

The Krein formula in almost Pontryagin spaces. A proof via orthogonal coupling

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Abstract: A new proof is provided for the Krein formula for generalized resolvents in the context of symmetric operators or relations with defect numbers $(1, 1)$ in an almost Pontryagin space. The new proof is geometric and uses the orthogonal coupling of the almost Pontryagin spaces induced by the Q -function and the parameter function in the Krein formula.

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1 Introduction

Large classes of classical problems in analysis such as interpolation problems, continuation problems of positive definite functions, and power moment problems can be solved using of Krein's resolvent formula. This method is based on the fact that solutions can be expressed in terms of resolvents of selfadjoint extensions of a certain symmetric operator acting in a space built from given data.

In order to briefly review Krein's formula let S be a closed symmetric linear operator in a Hilbert space \mathcal{H} and assume that S has defect numbers $(1, 1)$. Recall that there is a holomorphic family $\chi(z) \in \ker(S^* - z)$ and a Q -function q defined as a solution of the equation

$$\frac{q(z) - q(w)^*}{z - \bar{w}} = \chi(w)^* \chi(z). \quad (1.1)$$

The Krein formula, cf. [16], establishes a one-to-one correspondence between the set of all selfadjoint extensions A of S in \mathcal{H} and the set of parameters $\tau \in \mathbb{R} \cup \{\infty\}$ as follows

$$(A - z)^{-1} = (\mathring{A} - z)^{-1} - [\cdot, \chi(\bar{z})] \frac{1}{q(z) + \tau} \chi(z), \quad z \in \rho(A) \cap \rho(\mathring{A}). \quad (1.2)$$

In fact, it establishes a one-to-one correspondence between the set of all selfadjoint exit space extensions A of S and the set of all Nevanlinna functions $\tau(z)$ when the denominator $q(z) + \tau$ in (1.2) is replaced by $q(z) + \tau(z)$, cf. [1].

There is a huge literature on Krein's formula and its applications. We give only a very brief account on the history of Krein's formula and its generalizations. M.G. Krein considered in [15] densely defined symmetric operators in Hilbert spaces which have finite and equal defect numbers; the case of defect $(1, 1)$ goes back to [16]. In [23] the case of infinite equal defect numbers was settled by Sh.N. Saakjan. Linear relations (i.e., multi-valued linear operators)

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came into play in [21]. A general treatment in the setting of boundary relations was given rather recently in [7]. Concerning the early history, one should also mention M.A. Naimark [22] and A.V. Straus [24], who considered the problem in a somewhat different language. The move away from the positive definite regime to symmetric operators with equal (possibly infinite) defect numbers in Pontryagin spaces was made by M.G. Krein and H. Langer in [19, 20]. The case of standard symmetric relations in Krein spaces was investigated in [10]. Having available Krein's formula in indefinite inner product spaces significantly widens the range of applications, e.g. [17, 18] or [6]. However, the usual approach still requires that the data space is not degenerated.

The present paper is concerned with a variant of (1.2) in the context of symmetric operators or relations with defect numbers $(1, 1)$ in an almost Pontryagin space. Basically an almost Pontryagin space is a Pontryagin space to which a degenerated linear space has been added orthogonally; for a precise formulation, see [13]. The theory of such spaces has been specifically introduced to provide an abstract framework for classical problems from analysis giving rise to a degenerated data space. An earlier treatment of an appropriate version of the Krein formula in the context of degeneracies, including some applications, was given in [14]. There the treatment was rather ad hoc; the aim here is to give a proof relying on structural and geometric ideas. Our present approach is based on general algebraic constructions and it is hoped that the same method can be used to understand the case of higher defect.

The geometric intuition underlying our approach for the degenerated case is quite the same as in some proofs dealing with the case of Hilbert or Pontryagin spaces, cf. [11] and [8]. However, the actual formulas and the tools being employed here are often highly specific for the degenerated situation. For instance, we use orthogonal couplings, i.e., decompositions of a space into two orthogonal summands which have a nontrivial intersection, cf. [3, §4], and a strong duality between Q -functions and h_0 -resolvents, cf. [5, Theorem III.8]. Both notions have no analogs in the nondegenerated context.

The contents of this paper are, after this introduction, divided into three sections. In Section 2 we explain some notation, briefly review the notion of orthogonal coupling, and extend it to relations; furthermore, we recall a regularity condition such that the notion of a Q -function can be developed. In Section 3 we show that additive decompositions of a Q -function correspond to orthogonal couplings of the considered space and symmetry. In Section 4 we formulate and prove the discussed variant of Krein's formula in almost Pontryagin spaces by combining these facts with results established in our earlier work [3, 4, 5].

2 Preliminaries

2.1 Some notions and notations

Here we briefly review the most important notions for linear relations in almost Pontryagin spaces; the presentation is rather informal. For details concerning the theory of Pontryagin spaces we refer to [12], for linear relations in Pontryagin spaces to [2] and [9], for almost Pontryagin spaces to [13], and for orthogonal couplings, compressed resolvents, and Q -functions to [3] and [5].

A Pontryagin space \mathcal{A} is a linear space with inner product $[\cdot, \cdot]$, which can

be decomposed into the direct and orthogonal sum of a Hilbert space and a finite dimensional negative definite subspaces. An almost Pontryagin space admits finite degeneracy: it is a Pontryagin space to which a finite dimensional degenerated linear space has been added orthogonally. For a precise formulation, see [13, Proposition 2.5]. Moreover \mathcal{A}° denotes the isotropic part of \mathcal{A} , i.e., $\mathcal{A}^\circ = \mathcal{A} \cap \mathcal{A}^\perp$, and $\text{ind}_0(\mathcal{A}) = \dim \mathcal{A}^\circ$ is called the degree of degeneracy of \mathcal{A} .

In general, a pair (ι, \mathcal{P}) is called a canonical Pontryagin space extension of an almost Pontryagin space \mathcal{A} if \mathcal{P} is a Pontryagin space, and the extension embedding $\iota : \mathcal{A} \rightarrow \mathcal{P}$ is an injective morphism and $\dim \mathcal{P}/\iota(\mathcal{A}) = \text{ind}_0 \mathcal{A}$. Canonical Pontryagin space extensions are in some sense minimal among all Pontryagin spaces which contain \mathcal{A} as a closed subspace. Canonical Pontryagin space extensions are unique up to isomorphisms and will be denoted by $(\iota_{\text{ext}}, \mathfrak{P}_{\text{ext}}(\mathcal{A}))$; for a particular construction, see [3].

A linear relation T in an almost Pontryagin space \mathcal{A} is a linear subspace of the product $\mathcal{A} \times \mathcal{A}$; it is closed when it is closed as a subspace. The set $\gamma(T)$ of points of regular type of T is an open set defined by

$$\gamma(T) = \{\lambda \in \mathbb{C} : (T - z)^{-1} \text{ is a bounded operator}\},$$

and on connected components of $\gamma(T)$ the defect numbers $\dim(\mathcal{A}/\text{ran}(T - z))$ are constant. The adjoint T^* of the relation T in \mathcal{A} is the linear relation in \mathcal{A} defined by

$$T^* = \{(x, y) \in \mathcal{A}^2 : [y, a] - [x, b] = 0, (a, b) \in T\}.$$

Note that T^* always contains $\mathcal{A}^\circ \times \mathcal{A}^\circ$. A linear relation is symmetric in \mathcal{A} when $T \subseteq T^*$; this is the usual definition. In this case the set $\gamma(T)$ is either connected or splits into two connected components $\gamma(T) \cap \mathbb{C}^+$ and $\gamma(T) \cap \mathbb{C}^-$. Furthermore when T is closed the resolvent set $\rho(T)$ is defined as

$$\rho(T) = \{z \in \mathbb{C} : (T - z)^{-1} \text{ is a bounded everywhere defined operator}\},$$

so that $\rho(T)$ is open and

$$\rho(T) = \{z \in \gamma(T) : \text{ran}(T - z) \text{ is dense in } \mathcal{A}\}.$$

A linear relation S is called selfadjoint if S is closed, symmetric, and has zero defect numbers. For a closed symmetric relation S in an almost Pontryagin space \mathcal{A} introduce the linear relation S_{fac} by

$$S_{\text{fac}} = S/(\mathcal{A}^\circ \times \mathcal{A}^\circ),$$

so that S_{fac} is closed and symmetric in the Pontryagin space $\mathcal{A}/\mathcal{A}^\circ$. Assume that S has defect numbers $(1, 1)$. Then S_{fac} is selfadjoint in $\mathcal{A}/\mathcal{A}^\circ$ and $\rho(S_{\text{fac}}) \neq \emptyset$ if and only if

$$\exists z_+ \in \mathbb{C}^+, z_- \in \mathbb{C}^- : \text{ran}(S - z_\pm) + \mathcal{A}^\circ = \mathcal{A}; \quad (2.1)$$

and if this is the case, then

$$\rho(S_{\text{fac}}) = \{z \in \mathbb{C} : \text{ran}(S - z) + \mathcal{A}^\circ = \mathcal{A}\},$$

see [5, Lemma II.3].

2.2 Orthogonal coupling of symmetric relations

Loosely speaking the orthogonal coupling of a pair of inner product spaces is an orthogonal sum with a certain, possibly partial, overlap of isotropic parts. A formal definition goes as follows. Let \mathcal{A}_1 and \mathcal{A}_2 be almost Pontryagin spaces, let α be a linear subspace of $\mathcal{A}_1^\circ \times \mathcal{A}_2^\circ$, and set

$$\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2 := (\mathcal{A}_1[\dot{+}]\mathcal{A}_2)/\alpha,$$

cf. [3, §4]. Here $\mathcal{A}_1[\dot{+}]\mathcal{A}_2$ denotes the direct and orthogonal sum of the spaces \mathcal{A}_1 and \mathcal{A}_2 . Associated with this construction are several canonical maps: the embeddings $\iota_j : \mathcal{A}_j \rightarrow \mathcal{A}_1[\dot{+}]\mathcal{A}_2$, $j = 1, 2$, and the canonical projection $\pi : \mathcal{A}_1[\dot{+}]\mathcal{A}_2 \rightarrow (\mathcal{A}_1[\dot{+}]\mathcal{A}_2)/\alpha$. This leads to canonical maps from \mathcal{A}_j into $\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$, namely

$$\iota_1^\alpha(x_1) := (\pi \circ \iota_1)(x_1) = (x_1 \dot{+} 0)/\alpha, \quad \iota_2^\alpha(x_2) := (\pi \circ \iota_2)(x_2) = (0 \dot{+} x_2)/\alpha.$$

The maps ι_j are isometric homeomorphisms onto their ranges. They have continuous left inverses, namely the projections $\pi_j : \mathcal{A}_1[\dot{+}]\mathcal{A}_2 \rightarrow \mathcal{A}_j$, and these are jointly injective and satisfy $\pi_j \circ \iota_i = 0$, $i \neq j$. The map π is surjective, continuous, open and isometric, and maps closed subspaces to closed subspaces.

The corresponding construction of orthogonal couplings on the level of linear relations can be found in the following definition.

2.1 Definition. Let \mathcal{A}_1 and \mathcal{A}_2 be almost Pontryagin spaces and let α be a linear subspace of $\mathcal{A}_1^\circ \times \mathcal{A}_2^\circ$. Moreover, let $S_1 \subseteq (\mathcal{A}_1)^2$ and $S_2 \subseteq (\mathcal{A}_2)^2$ be linear relations. Then the orthogonal coupling $S_1 \boxplus_\alpha S_2 \subseteq (\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2)^2$ of S_1 and S_2 is defined as

$$S_1 \boxplus_\alpha S_2 := (\iota_1^\alpha \times \iota_1^\alpha)(S_1) + (\iota_2^\alpha \times \iota_2^\alpha)(S_2),$$

which is a linear relation. ◇

In the next lemma we show that several properties may be transferred from S_1 and S_2 to the orthogonal coupling $S_1 \boxplus_\alpha S_2$.

2.2 Lemma. *Let \mathcal{A}_1 and \mathcal{A}_2 be almost Pontryagin spaces, and let α be a linear subspace of $\mathcal{A}_1^\circ \times \mathcal{A}_2^\circ$. Moreover, let $S_1 \subseteq (\mathcal{A}_1)^2$ and $S_2 \subseteq (\mathcal{A}_2)^2$ be linear relations.*

- (i) *If S_1 and S_2 are both closed, so is $S_1 \boxplus_\alpha S_2$.*
- (ii) *Let $z \in \mathbb{C}$ and assume that $\text{ran}(S_1 - z)$ and $\text{ran}(S_2 - z)$ are both closed. Then also $\text{ran}(S_1 \boxplus_\alpha S_2 - z)$ is closed.*
- (iii) *If S_1 and S_2 are both symmetric (isometric), so is $S_1 \boxplus_\alpha S_2$.*

Proof. (i) Assume that S_j is closed, $j = 1, 2$. Then also $(\iota_j \times \iota_j)(S_j)$ is closed and hence, due to the existence of left inverses with the above mentioned properties, also the sum

$$S_1[\dot{+}]S_2 := (\iota_1 \times \iota_1)(S_1) + (\iota_2 \times \iota_2)(S_2) = (\pi_1 \times \pi_1)^{-1}(S_1) \cap (\pi_2 \times \pi_2)^{-1}(S_2)$$

is closed. Since $\pi \times \pi$ maps closed subspaces to closed subspaces, it follows that $S_1 \boxplus_\alpha S_2$ is closed.

(ii) Note that for each $z \in \mathbb{C}$,

$$\begin{aligned} \operatorname{ran}(S_1 \boxplus_\alpha S_2 - z) &= \operatorname{ran}((\pi \times \pi)(S_1[+]S_2 - z)) \\ &= \pi(\operatorname{ran}(S_1[+]S_2 - z)) \\ &= \pi(\iota_1(\operatorname{ran}(S_1 - z))[+] \iota_2(\operatorname{ran}(S_2 - z))). \end{aligned} \quad (2.2)$$

Item (ii) follows.

(iii) This statement holds since all canonical maps which are involved are isometric. \square

2.3 Q -functions of symmetric relations with defect $(1, 1)$

Let S be *any* closed symmetric relation in a Pontryagin space and assume that S has equal defect numbers. Then one may define a Q -function for S by choosing *some* canonical selfadjoint extension of S and *some* family of defect elements generated by that extension.

Now assume that S is a closed symmetric relation in an almost Pontryagin space and assume that S has defect numbers $(1, 1)$. In this case the definition of a Q -function is somewhat more involved: a certain regularity assumption must be satisfied by S and not all choices of selfadjoint extensions and defect families are suitable.

(A) *Situation in which Q -functions can be defined.* Let \mathcal{A} be an almost Pontryagin space with $\Delta := \operatorname{ind}_0 \mathcal{A} > 0$ and let $S \subseteq \mathcal{A}^2$ be a closed symmetric relation with defect index $(1, 1)$ which satisfies the regularity conditions

$$\forall h \in \mathcal{A}^\circ : S \cap (\operatorname{span}\{h\} \times \operatorname{span}\{h\}) = \{0\}, \quad (2.3)$$

$$\exists z_+ \in \mathbb{C}^+, z_- \in \mathbb{C}^- : \operatorname{ran}(S - z_\pm) + \mathcal{A}^\circ = \mathcal{A}. \quad (2.4)$$

(B) *Choices to be made.* Assume that the space \mathcal{A} and the relation S are given according to **(A)**. Then the following existence statements are valid, see [14, §2] (putting together Proposition 1, Corollary 1, and Proposition 2 of this reference).

(i) There exist elements h_l , $l = 0, \dots, \Delta - 1$, such that $\{h_0, \dots, h_{\Delta-1}\}$ is a basis of \mathcal{A}° , and such that

$$S \cap (\mathcal{A}^\circ)^2 = \operatorname{span}\{(h_l, h_{l+1}) : l = 0, \dots, \Delta - 2\}. \quad (2.5)$$

(ii) There exist selfadjoint relations $\mathring{A} \subseteq \mathfrak{P}_{\text{ext}}(\mathcal{A})^2$ with nonempty resolvent set which satisfy $\mathring{A} \supseteq S \cup \{(0, h_0)\}$. For each such relation \mathring{A} there exists a family $(\chi(z))_{z \in \rho(\mathring{A})}$ of elements $\chi(z) \in \mathfrak{P}_{\text{ext}}(\mathcal{A})$, $z \in \rho(\mathring{A})$, such that

$$\chi(z) \perp \operatorname{ran}(S - \bar{z}), \quad z \in \rho(\mathring{A}), \quad (2.6)$$

$$\chi(z) = (I + (z - w)(\mathring{A} - z)^{-1})\chi(w), \quad z, w \in \rho(\mathring{A}), \quad (2.7)$$

$$[\chi(z), h_l] = z^l, \quad z \in \rho(\mathring{A}), \quad l = 0, \dots, \Delta - 1. \quad (2.8)$$

The element h_0 is uniquely determined up to scalar multiples. Once a choice of h_0 is made, the elements $h_1, \dots, h_{\Delta-1}$ are unique.

2.3 Remark. Let S be any relation in \mathcal{A} , and assume that a basis $\{h_0, \dots, h_{\Delta-1}\}$ of \mathcal{A}° is given such that (2.5) holds. Then S satisfies (2.3).

To see this, let $h = \sum_{j=0}^{\Delta-1} \alpha_j h_j \in \mathcal{A}^\circ$, and assume that $(\lambda h, \mu h) \in S$ where $(\lambda, \mu) \neq (0, 0)$. Then we can write

$$(\lambda h, \mu h) = \sum_{j=0}^{\Delta-2} \beta_j (h_j, h_{j+1})$$

with some β_j . This yields $\sum_{j=0}^{\Delta-2} \mu \beta_j h_j = \sum_{j=0}^{\Delta-2} \lambda \beta_j h_{j+1}$, and comparing coefficients implies $\beta_j = 0$, $j = 0, \dots, \Delta - 2$, i.e., $h = 0$. \diamond

The following construction of Q -functions in almost Pontryagin spaces first appeared in [14] and was further developed and extended in [5, Part II].

2.4 Definition. Let \mathcal{A}, S be given according to **(A)** and let h_0, \mathring{A}, χ be chosen according to **(B)**. Then a function q with

$$\frac{q(z) - q(\bar{w})}{z - \bar{w}} = [\chi(z), \chi(w)], \quad z, w \in \rho(\mathring{A}), \quad (2.9)$$

is called a Q -function of S , or more specifically, a Q -function of S built with h_0, \mathring{A}, χ . \diamond

3 The sum of Q -functions as a Q -function

Let \mathcal{P}_1 and \mathcal{P}_2 be Pontryagin spaces and let S_1 and S_2 be closed symmetric relations with defect index $(1, 1)$ in these spaces. Moreover, let q_1 and q_2 be Q -functions of S_1 and S_2 , respectively. Then $S_1[+]S_2 \subseteq (\mathcal{P}_1[+] \mathcal{P}_2)^2$ is a closed symmetric relation and has defect index $(2, 2)$. The sum $q_1 + q_2$ is a Q -function of some symmetric extension of $S_1[+]S_2$ with defect $(1, 1)$.

In the case of almost Pontryagin spaces there is a significant difference. Now the direct and orthogonal sum has to be replaced by an orthogonal coupling with overlap and, as a consequence, the orthogonal coupling of S_1 and S_2 already has defect $(1, 1)$.

3.1 Theorem. Let \mathcal{A}_1, S_1 and \mathcal{A}_2, S_2 be given according to **(A)** and let q_j be a Q -function of S_j , $j = 1, 2$. Assume that

$$\text{ind}_0 \mathcal{A}_1 = \text{ind}_0 \mathcal{A}_2 > 0,$$

and that $q_1 + q_2$ does not vanish identically. Then there exists a bijective map α from \mathcal{A}_1° onto \mathcal{A}_2° , such that:

- (i) The relation $S := S_1 \boxplus_\alpha S_2$ acting in the almost Pontryagin space $\mathcal{A} := \mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$ is closed and symmetric with defect index $(1, 1)$, and satisfies the regularity conditions (2.3) and (2.4) in **(A)**.
- (ii) The function $q := q_1 + q_2$ is a Q -function of S .

Proof. For $j \in \{1, 2\}$, let $h_0^j, \mathring{A}_j, \chi_j$ be the data according to **(B)** for S_j such that q_j is a Q -function of S_j built with these data. These data will be used to construct corresponding data for S so that **(A)** and **(B)** are satisfied for S .

Step 1. We define α and h_0, \mathring{A}, η . Let α be the following subspace

$$\alpha := \text{span} \{ (h_l^1, -h_l^2) : l = 0, \dots, \Delta - 1 \};$$

then α is (the graph of) a bijective map of \mathcal{A}_1° onto \mathcal{A}_2° .

The space $\mathcal{A} := \mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$ is an almost Pontryagin space with $\text{ind}_0 \mathcal{A} = \Delta$, cf. [3, Remark 4.6]. Since α is a bijective function, the canonical mappings $\iota_j^\alpha : \mathcal{A}_j \rightarrow \mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$ are injective, cf. [3, Remark 4.3(ii)]. Let $\tilde{\iota}_j^\alpha : \mathfrak{P}_{\text{ext}}(\mathcal{A}_j) \rightarrow \mathfrak{P}_{\text{ext}}(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2)$ be chosen according to [3, Proposition 5.7]. We have embeddings of all involved spaces into $\mathfrak{P}_{\text{ext}}(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2)$, cf. [3, Remark 5.8]:

$$\begin{array}{ccccc} & & \mathfrak{P}_{\text{ext}}(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2) & & \\ & \tilde{\iota}_1^\alpha \nearrow & \uparrow \iota_{\text{ext}} & \nwarrow \tilde{\iota}_2^\alpha & \\ \mathfrak{P}_{\text{ext}}(\mathcal{A}_1) & & \mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2 & & \mathfrak{P}_{\text{ext}}(\mathcal{A}_2) \\ & \iota_{\text{ext}} \nwarrow & \nearrow \iota_1^\alpha & \nwarrow \iota_2^\alpha & \nearrow \iota_{\text{ext}} \\ & \mathcal{A}_1 & & & \mathcal{A}_2 \end{array}$$

In order to simplify the notation, we set

$$\mathcal{P}_j := \mathfrak{P}_{\text{ext}}(\mathcal{A}_j), \quad j = 1, 2, \quad \mathcal{P} := \mathfrak{P}_{\text{ext}}(\mathcal{A}) = \mathfrak{P}_{\text{ext}}(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2),$$

and we drop the explicit notation of $\iota_{\text{ext}}, \iota_j^\alpha, \tilde{\iota}_j^\alpha$. As a consequence we consider all spaces $\mathcal{A}_j, \mathcal{P}_j$ and $\mathcal{A} = \mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$ as subspaces of \mathcal{P} . Using this abuse of language, the definition of α gives $h_l^1 = h_l^2, l = 0, \dots, \Delta - 1$, and we set

$$h_l := h_l^j, \quad l = 0, \dots, \Delta - 1, \quad j = 1, 2.$$

Further, we have $S = \text{span}(S_1 \cup S_2)$, and by Lemma 2.2 the relation S is closed and symmetric.

Choose a concrete realization of \mathcal{A} as in [3, Remark 4.8] and, correspondingly, a concrete realization of \mathcal{P} as in [3, Remark 5.8]. Moreover, let $P_{\mathcal{A}}, P_{\mathcal{A}_{1,r}}$, etc., be the projections of \mathcal{P} onto $\mathcal{A}, \mathcal{A}_{1,r}$, etc., introduced there.

Now we define elements $\eta(z)$ and a relation $\mathring{A} \subseteq \mathcal{P}^2$ as

$$\eta(z) := \chi_1(z) + P_{\mathcal{A}}\chi_2(z), \quad z \in \rho(\mathring{A}_1) \cap \rho(\mathring{A}_2),$$

$$\mathring{A} := \text{cls} \left(S \cup \{ (0, h_0) \} \cup \{ (\eta(z) - \eta(w), z\eta(z) - w\eta(w)) : z, w \in \rho(\mathring{A}_1) \cap \rho(\mathring{A}_2) \} \right).$$

Step 2. We show that $q_1 + q_2$ satisfies the kernel relation (2.9) with η , and that (2.8) holds for h_0, \mathring{A}, η . First note that

$$[\chi_1(z), h_l] = z^l = [\chi_2(z), h_l], \quad l = 0, \dots, \Delta - 1,$$

and hence

$$[\chi_1(z), h] = [\chi_2(z), h], \quad h \in \mathcal{A}^\circ.$$

Therefore [3, Lemma 5.9(iv)] may be applied, which yields that

$$\begin{aligned}
[\eta(z), \eta(w)] &= [\chi_1(z), \chi_1(w)] + [\chi_2(z), \chi_2(w)] \\
&= \frac{q_1(z) - \overline{q_1(w)}}{z - \overline{w}} + \frac{q_2(z) - \overline{q_2(w)}}{z - \overline{w}} \\
&= \frac{(q_1 + q_2)(z) - \overline{(q_1 + q_2)(w)}}{z - \overline{w}};
\end{aligned} \tag{3.1}$$

recall also [3, Lemma 5.9(iii)]. Moreover, observe that by [3, Lemma 5.9(i), (iii)], and Remark 4.3(i)]

$$[\eta(z), y_j] = [\chi_j(z), y_j], \quad y_j \in \mathcal{A}_j, j = 1, 2, \quad z \in \rho(\mathring{A}_1) \cap \rho(\mathring{A}_2). \tag{3.2}$$

In particular, (2.8) holds.

Step 3. We show that \mathring{A} is selfadjoint with nonempty resolvent set, and that (2.6), (2.7) hold. We already noted that S is symmetric. By (3.1) and a standard computation, the relation

$$\{(\eta(z) - \eta(w), z\eta(z) - w\eta(w)) : z, w \in \rho(\mathring{A}_1) \cap \rho(\mathring{A}_2)\} \tag{3.3}$$

is symmetric. If $(x_j, y_j) \in S_j$, then by (3.2)

$$[\eta(z), y_j - \overline{z}x_j] = [\chi_j(z), y_j - \overline{z}x_j] = 0. \tag{3.4}$$

Since $S = \text{span}(S_1 \cup S_2)$, the span of S with (3.3) is symmetric. Further, (2.6) follows remembering (2.2).

From (3.2) we see that also $h_0 \perp \text{dom } \mathring{A}$, and together conclude that \mathring{A} is symmetric. To show that \mathring{A} is actually selfadjoint, we use [5, Lemma 2.12]. For each $z \in \rho(S_{\text{fac}}) \cap \rho(\mathring{A}_1) \cap \rho(\mathring{A}_2)$

$$\begin{aligned}
\mathfrak{P}_{\text{ext}}(\mathcal{A}) &\supseteq \text{ran}(\mathring{A} - z) \\
&\supseteq \text{ran}(S - z) + \text{span}\{h_0\} \\
&\quad + \text{span}\{\eta(w) : w \in \rho(\mathring{A}_1) \cap \rho(\mathring{A}_2), w \neq z\} \\
&= \mathcal{A} + \text{span}\{\eta(w) : w \in \rho(\mathring{A}_1) \cap \rho(\mathring{A}_2), w \neq z\} \\
&= \mathfrak{P}_{\text{ext}}(\mathcal{A}),
\end{aligned}$$

where for the last equality [5, Lemma II.27] is used. Hence it follows that \mathring{A} is indeed selfadjoint and that $\rho(\mathring{A}) \supseteq \rho(S_{\text{fac}}) \cap \rho(\mathring{A}_1) \cap \rho(\mathring{A}_2)$. Finally note that (2.7) is built in the definition of \mathring{A} .

Step 4. We show that S has defect index $(1, 1)$, and that (2.3), (2.4), (2.5) hold. Start with investigating $S \cap (\mathcal{A}^\circ)^2$. Assume that $(x_1, y_1) \in S_1$, $(x_2, y_2) \in S_2$ and that $(x_1 + x_2, y_1 + y_2) \in (\mathcal{A}^\circ)^2$. Then we have

$$P_{\mathcal{A}_1, r} x_1 = P_{\mathcal{A}_1, r} (x_1 + x_2) - P_{\mathcal{A}_1, r} x_2 = 0,$$

and hence $x_1 \in \mathcal{A}^\circ$. In the same way it follows that $x_2, y_1, y_2 \in \mathcal{A}^\circ$. Thus

$$(x_j, y_j) \in S_j \cap (\mathcal{A}^\circ)^2 = \text{span}\{(h_l, h_{l+1}) : l = 0, \dots, \Delta - 2\}, \quad j = 1, 2,$$

and hence also $(x_1 + x_2, y_1 + y_2) \in \text{span}\{(h_l, h_{l+1}) : l = 0, \dots, \Delta - 2\}$. We conclude that

$$S \cap (\mathcal{A}^\circ)^2 = \text{span}\{(h_l, h_{l+1}) : l = 0, \dots, \Delta - 2\},$$

i.e., (2.5) holds. By Remark 2.3, S satisfies (2.3).

It is easily seen that S satisfies the condition (2.4). In fact, for each $z \in \rho(S_{1,\text{fac}}) \cap \rho(S_{2,\text{fac}})$ we have

$$\begin{aligned} & \text{ran}(S - z) + \text{span}\{h_0\} \\ &= \text{ran}(S_1 - z) + \text{ran}(S_2 - z) + \text{span}\{h_0\} \\ &= [\text{ran}(S_1 - z) + \text{span}\{h_0\}] + [\text{ran}(S_2 - z) + \text{span}\{h_0\}] \\ &= \mathcal{A}_1 + \mathcal{A}_2 = \mathcal{A}. \end{aligned}$$

This observation also shows that the defect indices of S do not exceed 1, and that (2.4) holds. On the other hand, since $q_1 + q_2$ is not identically zero also $\eta(z)$ does not vanish identically, and we conclude from (2.6) that the defect indices of S are equal to 1.

Step 5. Conclusion. Putting together Steps 2, 3, and 4, we have shown that S has the properties **(A)**, that h_0, A, η have the properties **(B)**, and that $q_1 + q_2$ is a Q -function of S . \square

Before the statement of the next theorem, here is a small detour to negative indices. For the following definition, see [5, Definition II.10, Definition II.12].

3.2 Definition. Let f be a function which is meromorphic on $\mathbb{C} \setminus \mathbb{R}$ with domain of holomorphy $\rho(f)$. Then $\text{ind}_- f$ is defined as the supremum of the numbers of negative squares of quadratic forms

$$Q_f(\xi_1, \dots, \xi_n) := \sum_{i,j=1}^n N_f(z_i, z_j) \xi_i \bar{\xi}_j,$$

where $n \in \mathbb{N}_0$ and $z_1, \dots, z_n \in \rho(f)$. Here N_f stands for the Nevanlinna kernel

$$N_f(z, w) := \frac{f(z) - f(\bar{w})}{z - \bar{w}}.$$

Let $\Delta \in \mathbb{N}$. Then $\text{ind}_-^\Delta f$ is defined as the supremum of the numbers of negative squares of quadratic forms

$$Q_f^\Delta(\xi_1, \dots, \xi_n; \eta_0, \dots, \eta_{\Delta-1}) := \sum_{i,j=1}^n N_f(z_i, z_j) \xi_i \bar{\xi}_j + \sum_{k=0}^{\Delta-1} \sum_{i=1}^n \text{Re}(z_i^k \xi_i \bar{\eta}_k),$$

where $n \in \mathbb{N}_0$ and $z_1, \dots, z_n \in \rho(f)$. \diamond

As a direct consequence of the above definitions one sees the following decomposition of quadratic forms

$$Q_{q_1+q_2}^\Delta = Q_{q_1}^\Delta + N_{q_2}.$$

Therefore, in terms of Definition 3.2, one obtains the estimate

$$\text{ind}_-^\Delta(q_1 + q_2) \leq \text{ind}_-^\Delta q_1 + \text{ind}_- q_2.$$

In fact, Theorem 3.1 now provides an improvement of this estimate; remember here [5, (II.15)].

3.3 Corollary. *Let $q_1, q_2 \in \mathcal{N}_{<\infty}$ and $\Delta \in \mathbb{N}$. Then*

$$\text{ind}_-^\Delta(q_1 + q_2) \leq \text{ind}_-^\Delta q_1 + \text{ind}_-^\Delta q_2 - \Delta.$$

Proof. If one of the functions $q_1, q_2, q_1 + q_2$ vanishes identically this inequality holds trivially; recall [5, Remark II.13]. Hence, assume that

$$q_1, q_2, q_1 + q_2 \in \mathcal{N}_{<\infty} \setminus \{0\}.$$

By [5, Proposition II.14], q_j can be realized as a Q -function of some minimal symmetry $S_\Delta(q_j)$ acting in an almost Pontryagin space $\mathcal{A}_\Delta(q_j)$ with $\text{ind}_0 \mathcal{A}_\Delta(q_j) = \Delta$ and $\text{ind}_- \mathcal{A}_\Delta(q_j) = \text{ind}_- q_j - \Delta$. Theorem 3.1 tells us that $q_1 + q_2$ is a Q -function of the symmetry $S_\Delta(q_1) \boxplus_\alpha S_\Delta(q_2)$ acting in the almost Pontryagin space $\mathcal{A}_\Delta(q_1) \boxplus_\alpha \mathcal{A}_\Delta(q_2)$. Using [5, Proposition II.14] and [3, Remark 4.6], it follows that

$$\begin{aligned} \text{ind}_-^\Delta(q_1 + q_2) &\leq \text{ind}_- (\mathcal{A}_\Delta(q_1) \boxplus_\alpha \mathcal{A}_\Delta(q_2)) + \Delta \\ &= \text{ind}_- \mathcal{A}_\Delta(q_1) + \text{ind}_- \mathcal{A}_\Delta(q_2) + \Delta \\ &= \text{ind}_-^\Delta q_1 + \text{ind}_-^\Delta q_2 - \Delta, \end{aligned}$$

which is the desired result. \square

In Theorem 3.1 a pair of operators or relations and their Q -functions is combined via an orthogonal coupling. Its counterpart Theorem 3.4 shows how to write a given Q -function as a sum of Q -functions.

3.4 Theorem. *Let \mathcal{A}, S be given according to **(A)**, let $h_0 \in \mathcal{A}^\circ$ be according to (i) of **(B)**, and let q be a Q -function of S built with h_0 (and some \dot{A}, χ according to (ii) of **(B)**). Let \mathcal{A}_1 be a closed subspace of \mathcal{A} with $\mathcal{A}_1^\circ = \mathcal{A}^\circ$. Then \mathcal{A}_1 and $\mathcal{A}_2 := \mathcal{A}[-]\mathcal{A}_1$ are almost Pontryagin spaces and $\mathcal{A}_2^\circ = \mathcal{A}^\circ$.*

Let $M \subseteq \mathbb{C}$ be a subset which is symmetric with respect to \mathbb{R} and has nonempty interior, and assume that

$$(S - z)^{-1}(\mathcal{A}_1 \cap \text{ran}(S - z)) \subseteq \mathcal{A}_1, \quad z \in M. \quad (3.5)$$

Then the relations

$$S_1 := S \cap (\mathcal{A}_1)^2 \quad \text{and} \quad S_2 := S \cap (\mathcal{A}_2)^2$$

*are closed symmetric and are as in **(A)**. Explicitly: S_1 and S_2 have defect index $(1, 1)$, and satisfy the regularity conditions (2.3) and (2.4).*

*Let q_1 be a Q -function of S_1 built with h_0 (and some \dot{A}_1, χ_1 according to (ii) of **(B)**). Then $q_2 := q - q_1$ is a Q -function of S_2 built with h_0 (and some \dot{A}_2, χ_2 according to (ii) of **(B)**).*

Proof. The proof proceeds in several steps.

Step 1. Geometric situation. As closed subspaces of \mathcal{A} , the spaces \mathcal{A}_1 and \mathcal{A}_2 are themselves almost Pontryagin spaces. By assumption we have $\mathcal{A}_1^\circ = \mathcal{A}^\circ$. The quotient $\mathcal{A}/\mathcal{A}^\circ$ is a Pontryagin space and the image of \mathcal{A}_1 under the canonical projection $\pi : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{A}^\circ$ is a closed and nondegenerated subspace. Thus also its orthogonal complement $\pi(\mathcal{A}_1)^\perp$ is nondegenerated. We have $\mathcal{A}_2 = \pi^{-1}(\pi(\mathcal{A}_1)^\perp)$, and this shows that $\mathcal{A}_2^\circ = \mathcal{A}^\circ$.

Set $\mathcal{P} := \mathfrak{P}_{\text{ext}}(\mathcal{A})$ and choose a subspace $C \subseteq \mathcal{P}$ which is skewly linked with \mathcal{A}° . Moreover, set

$$\begin{aligned}\mathcal{A}_{1,r} &:= \mathcal{A}_1 \cap (\mathcal{A}^\circ + C)^\perp, & \mathcal{A}_{2,r} &:= \mathcal{A}_2 \cap (\mathcal{A}^\circ + C)^\perp, \\ \mathcal{P}_1 &:= \mathcal{A}_{1,r}[\dot{+}](\mathcal{A}^\circ + C), & \mathcal{P}_2 &:= \mathcal{A}_{2,r}[\dot{+}](\mathcal{A}^\circ + C).\end{aligned}$$

It will now be shown that

$$\mathcal{A}_1 = \mathcal{A}_{1,r}[\dot{+}]\mathcal{A}^\circ. \quad (3.6)$$

The inclusion ‘ \supseteq ’ holds due to our assumption that $\mathcal{A}^\circ = \mathcal{A}_1^\circ$. Let us now show the inclusion ‘ \subseteq ’. Assume that $x \in \mathcal{A}_1$, and write $x = x_1 + x_2$ with $x_1 \in \mathcal{A}^\circ + C$ and $x_2 \in (\mathcal{A}^\circ + C)^\perp$. Since $x \perp \mathcal{A}_1^\circ = \mathcal{A}^\circ$, it follows that $x_1 \perp \mathcal{A}^\circ$. Hence $x_1 \in \mathcal{A}^\circ \subseteq \mathcal{A}_1$, and therefore $x_2 \in \mathcal{A}_{1,r}$. Thus (3.6) holds.

The equality (3.6) implies that we may consider \mathcal{P}_1 as $\mathfrak{P}_{\text{ext}}(\mathcal{A}_1)$ and that $\mathcal{A}_{2,r} = \mathcal{P}_1^\perp$. In the same way, it follows that $\mathcal{A}_2 = \mathcal{A}_{2,r}[\dot{+}]\mathcal{A}^\circ$, \mathcal{P}_2 may be considered as $\mathfrak{P}_{\text{ext}}(\mathcal{A}_2)$, and $\mathcal{A}_{1,r} = \mathcal{P}_2^\perp$.

Finally, let $P_{\mathcal{A}}$, $P_{\mathcal{A}_{1,r}}$, etc., have the same meaning as in the proof of Theorem 3.1.

Step 2. The relations S_1 and S_2 . It is clear that S_j is a closed symmetric relation in \mathcal{A}_j . Moreover, since $S_j \subseteq S$, the regularity condition (2.3) is satisfied by S_j . In fact,

$$S_j \cap (\mathcal{A}_j^\circ)^2 = S \cap (\mathcal{A}^\circ)^2 = \text{span}\{(h_l, h_{l+1}) : l = 0, \dots, \Delta - 2\}.$$

From [4, Proposition 3.2] we obtain

$$\text{ran}(S_1 - z) = \mathcal{A}_1 \cap \text{ran}(S - z), \quad z \in \gamma(S). \quad (3.7)$$

Since $h_0 \in \mathcal{A}_1$, it follows that

$$\begin{aligned}\text{ran}(S_1 - z) + \text{span}\{h_0\} &= (\mathcal{A}_1 \cap \text{ran}(S - z)) + \text{span}\{h_0\} \\ &= \mathcal{A}_1 \cap (\text{ran}(S - z) + \text{span}\{h_0\}) \\ &= \mathcal{A}_1, \quad z \in \gamma(S) \cap \rho(S_{\text{fac}}).\end{aligned} \quad (3.8)$$

The set $\rho(S_{\text{fac}})$ is nonempty, because S satisfies (2.4), remember here (2.1). We see that S_1 satisfies (2.4) and the defect indices of S_1 do not exceed 1. Let $z \in [\gamma(S) \cap \rho(S_{\text{fac}})] \setminus \{0\}$. By [14, Corollary 2], we have $h_0 \notin \text{ran}(S - z)$ and hence also $h_0 \notin \text{ran}(S_1 - z)$. Thus the defect indices of S_1 are equal to 1.

To show similar facts for S_2 , choose an extension $\mathring{A} \subseteq \mathcal{P}^2$ of S according to (ii) of **(B)**. We claim that

$$(S - z)^{-1}(\mathcal{A}_2 \cap \text{ran}(S - z)) \subseteq \mathcal{A}_2, \quad z \in \rho(\mathring{A}) \cap \rho(S_{\text{fac}}). \quad (3.9)$$

Once this claim is established, the same argument as above will show that S_2 satisfies (2.4) and has defect indices (1, 1).

In order to verify (3.9) note that the relations (3.5), (3.7), and (3.8) imply that

$$(\mathring{A} - z)^{-1}\mathcal{A}_1 \subseteq \mathcal{A}_1, \quad z \in \rho(\mathring{A}) \cap \rho(S_{\text{fac}}).$$

Thus also

$$(\mathring{A} - z)^{-1}(\mathcal{P}[-]\mathcal{A}_1) \subseteq \mathcal{P}[-]\mathcal{A}_1, \quad z \in \rho(\mathring{A}) \cap \rho(S_{\text{fac}}).$$

By the choice of \mathring{A} , we have $(\mathring{A} - z)^{-1}\mathcal{A} \subseteq \mathcal{A}$. Since $\mathcal{A}_2 = \mathcal{A} \cap (\mathcal{P}[-]\mathcal{A}_1)$ it now follows that $(\mathring{A} - z)^{-1}\mathcal{A}_2 \subseteq \mathcal{A}_2$. In particular, (3.9) holds.

Step 3. Computation of the Nevanlinna kernel of q_2 . Let \mathring{A}, χ and \mathring{A}_1, χ_1 be data such that q is a Q -function of S built with h_0, \mathring{A}, χ and q_1 is a Q -function of S_1 built with $h_0, \mathring{A}_1, \chi_1$.

We have $(I - P_{\mathcal{A}_1})\mathcal{P}_1 = C$, in particular $(I - P_{\mathcal{A}_1})\mathcal{P}_1$ is a neutral subspace. Using this fact, we can compute

$$\begin{aligned} & [\chi_1(z), \chi_1(w)] \\ &= [P_{\mathcal{A}_1}\chi_1(z) + (I - P_{\mathcal{A}_1})\chi_1(z), P_{\mathcal{A}_1}\chi_1(w) + (I - P_{\mathcal{A}_1})\chi_1(w)] \\ &= [P_{\mathcal{A}_1}\chi_1(z), P_{\mathcal{A}_1}\chi_1(w)] + [P_{\mathcal{A}_1}\chi_1(z), (I - P_{\mathcal{A}_1})\chi_1(w)] \\ &\quad + [(I - P_{\mathcal{A}_1})\chi_1(z), P_{\mathcal{A}_1}\chi_1(w)] \\ &= -[P_{\mathcal{A}_1}\chi_1(z), P_{\mathcal{A}_1}\chi_1(w)] + [P_{\mathcal{A}_1}\chi_1(z), \chi_1(w)] + [\chi_1(z), P_{\mathcal{A}_1}\chi_1(w)]. \end{aligned} \tag{3.10}$$

Let $z \in \rho(\mathring{A}_1) \cap \rho(\mathring{A}) \cap \rho(S_{\text{fac}})$, in which case $\text{ran}(S_1 - \bar{z}) + \text{span}\{h_0\} = \mathcal{A}_1$. Since

$$\chi(z) \perp \text{ran}(S - \bar{z}) \supseteq \text{ran}(S_1 - \bar{z}), \quad \chi_1(z) \perp \text{ran}(S_1 - \bar{z}),$$

and

$$[\chi(z), h_0] = 1 = [\chi_1(z), h_0],$$

it follows that

$$[\chi(z), x] = [\chi_1(z), x], \quad x \in \mathcal{A}_1, \quad z \in \rho(\mathring{A}_1) \cap \rho(\mathring{A}) \cap \rho(S_{\text{fac}}). \tag{3.11}$$

Using this observation and the above formula (3.10) for $[\chi_1(z), \chi_1(w)]$, we obtain for $z, w \in \rho(\mathring{A}_1) \cap \rho(\mathring{A}) \cap \rho(S_{\text{fac}})$, $z \neq \bar{w}$,

$$\begin{aligned} & [\chi(z) - P_{\mathcal{A}_1}\chi_1(z), \chi(w) - P_{\mathcal{A}_1}\chi_1(w)] \\ &= [\chi(z), \chi(w)] - [\chi(z), P_{\mathcal{A}_1}\chi_1(w)] \\ &\quad - [P_{\mathcal{A}_1}\chi_1(z), \chi(w)] + [P_{\mathcal{A}_1}\chi_1(z), P_{\mathcal{A}_1}\chi_1(w)] \\ &= [\chi(z), \chi(w)] - [\chi_1(z), P_{\mathcal{A}_1}\chi_1(w)] \\ &\quad - [P_{\mathcal{A}_1}\chi_1(z), \chi_1(w)] + [P_{\mathcal{A}_1}\chi_1(z), P_{\mathcal{A}_1}\chi_1(w)] \\ &= [\chi(z), \chi(w)] - [\chi_1(z), \chi_1(w)] \\ &= \frac{q(z) - \overline{q(w)}}{z - \bar{w}} - \frac{q_1(z) - \overline{q_1(w)}}{z - \bar{w}} = \frac{q_2(z) - \overline{q_2(w)}}{z - \bar{w}}. \end{aligned} \tag{3.12}$$

Step 4. Construction of \mathring{A}_2, χ_2 . Introduce the elements $\chi_2(z)$ and the relation \mathring{A}_2 by

$$\chi_2(z) := \chi(z) - P_{\mathcal{A}_1}\chi_1(z), \quad z \in \rho(\mathring{A}) \cap \rho(\mathring{A}_1) \cap \rho(S_{\text{fac}}),$$

and

$$\begin{aligned} \mathring{A}_2 &:= \text{cls} \left(S_2 \cup \{(0, h_0)\} \cup \right. \\ &\quad \left. \cup \left\{ (\chi_2(z) - \chi_2(w), z\chi_2(z) - w\chi_2(w)) : z, w \in \rho(\mathring{A}) \cap \rho(\mathring{A}_1) \cap \rho(S_{\text{fac}}) \right\} \right). \end{aligned}$$

First it will be shown that \mathring{A}_2 is a relation in \mathcal{P}_2 . Note that $\ker P_{\mathcal{A}_1} \subseteq \mathcal{P}_2$ and hence

$$[P_{\mathcal{A}_1}y, x] = [y, x], \quad y \in \mathcal{P}, \quad x \in \mathcal{P}_2^\perp.$$

Together with (3.11) and the fact that $\mathcal{P}_2^\perp = \mathcal{A}_{1,r} \subseteq \mathcal{A}_1$, we obtain

$$[\chi_2(z), x] = [\chi(z), x] - [P_{\mathcal{A}_1}\chi_1(z), x] = 0, \quad x \in \mathcal{P}_2^\perp.$$

Thus $\chi_2(z) \in \mathcal{P}_2^{\perp\perp} = \mathcal{P}_2$ and $\mathring{A}_2 \subseteq (\mathcal{P}_2)^2$.

The next aim is to show that \mathring{A}_2 is symmetric in \mathcal{P}_2 . Note that the relation S_2 is symmetric in \mathcal{P}_2 . By (3.12) a standard computation shows that the relation

$$\text{span} \{ (\chi_2(z) - \chi_2(w), z\chi_2(z) - w\chi_2(w)) : z, w \in \rho(\mathring{A}) \cap \rho(\mathring{A}_1) \cap \rho(S_{\text{fac}}) \}$$

is symmetric. Moreover, since $\mathcal{A}_1 \perp \mathcal{A}_2$, we have

$$[\chi_2(z), x] = [\chi(z), x], \quad x \in \mathcal{A}_2.$$

Hence it follows that

$$\chi_2(z) \perp \text{ran}(S_2 - \bar{z}), \quad [\chi_2(z), h_l] = z^l, \quad l = 0, \dots, \Delta - 1.$$

Combining these facts, one sees that the relation \mathring{A}_2 is symmetric.

To show that the symmetric relation \mathring{A} is actually selfadjoint, we again use [5, Lemma 2.12]. For each $z \in \rho(\mathring{A}) \cap \rho(\mathring{A}_1) \cap \rho(S_{\text{fac}})$

$$\begin{aligned} \mathcal{P}_2 &\supseteq \text{ran}(\mathring{A}_2 - z) \\ &\supseteq \text{ran}(S_2 - z) + \text{span}\{h_0\} + \text{span} \{ \chi_2(w) : w \in \rho(\mathring{A}) \cap \rho(\mathring{A}_1), w \neq z \} \\ &= \mathcal{A}_2 + \text{span} \{ \chi_2(w) : w \in \rho(\mathring{A}) \cap \rho(\mathring{A}_1), w \neq z \} = \mathcal{P}_2. \end{aligned}$$

where for the last equality [5, Lemma II.27] is used. Hence \mathring{A}_2 is selfadjoint and $\rho(\mathring{A}) \supseteq \rho(\mathring{A}) \cap \rho(\mathring{A}_1) \cap \rho(S_{\text{fac}})$.

Step 5. Conclusion. We have shown that (2.3) and (2.4) are satisfied by S_2 and \mathcal{A}_2 . Furthermore, the validity of (2.6), (2.7), and (2.8) is built into the definition of \mathring{A}_2 . Putting together these facts, it follows that **(A)** and **(B)** are satisfied in the present context. Thus \mathring{A}_2 and $\chi_2(z)$ qualify for the definition of a Q -function of S_2 . By the computation (3.12) the function q_2 is indeed such a Q -function; cf. Definition 2.4. \square

4 The Krein formula and orthogonal coupling

This section is devoted to a description of the generalized resolvents of a symmetric relation. Recall the following definition; see [5, Definition I.11].

4.1 Definition. Let \mathcal{A} be an almost Pontryagin space and $S \subseteq \mathcal{A}^2$ a closed symmetric relation with $\gamma(S) \neq \emptyset$. Moreover, let $\mathring{\mathcal{A}}$ be an almost Pontryagin space with $\mathring{\mathcal{A}} \supseteq \mathcal{A}$ and $A \subseteq \mathring{\mathcal{A}}^2$ a selfadjoint relation with $\rho(A) \neq \emptyset$ and $A \supseteq S$. Then we call the function family

$$R_{x,z}(z) := [(A - z)^{-1}x, y], \quad x, y \in \mathcal{A}, \quad (4.1)$$

the generalized resolvent of S induced by A . The negative index of a generalized resolvent $R_{x,y}$ is

$$\text{ind}_- R_{x,y} := \text{ind}_- \text{cls} \left(\mathcal{A} \cup \bigcup_{z \in \rho(A)} (A - z)^{-1}(\mathcal{A}) \right).$$

◇

In view of applications it is practical to also call the family $[(S_{\text{fac}} - z)^{-1}x_{/\mathcal{A}^\circ}, y_{/\mathcal{A}^\circ}]$, $x, y \in \mathcal{A}$, a generalized resolvent of S , provided $\rho(S_{\text{fac}}) \neq \emptyset$. It may or may not be the case that this family is represented as in (4.1).

The following theorem is about the Krein formula in the setting of almost Pontryagin spaces. The present alternative proof runs along more geometric lines than the one in [14]. It is obtained by combining what we have shown so far.

4.2 Theorem. *Let \mathcal{A}, S be given according to **(A)**, let h_0, \mathring{A}, χ be given according to **(B)**, and denote by q a Q -function of S built with these data. Denote by S_{fac} the selfadjoint relation $S_{/\mathcal{A}^\circ}$ acting in the Pontryagin space $\mathcal{A}_{/\mathcal{A}^\circ}$. Then the set of all generalized resolvents of S is equal to the set of all function families given by*

$$R_{x,y}(z) = [(S_{\text{fac}} - z)^{-1}x_{/\mathcal{A}^\circ}, y_{/\mathcal{A}^\circ}] - [x, \chi(\bar{z})] \frac{1}{q(z) + \tau(z)} [\chi(z), y], \quad (4.2)$$

$$x, y \in \mathcal{A}, \quad z \in \rho(A) \cap \rho(S_{\text{fac}}), \quad \tau(z) + q(z) \neq 0,$$

where the parameter τ runs through the class $(\mathcal{N}_{<\infty} \setminus \{-q\}) \cup \{\infty\}$. In addition, we have

$$\text{ind}_- R_{x,y}(z) = \text{ind}_- \mathcal{A} + \text{ind}_-^\Delta \tau, \quad \tau \neq \infty. \quad (4.3)$$

In the proof we use the following elementary fact shown in [14, Proposition 3].

4.3 Lemma. *Let \mathcal{A}, S be given according to **(A)**, let h_0, \mathring{A}, χ be given according to **(B)**, and denote by q a Q -function of S built with these data. Then, for $x, y \in \mathcal{A}$ and $z \in \rho(A) \cap \rho(S_{\text{fac}})$, we have*

$$[(A - z)^{-1}x, y] = [(S_{\text{fac}} - z)^{-1}x_{/\mathcal{A}^\circ}, y_{/\mathcal{A}^\circ}] + [x, \chi(\bar{z})] [(A - z)^{-1}h_0, h_0] [\chi(z), y].$$

Proof of Theorem 4.2. The proof will be given in a number of steps.

Step 1. Let the parameter $\tau \in \mathcal{N}_{<\infty} \setminus \{-q\}$ be given. By [5, Theorem II.15, Proposition II.14] we can consider τ as a Q -function of a symmetry $S_\Delta(\tau)$ acting in an almost Pontryagin space $\mathcal{A}_\Delta(\tau)$ with $\text{ind}_0 \mathcal{A}_\Delta(\tau) = \Delta$ and $\text{ind}_- \mathcal{A}_\Delta(\tau) = \text{ind}_-^\Delta \tau - \Delta$. By Theorem 3.1 the function $q + \tau$ is a Q -function of the symmetry $S \boxplus_\alpha S_\Delta(\tau)$ acting in the space $\mathcal{A} \boxplus_\alpha \mathcal{A}_\Delta(\tau)$. Let $h_0 \in \mathcal{A} \boxplus_\alpha \mathcal{A}_\Delta(\tau)$ be an element according to (i) of **(B)** such that $q + \tau$ is a Q -function of $S \boxplus_\alpha S_\Delta(\tau)$ built with the data h_0 and some choice of a selfadjoint extension and defect elements. Then, by [5, Theorem III.8], there exists a selfadjoint extension A of $S \boxplus_\alpha S_\Delta(\tau)$ acting in $\mathfrak{P}_{\text{ext}}(\mathcal{A} \boxplus_\alpha \mathcal{A}_\Delta(\tau))$, such that

$$-\frac{1}{q(z) + \tau(z)} = [(A - z)^{-1}h_0, h_0], \quad z \in \rho(A).$$

By Lemma 4.3 the function family given by

$$\begin{aligned} & [(S_{\text{fac}} - z)^{-1}x_{/\mathcal{A}^\circ}, y_{/\mathcal{A}^\circ}] - [x, \chi(\bar{z})] \frac{1}{q(z) + \tau(z)} [\chi(z), y], \\ & x, y \in \mathcal{A}, \quad z \in \rho(A) \cap \rho(S_{\text{fac}}), q(z) + \tau(z) \neq 0, \end{aligned}$$

is the generalized resolvent of S induced by the selfadjoint extension A of S acting in the space $\mathfrak{P}_{\text{ext}}(\mathcal{A} \boxplus_{\alpha} \mathcal{A}_{\Delta}(\tau))$.

Step 2. Let a generalized resolvent $R_{x,y}(z)$ be given. If

$$R_{x,y}(z) = [(S_{\text{fac}} - z)^{-1}x_{/\mathcal{A}^\circ}, y_{/\mathcal{A}^\circ}],$$

then (4.2) holds with the parameter $\tau := \infty$. Assume throughout the following that $R_{x,y}$ does not coincide with $[(S_{\text{fac}} - z)^{-1}x_{/\mathcal{A}^\circ}, y_{/\mathcal{A}^\circ}]$. Then, by [14, Lemma 2], there exists an \mathcal{A} -minimal selfadjoint extension A of S acting in some Pontryagin space $\mathcal{P} \supseteq \mathcal{A}$ which induces the generalized resolvent $R_{x,y}(z)$. Set

$$\tilde{A} := \mathcal{P}[-]\mathcal{A}^\circ, \quad r(z) := [(A - z)^{-1}h_0, h_0].$$

By [5, Proposition III.9] there exists a closed symmetric relation $\tilde{S} \subseteq \tilde{\mathcal{A}}^2$ with defect $(1, 1)$ which extends S and satisfies (2.3) and (2.4), such that $-r(z)^{-1}$ is a Q -function of \tilde{S} built with h_0 and some selfadjoint relation and defect family.

By [14, Corollary 2] we have $h_0 \notin \text{ran}(\tilde{S} - z)$, $z \in \gamma(\tilde{S}) \cap \rho(\tilde{S}_{\text{fac}})$. Thus $\mathcal{A} \cap \text{ran}(\tilde{S} - z)$ is a subspace of \mathcal{A} with codimension 1. Since S has defect index $(1, 1)$, it follows that

$$\mathcal{A} \cap \text{ran}(\tilde{S} - z) = \text{ran}(S - z), \quad z \in \gamma(\tilde{S}) \cap \rho(\tilde{S}_{\text{fac}}) \cap \gamma(S), \quad (4.4)$$

and hence

$$(\tilde{S} - z)^{-1}(\mathcal{A} \cap \text{ran}(\tilde{S} - z)) = \text{dom } S \subseteq \mathcal{A}, \quad z \in \gamma(\tilde{S}) \cap \rho(\tilde{S}_{\text{fac}}) \cap \gamma(S).$$

Since $\gamma(\tilde{S}) \cap \rho(\tilde{S}_{\text{fac}}) \cap \gamma(S) \neq \emptyset$, the equality (4.4) also implies that $\tilde{S} \cap (\mathcal{A})^2 = S$.

We have checked all necessary hypotheses to apply Theorem 3.4 with $\tilde{\mathcal{A}}, \tilde{S}, h_0$, the Q -function $-r(z)^{-1}$ of \tilde{S} , the closed subspace \mathcal{A} , the set $M := \gamma(\tilde{S}) \cap \rho(\tilde{S}_{\text{fac}}) \cap \gamma(S)$, and the Q -function q of S . The conclusion of this theorem yields in particular that

$$\tau(z) := -\frac{1}{r(z)} - q(z) \in \mathcal{N}_{<\infty}.$$

Clearly, $\tau \neq -q$. Due to Lemma 4.3 we see that (4.2) holds with this parameter function τ .

Step 3. To show (4.3) note that by definition $\text{ind}_- R_{x,y}(z)$ is the negative index of a Pontryagin space in which an \mathcal{A} -minimal selfadjoint extension of S , inducing $R_{x,y}(z)$ as a generalized resolvent, acts. We claim that the selfadjoint relation constructed in Step 1 of this proof is \mathcal{A} -minimal. Once this claim is established, the equality (4.3) will follow from [3, Remark 4.6]:

$$\begin{aligned} \text{ind}_- \mathfrak{P}_{\text{ext}}(\mathcal{A} \boxplus_{\alpha} \mathcal{A}_{\Delta}(\tau)) &= \text{ind}_- \mathcal{A} + \text{ind}_- \mathcal{A}_{\Delta}(\tau) + \Delta \\ &= \text{ind}_- \mathcal{A} + \text{ind}_-^{\Delta} \tau. \end{aligned}$$

Let $\chi_2(z) \in \mathfrak{P}_{\text{ext}}(\mathcal{A}_\Delta(\tau))$ be the defect element which gives rise to the Q -function τ . Then, as we saw in the proof of Theorem 3.1, cf. (3.1), the defect elements in $\mathfrak{P}_{\text{ext}}(\mathcal{A} \boxplus_\alpha \mathcal{A}_\Delta(\tau))$ which give rise to the Q -function $q + \tau$ are equal to $P_{\mathcal{A} \boxplus_\alpha \mathcal{A}_\Delta(\tau)} \chi(z) + \chi_2(z)$; remember [3, Lemma 5.9(iii)]. By the construction of A in the proof of [5, Theorem III.8], ‘ $-Q^{-1} \subseteq \mathcal{R}$ ’, we have

$$(A - z)^{-1} h_0 = -\frac{1}{q(z) + \tau(z)} (P_{\mathcal{A} \boxplus_\alpha \mathcal{A}_\Delta(\tau)} \chi(z) + \chi_2(z)).$$

Since $\chi(z) \in \mathfrak{P}_{\text{ext}}(\mathcal{A})$, we have $P_{\mathcal{A} \boxplus_\alpha \mathcal{A}_\Delta(\tau)} \chi(z) \in \mathcal{A}$. This shows that

$$\chi_2(z) \in \text{span}(\mathcal{A} \cup \{(A - z)^{-1} h_0\}).$$

Since $S_\Delta(\tau)$ is minimal, cf. [5, Theorem II.15], we conclude that

$$\begin{aligned} \mathfrak{P}_{\text{ext}}(\mathcal{A} \boxplus_\alpha \mathcal{A}_\Delta(\tau)) &\supseteq \text{cls}(\mathcal{A} \cup \{(A - z)^{-1} h_0 : z \in \rho(A)\}) \\ &\supseteq \text{cls}(\mathcal{A} \cup \{\chi_2(z) : z \in \rho(A)\}) \\ &\supseteq \mathcal{A} + \mathfrak{P}_{\text{ext}}(\mathcal{A}_\Delta(\tau)) \\ &= \mathfrak{P}_{\text{ext}}(\mathcal{A} \boxplus_\alpha \mathcal{A}_\Delta(\tau)). \end{aligned}$$

Hence the minimality has been established. \square

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