

## Distributional representations of $\mathcal{N}_{\kappa}^{(\infty)}$ -functions

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The subclasses  $\mathcal{N}_{\kappa}^{(\infty)}$  of the classes  $\mathcal{N}_{\kappa}$  of generalized Nevanlinna functions appear in the context of Pontryagin space models, where they correspond to model relations having a particular spectral behaviour. Applications are found, for instance, in the investigation of differential expressions with singular coefficients. We study representations of  $\mathcal{N}_{\kappa}^{(\infty)}$ -functions as Cauchy-type integrals in a distributional sense and characterize the class of distributions occurring in such representations. We make explicit how the Pontryagin space model of an  $\mathcal{N}_{\kappa}^{(\infty)}$ -function is related to the multiplication operator in the  $L^2$ -space of the measure which describes the action of the representing distribution away from infinity. Moreover, we determine the distributional representations of a pair of functions associated with a symmetric generalized Nevanlinna function.

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### 1 Introduction

A function  $q$  is said to belong to the *Nevanlinna class*  $\mathcal{N}_0$  if it is analytic in  $\mathbb{C} \setminus \mathbb{R}$ , satisfies  $q(\bar{z}) = \overline{q(z)}$ ,  $z \in \mathbb{C} \setminus \mathbb{R}$ , and

$$\operatorname{Im} q(z) \cdot \operatorname{Im} z \geq 0, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

This class of functions has been intensively studied in various contexts of analysis. For instance, it plays an important role in the spectral theory of symmetric and self-adjoint operators in a Hilbert space or in classical problems like the power moment problem.

A fact which lies at the very core of the subject is that a function  $q \in \mathcal{N}_0$  can be represented as a Cauchy-type integral. This result goes back to the early stages of modern analysis; it is commonly attributed to the work of G. Herglotz from the 1910s.

**Theorem 1.1 (Herglotz integral representation)** *A function  $q$  belongs to the Nevanlinna class  $\mathcal{N}_0$  if and only if it can be represented as*

$$q(z) = a + bz + \int_{\mathbb{R}} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (1.1)$$

where  $a \in \mathbb{R}$ ,  $b \geq 0$ , and  $\mu$  is a positive Borel measure on the real line with

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty. \quad (1.2)$$

In many applications to differential operators the measure  $\mu$  plays the role of a spectral measure.

In the theory of spaces with an indefinite metric, in particular in the spectral theory of symmetric and self-adjoint operators in a Pontryagin space, an indefinite analogue of the class  $\mathcal{N}_0$  occurs.

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**Definition 1.2** A function  $q$  is said to belong to the *generalized Nevanlinna class*  $\mathcal{N}_{<\infty}$  if it is meromorphic in  $\mathbb{C} \setminus \mathbb{R}$ , satisfies  $q(\bar{z}) = \overline{q(z)}$ ,  $z \in \rho(q)$  (where  $\rho(q)$  denotes the domain of analyticity of  $q$ ), and the kernel

$$K_q(w, z) := \frac{q(z) - \overline{q(w)}}{z - \bar{w}}, \quad z, w \in \rho(q),$$

has a finite number of negative squares. By this we mean that the supremum of the numbers of negative squares of quadratic forms ( $n \in \mathbb{N}$  and  $z_1, \dots, z_n \in \rho(q)$ )

$$Q(\xi_1, \dots, \xi_n) := \sum_{i,j=1}^n K_q(z_j, z_i) \xi_i \bar{\xi}_j$$

is finite.

If  $q \in \mathcal{N}_{<\infty}$ , we denote by  $\text{ind}_- q$  the actual number of negative squares of  $K_q$ , that is, the maximum of the numbers of negative squares of forms  $Q(\xi_1, \dots, \xi_n)$ ,  $n \in \mathbb{N}$ ,  $z_1, \dots, z_n \in \rho(q)$ . Moreover, we set

$$\mathcal{N}_\kappa := \{q \in \mathcal{N}_{<\infty} : \text{ind}_- q = \kappa\}.$$

A systematic study of the class  $\mathcal{N}_{<\infty}$  was given by M. G. Kreĭn and H. Langer in [39] and subsequent papers. Over the years a vast literature accumulated, dealing in particular with Kreĭn's resolvent formula, boundary triples and Weyl functions, or analytic properties and operator representations. To mention just a few out of the recent literature, we refer to [1, 2, 3, 6, 8, 12, 26, 30, 36].

A central result is the integral representation of generalized Nevanlinna functions given in [39, Satz 3.1], which is the indefinite analogue of the Herglotz integral representation (1.1). Using the work [31] on distributional models for cyclic self-adjoint operators in a Pontryagin space, one obtains a distributional form of this integral representation. This fact is mentioned in [31, Introduction, p. 253], established in full generality in [35, Corollary 3.5] and refined in [33, Proposition 5.4].

In various contexts, in particular in connection with spectral properties of operator models for differential operators, subclasses of  $\mathcal{N}_0$  and their indefinite analogues occur, for instance, the Kac subclasses, see, e.g. [27], or the Stieltjes class, see, e.g. [34].

A class that is specific to the indefinite situation is the class  $\mathcal{N}_{<\infty}^{(\infty)}$  (which is sometimes denoted by  $\mathcal{N}_{<\infty}^\infty$  in the literature). Functions in this class are characterized by, in a certain sense, maximal possible growth at infinity; we recall its precise definition in Definition 3.5 below. They are also characterized by the property that infinity is the only generalized pole of non-positive type (see Remark 3.6 below).

Functions from the class  $\mathcal{N}_{<\infty}^{(\infty)}$  typically appear as singular Weyl functions of problems with singular perturbations. For abstract models see we refer to [7, 11, 14]; for differential equations with singular coefficients (fast growing potentials or derivatives of  $\delta$ -distributions) see, e.g. [10, 18, 22, 23, 38, 43, 45]; for perturbations with infinite coupling see [13]. Other abstract operator models for  $\mathcal{N}_{<\infty}^{(\infty)}$ -functions based on reproducing kernel Pontryagin spaces were studied in [15, 16, 17].

This class  $\mathcal{N}_{<\infty}^{(\infty)}$  lies in the focus of our present study. Our aim is to investigate representations of functions in this class with distributions (or more precisely distributional densities) on  $\mathbb{R} = \mathbb{R} \cup \{\infty\}$ , continuing the line of research from [31], [35, Corollary 3.5], [33, Proposition 5.4]. The main results in the present paper are

1. Theorem 3.9, where we characterize those distributions appearing in representations of  $\mathcal{N}_{<\infty}^{(\infty)}$ -functions (these distributions behave like measures on  $\mathbb{R}$  and only the behaviour at  $\infty$  is more complicated);
2. Theorem 4.4, where we determine the representing distributions of functions  $q_+$  and  $q_-$  naturally associated with a symmetric generalized Nevanlinna function, paying particular attention to the case when  $q \in \mathcal{N}_{<\infty}^{(\infty)}$ ;
3. Theorem 5.3, where we make explicit the connection between the ‘‘measure part’’ of the representing distribution, the algebraic eigenspace at  $\infty$  of the corresponding self-adjoint relation (for which  $\infty$  is in general a singular critical point), and the multiplication operator in a certain  $L^2$ -space.

Besides their intrinsic interest, these results are an essential tool for our forthcoming work on direct and inverse spectral theorems for two-dimensional canonical systems and Sturm–Liouville equations with two singular endpoints. These theorems will be presented in [44], and they rely heavily on the present results. In particular, the results from Theorem 3.9 are used to construct spectral measures for such canonical systems and to prove a connection between the growth of the spectral measure at  $\infty$  and the growth of the Hamiltonian of the canonical system. The construction of a corresponding Fourier transform relies heavily on Theorem 5.3. Further, Theorem 4.4 is used to relate the spectral measures of canonical systems with diagonal Hamiltonians to those of certain Sturm–Liouville equations.

Let us briefly outline the organization of the manuscript. In Section 2 we undertake a systematic study of the class  $\mathcal{F}_{\{\infty\}}$  of distributional densities on  $\overline{\mathbb{R}}$  that behave like measures on  $\mathbb{R}$ . We include this material to provide the reader with a polished form of the analytic setup and the representation theorem itself, which cannot be found elsewhere. In the following three sections we formulate and prove our main results. Thereby Section 3 is devoted to the actual representations of  $\mathcal{N}_{<\infty}^{(\infty)}$ -functions, in Section 4 we investigate the symmetric case, and in Section 5 we turn to the operator-theoretic aspects.

We aim to keep proofs as elementary as possible. This is not always the most efficient approach. However, we believe that clearly isolating what has to be imported and being detailed in what has to be done is for the benefit of the reader, and hence find it worth to proceed in this way<sup>1</sup>. We add some more detailed notices on alternative ways of proof in the course of the exposition.

## 2 The class $\mathcal{F}_{\{\infty\}}$ of distributional densities

When  $\Omega$  is an open subset of  $\mathbb{R}^n$ , the space of test functions,  $\mathcal{D}(\Omega)$ , and its dual, the space of distributions  $\mathcal{D}'(\Omega)$ , are studied in many textbooks. We refer, for example, to [5], [21] or [28], and take this theory for granted. The notion of distributions and distributional densities on a manifold seems to be much less standard. Our reference is the classic book [29]; an intrinsic geometric approach can be found in [25].

The manifold that appears in the context of representations of generalized Nevanlinna functions is  $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ , the one-point compactification of  $\mathbb{R}$ . We consider  $\overline{\mathbb{R}}$  as a  $C^\infty$ -manifold in the usual way via the charts (here, and in the following, we understand  $\frac{1}{\infty} := 0$  and  $\frac{1}{0} := \infty$ )

$$\Lambda_0 : \begin{cases} \overline{\mathbb{R}} \setminus \{\infty\} & \rightarrow \mathbb{R}, \\ x & \mapsto x, \end{cases} \quad \Lambda_\infty : \begin{cases} \overline{\mathbb{R}} \setminus \{0\} & \rightarrow \mathbb{R}, \\ x & \mapsto \frac{1}{x}. \end{cases}$$

We want to emphasize the viewpoint of linear functionals and hence work with distributional densities rather than distributions. Let us recall the definition from [29, p. 145] (we formulate it only for the particular manifold  $\overline{\mathbb{R}}$  using the particular charts  $\Lambda_0$  and  $\Lambda_\infty$ ).

**Definition 2.1** A *distributional density*  $\phi$  on  $\overline{\mathbb{R}}$  is a pair  $(\phi_0, \phi_\infty)$  where  $\phi_0$  and  $\phi_\infty$  belong to  $\mathcal{D}'(\mathbb{R})$  and are related by the transformation law<sup>2</sup>

$$\phi_\infty(f) = \phi_0(f \circ (\Lambda_\infty \circ \Lambda_0^{-1})), \quad f \in \mathcal{D}(\mathbb{R}), \quad 0 \notin \text{supp } f. \quad (2.1)$$

We denote the set of all distributional densities on  $\overline{\mathbb{R}}$  by  $\tilde{\mathcal{D}}'(\overline{\mathbb{R}})$ .

Throughout this paper, we drop the explicit notation of  $\Lambda_0$  and consider the auxiliary Euclidean space  $\mathbb{R}$  as a subset of the manifold  $\overline{\mathbb{R}}$ .

Each distributional density  $\phi$  induces a linear functional on  $C^\infty(\overline{\mathbb{R}})$  in a canonical way: choose a partition of unity  $\chi^0, \chi^\infty \in C^\infty(\overline{\mathbb{R}})$  subordinate to the open cover  $\{\mathbb{R} \setminus \{\infty\}, \mathbb{R} \setminus \{0\}\}$  of  $\overline{\mathbb{R}}$ , and define

$$\phi(g) := \phi_0(\chi^0 g) + \phi_\infty((\chi^\infty g) \circ \Lambda_\infty^{-1}), \quad g \in C^\infty(\overline{\mathbb{R}}). \quad (2.2)$$

Due to the transformation law (2.1), this definition is independent of the choice of  $\chi^0, \chi^\infty$ .

<sup>1</sup> In this context, we should mention that in the earlier literature on the subject proofs are often not carried out in detail.

<sup>2</sup> On comparing (2.1) with the requirement [29, (6.3.4)], one may wonder why the factor “ $|\det \psi'|$ ” disappeared. In fact, it did not. This is due to the definition of “ $f^*$ ” in [29]; the example [29, 6.1.3] may be enlightening.

**Remark 2.2** Let  $\phi = (\phi_0, \phi_\infty) \in \tilde{\mathcal{D}}'(\overline{\mathbb{R}})$ . It is another consequence of the transformation law (2.1) that both distributions  $\phi_0$  and  $\phi_\infty$  have finite order. To see this, choose  $N_0, N_\infty \in \mathbb{N}_0$  and  $C_0, C_\infty > 0$  such that

$$\begin{aligned} |\phi_0(f)| &\leq C_0 \|f\|_{[-2,2],N_0}, & f \in \mathcal{D}(\mathbb{R}), \text{ supp } f \subseteq [-2,2], \\ |\phi_\infty(f)| &\leq C_\infty \|f\|_{[-1,1],N_\infty}, & f \in \mathcal{D}(\mathbb{R}), \text{ supp } f \subseteq [-1,1]; \end{aligned}$$

here, for a compact subset  $K$  of  $\mathbb{R}$  and  $n \in \mathbb{N}_0$ , we set

$$\|f\|_{K,n} := \max \left\{ \sup_{x \in K} |f^{(k)}(x)| : k = 0, \dots, n \right\}, \quad f \in \mathcal{D}(\mathbb{R}).$$

Choose a partition of unity  $\chi^0, \chi^\infty \in C^\infty(\overline{\mathbb{R}})$  subordinate to the open cover  $\{(-2, 2), \overline{\mathbb{R}} \setminus [-1, 1]\}$  of  $\overline{\mathbb{R}}$ , and set  $N := \max\{N_0, N_\infty\}$ . Then, for each  $T \geq 2$ , we find a constant  $C_T > 0$  such that, for  $f \in \mathcal{D}(\mathbb{R})$  with  $\text{supp } f \subseteq [-T, T]$ ,

$$\begin{aligned} |\phi_0(f)| &\leq |\phi_0(\chi^0 f)| + |\phi_\infty((\chi^\infty f) \circ \Lambda_\infty^{-1})| \\ &\leq C_0 \|\chi^0 f\|_{[-2,2],N_0} + C_\infty \|(\chi^\infty f) \circ \Lambda_\infty^{-1}\|_{[-1,1],N_\infty} \leq C_T \|f\|_{[-T,T],N}. \end{aligned}$$

The dependence on  $T$  arises from the derivatives of the map  $\Lambda_\infty$ . This estimate shows that  $\text{ord } \phi_0 \leq N$ ; here we denote by ‘‘ord  $\phi_0$ ’’ the order of the distribution  $\phi_0$  in the classical sense; see, e.g. [28]. The fact that  $\text{ord } \phi_\infty \leq N$  is seen in the same way.

Let us introduce the *order* of a distributional density:

$$\text{ord } \phi := \max\{\text{ord } \phi_0, \text{ord } \phi_\infty\}, \quad \phi = (\phi_0, \phi_\infty) \in \tilde{\mathcal{D}}'(\overline{\mathbb{R}}).$$

In connection with generalized Nevanlinna functions, a subset  $\mathcal{F}(\overline{\mathbb{R}})$  of  $\tilde{\mathcal{D}}'(\overline{\mathbb{R}})$  appears, which is defined in Definition 2.3 (iii) below. Thereby, we call a distribution  $\phi \in \mathcal{D}'(\mathbb{R})$  *real* if it assigns real values to real-valued test functions.

**Definition 2.3**

- (i) Let  $\phi \in \mathcal{D}'(\mathbb{R})$  and let  $M \subseteq \mathbb{R}$  be an open set. Then we say that  $\phi$  is a *positive measure on  $M$*  if there exists a positive (possibly unbounded) Borel measure  $\mu$  on  $M$  such that

$$\phi(f) = \int_M f \, d\mu \quad \text{for all } f \in \mathcal{D}(\mathbb{R}) \text{ with } \text{supp } f \subseteq M.$$

Here, and in the rest of the paper, we include in the notion of a Borel measure the requirement that compact sets have finite measure.

- (ii) Let  $\phi = (\phi_0, \phi_\infty) \in \tilde{\mathcal{D}}'(\overline{\mathbb{R}})$  and let  $M \subseteq \overline{\mathbb{R}}$  be an open set. We say that  $\phi$  is a *positive measure on  $M$*  if  $\phi_0$  is a positive measure on  $M \cap \mathbb{R}$  and  $\phi_\infty$  is a positive measure on  $\Lambda_\infty(M \setminus \{0\})$ .
- (iii) Let  $\phi = (\phi_0, \phi_\infty) \in \tilde{\mathcal{D}}'(\overline{\mathbb{R}})$ . We say that  $\phi$  belongs to  $\mathcal{F}(\overline{\mathbb{R}})$  if  $\phi_0$  and  $\phi_\infty$  are real and there exists a finite subset  $F \subseteq \overline{\mathbb{R}}$  such that  $\phi$  is a positive measure on  $\overline{\mathbb{R}} \setminus F$ .
- (iv) If  $\phi \in \mathcal{F}(\overline{\mathbb{R}})$ , we denote by  $s(\phi)$  the smallest of all sets  $F$  such that  $\phi$  is a positive measure on  $\overline{\mathbb{R}} \setminus F$ .

To justify item (iv) of this definition, note that  $\phi$  being a positive measure on  $\overline{\mathbb{R}} \setminus F_1$  and  $\overline{\mathbb{R}} \setminus F_2$  implies that  $\phi$  is a positive measure on  $\overline{\mathbb{R}} \setminus (F_1 \cap F_2)$ . Moreover, note that a measure  $\mu$  as in (i) is uniquely determined by  $\phi$ .

The subclass  $\mathcal{F}_{\{\infty\}}$  of  $\mathcal{F}(\overline{\mathbb{R}})$ , which is defined below, is one of the central objects in our present study.

**Definition 2.4** We denote by  $\mathcal{F}_{\{\infty\}}$  the set of all  $\phi = (\phi_0, \phi_\infty) \in \tilde{\mathcal{D}}'(\overline{\mathbb{R}})$  that are a positive measure on  $\mathbb{R}$ . If  $\phi \in \mathcal{F}_{\{\infty\}}$ , we denote by  $\mu_\phi$  the unique positive Borel measure on  $\mathbb{R}$  that satisfies

$$\phi_0(f) = \int_{\mathbb{R}} f(x) \frac{d\mu_\phi(x)}{1+x^2}, \quad f \in \mathcal{D}(\mathbb{R}). \quad (2.3)$$

**Remark 2.5** On first sight, introducing the factor  $(1 + x^2)^{-1}$  in the definition of  $\mu_\phi$  may seem artificial, but in fact it is not. First, this density is necessary in order to include the Herglotz integral representation of  $\mathcal{N}_0$ -functions into our framework (cf. (1.2)); we give more details in Example 3.4. Second, we want to have the same Stieltjes-type inversion formula as in the positive definite case, cf. (3.7). Third, from the viewpoint of manifold theory adding this density means to pass from distributional densities to distributions, cf. [29, p. 145].

Note that, for each  $\phi = (\phi_0, \phi_\infty) \in \mathcal{F}_{\{\infty\}}$ , we have  $\text{ord } \phi_0 = 0$  and therefore  $\text{ord } \phi = \text{ord } \phi_\infty$ .

We also use another characteristic of  $\phi \in \mathcal{F}_{\{\infty\}}$  related to the order, namely

$$\text{ord}' \phi := \min \left\{ \text{ord } \psi : \psi = \phi + \sum_{k=0}^N a_k \delta_\infty^{(k)}, N \in \mathbb{N}_0, a_k \in \mathbb{R} \right\}; \quad (2.4)$$

here we denote by  $\delta_\infty^{(k)}$  the  $k$ th derivative of the Dirac distribution density concentrated at the point  $\infty$ , i.e.

$$\delta_\infty^{(k)} := (0, \delta_0^{(k)}) \in \tilde{\mathcal{D}}'(\overline{\mathbb{R}}),$$

where  $\delta_0$  is the standard Dirac distribution at 0 and  $\delta_0^{(k)}$  its  $k$ th derivative.

**Lemma 2.6** Let  $\phi = (\phi_0, \phi_\infty) \in \mathcal{F}_{\{\infty\}}$ . Then there exists a constant  $C > 0$  such that

$$|\phi_\infty(f)| \leq C \|f\|_{\text{supp } f, \text{ord}' \phi}, \quad f \in \mathcal{D}(\mathbb{R}), 0 \notin \text{supp } f. \quad (2.5)$$

*Proof.* Choose  $\psi$  of the form  $\phi + \sum_{k=0}^N a_k \delta_\infty^{(k)}$  with some  $N \in \mathbb{N}_0$  and  $a_k \in \mathbb{R}$  such that  $\text{ord } \psi = \text{ord}' \phi$ . Moreover, choose a partition of unity  $\chi^0, \chi^\infty \in C^\infty(\overline{\mathbb{R}})$  subordinate to the open cover  $\{(-2, 2), \overline{\mathbb{R}} \setminus [-1, 1]\}$  of  $\overline{\mathbb{R}}$ . Then we find a constant  $C_\infty > 0$  such that, for each  $f \in \mathcal{D}(\mathbb{R})$  with  $0 \notin \text{supp } f$ ,

$$\begin{aligned} |\phi_\infty(f)| &\leq |\phi_0((\chi^\infty f) \circ \Lambda_\infty)| + \underbrace{|\phi_\infty(\chi^0 f)|}_{=\psi_\infty(\chi^0 f)} \\ &\leq \|f\|_{[-1,1],0} \int_{[-1,1]} \frac{d\mu_\phi(x)}{1+x^2} + C_\infty \|f\|_{[-2,2], \text{ord } \psi_\infty}. \end{aligned}$$

□

Of course, using the transformation law (2.2) we also find an estimate

$$|\phi_\infty(f)| \leq C_\varepsilon \|f\|_{\text{supp } f, 0}, \quad f \in \mathcal{D}(\mathbb{R}), [-\varepsilon, \varepsilon] \cap \text{supp } f = \emptyset.$$

In contrast to (2.5), here the constant  $C_\varepsilon$  depends on  $\varepsilon$ .

The class of measures that may appear as  $\mu_\phi$  with some  $\phi \in \mathcal{F}_{\{\infty\}}$  turns out to be the one that is introduced in the following definition.

**Definition 2.7** Let  $\mu$  be a scalar-valued positive Borel measure on the real line. Then we say that  $\mu$  belongs to the class  $\mathbb{M}$  if there exists a number  $n \in \mathbb{N}_0$  such that

$$\int_{\mathbb{R}} \frac{d\mu(x)}{(1+x^2)^{n+1}} < \infty. \quad (2.6)$$

If  $\mu \in \mathbb{M}$ , we denote by  $\Delta(\mu)$  the minimal non-negative integer  $n$  such that (2.6) holds.

In the next theorem we make the relation between  $\mathcal{F}_{\{\infty\}}$  and  $\mathbb{M}$  explicit.

**Theorem 2.8** The following statements hold.

- (i) Let  $\mu \in \mathbb{M}$ . Then there exists a distributional density  $\phi \in \mathcal{F}_{\{\infty\}}$  with  $\mu_\phi = \mu$ .

(ii) Let  $\phi \in \mathcal{F}_{\{\infty\}}$ . Then  $\mu_\phi \in \mathbb{M}$  and

$$\text{ord}' \phi = \min \left\{ n \in \mathbb{N}_0 : \int_{\mathbb{R}} \frac{d\mu_\phi(x)}{(1+|x|)^{n+2}} < \infty \right\}. \quad (2.7)$$

In particular,

$$\Delta(\mu_\phi) = \left\lfloor \frac{\text{ord}' \phi + 1}{2} \right\rfloor \leq \left\lfloor \frac{\text{ord} \phi + 1}{2} \right\rfloor; \quad (2.8)$$

here  $\lfloor x \rfloor$  denotes the largest integer less than or equal to  $x$ .

(iii) Let  $\phi \in \mathcal{F}_{\{\infty\}}$  and  $\psi \in \tilde{\mathcal{D}}'(\overline{\mathbb{R}})$ . Then  $\psi \in \mathcal{F}_{\{\infty\}}$  and  $\mu_\psi = \mu_\phi$  if and only if there exist  $N \in \mathbb{N}$  and  $a_0, \dots, a_N \in \mathbb{R}$  with

$$\psi = \phi + \sum_{k=0}^N a_k \delta_\infty^{(k)}. \quad (2.9)$$

$\Rightarrow$  Notice: Analogues of some parts of this theorem are stated (mostly without proofs) in [31] and, somewhat more elaborate, in [37]. However, there one works in the vicinity of a finite point instead of  $\infty$  and with compactly supported distributions. Instead of carrying out the techniques necessary to reduce to the (anyway not explicitly given) arguments in [31], we prefer to give self-contained proofs for all assertions.  $\diamond$

Before we come to the proof of the theorem, we show a lemma, which is also used in later sections. The procedure of defining a distributional density connected with a given measure from  $\mathbb{M}$  is similar to a standard method of defining distributions associated with certain non-integrable functions.

**Lemma 2.9** Let  $\mu \in \mathbb{M}$  and let  $n \in \mathbb{N}_0$  such that

$$\int_{\mathbb{R}} \frac{d\mu(x)}{(1+|x|)^{n+2}} < \infty. \quad (2.10)$$

Moreover, let  $\chi^0, \chi^\infty \in C^\infty(\overline{\mathbb{R}})$  be a partition of unity subordinate to the open cover  $\{(-2, 2), \overline{\mathbb{R}} \setminus [-1, 1]\}$ . Define

$$\begin{aligned} \phi_0(f) &:= \int_{\mathbb{R}} f(x) \frac{d\mu(x)}{1+x^2}, \\ \phi_\infty(f) &:= \int_{\mathbb{R} \setminus \{0\}} \left[ f\left(\frac{1}{x}\right) - \chi^\infty(x) \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} \left(\frac{1}{x}\right)^k \right] \frac{d\mu(x)}{1+x^2} \end{aligned} \quad (2.11)$$

for  $f \in \mathcal{D}(\mathbb{R})$ . Then  $\phi = (\phi_0, \phi_\infty) \in \mathcal{F}_{\{\infty\}}$ ,  $\mu_\phi = \mu$  and  $\text{ord} \phi \leq n$ .

**Proof.** First we show that the integral in the definition of  $\phi_\infty$  is well defined and represents a distribution. Let  $f \in \mathcal{D}(\mathbb{R})$ . For small values of  $x$ , the second summand in the integral is not present and we obtain the immediate estimate

$$\int_{[-1,1] \setminus \{0\}} \left| f\left(\frac{1}{x}\right) - \chi^\infty(x) \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} \left(\frac{1}{x}\right)^k \right| \frac{d\mu(x)}{1+x^2} \leq \|f\|_\infty \int_{[-1,1] \setminus \{0\}} \frac{d\mu(x)}{1+x^2}. \quad (2.12)$$

For large values of  $x$ , we use Taylor's theorem to estimate

$$\begin{aligned} & \int_{\mathbb{R} \setminus (-2,2)} \left| f\left(\frac{1}{x}\right) - \chi^\infty(x) \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} \left(\frac{1}{x}\right)^k \right| \frac{d\mu(x)}{1+x^2} \\ & \leq \|f^{(n)}\|_\infty \cdot \frac{1}{n!} \int_{\mathbb{R} \setminus (-2,2)} \frac{1}{|x|^n} \cdot \frac{d\mu(x)}{1+x^2}. \end{aligned} \quad (2.13)$$

The integral on the right-hand side converges because of (2.10). On the remaining intervals  $(-2, -1)$  and  $(1, 2)$  we have the straightforward estimate

$$\begin{aligned} & \int_{(-2,2) \setminus [-1,1]} \left| f\left(\frac{1}{x}\right) - \chi^\infty(x) \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} \left(\frac{1}{x}\right)^k \right| \frac{d\mu(x)}{1+x^2} \\ & \leq \left( \|f\|_\infty + \sum_{k=0}^{n-1} \frac{\|f^{(k)}\|_\infty}{k!} \right) \int_{(-2,2) \setminus [-1,1]} \frac{d\mu(x)}{1+x^2}. \end{aligned} \tag{2.14}$$

Putting together (2.12), (2.13) and (2.14) we obtain that  $\phi_\infty$  is a distribution, and that  $\text{ord } \phi_\infty \leq n$ . Since, clearly,  $\text{ord } \phi_0 = 0$ , we have  $\text{ord } \phi \leq n$ .

Next we show the transformation rule (2.1). Let  $f \in \mathcal{D}(\mathbb{R})$  with  $0 \notin \text{supp } f$  be given. Then  $f^{(k)}(0) = 0$ ,  $k \geq 0$ , and hence (remember that we suppress the notation of  $\Lambda_0$ )

$$\phi_\infty(f) = \int_{\mathbb{R} \setminus \{0\}} f\left(\frac{1}{x}\right) \frac{d\mu(x)}{1+x^2} = \int_{\mathbb{R} \setminus \{0\}} [f \circ \Lambda_\infty](x) \frac{d\mu(x)}{1+x^2} = \phi_0(f \circ \Lambda_\infty).$$

Thus we have  $\phi \in \mathcal{F}_{\{\infty\}}$ . It follows from the definition of  $\phi_0$  that  $\mu_\phi = \mu$ . □

*Proof of Theorem 2.8.* Item (i) follows immediately from Lemma 2.9.

Next we prove (iii). If  $\psi$  is of the form (2.9), then

$$\psi = \left( \phi_0, \phi_\infty + \sum_{k=0}^N a_k \delta_0^{(k)} \right),$$

and hence  $\psi \in \mathcal{F}_{\{\infty\}}$  and  $\mu_\psi = \mu_\phi$ .

For the converse, assume that  $\psi = (\psi_0, \psi_\infty) \in \mathcal{F}_{\{\infty\}}$  and  $\mu_\psi = \mu_\phi$ . This clearly implies that  $\psi_0 = \phi_0$ . Moreover, for each function  $f \in \mathcal{D}(\mathbb{R})$  with  $0 \notin \text{supp } f$ , we have

$$\phi_\infty(f) = \phi_0(f \circ \Lambda_\infty) = \psi_0(f \circ \Lambda_\infty) = \psi_\infty(f).$$

Thus  $\text{supp}(\phi_\infty - \psi_\infty) \subseteq \{0\}$ , and it follows that  $\phi_\infty - \psi_\infty = \sum_{k=0}^N a_k \delta_0^{(k)}$  with some  $N \in \mathbb{N}$  and  $a_0, \dots, a_N \in \mathbb{R}$ . This shows that  $\phi$  and  $\psi$  are related by (2.9).

Finally, we come to the proof of (ii). Let again  $\chi^0, \chi^\infty \in C^\infty(\overline{\mathbb{R}})$  be a partition of unity subordinate to the open cover  $\{(-2, 2), \overline{\mathbb{R}} \setminus [-1, 1]\}$ . Moreover, choose a function  $\psi \in C^\infty(\mathbb{R})$  such that

$$\psi|_{(-\infty, 1)} = 0, \quad \psi|_{(2, \infty)} = 1, \quad \psi(x) \in [0, 1], \quad x \in \mathbb{R}.$$

Set  $n := \text{ord}' \phi$  and, for each  $T \geq 4$ , define

$$f_T(x) := \begin{cases} \chi^\infty(x) \cdot \frac{1}{x^n} \cdot \psi\left(\frac{T}{x}\right), & x \in \mathbb{R} \setminus \{0\}, \\ 0, & x = 0. \end{cases}$$

Then  $f_T \in C^\infty(\mathbb{R})$  and

$$\text{supp } f_T \subseteq [1, T], \quad f_T|_{[2, \frac{T}{2}]} = \frac{1}{x^n}, \quad f_T(x) \geq 0, \quad x \in \mathbb{R}.$$

By Lemma 2.6 there exists a constant  $C > 0$  such that

$$|\phi_\infty(g)| \leq C \|g\|_{[0,1],n}, \quad g \in C^\infty(\mathbb{R}) \text{ with } \text{supp } g \subseteq (0, 1].$$

Since  $\text{supp}[f_T \circ \Lambda_\infty] \subseteq [\frac{1}{T}, 1]$ , it follows that

$$\begin{aligned} \int_{[\frac{1}{2}, \frac{x}{2}]} \frac{1}{x^n} \cdot \frac{d\mu_\Phi(x)}{1+x^2} &\leq \int_{\mathbb{R}} f_T(x) \frac{d\mu_\Phi(x)}{1+x^2} = \Phi_0(f_T) \\ &= \Phi_\infty(f_T \circ \Lambda_\infty) \leq C \|f_T \circ \Lambda_\infty\|_{[0,1],n}. \end{aligned} \quad (2.15)$$

In order to estimate the last expression, we write explicitly:

$$[f_T \circ \Lambda_\infty](x) = \chi^\infty\left(\frac{1}{x}\right) \cdot x^n \cdot \psi(Tx), \quad x \in \text{supp}[f_T \circ \Lambda_\infty],$$

which clearly implies that  $\|f_T \circ \Lambda_\infty\|_\infty \leq 1$ . Next we compute, for  $k = 1, \dots, n$  and  $x \in [\frac{1}{T}, 1]$ ,

$$[f_T \circ \Lambda_\infty]^{(k)}(x) = \sum_{\substack{i,j,l \geq 0 \\ i+j+l=k}} \binom{k}{i,j,l} \left[ \frac{d^i}{dx^i} \chi^\infty\left(\frac{1}{x}\right) \right] \frac{n!}{(n-j)!} x^{n-j} \cdot T^l \psi^{(l)}(Tx).$$

Since  $\chi^\infty|_{[2,\infty)} = 1$ , we have

$$\frac{d^j}{dx^j} \chi^\infty\left(\frac{1}{x}\right) = 0, \quad x < \frac{1}{2}, \quad j \geq 1.$$

Thus

$$c_1 := \sup \left\{ \left| \frac{d^j}{dx^j} \chi^\infty\left(\frac{1}{x}\right) \right| : x \in (0, 1], j = 0, \dots, n \right\} < \infty.$$

Since  $\psi|_{[2,\infty)} = 1$  and  $\psi|_{(-\infty,1]} = 0$ , we have

$$\psi^{(l)}(x) = 0, \quad x \in \mathbb{R} \setminus (1, 2), \quad l \geq 1.$$

Using the fact that  $\|\psi\|_\infty = 1$  we obtain

$$c_2 := \sup \{ |\psi^{(l)}(x)| : x \in \mathbb{R}, l = 0, \dots, n \} < \infty.$$

Moreover,  $\psi^{(l)}(Tx) = 0$  for  $x \geq \frac{2}{T}, l \geq 1$ , and hence

$$|x^{n-j} T^l \psi^{(l)}(Tx)| \leq \left(\frac{2}{T}\right)^{n-j} T^l c_2 = 2^{n-j} c_2 T^{l+j-n}, \quad x \in \left[\frac{1}{T}, 1\right], l \geq 1.$$

Since  $l + j = k - i \leq k \leq n$ , we have  $T^{l+j-n} \leq 1$ .

Putting together these estimates we can deduce that, for  $k = 1, \dots, n$  and  $x \in [\frac{1}{T}, 1]$ ,

$$|[f_T \circ \Lambda_\infty]^{(k)}(x)| \leq c_1 n! \cdot \left[ \sum_{\substack{i,j,l \geq 0 \\ i+j+l=k}} \binom{k}{i,j,l} \right] \cdot \max\{1, 2^n c_2\}.$$

The right-hand side of (2.15) is therefore bounded independently of  $T$ . This shows that

$$\int_{[2,\infty)} \frac{1}{x^n} \cdot \frac{d\mu_\Phi(x)}{1+x^2} < \infty.$$

The same argument applies on the negative semi-axis, and we see that

$$\int_{\mathbb{R}} \frac{d\mu_\Phi(x)}{(1+|x|)^{2+n}} < \infty.$$



It follows that  $\mu_\phi \in \mathbb{M}$  and that the inequality “ $\geq$ ” in (2.7) holds.

For the reverse inequality, define a distributional density  $\psi$  as in Lemma 2.9 with  $\mu$  replaced by  $\mu_\phi$  and  $n$  minimal so that (2.10) holds. Then  $\mu_\psi = \mu_\phi$  and hence, by the already proved item (iii),  $\phi$  and  $\psi$  differ only by a term  $\sum_{k=0}^N a_k \delta_\infty^{(k)}$ . This, together with Lemma 2.9 and the definition of  $\text{ord}' \phi$  in (2.4), shows that

$$\text{ord}' \phi \leq \text{ord} \psi \leq n.$$

The relation (2.8) is immediate from (2.7). □

**Remark 2.10** If  $\mu \in \mathbb{M}$  and  $n \in \mathbb{N}_0$  is minimal such that (2.10) is valid, then the distributional density  $\phi$  from Lemma 2.9 satisfies

$$\text{ord} \phi = \text{ord}' \phi = n;$$

this follows from the last part of the proof of Theorem 2.8.

### 3 Representations of $\mathcal{N}_\kappa^{(\infty)}$ -functions

The class  $\mathcal{F}(\overline{\mathbb{R}})$  can be used to represent functions in  $\mathcal{N}_{<\infty}$ . Let us recall this fact in the formulation of [33, Proposition 5.4]. For this we need some more notation. First, we denote by  $\mathbb{R}(z)$  the set of all rational functions with real coefficients. Second, for each  $z \in \mathbb{C} \setminus \mathbb{R}$ , let  $\beta_z : \overline{\mathbb{R}} \rightarrow \mathbb{C}$  be defined by

$$\beta_z(x) := \begin{cases} \frac{1+xz}{x-z}, & x \in \mathbb{R}, \\ z, & x = \infty. \end{cases} \quad (3.1)$$

Clearly,

$$(\beta_z \circ \Lambda_\infty^{-1})(x) = \frac{x+z}{1-xz}, \quad x \in \mathbb{R},$$

and we see that  $\beta_z \in C^\infty(\overline{\mathbb{R}})$  for each  $z \in \mathbb{C} \setminus \mathbb{R}$ .

**Theorem 3.1** ([33]) *Let  $\phi \in \mathcal{F}(\overline{\mathbb{R}})$  and  $r \in \mathbb{R}(z)$ . Then the function*

$$q(z) := r(z) + \phi(\beta_z) \quad (3.2)$$

*belongs to  $\mathcal{N}_{<\infty}$ .*

*Conversely, let  $q \in \mathcal{N}_{<\infty}$  be given. Then there exist unique  $\phi \in \mathcal{F}(\overline{\mathbb{R}})$  and  $r \in \mathbb{R}(z)$  such that*

- (i) *the representation (3.2) holds;*
- (ii)  *$r$  is analytic on  $\mathbb{R}$  and  $r(z) = O(1)$  as  $|z| \rightarrow \infty$ .*

**Definition 3.2** Let  $\phi \in \mathcal{F}(\overline{\mathbb{R}})$  and  $r \in \mathbb{R}(z)$  be given. Then we write  $q_{r,\phi}$  for the expression on the right-hand side of (3.2); if  $r = 0$ , we just write  $q_\phi$ .

Let  $q \in \mathcal{N}_{<\infty}$  be given. Then we write  $r_q$  and  $\phi_q$  for the unique  $r \in \mathbb{R}(z)$  and  $\phi \in \mathcal{F}(\overline{\mathbb{R}})$ , respectively, such that (i) and (ii) in Theorem 3.1 hold.

**Remark 3.3** Since  $\beta_i(x) = i$  for all  $x \in \overline{\mathbb{R}}$ , we have  $q_\phi(i) = \phi(\beta_i) = i\phi(\mathbb{1}_{\overline{\mathbb{R}}}) \in i\mathbb{R}$  when  $\phi \in \mathcal{F}(\overline{\mathbb{R}})$ ; here  $\mathbb{1}_{\overline{\mathbb{R}}}$  denotes the constant function on  $\overline{\mathbb{R}}$  with value 1.

For the purpose of illustration let us elaborate on the case when  $q \in \mathcal{N}_0$  and make explicit the relation between the Herglotz integral representation (1.1) and the distributional representation (3.2).

**Example 3.4** Let  $\mu$  be a positive Borel measure on  $\mathbb{R}$  with

$$\int_{\mathbb{R}} \frac{d\mu(x)}{1+x^2} < \infty \quad (3.3)$$

and let  $a \in \mathbb{R}$  and  $b \geq 0$ . Define a distributional density  $\phi = (\phi_0, \phi_\infty) \in \mathcal{F}_{\{\infty\}}$  as in Lemma 2.9 with the given measure  $\mu$  and the number  $n = 0$ . Note that, since  $n = 0$ , the partition of unity does not enter the formulae in (2.11). Now, set

$$\psi := \phi + b\delta_\infty.$$

Let  $\chi^0, \chi^\infty \in C^\infty(\overline{\mathbb{R}})$  be some partition of unity subordinate to the open cover  $\{\overline{\mathbb{R}} \setminus \{\infty\}, \overline{\mathbb{R}} \setminus \{0\}\}$ . Then we obtain from (2.2) and (2.11) that

$$\begin{aligned} \Psi(\beta_z) &= \phi_0(\chi^0\beta_z) + \phi_\infty((\chi^\infty\beta_z) \circ \Lambda_\infty^{-1}) + b(\chi^\infty\beta_z)(\infty) \\ &= \int_{\mathbb{R}} \chi^0(x)\beta_z(x) \frac{d\mu(x)}{1+x^2} + \int_{\mathbb{R} \setminus \{0\}} \chi^\infty(x)\beta_z(x) \frac{d\mu(x)}{1+x^2} + bz \\ &= \int_{\mathbb{R}} \frac{1+xz}{x-z} \cdot \frac{d\mu(x)}{1+x^2} + bz = bz + \int_{\mathbb{R}} \left( \frac{1}{x-z} - \frac{x}{1+x^2} \right) d\mu(x). \end{aligned}$$

To fully match (1.1) with (3.2), it remains to notice that a real rational function that has the properties in Theorem 3.1 (ii) and has no non-real poles (which holds for  $\mathcal{N}_0$ -functions) must be equal to a real constant.

The subclass  $\mathcal{N}_{<\infty}^{(\infty)}$  of  $\mathcal{N}_{<\infty}$ , which is defined below, is the central object in the rest of the paper.

**Definition 3.5** Let  $q \in \mathcal{N}_{<\infty}$ . Then we say that  $q \in \mathcal{N}_{<\infty}^{(\infty)}$  if

$$\lim_{z \rightarrow i\infty} \frac{q(z)}{z^{2\kappa-1}} \in (-\infty, 0) \quad \text{or} \quad \lim_{z \rightarrow i\infty} \left| \frac{q(z)}{z^{2\kappa-1}} \right| = \infty, \quad (3.4)$$

where  $\kappa := \text{ind}_- q$ . Here we denote by  $\dot{\rightarrow}$  the non-tangential limit towards  $i\infty$ .

Moreover, we set  $\mathcal{N}_\kappa^{(\infty)} := \{q \in \mathcal{N}_{<\infty}^{(\infty)} : \text{ind}_- q = \kappa\}$ ,  $\kappa \in \mathbb{N}_0$ .

The significance in this definition is not that the asymptotic relation (3.4) holds with *some*  $\kappa$ , but that it holds *exactly* with  $\kappa = \text{ind}_- q$ . In fact, (3.4) always holds with some  $\kappa \in \mathbb{Z}$ ,  $\kappa \leq \text{ind}_- q$ .

**Remark 3.6** The origin of the class  $\mathcal{N}_{<\infty}^{(\infty)}$  lies in a spectral property of corresponding operator models, and also the choice of notation is explained by this fact. Namely, using the language of [40] and [42]<sup>3</sup>, a generalized Nevanlinna function  $q$  belongs to the class  $\mathcal{N}_{<\infty}^{(\infty)}$  if and only if the point  $\infty$  is a generalized pole of non-positive type with the maximal possible multiplicity, namely  $\text{ind}_- q$ .

The multiplicities of generalized poles of non-positive type in  $\overline{\mathbb{R}}$  and the multiplicities of poles located in  $\mathbb{C}^+$  together sum up to  $\text{ind}_- q$ , cf. [40, Theorem 3.5]. Hence a generalized Nevanlinna function  $q$  belongs to  $\mathcal{N}_{<\infty}^{(\infty)}$  if and only if  $q$  neither has finite generalized poles of non-positive type nor non-real poles. Using the analytic characterization [42, Theorem 3.1] of generalized poles of non-positive type, we thus obtain the following equivalence for  $q \in \mathcal{N}_{<\infty}$ :

$$q \in \mathcal{N}_{<\infty}^{(\infty)} \quad \iff \quad \begin{array}{l} q \text{ has no non-real poles and} \\ \lim_{z \rightarrow x} (z-x)q(z) \in (-\infty, 0] \quad \text{for every } x \in \mathbb{R}. \end{array}$$

Using the analytic characterization [42, Theorem 3.2] of the multiplicity of a generalized pole of non-positive type, we see that, for a function  $q \in \mathcal{N}_{<\infty}^{(\infty)}$ , the negative index  $\text{ind}_- q$  can be recovered from the asymptotics of  $q$  at  $i\infty$ ; namely,  $\text{ind}_- q$  is the unique number  $\kappa \in \mathbb{N}_0$  such that (3.4) holds and

$$\lim_{z \rightarrow i\infty} \frac{q(z)}{z^{2\kappa+1}} = 0.$$

<sup>3</sup> As common in the recent literature, we substitute the terminology “negative type” in these references by “non-positive type”.

Let us recall some properties of  $\mathcal{N}_{<\infty}^{(\infty)}$ -functions. Thereby, we denote by  $\mathbb{R}[z]$  the set of all polynomials in the variable  $z$  with real coefficients. The proof is immediate from what we said above; we skip the details.

**Lemma 3.7** *We have  $\mathcal{N}_0 = \mathcal{N}_0^{(\infty)} \subseteq \mathcal{N}_{<\infty}^{(\infty)}$  and  $\mathbb{R}[z] \subseteq \mathcal{N}_{<\infty}^{(\infty)}$ . If  $q_1, q_2 \in \mathcal{N}_{<\infty}^{(\infty)}$ , then also  $q_1 + q_2 \in \mathcal{N}_{<\infty}^{(\infty)}$  and*

$$\text{ind}_-(q_1 + q_2) \leq \max \{ \text{ind}_- q_1, \text{ind}_- q_2 \}.$$

*In this relation strict inequality may occur only if  $\text{ind}_- q_1 = \text{ind}_- q_2$  and for both functions the second relation in (3.4) holds.*

An example where the strict inequality in the previous lemma occurs is  $q_1(z) = z^2 + z$ ,  $q_2(z) = -z^2$ .

**Remark 3.8** A function  $q$  belongs to  $\mathcal{N}_{<\infty}^{(\infty)}$  if and only if it can be written as

$$q(z) = (z^2 + 1)^m q_0(z) + p(z) \tag{3.5}$$

where  $m \in \mathbb{N}_0$ ,  $q_0 \in \mathcal{N}_0$  and  $p \in \mathbb{R}[z]$ ; see, e.g. [13].

In the next theorem we show in detail that functions of the class  $\mathcal{N}_{<\infty}^{(\infty)}$  correspond precisely to distributional densities of the class  $\mathcal{F}_{\{\infty\}}$ . In its formulation remember the notation in Definition 3.2.

**Theorem 3.9** *The classes  $\mathcal{N}_{<\infty}^{(\infty)}$ ,  $\mathcal{F}_{\{\infty\}}$  and  $\mathbb{M}$  are related as follows.*

(i) *Let  $\phi = (\phi_0, \phi_\infty) \in \mathcal{F}_{\{\infty\}}$  and  $r \in \mathbb{R}$ ; then  $q_{r,\phi} \in \mathcal{N}_{<\infty}^{(\infty)}$ . Moreover, define the functions*

$$g_T(x) := x^{\text{ord } \phi} \chi^0(Tx), \quad x \in \mathbb{R},$$

*where  $\chi^0$  denotes a  $C^\infty(\mathbb{R})$ -function with  $\text{supp } \chi^0 \subseteq (-2, 2)$ ,  $\chi^0|_{[-1,1]} = 1$  and  $\chi^0(x) \in [0, 1]$ ,  $x \in \mathbb{R}$ . Then*

$$\text{ind}_- q_{r,\phi} = \left\lfloor \frac{\text{ord } \phi}{2} \right\rfloor + \begin{cases} 0 & \text{if } \text{ord } \phi \text{ is even and } \lim_{T \rightarrow \infty} \phi_\infty(g_T) \geq 0, \\ 1 & \text{otherwise.} \end{cases} \tag{3.6}$$

*The limit on the right-hand side of (3.6) always exists.*

(ii) *Let  $q \in \mathcal{N}_{<\infty}^{(\infty)}$ . Then  $\phi_q \in \mathcal{F}_{\{\infty\}}$  and  $r_q$  is a real constant:  $r_q = \text{Re } q(i)$ . The measure  $\mu_{\phi_q}$  can be recovered by means of Stieltjes' inversion formula: for each compact interval  $[a, b] \subseteq \mathbb{R}$ ,*

$$\mu_{\phi_q}([a, b]) = \frac{1}{\pi} \lim_{\delta \searrow 0} \lim_{\varepsilon \searrow 0} \int_{a-\delta}^{b+\delta} \text{Im } q(x + i\varepsilon) dx. \tag{3.7}$$

*For the mass at a point  $c \in \mathbb{R}$  one has*

$$\mu_{\phi_q}(\{c\}) = - \lim_{z \hat{\rightarrow} c} (z - c)q(z), \tag{3.8}$$

*where  $z \hat{\rightarrow} c$  denotes the non-tangential limit.*

(iii) *Let  $\phi \in \mathcal{F}_{\{\infty\}}$  and  $\psi \in \widetilde{\mathcal{D}}'(\overline{\mathbb{R}})$ . Then  $\psi \in \mathcal{F}_{\{\infty\}}$  and  $\mu_\phi = \mu_\psi$  if and only if  $p := q_\phi - q_\psi$  is a real polynomial with  $\text{Re } p(i) = 0$ .*

(iv) *Let  $q_1, q_2 \in \mathcal{N}_{<\infty}^{(\infty)}$ . Then  $\mu_{\phi_{q_1}} = \mu_{\phi_{q_2}}$  if and only if  $q_1 - q_2$  is a real polynomial.*

(v) *Let  $\mu \in \mathbb{M}$ . Then*

$$\{ \text{ind}_- q_\phi : \phi \in \mathcal{F}_{\{\infty\}}, \mu_\phi = \mu \} = \{ n \in \mathbb{N}_0 : n \geq \Delta(\mu) \}.$$

To shorten notation, we sometimes write

$$\mu_q := \mu_{\phi_q} \quad \text{for } q \in \mathcal{N}_{<\infty}^{(\infty)}. \quad (3.9)$$

$\Rightarrow$  *Notice:* For the reader with experience in the theory of Pontryagin space operator models, some of these assertions are of course not surprising. Indeed, for some parts, a proof alternative to the one given below could proceed via analysing the operator model. It would require to put together knowledge (explicitly and implicitly) contained in [42], [31] and [35]. We prefer to give an approach as elementary as possible for all parts of the theorem; all we need to know about the operator model is what we summarized in Remark 3.6.  $\diamond$

*Proof of Theorem 3.9.* To start with, let  $\phi = (\phi_0, \phi_\infty) \in \mathcal{F}_{\{\infty\}}$  and  $r \in \mathbb{R}$  be given, and let us show that  $q_{r,\phi} \in \mathcal{N}_{<\infty}^{(\infty)}$ . Clearly, this function has no non-real poles. Let  $x_0 \in \mathbb{R}$ , choose  $T > 0$  such that  $|x_0| < T$ , and choose a partition of unity  $\chi^0, \chi^\infty \in C^\infty(\overline{\mathbb{R}})$  subordinate to the open cover

$$\{(-2T, 2T), \overline{\mathbb{R}} \setminus [-T, T]\}.$$

Then we have

$$q_{r,\phi}(z) = r + \phi_0(\chi^0 \beta_z) + \phi_\infty((\chi^\infty \beta_z) \circ \Lambda_\infty^{-1}).$$

The second summand can be written as

$$\phi_0(\chi^0 \beta_z) = \int_{(-2T, 2T)} \beta_z(t) \cdot \chi^0(t) \frac{d\mu_\phi(t)}{1+t^2}$$

and hence belongs to the class  $\mathcal{N}_0$ . The first and third summands are analytic on  $\mathbb{C} \setminus [-T, T]$ . Thus the point  $x_0$  is not a generalized pole of non-positive type for any of the summands, and hence also not for their sum. We conclude that  $q_{r,\phi}$  has no finite generalized poles of non-positive type, and hence belongs to the class  $\mathcal{N}_{<\infty}^{(\infty)}$ .

We come to the proof of (ii). Let  $q \in \mathcal{N}_{<\infty}^{(\infty)}$  be given and consider the representation  $q(z) = r_q(z) + \phi_q(\beta_z)$  guaranteed by Theorem 3.1. Since  $q$  has no non-real poles, the function  $r_q$  is a real constant. Moreover,  $r_q = \operatorname{Re} q(i)$  because  $\phi_q(\beta_i) \in i\mathbb{R}$  by Remark 3.3.

We have to show that  $\phi_q \in \mathcal{F}_{\{\infty\}}$ . To this end, let  $\phi_0, \phi_\infty$  so that  $\phi_q = (\phi_0, \phi_\infty)$  and let  $T > 0$  be such that  $s(\phi_q) \setminus \{\infty\} \subseteq (-T, T)$ . Choose a partition of unity  $\chi^0, \chi^\infty \in C^\infty(\overline{\mathbb{R}})$  that is subordinate to the open cover  $\{(-T-1, T+1), \overline{\mathbb{R}} \setminus [-T, T]\}$  and set

$$q_0(z) := \phi_0(\chi^0 \beta_z), \quad q_\infty(z) := \phi_\infty((\chi^\infty \beta_z) \circ \Lambda_\infty^{-1}),$$

so that  $q = q_0 + q_\infty$ . The function  $q_\infty$  is analytic on some neighbourhood of  $[-T, T]$ , and hence, for every  $x_0 \in [-T, T]$ ,

$$\lim_{z \rightarrow x_0} (z - x_0)q_0(z) = \lim_{z \rightarrow x_0} (z - x_0)q(z) \in (-\infty, 0]$$

since  $q$  has no finite generalized poles of non-positive type. The function  $q_0$  itself is analytic in  $\mathbb{C} \setminus \operatorname{supp} \chi^0$ , and it follows that

$$\lim_{z \rightarrow x_0} (z - x_0)q_0(z) = 0 \quad \text{for } x_0 \in (-\infty, -T-1] \cup [T+1, \infty).$$

The relations  $\beta_z^{(n)}(x) = (-1)^n n!(1+z^2)(x-z)^{-(n+1)}$ ,  $n \geq 1$ , (here differentiation is with respect to  $x$ ) imply that

$$\sup_{y \geq 1} \|\beta_{iy}\|_{\operatorname{supp} \chi_0, \operatorname{ord} \phi_0} < \infty.$$

Moreover,

$$\lim_{y \rightarrow +\infty} \frac{1}{y} q_0(iy) = 0.$$

Let  $x_0 \in (-T-1, -T) \cup (T, T+1)$ . Choose  $\varepsilon > 0$  with  $[x_0 - 2\varepsilon, x_0 + 2\varepsilon] \subseteq (-T-1, -T) \cup (T, T+1)$ , and a partition of unity  $\tilde{\chi}^0, \tilde{\chi}^\infty \in C^\infty(\overline{\mathbb{R}})$  subordinate to the open cover  $\{(x_0 - 2\varepsilon, x_0 + 2\varepsilon), \overline{\mathbb{R}} \setminus [x_0 - \varepsilon, x_0 + \varepsilon]\}$ . Then the function  $(\tilde{\chi}^\infty \chi^0 \phi_0)(\beta_z)$  is analytic in a neighbourhood of  $[x_0 - \varepsilon, x_0 + \varepsilon]$ . Since  $\phi$  is a positive measure on  $(-T-1, -T) \cup (T, T+1)$ , say  $\mu$ , we have

$$(\tilde{\chi}^0 \chi^0 \phi_0)(\beta_z) = \int_{\mathbb{R}} \beta_z(x) \cdot \tilde{\chi}^0(x) \chi^0(x) d\mu(x),$$

and hence

$$\lim_{z \rightarrow x_0} (z - x_0) q_0(z) = \lim_{z \rightarrow x_0} (z - x_0) (\tilde{\chi}^0 \chi^0 \phi_0)(\beta_z) = -\mu(\{x_0\}) \in (-\infty, 0].$$

Altogether, we see that the function  $q_0$  has no generalized poles of non-positive type. Clearly, it also has no non-real poles, and it follows that  $q_0 \in \mathcal{N}_0$ . Thus we find a positive measure  $\sigma$  supported in  $[-T-1, T+1]$  such that

$$\phi_0(\chi^0 \beta_z) = q_0(z) = \int_{\mathbb{R}} \beta_z(x) d\sigma(x).$$

Since a compactly supported distribution is uniquely determined by its Cauchy transform (see, e.g. [5, Theorem 2.3.3]), it follows that  $\phi_0$  is a measure on  $(-T, T)$  and equals  $\sigma$  there. Letting  $T$  tend to infinity we can deduce that  $\phi_0$  is a positive measure on  $\mathbb{R}$ .

The fact that  $\mu_{\phi_q}$  can be recovered by means of the Stieltjes inversion formula in (3.7) was proved in [33, Lemma 5.5]. On comparing the present formulation with this reference, remember that we included the density  $\frac{1}{1+x^2}$  in the definition of  $\mu_{\phi_q}$ . Formula (3.8) can be proved in a similar way as [33, Lemma 5.5] by choosing a partition of unity and reducing to the positive definite case. This completes the proof of (ii).

Next, we show item (iii). Consider the function  $\check{\beta}_z := \beta_z \circ \Lambda_\infty^{-1}$ , i.e.

$$\check{\beta}_z(x) = \frac{x+z}{1-xz}, \quad x \in \mathbb{R}.$$

A simple computation gives (again differentiation is with respect to  $x$ )

$$\check{\beta}_z(0) = z, \quad \check{\beta}_z^{(n)}(0) = (1+z^2)n!z^{n-1}, \quad n \in \mathbb{N}.$$

We see that  $\{\check{\beta}_z^{(n)}(0) : n \in \mathbb{N}_0\}$  is a basis for the real vector space

$$\{b_m z^m + \dots + b_1 z + b_0 : m \in \mathbb{N}_0, b_j \in \mathbb{R}, b_0 + b_2 + \dots = 0\} = \{p \in \mathbb{R}[z] : \operatorname{Re} p(i) = 0\}.$$

Hence a function  $p$  is a real polynomial with  $\operatorname{Re} p(i) = 0$  if and only if it can be represented as

$$p(z) = \left( \sum_{k=0}^N a_k \delta_\infty^{(k)} \right) (\beta_z)$$

with some  $N \in \mathbb{N}_0$  and  $a_0, \dots, a_N \in \mathbb{R}$ . Now the assertion follows from Theorem 2.8 (iii) and the uniqueness statement in the converse part of Theorem 3.1.

Item (iv) follows immediately from (iii) and the representations  $q_i(z) = r_{q_i} + q_{\phi_{q_i}}(z)$ ,  $i = 1, 2$ , with  $r_{q_i} \in \mathbb{R}$ .

For the proof of (v), let  $\mu \in \mathbb{M}$  be given and set  $\Delta := \Delta(\mu)$ . Consider the function

$$q(z) := (1+z^2)^\Delta \int_{\mathbb{R}} \beta_z(x) \frac{d\mu(x)}{(1+x^2)^{\Delta+1}}. \tag{3.10}$$

The integral term belongs to the class  $\mathcal{N}_0$  and hence has no generalized poles of non-positive type. It follows that also  $q$  has no finite generalized poles of non-positive type. Clearly,  $q$  has no non-real poles, and hence  $q \in \mathcal{N}_{<\infty}^{(\infty)}$ . Let  $a, b \in \mathbb{R}$  with  $a < b$ , choose  $T > 0$  such that  $[a, b] \subseteq (-T, T)$ , and choose a partition of unity  $\chi^0, \chi^\infty \in C^\infty(\overline{\mathbb{R}})$  subordinate to the open cover

$$\{(-2T, 2T), \overline{\mathbb{R}} \setminus [-T, T]\}.$$

Using  $\beta_z(t) = (\frac{1}{t-z} - \frac{t}{1+t^2})(1+t^2)$  we can rewrite  $q$  as follows

$$\begin{aligned} q(z) &= (1+z^2)^\Delta \int_{\mathbb{R}} \frac{1}{t-z} \cdot \frac{\chi^0(t) d\mu(t)}{(1+t^2)^\Delta} \\ &\quad - (1+z^2)^\Delta \int_{\mathbb{R}} \frac{t}{1+t^2} \cdot \frac{\chi^0(t) d\mu(t)}{(1+t^2)^\Delta} + (1+z^2)^\Delta \int_{\mathbb{R}} \beta_z(t) \frac{\chi^\infty(t) d\mu(t)}{(1+t^2)^{\Delta+1}}. \end{aligned}$$

The second and third summands are analytic in a neighbourhood of  $(-T, T)$ . The integral in the first summand is a Cauchy integral of a finite measure. Using (3.7) and the Stieltjes–Lifšic inversion formula (e.g. as in [24, Theorem 1.2.4]) we obtain

$$\begin{aligned} \mu_{\phi_q}([a, b]) &= \frac{1}{\pi} \lim_{\delta \searrow 0} \lim_{\varepsilon \searrow 0} \int_{a-\delta}^{b+\delta} \operatorname{Im} q(x+i\varepsilon) dx \\ &= \frac{1}{\pi} \lim_{\delta \searrow 0} \lim_{\varepsilon \searrow 0} \int_{a-\delta}^{b+\delta} \operatorname{Im} \left[ (1+(x+i\varepsilon)^2)^\Delta \int_{\mathbb{R}} \frac{1}{t-(x+i\varepsilon)} \cdot \frac{\chi^0(t) d\mu(t)}{(1+t^2)^\Delta} \right] dx \\ &= \int_{[a, b]} (1+x^2)^\Delta \cdot \frac{\chi^0(x) d\mu(x)}{(1+x^2)^\Delta} = \mu([a, b]). \end{aligned}$$

Since  $a$  and  $b$  were arbitrary, this shows that  $\mu_{\phi_q} = \mu$ . Moreover,  $\operatorname{Re} q(i) = 0$  and hence  $q = q_{\phi_q}$ . By the already proved item (iii) we have

$$\{q_\phi : \phi \in \mathcal{F}_{\{\infty\}}, \mu_\phi = \mu\} = \{q+p : p \in \mathbb{R}[z], \operatorname{Re} p(i) = 0\}.$$

The negative index of a function of the form  $q+p$  with  $q$  from (3.10) and  $p \in \mathbb{R}[z]$  was computed in [39, Lemma 3.3], namely, write  $p(z) = b_m z^m + \dots + b_1 z + b_0$  with  $b_m \neq 0$  if  $p \neq 0$  and set  $m = -1$  if  $p = 0$ ; then

$$\operatorname{ind}_-(q+p) = \max \left\{ \Delta, \left\lfloor \frac{m+1}{2} \right\rfloor \right\} - \begin{cases} 1, & m > 2\Delta, m \text{ odd}, b_m > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (3.11)$$

From this formula it is obvious that  $\operatorname{ind}_-(q+p)$  covers the set  $\{n \in \mathbb{N}_0 : n \geq \Delta\}$  when  $p$  varies through all real polynomials with  $\operatorname{Re} p(i) = 0$ .

Finally, we turn to the formula (3.6) for the negative index. Clearly, it is enough to consider the case when  $r = 0$ . Let  $\phi \in \mathcal{F}_{\{\infty\}}$  be given and set  $\Delta := \Delta(\mu_\phi)$ . Moreover, fix a partition of unity  $\chi^0, \chi^\infty \in C^\infty(\overline{\mathbb{R}})$  subordinate to the open cover  $\{(-2, 2), \overline{\mathbb{R}} \setminus [-1, 1]\}$ .

Set  $n := \operatorname{ord}' \phi$ . By Theorem 2.8 (ii),  $n$  satisfies (2.10). Hence we can apply Lemma 2.9 with  $\mu$  replaced by  $\mu_\phi$ . This yields a distributional density, which we call  $\psi$ . Then  $\mu_\psi = \mu_\phi$  and, by Remark 2.10, we have

$$\operatorname{ord} \psi = n = \operatorname{ord}' \phi.$$

Set again  $\check{\beta}_z(t) := \beta_z(\frac{1}{t})$ . A short computation shows that

$$\check{\beta}_z\left(\frac{1}{x}\right) - \sum_{k=0}^{n-1} \frac{\check{\beta}^{(k)}(0)}{k!} \left(\frac{1}{x}\right)^k = (1+z^2) z^{n-1} \frac{1}{x^{n-1}} \cdot \frac{1}{x-z}, \quad x \in \mathbb{R} \setminus \{0\}.$$

Hence

$$\begin{aligned} q_\psi(z) &= \psi(\beta_z) = \psi_0(\chi^0 \beta_z) + \psi_\infty((\chi^\infty \beta_z) \circ \Lambda_\infty^{-1}) \\ &= \int_{\mathbb{R}} \beta_z(x) \frac{\chi^0(x) d\mu_\phi(x)}{1+x^2} + (1+z^2)z^{n-1} \int_{\mathbb{R} \setminus \{0\}} \frac{1}{x-z} \cdot \frac{\chi^\infty(x) d\mu_\phi(x)}{x^{n-1}(1+x^2)}. \end{aligned}$$

Using the dominated convergence theorem for each of the integrals we obtain

$$\lim_{y \rightarrow +\infty} \frac{1}{y^{n+1}} q_\psi(iy) = 0. \quad (3.12)$$

Consider the function  $q$  defined by (3.10) with  $\mu$  replaced by  $\mu_\phi$ . The integral in (3.10) belongs to the class  $\mathcal{N}_0$  and (obviously) has no linear term in its Herglotz integral representation, which implies that

$$\lim_{y \rightarrow +\infty} \frac{1}{y^{2\Delta+1}} q(iy) = 0. \quad (3.13)$$

Since  $\operatorname{Re} q(i) = 0$ , we can write  $q(z) = \theta(\beta_z)$  with some  $\theta \in \mathcal{F}_{\{\infty\}}$ . As  $\mu_\theta = \mu_\phi = \mu_\psi$  by the previous part of the proof, it follows from Theorem 2.8 (iii) that there exist  $N, M \in \mathbb{N}_0 \cup \{-1\}$  and  $a_k, b_k \in \mathbb{R}$  such that

$$\phi = \theta + \sum_{k=0}^N a_k \delta_\infty^{(k)}, \quad \psi = \theta + \sum_{k=0}^M b_k \delta_\infty^{(k)}, \quad (3.14)$$

where we set  $N = -1$  and  $a_0 = 0$  if the sum in the expression for  $\phi$  is not present, and otherwise, we choose  $N$  such that  $a_N \neq 0$ ; in a similar way  $M$  and  $b_M$  are chosen. From this we obtain the following representations:

$$q_\phi(z) = q(z) + a_0 z + \sum_{k=1}^N a_k (1+z^2) k! z^{k-1}, \quad (3.15)$$

$$q_\psi(z) = q(z) + b_0 z + \sum_{k=1}^M b_k (1+z^2) k! z^{k-1}. \quad (3.16)$$

The number  $2\Delta$  equals either  $\operatorname{ord}' \phi$  or  $\operatorname{ord}' \phi + 1$  depending whether  $\operatorname{ord}' \phi$  is even or odd. The relation (3.16), together with (3.12) and (3.13), implies that  $M \leq 2\Delta - 1$ . Using (3.15) and again [39, Lemma 3.3], cf. (3.11), we obtain

$$\operatorname{ind}_- q_\phi = \max \left\{ \Delta, \left\lfloor \frac{N}{2} \right\rfloor + 1 \right\} - \begin{cases} 1, & N \geq 2\Delta, N \text{ even}, a_N > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (3.17)$$

We have to relate the right-hand side to the order of  $\phi$ . From (3.14) we have

$$\phi = \psi + \sum_{k=0}^N a_k \delta_\infty^{(k)} - \sum_{k=0}^M b_k \delta_\infty^{(k)}, \quad (3.18)$$

and (2.8) implies that

$$\operatorname{ord} \psi = \operatorname{ord}' \phi \in \{2\Delta - 1, 2\Delta\}. \quad (3.19)$$

Moreover,

$$N \leq \operatorname{ord} \phi \quad (3.20)$$

since otherwise, we would have  $M \leq 2\Delta - 1 \leq \operatorname{ord} \psi \leq \operatorname{ord} \phi < N$ , which contradicts (3.18). The following implications give some preliminary information about  $\operatorname{ord} \phi$ :

$$N \leq 2\Delta - 1 \implies \operatorname{ord} \phi = \operatorname{ord} \psi, \quad (3.21)$$

$$N \geq 2\Delta \implies \operatorname{ord} \phi = N. \quad (3.22)$$

To show (3.21), assume that  $N \leq 2\Delta - 1$  and  $\text{ord } \phi \neq \text{ord } \psi$ ; then  $\text{ord } \phi > \text{ord } \psi \geq 2\Delta - 1 \geq M, N$ , which contradicts (3.18). Likewise, to show (3.22), assume that  $N \geq 2\Delta$  and  $\text{ord } \phi \neq N$ ; then, by (3.20),  $N > \text{ord } \phi \geq \text{ord } \psi$  and  $N \geq 2\Delta > M$ , which again is a contradiction to (3.18).

Next we consider the limit  $\lim_{T \rightarrow \infty} \phi_\infty(g_T)$  that appears on the right-hand side of (3.6). Note that

$$\delta_0^{(k)}(g_T) = (-1)^k \delta_0(g_T^{(k)}) = \begin{cases} (-1)^k k! & \text{if } k = \text{ord } \phi, \\ 0 & \text{otherwise.} \end{cases}$$

Since  $n = \text{ord}' \phi \leq \text{ord } \phi$ , the regularizing term in the formula for  $\phi_\infty$  in (2.11) is not present if we apply  $\psi_\infty$  to  $g_T$ , and hence

$$\psi_\infty(g_T) = \int_{\mathbb{R} \setminus \{0\}} \frac{1}{x^{\text{ord } \phi}} \chi^0\left(\frac{T}{x}\right) \frac{d\mu_\phi(x)}{1+x^2}.$$

The dominated convergence theorem implies that this expression tends to 0 as  $T \rightarrow \infty$ . From this and (3.18) it follows that

$$\begin{aligned} \lim_{T \rightarrow \infty} \phi_\infty(g_T) &= \lim_{T \rightarrow \infty} \left( \sum_{k=0}^N a_k \delta_0^{(k)}(g_T) - \sum_{k=0}^M b_k \delta_0^{(k)}(g_T) \right) \\ &= (-1)^N N! a_N \delta_{N, \text{ord } \phi} - (-1)^M M! b_M \delta_{M, \text{ord } \phi}, \end{aligned} \quad (3.23)$$

where  $\delta_{i,j}$  denotes the usual Kronecker delta.

In order to show (3.6), we distinguish two cases. First, assume that  $N < \text{ord } \phi$ . Then  $N \leq 2\Delta - 1$  by (3.22), and hence  $\lfloor \frac{N}{2} \rfloor + 1 \leq \Delta$ . This, together with (3.17), implies that

$$\text{ind}_- q_\phi = \max \left\{ \Delta, \left\lfloor \frac{N}{2} \right\rfloor + 1 \right\} = \Delta$$

and hence  $\text{ord } \phi = \text{ord } \psi$  by (3.21). From the relations  $N \leq 2\Delta - 1$  and (3.21) we obtain that  $\text{ord } \phi = \text{ord } \psi$ , and hence, with (3.19),

$$\begin{aligned} \text{ind}_- q_\phi = \Delta &= \begin{cases} \frac{\text{ord } \phi}{2} & \text{if } \text{ord } \phi \text{ is even,} \\ \frac{\text{ord } \phi + 1}{2} & \text{otherwise,} \end{cases} \\ &= \left\lfloor \frac{\text{ord } \phi}{2} \right\rfloor + \begin{cases} 0 & \text{if } \text{ord } \phi \text{ is even,} \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

If  $\text{ord } \phi$  is even, then  $M, N \leq 2\Delta - 1 < \text{ord } \phi$  and therefore  $\lim_{T \rightarrow \infty} \phi_\infty(g_T) = 0$ , which completes the proof of (3.6) in the case when  $N < \text{ord } \phi$ .

Second, assume that  $N = \text{ord } \phi$  (note that  $N > \text{ord } \phi$  is excluded by (3.20)). Then

$$\left\lfloor \frac{N}{2} \right\rfloor + 1 = \left\lfloor \frac{\text{ord } \phi}{2} \right\rfloor + 1 \geq \left\lfloor \frac{\text{ord } \psi}{2} \right\rfloor + 1 \geq \left\lfloor \frac{2\Delta - 1}{2} \right\rfloor + 1 = \Delta$$

by (3.19). Now it follows from (3.17) that

$$\text{ind}_- q_\phi = \left\lfloor \frac{\text{ord } \phi}{2} \right\rfloor + \begin{cases} 0, & N \geq 2\Delta, N \text{ even, } a_N > 0, \\ 1, & \text{otherwise.} \end{cases}$$

If  $N$  is even, then  $N = \text{ord } \phi \geq \text{ord } \psi \geq 2\Delta - 1$  and hence  $N \geq 2\Delta$ ,  $M < \text{ord } \phi$  and  $\lim_{T \rightarrow \infty} \phi_\infty(g_T) = N! a_N$ . Therefore

$$\begin{aligned} N \geq 2\Delta, N \text{ even, } a_N > 0 &\iff N \text{ even, } a_N > 0 \\ &\iff \text{ord } \phi \text{ even, } \lim_{T \rightarrow \infty} \phi_\infty(g_T) \geq 0. \end{aligned}$$

This completes the proof of (3.6) also in the case when  $N = \text{ord } \phi$ . □



$\Rightarrow$  *Notice:* In connection with the proof of item (v) and the relation (3.6) again one observation is in order. We use the function  $q_\psi$  to compute the order and the function  $q$  to compute the negative index. Therefore we have to relate these two functions, which is done in (3.16). Alternatively, one could carry out an argument modelled after the proof of [39, Lemma 3.3] (which is elementary) to show an analogue of the formula (3.11) for  $q_\psi$  in place of  $q$ . Yet alternatively, one could proceed (with the usual technical efforts of switching the roles of 0 and  $\infty$  and reducing to compactly supported distributions) via the operator model constructed in [31], appeal to the formula for negative index of spaces in [31, Theorem 2.5] and to the relation with negative index of functions provided in [35, Corollary 3.5].  $\diamond$

#### 4 Distributional densities associated with symmetric $\mathcal{N}_\kappa$ -functions

We call a generalized Nevanlinna function  $q \in \mathcal{N}_{<\infty}$  *symmetric* if it is an odd function, i.e.  $q(-z) = -q(z)$ . Further, we call a distributional density  $\phi = (\phi_0, \phi_\infty)$  *symmetric* if both  $\phi_0$  and  $\phi_\infty$  annihilate all odd test functions. The following fact was shown in [33, Theorem 5.9 (ii)].

**Theorem 4.1** ([33]) *Let  $q \in \mathcal{N}_{<\infty}$ , and let  $r \in \mathbb{R}(z)$  and  $\phi \in \mathcal{F}(\overline{\mathbb{R}})$  be the unique data in the distributional representation (3.2). Then  $q$  is symmetric if and only if  $r$  is odd and  $\phi$  is symmetric.*

If  $q \in \mathcal{N}_{<\infty}$  is symmetric, then two functions  $q_+$  and  $q_-$  are well defined by

$$q_+(z^2) = zq(z), \quad q_-(z^2) = \frac{q(z)}{z}. \quad (4.1)$$

In the positive definite case, symmetric Nevanlinna functions appear in the spectral theory of Kreĭn strings. The functions  $q_\pm$  then belong to the Stieltjes class (or inverse Stieltjes class, respectively). From a function-theoretic point of view, these functions were studied in [32, §5]; an operator-theoretic interpretation (splitting of the model space) was (implicitly) given in [4] and was exploited, e.g. for the study of Gaussian processes in [20].

Let us return to the indefinite case. It was shown in [33, Proposition 3.2] that, for symmetric  $q \in \mathcal{N}_{<\infty}$ , the functions  $q_+$  and  $q_-$  also belong to the class  $\mathcal{N}_{<\infty}$  and that

$$\text{ind}_- q = \text{ind}_- q_+ + \text{ind}_- q_-.$$

In Theorem 4.4 below we determine the distributional representations of  $q_+$  and  $q_-$  explicitly in terms of the representation of  $q$ . These formulae are closely related to the integral representations obtained in [32] (for Stieltjes class functions), in [39] (for functions of the class  $\mathcal{N}_\kappa^+$ ) and in [9] (for operator-valued functions with spectral gaps).

Before we formulate and prove Theorem 4.4, we show two preparatory results on distributional densities  $\phi$  where we introduce the decomposition of  $\phi$  corresponding to the construction of  $q_\pm$ . For this, we define the functions

$$h_0(t) := \begin{cases} \frac{1+t^2}{1+t^4}, & t \in \mathbb{R}, \\ 0, & t = \infty, \end{cases} \quad h_1(t) := \begin{cases} \frac{t^2(1+t^2)}{1+t^4}, & t \in \mathbb{R}, \\ 1, & t = \infty. \end{cases} \quad (4.2)$$

Clearly,  $h_0, h_1 \in C^\infty(\overline{\mathbb{R}})$  and

$$h_1\left(\frac{1}{t}\right) = h_0(t), \quad t \in \overline{\mathbb{R}}. \quad (4.3)$$

Moreover, for a real-valued function  $g \in C^\infty(\mathbb{R})$  that satisfies  $\lim_{|x| \rightarrow \infty} |g(x)| = \infty$ , we denote by  $C_g$  the operator of composition with  $g$ , i.e.  $[C_g(f)](x) = f(g(x))$ ,  $x \in \mathbb{R}$ , for  $f \in \mathcal{D}(\mathbb{R})$ . Then, for each distribution  $\phi \in \mathcal{D}'(\mathbb{R})$ , also the map  $\phi \circ C_g$  belongs to  $\mathcal{D}'(\mathbb{R})$ . If  $g \in C^\infty(\overline{\mathbb{R}})$ , then one can define  $C_g(f)$  in a similar way for  $f \in C^\infty(\overline{\mathbb{R}})$ , so that  $C_g(f) \in C^\infty(\overline{\mathbb{R}})$ . Moreover, set  $\tau(t) := t^2$ ,  $t \in \mathbb{R}$ , and extend it to  $\overline{\mathbb{R}}$  by  $\tau(\infty) = \infty$ .

**Lemma 4.2** *Let  $\phi = (\phi_0, \phi_\infty) \in \tilde{\mathcal{D}}'(\overline{\mathbb{R}})$ . Then  $\phi^+ := (\phi_0^+, \phi_\infty^+)$  and  $\phi^- := (\phi_0^-, \phi_\infty^-)$  belong to  $\tilde{\mathcal{D}}'(\overline{\mathbb{R}})$  where*

$$\begin{aligned}\phi_0^+ &:= (h_1\phi_0) \circ C_\tau, & \phi_\infty^+ &:= (h_0\phi_\infty) \circ C_\tau, \\ \phi_0^- &:= (h_0\phi_0) \circ C_\tau, & \phi_\infty^- &:= (h_1\phi_\infty) \circ C_\tau.\end{aligned}\tag{4.4}$$

Their action on  $C^\infty(\overline{\mathbb{R}})$  as linear functionals is given by

$$\phi^+(f) = \phi(h_1 C_\tau(f)), \quad \phi^-(f) = \phi(h_0 C_\tau(f)), \quad f \in C^\infty(\overline{\mathbb{R}}),\tag{4.5}$$

and we have  $\text{supp } \phi^\pm \subseteq [0, \infty) \cup \{\infty\}$ .

**Proof.** As we noticed before the lemma, the components of  $\phi^+$  and  $\phi^-$  are distributions on  $\mathbb{R}$ . We have to check the transformation law (2.1). To this end, let  $f \in \mathcal{D}(\mathbb{R})$  with  $0 \notin \text{supp } f$  be given. Then, with (4.3), we have

$$\begin{aligned}\phi_0^+(f \circ \Lambda_\infty) &= (h_1\phi_0)[f \circ \Lambda_\infty \circ \tau] = \phi_0[h_1 \cdot (f \circ \tau \circ \Lambda_\infty)] \\ &= \phi_0[(h_0 \cdot (f \circ \tau)) \circ \Lambda_\infty] = \phi_\infty[h_0 \cdot (f \circ \tau)] \\ &= (h_0\phi_\infty)[f \circ \tau] = \phi_\infty^+(f).\end{aligned}$$

The validity of the transformation law for  $\phi^-$  is seen in the same way.

For the proof of (4.5), choose a partition of unity  $\chi^0, \chi^\infty \in C^\infty(\overline{\mathbb{R}})$  subordinate to the open cover  $\{\mathbb{R} \setminus \{\infty\}, \mathbb{R} \setminus \{0\}\}$ . Set  $\tilde{\chi}^0 = \chi^0 \circ \tau, \tilde{\chi}^\infty = \chi^\infty \circ \tau$ . Then  $\tilde{\chi}^0, \tilde{\chi}^\infty \in C^\infty(\overline{\mathbb{R}})$  is a partition of unity subordinate to the same open cover. This, together with

$$\begin{aligned}\phi_0^+(\chi^0 f) &= (h_1\phi_0)[(\chi^0 \circ \tau) \cdot (f \circ \tau)] = \phi_0[h_1 \cdot \tilde{\chi}^0 \cdot (f \circ \tau)], \\ \phi_\infty^+(\chi^\infty f \circ \Lambda_\infty^{-1}) &= (h_0\phi_\infty)[(\chi^\infty f) \circ \Lambda_\infty^{-1} \circ \tau] = (h_0\phi_\infty)[(\chi^\infty f) \circ \tau \circ \Lambda_\infty^{-1}] \\ &= (h_0\phi_\infty)[(\tilde{\chi}^\infty \cdot (f \circ \tau)) \circ \Lambda_\infty^{-1}] = \phi_\infty[h_0 \cdot ((\tilde{\chi}^\infty \cdot (f \circ \tau)) \circ \Lambda_\infty^{-1})] \\ &= \phi_\infty[(h_1 \cdot \tilde{\chi}^\infty \cdot (f \circ \tau)) \circ \Lambda_\infty^{-1}],\end{aligned}$$

yields

$$\begin{aligned}\phi^+(f) &= \phi_0^+(\chi^0 f) + \phi_\infty^+(\chi^\infty f \circ \Lambda_\infty^{-1}) \\ &= \phi_0[\tilde{\chi}^0 \cdot h_1 \cdot (f \circ \tau)] + \phi_\infty[(\tilde{\chi}^\infty \cdot h_1 \cdot (f \circ \tau)) \circ \Lambda_\infty^{-1}] = \phi[h_1 \cdot (f \circ \tau)].\end{aligned}$$

The second relation in (4.5) is seen in the same way.

If  $\text{supp } f \subseteq (-\infty, 0)$ , then  $C_\tau(f)$  vanishes identically, and hence  $\phi^\pm(f) = 0$ . This implies that  $\text{supp } \phi^\pm \subseteq [0, \infty) \cup \{\infty\}$ .  $\square$

In the next lemma we consider the case when  $\phi$  belongs to the class  $\mathcal{F}_{\{\infty\}}$  and relate the corresponding measures.

**Lemma 4.3** *Let  $\phi \in \mathcal{F}_{\{\infty\}}$ . Then also  $\phi^+, \phi^- \in \mathcal{F}_{\{\infty\}}$ , and the measures  $\mu_\phi, \mu_{\phi^+}$  and  $\mu_{\phi^-}$  are related as follows ( $\tau : \mathbb{R} \rightarrow \mathbb{R}$  denotes again the map  $\tau(t) := t^2$  and  $\mu_\phi^\tau$  the corresponding image measure):*

$$\mu_{\phi^+} \ll \mu_\phi^\tau, \quad \frac{d\mu_{\phi^+}}{d\mu_\phi^\tau}(t) = t \cdot \mathbf{1}_{(0, \infty)},\tag{4.6}$$

$$\mu_{\phi^-} \ll \mu_\phi^\tau, \quad \frac{d\mu_{\phi^-}}{d\mu_\phi^\tau}(t) = \mathbf{1}_{[0, \infty)},\tag{4.7}$$

where, e.g.  $\mathbb{1}_{(0,\infty)}$  denotes the characteristic function of  $(0, \infty)$ .

Assume, in addition, that  $\mu_\phi$  is absolutely continuous with respect to the Lebesgue measure  $\lambda$  on some interval  $(a, b)$  with  $0 \leq a < b$ . Then  $\mu_{\phi^+}$  and  $\mu_{\phi^-}$  are absolutely continuous with respect to  $\lambda$  on the interval  $(a^2, b^2)$ , and

$$\begin{aligned} \frac{d\mu_{\phi^+}}{d\lambda}(t) &= \sqrt{t} \frac{d\mu_\phi}{d\lambda}(\sqrt{t}), \\ \frac{d\mu_{\phi^-}}{d\lambda}(t) &= \frac{1}{\sqrt{t}} \cdot \frac{d\mu_\phi}{d\lambda}(\sqrt{t}), \end{aligned} \quad t \in (a^2, b^2). \quad (4.8)$$

**Proof.** Write  $\phi = (\phi_0, \phi_\infty)$  and let  $f \in \mathcal{D}(\mathbb{R})$ . Then

$$\begin{aligned} \phi_0^+(f) &= \phi_0(h_1 \cdot (f \circ \tau)) = \int_{\mathbb{R}} h_1(s) f(s^2) \frac{d\mu_\phi(s)}{1+s^2} \\ &= \int_{\mathbb{R}} \frac{s^2}{1+s^4} f(s^2) d\mu_\phi(s) = \int_{\tau(\mathbb{R})} \frac{t}{1+t^2} f(t) d\mu_\phi^\tau(t) \\ &= \int_{\mathbb{R}} f(t) \cdot t \cdot \mathbb{1}_{(0,\infty)} \frac{d\mu_\phi^\tau(t)}{1+t^2}, \end{aligned}$$

and

$$\begin{aligned} \phi_0^-(f) &= \phi_0(h_0 \cdot (f \circ \tau)) = \int_{\mathbb{R}} h_0(s) f(s^2) \frac{d\mu_\phi(s)}{1+s^2} \\ &= \int_{\mathbb{R}} \frac{1}{1+s^4} f(s^2) d\mu_\phi(s) = \int_{\tau(\mathbb{R})} \frac{1}{1+t^2} f(t) d\mu_\phi^\tau(t) \\ &= \int_{\mathbb{R}} f(t) \cdot \mathbb{1}_{[0,\infty)} \frac{d\mu_\phi^\tau(t)}{1+t^2}. \end{aligned}$$

Assume now that  $\mu_\phi$  is absolutely continuous with respect to  $\lambda$  on  $(a, b)$  where  $0 \leq a < b$ . Let  $c, d$  be arbitrary such that  $a \leq c < d \leq b$ . It follows from (4.6) that

$$\begin{aligned} \mu_{\phi^+}((c, d)) &= \int_{(c,d)} t d\mu_\phi^\tau(t) = \int_{\tau^{-1}((c,d))} s^2 d\mu_\phi(s) \\ &= 2 \int_{(\sqrt{c}, \sqrt{d})} s^2 d\mu_\phi(s) = 2 \int_{(\sqrt{c}, \sqrt{d})} s^2 \frac{d\mu_\phi}{d\lambda}(s) ds \\ &= 2 \int_{(c,d)} t \frac{d\mu_\phi}{d\lambda}(\sqrt{t}) \frac{dt}{2\sqrt{t}} = \int_{(c,d)} \sqrt{t} \frac{d\mu_\phi}{d\lambda}(\sqrt{t}) dt, \end{aligned}$$

which implies the first relation in (4.8). The second relation is shown in a similar way.  $\square$

Let us note explicitly that, in the situation of the above lemma,  $\text{supp } \mu_\phi^\tau \subseteq [0, \infty)$ . Hence, we could also write  $d\mu_{\phi^+}(s) = s d\mu_\phi^\tau(s)$  and  $\mu_{\phi^-} = \mu_\phi^\tau$ .

The next theorem provides the distributional representations for the functions  $q_\pm$  in terms of the representation of a function  $q$  from the class  $\mathcal{N}_{<\infty}$ .

**Theorem 4.4** *Let  $q \in \mathcal{N}_{<\infty}$  be symmetric and let  $q(z) = r(z) + \phi(\beta_z)$  with  $r \in \mathbb{R}(z)$  and  $\phi \in \widetilde{\mathcal{D}}'(\mathbb{R})$  be its distributional representation (3.2). Moreover, let  $q_+$  and  $q_-$  be the functions defined in (4.1). Then the unique distributional representations of  $q_+$  and  $q_-$  are*

$$q_+(\lambda) = r^+(\lambda) + \phi^+(\beta_\lambda), \quad q_-(\lambda) = r^-(\lambda) + \phi^-(\beta_\lambda),$$

where  $\phi^+$  and  $\phi^-$  are defined as in Lemma 4.2 and

$$r^+(\lambda) := \sqrt{\lambda} r(\sqrt{\lambda}) - \phi(h_0), \quad r^-(\lambda) := \frac{r(\sqrt{\lambda})}{\sqrt{\lambda}} + \phi(h_0)$$

with  $h_0$  as in (4.2).

Assume, in addition, that  $q \in \mathcal{N}_{<\infty}^{(\infty)}$ . Then  $q_+, q_- \in \mathcal{N}_{<\infty}^{(\infty)}$ , and the corresponding measures satisfy

$$\frac{d\mu_{q_+}}{d\mu_q^\tau}(t) = t \cdot \mathbb{1}_{(0, \infty)}, \quad \frac{d\mu_{q_-}}{d\mu_q^\tau}(t) = \mathbb{1}_{[0, \infty)}, \quad (4.9)$$

where we use the notation from (3.9) and where  $\tau : \mathbb{R} \rightarrow \mathbb{R}$  denotes again the map  $\tau(t) := t^2$  and  $\mu_q^\tau$  the corresponding image measure of  $\mu_q$ .

**Proof.** Set  $\hat{\beta}_z(t) := \beta_z(-t)$ . Since  $\phi$  is symmetric, we have  $\phi(\beta_z) = \phi(\hat{\beta}_z)$ . Hence, we may express  $q_+$  as follows

$$q_+(z^2) = zq(z) = zr(z) + \frac{z}{2} \left( \phi(\beta_z) + \phi(\hat{\beta}_z) \right) = zr(z) + \phi\left(\frac{z}{2}(\beta_z + \hat{\beta}_z)\right).$$

For  $t \in \mathbb{R}$  we have

$$\begin{aligned} \frac{z}{2}(\beta_z(t) + \hat{\beta}_z(t)) &= \frac{z}{2} \left( \frac{1+tz}{t-z} + \frac{1-tz}{-t-z} \right) = \frac{z^2(1+t^2)}{t^2-z^2} \\ &= \frac{t^2(1+t^2)}{1+t^4} \beta_{z^2}(t^2) - \frac{1+t^2}{1+t^4} = h_1(t) C_\tau(\beta_{z^2})(t) - h_0(t), \end{aligned}$$

and this relation extends also to  $t = \infty$ . Hence

$$q_+(z^2) = zr(z) + \phi(h_1 C_\tau(\beta_{z^2})) - \phi(h_0) = [zr(z) - \phi(h_0)] + \phi^+(\beta_{z^2}),$$

which shows the required representation of  $q_+$ . The representation of  $q_-$  is proved in a similar way.

Note that the function  $\sqrt{\lambda} r(\sqrt{\lambda})$  is rational and symmetric with respect to  $\mathbb{R}$  since  $r$  is rational, symmetric with respect to  $\mathbb{R}$  and odd. Moreover, we have  $r(z) = O(\frac{1}{z})$  when  $|z| \rightarrow \infty$ , and hence the function  $\sqrt{\lambda} r(\sqrt{\lambda})$  remains bounded at infinity.

If  $q \in \mathcal{N}_{<\infty}^{(\infty)}$ , then  $r$  must vanish identically (as an odd and constant function), and  $\phi \in \mathcal{F}_{\{\infty\}}$ . Thus also  $\phi^+$  and  $\phi^-$  belong to  $\mathcal{F}_{\{\infty\}}$  by Lemma 4.3, and the corresponding rational summands in the representations of  $q_+$  and  $q_-$  are constant. This shows that  $q_+$  and  $q_-$  belong to  $\mathcal{N}_{<\infty}^{(\infty)}$ . The relations in (4.9) follow directly from (4.6) and (4.7).  $\square$

## 5 The operator model associated with $\phi \in \mathcal{F}_{\{\infty\}}$

With a distributional density  $\phi \in \mathcal{F}(\overline{\mathbb{R}})$  a model space and an operator are associated; see [31], [35]. Let us recall the definitions for the present case of interest, that is, for  $\phi \in \mathcal{F}_{\{\infty\}}$ .

We denote by  $\mathcal{B}_p(\phi)$ ,  $p \in [1, \infty)$ , the linear space of all complex-valued functions  $f$  on  $\overline{\mathbb{R}}$  for which there exists a  $T > 0$  such that

$$f|_{\overline{\mathbb{R}} \setminus [-T, T]} \in C^\infty(\overline{\mathbb{R}} \setminus [-T, T]) \quad \text{and} \quad f|_{(-2T, 2T)} \in L^p\left(\frac{d\mu_\phi(x)}{1+x^2}\right).$$

The action of a distributional density  $\phi \in \mathcal{F}_{\{\infty\}}$  extends naturally from  $C^\infty(\overline{\mathbb{R}})$  to  $\mathcal{B}_1(\phi)$ . Namely, if  $f \in \mathcal{B}_1(\phi)$ , choose  $T > 0$  as above and a partition of unity  $\chi^0, \chi^\infty \in C^\infty(\overline{\mathbb{R}})$  subordinate to the open cover  $\{(-2T, 2T), \overline{\mathbb{R}} \setminus [-T, T]\}$ . Then

$$\phi(f) := \int_{(-2T, 2T)} \chi^0(x) f(x) \frac{d\mu_\phi(x)}{1+x^2} + \phi_\infty((\chi^\infty f) \circ \Lambda_\infty^{-1})$$

is well defined, does not depend on the partition of unity, and  $\phi$  is a linear functional on  $\mathcal{B}_1(\phi)$ .

**Definition 5.1** Let  $\phi \in \mathcal{F}_{\{\infty\}}$  be given. Then we define an inner product  $[\cdot, \cdot]_\phi$  on  $\mathcal{B}_2(\phi)$  by

$$[f, g]_\phi := \phi(f\bar{g}), \quad f, g \in \mathcal{B}_2(\phi).$$

The Pontryagin space completion of the inner product space  $\langle \mathcal{B}_2(\phi), [\cdot, \cdot]_\phi \rangle$  (which exists as shown in the references named above) is the model space associated with  $\phi$  and is denoted by  $\Pi(\phi)$ .

The linear relation

$$A_\phi := \text{Clos} \{ (f; g) \in \mathcal{B}_2(\phi)^2 : g(t) = tf(t), t \in \mathbb{R} \}$$

is the model relation associated with  $\phi$ . Here ‘‘Clos’’ denotes the closure in  $\Pi(\phi) \times \Pi(\phi)$ .

It was shown in [35, Proposition 3.1] that  $A_\phi$  is self-adjoint and has non-empty resolvent set. Moreover, the point  $\infty$  is the only possible critical point of  $A_\phi$  and, at the same time, the only possible point of non-positive type of  $A_\phi$ . The use of the terminology ‘‘model relation’’ originates from the fact that  $\phi$  can be recovered from the spectral function of  $A_\phi$  by a type of Stieltjes inversion formula. Moreover, the function  $q_\phi$  is a  $Q$ -function of  $A_\phi$ .

We denote by  $\text{mul}(A_\phi^n)$  the multi-valued part of the relation  $A_\phi^n$  and set

$$\mathcal{E}_{A_\phi}(\infty) := \bigcup_{n=1}^{\infty} \text{mul}(A_\phi^n) \subseteq \Pi(\phi).$$

**Remark 5.2** Let  $\phi \in \mathcal{F}_{\{\infty\}}$  and denote by  $\mathcal{E}_{A_\phi}(\infty)^\circ$  the isotropic part of  $\mathcal{E}_{A_\phi}(\infty)$ . Then

$$\dim(\mathcal{E}_{A_\phi}(\infty)^\circ) = \Delta(\mu_\phi). \tag{5.1}$$

In particular, we see that the point  $\infty$  is a singular critical point of  $A_\phi$  if and only if  $\Delta(\mu_\phi) > 0$ .

The relation (5.1) follows from the explicit form of the model operator given in [31, §3.2] with the help of the transformation ‘‘Inv’’ from [35, §2] to exchange the roles of 0 and  $\infty$ . It can also be deduced from the Stieltjes inversion formula (3.7) and [13, Theorem 4.2], where it was shown that the minimal possible  $m$  in (3.5) is equal to  $\dim(\mathcal{E}_{A_\phi}(\infty)^\circ)$ .

It is an important fact that the model relation  $A_\phi$  is closely related to the multiplication operator  $M_x$  by the independent variable in the space  $L^2((1+x^2)^{-1}d\mu_\phi(x))$ ; the next theorem contains the precise statement. Its proof is not difficult, given the knowledge from [35, Proposition 3.1].

**Theorem 5.3** Let  $\phi \in \mathcal{F}_{\{\infty\}}$  be given. Then the map

$$\text{id} : \{ f \in \mathcal{B}_2(\phi) : \text{supp } f \subseteq \mathbb{R} \} \rightarrow L^2\left(\frac{d\mu_\phi(x)}{1+x^2}\right)$$

extends to an isometric, continuous and surjective map (orthogonal companions are understood w.r.t. the indefinite inner product  $[\cdot, \cdot]_\phi$ )

$$\psi(\phi) : \mathcal{E}_{A_\phi}(\infty)^\perp \rightarrow L^2\left(\frac{d\mu_\phi(x)}{1+x^2}\right).$$

Moreover,

$$(\psi(\phi) \times \psi(\phi))(A_\phi \cap (\mathcal{E}_{A_\phi}(\infty)^\perp)^2) = M_x$$

where  $M_x$  denotes the graph of the operator of multiplication by the independent variable.

**Proof.** Set

$$\mathcal{L}_N := \{ f \in \mathcal{B}_2(\phi) : \text{supp } f \subseteq [-N, N] \}, \quad N \in \mathbb{N},$$

$$\mathcal{L} := \bigcup_{N \in \mathbb{N}} \mathcal{L}_N, \quad \mathcal{A} := \text{Clos } \mathcal{L}.$$

It follows from the definitions of the inner product in  $\mathcal{B}_2(\phi)$  and the measure  $\mu_\phi$  that the identity maps  $\mathcal{L}$  isometrically onto a dense subspace of  $L^2((1+x^2)^{-1}d\mu_\phi(x))$ . It therefore extends to a continuous, isometric and surjective map from  $\mathcal{A}$  onto  $L^2((1+x^2)^{-1}d\mu_\phi(x))$ , cf. [46, Proposition 2.1]; we denote this extension by  $\psi(\phi)$ .

Denote by  $E$  the spectral family associated with  $A_\phi$ ; see, e.g. [41]. Since the only possible critical point of  $A_\phi$  and the only possible point of non-positive type of  $A_\phi$  is  $\infty$ , the spectral projector  $E(\Delta)$  is defined for every bounded Borel set. The space  $\mathcal{L}_N$  is complete with respect to  $[\cdot, \cdot]_\phi$  and hence a closed subspace of  $\Pi(\phi)$ . By [35, Proposition 3.1] we have  $E([-N, N])f = \mathbb{1}_{[-N, N]} \cdot f$ ,  $f \in \mathcal{B}_2(\phi)$ , where  $\mathbb{1}_{[-N, N]}$  denotes the characteristic function of  $[-N, N]$ . Hence  $E([-N, N])\mathcal{B}_2(\phi) = \mathcal{L}_N$ . Since  $\mathcal{L}_N$  is closed, it follows that  $\text{ran } E([-N, N]) = \mathcal{L}_N$ . We conclude that

$$\mathcal{L} = \bigcup_{N \in \mathbb{N}} \text{ran } E([-N, N]),$$

and hence that  $\mathcal{A} = \mathcal{E}_{A_\phi}(\infty)^\perp$ . This shows the first assertion.

Denote by  $M_x^{(N)}$  the multiplication operator in the space  $\mathcal{L}_N$ , which clearly has non-empty resolvent set and satisfies  $A_\phi \cap \mathcal{L}_N^2 \supseteq M_x^{(N)}$ . Since each space  $\mathcal{L}_N$  is invariant under resolvents of  $A_\phi$ , also  $A_\phi \cap \mathcal{L}_N^2$  has non-empty resolvent set, and hence

$$A_\phi \cap \mathcal{L}_N^2 = M_x^{(N)}.$$

Again, since  $\mathcal{L}_N$  is invariant under resolvents of  $A_\phi$ , we have

$$(\psi(\phi) \times \psi(\phi))(A_\phi \cap \mathcal{A}^2) = \text{Clos} \left( \bigcup_{N \in \mathbb{N}} (\psi(\phi) \times \psi(\phi))(A_\phi \cap \mathcal{L}_N^2) \right).$$

Here the closure is taken in  $L^2((1+x^2)^{-1}d\mu_\phi(x))$ . As we have seen above, the right-hand side of the above relation is further equal to

$$\text{Clos} \left( \bigcup_{N \in \mathbb{N}} M_x^{(N)} \right) = M_x,$$

and the second assertion follows. □

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