if } M > 0.$$

Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are admissible for comparison, so are $\mathcal{C}_{1,\mu_0}$ and $\mathcal{C}_2$, and it follows that $\mathcal{K} \in \mathcal{C}_{1,\mu_0}$. Since $\mathcal{K} \in \mathbb{G}(\lambda, c) \setminus \mathcal{C}_1 \cap \mathbb{G}(\lambda, c)$, we have $\mathcal{K} \notin \mathcal{C}_1$. Hence, we must have $M > 0$ and $\mathcal{K} = \mathcal{K}_{m_0}$ for some $m_0 \in \{1, \dots, M\}$. Thus $\mu_0 \in \mathcal{M}(\mu_1, \mathcal{C}_1)$, in particular, $\operatorname{ind} \mathcal{C}_1 < \infty$. Moreover, $\bigcup_{\mathcal{H} \in \mathcal{C}_1} \mathcal{H} \subseteq \mathcal{K} \in \mathbb{G}(\lambda, c)$, i.e., $\mathcal{C}_1 = \mathcal{C}_1 \cap \mathbb{G}(\lambda, c)$. Remembering (5.6), we obtain

$$\mathcal{C}_1 = \mathcal{C}_1 \cap \mathbb{G}(\lambda, c) \subseteq \mathsf{P}[\mathcal{C}_1; \mu_2] \subseteq \mathcal{C}_1,$$

and hence equality prevails throughout.

Let $\mathcal{H}' \in \mathcal{C}_2$ with

$$\bigcup_{\mathcal{H} \in \mathcal{C}_1} \mathcal{H} \subseteq \mathcal{H}' \subseteq \mathcal{K}.$$

Since the norms $\|.\|_{\mu_2}$ and $\|.\|_{\mu_1+\mu_0}$ are equivalent on $\mathcal{K}$, we have $\mathcal{H}' \in \mathcal{C}_{1,\mu_0}$. Thus

$$\{\mathcal{H}' \in \mathcal{C}_2 : \mathcal{H}' \subseteq \mathcal{K}\} = \mathcal{C}_1 \dot{\cup} \{\mathcal{K}_m : m \in \mathbb{N}, m \leq m_0\}.$$

Since the choice of $\mathcal{K} \in \mathcal{C}_2 \cap \mathbb{G}(\lambda, c) \setminus \mathcal{C}_1 \cap \mathbb{G}(\lambda, c)$ was arbitrary, (3.6) follows. ❏

6 Stability and inclusion results for whole chains

In this section we present some instances when our growth dependent stability result can be applied to obtain knowledge about whole chains rather than for beginning sections. Those examples, corollaries and supplements of Theorem 3.9 cover several known stability results.

6.1 Using a priori knowledge on measures and/or chains

The following fact is a trivial corollary of Theorem 3.9.

6.1 Corollary. *Let $\mu_1, \mu_2 \in \mathbb{M}_+(\mathbb{R})$, and let $\mathcal{C}_i \in \mathsf{Chains}[\mu_i]$, $i = 1, 2$, be admissible for comparison. Let λ be a growth function, $c \in \mathbb{R}\cup\{\infty\}$, $\nu \in \mathbb{M}_+(\mathbb{R})$, and assume that* (3.4) *and* (3.5) *hold. Assume in addition that*

$$\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathbb{G}(\lambda, c). \tag{6.1}$$

Then one of the following three alternatives must take place.

[I] $\mathcal{C}_1 = \mathcal{C}_2$.

[II_1] $\operatorname{ind}\mathcal{C}_2 < \infty$ *and*

$$\begin{aligned} &\exists\, N \in \mathbb{N}\cup\{\infty\}\ \exists\, \mathcal{H}_n \text{ for } n \in \mathbb{N}_0, n < N : \\ &\qquad \mathcal{H}_{n-1} \subsetneq \mathcal{H}_n,\ \dim \mathcal{H}_n/\mathcal{H}_{n-1} = 1, \quad n \in \mathbb{N}, n < N, \end{aligned} \tag{6.2}$$

such that $\mathcal{C}_1 = \mathcal{C}_2 \dot\cup \{\mathcal{H}_n : n \in \mathbb{N}_0, n < N\}$.

[II_2] $\operatorname{ind}\mathcal{C}_1 < \infty$ *and* (6.2) *such that* $\mathcal{C}_2 = \mathcal{C}_1 \dot\cup \{\mathcal{H}_n : n \in \mathbb{N}_0, n < N\}$.

In addition:

(*a*) *If "$N < \infty$" in* [II_1] *or* [II_2], *then the corresponding larger chain has index zero.*

(*b*) *If "$N = \infty$" in* [II_1] *or* [II_2], *then the corresponding larger chain has nonzero index.*

Proof. Observe that (6.1) yields $\mathcal{C}_i \cap \mathbb{G}(\lambda, c) = \mathcal{C}_i$, $i = 1, 2$. Moreover, "$N < \infty$" in (6.2) implies that the corresponding larger chain has a maximal element, whereas "$N = \infty$" implies that it has no. ❑

The justification to formulate this corollary is that there are situations when the required a priori knowledge (6.1) is available from accessible properties of the measures or chains under consideration.

6.2 Example. Let $\mu \in \mathbb{M}_+(\mathbb{R})$ with $\int_{\mathbb{R}} \frac{d\mu(x)}{1+x^2} < \infty$, and assume that $\operatorname{supp}\mu$ is semibounded (from above or from below). Let $\mathcal{C} \in \mathsf{Chains}[\mu]$ be the chain with $1 \in \mathcal{H} + z\mathcal{H}$, $\mathcal{H} \in \mathcal{C}$. Then $\mathcal{C} \subseteq \mathbb{G}(r^{\frac{1}{2}}, \infty)$.

Hence, under the a priori assumption that

$$\operatorname{supp}\mu_1, \operatorname{supp}\mu_2 \text{ semibounded}, \qquad 1 \in \mathcal{H} + z\mathcal{H}, \mathcal{H} \in \mathcal{C}_1 \cup \mathcal{C}_2, \tag{6.3}$$

Corollary 6.1 applies with $\lambda(r) := r^{\frac{1}{2}}$ and $c := \infty$. ◊

The bounded type property mentioned in Definition 2.14, (2/c), and Definition 3.1, (2), gives rise to two other situations where a priori knowledge is available.

6.3 Example. Let $\mathcal{C}$ be a chain for some measure μ, and assume that there exists a function $f \in \bigcup_{\mathcal{H}\in\mathcal{C}} \mathcal{H}$ which does not vanish identically and is of bounded type in $\mathbb{C}^+$. Let $\lambda(r) := r$ and $c := 0$. Then the triple $[\mathcal{C}; \lambda, c]$ possesses the property

[A] For each chain $\mathcal{C}'$ which is admissible for comparison with $\mathcal{C}$, we have $\mathcal{C}' \subseteq \mathbb{G}(\lambda, c)$.

Hence, under the a priori assumption that

$$\exists f \in \Big(\bigcup_{\mathcal{H}\in\mathcal{C}} \mathcal{H}\Big) \setminus \{0\} : \ f \text{ bounded type in } \mathbb{C}^+, \tag{6.4}$$

Corollary 6.1 applies with $\lambda(r) := r$ and $c := 0$. ♢

6.4 Example. Let $\mathcal{C}$ be a chain for some measure μ, let λ be a growth function with $r = O(\lambda(r))$, and let $c \in \mathbb{R} \cup \{\infty\}$. Assume that there exists a function $f \in \bigcup_{\mathcal{H}\in\mathcal{C}} \mathcal{H}$ which does not vanish identically and satifies

$$\limsup_{|z|\to\infty} \frac{\ln|f(z)|}{\lambda(|z|)} < \infty, \quad \limsup_{x\to\pm\infty} \frac{\ln|f(x)|}{\lambda(|x|)} \le c. \tag{6.5}$$

Then the triple $[\mathcal{C}; \lambda, c]$ possesses the property [A]. This follows, e.g., from the argument in the proof of [KW05a, Theorem 3.10] using that the indicator $h(f, \lambda; \theta)$ is continuous.

Hence, under the a priori assumption that

$$r = O(\lambda(r)), \qquad \exists f \in \Big(\bigcup_{\mathcal{H}\in\mathcal{C}} \mathcal{H}\Big) \setminus \{0\} : (6.5) \text{ holds}, \tag{6.6}$$

Corollary 6.1 applies with $\lambda(r)$ and c. ♢

Let us point out the essential difference in the nature of the conditions (6.3) and (6.4)/(6.6). The first requests knowlegde on *both* measures and chains, wheras the latter request knowledge only on *one* chain. Taking a perturbation viewpoint this means to either require knowledge about the unperturbed *and* the perturbed objects, or *only* about the unperturbed ones.

The reason that (6.4)/(6.6) are in this sense much better than (6.3), is the property [A]. In fact, we may say that whenever [A] is present, Corollary 6.1 applies. Even more, we can improve the additional property (b).

6.5 Proposition. *Let $\mu_1, \mu_2 \in \mathbb{M}_+(\mathbb{R})$, and let $\mathcal{C}_i \in \mathsf{Chains}[\mu_i]$, $i = 1, 2$, be admissible for comparison. Let λ be a growth function, $c \in \mathbb{R}\cup\{\infty\}$, $\nu \in \mathbb{M}_+(\mathbb{R})$, and assume that* (3.4) *and* (3.5) *hold. Assume that the triple $[\mathcal{C}_1; \lambda, c]$ possesses the property* [A].

Then one of the alternatives [I], [II_1], [II_2] *must take place. Moreover, the addition* (a) *holds and:*

(b_+) *If "$N = \infty$" in* [II_1] *or* [II_2]*, then the corresponding larger chain has infinite index.*

Proof. The property [A], applied with the chains $\mathcal{C}' := \mathcal{C}_1$ and $\mathcal{C}' := \mathcal{C}_2$ implies that the hypothesis (6.1) is fullfilled. Hence one of [I], [II_1], [II_2] takes place.

We need to consider the case that "$N = \infty$" in [II_1]. Assume on the contrary that $\operatorname{ind}\mathcal{C}_1 < \infty$. Choose $\mu_0 \in \mathcal{M}(\mu_1, \mathcal{C}_1)$ with $|\operatorname{supp}\mu_0| = \operatorname{ind}\mathcal{C}_1$. By Theorem 4.10 there exists a de Branges space $\mathcal{K}$ such that

$$\mathcal{C}_{1,\mu_0} := \mathcal{C}_1 \dot\cup \{\mathcal{K}\} = \mathcal{C}_2 \dot\cup \{\mathcal{H}_n : n \in \mathbb{N}_0\} \dot\cup \{\mathcal{K}\}. \tag{6.7}$$

Due to [A] we have $\mathcal{C}_{1,\mu_0} \subseteq \mathbb{G}(\lambda, c)$. The measures $\mu_2, \mu_1 + \mu_0$ and $\nu + \mu_0$ satisfy (3.4) and (3.5). Hence, Corollary 6.1 applies with these measures and the chains $\mathcal{C}_2$ and $\mathcal{C}_{1,\mu_0}$. Since $\mathcal{C}_2 \subsetneq \mathcal{C}_1 \subseteq \mathcal{C}_{1,\mu_0}$, the respective alternative [II_1] must occur. This contradicts the form (6.7) of $\mathcal{C}_{1,\mu_0}$. ❑

As a consequence of Proposition 6.5 we reobtain a certain part of a stability result on the type of a measure shown in [BS11, Corollary 1.5], see also the discussion in [Pol13, Theorem 11]. The part which we cover is enlarging the measure and limited redistribution of mass, what we do not cover is shifting of mass.

Moreover, we restrict ourselves to consideration of measures μ with $\int_{\mathbb{R}} \frac{d\mu(x)}{1+x^2} < \infty$. The case of power bounded measures and admitting a polynomial factor in the perturbation could be treated analogously making use of the theory developed in [LW13b] and [LW13a]. This, however, would require recalling a serious amount of notation, and we feel that it is beyond the scope of the present paper.

To formulate the stability result, let us recall (one possible) definition of the type of a measure. Let $\mu \in \mathbb{M}_+(\mathbb{R})$ and assume that $\int_{\mathbb{R}} \frac{d\mu(x)}{1+x^2} < \infty$. Moreover, let $\mathcal{C} \in \mathsf{Chains}[\mu]$ be the chain with $1 \in \mathcal{H} + z\mathcal{H}$, $\mathcal{H} \in \mathcal{C}$. Then the *type of the measure* μ is defined as the supremum of the exponential types of all functions in $\bigcup_{\mathcal{H}\in\mathcal{C}} \mathcal{H}$.

6.6 Corollary (part of [BS11, Corollary 1.5])**.** *Let $\mu_1, \mu_2 \in \mathbb{M}_+(\mathbb{R})$ with $\int_{\mathbb{R}} \frac{d\mu_i(x)}{1+x^2} < \infty$, $i = 1, 2$, and let $\mathcal{C}_i \in \mathsf{Chains}[\mu_i]$ be the chains with $1 \in \mathcal{H} + z\mathcal{H}$, $\mathcal{H} \in \mathcal{C}_i$. Assume that there exists a measure $\nu \in \mathbb{M}_+(\mathbb{R})$ with*

(1) $\exists\, \delta > 0 : \displaystyle\int_{\mathbb{R}} e^{\delta|x|}\, d\mu < \infty,$

(2) $\exists\, \epsilon \in (0, 1) : \ |\mu_1 - \mu_2| \le (1 - \epsilon)(\mu_1 + \mu_2) + \nu.$

Then μ_1 and μ_2 have equal types.

Proof. Proposition 6.5 yields that one of [I], [II_1], [II_2] must take place. In each of these cases the classical formula of de Branges–Krein for computing the type of a measure in terms of the associated canonical system yields that the types of μ_1 and μ_2 are equal. ❑

6.2 Compactly supported perturbations

The situation that the measure ν is compactly supported is of particular interest (and will be of importance in forthcoming work). In this case already a very weak a priori assumption on one of the chains is enough to ensure that one of the alternatives [I], [II_1], [II_2] must take place, and that even finer additions hold than in Proposition 6.5.

6.7 Theorem. *Let $\mu_1, \mu_2 \in \mathbb{M}_+(\mathbb{R})$, let $\mathcal{C}_i \in \mathsf{Chains}[\mu_i]$, $i = 1, 2$, be admissible for comparison, and let $\nu \in \mathbb{M}_+(\mathbb{R})$. Assume that*

$$\operatorname{supp}\nu \textit{ compact}, \quad (3.5) \textit{ holds}, \quad \exists\, f \in \Big(\bigcup_{\mathcal{H}\in\mathcal{C}_1\cup\mathcal{C}_2} \mathcal{H}\Big)\setminus\{0\} : f \textit{ is of finite order.}$$

Then one of the alternatives [I], [II_1], [II_2] *must take place. Moreover, the addition* (a) *holds and:*

(b_{++}) *If "$N = \infty$" in* [II_1] *or* [II_2], *then measure corresponding to the larger chain is not discrete.*

(c) *If $\mu_1 \ll \mu_2$ then $\mathcal{C}_1 \subseteq \mathcal{C}_2$, and if $\mu_2 \ll \mu_1$ then $\mathcal{C}_2 \subseteq \mathcal{C}_1$.*

Proof. Since $\operatorname{supp}\nu$ is compact, we have $p(\lambda,\nu) = \infty$ for every growth function λ, and hence (3.4) always holds. Choose $f \in \big(\bigcup_{\mathcal{H}\in\mathcal{C}_1\cup\mathcal{C}_2}\mathcal{H}\big) \setminus \{0\}$ with finite order, say ρ_f, and choose $\rho > \max\{1,\rho_f\}$. By Example 6.4 the triple $[\mathcal{C}_1; r^\rho, 0]$ possesses the property [A]. Hence, Proposition 6.5 is applicable, and yields that one of the alternatives [I], [II_1], [II_2] must take place and that the additions (a) and (b_+) hold.

Consider the case that [II_1] takes place with "$N = \infty$". Then we know from (b_+) that $\operatorname{ind}\mathcal{C}_1 = \infty$. Moreover, since $\operatorname{ind}\mathcal{C}_2 < \infty$, the measure μ_2 is discrete. To show (b_{++}) assume on the contrary that μ_1 is discrete. Denote $K := \operatorname{supp}\nu$, then by (5.7)

$$\frac{\epsilon}{2-\epsilon}\mu_2(\{x\}) \leq \mu_1(\{x\}) \leq \frac{2-\epsilon}{\epsilon}\mu_2(\{x\}), \quad x \in \mathbb{R}\setminus K. \tag{6.8}$$

Choose $\sigma_2 \in \mathcal{M}(\mu_2,\mathcal{C}_2)$, and set

$$\sigma_1 := \mathbb{1}_K\mu_2 + \sigma_2.$$

Remembering that $\epsilon \in (0,1)$, we obtain

$$\frac{\epsilon}{2-\epsilon}\big(\mu_2+(\mathbb{1}_K\mu_1+\sigma_2)\big) \leq \underbrace{\mathbb{1}_{K^c}\mu_1+\mathbb{1}_K\mu_2+\mathbb{1}_K\mu_1+\sigma_2}_{=\mu_1+\sigma_1} \leq \frac{2-\epsilon}{\epsilon}\big(\mu_2+(\mathbb{1}_K\mu_1+\sigma_2)\big)$$

By equivalence of norms we have $\mathcal{C}_{2,\mathbb{1}_K\mu_1+\sigma_2} \in \mathsf{Chains}[\mu_1+\sigma_1]$. However, $\mathcal{C}_{1,\sigma_1}$ and $\mathcal{C}_{2,\mathbb{1}_K\mu_1+\sigma_2}$ both contain $\mathcal{C}_2$, and hence are equal. Since $\mathbb{1}_K\mu_1 + \sigma_2 \in \mathcal{M}(\mu_2,\mathcal{C}_2)$, equivalence of norms yields $\sigma_1 \in \mathcal{M}(\mu_1,\mathcal{C}_1)$. Thus $\operatorname{ind}\mathcal{C}_1 < \infty$, and we have reached a contradiction. The case that [II_2] takes place with "$N = \infty$" is treated in the same way.

To show (c) assume on the contrary that $\mu_1 \ll \mu_2$ and $\mathcal{C}_2 \subsetneq \mathcal{C}_1$. Then $\operatorname{ind}\mathcal{C}_2 < \infty$ and hence μ_2 is discrete. By absolute continuity, also μ_1 is discrete and $\operatorname{supp}\mu_1 \subseteq \operatorname{supp}\mu_2$. In the same way as in the first part of the proof, we obtain (6.8).

Since $K \cap \operatorname{supp}\mu_1$ is finite, we find $\gamma_0 \in (0,1)$ and $\gamma_1 > 1$ such that

$$\gamma_0\mu_2(\{x\}) \leq \mu_1(\{x\}) \leq \gamma_1\mu_2(\{x\}), \quad x \in \operatorname{supp}\mu_1. \tag{6.9}$$

Set $\sigma := \mathbb{1}_{(\operatorname{supp}\mu_1)^c}\mu_2$. Since $K\cap\operatorname{supp}\mu_2$ is finite, (6.8) implies that $|\operatorname{supp}\sigma| < \infty$. From (6.9) we obtain

$$\frac{1}{\gamma_1}(\mu_1+\sigma) \leq \mu_2 \leq \frac{1}{\gamma_0}(\mu_1+\sigma).$$

Equivalence of norms implies that $\mathcal{C}_2 \in \mathsf{Chains}[\mu_1+\sigma]$. However, $\mathcal{C}_2 \subsetneq \mathcal{C}_1 \subseteq \mathcal{C}_{1,\sigma}$, and we have reached a contradiction. The case that $\mu_2 \ll \mu_1$ is treated in the same way. ❑

The following inclusion result was obtained in [Win00] from a Gelfand-Levitan type construction (see Lemma 2.3 and the paragraph at the bottom of p.243 in [Win00]): *Let $\mu_1,\mu_2 \in \mathbb{M}_+(\mathbb{R})$ with $\int_{\mathbb{R}}\frac{d\mu_i(x)}{1+x^2} < \infty$, $i = 1,2$, and let $\mathcal{C}_i \in \mathsf{Chains}[\mu_i]$ be the chains with $1 \in \mathcal{H}+z\mathcal{H}$, $\mathcal{H}\in\mathcal{C}_i$. Assume that* (*i*) *there exists a bounded interval* E *and* $c > 0$ *such that* $\mu_2|_{\mathbb{R}\setminus E} \ll \mu_1|_{\mathbb{R}\setminus E}$ *and* $\frac{d|\mu_1-\mu_2|}{d\mu_1}(x) \leq \frac{c}{1+x^2}$ *for* μ_1*-a.a* $x \in \mathbb{R} \setminus E$*;*

(*ii*) $\mu_1 \ll \mu_2$.

Then $\mathcal{C}_1 \subseteq \mathcal{C}_2$.

Note that strict inclusion may occur; examples are obtained by choosing μ_1 such that $\mathcal{C}_1$ has finite index and setting $\mu_2 := \mu_1 + \sigma$ where $\sigma \in \mathcal{M}(\mu_1, \mathcal{C}_1)$.

Using Theorem 6.7 we can show a slight improvement of this stability result[6].

6.8 Corollary ([Win00]). *Let* $\mu_1, \mu_2 \in \mathbb{M}_+(\mathbb{R})$ *with* $\int_{\mathbb{R}} \frac{d\mu_i(x)}{1+x^2} < \infty$, $i = 1, 2$, *and let* $\mathcal{C}_i \in \mathsf{Chains}[\mu_i]$ *be the chains with* $1 \in \mathcal{H} + z\mathcal{H}$, $\mathcal{H} \in \mathcal{C}_i$. *Assume that*

(1) *there exists a bounded interval* E *such that* $\mu_2|_{\mathbb{R}\setminus E} \ll \mu_1|_{\mathbb{R}\setminus E}$ *and* $\operatorname{ess\,sup}_{x\in\mathbb{R}\setminus E} \frac{d|\mu_1-\mu_2|}{d\mu_1}(x) < 1$*;*

(2) $\mu_1 \ll \mu_2$.

Then $\mathcal{C}_1 \subseteq \mathcal{C}_2$.

Proof. Set $\alpha := \operatorname{ess\,sup}_{x\in\mathbb{R}\setminus E} \frac{d|\mu_1-\mu_2|}{d\mu_1}(x)$ and $\nu := \mathbb{1}_E \cdot |\mu_1 - \mu_2|$, then

$$|\mu_1 - \mu_2| \leq \alpha \cdot \mu_1 + \nu.$$

In particular, (3.5) holds. Hence, Theorem 6.7 is applicable. ❑

6.3 A priori knowledge on sign

A stability result can also be shown when the unperturbed measure (in the statement below this is μ_2) is everywhere larger than the perturbed one.

6.9 Proposition. *Let* $\mu_1, \mu_2 \in \mathbb{M}_+(\mathbb{R})$, *and let* $\mathcal{C}_i \in \mathsf{Chains}[\mu_i]$, $i = 1, 2$, *be admissible for comparison. Let* λ *be a growth function and* $c \in \mathbb{R}\cup\{\infty\}$. *Assume that*

(1) $\operatorname{ind}\mathcal{C}_2 > 0$ *and* $\mathcal{C}_2 \subseteq \mathbb{G}(\lambda, c)$*;*

(2) *there exists* $\nu \in \mathbb{M}_+(\mathbb{R})$ *with* (3.4) *and* $\epsilon \in (0, 1)$ *with* $\epsilon\mu_2 - \nu \leq \mu_1 \leq \mu_2$.

Then one of the alternatives [I], [II$_2$] *must take place.*

Proof. By assumption we have $\mathcal{C}_2 \cap \mathbb{G}(\lambda, c) = \mathcal{C}_2$. Hence, Theorem 3.9 yields that either

(A) $\mathcal{C}_2 \subseteq \mathcal{C}_1 \cap \mathbb{G}(\lambda, c)$

or

(B) $\mathcal{C}_1 \cap \mathbb{G}(\lambda, c) \subsetneq \mathcal{C}_2 \cap \mathbb{G}(\lambda, c)$.

Consider the case (A). From Theorem 3.5, (2), and Theorem 3.9, (1), we obtain

$$\mathcal{C}_2 \subseteq \mathcal{C}_1 \cap \mathbb{G}(\lambda, c) \subseteq \mathsf{P}[\mathcal{C}_1; \mu_2] \subseteq \mathcal{C}_2$$

and hence equality must hold throughout. By assumption $\operatorname{ind}\mathcal{C}_2 > 0$, and Theorem 3.5, (3), implies that $\mathcal{C}_1 = \mathcal{C}_2$.

Consider the case (B). Theorem 3.9, (3), applies and yields that $\operatorname{ind}\mathcal{C}_1 < \infty$, that $\mathcal{C}_1 = \mathcal{C}_1 \cap \mathbb{G}(\lambda, c) \subsetneq \mathcal{C}_2$, and that $\mathcal{C}_2 \setminus \mathcal{C}_1$ is of the desired form. ❑

[6]Notice: we obtain only the inclusion result, not the formula [Win00, (3.30)] for the elements of $\mathcal{C}_1$ as de Branges subspaces of $L^2(\mu_2)$.

6.4 The closure of polynomials

Theorem 3.9 can be employed to obtain results about density of polynomials in a space $L^2(\mu)$ and properties of their closure if they are not dense.

Recall the notion of determinacy of a measure and of its index of determinacy. For practical reasons, we do not use the definition given in [BD95], but the equivalent property provided in [BD95, Theorems 3.6,3.9].

6.10 Definition. Let $\mu \in \mathbb{M}_+(\mathbb{R})$ and assume that μ possesses all power moments.

(1) The measure μ is called *determinate*, if it is uniquely determined by the sequence of its power moments. It is called *indeterminate* otherwise.

(2) The measure μ is said to have *infinite index of determinacy* if every measure $\mu + \mu_0$ with $\mu_0 \in \mathbb{M}_+^{\mathrm{fin}}(\mathbb{R})$, is determinate.

(3) Assume that μ is determinate but does not have infinite index of determinacy. Then the *index of determinacy of* μ is the minimal number $N \in \mathbb{N}$ such that there exists $\mu_0 \in \mathbb{M}_+^{\mathrm{fin}}(\mathbb{R})$, $|\operatorname{supp} \mu_0| = N$, with $\mu + \mu_0$ being indeterminate.

We denote the index of determinacy of μ by $\operatorname{ind} \mu$ (thereby including the possible value ∞). Moreover, we set $\operatorname{ind} \mu := 0$ if μ is indeterminate. $\Diamond$

Reformulated in terms of our present notation, we may say the following: *Let $\mu \in \mathbb{M}_+(\mathbb{R})$ with $|\operatorname{supp} \mu| = \infty$ be given, and assume that μ possesses all power moments and is determinate. Then $\mathcal{C} := \{\mathbb{C}[z]_n : n \in \mathbb{N}\} \in \mathsf{Chains}[\mu]$ and $\operatorname{ind} \mathcal{C} = \operatorname{ind} \mu$.*

6.11 Proposition. *Let $\mu_1 \in \mathbb{M}_+(\mathbb{R})$ possess all power moments, and assume that either $|\operatorname{supp} \mu_1| < \infty$ or μ_1 has infinite index of determinacy. Let λ be a growth function and $c \in [0, \infty]$. Let $\mu_2 \in \mathbb{M}_+(\mathbb{R})$, and assume that* (3.4) *and* (3.5) *hold.*

Then μ_2 possesses all power moments, and each space $\mathcal{H} \in \mathsf{Sub}[\mu_2]$ with $1 \in \mathcal{H} + z\mathcal{H}$ is either finite-dimensional or does not belong to the growth class $\mathbb{G}(\lambda, c)$.

Proof. The case that $\mu_2 = 0$ is trivial. Hence, assume that $\mu_2 \neq 0$. Moreover, notice that, for every polynomial $p \in \mathbb{C}[z]$,

$$\limsup_{|z| \to \infty} \frac{\ln |p(z)|}{\lambda(|z|)} = 0. \tag{6.10}$$

Consider the chain $\mathcal{C}_1$ for μ_1 which consists of polynomials, i.e.,

$$\mathcal{C}_1 := \begin{cases} \{\mathbb{C}[z]_n : n = 1, \ldots, |\operatorname{supp} \mu_1|\}, & |\operatorname{supp} \mu_1| < \infty, \\ \{\mathbb{C}[z]_n : n = 1, 2, \ldots\} \qquad\qquad , & \operatorname{ind} \mu_1 = \infty. \end{cases}$$

Since $c \geq 0$, we have $\mathcal{C}_1 \subseteq \mathbb{G}(\lambda, c)$.

Since $p(\lambda, \nu) > 0$, we see from (6.10) that $\mathbb{C}[z] \subseteq L^2(\nu)$, i.e., ν possesses all power moments. Now (5.4) yields that also μ_2 possesses all power moments. Let $\mathcal{C}_2 \in \mathsf{Chains}[\mu_2]$ be the chain with $\operatorname{span}\{1\} \in \mathcal{C}_2$. Theorem 3.9 yields that either

(A) $\mathcal{C}_2 \cap \mathbb{G}(\lambda, c) \subseteq \mathcal{C}_1$

or

(B) $\mathcal{C}_1 \subsetneq \mathcal{C}_2 \cap \mathbb{G}(\lambda, c)$.

If the alternative (A) takes place, the assertion of the present proposition follows immediately. Assume that (B) takes place. By Theorem 3.9, $(3/c)$, we must have $\operatorname{ind} \mathcal{C}_1 < \infty$. Our assumption says that this may happen only if $|\operatorname{supp} \mu_1| < \infty$. By Theorem 3.9, $(3/b)$, we have

$$\mathcal{C}_2 \cap \mathbb{G}(\lambda, c) = \mathcal{C}_1 \dot{\cup} \big\{\mathcal{H}_n : n \in \mathbb{N}_0, n < N\big\}$$

where $N \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{H}_{n-1} \subsetneq \mathcal{H}_n$, $\dim \mathcal{H}_n / \mathcal{H}_{n-1} = 1$, $n \in \mathbb{N}, n < N$. This show that each member of $\mathcal{C}_2 \cap \mathbb{G}(\lambda, c)$ is finite-dimensional. ❑

Using $\lambda(r) := r$ and $c := 0$ in Proposition 6.11, we obtain a slightly refined version of Yuditskii's theorem [Yud00].

6.12 Corollary ([Yud00, Theorem])**.** *Let $\mu_1 \in \mathbb{M}_+(\mathbb{R})$ possess all power moments and have infinite index of determinacy. Let $\mu_2 \in \mathbb{M}_+(\mathbb{R})$, and assume that*

$$|\mu_1 - \mu_2| \leq (1 - \epsilon)(\mu_1 + \mu_2) + \nu$$

with some $\epsilon \in (0, 1)$ and $\nu \in \mathbb{M}_+(\mathbb{R})$ satisfying

$$\exists \delta > 0 : \int_{\mathbb{R}} e^{\delta |x|} \, d\nu(x) < \infty.$$

Then $\mathbb{C}[z]$ is a dense subspace of $L^2(\mu_2)$.

Proof. If $\mathbb{C}[z]$ were not dense, its closure would be a space contained in $\mathbb{G}(r, 0)$. ❑

Another interesting consequence of Proposition 6.11 is the case "$\mu_1 = 0$".

6.13 Corollary. *Let λ be a growth function, $c \in [0, \infty]$, $\mu \in \mathbb{M}_+(\mathbb{R})$ with $|\operatorname{supp} \mu| = \infty$, and assume that*

$$c < \frac{1}{2} p(\lambda, \mu) \text{ or } p(\lambda, \mu) = \infty.$$

Then either $\mathbb{C}[z]$ is dense in $L^2(\mu)$ or $\operatorname{Clos}_{L^2(\mu)} \mathbb{C}[z] \notin \mathbb{G}(\lambda, c)$. ❑

6.14 Remark. Using $\lambda(r) := r$ and $c := 0$ in Corollary 6.13 gives the classical fact (see, e.g., [Fre69, Satz 5.2]) that a measure $\mu \in \mathbb{M}_+(\mathbb{R})$ is determinate provided that

$$\exists \delta > 0 : \int_{\mathbb{R}} e^{\delta |x|} \, d\mu < \infty.$$

$\Diamond$

References

[Bar06] A.D. Baranov. “Completeness and Riesz bases of reproducing kernels in model subspaces”. In: *Int. Math. Res. Not.* (2006), Art. ID 81530, 34.

[BBH07] A.D. Baranov, A. Borichev, and V. Havin. “Majorants of meromorphic functions with fixed poles”. In: *Indiana Univ. Math. J.* 56.4 (2007), pp. 1595–1628.

[BD95] C. Berg and A.J. Duran. “The index of determinacy for measures and the l^2-norm of orthonormal polynomials”. In: *Trans. Amer. Math. Soc.* 347.8 (1995), pp. 2795–2811.

[BP07] C. Berg and H.L. Pedersen. “Logarithmic order and type of indeterminate moment problems”. In: *Difference equations, special functions and orthogonal polynomials.* With an appendix by Walter Hayman. World Sci. Publ., Hackensack, NJ, 2007, pp. 51–79.

[Bra59a] L. de Branges. “Some Hilbert spaces of entire functions”. In: *Proc. Amer. Math. Soc.* 10 (1959), pp. 840–846.

[Bra59b] L. de Branges. “Some mean squares of entire functions”. In: *Proc. Amer. Math. Soc.* 10 (1959), p. 833 839.

[Bra61] L. de Branges. “Some Hilbert spaces of entire functions. II”. In: *Trans. Amer. Math. Soc.* 99 (1961), pp. 118–152.

[Bra68] L. de Branges. *Hilbert spaces of entire functions.* Englewood Cliffs, N.J.: Prentice-Hall Inc., 1968, pp. ix+326.

[BS11] A. Borichev and M. Sodin. “Weighted exponential approximation and non-classical orthogonal spectral measures”. In: *Adv. Math.* 226.3 (2011), pp. 2503–2545.

[DK78a] H. Dym and N. Kravitsky. “On recovering the mass distribution of a string from its spectral function”. In: *Topics in functional analysis (essays dedicated to M. G. Kreĭn on the occasion of his 70th birthday).* Vol. 3. Adv. in Math. Suppl. Stud. Academic Press, New York-London, 1978, pp. 45–90.

[DK78b] H. Dym and N. Kravitsky. “On the inverse spectral problem for the string equation”. In: *Integral Equations Operator Theory* 1.2 (1978), pp. 270–277.

[DM70] H. Dym and H.P. McKean. “Application of de Branges spaces of integral functions to the prediction of stationary Gaussian processes”. In: *Illinois J. Math.* 14 (1970), pp. 299–343.

[Dym70] H. Dym. “An introduction to de Branges spaces of entire functions with applications to differential equations of the Sturm-Liouville type”. In: *Advances in Math.* 5 (1970), pp. 395–471.

[Fre69] G. Freud. *Orthogonale Polynome.* Lehrbücher und Monographien aus dem Gebiete der Exakten Wissenschaften, Mathematische Reihe, Band 33. Birkhäuser Verlag, Basel-Stuttgart, 1969, p. 294.

[GL51] I. M. Gel′fand and B. M. Levitan. “On the determination of a differential equation from its spectral function”. In: *Izvestiya Akad. Nauk SSSR. Ser. Mat.* 15 (1951), pp. 309–360.

[Gol62] A. A. Gol′dberg. “The integral over a semi-additive measure and its application to the theory of entire functions. II”. In: *Mat. Sb. (N.S.)* 61.103 (1962), pp. 334–349.

[HM03a] V. Havin and J. Mashreghi. “Admissible majorants for model subspaces of H^2. I. Slow winding of the generating inner function”. In: *Canad. J. Math.* 55.6 (2003), pp. 1231–1263.

[HM03b] V. Havin and J. Mashreghi. “Admissible majorants for model subspaces of H^2. II. Fast winding of the generating inner function”. In: *Canad. J. Math.* 55.6 (2003), pp. 1264–1301.

[KW05a] M. Kaltenbäck and H. Woracek. “De Branges spaces of exponential type: general theory of growth”. In: *Acta Sci. Math. (Szeged)* 71.1-2 (2005), pp. 231–284.

[KW05b] M. Kaltenbäck and H. Woracek. “Hermite-Biehler functions with zeros close to the imaginary axis”. In: *Proc. Amer. Math. Soc.* 133.1 (2005), 245–255 (electronic).

[Lev80] B.Ja. Levin. *Distribution of zeros of entire functions.* Revised. Vol. 5. Translations of Mathematical Monographs. Translated from the Russian by R. P. Boas, J. M. Danskin, F. M. Goodspeed, J. Korevaar, A. L. Shields and H. P. Thielman. Providence, R.I.: American Mathematical Society, 1980, pp. xii+523.

[LG86] P. Lelong and L. Gruman. *Entire functions of several complex variables.* Vol. 282. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Berlin: Springer-Verlag, 1986, pp. xii+270.

[LW13a] M. Langer and H. Woracek. “Indefinite Hamiltonian systems whose Titchmarsh–Weyl coefficients have no finite generalized poles of non-positive type”. In: *Oper. Matrices* 7.3 (2013), pp. 477–555.

[LW13b] M. Langer and H. Woracek. “The exponential type of the fundamental solution of an indefinite Hamiltonian system”. In: *Complex Anal. Oper. Theory* 7.1 (2013), pp. 285–312.

[OS02] J. Ortega-Cerdà and K. Seip. “Fourier frames”. In: *Ann. of Math. (2)* 155.3 (2002), pp. 789–806.

[Pit72] L.D. Pitt. “On problems of trigonometrical approximation from the theory of stationary Gaussian processes”. In: *J. Multivariate Anal.* 2 (1972), pp. 145–161.

[Pol13] A. Poltoratski. “A problem on completeness of exponentials”. In: *Ann. of Math. (2)* 178.3 (2013), pp. 983–1016.

[Rem02] C. Remling. “Schrödinger operators and de Branges spaces”. In: *J. Funct. Anal.* 196.2 (2002), pp. 323–394.

[Rub96] L.A. Rubel. *Entire and meromorphic functions.* Universitext. With the assistance of James E. Colliander. New York: Springer-Verlag, 1996, pp. viii+187.

[Win00] H. Winkler. “Small perturbations of canonical systems”. In: *Integral Equations Operator Theory* 38.2 (2000), pp. 222–250.

[Win95] H. Winkler. “The inverse spectral problem for canonical systems”. In: *Integral Equations Operator Theory* 22.3 (1995), pp. 360–374.

[Wor14] H. Woracek. *Reproducing kernel almost Pontryagin spaces*. 40 pp., ASC Report 14. http://www.asc.tuwien.ac.at/preprint/2014/asc14x2014.pdf. Vienna University of Technology, 2014.

[Yud00] P. Yuditskii. “Analytic perturbation preserves determinacy of infinite index”. In: *Math. Scand.* 86.2 (2000), pp. 288–292.

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