The exponential type of the fundamental solution of an indefinite Hamiltonian system

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Abstract

The fundamental solution of a Hamiltonian system whose Hamiltonian H is positive definite and locally integrable is an entire function of exponential type. Its exponential type can be computed as the integral over $\sqrt{\det H}$. We show that this formula remains true in the indefinite (Pontryagin space) situation, where the Hamiltonian is permitted to have finitely many inner singularities. As a consequence, we obtain a statement on non-cancellation of exponential growth for a class of entire matrix functions.

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1 Introduction

Consider a 2×2-Hamiltonian system of the form

$$\frac{d}{dx} y(x) = zJH(x)y(x), \quad x \in (a,b), \tag{1.1}$$

where $-\infty < a < b \leq \infty$, the Hamiltonian H takes real and non-negative 2×2-matrices as values, is locally integrable on (a,b) and does not vanish identically on any set of positive measure. Moreover, $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $z \in \mathbb{C}$.

Assume that Weyl's limit circle case prevails at the endpoint a, i.e. that $\int_a^{a+\varepsilon} \operatorname{tr} H(t)\, dt < \infty$ for some $\varepsilon > 0$, and denote by $\omega(x;z) = (\omega_{ij}(x;z))_{i,j=1}^2$, $x \in [a,b)$, the (transpose of the) fundamental solution of (1.1), i.e. the unique solution of the initial value problem

$$\begin{cases} \dfrac{\partial}{\partial x} \omega(x;z)J = z\omega(x;z)H(x), & x \in (a,b), \\ \omega(a;z) = I. \end{cases} \tag{1.2}$$

Recall that a function that is analytic on a simply connected domain D is said to be of *bounded type on* D if it can be written as the quotient of two functions that are bounded and analytic on D; see, e.g. [RR94, Definition 3.15 and Theorem 3.20] or [dB68, Section 8]. Moreover, an entire function f is said to be *of exponential type* if

$$\operatorname{et} f := \limsup_{z \to \infty} \frac{\log |f(z)|}{|z|} < \infty;$$

in this case the number $\operatorname{et} f$ is called the *exponential type of* f. By a theorem of M. G. Kreĭn an entire function that is of bounded type on the upper and lower half-planes is of exponential type; see, e.g. [K47] or [RR94, Theorem 6.17].

The following statement is a classical result; see, e.g. [dB61, Theorem X] or [KL]; for particular situations like strings or Sturm–Liouville equations see also [GK70] or [T46].

1.1 Theorem (Exponential type of $\omega(x;\cdot)$, cf. [dB61])**.** *Let $(\omega(x;\cdot))_{x\in[a,b)}$ be the solution of (1.2), and let $x\in[a,b)$ be fixed.*

(*i*) *The functions $\omega_{ij}(x;\cdot)$, $i,j=1,2$, are entire functions which take real values along the real line and are of bounded type on the upper and lower half-planes.*

(*ii*) *The exponential types $\operatorname{et}\omega_{ij}(x;\cdot)$ of the functions $\omega_{ij}(x;\cdot)$, $i,j=1,2$, coincide.*

(*iii*) *The exponential type can be computed from H by means of the formula*

$$\operatorname{et}\omega_{ij}(x;\cdot)=\int_a^x\sqrt{\det H(t)}\,dt. \tag{1.3}$$

❑

Whereas items (*i*) and (*ii*) of this statement are rather easy to see, the proof of (*iii*) requires significant effort.

In the recent series of papers [KW/IV]–[KW/VI] an indefinite (Pontryagin space) analogue of Hamiltonian systems, their operator models and spectral theory were developed. Very roughly speaking, an indefinite Hamiltonian system is a system of the form (1.1) where the Hamiltonian H is permitted to have a finite number of singularities in (a,b) where it is not integrable but satisfies certain growth conditions, plus finitely many scalar parameters assigned to each singularity which correspond to point-interaction and interface conditions. The precise definition of this notion is somewhat elaborate, cf. [KW/IV, Definition 8.1]. Since for our present purposes the above rough picture is sufficient, we do not go into more details.

Although, due to the presence of singularities, the initial value problem (1.2) is not uniquely solvable, one can single out one 'fundamental solution' $\omega(x;z)$ which is meaningful in the sense that it allows one to construct a Titchmarsh–Weyl coefficient and to prove direct and inverse spectral theorems; see [KW/V, Theorem 5.1], [KW/VI, Theorems 1.3, 1.4]. For a fixed x such a 'fundamental solution' $\omega(x;z)$ belongs to a certain class $\mathcal{M}_{<\infty}$ of entire 2×2-matrix functions; see Definition 2.1 below. In [KW/III] maximal chains of matrices from $\mathcal{M}_{<\infty}$ were introduced axiomatically (see Definition 2.4 below), and in [KW/V], [KW/VI] it was shown that these chains of matrices are exactly the 'fundamental solutions' of an indefinite Hamiltonian. In particular, such a maximal chain of matrices satisfies the first equation in (1.2) between the singularities with some H.

Our aim in the present paper is to prove the analogue of Theorem 1.1 in this more general situation. Thereby, the properties analogous to (*i*) and (*ii*) are again easy to see (and in essence known). The hard part is to show equality (1.3). This is the main result of this paper and given in Theorem 4.1.

In Theorem 5.18 we prove 'non-cancellation of exponential growth' for matrices from the class $\mathcal{M}_{<\infty}$, namely the equality $\operatorname{et}(W_1W_2)=\operatorname{et}W_1+\operatorname{et}W_2$ for $W_1,W_2\in\mathcal{M}_{<\infty}$. In order to show this theorem, we consider the following problem in Section 5. For a matrix $W\in\mathcal{M}_{<\infty}$ there exists an essentially unique finite maximal chain of matrices $(\omega(x;\cdot))_{x\in I}$ such that $W=\omega(\sup I;\cdot)$; this chain is called the chain going down from W. In Section 5 we construct a chain going down from W_1W_2 using the chains going down from W_1 and W_2

for two matrices $W_1, W_2 \in \mathcal{M}_{<\infty}$. Since some cancellation may occur, one has to distinguish several cases for which the procedure of constructing the chain going down from W_1W_2 varies.

Let us briefly outline the organisation of this paper. In Section 2 we recall the definitions of the classes $\mathcal{M}_{<\infty}$, $\mathcal{N}_{<\infty}$ (the latter being the class of generalized Nevanlinna functions), the notion of maximal chains of matrices, and provide an explicit proof of the properties analogous to Theorem 1.1 (i), (ii). In Section 3 we investigate two transformations of matrices, which are important tools for the proof of Theorem 4.1; this is a supplement to [KW/II], [KW/III], where these transformations were already studied and proved to be useful. Section 4 contains our main result, Theorem 4.1, and its proof. In Section 5 we discuss pasting of maximal chains. This is needed to prove Theorem 5.18 in full generality, and supplements [KW/II], where a generic case of pasting was considered. After these preparations we formulate and prove Theorem 5.18.

2 Chains of matrices and exponential type

This section is divided into three subsections. In the first we recall the notions of the classes $\mathcal{M}_{<\infty}$ and $\mathcal{N}_{<\infty}$, and in the second the notion of maximal chains of matrices. Finally, in the third subsection we turn to the exponential type of matrices from the class $\mathcal{M}_{<\infty}$.

a. Matrices of the class $\mathcal{M}_{<\infty}$.
If W is an entire 2×2-matrix-valued function which satisfies $W(z)JW(\overline{z})^* = J$ for $z \in \mathbb{C}$, then a kernel H_W is defined by

$$H_W(w, z) := \frac{W(z)JW(w)^* - J}{z - \overline{w}}, \qquad z, w \in \mathbb{C},$$

where J is as in the paragraph after (1.1). For $z = \overline{w}$ this formula has to be interpreted appropriately as a derivative, which is possible by analyticity.

2.1 Definition. Let $W = (w_{ij})_{i,j=1}^2$ be a 2×2-matrix-valued function and let $\kappa \in \mathbb{N}_0$. We write $W \in \mathcal{M}_\kappa$ if

- **(M1)** the entries w_{ij} of W are entire functions which take real values along the real line;
- **(M2)** $\det W(z) = 1$ for $z \in \mathbb{C}$, and $W(0) = I$;
- **(M3)** the kernel H_W has κ negative squares on $\mathbb{C}$.

Note that the conditions (M1) and (M2) together imply that $W(z)JW(\overline{z})^* = J$. //

We need some more generic notation. Set

$$\mathcal{M}_{<\infty} := \bigcup_{\nu \in \mathbb{N} \cup \{0\}} \mathcal{M}_\nu,$$

and write $\operatorname{ind}_- W = \kappa$ to express that $W \in \mathcal{M}_\kappa$. Define a map $\mathfrak{t}\colon \mathcal{M}_{<\infty} \to \mathbb{R}$ by (primes always denote differentiation with respect to the complex variable z)

$$\mathfrak{t}(W) := \operatorname{tr}\big(W'(0)J\big), \qquad W \in \mathcal{M}_{<\infty}.$$

Each matrix $W \in \mathcal{M}_{<\infty}$ generates, by means of the kernel H_W, a reproducing kernel Pontryagin space whose elements are 2-vector-valued entire functions, cf. [ADSR]; we denote this space by $\mathfrak{K}(W)$.

2.2. Note that $\mathfrak{K}(W)$ is finite-dimensional if and only if W is a polynomial. Since $H_{W^{-1}}(w,z) = -W(z)^{-1}H_W(w,z)(W(w)^{-1})^*$, the kernel $H_{W^{-1}}$ has finitely many negative squares if and only if $\mathfrak{K}(W)$ is finite-dimensional. Hence $W^{-1} \in \mathcal{M}_{<\infty}$ if and only if W is a polynomial.

It is often essential that matrices of the class $\mathcal{M}_{<\infty}$ are related to de Branges–Pontryagin spaces. Let $W = (w_{ij})_{i,j=1}^2 \in \mathcal{M}_{<\infty}$ be given and assume that the constant $(1,0)^T$ does not belong to $\mathfrak{K}(W)$. Then the projection of $\mathfrak{K}(W)$ onto its second component is an isometric isomorphism from $\mathfrak{K}(W)$ onto the de Branges–Pontryagin space $\mathfrak{P}(w_{22}+iw_{21})$ generated by the function $w_{22}+iw_{21}$, cf. [KW/I, §8, §9].

Note also that solutions $\omega(x;\cdot)$ of (1.2) are in $\mathcal{M}_0$ for every $x \in [a,b)$. Chains of matrices from $\mathcal{M}_{<\infty}$, which are considered in the next subsections, are therefore a generalization of fundamental solutions.

Let us turn to the class $\mathcal{N}_{<\infty}$ of generalized Nevanlinna functions. If $q\colon D \to \mathbb{C}$ is an analytic function defined on some open subset D of the complex plane, we define a kernel N_q by

$$N_q(w,z) := \frac{q(z) - \overline{q(w)}}{z - \overline{w}}\,, \qquad z, w \in D.$$

Again, for $z = \overline{w}$, this formula has to be interpreted appropriately.

2.3 Definition. Let q be a complex-valued function and let $\kappa \in \mathbb{N}_0$. We write $q \in \mathcal{N}_\kappa$ if

(N1) q is meromorphic on $\mathbb{C} \setminus \mathbb{R}$ and real, i.e. $q(\overline{z}) = \overline{q(z)}$;

(N2) the kernel N_q has κ negative squares on the domain of holomorphy of q.

As above, we set $\mathcal{N}_{<\infty} := \bigcup_{\kappa\in\mathbb{N}\cup\{0\}} \mathcal{N}_\kappa$, and write $\operatorname{ind}_- q = \kappa$ to express that $q \in \mathcal{N}_{<\infty}$ belongs to $\mathcal{N}_\kappa$. //

Matrices of the class $\mathcal{M}_{<\infty}$ give rise to generalized Nevanlinna functions as follows: for a 2×2-matrix-valued function $W(z) = (w_{ij}(z))_{i,j=1}^2$ and a scalar function $\tau(z)$, we denote by $W \star \tau$ the scalar function

$$(W \star \tau)(z) := \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_{21}(z)\tau(z) + w_{22}(z)}$$

wherever this expression is defined. We allow also the parameter $\tau = \infty$, in which case we set $W \star \tau := w_{21}^{-1}w_{11}$. A straightforward computation shows that

$$(W_1W_2) \star \tau = W_1 \star (W_2 \star \tau).$$

Rewriting the kernel $N_{W\star\tau}$ shows that $W \star \tau \in \mathcal{N}_{<\infty}$ whenever $W \in \mathcal{M}_{<\infty}$ and $\tau \in \mathcal{N}_{<\infty} \cup \{\infty\}$. In fact, $\operatorname{ind}_- W \star \tau \le \operatorname{ind}_- W + \operatorname{ind}_- \tau$ if we set $\operatorname{ind}_- \infty = 0$, cf. [KW/V, §2e].

b. Chains of matrices.

In this subsection we consider generalizations of fundamental solutions of Hamiltonian systems, i.e. solutions of (1.2), to an indefinite situation. These generalizations of fundamental solutions are introduced axiomatically as objects of their own right, cf. [KW/III].

The fundamental solution of a classical (positive) Hamiltonian system (1.2) is a chain of matrices $(\omega(x;\cdot\,))_{x\in[a,b)}$ where $\omega(x;\cdot\,)\in\mathcal{M}_0$ for every $x\in[a,b)$. In the indefinite setting we allow $\omega(x;\cdot\,)$ to be in $\mathcal{M}_{<\infty}$, and the chain may have a finite number of singularities.

2.4 Definition. A mapping $\omega\colon I\to\mathcal{M}_{<\infty}$ is called a *maximal chain of matrices* (or *maximal chain*, for short) if the following axioms are satisfied.

- **(W1)** The set I is of the form $I=[\sigma_0,\sigma_{n+1})\setminus\{\sigma_1,\ldots,\sigma_n\}$ for some numbers $n\in\mathbb{N}\cup\{0\}$ and $\sigma_0,\ldots,\sigma_{n+1}\in\mathbb{R}\cup\{+\infty\}$ with $\sigma_0<\sigma_1<\ldots<\sigma_{n+1}$.
- **(W2)** The function ω is not constant on any interval contained in I.
- **(W3)** For all $s,t\in I$, $s\leq t$, the matrix $\omega(s,t):=\omega(s)^{-1}\omega(t)$ belongs to $\mathcal{M}_{<\infty}$, and
$$\operatorname{ind}_-\omega(t)=\operatorname{ind}_-\omega(s)+\operatorname{ind}_-\omega(s,t).$$
- **(W4)** Let $t\in I$ and $W\in\mathcal{M}_{<\infty}$, $W\neq I$. If $W^{-1}\omega(t)\in\mathcal{M}_{<\infty}$ and $\operatorname{ind}_-\omega(t)=\operatorname{ind}_-W+\operatorname{ind}_-W^{-1}\omega(t)$, then there exists a number $s\in I$ such that $W=\omega(s)$.
- **(W5)** We have $\lim_{t\nearrow\sigma_{n+1}}\mathfrak{t}(\omega(t))=+\infty$. If I is not connected, i.e. $n>0$, then there exist numbers $s,t\in(\sigma_n,\sigma_{n+1})$ such that $\omega(s,t)$ is not a linear polynomial.

The points $\sigma_1,\ldots,\sigma_n$ are called the *singularities* of ω, and we refer to $\omega(s,t)=\omega(s,t;\cdot\,)$ as the *transfer matrix from s to t*. If we want to be specific about the domain of ω, we also write $\omega=(\omega(x))_{x\in I}=(\omega(x;z))_{x\in I}$. Observe the notational difference between $\omega(x)=\omega(x;z)$ and $\omega(s,t)=\omega(s,t;z)$. //

The limit condition in (W5) means that at the endpoint σ_{n+1} Weyl's limit point case prevails. It becomes clear from relation (2.1) below that this is a generalization of the notion in the definite setting. Moreover, it can be shown that $\lim_{x\searrow\sigma_0}\omega(x;z)=I$, cf. [KW/III, Lemma 3.5 (v)]. Condition (W4) is a maximality condition, which guarantees that there are no unnecessary 'holes' in the chain.

The bounded analogues of maximal chains of matrices, i.e. chains where also the right endpoint is in limit circle case, are defined as follows.

2.5 Definition. A mapping $\omega\colon I\to\mathcal{M}_{<\infty}$ is called a *finite maximal chain of matrices* (or *finite maximal chain*, for short) if

- **(W1$_f$)** the set I is of the form $I=[\sigma_0,\sigma_{n+1}]\setminus\{\sigma_1,\ldots,\sigma_n\}$ for some numbers $n\in\mathbb{N}\cup\{0\}$ and $\sigma_0,\ldots,\sigma_{n+1}\in\mathbb{R}$ with $\sigma_0<\sigma_1<\ldots<\sigma_n<\sigma_{n+1}$

and ω satisfies the axioms (W2), (W3) and (W4) from above. Again, $\sigma_1,\ldots,\sigma_n$ are called the *singularities* of the chain ω. //

Two chains of matrices share many of their properties if they differ only in the choice of scale. More precisely: we say that $\omega_1 = (\omega_1(x;z))_{x\in I_1}$ and $\omega_2 = (\omega_2(x;z))_{x\in I_2}$ are *reparameterizations* of each other, and write $\omega_1 \sim \omega_2$, if there exists a strictly increasing bijection $\varphi\colon I_1 \to I_2$ such that $\omega_1 = \omega_2 \circ \varphi$.

Trivially, if $\omega = (\omega(x;z))_{x\in I}$ is a finite maximal chain, then $\omega(\sup I;\cdot\,) \in \mathcal{M}_{<\infty}$. Far from trivial is the following converse result; see [KW/II, Theorem 7.1].

2.6. Let $W \in \mathcal{M}_{<\infty}$. Then there exists a finite maximal chain $\omega = (\omega(x;z))_{x\in I}$ such that $\omega(\sup I;\cdot\,) = W$. This chain is unique up to reparameterization.

We refer to a chain having this property as a 'chain going down from W'.

A (finite) maximal chain ω is called *properly parameterized* if for each compact interval $J \subseteq \{\inf I\} \cup I$ the functions $\mathfrak{t}\circ\omega|_J$ and $(\mathfrak{t}\circ\omega|_J)^{-1}$ are absolutely continuous. Each equivalence class of (finite) maximal chains modulo reparameterization contains properly parameterized chains.

With an 'indefinite Hamiltonian system' as defined in [KW/VI] there is associated a properly parameterized (finite) maximal chain of matrices which plays the role of the fundamental solution in the positive definite setting; see [KW/V, §5]. Conversely, each properly parameterized (finite) maximal chain of matrices arises in this way; see [KW/VI]. In particular, and this is all we need to know in the present paper, each properly parameterized (finite) maximal chain of matrices satisfies a canonical differential equation

$$\frac{\partial}{\partial x}\omega(x;z)J = z\omega(x;z)H(x), \quad x \in I,$$

with some Hamiltonian function H having singularities at $\sigma_1,\dots,\sigma_n$; see [KW/VI, Definition 2.3 and §3]. It follows easily that if $[x_1,x_2] \subseteq I$, then

$$\mathfrak{t}\big(\omega(x_2;\cdot\,)\big) - \mathfrak{t}\big(\omega(x_1;\cdot\,)\big) = \int_{x_1}^{x_2} \operatorname{tr} H(x)dx. \tag{2.1}$$

Finally, we recall the notion of indivisible intervals. For $l,\phi \in \mathbb{R}$, set

$$W_{(l,\phi)}(z) := \begin{pmatrix} 1 - lz\sin\phi\cos\phi & lz\cos^2\phi \\ -lz\sin^2\phi & 1 + lz\sin\phi\cos\phi \end{pmatrix}.$$

Let $\omega = (\omega(x;z))_{x\in I}$ be a (finite) maximal chain. Then a non-empty open interval $(s,t) \subseteq I$ is called *indivisible* of *type* $\phi \in [0,\pi)$ if for all $s',t' \in (s,t)$,

$$\omega(s',t') = W_{(l(s',t'),\phi)},$$

with some $l(s',t') > 0$ for $s',t' \in (s,t)$, $s' < t'$. The number

$$\sup\big\{l(s',t')\colon s',t' \in (s,t),\ s' < t'\big\} \in (0,\infty]$$

is called the *length* of the indivisible interval (s,t).

If the intersection of two indivisible intervals is non-empty, then their types coincide and their union is again indivisible. Hence, each indivisible interval is contained in a maximal indivisible interval.

Let us note that the simplest type of a singularity σ in a maximal chain ω is an indivisible interval of 'negative length', which means that there exist points $s_- < \sigma < s_+$ with $\omega(s_-,s_+) = W_{(l,\phi)}$ with some $l < 0$.

c. The exponential type of matrices in $\mathcal{M}_{<\infty}$.

We provide an explicit proof of the indefinite analogues of Theorem 1.1 $(i), (ii)$.

2.7 Proposition. *Let $W = (w_{ij})_{i,j=1}^2 \in \mathcal{M}_{<\infty}$. Then the functions w_{ij} are real along the real line, of bounded type in the upper and lower half-planes and have the same exponential type.*

Proof. Since $W \in \mathcal{M}_{<\infty}$, clearly, $w_{ij}(z)$ is real whenever $z \in \mathbb{R}$ for all $i, j = 1, 2$.

Assume first that no entry of W vanishes identically and that the constant $(1,0)^T$ does not belong to the reproducing kernel Pontryagin space $\mathfrak{K}(W)$ generated by the kernel H_W. Then the function 1 is associated with the de Branges–Pontryagin space $\mathfrak{P}(w_{22} + iw_{21})$, cf. [KW/I, Proposition 10.3]. Hence, all functions associated with $\mathfrak{P}(w_{21}+iw_{22})$ are of bounded type in the upper half-plane. In particular, this applies to w_{22} and w_{21}. Since w_{21} and w_{22} are real, they are of bounded type also in the lower half-plane.

Each of the quotients

$$\frac{w_{11}}{w_{21}}, \ \frac{w_{12}}{w_{22}}, \ \frac{w_{12}}{w_{11}}, \ \frac{w_{21}}{w_{22}}$$

belongs to the class $\mathcal{N}_{<\infty}$ and is neither constant equal to 0 nor constant equal to ∞. Hence, each of these quotients is of bounded type in the upper and lower half planes, and has zero mean type; see [KW/I, Proposition 2.4] (for the definition of mean type and its connection with the exponential type of an entire function see, e.g. [dB68, Sections 9 and 10]). This implies that all entries w_{ij} are of bounded type and that their exponential types coincide.

If one entry of W vanishes identically, then all other entries are polynomials, cf. [KW/I, Corollary 9.8]. Hence, trivially, all entries are of bounded type and zero exponential type.

If the constant $(1,0)^T$ belongs to $\mathfrak{K}(W)$, consider the matrix $-JWJ$ instead. Since at most one constant can belong to $\mathfrak{K}(W)$, cf. [KW/I, Corollary 8.4], we can apply the above said to $-JWJ$. ❑

2.8. Let $W = (w_{ij})_{i,j=1}^2 \in \mathcal{M}_{<\infty}$. Then we denote the common exponential type of the functions w_{ij}, $i, j = 1, 2$ by $\operatorname{et} W$.

3 Two transformations of matrices

We employ two transformations of matrices. These transformations already appeared in previous work and have proved to be useful there, cf. [KW/II, KW/III]. For our present purposes we need to provide more properties of these transformations, supplementing the properties established earlier.

a. The transformation $\mathcal{T}^a$.

3.1 Definition. Let $a \in \mathbb{R}$. For $W \in \mathcal{M}_{<\infty}$, we define

$$(\mathcal{T}^a W)(z) := W(z+a)W(a)^{-1}\,.$$

//

¿From the fact that $W(a)^*JW(a) = J$, which can easily be shown for $W \in \mathcal{M}_{<\infty}$, it follows that $\mathcal{T}^a$ maps $\mathcal{M}_{<\infty}$ into itself and preserves negative indices.

In order to show the next lemma, we use similar arguments as in [KW/II, Lemma 10.2] where a variant for truncated chains was proved. However, for the convenience of the reader, we provide a complete proof.

3.2 Lemma. *Let $W \in \mathcal{M}_{<\infty}$, and let $\omega = (\omega(x;z))_{x\in I}$ be a chain going down from W. Moreover, let $a \in \mathbb{R}$. Then a chain going down from $\mathcal{T}^aW$ is given by $(\mathcal{T}^a\omega(x;z))_{x\in I}$.*

Proof. Write $I = [\sigma_0, \sigma_{n+1}] \setminus \{\sigma_1, \dots, \sigma_n\}$, and set $\tilde{W} := \mathcal{T}^aW$ and $\tilde{\omega}(x;z) := \mathcal{T}^a\omega(x;z)$. In order to show that $(\tilde{\omega}(x;z))_{x\in I}$ is a finite maximal chain we use [KW/V, Proposition 3.10]. Property (i) that is required in that proposition is trivially satisfied. Since $\mathcal{T}^a$ preserves negative indices, the function

$$x \mapsto \operatorname{ind}_- \tilde{\omega}(x;z), \quad x \in I,$$

shares the properties of $x \mapsto \operatorname{ind}_- \omega(x;z)$ to be non-decreasing, constant on each component of I and taking different values on different components. Moreover, since

$$\tilde{\omega}(x;z)^{-1}\tilde{W}(z) = \omega(x;a) \cdot \omega(x;z+a)^{-1}W(z+a) \cdot W(a)^{-1},$$

we have $\tilde{\omega}(x;z)^{-1}\tilde{W}(z) \in \mathcal{M}_{<\infty}$ and

$$\begin{aligned} &\operatorname{ind}_- \left[\tilde{\omega}(x;z)^{-1}\tilde{W}(z)\right] = \operatorname{ind}_- \omega(x;z)^{-1}W(z) \\ &= \operatorname{ind}_- W - \operatorname{ind}_- \omega(x;z) = \operatorname{ind}_- \tilde{W} - \operatorname{ind}_- \tilde{\omega}(x;z). \end{aligned}$$

Thus property (ii) required in [KW/V, Proposition 3.10] holds.

Since $\tilde{\omega}(x;z)$ depends continuously on $x \in I$ with respect to locally uniform convergence, the function $x \mapsto \mathfrak{t}(\tilde{\omega}(x;z))$ is continuous. If $s, t \in I$, $s < t$, are such that $[s,t] \subseteq I$, then

$$\mathfrak{t}(\tilde{\omega}(t;z)) = \mathfrak{t}\big(\tilde{\omega}(s;z) \cdot \tilde{\omega}(s;z)^{-1}\tilde{\omega}(t;z)\big) = \mathfrak{t}(\tilde{\omega}(s;z)) + \mathfrak{t}\big(\tilde{\omega}(s;z)^{-1}\tilde{\omega}(t;z)\big).$$

Since $\tilde{\omega}(s;z)^{-1}\tilde{\omega}(t;z) \in \mathcal{M}_0 \setminus \{I\}$, we have $\mathfrak{t}(\tilde{\omega}(s;z)^{-1}\tilde{\omega}(t;z)) > 0$. Thus the function $x \mapsto \mathfrak{t}(\tilde{\omega}(x;z))$ is strictly increasing on each component of I.

Let $i \in \{1, \dots, n\}$, and denote by q_{σ_i} the intermediate Weyl coefficient of ω at the singularity σ_i, i.e. the limit $q_{\sigma_i}(z) := \lim_{x\to\sigma_i} \omega(x;z) \star \tau_x$, which exists for every function $\tau_x \colon (\sigma_{i-1}, \sigma_i) \cup (\sigma_i, \sigma_{i+1}) \to \mathbb{R} \cup \{\infty\}, x \mapsto \tau_x$ and is independent of τ_x; see [KW/III, Proposition 5.1, Theorem 5.6]. Then, for each $\tau \in \mathbb{R} \cup \{\infty\}$,

$$\begin{aligned} \lim_{x\nearrow\sigma_i} \tilde{\omega}(x;z) \star \tau &= \lim_{x\nearrow\sigma_i} \omega(x;z+a) \star \underbrace{\big(\omega(x;a)^{-1} \star \tau\big)}_{\in\mathbb{R}\cup\{\infty\} \text{ for every } x} \\ &= q_{\sigma_i}(z+a) = \lim_{x\searrow\sigma_i} \omega(x;z+a) \star \big(\omega(x;a)^{-1} \star \tau\big) = \lim_{x\searrow\sigma_i} \tilde{\omega}(x;z) \star \tau. \end{aligned}$$

We see that the limits $\lim_{x\nearrow\sigma_i} \tilde{\omega}(x;z) \star \tau$ and $\lim_{x\searrow\sigma_i} \tilde{\omega}(x;z) \star \tau$ exist independently of τ and coincide. This implies that (iii) and (iv) required in [KW/V, Proposition 3.10] hold true. Now that proposition directly implies the assertion. ❏

3.3 Remark. The same computation as in the positive definite case, cf. [W95], shows that $\tilde{\omega}(x;z) := \mathcal{T}^a\omega(x;z)$ satisfies the differential equation

$$\frac{\partial}{\partial x}\tilde{\omega}(x;z)J = z\tilde{\omega}(x;z)\tilde{H}(x), \quad x \in I,$$

with $\tilde{H}(x) := \omega(x;a)H(x)\omega(x;a)^T$. //

The reproducing kernel spaces generated by W and $\mathcal{T}^a W$ are related in a simple way.

3.4 Lemma. *Let $W = (w_{ij})_{i,j=1}^2 \in \mathcal{M}_{<\infty}$, $a \in \mathbb{R}$, and set $\tilde{W} = (\tilde{w}_{ij})_{i,j=1}^2 := \mathcal{T}^a W$. Then the map*

$$\Lambda^a\colon \big(F_1(z), F_2(z)\big)^T \mapsto \big(F_1(z+a), F_2(z+a)\big)^T$$

is an isometric isomorphism from $\mathfrak{K}(W)$ onto $\mathfrak{K}(\tilde{W})$, and the map

$$\lambda^a\colon F(z) \mapsto F(z+a)$$

is an isometric isomorphism from $\mathfrak{P}(w_{22}+iw_{21})$ onto $\mathfrak{P}(\tilde{w}_{22}+i\tilde{w}_{21})$.

Proof. Since $W(a)^*JW(a) = J$, we have

$$H_{\tilde{W}}(w,z) = H_W(w+a, z+a).$$

Thus the map $\Phi^a\colon H_W(w,z)(\alpha,\beta)^T \mapsto H_W(w,z+a)(\alpha,\beta)^T$, $(\alpha,\beta)^T \in \mathbb{C}^2$, extends by linearity and isometry to an isometric isomorphism from $\mathfrak{K}(W)$ onto $\mathfrak{K}(\tilde{W})$. Since in both spaces point evaluation is continuous, this extension is nothing but the map Λ^a.

The reproducing kernels of the spaces $\mathfrak{P}(w_{22}+iw_{21})$ and $\mathfrak{P}(\tilde{w}_{22}+i\tilde{w}_{21})$ are given by

$$(0,1)H_W(w,z)(0,1)^T \quad \text{and} \quad (0,1)H_{\tilde{W}}(w,z)(0,1)^T,$$

respectively. Hence, the same argument as used above yields the assertion concerning isomorphy of these spaces. ❑

As an immediate consequence of this lemma, we obtain that for a given constant function $(\cos\alpha, \sin\alpha)^T$:

$$\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix} \in \mathfrak{K}(W) \quad \Longleftrightarrow \quad \begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix} \in \mathfrak{K}(\mathcal{T}^a W). \tag{3.1}$$

In the present context the following simple observation is important.

3.5 Lemma. *Let $a \in \mathbb{R}$. If $W \in \mathcal{M}_{<\infty}$, then*

$$\operatorname{et}(\mathcal{T}^a W) = \operatorname{et} W. \tag{3.2}$$

If $W_1, W_2 \in \mathcal{M}_{<\infty}$ are such that $W_1^{-1}W_2 \in \mathcal{M}_{<\infty}$, then

$$\operatorname{et}\big[(\mathcal{T}^a W_1)^{-1}(\mathcal{T}^a W_2)\big] = \operatorname{et}\big[W_1^{-1}W_2\big]. \tag{3.3}$$

Proof. To see (3.2), note that $(\mathcal{T}^a W)(z) = W(z+a)W(a)^{-1}$ implies that $\operatorname{et}(\mathcal{T}^a W) \leq \operatorname{et} W$. The reverse inequality follows from writing $W(z) = (\mathcal{T}^a W)(z-a)W(a)$. Relation (3.3) follows in the same way from

$$(\mathcal{T}^a W_1)(z)^{-1}(\mathcal{T}^a W_2)(z) = W_1(a) \cdot W_1(z+a)^{-1} W_2(z+a) \cdot W_2(a)^{-1}.$$

❑

b. The transformation $\mathcal{T}_m$.

3.6 Definition. Let $m \in \mathbb{R}$. For $W = (w_{ij})_{i,j=1}^2 \in \mathcal{M}_{<\infty}$ we set

$$\alpha(W) := 1 - m w_{21}'(0), \quad \beta(W) := m \frac{w_{21}''(0)}{2} + m w_{21}'(0) w_{11}'(0) - 2 w_{11}'(0).$$

If $\alpha(W) \neq 0$, define

$$(\mathcal{T}_m W)(z) := \begin{pmatrix} 1 & -\frac{m}{z} \\ 0 & 1 \end{pmatrix} W(z) \begin{pmatrix} \frac{1}{\alpha(W)} & m\frac{\beta(W)}{\alpha(W)} + \frac{m}{z} \\ 0 & \alpha(W) \end{pmatrix}.$$

//

The relation between the chain going down from a matrix W and the chain going down from its transformed $\mathcal{T}_m W$ is not as simple as in the case of $\mathcal{T}^a$. Let us comprehensively recall the content of [KW/III, Theorem 4.4].

3.7. Relation between chains going down from W and $\mathcal{T}_m W$.
Let $W \in \mathcal{M}_{<\infty}$ and let $\omega = (\omega(x;z))_{x \in I}$ be a chain going down from W. Moreover, let $m \in \mathbb{R}$, and assume that $\alpha(W) \neq 0$. Let $\tilde{\omega} = (\tilde{\omega}(x;z))_{x \in \tilde{I}}$ be a chain going down from $\tilde{W} := \mathcal{T}_m W$. Then there exists a strictly increasing map

$$\iota \colon \big\{ x \in I \colon \alpha(\omega(x;\cdot)) \neq 0 \big\} \to \tilde{I}$$

such that

$$\tilde{\omega}(\iota x; z) = \mathcal{T}_m \omega(x;z), \qquad x \in I \text{ for which } \alpha(\omega(x;\cdot)) \neq 0.$$

The domain and range of ι cover I and $\tilde{I}$, respectively, with possible exception of finitely many indivisible intervals of type 0 and finitely many points. At zeros of $x \mapsto \alpha(\omega(x;\cdot))$ new singularities arise. We have $\inf I, \sup I \in \operatorname{dom} \iota$, and $\inf \tilde{I}, \sup \tilde{I} \in \operatorname{ran} \iota$. The parameterization of $\tilde{\omega}$ can be chosen such that ι is a translation on each component of its domain. //

Also the relation between the respective Hamiltonians, when we deal with properly parameterized chains, is not so straightforward. In order to shorten notation, we always write $\alpha(x) := \alpha(\omega(x;\cdot))$ and $\beta(x) := \beta(\omega(x;\cdot))$.

3.8 Remark. Let ω be a properly parameterized chain going down from some matrix $W \in \mathcal{M}_{<\infty}$ and let H be the Hamiltonian in the canonical differential equation for ω. Let $m \in \mathbb{R}$ and assume that $\alpha(W) \neq 0$. Set $\tilde{W} := \mathcal{T}_m W$ and let $\tilde{\omega}$ be a chain going down from $\tilde{W}$ being parameterized in such a way that the map ι from 3.7 is a translation on each component of its domain.

Comparing the power series coefficients of z^1 and z^2 in the equation $\frac{\partial}{\partial x}\omega(x;z)J = z\omega(x;z)H(x)$ we obtain relations among $\frac{\partial}{\partial x}\omega'_{ij}(x;0)$, $\frac{\partial}{\partial x}\omega''_{ij}(x;0)$ and $H(x)$. These can be used to show that

$$\frac{\partial}{\partial x}\alpha(x) = mh_{22}(x), \qquad \frac{\partial}{\partial x}\beta(x) = h_{12}(x)\alpha(x). \tag{3.4}$$

Using (3.4) and the same computation as in the positive definite case, cf. [W95], we arrive at

$$\frac{\partial}{\partial y}\tilde{\omega}(y;z)J = z\tilde{\omega}(y;z)\tilde{H}(y), \qquad y \in \operatorname{ran}\iota,$$

where

$$\tilde{H}(\iota x) := \begin{pmatrix} \alpha(x) & -m\dfrac{\beta(x)}{\alpha(x)} \\ 0 & \dfrac{1}{\alpha(x)} \end{pmatrix} H(x) \begin{pmatrix} \alpha(x) & -m\dfrac{\beta(x)}{\alpha(x)} \\ 0 & \dfrac{1}{\alpha(x)} \end{pmatrix}^T, \quad x \in \operatorname{dom}\iota. \tag{3.5}$$

Since ι is a translation componentwise and $\tilde{H}(\iota x) = 0$ if and only if $H(x) = 0$, the function $\tilde{H}|_{\operatorname{ran}\iota}$ does not vanish identically on any set of positive measure.

The set $\tilde{I} \setminus \operatorname{ran}\iota$ consists of at most finitely many intervals and finitely many points. Hence, we can choose a reparameterization $\varphi\colon \tilde{I} \to \tilde{I}$ such that $\varphi|_{\operatorname{ran}\iota} = \operatorname{id}_{\operatorname{ran}\iota}$ and such that $\tilde{\omega} \circ \varphi$ is properly parameterized. //

3.9 Remark.

(i) Assume that W is not a matrix polynomial and let $m \in \mathbb{R}$ be such that $\alpha(W) \neq 0$. Then we have

$$(1,0)^T \in \mathfrak{K}(W) \iff (1,0)^T \in \mathfrak{K}(\mathcal{T}_m W).$$

To see this, recall from [KW/II, Theorem 5.7] that

$$(1,0)^T \in \mathfrak{K}(W) \iff \neg\Big[\lim_{y\to+\infty} \frac{1}{y}\big(W \star \infty\big)(iy) = 0\Big].$$

Since $(\mathcal{T}_m W) \star \infty = (W \star \infty) - \frac{m}{z}$, the property on the right-hand side is inherited.

(ii) Let $W = (w_{ij})_{i,j=1}^2 \in \mathcal{M}_{<\infty}$, $m \in \mathbb{R}$ with $\alpha(W) \neq 0$, and write $\mathcal{T}_m W = (\tilde{w}_{ij})_{i,j=1}^2$. Then the de Branges–Pontryagin spaces $\mathfrak{P} := \mathfrak{P}(w_{22} + iw_{21})$ and $\tilde{\mathfrak{P}} := \mathfrak{P}(\tilde{w}_{22} + i\tilde{w}_{21})$ coincide as sets. Their inner products $[\cdot,\cdot]$ and $[\cdot,\cdot]_\sim$, respectively, are related as follows:

$$[F,G]_\sim = [F,G] + mF(0)\overline{G(0)}, \qquad F,G \in \tilde{\mathfrak{P}}.$$

For a proof see [KW/III, Theorem 4.4].

//

Again, for our present purposes, it is important to observe the following lemma about the exponential type of transformed matrices.

3.10 Lemma. *Let $m \in \mathbb{R}$. If $W \in \mathcal{M}_{<\infty}$ and $\alpha(W) \neq 0$, then*

$$\operatorname{et}(\mathcal{T}_m W) = \operatorname{et} W. \tag{3.6}$$

If $W_1, W_2 \in \mathcal{M}_{<\infty}$ are such that $\alpha(W_1), \alpha(W_2) \neq 0$ and $W_1^{-1} W_2 \in \mathcal{M}_{<\infty}$, then

$$\operatorname{et}\left[(\mathcal{T}_m W_1)^{-1}(\mathcal{T}_m W_2)\right] = \operatorname{et}\left[W_1^{-1} W_2\right].$$

Proof. To see these relations, note that the transformation changes the left lower entry only by a constant and non-zero factor. ❑

4 Computing the exponential type

The next statement is our main result. It is the precise analogue of Theorem 1.1 (*iii*) in the indefinite situation, i.e. for (possibly finite) maximal chains of matrices as introduced in Definition 2.4, Definition 2.5.

4.1 Theorem. *Let $\omega = (\omega(x; z))_{x \in I}$ be a properly parameterized (possibly finite) maximal chain of matrices and let H be the Hamiltonian defined on I such that ω satisfies the canonical differential equation*

$$\begin{cases} \dfrac{\partial}{\partial x}\omega(x; z)J = z\omega(x; z)H(x), \quad x \in I, \\ \omega(\inf I; z) = I. \end{cases}$$

Then, for each $x \in I$, the function $\sqrt{\det H(t)}$ is defined a.e. and integrable on $[\inf I, x]$. Moreover,

$$\operatorname{et} \omega(x; \cdot\,) = \int\limits_{\inf I}^{x} \sqrt{\det H(t)}\, dt,$$

where $\operatorname{et} \omega(x; \cdot\,)$ is defined as in 2.8.

Our method to prove Theorem 4.1 is of inductive nature. To carry out this argument, we need to name one condition for a finite maximal chain ω:

(ET) Whenever $\omega \circ \varphi$ is a proper reparameterization of ω defined on the set I_φ and H_φ denotes the Hamiltonian function in the canonical differential equation for $\omega \circ \varphi$, then

$$\sqrt{\det H_\varphi} \in L^1\big([\inf I_\varphi, \sup I_\varphi]\big),$$
$$\operatorname{et}\big(\omega \circ \varphi(\sup I_\varphi)\big) = \int\limits_{\inf I_\varphi}^{\sup I_\varphi} \sqrt{\det H_\varphi(t)}\, dt, \qquad x \in I_\varphi. \tag{4.1}$$

If $\omega \circ \varphi_1$ and $\omega \circ \varphi_2$ are both proper reparameterizations of ω, then they together do or do not satisfy (4.1). Hence, in order to establish (ET) for some chain ω it is enough to find one proper reparameterization with (4.1).

4.2 Remark. The statement in Theorem 1.1 (*iii*), i.e. the definite analogue of Theorem 4.1, in essence says that each chain going down from a matrix $W \in \mathcal{M}_0$ satisfies (ET). In order to prove Theorem 4.1, it is our task to show that in fact all finite maximal chains satisfy (ET). //

Proof (of Theorem 4.1).

Step 1: Reductions.

We carry out a couple of reductions. First we reduce to (ET). Assume that $\omega = (\omega(x;z))_{x\in I}$ is a maximal chain, and let $x \in I$ be given. Then $\omega|_{I\cap[\inf I,x]}$ is a finite maximal chain, cf. [KW/V, Remark 3.15]. If we know that this finite maximal chain satisfies (ET), then the asserted formula for exponential type holds for the given point x. Since $x \in I$ was arbitrary, the proof of Theorem 4.1 is completed once we have shown that each finite maximal chain satisfies (ET).

Next we take care of the case that ω is a chain going down from a polynomial matrix. In this case, ω is composed of a finite union of indivisible intervals, and hence $\det H = 0$ a.e. on I. Thus, trivially, (ET) holds.

Third, we show that it is sufficient to consider the case that ω is a chain going down from a matrix W with $(1,0)^T \notin \mathfrak{K}(W)$. Assume that this condition is not satisfied. Then we consider the chain $\tilde{\omega}(x;\cdot\,) := -J\omega(x;\cdot\,)J$ which is going down from the matrix $\tilde{W} := -JWJ$. This chain now satisfies the stated condition, and we have $\operatorname{et} W = \operatorname{et} \tilde{W}$ and $\det H = \det \tilde{H}$ where H and $\tilde{H}$ denote the respective Hamiltonians.

Step 2: (ET) *is preserved under $\mathcal{T}^a$.*

Assume that $\omega = (\omega(x;z))_{x\in I}$ is a chain going down from some matrix $W \in \mathcal{M}_{<\infty}$ and that ω satisfies (ET). Without loss of generality assume that ω is properly parameterized, so that (4.1) holds for ω itself, and let H be the Hamiltonian in the canonical differential equation for ω.

By Lemma 3.2 a chain going down from $\mathcal{T}^a W$ is given by $\tilde{\omega} := (\mathcal{T}^a\omega(x;z))_{x\in I}$. By Remark 3.3 it satisfies a canonical differential equation with Hamiltonian $\tilde{H}(x) = \omega(x;a)H(x)\omega(x;a)^T$. The Hamiltonian $\tilde{H}$ shares the property of H not to vanish on any set of positive measure. Hence, $\tilde{\omega}$ is properly parameterized. It is obvious that $\det \tilde{H} = \det H$, and (3.2) says that $\operatorname{et} \tilde{\omega}(x;\cdot\,) = \operatorname{et} \omega(x;\cdot\,)$. We conclude that $\tilde{\omega}$ satisfies (4.1).

Step 3: (ET) *is preserved under $\mathcal{T}_m$.*

Assume that $\omega = (\omega(x;z))_{x\in I}$ is a properly parameterized chain going down from some matrix $W \in \mathcal{M}_{<\infty}$ which satisfies (ET), and let H be the corresponding Hamiltonian. Let $m \in \mathbb{R}$ be such that $\alpha(W) \neq 0$, and let $\tilde{\omega} = (\tilde{\omega}(x;z))_{x\in\tilde{I}}$ be a chain going down from $\mathcal{T}_m W$ which is parameterized properly and such that the map ι in 3.7 is a translation on each component. Then the Hamiltonian $\tilde{H}$ corresponding to $\tilde{\omega}$ is, on $\operatorname{ran}\iota$, given by (3.5) and hence $\det \tilde{H}(\iota x) = \det H(x)$. Moreover, we know that domain and range of ι cover I and $\tilde{I}$, respectively, with possible exception of finitely many indivisible intervals and single points. Together with (3.6) it follows that

$$\begin{aligned}
\operatorname{et} \mathcal{T}_m W = \operatorname{et} W &= \int_I \sqrt{\det H(x)}\,dx = \int_{\operatorname{dom}\iota} \sqrt{\det H(x)}\,dx \\
&= \int_{\operatorname{dom}\iota} \sqrt{\det \tilde{H}(\iota x)}\,dx = \int_{\operatorname{ran}\iota} \sqrt{\det \tilde{H}(y)}\,dy \\
&= \int_{\tilde{I}} \sqrt{\det \tilde{H}(y)}\,dy,
\end{aligned}$$

which shows that $\tilde{\omega}$ satisfies (4.1).

Step 4: A step-by-step procedure.
Assume that ω is a chain going down from some matrix $W = (w_{ij})_{i,j=1}^2 \in \mathcal{M}_{<\infty}$ which is not a polynomial and satisfies $(1,0)^T \notin \mathfrak{K}(W)$. We denote the de Branges–Pontryagin space generated by the function $w_{22} + iw_{21}$ by $\mathfrak{P}(W)$ and its inner product by $[\cdot,\cdot]_W$. Similar notation is applied to all other occurring matrices.

By [KW/III, Theorem 3.3], there exist points $a_1, \dots, a_N \in \mathbb{R}$ and a number $m_0 > 0$ such that, for all $m \geq m_0$, the inner product

$$(F,G) := [F,G]_W + m \sum_{k=1}^{N} F(a_k)\overline{G(a_k)}, \qquad F,G \in \mathfrak{P}(W), \tag{4.2}$$

turns the linear space $\mathfrak{P}(W)$ into a de Branges–Hilbert space.

This transformation of the inner product can be reproduced on the level of $\mathfrak{K}(W)$ by iteratively applying the transformations $\mathcal{T}^{\pm a_k}$ and $\mathcal{T}_m$. To see this, note that (equalities do not include equality of inner products)

$$\mathfrak{P}(\mathcal{T}^{-a}\mathcal{T}_m\mathcal{T}^a W) = \big\{F(z-a)\colon F \in \mathfrak{P}(\mathcal{T}_m\mathcal{T}^a W)\big\},$$
$$\mathfrak{P}(\mathcal{T}_m\mathcal{T}^a W) = \mathfrak{P}(\mathcal{T}^a W) = \big\{F(z+a)\colon F \in \mathfrak{P}(W)\big\},$$

and hence that $\mathfrak{P}(\mathcal{T}^{-a}\mathcal{T}_m\mathcal{T}^a W) = \mathfrak{P}(W)$. Concerning inner products, we compute, for $F, G \in \mathfrak{P}(W)$,

$$\begin{aligned}
[F(z),G(z)]_{\mathcal{T}^{-a}\mathcal{T}_m\mathcal{T}^a W} &= [F(z+a),G(z+a)]_{\mathcal{T}_m\mathcal{T}^a W} \\
&= [F(z+a),G(z+a)]_{\mathcal{T}^a W} + mF(a)\overline{G(a)} \\
&= [F(z),G(z)]_W + mF(a)\overline{G(a)}.
\end{aligned}$$

Let us consider the matrix

$$\tilde{W} := (\mathcal{T}^{a_N}\mathcal{T}_m\mathcal{T}^{a_N}) \cdots (\mathcal{T}^{a_1}\mathcal{T}_m\mathcal{T}^{a_1})W, \tag{4.3}$$

where $m \geq m_0$ is chosen such that all transformations are defined. Such a choice is possible, since the transform $\mathcal{T}_m M$ of a matrix M can be undefined for at most one value of m. Then the spaces $\mathfrak{P}(W)$ and $\mathfrak{P}(\tilde{W})$ coincide as sets, and the inner product of $\mathfrak{P}(\tilde{W})$ is nothing but (4.2).

The property $(1,0)^T \notin \mathfrak{K}(W)$ is preserved when the transformations $\mathcal{T}^{\pm a_i}$ and $\mathcal{T}_m$ are applied. Hence, projection onto the second component is an isometric isomorphism of $\mathfrak{K}(\tilde{W})$ onto $\mathfrak{P}(\tilde{W})$. In particular, $\mathfrak{K}(\tilde{W})$ is positive definite, i.e. $\tilde{W} \in \mathcal{M}_0$. Thus, as we have noted in Remark 4.2 above, a chain going down from $\tilde{W}$ satisfies (ET). By what we have shown in Steps 2 and 3 above, reversing the transformations (4.3) yields that the chain ω we started with satisfies (ET). ❑

5 Pasting of chains

If $W_-, W_+ \in \mathcal{M}_{<\infty}$, then also their product $W := W_- W_+$ belongs to the class $\mathcal{M}_{<\infty}$. In fact, we have $\operatorname{ind}_- W \leq \operatorname{ind}_- W_- + \operatorname{ind}_- W_+$, which follows immediately from rewriting the kernel H_W; see, e.g. [KW/V, (2.19)]. Let ω^-

and ω^+ be chains going down from W_- and W_+, respectively. It is our aim in this section to construct a chain ω that is going down from W. In the generic case, the chain ω can be obtained simply by appending ω^+ to ω^-. More precisely, recall the following fact which has been shown in [KW/II, Section 7].

5.1 Proposition ([KW/II]). *Let $W_-, W_+ \in \mathcal{M}_{<\infty}$ be given, and let $\omega^- = (\omega^-(x;z))_{x\in I^-}$ and $\omega^+ = (\omega^+(x;z))_{x\in I^+}$ be chains going down from W_- and W_+, respectively. Assume that the following condition holds:*

(link) *If ω^- ends with an indivisible interval and ω^+ starts with an indivisible interval, then the types of these intervals are different.*

Set $l := \max I^- - \min I^+$ and define a map $\omega\colon I^- \cup (I^+ + l) \to \mathcal{M}_{<\infty}$ by

$$\omega(x) := \begin{cases} \omega^-(x), & x \in I^-, \\ W_-\omega^+(x-l), & x \in I^+ + l. \end{cases}$$

Then ω is a chain going down from W_-W_+. ❑

If the condition (link) fails to hold, the relation among the chains ω^-, ω^+ and ω is not so straightforward. Still, ω is obtained in a way by plugging together ω^- and ω^+, but some cancellation may happen. This means that not all matrices of ω^- and ω^+ necessarily appear in the chain ω. We speak of 'cancellation' since it is always an end section of ω^- and a corresponding beginning section of ω^+ which might vanish when plugging them together. Intuitively, one can understand this phenomenon as cancellation or merging of singularities (of the most simple, finite-dimensional, kind).

5.2 Example. Define two matrices W_- and W_+ by

$$W_-(z) = \begin{pmatrix} \cos z & -2z\cos z + \sin z \\ -\sin z & 2z\sin z + \cos z \end{pmatrix}, \qquad W_+(z) = \begin{pmatrix} 1 & 2z \\ 0 & 1 \end{pmatrix}.$$

Then

$$W(z) = W_-(z)W_+(z) = \begin{pmatrix} \cos z & \sin z \\ -\sin z & \cos z \end{pmatrix}.$$

One can show that

$$\omega_-(x;z) = \begin{cases} \begin{pmatrix} \cos xz & \sin xz \\ -\sin xz & \cos xz \end{pmatrix}, & x \in [0,1], \\ \begin{pmatrix} \cos z & \sin z \\ -\sin z & \cos z \end{pmatrix} \begin{pmatrix} 1 & \dfrac{x-1}{2-x} z \\ 0 & 1 \end{pmatrix}, & x \in (1,2) \cup (2,3], \end{cases}$$

$$\omega_+(x;z) = \begin{pmatrix} 1 & xz \\ 0 & 1 \end{pmatrix}, \qquad x \in [0,2],$$

are chains going down from W_- and W_+, respectively. The corresponding

Hamiltonians are given by

$$H_-(x) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & x \in [0,1], \\ \begin{pmatrix} (x-2)^{-2} & 0 \\ 0 & 0 \end{pmatrix}, & x \in (1,2) \cup (2,3], \end{cases}$$

$$H_+(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad x \in [0,2],$$

On the other hand, a chain going down from W is

$$\omega(x;z) = \begin{pmatrix} \cos xz & \sin xz \\ -\sin xz & \cos xz \end{pmatrix}, \qquad x \in [0,1],$$

and the corresponding Hamiltonian

$$H(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad x \in [0,1].$$

The chain ω is not obtained by appending ω_- and ω_+ as in Proposition 5.1. The parts $\big(\omega_-(x;\cdot)\big)_{x\in[1,3]\setminus\{2\}}$ and ω_+ of the chains ω_- and ω_+ do not contribute to ω. //

In the following discussion we make explicit how ω is constructed from ω^- and ω^+ in the case that condition (link) does not hold. For the rest of this section, let $W_-, W_+ \in \mathcal{M}_{<\infty}$ and respective chains ω^-, ω^+ be fixed and assume that (link) fails. Moreover, let ω be a chain going down from W_-W_+. Denote the domains of ω^-, ω^+ and ω by I^-, I^+ and I, and set

$$a^\pm := \min I^\pm,\ b^\pm := \max I^\pm, \quad a := \min I,\ b := \max I.$$

First we have to formalize how far indivisible intervals reach into ω^- from the right and into ω^+ from the left endpoint.

5.3 Definition. We define a strictly decreasing finite or infinite sequence (γ_n^-) of points in $[a^-, b^-]$ by the following inductive algorithm.

(i_-) Set $\gamma_0^- := b^-$.

(ii_-) Let $n \in \mathbb{N}_0$ and assume that γ_n^- is already defined. If γ_n^- is right endpoint of some indivisible interval, let γ_{n+1}^- be such that $(\gamma_{n+1}^-, \gamma_n^-)$ is maximal indivisible.

(iii_-) Let $n \in \mathbb{N}_0$ and assume that γ_n^- is already defined. If γ_n^- is not right endpoint of some indivisible interval, terminate.

Let $\alpha_n^- \in [0,\pi)$ be the type of the indivisible interval $(\gamma_n^-, \gamma_{n-1}^-)$, and set $\gamma^- := \inf \gamma_n^-$.

In a completely symmetric manner, we define a strictly increasing finite or infinite sequence (γ_n^+) of points in $[a^+, b^+]$ by the following rules.

(i_+) Set $\gamma_0^+ := a^+$.

(ii_+) Let $n \in \mathbb{N}_0$, and assume that γ_n^+ is already defined. If γ_n^+ is left endpoint of some indivisible interval, let γ_{n+1}^+ be such that $(\gamma_n^+, \gamma_{n+1}^+)$ is maximal indivisible.

(iii_+) Let $n \in \mathbb{N}_0$, and assume that γ_n^+ is already defined. If γ_n^+ is not left endpoint of some indivisible interval, terminate.

Also, let $\alpha_n^+ \in [0, \pi)$ be the type of the indivisible interval $(\gamma_{n-1}^+, \gamma_n^+)$, and set $\gamma^+ := \sup \gamma_n^+$. //

We agree that writing down a number $\gamma_n^\pm$ always includes the requirement that this number is actually defined, and we set $N^\pm := \sup\{n \in \mathbb{N}_0 \colon \gamma^\pm \text{ defined}\} \in \mathbb{N}_0 \cup \{+\infty\}$. Let us list some obvious properties of the sequences $(\gamma_n^\pm)$.

5.4. Properties of $(\gamma_n^\pm)$.
We formulate statements for the sequence (γ_n^-). Symmetric analogues hold for the sequence (γ_n^+).

(i) A point γ_n^- may happen to be a singularity. However, two consecutive points $\gamma_n^-, \gamma_{n+1}^-$ cannot simultaneously be singularities, cf. [KW/III].

(ii) No point γ_n^- is inner point of an indivisible interval.

(iii) Consider three consecutive points $\gamma_{n+1}^-, \gamma_n^-, \gamma_{n-1}^-$. If $\gamma_n^- \in I^-$, then $\alpha_n^- \neq \alpha_{n+1}^-$. If γ_n^- is a singularity, then $\alpha_n^- = \alpha_{n+1}^-$, cf. [KW/III].

(iv) If the sequence (γ_n^-) is infinite, then the point γ^- may be right endpoint of an indivisible interval. If the sequence (γ_n^-) is finite, then $\gamma^- = \gamma_{N^-}^-$ and γ^- cannot be right endpoint of an indivisible interval.

(v) The transfer matrix $\omega(x, b^-) = \omega(x, b^-; \cdot)$ is a polynomial if and only if there exists an $n \in \mathbb{N}_0$ such that $\gamma_n^- \leq x$. If the sequence (γ_n^-) is infinite, this means that $\gamma^- < x$; if it is finite, this means that $\gamma^- \leq x$.

//

It is practical to notationally specify those points $\gamma_n^\pm$ that are not singularities. Let $(\mathring{\gamma}_l^-)$ be the strictly decreasing finite or infinite sequence of all points γ_n^- that belong to I^-. We use indices so that the sequence is $\mathring{\gamma}_0^- > \mathring{\gamma}_1^- > \dots$, i.e. we leave no gaps in the sequence of indices. Again, when writing $\mathring{\gamma}_l^-$, we implicitly require that this number is defined. Moreover, we set $L^- := \sup\{l \in \mathbb{N}_0 \colon \mathring{\gamma}_l^- \text{ defined}\} \in \mathbb{N}_0 \cup \{+\infty\}$, and let $n^-(l) \in \mathbb{N}_0$ be such that $\mathring{\gamma}_l^- = \gamma_{n^-(l)}^-$.

Similarly, let $(\mathring{\gamma}_l^+)$ be the strictly increasing sequence of all points γ_n^+ that belong to I^+, and define L^+ and $n^+(l)$ accordingly.

Let us collect some obvious properties of these sequences.

5.5. Properties of $(\mathring{\gamma}_l^\pm)$.
Again, we formulate the statements only for the sequences built from ω^-.

(i) Since any chain contains only finitely many singularities, the sequences (γ_n^-) and $(\mathring{\gamma}_l^-)$ are either both finite or both infinite.

(ii) If the sequence $(\mathring{\gamma}_l^-)$ is finite, then $\gamma^- = \gamma_{N^-}^-$ and there are two possibilities: if $\gamma_{N^-}^- \in I^-$, then $n(L^-) = N^-$, i.e. $\mathring{\gamma}_{L^-}^- = \gamma_{N^-}^-$; if $\gamma_{N^-}^-$ is a singularity, then $n(L^-) = N^- - 1$ and $\mathring{\gamma}_{L^-}^- > \gamma_{N^-}^-$.

(*iii*) Because of 5.4 (*i*) and (*iii*) the interval $(\mathring{\gamma}_l^-, \mathring{\gamma}_{l-1}^-)$ is the union of at most two indivisible intervals, and we may define $\mathring{\alpha}_l^-$ as the type of this (these) indivisible interval(s).

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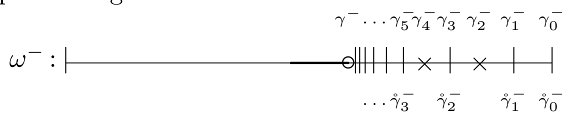
5.6 Remark. We use pictures to describe the structure of a chain. These are (hopefully) helpful and intuitive; still a little explanation is required. For example we might draw:



| . . . this point belongs to I
× . . . this point is a singularity
∘ . . . this point may or may not be a singularity
▬ . . . this piece of ω^- does not end indivisibly towards the indicated endpoint

The two possibilities from 5.5 (*ii*) can thus be visualized as follows:

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The first step towards understanding how ω is built is to see that cancellation can happen only within the range of indivisible intervals.

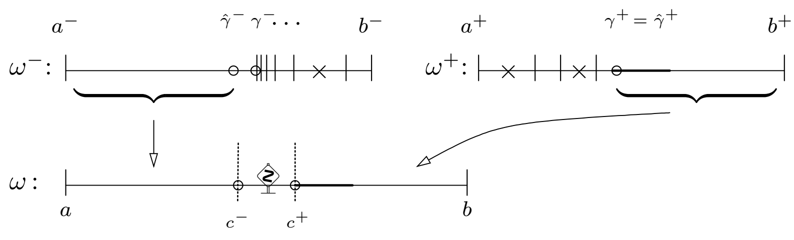
If γ^- is not right endpoint of an indivisible interval, set $\hat{\gamma}_- := \gamma^-$. Otherwise, let $\hat{\gamma}^-$ be such that $(\hat{\gamma}^-, \gamma^-)$ is maximal indivisible. Similarly, if γ^+ is not left endpoint of an indivisible interval, set $\hat{\gamma}_+ := \gamma^+$, and otherwise let $\hat{\gamma}^+$ be such that $(\gamma^+, \hat{\gamma}^+)$ is maximal indivisible.

The following lemma shows that the part $\omega^-|_{I^- \cap [a^-, \hat{\gamma}^-]}$ appears in ω and that also the part $\omega^+|_{I^+ \cap [\hat{\gamma}^+, b^+]}$ premultiplied by W_- can be found in ω.

5.7 Lemma. *There exist numbers $c^-, c^+ \in I$ such that*

$$\omega|_{I \cap [a, c^-]} \sim \omega^-|_{I^- \cap [a^-, \hat{\gamma}^-]}, \quad \omega|_{I \cap [c^+, b]} \sim W_- \omega^+|_{I^+ \cap [\hat{\gamma}^+, b^+]}.$$

For example, we may have:



. . . cancellation possible! Still to be determined.

Before we prove this lemma we show the following preliminary observation.

5.8. Let $W, M \in \mathcal{M}_{<\infty}$ and assume that W is not a polynomial. Moreover, let ζ and ϖ be chains going down from W and WM, respectively. Then ζ starts with an indivisible interval of type α if and only if ϖ does.

The assertion obtained when 'starts' is replaced by 'ends' and 'WM' is replaced by 'MW' also holds true.

Proof. For definiteness, let us assume that $\alpha = 0$. This can always be achieved by using rotation isomorphisms; see [KW/V, §3.c].

Step 1. Let $\nu = (\nu(x;\cdot\,))_{x\in I}$ be a chain, and assume that the subset $\tilde{I}$ of all points of I for which $\nu(x;\cdot\,)$ is not a polynomial is not empty. Then, by the construction in [KW/II, §7] and [KW/II, Theorem 5.7], the following equivalences hold:

ν starts with an indivisible interval of type 0

$$\iff \quad \forall\,\tau \in \mathcal{N}_{<\infty}\,\forall\,x \in \tilde{I}\colon\ \neg\Big[\lim_{y\to+\infty}\frac{1}{y}\Big(\nu(x;\cdot\,)\star\tau\Big)(iy) = 0\Big] \tag{5.1}$$

$$\iff \quad \exists\,\tau \in \mathcal{N}_{<\infty}\,\exists\,x \in \tilde{I}\colon\ \neg\Big[\lim_{y\to+\infty}\frac{1}{y}\Big(\nu(x;\cdot\,)\star\tau\Big)(iy) = 0\Big]. \tag{5.2}$$

Step 2. We show that WM is not a polynomial. Assume on the contrary that $P := WM$ is a polynomial. Then we can write $W^{-1} = MP^{-1}$. However, $P^{-1} \in \mathcal{M}_{<\infty}$ but $W^{-1} \notin \mathcal{M}_{<\infty}$, since by assumption W is not a polynomial; see 2.2. We have reached a contradiction, and conclude that WM is not a polynomial.

Step 3. Assume that ζ starts with an indivisible interval of type 0. Then (5.1) with $\tau = M\star\infty$ and x equal to the right endpoint of the chain ζ implies that

$$\neg\Big[\lim_{y\to+\infty}\frac{1}{y}\Big(W\star(M\star\infty)\Big)(iy) = 0\Big].$$

Since $W\star(M\star\infty) = (WM)\star\infty$, this together with (5.2) implies that ϖ starts with an indivisible interval of type 0. The converse follows in the same way.

Step 4. To see that the same assertion holds for indivisible intervals at the end of the chains instead of at their beginning, we only need to reverse the chain, cf. [KW/V, §3.c]. ❑

Proof (of Lemma 5.7). Let $x \in I^- \cap [a^-, \hat{\gamma}^-)$ be given. We show that there exists a point $y \in I \cap [x, \gamma^-]$ which is not inner point of an indivisible interval and has the property that $\omega(y, b^-)$ is not a polynomial. First, assume that γ^- is not right endpoint of an indivisible interval, so that $\hat{\gamma}^- = \gamma^-$. In this case we can choose any point $y \in I$ with $x \le y < \gamma^-$ such that y is not inner point of an indivisible interval. Second, assume that γ^- is right endpoint of an indivisible interval. Then $\hat{\gamma}^- < \gamma^-$ and the sequence (γ_n^-) is infinite. If $\gamma^- \in I$, choose $y := \gamma^-$. If γ^- is a singularity, then necessarily $\hat{\gamma}^- \in I$, and we can choose $y := \hat{\gamma}^-$.

Consider the chain ϖ going down from $\omega^-(y,b^-)W_+$. By 5.8, ϖ starts with an indivisible interval of some type α if and only the chain $\zeta = (\omega^-(y,t))_{t\in I^-\cap[y,b^-]}$ does. Since y is not inner point of an indivisible interval, it cannot happen that ζ ends with an indivisible interval and $\omega^-|_{I\cap[a^-,y]}$ starts with an indivisible interval of the same type. Hence, the chains $\omega^-|_{I\cap[a^-,y]}$ and ϖ satisfy (link), and we conclude that $\omega^-(y)$ appears as a member of ω because ω is a chain going down from $\omega^-(y)\omega^-(y,b^-)W_+ = W_-W_+$. Thus $\omega^-|_{I^-\cap[a^-,y]}$ is a beginning section of ω .

Passing to the limit $x \nearrow \hat{\gamma}^-$ if necessary, gives that $\omega^-|_{I^-\cap[a^-,\hat{\gamma}^-]}$ is a beginning section of ω.

The assertion concerning the end section of ω is again seen by reversing the chains $\omega^\pm$ and ω. ❑

Next, we observe that cancellation cannot happen arbitrarily often.

5.9 Lemma. *One of the following two alternatives holds.*

(1°) *There exists* $l_0 \in \mathbb{N}$*, such that*

$$\begin{aligned} &\omega^-(\mathring{\gamma}_l^-, b^-)\omega^+(a^+, \mathring{\gamma}_l^+) = I, \quad l < l_0, \\ &\omega^-(\mathring{\gamma}_{l_0}^-, b^-)\omega^+(a^+, \mathring{\gamma}_{l_0}^+) \neq I. \end{aligned} \tag{5.3}$$

(2°) *At least one of the sequences* $(\mathring{\gamma}_l^-)$, $(\mathring{\gamma}_l^+)$ *is finite and has at most* $\operatorname{ind}_- W_- + \operatorname{ind}_- W_+ + 1$ *elements. Moreover,*

$$\omega^-(\mathring{\gamma}_l^-, b^-)\omega^+(a^+, \mathring{\gamma}_l^+) = I \qquad \text{for all } l \text{ for which } \mathring{\gamma}_l^-, \mathring{\gamma}_l^+ \text{ are defined.} \tag{5.4}$$

Proof. First of all, note that clearly the condition (5.3) is equivalent to

$$\begin{aligned} &\omega^-(\mathring{\gamma}_l^-, \mathring{\gamma}_{l-1}^-)\omega^+(\mathring{\gamma}_{l-1}^+, \mathring{\gamma}_l^+) = I, \qquad l < l_0, \\ &\omega^-(\mathring{\gamma}_{l_0}^-, \mathring{\gamma}_{l_0-1}^-)\omega^+(\mathring{\gamma}_{l_0-1}^+, \mathring{\gamma}_{l_0}^+) \neq I. \end{aligned} \tag{5.5}$$

Assume that for some $l_0 \in \mathbb{N}$ the first line in (5.5) holds. If $l < l_0$, then not both $\omega^-(\mathring{\gamma}_l^-, \mathring{\gamma}_{l-1}^-)$ and $\omega^+(\mathring{\gamma}_{l-1}^+, \mathring{\gamma}_l^+)$ can belong to $\mathcal{M}_0$, since the functional $\mathfrak{t}$ is additive. Hence at least one of the intervals $(\mathring{\gamma}_l^-, \mathring{\gamma}_{l-1}^-)$ and $(\mathring{\gamma}_{l-1}^+, \mathring{\gamma}_l^+)$ must contain a singularity. However, the chain ω^- has at most $\operatorname{ind}_- W_-$ singularities, and the chain ω^+ at most $\operatorname{ind}_- W_+$ many. It follows that $l_0 \leq \operatorname{ind}_- W_- + \operatorname{ind}_- W_+ + 1$.

We conclude that, if $\omega(\mathring{\gamma}_l^-, b^-)\omega(a^+, \mathring{\gamma}_l^+) = I$ for all l, then $\mathring{\gamma}_l^-$ and $\mathring{\gamma}_l^+$ can both be defined at most up to $l = \operatorname{ind}_- W_- + \operatorname{ind}_- W_+$. ❑

Note that $\mathring{\alpha}_l^- = \mathring{\alpha}_l^+$ for all $l < l_0$ in case (1°) and for all l in Case (2°).

The case when (1°) holds is easily dealt with.

5.10. Case (1°).
First we determine the chain ϖ going down from

$$M := \omega^-(\mathring{\gamma}_{l_0}^-, \mathring{\gamma}_{l_0-1}^-)\omega^+(\mathring{\gamma}_{l_0-1}^+, \mathring{\gamma}_{l_0}^+) = \omega^-(\mathring{\gamma}_{l_0}^-, b^-)\omega^+(a^+, \mathring{\gamma}_{l_0}^+).$$

The chains ϖ^- and ϖ^+ going down from $\omega^-(\mathring{\gamma}_{l_0}^-, \mathring{\gamma}_{l_0-1}^-)$ and $\omega^+(\mathring{\gamma}_{l_0-1}^+, \mathring{\gamma}_{l_0}^+)$, respectively, consist of just one indivisible interval or of two indivisible intervals with a singularity in between. The type of the indivisible interval(s) in ϖ^- is $\mathring{\alpha}_{l_0}^-$, in ϖ^+ it is $\mathring{\alpha}_{l_0}^+$.

If $\mathring{\alpha}_{l_0}^- \neq \mathring{\alpha}_{l_0}^+$, then ϖ^- and ϖ^+ satisfy (link), and hence ϖ is obtained by pasting these two chains. If $\mathring{\alpha}_{l_0}^- = \mathring{\alpha}_{l_0}^+$, for definiteness say $\mathring{\alpha}_{l_0}^- = \mathring{\alpha}_{l_0}^+ = 0$, then

$$\omega^-(\mathring{\gamma}_{l_0}^-, \mathring{\gamma}_{l_0-1}^-; z) = \begin{pmatrix} 1 & p_-(z) \\ 0 & 1 \end{pmatrix}, \quad \omega^+(\mathring{\gamma}_{l_0-1}^+, \mathring{\gamma}_{l_0}^+; z) = \begin{pmatrix} 1 & p_+(z) \\ 0 & 1 \end{pmatrix}$$

with some polynomials $p_\pm$. Hence, also their product is of this form. The chain ϖ thus consists either of just one indivisible interval of type 0, namely if $p_-(z) + p_+(z) = \ell z$ with $\ell > 0$, and otherwise of two such intervals with a singularity in between.

In any case, and this is the presently important observation, the chain ϖ starts with an indivisible interval of type $\mathring{\alpha}_{l_0}^-$ and ends with one of type $\mathring{\alpha}_{l_0}^+$. Thus both pairs of chains

$$\omega^-|_{I^-\cap[a^-,\mathring{\gamma}_{l_0}^-]},\ \varpi \qquad \text{and} \qquad \varpi,\ \omega^+|_{I^+\cap[\mathring{\gamma}_{l_0}^+,b^+]}$$

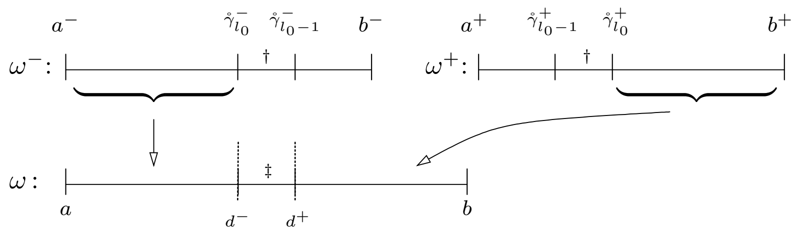
satisfy (link). This implies that the three chains $\omega^-|_{I^-\cap[a^-,\mathring{\gamma}_{l_0}^-]}$, ϖ, $\omega^+|_{I^+\cap[\mathring{\gamma}_{l_0}^+,b^+]}$ can be pasted, which yields a chain that goes down from

$$\begin{aligned}\omega^-(\mathring{\gamma}_{l_0}^-)M\omega^+(\mathring{\gamma}_{l_0}^+,b^+) &= \omega^-(\mathring{\gamma}_{l_0}^-)\omega^-(\mathring{\gamma}_{l_0}^-,b^-)\omega^+(a^+,\mathring{\gamma}_{l_0}^+)\omega^+(\mathring{\gamma}_{l_0}^+,b^+)\\ &= W_-W_+ = W.\end{aligned}$$

Since the chain going down from W is unique up to reparameterization, there exist $d^-,d^+\in I$ such that

$$\omega|_{I\cap[a,d^-]}\sim\omega^-|_{I^-\cap[a^-,\mathring{\gamma}_{n_0}^-]},\quad \omega|_{I\cap[d^-,d^+]}\sim\varpi,\quad \omega|_{I\cap[d^+,b]}\sim\omega^+|_{I^+\cap[\mathring{\gamma}_{n_0}^+,b^+]}.$$

We may indicate this situation as follows:



† . . . union of at most two indivisible intervals containing at most one singularity

‡ . . . union of at most four indivisible intervals containing at most two singularities

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5.11 Remark.

(*i*) Cancellation can only happen step by step. To see this, assume that, for some $l > l_0$, we have $\omega^-(\mathring{\gamma}_l^-,b^-)\omega^+(a^+,\mathring{\gamma}_l^+) = I$ but $\omega^-(\mathring{\gamma}_{l_0}^-,b^-)\omega^+(a^+,\mathring{\gamma}_{l_0}^+) \neq I$. By the consideration above, the matrix $\omega(\mathring{\gamma}_l^-,b^-)\omega^+(a^+,\mathring{\gamma}_l^+)$ is the transfer matrix between two different points of the chain ω, which cannot be equal to I, a contradiction. Hence $\omega^-(\mathring{\gamma}_{l_0}^-,b^-)\omega^+(a^+,\mathring{\gamma}_{l_0}^+) \neq I$ for all $l \geq l_0$.

(*ii*) The points $\hat{\gamma}^-$ and $\hat{\gamma}^+$ in Lemma 5.7 can always be replaced by γ^- and γ^+. To see this, note first that $\hat{\gamma}^- = \gamma^-$ unless the sequence (γ_n^-) is infinite. In the latter case, however, even $\omega^-|_{[a^-,\gamma_n^-]}$ with $n := \operatorname{ind}_- W_- + \operatorname{ind}_- W_+ + 1$ is a beginning section of ω. The same argument works for '$.^+$' instead of '$.^-$'.

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Assume now that the alternative (2°) holds. Then the situation is more involved, and we have to distinguish some subcases. For definiteness, let us assume that $(\mathring{\gamma}_l^-)$ is the 'shorter' sequence, i.e. $L^- \leq L^+$. The case that $(\mathring{\gamma}_l^+)$ is the shorter one is treated completely similarly; we do not give details.

5.12. Case (2° *a*)*:* $\mathring{\gamma}^-_{L^-} = \gamma^-$.
In this case $\mathring{\gamma}^-_{L^-}$ is not right endpoint of an indivisible interval, and hence the chains

$$\omega^-|_{I^-\cap[a^-,\mathring{\gamma}^-_{L^-}]}, \quad \omega^+(\mathring{\gamma}^+_{L^-},\cdot)|_{I^+\cap[\mathring{\gamma}^+_{L^-},b^+]}$$

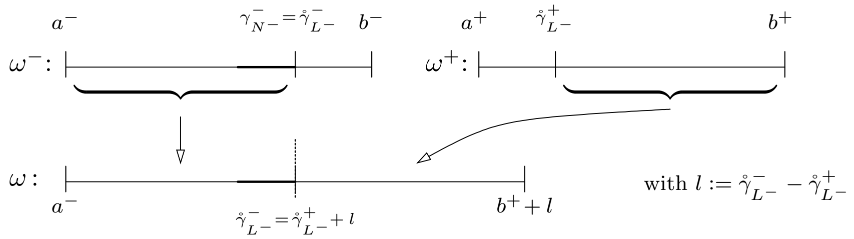
satisfy (link). The pasting of these two chains is a chain going down from

$$\begin{aligned}\omega^-(\mathring{\gamma}^-_{L^-})\omega^+(\mathring{\gamma}^+_{L^-},b^+) &= \omega^-(\mathring{\gamma}^-_{L^-})\omega^-(\mathring{\gamma}^-_{L^-},b^-)\omega^+(a^+,\mathring{\gamma}^+_{L^-})\omega^+(\mathring{\gamma}^+_{L^-},b^+)\\ &= W_-W_+ = W,\end{aligned}$$

where in the first equality we used (5.4). Hence this chain is a reparameterization of ω, which implies that ω, parameterized appropriately, has the form

$$\omega(x;z) = \begin{cases}\omega^-(x;z), & x \in I^- \cap [a^-,\mathring{\gamma}^-_{L^-}],\\ \omega^-(\mathring{\gamma}^-_{L^-};z)\omega^+(\mathring{\gamma}^+_{L^-},x-l;z), & x \in (I^+ \cap [\mathring{\gamma}^+_{L^-},b^+]) + l,\end{cases} \tag{5.6}$$

where the shift l is given by $l := \mathring{\gamma}^-_{L^-} - \mathring{\gamma}^+_{L^-}$. Note that by (5.4), we have $\omega^-(\mathring{\gamma}^-_{L^-};z)\omega^+(\mathring{\gamma}^+_{L^-},x-l;z) = W_-(z)\omega^+(x-l;z)$.



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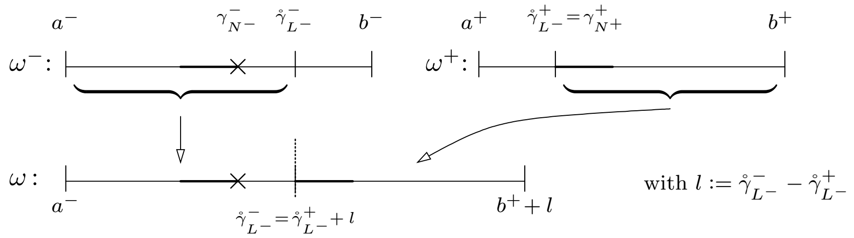
5.13. Case (2° *b*)*:* $\mathring{\gamma}^-_{L^-} > \gamma^-$, $\mathring{\gamma}^+_{L^-} = \gamma^+$.
In this case $\gamma^+_{L^-}$ is not left endpoint of an indivisible interval, and we obtain (5.6) as in the previous case.



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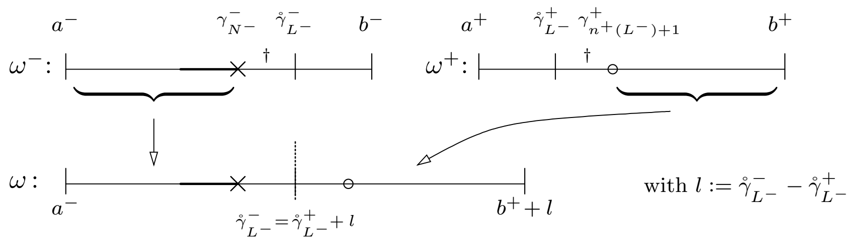
5.14. Case (2° *c*)*:* $\mathring{\gamma}^-_{L^-} > \gamma^-$, $\mathring{\gamma}^+_{L^-} < \gamma^+$, $\alpha^-_{N^-} \neq \alpha^+_{n^+(L^-)+1}$.
The types of the two indivisible intervals are different, and hence the chains $\omega^-|_{I^-\cap[a^-,\mathring{\gamma}^-_{L^-}]}$ and $\omega^+(\mathring{\gamma}^+_{L^-},\cdot)|_{I^+\cap[\mathring{\gamma}^+_{L^-},b^+]}$ satisfy (link). Thus again (5.6) holds.



† . . . indivisible intervals having different types

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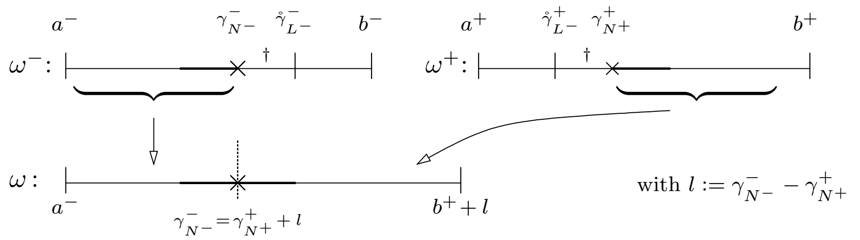
5.15. Case (2° *d*)*:* $\mathring{\gamma}^-_{L^-} > \gamma^-$, $\mathring{\gamma}^+_{L^-} < \gamma^+$, $\alpha^-_{N^-} = \alpha^+_{n^+(L^-)+1}$, $L^- = L^+$.
We know that $\omega^-|_{I^-\cap[a^-,\gamma^-]}$ is a beginning section of ω and $W_-\omega^+|_{I^+\cap[\gamma^+,b^+]}$ is an end section. In the present case this beginning section ends at a singularity σ_-, and this end section starts at a singularity σ_+. We show that $\sigma_- = \sigma_+$. To this end, we compute the respective intermediate Weyl coefficients (for definiteness, assume that $\alpha^-_{N^-} = \alpha^+_{n^+(L^-)+1} = 0$):

$$\begin{aligned} q_{\sigma_-} &= \lim_{x\nearrow\sigma_-} \omega(x;\cdot\,)\star\infty = \lim_{x\nearrow\gamma^-} \omega^-(x;\cdot\,)\star\infty \\ &= \lim_{x\searrow\gamma^-} \omega^-(x;\cdot\,)\star\infty = \omega^-(\mathring{\gamma}^-_{L^-};\cdot\,)\star\infty \end{aligned}$$

since $\omega^-(x;z)\star\infty$ is constant in x on the interval $(\gamma^-,\mathring{\gamma}^-_{L^-})$;

$$\begin{aligned} q_{\sigma_+} &= \lim_{x\searrow\sigma_+} \omega(x;\cdot\,)\star\infty = \lim_{x\searrow\gamma^+} W_-\omega^+(x;\cdot\,)\star\infty \\ &= \lim_{x\searrow\gamma^+} \omega^-(\mathring{\gamma}^-_{L^-};\cdot\,)\omega^+(\mathring{\gamma}^+_{L^-},x;\cdot\,)\star\infty \\ &= \omega^-(\mathring{\gamma}^-_{L^-};\cdot\,)\star\Big(\lim_{x\searrow\gamma^+} \omega^+(\mathring{\gamma}^+_{L^-},x;\cdot\,)\star\infty\Big) \\ &= \omega^-(\mathring{\gamma}^-_{L^-};\cdot\,)\star\Big(\lim_{x\nearrow\gamma^+} \omega^+(\mathring{\gamma}^+_{L^-},x;\cdot\,)\star\infty\Big) = \omega^-(\mathring{\gamma}^-_{L^-};\cdot)\star\infty, \end{aligned}$$

where for the third equality we used (5.4) and for the last equality the fact that $(\mathring{\gamma}^+_{L^-},\gamma^+)$ is an indivisible interval of type 0. We see that $q_{\sigma_-} = q_{\sigma_+}$. Since the negative index of the intermediate Weyl coefficient at a singularity equals the negative index of the matrices in the component to the left of this singularity, this implies that $\sigma_- = \sigma_+$. Thus ω is exhausted by the beginning section $\omega^-|_{I^-\cap[a^-,\gamma^-]}$ and the end section $W_-\omega^+|_{I^+\cap[\gamma^+,b^+]}$.



† . . . indivisible intervals having the same type

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In order to settle the last remaining case, namely '$\mathring{\gamma}^-_{L^-} > \gamma^-$, $\mathring{\gamma}^+_{L^-} < \gamma^+$, $\alpha^-_{N^-} = \alpha^+_{n^+(L^-)+1}$, $L^- < L^+$', we need the following preliminary observation, which uses a similar argument as the observation 5.8.

5.16. Let $W, P \in \mathcal{M}_{<\infty}$ *and assume that* W *is not a polynomial and that* P *is of the form*

$$P(z) = \begin{pmatrix} 1 & p(z) \\ 0 & 1 \end{pmatrix} \tag{5.7}$$

where p is a polynomial. Let ζ, η and ϖ be the chains going down from W, P and PW, respectively. Assume that ζ starts with an indivisible interval of type 0 whose right endpoint is a singularity which itself is not left endpoint of an indivisible interval. Then ϖ starts in exactly the same way as ζ.

The assertion obtained when 'starts' and 'ends' are exchanged and 'PW' is replaced by 'WP' also holds true.

Proof. We carry out the proof in two steps.

Step 1. Let $\nu = (\nu(x;\cdot))_{x\in I}$ be a chain and assume that the subset $\tilde{I}$ of all points of I for which $\nu(x;\cdot)$ is not a polynomial is not empty. Then, again referring to the construction in [KW/II, §7] and [KW/II, Theorem 5.7], the following equivalences hold:

ν starts with an indivisible interval of type 0 whose right endpoint is a singularity which itself is not left endpoint of an indivisible interval

$$\iff \forall\,\tau\in\mathcal{N}_{<\infty}\,\forall\,x\in\tilde{I}\,\forall\,p\in\mathbb{R}[z]\colon\ \neg\Big[\lim_{y\to+\infty}\frac{1}{y}\Big(p+\nu(x;\cdot)\star\tau\Big)(iy)=0\Big]$$

$$\iff \exists\,\tau\in\mathcal{N}_{<\infty}\,\exists\,x\in\tilde{I}\,\forall\,p\in\mathbb{R}[z]\colon\ \neg\Big[\lim_{y\to+\infty}\frac{1}{y}\Big(p+\nu(x;\cdot)\star\tau\Big)(iy)=0\Big].$$

Step 2. Assume now that we are in the situation of the statement. Then

$$(PW)\star\infty = P\star(W\star\infty) = p+(W\star\infty).$$

The assertion follows from the equivalences in Step 1. The assertion with 'starts' exchanged with 'ends' and 'PW' replaced by 'WP' follows by considering the reversed chains. ❑

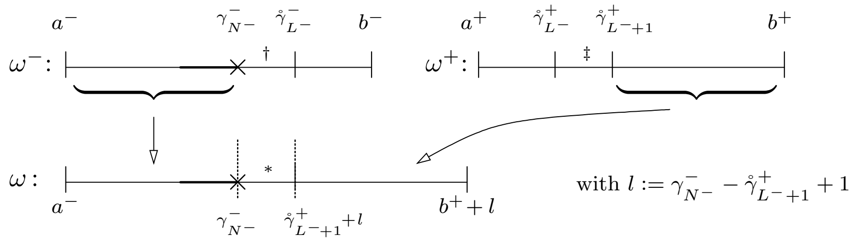
5.17. Case (2° *e*)*:* $\mathring{\gamma}^-_{L^-}>\gamma^-$, $\mathring{\gamma}^+_{L^-}<\gamma^+$, $\alpha^-_{N^-}=\alpha^+_{n^+(L^-)+1}$, $L^-<L^+$.
Without loss of generality let us assume that $\alpha^-_{N^-}=0$. Then the interval $(\mathring{\gamma}^+_{L^-},\mathring{\gamma}^+_{L^-+1})$ is either one indivisible interval of type 0 or consists of two indivisible intervals both of type 0 with a singularity at $\gamma^+_{n^+(L^-)+1}\in(\mathring{\gamma}^+_{L^-},\mathring{\gamma}^+_{L^-+1})$. In both cases the transfer matrix $\omega^+(\mathring{\gamma}^+_{L^-},\mathring{\gamma}^+_{L^-+1};\cdot)$ is of the form (5.7) where p is a polynomial. Hence we can apply 5.16 to the matrices $\omega^-(\mathring{\gamma}^-_{L^-};\cdot)$ and $\omega^+(\mathring{\gamma}^+_{L^-},\mathring{\gamma}^+_{L^-+1};\cdot)$. It follows that the chain ϖ going down from $\omega^-(\mathring{\gamma}^-_{L^-};\cdot)\omega^+(\mathring{\gamma}^+_{L^-},\mathring{\gamma}^+_{L^-+1};\cdot)$ ends with an indivisible interval of type $\alpha^-_{N^-}$ whose left endpoint is a singularity σ_+ of ϖ. We already know that $\omega^-|_{I^-\cap[a^-,\gamma^-]}$ is a beginning section of ϖ. This section ends at a singularity σ_- of ϖ. We compute the respective intermediate Weyl coefficients in a similar way as in Case (2° *d*):

$$\begin{aligned}
q_{\sigma_-} &= \lim_{x\nearrow\sigma_-}\varpi(x;\cdot)\star\infty = \lim_{x\nearrow\gamma^-}\omega^-(x;\cdot)\star\infty\\
&= \lim_{x\searrow\gamma^-}\omega^-(x;\cdot)\star\infty = \omega^-(\mathring{\gamma}^-_{L^-};\cdot)\star\infty,\\
q_{\sigma_+} &= \lim_{x\searrow\sigma_+}\varpi(x;\cdot)\star\infty = \omega^-(\mathring{\gamma}^-_{L^-};\cdot)\omega^+(\mathring{\gamma}^+_{L^-},\mathring{\gamma}^+_{L^-+1};\cdot)\star\infty\\
&= \omega^-(\mathring{\gamma}^-_{L^-};\cdot)\star\infty.
\end{aligned}$$

Equality of intermediate Weyl coefficients implies that $\sigma_- = \sigma_+$, i.e. the chain ϖ is exhausted by the beginning section $\omega^-|_{I^-\cap[a^-,\gamma^-]}$ and to the right of it an indivisible interval or two indivisible intervals of the same type with a singularity in between. We see that the chains ϖ and $\omega^+(\mathring{\gamma}^+_{L^-+1},\cdot\,;\cdot\,)|_{I^+\cap[\mathring{\gamma}^+_{L^-+1},b^+]}$ satisfy (link). The pasting of these two chains yields a chain that goes down from

$$\begin{aligned}&\omega^-(\mathring{\gamma}^-_{L^-})\omega^+(\mathring{\gamma}^+_{L^-},\mathring{\gamma}^+_{L^-+1})\omega^+(\mathring{\gamma}^+_{L^-+1},b^+)\\&=\omega^-(\mathring{\gamma}^-_{L^-})\omega^-(\mathring{\gamma}^-_{L^-},b^-)\omega^+(a^+,\gamma^+_{\mathrm{L}^-})\omega^+(\mathring{\gamma}^+_{\mathrm{L}^-},\mathrm{b}^+)\\&=W_-W_+=W,\end{aligned}$$

where we used (5.4) for $l=L^-$. Hence this chain is a reparameterization of ω.



† . . . indivisible interval
‡ . . . one or two indivisible intervals having the same type
∗ . . . one or two indivisible intervals having the same type as those above

//

With these considerations and Theorem 4.1 we can prove a result about non-cancellation of exponential growth for products of matrices from $\mathcal{M}_{<\infty}$, which on first sight may seem surprising.

5.18 Theorem. *Let $W_1, W_2 \in \mathcal{M}_{<\infty}$. Then* $\operatorname{et}(W_1W_2) = \operatorname{et}W_1 + \operatorname{et}W_2$.

Proof. Let $\omega_1 = (\omega_1(x;\cdot\,))_{x\in I_1}$, $\omega_2 = (\omega_2(x;\cdot\,))_{x\in I_1}$ and $\omega = (\omega(x;\cdot\,))_{x\in I_1}$ be properly parameterized chains going down from W_1, W_2 and W_1W_2, respectively, and let H_1, H_2 and H be the respective Hamiltonians in the canonical differential equation satisfied by these chains. By Theorem 4.1, we have

$$\operatorname{et}W_j = \int_{I_j}\sqrt{\det H_j(x)}\,dx,\ \ j=1,2,\qquad \operatorname{et}W = \int_I\sqrt{\det H(x)}\,dx.$$

If the chains ω_1 and ω_2 satisfy (link), then H is simply a reparameterization of the Hamiltonian that is obtained by appending H_2 to H_1, and hence

$$\int_I\sqrt{\det H(t)}\,dt = \int_{I_1}\sqrt{\det H_1(t)}\,dt + \int_{I_2}\sqrt{\det H_2(t)}\,dt. \tag{5.8}$$

Assume now that (link) does not hold. Then H is obtained from H_1 and H_2 by possibly deleting some indivisible intervals from H_1 and H_2, plugging together the resulting Hamiltonians, and possibly inserting some indivisible intervals. This, however, does not change the values of the integrals in (5.8). Hence, this equality remains true. ❑

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