Symmetry in de Branges almost Pontryagin spaces

Henrik Winkler and Harald Woracek

Abstract. In many examples of de Branges spaces symmetry appears naturally. Presence of symmetry gives rise to a decomposition of the space into two parts, the 'even' and the 'odd' part, which themselves can be regarded as de Branges spaces. The converse question is to decide whether a given space is the 'even' part or the 'odd' part of some symmetric space, and, if yes, to describe the totality of all such symmetric spaces. We consider this question in an indefinite (almost Pontryagin space) setting, and give a complete answer. Interestingly, it turns out that the answers for the 'even' and 'odd' cases read quite differently; the latter is significantly more complex.

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1. Introduction

In the 1960's L.de Branges axiomatically introduced a particular kind of Hilbert spaces of entire functions and developed their structure theory, cf. [dB68]. These spaces can be viewed as weighted analogues of the Paley-Wiener spaces of Fourier transforms of square integrable functions supported on a compact interval[1]. Ever since, de Branges's spaces were intensively studied, and interest is still growing. Besides the intrinsic beauty of the theory, one reason is that they appear in many places in complex analysis, functional analysis, or differential equations. For example, when dealing with power moment problems and Jacobi operators ([BS99, Akh61]), classical functions like Gauß hypergeometric functions or Dirichlet L-functions ([dB68, Lag06]), bases of exponentials in weighted L^2-spaces ([OCS02]), Beurling-Malliavin type theorems ([HM03a, HM03b]), Schrödinger operators ([Rem02]), and

[1] This fact, already stated by L.de Branges, recently was made explicit in [LS02].

many others. De Branges' structure theory has important implications to all these fields. In particular, it can be seen as the mother of several inverse spectral theorems for different kinds of differential equations.

Over the past decade a generalization of de Branges theory to an indefinite setting was developed, cf. [KW99a]–[KW10]: The axioms of a de Branges space $\mathcal{H}$ remain the same, only the requirement that $\mathcal{H}$ is a Hilbert space is weakened to assuming that $\mathcal{H}$ is an almost Pontryagin space (that is, a direct and orthogonal sum of a Hilbert space with a finite-dimensional negative semidefinite space). On first sight this may seem a minor generalization; for several reasons it is not: (1) Passing to the indefinite situation creates a deep theory; significant effort is needed to establish the analogues of de Branges' theorems. (2) The indefinite theory has a broad variety of applications, e.g., to indefinite versions of power moment problems ([KL79, KL80][2]), or differential equations with inner singularities or singular endpoints ([LW]). (3) Some classical, i.e., 'positive definite', questions can be solved by making a detour via the indefinite world (e.g. [Wor12]).

A de Branges almost Pontryagin space $\mathfrak{L}$ is called symmetric, if the assignment

$$\mathfrak{i} : F(z) \mapsto F(-z)$$

maps $\mathfrak{L}$ isometrically into itself (since $\mathfrak{i}$ is involutory and has closed graph, it thus induces an isometric isomorphism of $\mathfrak{L}$ onto itself). This notion appeared already at a very early stage, cf. [dB62]. Symmetry arises for example from functional equations (e.g. [KW05, Example 3.2]), or in the context of Schrödinger operators or Kreĭn strings where symmetry is implemented in the construction by writing the spectral parameter as 'λ^2'. As a general rule, symmetry is connected with properties of corresponding operator models. This statement in fact holds on a very general level, namely within the framework of unitary colligations. For the Hilbert space situation this goes back to [Lub76], the Pontryagin space situation is treated in [AADR02], and a more specific version for reproducing kernel spaces (using the language of Q-functions rather than colligations) can be found in [KWW06b]. A straight connection with these results, however, can only be made if the de Branges space under consideration is nondegenerated.

Due to the presence of the isometric involution $\mathfrak{i}$, a symmetric de Branges space decomposes into the direct and orthogonal sum of its subspaces $\mathfrak{L}_e$ and $\mathfrak{L}_o$ consisting of all even or odd, respectively, functions in $\mathfrak{L}$. It is an important fact that $\mathfrak{L}_e$ and $\mathfrak{L}_o$ themselves can be considered as de Branges spaces: Set

$$\mathfrak{L}^{\mathsf{ev}} := \big\{H(\sqrt{z}) : H \in \mathfrak{L}, H \text{ even}\big\}, \quad [F,G]_{\mathfrak{L}^{\mathsf{ev}}} := [F(z^2), G(z^2)]_{\mathfrak{L}}\,,$$
$$\mathfrak{L}^{\mathsf{od}} := \big\{\frac{1}{\sqrt{z}}H(\sqrt{z}) : H \in \mathfrak{L}, H \text{ odd}\big\}, \quad [F,G]_{\mathfrak{L}^{\mathsf{od}}} := [zF(z^2), zG(z^2)]_{\mathfrak{L}}\,.$$

[2]Making explicit the connection with de Branges almost Pontryagin spaces is work in progress.

Then one can show that $\mathfrak{L}^{\mathsf{ev}}$ and $\mathfrak{L}^{\mathsf{od}}$ are de Branges spaces. By their definition, they are isomorphic to $\mathfrak{L}_e$ and $\mathfrak{L}_o$.

A converse question suggests itself: *Given a de Branges space $\mathfrak{P}$, does there exist a symmetric de Branges space $\mathfrak{L}$ with $\mathfrak{L}^{\mathsf{ev}} = \mathfrak{P}$ (or with $\mathfrak{L}^{\mathsf{od}} = \mathfrak{P}$, respectively)? If yes, what is the totality of all such spaces?*

In the present paper we give a complete answer to these questions. Our main results are Theorem 3.3 and Theorem 4.3. Comprehensively formulated, these results state the following: Let $\mathfrak{P}$ be a de Branges almost Pontryagin space.

(1_{ev}) There exists $\mathfrak{L}$ with $\mathfrak{L}^{\mathsf{ev}} = \mathfrak{P}$ if and only if the quadratic form

$$[F,G]_s := [zF(z), G(z)]_{\mathfrak{P}}\,, \tag{1.1}$$

defined for all $F,G \in \mathfrak{P}$ with $zF(z), zG(z) \in \mathfrak{P}$, has a finite number of negative squares.

(2_{ev}) The set of all spaces $\mathfrak{L}$ with $\mathfrak{L}^{\mathsf{ev}} = \mathfrak{P}$ forms a one-parameter family $\{\mathfrak{L}_\tau : \tau \in \mathbb{R} \cup \{\infty\}\}$.

(1_{od}) Just the same as for the 'ev'-case[3]: There exists $\mathfrak{L}$ with $\mathfrak{L}^{\mathsf{od}} = \mathfrak{P}$ if and only if the quadratic form (1.1) has a finite number of negative squares.

(2_{ev}) The set of all spaces $\mathfrak{L}$ with $\mathfrak{L}^{\mathsf{od}} = \mathfrak{P}$ forms a two-parameter family $\{\mathfrak{L}_{l,q} : l \in \mathbb{R}, q \in \mathbb{R} \cup \{\infty\}\}$.

Some partial results in this direction have already been obtained in earlier work. First, in the Hilbert space situation this was done (in the language of Hermite-Biehler functions) already in [dB68]. Second, (1_{ev}) was shown already in [KWW06a], where also a description of all spaces $\mathfrak{L}$ with $\mathfrak{L}^{\mathsf{ev}} = \mathfrak{P}$ was given. However, this description is very implicit. Third, if all considerations are restricted to nondegenerated spaces $\mathfrak{P}$ and $\mathfrak{L}$, the facts corresponding to the above mentioned ones can be deduced rather easily from some results on indefinite Hermite-Biehler functions shown in [KWW06a] and [PW07] (though this is not stated explicitly there).

The main achievements and novelties of our present work are that: (1) We obtain full understanding of degenerated spaces; (2) We exhaustively treat the 'od'-case (which is the more complex one); (3) We obtain explicit descriptions of the families as indicated in (2_{ev}) and (2_{od}).

To close this introduction, let us briefly comment on the organisation of the manuscript. In Section 2 we recall some facts about de Branges spaces, and provide some preliminaries concerning perturbations of inner products which are extensively used throughout. In Section 3 we deal with the 'ev'-case. This case is simpler than the 'od'-case, however, the methods are similar and the discussion may serve as a model for the proof of the 'od'-case. The most extensive part of the paper is Section 4, where we then settle the 'od'-case.

[3]The fact that the conditions (1_{ev}) and (1_{od}) coincide expresses strong symmetry. Responsible for its presence is that we consider the indefinite situation. In the positive definite setting the answers would be quite different.

2. Preliminaries

2.1. Symmetric and seminbounded dB-spaces

Before we can state the definition of a de Branges space, we need to recall the notion of reproducing kernel almost Pontryagin spaces. Denote by $\mathbb{H}(\mathbb{C})$ the linear space of all entire functions, and let χ_w, $w \in \mathbb{C}$, be the point evaluation functional

$$\chi_w : \begin{cases} \mathbb{H}(\mathbb{C}) & \to & \mathbb{C} \\ F & \mapsto & F(w) \end{cases} .$$

2.1. Definition. Let $\mathcal{L}$ be a linear space and $[.,.]$ an inner product on $\mathcal{L}$.

(i) Assume that $\mathcal{O}$ is a Hilbert space topology on $\mathcal{L}$. Then the triple $\langle \mathcal{L}, [.,.], \mathcal{O} \rangle$ is called an almost Pontryagin space, if $[.,.]$ is $\mathcal{O} \times \mathcal{O}$-continuous, and if there exists an $\mathcal{O}$-closed linear subspace $\mathcal{M}$ of $\mathcal{L}$ with finite codimension, such that $\langle \mathcal{M}, [.,.] \rangle$ is a Hilbert space.

(ii) The inner product space $\langle \mathcal{L}, [.,.] \rangle$ is called a reproducing kernel almost Pontryagin space of entire functions, if $\mathcal{L} \subseteq \mathbb{H}(\mathbb{C})$ and if there exists a Hilbert space topology $\mathcal{O}$ on $\mathcal{L}$, such that $\langle \mathcal{L}, [.,.], \mathcal{O} \rangle$ is an almost Pontryagin space and that each point evaluation functional $\chi_w|_{\mathcal{L}}$, $w \in \mathbb{C}$, is $\mathcal{O}$-continuous.

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For a given inner product space $\langle \mathcal{L}, [.,.] \rangle$, there may exist several different Hilbert space topologies which turn $\mathcal{L}$ into an almost Pontryagin space. Uniqueness prevails only if $\mathcal{L}$ is nondegenerated. However, if $\mathcal{L} \subseteq \mathbb{H}(\mathbb{C})$, then there exists at most one Hilbert space topology on $\mathcal{L}$ which in addition makes all point evaluations continuous, cf. [KWW05, §5]. This says that the topology of a reproducing kernel almost Pontryagin space is uniquely determined by its inner product (and hence there is no need to include it into the notation).

Let $\langle \mathcal{L}, [.,.] \rangle$ be a reproducing kernel almost Pontryagin space of entire functions. If $\mathcal{L}$ is nondegenerated, there exists a reproducing kernel in the classical sense, i.e. a family of elements $K(w,.) \in \mathcal{L}$, $w \in \mathbb{C}$, with the property that

$$[F, K(w,.)] = F(w), \quad F \in \mathcal{L},\ w \in \mathbb{C}\,.$$

If $\mathcal{L}$ is degenerated, such a family clearly cannot exist.

2.2. Definition. An inner product space $\langle \mathfrak{P}, [.,.] \rangle$ is called a de Branges space (dB-space, for short), if it satisfies the following axioms.

(dB1) $\langle \mathfrak{P}, [.,.] \rangle$ is a reproducing kernel almost Pontryagin space of entire functions.

(dB2) If $F \in \mathfrak{P}$, then the function $F^{\#}(z) := \overline{F(\overline{z})}$ belongs to $\mathfrak{P}$. Moreover,

$$[F^{\#}, G^{\#}] = [G, F], \quad F, G \in \mathfrak{P}\,.$$

(dB3) If $F \in \mathfrak{P}$ and $z_0 \in \mathbb{C} \setminus \mathbb{R}$ with $F(z_0) = 0$, then

$$\frac{z - \overline{z_0}}{z - z_0} F(z) \in \mathfrak{P}\,.$$

If additionally $G \in \mathfrak{P}$ with $G(z_0) = 0$, then

$$\Big[\frac{z-\overline{z_0}}{z-z_0}F(z), \frac{z-\overline{z_0}}{z-z_0}G(z)\Big] = [F,G]\,.$$

In the present paper, we always require the additional condition

(dB4) For each $t \in \mathbb{R}$ there exists $F \in \mathfrak{P}$ with $F(t) \neq 0$.

The set of all dB-spaces will be denoted by $\mathcal{DB}$.

Moreover, we denote by $\mathcal{S}_{\mathfrak{P}}$ the operator of multiplication by the independent variable in the dB-space $\mathfrak{P}$ with maximal domain, i.e. $\operatorname{dom}\mathcal{S}_{\mathfrak{P}} := \{F \in \mathfrak{P} :\ F(z), zF(z) \in \mathfrak{P}\}$. //

Recall that the presence of (dB4) implies that the space $\mathfrak{P}$ is invariant with respect to division of real zeros, see, e.g., [KW99a, Lemma 4.1].

Next, we specify the subclasses of $\mathcal{DB}$ which are under investigation; namely symmetric and semibounded dB-spaces.

2.3. Definition. Let $\langle \mathfrak{P}, [.,.]\rangle$ be a dB-space.

(i) We say that $\mathfrak{P}$ is symmetric, if the map $\mathfrak{i} : F(z) \mapsto F(-z)$ leaves $\mathfrak{P}$ invariant and $\mathfrak{i}|_{\mathfrak{P}}$ is isometric with respect to the inner product of $\mathfrak{P}$.

(ii) We say that $\mathfrak{P}$ is semibounded, if the inner product on $\operatorname{dom}\mathcal{S}_{\mathfrak{P}}$ defined as

$$[F,G]_s := [\mathcal{S}_{\mathfrak{P}}F, G], \quad F, G \in \operatorname{dom}\mathcal{S}_{\mathfrak{P}}\,,$$

has a finite number of negative squares.

The subclass of $\mathcal{DB}$ containing all symmetric dB-spaces is denoted as $\mathcal{DB}^{\mathrm{sym}}$, the subclass containing all semibounded ones by $\mathcal{DB}^{\mathrm{sb}}$. //

Nondegenerated dB-spaces can be generated from entire function having certain properties.

2.4. Definition. Denote by $\mathcal{HB}_{<\infty}$ the set of all entire functions E, such that E and $E^{\#}$ have no common nonreal zeros, $E^{-1}E^{\#}$ is not constant, and the reproducing kernel

$$K_E(w,z) := \frac{i}{2}\frac{E(z)\overline{E(w)} - E^{\#}(z)E(\overline{w})}{z-\overline{w}}$$

has a finite number negative squares. Moreover, in the present paper, we always require the following two additional properties:

$$E(x) \neq 0,\ x \in \mathbb{R} \quad \text{and} \quad E(0) = 1\,.$$

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If $E \in \mathcal{HB}_{<\infty}$, we denote the actual number of negative squares of the kernel K_E by $\operatorname{ind}_- E$. Each function $E \in \mathcal{HB}_{<\infty}$ generates a reproducing kernel Pontryagin space via the kernel K_E; we denote this space by $\mathfrak{P}(E)$. The fact that $E \in \mathcal{HB}_0$ if and only if $E \in \mathcal{HB}_{<\infty}$ and $\operatorname{ind}_- E = 0$, is well-known, see, e.g. [dB68].

Throughout this paper, we agree on a generic notation applied to Hermite-Biehler functions: If we speak of functions E, A, B (or $\tilde{E}, \tilde{A}, \tilde{B}$, or similar), these functions always shall be related as

$$A := \frac{1}{2}(E + E^{\#}), \quad B := \frac{i}{2}(E - E^{\#}).$$

Using this notation, the reproducing kernel K_E can be rewritten as

$$K_E(w, z) = \frac{B(z)A(\overline{w}) - B(\overline{w})A(z)}{z - \overline{w}}. \tag{2.1}$$

2.5. *Remark.* Let E be an entire function. Then $E \in \mathcal{HB}_{<\infty}$ if and only if the functions A and B are linearly independent, have no common zeros, and the function $\frac{B}{A}$ belongs to $\mathcal{N}_{<\infty}$. //

Next, we specify two subclasses of $\mathcal{HB}_{<\infty}$; the classes of symmetric and semibounded Hermite-Biehler functions.

2.6. Definition. Denote

$$\mathcal{HB}_{<\infty}^{\mathrm{sym}} := \big\{E \in \mathcal{HB}_{<\infty} : E^{\#}(z) = E(-z)\big\}$$

$$\mathcal{HB}_{<\infty}^{\mathrm{sb}} := \big\{E \in \mathcal{HB}_{<\infty} : A \text{ has only finitely many zeros on } (-\infty, 0)\big\}$$

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The relation between dB-spaces and Hermite-Biehler functions is the following, cf. [KW99a, Theorem 5.3, Corollary 6.2], [KWW06a, Proposition 4.3, Proposition 4.4].

2.7. *Remark.*

(*i*) Let $E \in \mathcal{HB}_{<\infty}$. Then the space $\mathfrak{P}(E)$ is a de Branges Pontryagin space.

(*ii*) Let $\mathfrak{P}$ be a nondegenerated dB-space. Then there exists a function $E \in \mathcal{HB}_{<\infty}$ such that $\mathfrak{P} = \mathfrak{P}(E)$. The function E in this realization is not unique. It is uniquely determined only up to real scalar multiples of B (remember here that we included the requirement that '$E(0) = 1$' in the definition of the Hermite-Biehler class).

(*iii*) We have $\mathfrak{P} \in \mathcal{DB}^{\mathrm{sym}}$ if and only if there exists a function $E \in \mathcal{HB}_{<\infty}^{\mathrm{sym}}$ such that $\mathfrak{P} = \mathfrak{P}(E)$. In this case, the function $E \in \mathcal{HB}_{<\infty}^{\mathrm{sym}}$ in this representation is unique.

(*iv*) We have $\mathfrak{P} \in \mathcal{DB}^{\mathrm{sb}}$ if and only if there exists a function $E \in \mathcal{HB}_{<\infty}^{\mathrm{sb}}$ such that $E \in \mathcal{HB}_{<\infty}^{\mathrm{sb}}$.

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2.2. A perturbation of inner products

In the sequel it is important to be more precise when talking about equality of de Branges spaces, since we often face the situation that two de Branges spaces $\mathfrak{P}_1$ and $\mathfrak{P}_2$ contain the same functions, but carry different inner products.

2.8 (Notation). Let $\mathfrak{P}_1$ and $\mathfrak{P}_2$ be two inner product spaces. Then we write '$\mathfrak{P}_1 = \mathfrak{P}_2$', if $\mathfrak{P}_1$ and $\mathfrak{P}_2$ contain the same elements and their inner products coincide. We write '$\mathfrak{P}_1 \overset{\text{set}}{=} \mathfrak{P}_2$' if $\mathfrak{P}_1$ and $\mathfrak{P}_2$ contain the same elements (but their inner products might be different).

The same notation applies to inclusions instead of equalities. //

Next we introduce certain perturbations of the inner product on a de Branges space.

2.9. Definition. Let a dB-space $\langle \mathfrak{P}, [.,.]_{\mathfrak{P}} \rangle$, points $t_1, \dots, t_n > 0$ and weights $\gamma_1, \dots, \gamma_n \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$, be given. Then we define perturbed inner products on $\mathfrak{P}$ as

$$[\![F, G]\!]_{\mathfrak{P}} := [F, G]_{\mathfrak{P}} + \sum_{i=1}^{n} \gamma_i F(t_i)\overline{G(t_i)}\,, \tag{2.2}$$

$$[\![F, G]\!]_{\mathfrak{P}}^{\mathsf{ev}} := [F, G]_{\mathfrak{P}} + \sum_{i=1}^{n} \frac{\gamma_i}{2}\Big(F(\sqrt{t_i})\overline{G(\sqrt{t_i})} + F(-\sqrt{t_i})\overline{G(-\sqrt{t_i})}\Big)\,, \tag{2.3}$$

$$[\![F, G]\!]_{\mathfrak{P}}^{\mathsf{od}} := [F, G]_{\mathfrak{P}} + \sum_{i=1}^{n} \frac{\gamma_i}{2t_i}\Big(F(\sqrt{t_i})\overline{G(\sqrt{t_i})} + F(-\sqrt{t_i})\overline{G(-\sqrt{t_i})}\Big)\,, \tag{2.4}$$

$$[\![F, G]\!]_{\mathfrak{P}}^{\varepsilon} := [F, G]_{\mathfrak{P}} + \varepsilon F(0)\overline{G(0)}\,. \tag{2.5}$$

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By [KW99a, Lemma 3.2] each of

$$\langle \mathfrak{P}, [\![.,.]\!]_{\mathfrak{P}} \rangle, \quad \langle \mathfrak{P}, [\![.,.]\!]_{\mathfrak{P}}^{\mathsf{ev}} \rangle, \quad \langle \mathfrak{P}, [\![.,.]\!]_{\mathfrak{P}}^{\mathsf{od}} \rangle, \quad \langle \mathfrak{P}, [\![.,.]\!]_{\mathfrak{P}}^{\varepsilon} \rangle$$

is again a dB-space.

Let us show that the perturbations (2.2)–(2.5) are compatible with notions related to symmetry.

2.10. Lemma. *Let $\langle \mathfrak{P}, [.,.]_{\mathfrak{P}} \rangle$ and $\langle \mathfrak{Q}, [.,.]_{\mathfrak{Q}} \rangle$ be dB-spaces, and let points $t_1, \dots, t_n > 0$ and weights $\gamma_1, \dots, \gamma_n \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$, be given.*

(i) We have

$$\langle \mathfrak{P}, [.,.]_{\mathfrak{P}} \rangle \in \mathcal{DB}^{\mathrm{sb}} \iff \langle \mathfrak{P}, [\![.,.]\!]_{\mathfrak{P}} \rangle \in \mathcal{DB}^{\mathrm{sb}}\,.$$

(ii) We have

$$\begin{aligned} \langle \mathfrak{Q}, [.,.]_{\mathfrak{Q}} \rangle \in \mathcal{DB}^{\mathrm{sym}} &\iff \langle \mathfrak{Q}, [\![.,.]\!]_{\mathfrak{Q}}^{\mathsf{ev}} \rangle \in \mathcal{DB}^{\mathrm{sym}} \\ &\iff \langle \mathfrak{Q}, [\![.,.]\!]_{\mathfrak{Q}}^{\mathsf{od}} \rangle \in \mathcal{DB}^{\mathrm{sym}} \\ &\iff \langle \mathfrak{P}, [\![.,.]\!]_{\mathfrak{Q}}^{\varepsilon} \rangle \in \mathcal{DB}^{\mathrm{sym}}\,. \end{aligned}$$

(iii) Assume that $\mathfrak{Q} \in \mathcal{DB}^{\mathrm{sym}}$. Then

$$\begin{aligned} \langle \mathfrak{Q}, [.,.]_{\mathfrak{Q}} \rangle^{\mathsf{ev}} = \langle \mathfrak{P}, [.,.]_{\mathfrak{P}} \rangle &\iff \langle \mathfrak{Q}, [\![.,.]\!]_{\mathfrak{Q}}^{\mathsf{ev}} \rangle^{\mathsf{ev}} = \langle \mathfrak{P}, [\![.,.]\!]_{\mathfrak{P}} \rangle \\ \langle \mathfrak{Q}, [.,.]_{\mathfrak{Q}} \rangle^{\mathsf{od}} = \langle \mathfrak{P}, [.,.]_{\mathfrak{P}} \rangle &\iff \langle \mathfrak{Q}, [\![.,.]\!]_{\mathfrak{Q}}^{\mathsf{od}} \rangle^{\mathsf{od}} = \langle \mathfrak{P}, [\![.,.]\!]_{\mathfrak{P}} \rangle \\ &\iff \langle \mathfrak{Q}, [\![.,.]\!]_{\mathfrak{Q}}^{\varepsilon} \rangle^{\mathsf{od}} = \langle \mathfrak{P}, [.,.]_{\mathfrak{P}} \rangle \end{aligned}$$

Proof. To show (i) it is enough to note that $[\![.,.]\!]_s$ and $[.,.]_s$ are finite rank perturbations of each other. In fact,

$$[\![\mathcal{S}_{\mathfrak{P}}F,G]\!]_{\mathfrak{P}}=[\mathcal{S}_{\mathfrak{P}}F,G]_{\mathfrak{P}}+\sum_{i=1}^n\gamma_i t_i F(t_i)\overline{G(t_i)},\quad F,G\in\operatorname{dom}\mathcal{S}_{\mathfrak{P}}\,.$$

For (ii), assume first that $\langle\mathfrak{Q},[.,.]_{\mathfrak{Q}}\rangle\in\mathcal{D}B^{\mathrm{sym}}$. Since $\langle\mathfrak{Q},[.,.]_{\mathfrak{Q}}\rangle\overset{\mathrm{set}}{=}\langle\mathfrak{Q},[\![.,.]\!]_{\mathfrak{Q}}^{\mathsf{ev}}\rangle$, and $\mathfrak{i}$ maps $\mathfrak{Q}$ into itself, it is enough to check isometry. We compute

$$[\![\mathfrak{i}F,\mathfrak{i}G]\!]_{\mathfrak{Q}}^{\mathsf{ev}}=[\mathfrak{i}F,\mathfrak{i}G]_{\mathfrak{Q}}+$$

$$+\sum_{i=1}^n\frac{\gamma_i}{2}\Big((\mathfrak{i}F)(\sqrt{t_i})\overline{(\mathfrak{i}G)(\sqrt{t_i})}+(\mathfrak{i}F)(-\sqrt{t_i})\overline{(\mathfrak{i}G)(-\sqrt{t_i})}\Big)=$$

$$=[F,G]_{\mathfrak{Q}}+\sum_{i=1}^n\frac{\gamma_i}{2}\Big(F(-\sqrt{t_i})\overline{G(-\sqrt{t_i})}+F(\sqrt{t_i})\overline{G(\sqrt{t_i})}\Big)=[\![F,G]\!]_{\mathfrak{Q}}^{\mathsf{ev}}$$

Thus $\langle\mathfrak{Q},[\![.,.]\!]_{\mathfrak{Q}}^{\mathsf{ev}}\rangle$ is symmetric.

If we apply this fact with the points $t_1,\dots,t_n$ and the weights $\frac{\gamma_1}{t_1},\dots,\frac{\gamma_n}{t_n}$, we obtain the implication '$\langle\mathfrak{Q},[.,.]_{\mathfrak{Q}}\rangle\in\mathcal{D}B^{\mathrm{sym}}\Rightarrow\langle\mathfrak{Q},[\![.,.]\!]_{\mathfrak{Q}}^{\mathsf{od}}\rangle\in\mathcal{D}B^{\mathrm{sym}}$'. For the implication '$\langle\mathfrak{Q},[.,.]_{\mathfrak{Q}}\rangle\in\mathcal{D}B^{\mathrm{sym}}\Rightarrow\langle\mathfrak{Q},[\![.,.]\!]_{\mathfrak{Q}}^{\varepsilon}\rangle\in\mathcal{D}B^{\mathrm{sym}}$' compute

$$[\![\mathfrak{i}F,\mathfrak{i}G]\!]_{\mathfrak{Q}}^{\varepsilon}=[\mathfrak{i}F,\mathfrak{i}G]_{\mathfrak{Q}}+\varepsilon(\mathfrak{i}F)(0)\overline{(\mathfrak{i}G)(0)}=$$

$$=[F,G]_{\mathfrak{Q}}+F(0)\overline{G(0)}=[\![F,G]\!]_{\mathfrak{Q}}^{\varepsilon}\,.$$

Applying the already proved implications with the points $t_1,\dots,t_n$ and the weights $-\gamma_1,\dots,-\gamma_n$ or $-\varepsilon$, respectively, gives the converse implications.

It remains to show (iii). In order to establish the first asserted equivalence, it is enough to show that the map $F(z)\mapsto F(z^2)$ is $[\![.,.]\!]_{\mathfrak{P}}$-to-$[\![.,.]\!]_{\mathfrak{Q}}^{\mathsf{ev}}$–isometric. However, we compute

$$[\![F(z^2),G(z^2)]\!]_{\mathfrak{Q}}^{\mathsf{ev}}=[F(z^2),G(z^2)]_{\mathfrak{Q}}+$$

$$+\sum_{i=1}^n\frac{\gamma_i}{2}\Big(F((\sqrt{t_i})^2)\overline{G((\sqrt{t_i})^2)}+F((-\sqrt{t_i})^2)\overline{G((-\sqrt{t_i})^2)}\Big)=$$

$$=[F(z),G(z)]_{\mathfrak{P}}+\sum_{i=1}^n\gamma_i F(t_i)\overline{G(t_i)}\,.$$

Similar computations show that the map $F(z)\mapsto zF(z^2)$ is $[\![.,.]\!]_{\mathfrak{P}}$-to-$[\![.,.]\!]_{\mathfrak{Q}}^{\mathsf{od}}$– and $[.,.]_{\mathfrak{P}}$-to-$[\![.,.]\!]_{\mathfrak{Q}}^{\varepsilon}$–isometric, and this gives the other asserted equivalences. ❑

2.3. Structure of symmetric dB-spaces

Next, we state some facts on the structure of symmetric dB-spaces. Remember our convention that $E \in \mathcal{HB}_{<\infty}$ includes the requirement that $E(0) = 1$. Due to this, we have $E \in \mathcal{HB}^{\mathrm{sym}}_{<\infty}$ if and only if A is even and B is odd.

2.11. Lemma. *Let $E \in \mathcal{HB}^{\mathrm{sym}}_{<\infty}$. Then*

$$\overline{\operatorname{dom} \mathcal{S}_E} \neq \mathfrak{P}(E) \iff \Big(A \in \mathfrak{P}(E) \vee B \in \mathfrak{P}(E)\Big)$$

If in addition $\overline{\operatorname{dom} \mathcal{S}_E}$ is nondegenerated, then

$$A \in \mathfrak{P}(E) \iff \Big(\overline{\operatorname{dom} \mathcal{S}_E} \neq \mathfrak{P}(E) \wedge \overline{\operatorname{dom} \mathcal{S}_E}^{\mathsf{od}} = \mathfrak{P}(E)^{\mathsf{od}}\Big)$$

$$B \in \mathfrak{P}(E) \iff \Big(\overline{\operatorname{dom} \mathcal{S}_E} \neq \mathfrak{P}(E) \wedge \overline{\operatorname{dom} \mathcal{S}_E}^{\mathsf{ev}} = \mathfrak{P}(E)^{\mathsf{ev}}\Big)$$

Proof. Since $\mathfrak{i}(zF(z)) = -z\mathfrak{i}(F(z))$, symmetry of $\mathfrak{P}(E)$ implies that $\overline{\operatorname{dom} \mathcal{S}_E}$ is invariant under the isometric involution $\mathfrak{i}_{\mathcal{DB}}$. Hence, also the orthogonal companion $\mathfrak{P}(E)[-]\overline{\operatorname{dom} \mathcal{S}_E}$ has this property.

Assume that $\overline{\operatorname{dom} \mathcal{S}_E} \neq \mathfrak{P}(E)$. Then there exists $\alpha \in [0, \pi)$ such that $\mathfrak{P}(E)[-]\overline{\operatorname{dom} \mathcal{S}_E} = \operatorname{span}\{A \cos\alpha + B \sin\alpha\}$. It follows that, for some $\lambda \in \mathbb{C}$,

$$A(z) \cos\alpha - B(z) \sin\alpha = A(-z) \cos\alpha + B(-z) \sin\alpha = \lambda\big(A(z) \cos\alpha + B(z) \sin\alpha\big) .$$

Evaluating at $z = 0$ gives $\cos\alpha = 0$ or $\lambda = 1$. In the first case, $B \in \mathfrak{P}(E)$. In the second case, it follows that $\sin\alpha = 0$ and hence $A \in \mathfrak{P}(E)$. The converse implication is clear.

If $\overline{\operatorname{dom} \mathcal{S}_E}$ is nondegenerated, then it spans together with its orthogonal complement the whole space. Hence, the asserted equivalences are immediate from what we just showed and that fact that A is even and B is odd. ❑

The characterizations in this lemma do not depend on the inner product under consideration. Hence, we obtain the following corollary.

2.12. Corollary. *Let $E_1, E_2 \in \mathcal{HB}^{\mathrm{sym}}_{<\infty}$, and assume that $\mathfrak{P}(E_1) \overset{\mathrm{set}}{=} \mathfrak{P}(E_2)$, and that $\overline{\operatorname{dom} \mathcal{S}_{E_1}}$ is a nondegenerated subspace of $\mathfrak{P}(E_1)$ and $\overline{\operatorname{dom} \mathcal{S}_{E_2}}$ is a nondegererated subspace of $\mathfrak{P}(E_2)$. Then*

$$A_1 \in \mathfrak{P}(E_1) \iff A_2 \in \mathfrak{P}(E_2), \quad B_1 \in \mathfrak{P}(E_1) \iff B_2 \in \mathfrak{P}(E_2)$$

❑

An important fact in the structure theory of de Branges spaces is that isometric inclusions of nondegenerated spaces can be characterized via certain entire matrix functions. We denote by $\mathcal{M}_{<\infty}$ the set of all entire 2×2-matrix functions W, which satisfy $\det W(z) = 1$, $z \in \mathbb{C}$, and $W(0) = I$, and have the property that the reproducing kernel

$$H_W(w, z) := \frac{W(z)JW(w)^* - J}{z - \overline{w}}$$

has a finite number of negative squares. Here J denotes the signature matrix $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

If $W \in \mathcal{M}_{<\infty}$, we denote the actual number of negative squares of the kernel H_W by $\operatorname{ind}_- W$. Each function $W \in \mathcal{M}_{<\infty}$ generates a reproducing kernel Pontryagin space via the kernel H_W; we denote this space by $\mathfrak{K}(W)$. For the afore mentioned relation bewteen isometric inclusions and matrix functions of the class $\mathcal{M}_{<\infty}$ see [KW99a, Theorem 12.2].

In the present context we need the following observation:

2.13. Lemma. *Let $E, \tilde{E} \in \mathcal{H}B^{\mathrm{sym}}_{<\infty}$ be given. Assume that $\mathfrak{P}(E) \subseteq \mathfrak{P}(\tilde{E})$, and that each dB-space $\mathfrak{P}$ with $\mathfrak{P}(E) \subsetneq \mathfrak{P} \subsetneq \mathfrak{P}(\tilde{E})$ is degenerated. Let $W \in \mathcal{M}_{<\infty}$ be the unique matrix function with*

$$(\tilde{A}, \tilde{B}) = (A, B)W, \quad \operatorname{ind}_- W = \operatorname{ind}_{\mathfrak{P}}(\tilde{E}) - \operatorname{ind}_- \mathfrak{P}(E)\,.$$

Then W is of the form

$$W = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \quad \textit{or} \quad W = \begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix}, \tag{2.6}$$

where p is a polynomial of degree $\dim \mathfrak{P}(\tilde{E})/\mathfrak{P}(E)$, and $p(0) = 0$.

If $\mathfrak{P}$ is a dB-space with $\mathfrak{P}(E) \subseteq \mathfrak{P} \subseteq \mathfrak{P}(\tilde{E})$, then

$$\mathfrak{P} = \mathfrak{P}(E)[\dot{+}]\operatorname{span}\left\{C(z)z^k : 0 \leq k < \dim \mathfrak{P}/\mathfrak{P}(E)\right\},$$

where $C = A$ or $C = B$, depending whether in (2.6) the first or the second case takes place.

Proof. Assume that W is not a matrix polynomial. Consider the dB-space $\mathfrak{P}(E_W)$ associated with W as in [KW11, §2.e]. Then $\dim \mathfrak{P}(E_W) = \infty$, and hence $\mathfrak{P}(E_W)$ contains a nondegenerated subspace which is itself a dB-space. Correspondingly, there exists a factorization $W = W_1 W_2$ with $\operatorname{ind}_- W = \operatorname{ind}_- W_1 + \operatorname{ind}_- W_2$ and $W_1 \neq I$, $W_2 \neq I$. Setting $(\hat{A}, \hat{B}) := (A, B)W_1$, we obtain a nondegenerated dB-space $\mathfrak{P}$ with $\mathfrak{P}(E) \subsetneq \mathfrak{P} \subsetneq \mathfrak{P}(\tilde{E})$, namely $\mathfrak{P} := \mathfrak{P}(\hat{E})$. This is a contradiction, and we conclude that W must be a matrix polynomial.

For $\alpha \in [0, \pi)$ denote $N_\alpha := \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$. Again since W cannot be nontrivially factorized, it must be of the form

$$W = N_\alpha \begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix} N_{-\alpha}$$

with some $\alpha \in [0, \pi)$ and some polynomial p, $p(0) = 0$, see, e.g., [KW06b, Theorem 3.1]. The space $\mathfrak{K}(W)$ is spanned by the functions ($\xi_\alpha := (\cos\alpha, \sin\alpha)^T$)

$$\xi_\alpha, z\xi_\alpha, \ldots, z^{d-1}\xi_\alpha\,,$$

where $d := \deg p$. The Gram matrix of its inner product with respect to this basis is of the form

$$\mathrm{GM}_{\mathfrak{K}(W)} = \begin{pmatrix} 0 & \cdots & 0 & c_1 \\ \vdots & \iddots & \iddots & \vdots \\ 0 & \iddots & & \vdots \\ c_1 & \cdots & \cdots & c_d \end{pmatrix}$$

with some numbers $c_i \in \mathbb{R}$, $c_1 \neq 0$, see, e.g., [KW11, Proposition 2.8]. The map

$$\begin{pmatrix} f_+ \\ f_- \end{pmatrix} \mapsto Af_+ + Bf_-$$

is an isometry of $\mathfrak{K}(W)$ onto $\mathfrak{P}(\tilde{E})[-]\mathfrak{P}(E)$. In particular, we see that $d = \dim \mathfrak{K}(W) = \dim \mathfrak{P}(\tilde{E})/\mathfrak{P}(E)$, and that

$$\mathfrak{P}(\tilde{E})[-]\mathfrak{P}(E) = \operatorname{span}\left\{C(z)z^k : k = 0, \dots, d-1\right\},$$

where $C(z) := \cos\alpha A(z) + \sin\alpha B(z)$. By Lemma 2.11 we must have either $C = A$ or $C = B$, i.e. either $\alpha = 0$ or $\alpha = \frac{\pi}{2}$.

To show the last assertion, let a dB-space $\mathfrak{P}$ with $\mathfrak{P}(E) \subseteq \mathfrak{P} \subseteq \mathfrak{P}(\tilde{E})$ be given. Let n be the maximal integer such that $\mathfrak{P}$ contains a function F of the form

$$F(z) = C(z) \cdot \Big(z^n + \sum_{k=0}^{n-1} \alpha_k z^k\Big).$$

Write $z^n + \sum_{k=0}^{n-1} \alpha_k z^k = \prod_{l=1}^{n}(z - z_l)$, then each of the functions

$$F_m(z) := C(z) \prod_{l=1}^{m}(z - z_l), \quad m = 0, \dots, n,$$

belongs to $\mathfrak{P}$. Hence, $\operatorname{span}\left\{C(z)z^k : 0 \leq k \leq n\right\} \subseteq \mathfrak{P}[-]\mathfrak{P}(E)$. The converse inclusion holds trivially, and we conclude that $\dim \mathfrak{P}/\mathfrak{P}(E) = n + 1$. ❑

2.14. Lemma. *Let $\mathfrak{Q} \in \mathcal{DB}^{\mathrm{sym}}$, and assume that $\mathfrak{Q}$ is degenerated but $\overline{\operatorname{dom} \mathcal{S}_{\mathfrak{Q}}}$ is nondegenerated. Write $\overline{\operatorname{dom} \mathcal{S}_{\mathfrak{Q}}} = \mathfrak{P}(E)$. Then*

$$\mathfrak{Q}^\circ = \operatorname{span}\{A\} \quad \textit{or} \quad \mathfrak{Q}^\circ = \operatorname{span}\{B\}.$$

Proof. Choose points $t_1, \dots, t_n > 0$ and weights $\gamma_1, \dots, \gamma_n \in \mathbb{R}$ such that the corresponding perturbation $[\![.,.]\!]_{\mathfrak{Q}}^{\mathsf{ev}}$ turns $\mathfrak{Q}$ into a Hilbert space. Choose a symmetric dB-Hilbert space $\mathfrak{Q}_1$ with

$$\langle \mathfrak{Q}, [\![.,.]\!]_{\mathfrak{Q}}^{\mathsf{ev}} \rangle \subseteq \mathfrak{Q}_1, \quad \dim \mathfrak{Q}_1/\mathfrak{Q} = 4n.$$

Such a choice is possible by successively appending indivisible intervals of the types 0 and $\frac{\pi}{2}$ to $\langle \mathfrak{Q}, [\![.,.]\!]_{\mathfrak{Q}}^{\mathsf{ev}} \rangle$, cf. the construction in [KW03, Proof of Theorem 2.3].

Now we return to the original inner product on $\mathfrak{Q}$. Denote $\mathfrak{Q}_2 := \langle \mathfrak{Q}_1, [\![.,.]\!]_{\mathfrak{Q}_1}^{\mathsf{ev}} \rangle$, where $[\![.,.]\!]_{\mathfrak{Q}_1}^{\mathsf{ev}}$ is the perturbation constructed with the points $t_1, \dots, t_n$ and weights $-\gamma_1, \dots, -\gamma_n$. Then we have

$$\mathfrak{P}(E) \subseteq \mathfrak{Q} \subseteq \mathfrak{Q}_2,$$

and $\dim \mathfrak{Q}_2/\mathfrak{Q} = 4n$.

The chain of dB-spaces $\mathfrak{P}$ with $\mathfrak{Q} \subseteq \mathfrak{P} \subseteq \mathfrak{Q}_2$ consists of $4n + 1$ spaces, each of them contained in the next larger one with codimension 1. Since $[\![.,.]\!]_{\mathfrak{Q}_1}^{\mathsf{ev}}$ is a (at most) $2n$-dimensional perturbation of the Hilbert space inner product $[.,.]_{\mathfrak{Q}_1}$, the dimension of the isotropic part of each member of this chain cannot exceed $2n$. By [KW03, Theorem 2.3], the length of a subchain of subsequent members which are all degenerated is at most $4n$. Hence, there exists a nondegenerated dB-space $\mathfrak{P}(\tilde{E})$ with $\mathfrak{Q} \subseteq \mathfrak{P}(\tilde{E}) \subseteq \mathfrak{Q}_2$. Without loss of generality, assume that $\mathfrak{P}(\tilde{E})$ is the smallest one with this property.

By Lemma 2.13, we have $(\tilde{A}, \tilde{B}) = (A, B)W$ where W is one of the matrices (2.6), and, depending which case takes place

$$\mathfrak{Q} = \mathfrak{P}(E)[\dot{+}]\,\mathrm{span}\{A\} \quad \text{or} \quad \mathfrak{Q} = \mathfrak{P}(E)[\dot{+}]\,\mathrm{span}\{B\}\,.$$

It follows that $\mathfrak{Q}^\circ$ is spanned either by A or by B. ❑

3. The 'ev'-case

As we have already indicated in the introduction, we are going to show that for each space $\mathfrak{P} \in \mathcal{DB}^{\mathrm{sb}}$ there exists a symmetric dB-space $\mathfrak{Q}$ with $\mathfrak{Q}^{\mathsf{ev}} = \mathfrak{P}$, and that the totality of all such spaces is described as a one-parameter family.

Let us define one-parameter families of spaces in a general setting.

3.1. Definition. Let $\mathcal{L}$ be a reproducing kernel almost Pontryagin space of entire functions, and let C be an entire function which does not belong to $\mathcal{L}$. Then, for each parameter $\tau \in \mathbb{R} \cup \{\infty\}$, we define an inner product space $\mathfrak{L}_\tau$ as follows: If $\tau = \infty$, we set $\mathfrak{L}_\infty := \mathfrak{L}$. If $\tau \in \mathbb{R}$, the underlying linear space is

$$\mathfrak{L}_\tau :\overset{\mathrm{set}}{=} \mathfrak{L} \dot{+} \mathrm{span}\{C\}\,,$$

and the inner product $[.,.]_\tau$ of $\mathfrak{L}_\tau$ is defined by means of its Gram matrix with respect to this direct sum decomposition as

$$\mathrm{GM}_{[.,.]_\tau} := \begin{pmatrix} [.,.]_{\mathfrak{L}} & 0 \\ 0 & \tau \end{pmatrix}\,.$$

Moreover, the space $\mathfrak{L}_\tau$ is endowed with the product topology of the topology of $\mathcal{L}$ and the euclidean topology of $\mathbb{C}$. //

3.2. *Remark.* For further reference, we state some immediate geometric properties of the family $\mathfrak{L}_\tau$, $\tau \in \mathbb{R} \cup \{\infty\}$.

- (i) The space $\mathfrak{L}_\tau$ is an almost Pontryagin space.
- (ii) For each $\tau \in \mathbb{R}$ we have $\mathfrak{L}_\tau = \mathfrak{L}[\dot{+}]\,\mathrm{span}\{C\}$.
- (iii) Assume that $\mathfrak{L}$ is nondegenerated. Then $\mathfrak{L}_\tau$ is degenerated if and only if $\tau = 0$. In this case, $\mathfrak{L}_0^\circ = \mathrm{span}\{C\}$.
- (iv) For $\tau \in \mathbb{R}$, the inner product $[.,.]_\tau$ depends continuously on τ.
- (v) If $\tau, \tau' \in \mathbb{R} \cup \{\infty\}$ are such that $\mathfrak{L}_\tau = \mathfrak{L}_{\tau'}$, then $\tau = \tau'$.

//

3.3. Theorem. *Let $\mathfrak{P} \in \mathcal{DB}^{\mathrm{sb}}$ be given. Then there exists a symmetric dB-Hilbert space $\mathring{\mathfrak{Q}} = \mathfrak{P}(\mathring{E})$, points $t_1, \dots, t_m > 0$, and weights $\omega_1, \dots, \omega_m \in \mathbb{R}$, such that the following statement holds.*

For each inner product space $\langle \mathfrak{Q}, [.,.]_{\mathfrak{Q}} \rangle$, the properties (i) and (ii) are equivalent:

- (i) $\mathfrak{Q} \in \mathcal{DB}^{\mathrm{sym}}$ *and* $\mathfrak{Q}^{\mathsf{ev}} = \mathfrak{P}$.

(*ii*) *There exists a parameter* $\tau \in \mathbb{R} \cup \{\infty\}$, *such that*

$$\mathfrak{Q} = \langle \mathring{\mathfrak{Q}}_\tau, [\![.,.]\!]^{\mathsf{ev}}_{\mathring{\mathfrak{Q}}_\tau} \rangle \,,$$

where the family $\mathring{\mathfrak{Q}}_\tau$ *is constructed with the dB-Hilbert space* $\mathring{\mathfrak{Q}}$ *and the function* $\mathring{B}$, *and* $[\![.,.]\!]^{\mathsf{ev}}_{\mathring{\mathfrak{Q}}_\tau}$ *is the perturbation of the inner product of* $\mathring{\mathfrak{Q}}_\tau$ *buildt with the points* $t_1, \ldots, t_m$ *and weights* $\omega_1, \ldots, \omega_m$.

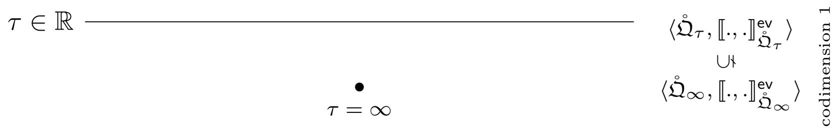
For each $\tau \in \mathbb{R}$ *we have* $\overline{\operatorname{dom} \mathcal{S}_{\mathring{\mathfrak{Q}}_\tau}} = \mathring{\mathfrak{Q}}_\infty$.

Assume in addition that:

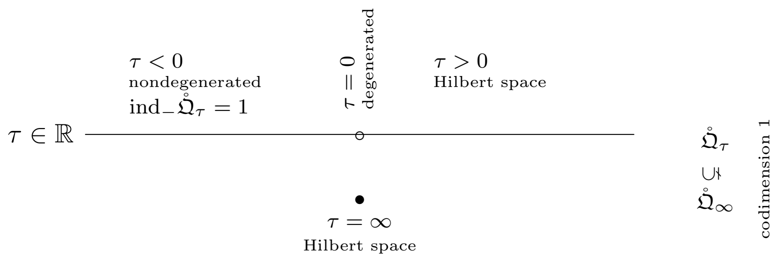
(A) *The family of all symmetric dB-spaces* $\mathfrak{Q}$ *with* $\mathfrak{Q}^{\mathsf{ev}} = \mathfrak{P}$ *contains a Hilbert space.*

Then $\mathring{\mathfrak{Q}}$ *can be chosen such that no perturbation is necessary (i.e.,* $m = 0$*).*

We can thus picture the totality of symmetric dB-spaces $\mathfrak{Q}$ with $\mathfrak{Q}^{\mathsf{ev}} = \mathfrak{P}$ as



If $\mathfrak{P}$ has property (A), we have the refined picture



In the rest of the section we give the proof Theorem 3.3[4]. Let us start with recalling three known facts[5]. First, the content of [KWW06a, Theorem 4.5, (ii_a), (ii_b)].

3.4. *Remark.* Let $E \in \mathcal{H}B^{\mathrm{sb}}_{<\infty}$ be given. For $\gamma \in \mathbb{R}$ set

$$(A_\gamma(z), B_\gamma(z)) := \big(A(z^2), \tfrac{B(z^2)}{z}\big) \begin{pmatrix} 1 & 0 \\ -\gamma z & 1 \end{pmatrix} . \tag{3.1}$$

(*i*) For each $\gamma \in \mathbb{R}$ the function E_γ belongs to the class $\mathcal{H}B^{\mathrm{sym}}_{<\infty}$, and we have $\mathfrak{P}(E_\gamma)^{\mathsf{ev}} = \mathfrak{P}(E)$.

(*ii*) If $\tilde{E} \in \mathcal{H}B^{\mathrm{sym}}_{<\infty}$ is such that $\mathfrak{P}(\tilde{E})^{\mathsf{ev}} = \mathfrak{P}(E)$, then there exists a unique parameter $\gamma \in \mathbb{R}$ with $\tilde{E} = E_\gamma$.

[4] Compared with the 'od'-case treated in the next section, the 'ev'-case is rather simple. However, the proof for the 'od'-case proceeds along the same lines and use the same ideas. Hence, it is worth to be explicit also in the 'ev'-case.

[5] We state these facts in exactly this way, in order to even more stress the analogy with the later argument for the 'od'-case.

(iii) If $\gamma, \gamma' \in \mathbb{R}$, then the functions E_γ and $E_{\gamma'}$ are related as

$$\big(A_{\gamma'}, B_{\gamma'}\big) = \big(A_\gamma, B_\gamma\big) \begin{pmatrix} 1 & 0 \\ -(\gamma-\gamma')z & 1 \end{pmatrix} .$$

//

Second, the structure of the reproducing kernel space generated by a matrix as in (3.1), see [KW11, Proposition 2.8] or [dB68].

3.5. *Remark.* Let $l \in \mathbb{R} \setminus \{0\}$, and consider the matrix function

$$T(z) := \begin{pmatrix} 1 & 0 \\ -lz & 1 \end{pmatrix} . \tag{3.2}$$

Then $T \in \mathcal{M}_{<\infty}$,

$$\operatorname{ind}_- T = \begin{cases} 0\,, & l > 0 \\ 1\,, & l < 0 \end{cases} ,$$

and the space $\mathfrak{K}(T)$ is given as

$$\mathfrak{K}(T) = \operatorname{span}\Big\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big\}, \quad \Big[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big]_{\mathfrak{K}(T)} = \frac{1}{l} .$$

//

Finally, the behaviour of dB-Pontryagin space, when (A, B) is multiplied with a matrix of the form (3.2), cf. [KW99a, Theorem 12.2, Proposition 13.5].

3.6. *Remark.* Let $E \in \mathcal{HB}_{<\infty}$ and $l \in \mathbb{R}$ be given, and set

$$\big(\tilde{A}(z), \tilde{B}(z)\big) := \big(A(z), B(z)\big) \begin{pmatrix} 1 & 0 \\ -lz & 1 \end{pmatrix} .$$

Then $\tilde{E} \in \mathcal{HB}_{<\infty}$. Moreover, we have:

(i) Assume that $B \notin \mathfrak{P}(E)$. Then

$$\mathfrak{P}(E_l) = \mathfrak{P}(E)[\dot{+}]_{E_l} \operatorname{span}\{B\}, \quad [B, B]_{E_l} = \frac{1}{l} .$$

(ii) Assume that $B \in \mathfrak{P}(E)$. Then

$$\overline{\operatorname{dom} \mathcal{S}_E} \subseteq \mathfrak{P}(\tilde{E}) \overset{\text{set}}{=} \begin{cases} \overline{\operatorname{dom} \mathcal{S}_E} \\ \mathfrak{P}(E) \end{cases} ,$$

where the bracket on the right just means that either the first or the second case takes place.

For later reference, let us remark that the analogous statements hold when considering

$$\big(\tilde{A}(z), \tilde{B}(z)\big) := \big(A(z), B(z)\big) \begin{pmatrix} 1 & lz \\ 0 & 1 \end{pmatrix} ,$$

provided the function B is replaced everywhere by the function A. //

Proof (of Theorem 3.3*; Part 1: Reduction).* We show that we may without loss of generality restrict to the case that $\mathfrak{P}$ has property (A).

Choose points $t_1, \ldots, t_n > 0$ and weights $\gamma_1, \ldots, \gamma_n > 0$, such that the corresponding perturbation $[\![.,.]\!]_{\mathfrak{P}}$ of $[.,.]_{\mathfrak{P}}$ turns $\mathfrak{P}$ into a Hilbert space. Write $\langle \mathfrak{P}, [\![.,.]\!]_{\mathfrak{P}} \rangle = \mathfrak{P}(E)$ with $E \in \mathcal{H}B^{\mathrm{sb}}_{<\infty}$, $\operatorname{ind}_- E = 0$, cf. Lemma 2.10, ($i$), and let the function $E_0 \in \mathcal{H}B^{\mathrm{sym}}_{<\infty}$ be defined by (3.1) with $\gamma = 0$. Then

$$\langle \mathfrak{P}(E_0), [.,.]_{E_0} \rangle^{\mathsf{ev}} = \langle \mathfrak{P}, [\![.,.]\!]_{\mathfrak{P}} \rangle \,,$$

cf. Remark 3.4, (i).

Next, choose points $t_{n+1}, \ldots, t_m > 0$ and weights $\gamma_{n+1}, \ldots, \gamma_m > 0$ such that the corresponding perturbation $[\![.,.]\!]^{\mathsf{ev}}_{E_0}$ of $[.,.]_{E_0}$ turns $\mathfrak{P}(E_0)$ into a Hilbert space. Moreover, let $[\![.,.]\!]'_{\mathfrak{P}}$ be the perturbation of $[\![.,.]\!]_{\mathfrak{P}}$ on $\mathfrak{P}$ using $t_{n+1}, \ldots, t_m$ and $\gamma_{n+1}, \ldots, \gamma_m$. Then, by Lemma 2.10, ($iii$),

$$\langle \mathfrak{P}(E_0), [\![.,.]\!]^{\mathsf{ev}}_{E_0} \rangle^{\mathsf{ev}} = \langle \mathfrak{P}, [\![.,.]\!]'_{\mathfrak{P}} \rangle \,. \tag{3.3}$$

Assuming it is already proved that the totality of all symmetric dB-spaces $\mathfrak{Q}'$ with $(\mathfrak{Q}')^{\mathsf{ev}} = \langle \mathfrak{P}, [\![.,.]\!]'_{\mathfrak{P}} \rangle$ is described by a family $\mathring{\mathfrak{Q}}_\tau$, $\tau \in \mathbb{R} \cup \{\infty\}$, we may return to the original inner product on $\mathfrak{P}$ by performing the perturbation with points $t_1, \ldots, t_n$ and weights $-\gamma_1, \ldots, -\gamma_m$, and in this way obtain the desired description of the family of all symmetric dB-spaces $\mathfrak{Q}$ with $\mathfrak{Q}^{\mathsf{ev}} = \langle \mathfrak{P}, [.,.]_{\mathfrak{P}} \rangle$. By (3.3), the space $\langle \mathfrak{P}, [\![.,.]\!]'_{\mathfrak{P}} \rangle$ satisfies the additional condition. ❑

From now on we assume that (A) holds, and fix a dB-Hilbert space $\mathring{\mathfrak{Q}}$ with $\mathring{\mathfrak{Q}}^{\mathsf{ev}} = \mathfrak{P}$. Since $\mathring{\mathfrak{Q}}$, and hence also $\mathfrak{P}$ is a Hilbert space, we may choose $\mathring{E} \in \mathcal{H}B^{\mathrm{sym}}$ and $E \in \mathcal{H}B^{\mathrm{sb}}$ with $\operatorname{ind}_- \mathring{E} = \operatorname{ind}_- E = 0$, such that $\mathring{\mathfrak{Q}} = \mathfrak{P}(\mathring{E})$ and $\mathfrak{P} = \mathfrak{P}(E)$. Let $\mathring{\gamma} \in \mathbb{R}$ be the unique parameter, such that $\mathring{E} = E_{\mathring{\gamma}}$, cf. Remark 3.4, ($ii$).

If $\mathring{B} \in \mathfrak{P}(\mathring{E})$, the space $\mathfrak{P}(\hat{E})$ with

$$(\hat{A}, \hat{B}) := (\mathring{A}, \mathring{B}) \begin{pmatrix} 1 & 0 \\ \frac{1}{[\mathring{A}, \mathring{A}]_{\mathring{E}}} z & 1 \end{pmatrix}$$

is again a dB-Hilbert space, in fact, $\mathfrak{P}(\hat{E}) = \overline{\operatorname{dom} \mathcal{S}_{\mathring{E}}}$. It again has the property that $\mathfrak{P}(\hat{E})^{\mathsf{ev}} = \mathfrak{P}$. Hence, we may assume from the start that $\mathring{B} \notin \mathring{\mathfrak{Q}}$.

Proof (of Theorem 3.3*; Part 2: Properties of $\mathring{\mathfrak{Q}}_\tau$).* If $\tau = \infty$, we have $\mathring{\mathfrak{Q}}_\tau = \mathring{\mathfrak{Q}}$, and hence this case is trivial. Next we deal with the case that $\tau \in \mathbb{R} \setminus \{0\}$. By Remark 3.4, we have $\mathring{E} = E_{\mathring{\gamma}}$ with some $\mathring{\gamma} \in \mathbb{R}$. Consider the space $\mathfrak{P}(E_{\mathring{\gamma}+\frac{1}{\tau}})$. Since

$$\big(A_{\mathring{\gamma}+\frac{1}{\tau}}, B_{\mathring{\gamma}+\frac{1}{\tau}}\big) = \big(A_{\mathring{\gamma}}, B_{\mathring{\gamma}}\big) \begin{pmatrix} 1 & 0 \\ -\frac{1}{\tau} & 1 \end{pmatrix} ,$$

we obtain from Remark 3.6, (i), that

$$\mathfrak{P}(E_{\mathring{\gamma}+\frac{1}{\tau}}) = \mathring{\mathfrak{Q}}[\dot{+}] \operatorname{span}\{\mathring{B}\}, \quad [\mathring{B}, \mathring{B}] = \tau \,.$$

Comparing with the definition of $\mathring{\mathfrak{Q}}_\tau$, we conclude that

$$\mathring{\mathfrak{Q}}_\tau = \mathfrak{P}(E_{\mathring{\gamma}+\frac{1}{\tau}}), \quad \tau \in \mathbb{R} \setminus \{0\} \,. \tag{3.4}$$

It follows from Remark 3.4 that $\mathring{\mathfrak{Q}}_\tau^{\mathsf{ev}} = \mathfrak{P}$. Clearly, $\mathring{\mathfrak{Q}}_\tau$ is a Hilbert space if $\tau > 0$, and a Pontryagin space with negative index 1 if $\tau < 0$ (a maximal negative subspace being spanned by $\mathring{B}$).

It remains to consider the case that $\tau = 0$. However, we have $\mathring{\mathfrak{Q}}_0 \overset{\text{set}}{=} \mathring{\mathfrak{Q}}_\tau$, $\tau \in \mathbb{R} \setminus \{0\}$, and

$$[F, G]_0 = \lim_{\tau \to 0} [F, G]_\tau, \quad F, G \in \mathring{\mathfrak{Q}}_0 \,.$$

This implies that $\mathring{\mathfrak{Q}}_0$ is a dB-space, and that $\mathring{\mathfrak{Q}}_0^{\mathsf{ev}} = \mathfrak{P}$. Moreover, obviously, $\mathring{\mathfrak{Q}}_0$ is degenerated (its symmetric part being spanned by $\mathring{B}$). ❑

Proof (of Theorem 3.3*; Part 3: The family $\mathring{\mathfrak{Q}}_\tau$ exhausts all).* Let a symmetric dB-space $\mathfrak{Q}$ with $\mathfrak{Q}^{\mathsf{ev}} = \mathfrak{P}$ be given. The case that $\mathfrak{Q}$ is nondegenerated is easily settled: By Remark 3.4 there exists a parameter $\gamma \in \mathbb{R}$ with $\mathfrak{Q} = \mathfrak{P}(E_\gamma)$. If $\gamma = \mathring{\gamma}$, we have $\mathfrak{Q} = \mathring{\mathfrak{Q}}_\infty$. Otherwise, by (3.4),

$$\mathfrak{P}(E_\gamma) = \mathring{\mathfrak{Q}}_{\frac{1}{\gamma - \mathring{\gamma}}} \,.$$

From now on assume that $\mathfrak{Q}$ is degenerated. This case is more involved.

Case '$\mathfrak{Q}$ degenerated'; $\mathfrak{Q}$ as a set: Choose points $s_1, \ldots, s_n > 0$ and weights $\delta_1, \ldots, \delta_n > 0$, such that the correspondingly perturbed inner product $[\![.,.]\!]_{\mathfrak{Q}}^{\mathsf{ev}}$ turns $\mathfrak{Q}$ into a Pontryagin space. Set $\overset{\infty}{\mathfrak{Q}} := \mathring{\mathfrak{Q}}_1$, then $\overset{\infty}{\mathfrak{Q}}$ is a Hilbert space and $\overset{\infty}{\mathfrak{Q}}^{\mathsf{ev}} = \mathfrak{P}$. Since all weights δ_i are positive, the corresponding perturbation $[\![.,.]\!]_{\overset{\infty}{\mathfrak{Q}}}^{\mathsf{ev}}\rangle$ of $[.,.]_{\overset{\infty}{\mathfrak{Q}}}$ turns $\langle\overset{\infty}{\mathfrak{Q}}$ again into Hilbert space. Let $[\![.,.]\!]_{\mathfrak{P}}$ be the corresponding perturbation of $[.,.]_{\mathfrak{P}}$, then we can write

$$\begin{aligned}
\langle \mathfrak{P}, [\![.,.]\!]_{\mathfrak{P}} \rangle &= \mathfrak{P}(E^+) \quad \text{with } E^+ \in \mathcal{H}B^{\text{sb}}_{<\infty}, \operatorname{ind}_- E^+ = 0 \,, \\
\langle \mathfrak{Q}, [\![.,.]\!]_{\mathfrak{Q}}^{\mathsf{ev}} \rangle &= \mathfrak{P}(\tilde{E}^+) \quad \text{with } \tilde{E}^+ \in \mathcal{H}B^{\text{sym}}_{<\infty}, \\
\langle \overset{\infty}{\mathfrak{Q}}, [\![.,.]\!]_{\overset{\infty}{\mathfrak{Q}}}^{\mathsf{ev}} \rangle &= \mathfrak{P}(\overset{\infty}{E}{}^+) \quad \text{with } \overset{\infty}{E}{}^+ \in \mathcal{H}B^{\text{sym}}_{<\infty}, \operatorname{ind}_- \overset{\infty}{E}{}^+ = 0 \,.
\end{aligned}$$

By Lemma 2.10, (iii),

$$\mathfrak{P}(\tilde{E}^+)^{\mathsf{ev}} = \mathfrak{P}(\overset{\infty}{E}{}^+)^{\mathsf{ev}} = \mathfrak{P}(E^+) \,.$$

From Remark 3.4 we obtain parameters $\tilde{\gamma} \in \mathbb{R}$ and $\overset{\infty}{\gamma} \in \mathbb{R}$, such that $\tilde{E}^+ = E^+_{\tilde{\gamma}}$ and $\overset{\infty}{E}{}^+ = E^+_{\overset{\infty}{\gamma}}$, and hence

$$(\tilde{A}^+, \tilde{B}^+) = (\overset{\infty}{A}{}^+, \overset{\infty}{B}{}^+) \begin{pmatrix} 1 & 0 \\ -(\tilde{\gamma} - \overset{\infty}{\gamma})z & 1 \end{pmatrix} .$$

Since $\mathring{B} \notin \mathring{\mathfrak{Q}}$, we have $\overset{\circ\circ}{B} \in \overset{\circ\circ}{\mathfrak{Q}}$. By Corollary 2.12, thus also $\overset{\circ\circ}{B}{}^{+} \in \mathfrak{P}(\overset{\circ\circ}{E}{}^{+})$. We may apply Remark 3.6, (ii), and it follows that

$$\overline{\operatorname{dom} \mathcal{S}_{\overset{\circ\circ}{E}{}^{+}}} \subseteq \mathfrak{P}(\tilde{E}^{+}) \overset{\text{set}}{=} \begin{cases} \overline{\operatorname{dom} \mathcal{S}_{\overset{\circ\circ}{E}{}^{+}}} \\ \mathfrak{P}(\overset{\circ\circ}{E}{}^{+}) \end{cases} .$$

Returning to the unperturbed inner products, yields that

$$\mathring{\mathfrak{Q}} \subseteq \mathfrak{Q} \overset{\text{set}}{=} \begin{cases} \mathring{\mathfrak{Q}} \\ \overset{\circ\circ}{\mathfrak{Q}} \end{cases} .$$

Case '$\mathfrak{Q}$ degenerated'; Finish of proof: If we had $\mathfrak{Q} \overset{\text{set}}{=} \mathring{\mathfrak{Q}}$, we would in fact have $\mathfrak{Q} = \mathring{\mathfrak{Q}}$. This contradicts the fact that $\mathfrak{Q}$ is degenerated. Hence, $\mathfrak{Q} \overset{\text{set}}{=} \overset{\circ\circ}{\mathfrak{Q}}$, and $\overline{\operatorname{dom} \mathcal{S}_{\mathfrak{Q}}} = \mathring{\mathfrak{Q}}$ is nondegenerated. Lemma 2.14 implies that either $\mathfrak{Q}^{\circ} = \operatorname{span}\{\mathring{A}\}$ or $\mathfrak{Q}^{\circ} = \operatorname{span}\{\mathring{B}\}$. The first case is ruled out since $\mathfrak{Q}^{\mathsf{ev}} = \mathfrak{P} = \mathring{\mathfrak{Q}}^{\mathsf{ev}}$. We see that $\mathfrak{Q} = \mathfrak{Q}_0$. ❑

4. The 'od'-case

Interestingly, the situation for the 'od'-branch is much more complex. Again, let us first introduce two-parameter families of inner product spaces on a general level.

4.1. Definition. Let $\mathcal{L}$ be a reproducing kernel almost Pontryagin space of entire functions, let $p \in \mathbb{R} \setminus \{0\}$, and let C and D be entire functions with $C(0) = 1$, $D(0) = -1$ which are linearly independent modulo $\mathcal{L}$. Then, for each pair of parameters $l \in \mathbb{R}$, $q \in \mathbb{R} \cup \{\infty\}$, we define an inner product space $\mathfrak{L}_{l,q}$ as follows.

(i) The underlying linear space of $\mathfrak{L}_{l,q}$ is

$$\mathfrak{L}_{l,q} :\overset{\text{set}}{=} \begin{cases} \mathfrak{L} \dot{+} \operatorname{span}\{C\} & , \quad q = \frac{p}{1-pl} \\ \mathfrak{L} \dot{+} \operatorname{span}\{C\} \dot{+} \operatorname{span}\{D\}, & q \neq \frac{p}{1-pl} \end{cases}$$

where $\frac{p}{1-pl}$ is understood as ∞ if $l = \frac{1}{p}$.

(ii) The inner product $[.,.]_{l,q}$ of $\mathfrak{L}_{l,q}$ is defined by means of its Gram matrix with respect to the direct sum decomposition written in (i):

$$\operatorname{GM}_{[.,.]_{l,q}} := \begin{cases} \begin{pmatrix} [.,.]_{\mathfrak{L}} & 0 \\ 0 & \frac{1}{p} - l \end{pmatrix} & , \quad q = \frac{p}{1-pl} \\[2ex] \left(\begin{array}{c|cc} [.,.]_{\mathfrak{L}} & 0 & 0 \\ \hline 0 & \frac{1}{p} - l & l \\ 0 & l & \frac{q(l^2 p) + (1+pl)}{q(1-pl) - p} \end{array}\right), & q \neq \frac{p}{1-pl} \end{cases}$$

where, in the second case, the right lower entry takes its natural (limit-) value if $q = \infty$.

Moreover, the space $\mathfrak{L}_{l,q}$ is endowed with the product topology of the topology of $\mathcal{L}$ and the euclidean topology of $\mathbb{C}$ or $\mathbb{C}^2$, respectively. //

4.2. *Remark.* Again, let us immediately state some simple geometric properties of the family $\mathfrak{L}_{l,q}$, $l \in \mathbb{R}$, $q \in \mathbb{R} \cup \{\infty\}$.

(i) The space $\mathfrak{L}_{l,q}$ is an almost Pontryagin space.

(ii) Assume that $\mathfrak{L}$ is nondegenerated. Then $\mathfrak{L}_{l,q}$ is degenerated if and only if $q = \infty$. We have

$$\mathfrak{L}_{l,\infty}^{\circ} = \begin{cases} \operatorname{span}\{C\} & , \quad q = \frac{p}{1-pl} \\ \operatorname{span}\{lC + (l - \frac{1}{p})D\}, & q \neq \frac{p}{1-pl} \end{cases}$$

(iii) The inner product $[.,.]_{l,q}$ depends continuously on (l,q), if (l,q) varies in either of the sets

$$M_s := \Big\{(l,q) \in \mathbb{R} \times (\mathbb{R} \cup \{\infty\}) : \, q = \frac{p}{1-pl}\Big\},$$
$$M_b := \Big\{(l,q) \in \mathbb{R} \times (\mathbb{R} \cup \{\infty\}) : \, q \neq \frac{p}{1-pl}\Big\}.$$

(iv) We have $\mathfrak{L}_{l,q} \overset{\text{set}}{=} \mathfrak{L}_{l',q'}$ whenever (l,q) and (l',q') either both belong to M_s, or both belong to M_b. If $(l,q) \in M_s$ and $(l',q') \in M_b$, then $\mathfrak{L}_{l,q} \overset{\text{set}}{\subseteq} \mathfrak{L}_{l',q'}$ and $\dim \mathfrak{L}_{l',q'}/\mathfrak{L}_{l,q} = 1$. If, in addition, $l = l'$, then $\mathfrak{L}_{l,q} \subseteq \mathfrak{L}_{l',q}$.

(v) If $l, l' \in \mathbb{R}$ and $q, q' \in \mathbb{R} \cup \{\infty\}$ are such that $\mathfrak{L}_{l,q} = \mathfrak{L}_{l',q'}$, then $l = l'$ and $q = q'$.

(vi) For $l \in \mathbb{R}$ denote by $\mathring{q}(l) \in \mathbb{R} \cup \{\infty\}$ the unique number such that $(l, \mathring{q}(l)) \in M_s$. Then

$$[.,.]_{l,q}|_{\mathfrak{L}_{l,\mathring{q}(l)} \times \mathfrak{L}_{l,\mathring{q}(l)}} = [.,.]_{l,\mathring{q}(l)}, \quad q \in \mathbb{R} \cup \{\infty\}.$$

If $l \neq l'$, then $[.,.]_{l,\mathring{q}(l)} \neq [.,.]_{l',\mathring{q}(l')}$.

//

4.3. Theorem. *Let $\mathfrak{P} \in \mathcal{D}B^{\text{sb}}$ be given. Then there exists a reproducing kernel Hilbert space $\mathfrak{L}$ of entire functions, a number $p > 0$, even entire functions C and D with $C(0) = 1$, $D(0) = -1$, which are linearly independent modulo $\mathfrak{L}$, points $t_1, \ldots, t_m > 0$ and weights $\omega_1, \ldots, \omega_m \in \mathbb{R}$, such that the following statement holds.*

For each inner product space $\langle \mathfrak{Q}, [.,.]_{\mathfrak{Q}} \rangle$, the properties ($i$) and ($ii$) are equivalent:

(i) *$\mathfrak{Q} \in \mathcal{D}B^{\text{sym}}$ and $\mathfrak{Q}^{\mathsf{od}} = \mathfrak{P}$..*

(ii) *There exist parameters $l \in \mathbb{R}$, $q \in \mathbb{R} \cup \{\infty\}$, such that*

$$\mathfrak{Q} = \langle \mathfrak{L}_{l,q}, [\![.,.]\!]_{\mathfrak{L}_{l,q}}^{\mathsf{od}} \rangle$$

where the family $\mathfrak{L}_{l,q}$ is constructed from $\mathfrak{L}, p, C, D$, and $[\![.,.]\!]_{\mathfrak{L}_{l,q}}^{\mathsf{od}}$ is the perturbation of the inner product of $\mathfrak{L}_{l,q}$ buildt with the points $t_1, \ldots, t_m$ and weights $\omega_1, \ldots, \omega_m$.

The choice of $\mathfrak{L}, p, C, D$ can be made such that

$$\mathfrak{L} = \begin{cases} \operatorname{ran} \mathcal{S}_{\mathfrak{L}_{l,q}}, & (l,q) \in M_s \\ \operatorname{ran} \mathcal{S}_{\mathfrak{L}_{l,q}} \cap \overline{\operatorname{dom} \mathcal{S}_{\mathfrak{L}_{l,q}}}, & (l,q) \in M_b \end{cases}$$

and, in case $(l,q) \in M_b$,

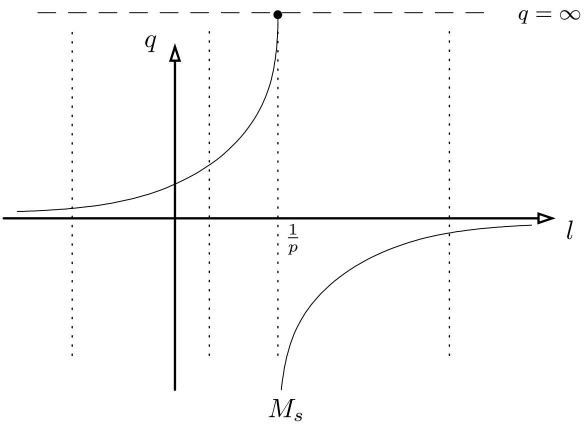
$$\operatorname{ran} \mathcal{S}_{\mathfrak{L}_{l,q}} = \mathfrak{L} \dot{+} \operatorname{span}\{C + D\}, \quad \overline{\operatorname{dom} \mathcal{S}_{\mathfrak{L}_{l,q}}} = \mathfrak{L} \dot{+} \operatorname{span}\{C\}.$$

Assume in addition that:

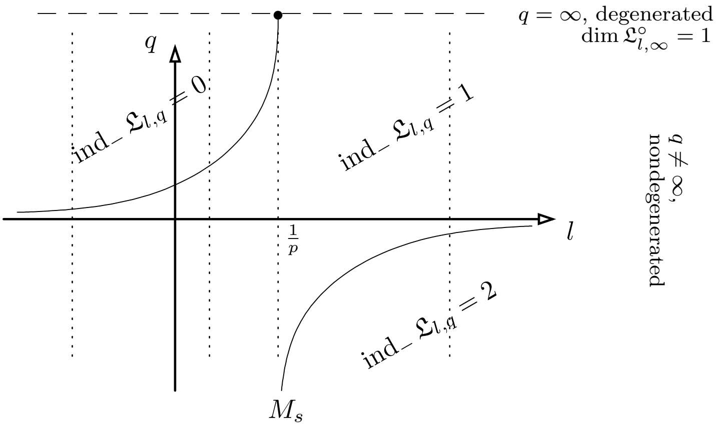
(A) *The family of all symmetric dB-spaces $\mathfrak{Q}$ with $\mathfrak{Q}^{\mathsf{od}} = \mathfrak{P}$ contains a Hilbert space.*

Then the choice of $\mathfrak{L}, p, C, D$ can be made in such a way that no perturbation is necessary.

We can thus picture the totality of symmetric dB-spaces $\mathfrak{Q}$ with $\mathfrak{Q}^{\mathsf{od}} = \mathfrak{P}$ as



If $\mathfrak{P}$ satisfies (A), we have the following refinement of this picture:



For the proof of this theorem, we mimic the proof of the 'ev'-case. First, the required replacements of Remark 3.4–Remark 3.6.

4.4. Lemma. *Let $E \in \mathcal{HB}^{\mathrm{sb}}_{<\infty}$ be given. For $\gamma, \delta \in \mathbb{R}$ set*

$$\big(A_{\gamma,\delta}(z), B_{\gamma,\delta}(z)\big) := \big(A(z^2), zB(z^2)\big) \begin{pmatrix} 1 & \delta z \\ \frac{\gamma}{z} & 1+\delta\gamma \end{pmatrix} . \tag{4.1}$$

Then the following hold:

(*i*) *For each $\gamma, \delta \in \mathbb{R}$ the function $E_{\gamma,\delta}$ belongs to the class $\mathcal{HB}^{\mathrm{sym}}_{<\infty}$, and we have $\mathfrak{P}(E_{\gamma,\delta})^{\mathsf{od}} = \mathfrak{P}(E)$.*

(*ii*) *If $\tilde{E} \in \mathcal{HB}^{\mathrm{sym}}_{<\infty}$ is such that $\mathfrak{P}(\tilde{E})^{\mathsf{od}} = \mathfrak{P}(E)$, then there exist unique parameters $\gamma, \delta \in \mathbb{R}$ with $\tilde{E} = E_{\gamma,\delta}$.*

(*iii*) *Let $\gamma, \delta \in \mathbb{R}$ and $\gamma', \delta' \in \mathbb{R}$ be given. Then the functions $E_{\gamma,\delta}$ and $E_{\gamma',\delta'}$ are related by*

$$\big(A_{\gamma',\delta'}, B_{\gamma',\delta'}\big) = \big(A_{\gamma,\delta}, B_{\gamma,\delta}\big) \begin{pmatrix} 1-\delta(\gamma'-\gamma) & z\big[(\delta'-\delta) - \delta\delta'(\gamma'-\gamma)\big] \\ \frac{\gamma'-\gamma}{z} & 1+\delta'(\gamma'-\gamma) \end{pmatrix} .$$

Proof. Let $\gamma, \delta \in \mathbb{R}$ be given. Clearly, $A_{\gamma,\delta}$ is even and $B_{\gamma,\delta}$ is odd. We can rewrite

$$\frac{B_{\gamma,\delta}(z)}{A_{\gamma,\delta}(z)} = \delta z - \Big[- \Big[z\frac{B(z^2)}{A(z^2)}\Big]^{-1} - \frac{\gamma}{z}\Big]^{-1} .$$

Using notation and results of [KWW06b, Theorem 4.1], since $\frac{B}{A} \in \mathcal{N}^{\mathrm{ep}}_{<\infty}$, it follows that $\frac{B_{\gamma,\delta}(z)}{A_{\gamma,\delta}(z)} \in \mathcal{N}_{<\infty}$. The matrix on the right hand side of (4.1) has determinant 1 for all $z \in \mathbb{C} \setminus \{0\}$, and hence $A_{\gamma,\delta}$ and $B_{\gamma,\delta}$ have no common zeros in $\mathbb{R} \setminus \{0\}$. Moreover, $A_{\gamma,\delta}(0) = 1$ and $B_{\gamma,\delta}(0) = 0$. It follows that $E_{\gamma,\delta} \in \mathcal{HB}^{\mathrm{sym}}_{<\infty}$.

From [KWW06a, Proposition 4.9] we obtain that the space $\mathfrak{P}(E_{\gamma,\delta})^{\mathsf{od}}$ is generated by the function $E^{\mathsf{od}}_{\gamma,\delta}$ with

$$A^{\mathsf{od}}_{\gamma,\delta}(z^2) = A_{\gamma,\delta}(z), \quad B^{\mathsf{od}}_{\gamma,\delta}(z^2) = \frac{B_{\gamma,\delta}(z)}{z} - B'_{\gamma,\delta}(0)A_{\gamma,\delta}(z) .$$

Substituting the definitions of $A_{\gamma,\delta}$ and $B_{\gamma,\delta}$, and remembering that $B'_{\gamma,\delta}(0) = \delta$, gives

$$\big(A^{\mathsf{od}}_{\gamma,\delta}, B^{\mathsf{od}}_{\gamma,\delta}\big) = (A, B) \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} ,$$

and hence $\mathfrak{P}(E^{\mathsf{od}}_{\gamma,\delta}) = \mathfrak{P}(E)$. This finishes the proof of (*i*).

For the proof of existence in (*ii*), let a function $\tilde{E} \in \mathcal{HB}^{\mathrm{sym}}_{<\infty}$ with $\mathfrak{P}(\tilde{E})^{\mathsf{od}} = \mathfrak{P}(E)$ be given. Since $\mathfrak{P}(E) = \mathfrak{P}(\tilde{E})^{\mathsf{od}} = \mathfrak{P}(\tilde{E}^{\mathsf{od}})$, there exists a number $\gamma \in \mathbb{R}$ with

$$(\tilde{A}^{\mathsf{od}}, \tilde{B}^{\mathsf{od}}) = (A, B) \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} .$$

This gives

$$\tilde{A}(z) = \tilde{A}^{\mathsf{od}}(z^2) = A(z^2) + \gamma B(z^2) ,$$

$$\frac{\tilde{B}(z)}{z} - \tilde{B}'(0)\tilde{A}(z) = \tilde{B}^{\mathsf{od}}(z^2) = B(z^2) .$$

The second relation rewrites as

$$\tilde{B}(z) = zB(z^2)+z\tilde{B}'(0)\big(A(z^2)+\gamma B(z^2)\big) = z\tilde{B}'(0)A(z^2)+\big(1+\tilde{B}'(0)\gamma\big)zB(z^2)\,,$$

and we see that $\tilde{E} = E_{\gamma,\tilde{B}'(0)}$.

For the proof of uniqueness, assume that $\gamma,\delta \in \mathbb{R}$ and $\gamma',\delta' \in \mathbb{R}$ are such that $\mathfrak{P}(E_{\gamma,\delta}) = \mathfrak{P}(E_{\gamma',\delta'})$. This implies that $E_{\gamma,\delta} = E_{\gamma',\delta'}$, remember Remark 2.7, (iii). We obtain the equations

$$A(z^2) + \gamma B(z^2) = A(z^2) + \gamma' B(z^2)\,,$$

$$\delta z A(z^2) + (1+\delta\gamma)zB(z^2) = \delta' z A(z^2) + (1+\delta'\gamma')zB(z^2)\,.$$

The first equation implies that $\gamma = \gamma'$ (note that B cannot vanish identically, cf. Remark 2.5). Since (again Remark 2.5) the functions A and B are linearly independent, the second equation implies $\delta = \delta'$.

The formula asserted in (iii) follows by a straightforward computation.

❏

4.5. Lemma. *Let $a,b,c,d \in \mathbb{R}$ with $ad-bc = 1$ be given, assume that not both of b and c are equal to zero, and consider the matrix function*

$$T(z) := \begin{pmatrix} a & bz \\ \frac{c}{z} & d \end{pmatrix}, \quad z \in \mathbb{C}\setminus\{0\}\,.$$

Then the kernel

$$K_T(w,z) := \frac{T(z)JT(w)^* - J}{z-\overline{w}}, \quad z,w \in \mathbb{C}\setminus\{0\} \tag{4.2}$$

has a finite number of negative squares, in fact,

$$\operatorname{ind}_- T = \begin{cases} 0\,, & c=0, ab>0 \;\vee\; b=0, cd>0 \;\vee \\ & b,c,d\neq 0, \operatorname{sgn} b = \operatorname{sgn} c = \operatorname{sgn} d \\[1ex] 1\,, & c=0, ab<0 \;\vee\; b=0, cd<0 \;\vee \\ & b,c,\neq 0, d=0 \;\vee \\ & b,c,d\neq 0, \operatorname{sgn} b \neq \operatorname{sgn} c \\[1ex] 2\,, & b,c,d\neq 0, \operatorname{sgn} b = \operatorname{sgn} c \neq \operatorname{sgn} d \end{cases}\,.$$

The reproducing kernel space $\mathfrak{K}(T)$ generated by the kernel K_T is given as

$$\mathfrak{K}(T) = \begin{cases} \operatorname{span}\big\{\binom{1}{0}\big\} & , \quad b\neq 0, c=0 \\ \operatorname{span}\big\{\binom{0}{\frac{1}{z}}\big\} & , \quad b=0, c\neq 0 \\ \operatorname{span}\big\{\binom{1}{0},\binom{0}{\frac{1}{z}}\big\}, & b\neq 0, c\neq 0 \end{cases}\,. \tag{4.3}$$

Its inner product is given by

$$\Big[\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\Big]_{\mathfrak{K}(T)} = \frac{1}{ab} \text{ if } c=0, \quad \Big[\begin{pmatrix}0\\\frac{1}{z}\end{pmatrix},\begin{pmatrix}0\\\frac{1}{z}\end{pmatrix}\Big]_{\mathfrak{K}(T)} = \frac{1}{cd} \text{ if } b=0\,,$$

and by the Gram matrix

$$\mathrm{GM}_{[.,.]_{\mathfrak{K}(T)}} = \begin{pmatrix} \frac{d}{b} & -1 \\ -1 & \frac{a}{c} \end{pmatrix}$$

if $b, c \neq 0$.

Proof. A computation shows that

$$K_T(w,z) = \begin{pmatrix} ab & \frac{bc}{\overline{w}} \\ \frac{bc}{z} & \frac{cd}{z\overline{w}} \end{pmatrix}, \quad z, w \in \mathbb{C} \setminus \{0\}.$$

Hence,

$$\operatorname{span}\left\{K_T(w,.)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} : \alpha,\beta\in\mathbb{C}\right\} = \operatorname{span}\left\{b\begin{pmatrix}a\\ \frac{c}{z}\end{pmatrix}, c\begin{pmatrix}b\\ \frac{d}{z}\end{pmatrix}\right\},$$

and we see that (4.3) holds. In particular, K_T has at most 2 negative squares.

In order to show the required formula for the Gram matrix of the inner product of $\mathfrak{K}(T)$, we distinguish the three cases.

Case 1; $c = 0$: In this case we have $K_T(w,z) = \begin{pmatrix} ab & 0 \\ 0 & 0\end{pmatrix}$, and hence

$$\begin{aligned}\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right] &= \frac{1}{(ab)^2}\left[\begin{pmatrix}ab\\0\end{pmatrix}, \begin{pmatrix}ab\\0\end{pmatrix}\right] = \frac{1}{(ab)^2}\left[K_T(1,.)\begin{pmatrix}1\\0\end{pmatrix}, K_T(1,.)\begin{pmatrix}1\\0\end{pmatrix}\right] = \\ &= \frac{1}{(ab)^2}\cdot\begin{pmatrix}1\\0\end{pmatrix}^* K_T(1,1)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{ab}.\end{aligned}$$

Case 2; $b = 0$: In this case we have $K_T(w,z) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{cd}{z\overline{w}}\end{pmatrix}$, and hence

$$\begin{aligned}\left[\begin{pmatrix}0\\ \frac{1}{z}\end{pmatrix}, \begin{pmatrix}0\\ \frac{1}{z}\end{pmatrix}\right] &= \frac{1}{(cd)^2}\left[\begin{pmatrix}0\\ \frac{cd}{z}\end{pmatrix} \begin{pmatrix}0\\ \frac{cd}{z}\end{pmatrix}\right] = \frac{1}{(cd)^2}\left[K_T(1,.)\begin{pmatrix}0\\1\end{pmatrix}, K_T(1,.)\begin{pmatrix}0\\1\end{pmatrix}\right] = \\ &= \frac{1}{(cd)^2}\cdot\begin{pmatrix}0\\1\end{pmatrix}^* K_T(1,1)\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{cd}.\end{aligned}$$

Case 3; $b, c \neq 0$: From $K_T(1,z)\binom{1}{0} = b\binom{a}{\frac{c}{z}}$ and $K_T(1,z)\binom{0}{1} = c\binom{b}{\frac{d}{z}}$, we obtain that

$$\left[b\begin{pmatrix}a\\ \frac{c}{z}\end{pmatrix}, b\begin{pmatrix}a\\ \frac{c}{z}\end{pmatrix}\right] = ab, \quad \left[b\begin{pmatrix}a\\ \frac{c}{z}\end{pmatrix}, c\begin{pmatrix}b\\ \frac{d}{z}\end{pmatrix}\right] = bc, \quad \left[c\begin{pmatrix}b\\ \frac{d}{z}\end{pmatrix}, c\begin{pmatrix}b\\ \frac{d}{z}\end{pmatrix}\right] = cd.$$

We have

$$\begin{pmatrix}1\\0\end{pmatrix} = \frac{d}{b}\cdot b\begin{pmatrix}a\\ \frac{c}{z}\end{pmatrix} - c\begin{pmatrix}b\\ \frac{d}{z}\end{pmatrix}, \quad \begin{pmatrix}0\\ \frac{1}{z}\end{pmatrix} = -b\begin{pmatrix}a\\ \frac{c}{z}\end{pmatrix} + \frac{a}{c}\cdot c\begin{pmatrix}b\\ \frac{d}{z}\end{pmatrix},$$

and hence

$$\begin{aligned}\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right] &= \left(\frac{d}{b}\right)^2\cdot ab - 2\frac{d}{b}\cdot bc + cd = \frac{d}{b}, \\ \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\ \frac{1}{z}\end{pmatrix}\right] &= -\frac{d}{b}\cdot ab + \frac{d}{b}\frac{a}{c}\cdot bc + bc - \frac{a}{c}\cdot cd = -1, \\ \left[\begin{pmatrix}0\\ \frac{1}{z}\end{pmatrix}, \begin{pmatrix}0\\ \frac{1}{z}\end{pmatrix}\right] &= ab - 2\frac{a}{c}\cdot bc + \left(\frac{a}{c}\right)^2\cdot cd = \frac{a}{c}.\end{aligned}$$

It remains to compute the negative index of $\mathfrak{K}(T)$. If $b = 0$ or $c = 0$, this is immediate. If $b, c \neq 0$, we apply Gundelfinger's rule, see, e.g., [Ioh74, p.48f.]:

The sequence of principal minors (in this rule the minor of order zero is formally understood as $+1$) of the Gram matrix is

$$+1,\ \frac{d}{b},\ \frac{1}{bc}\,,$$

and hence the desired formula follows. ❑

4.6. Lemma. *Let $E \in \mathcal{H}B^{\mathrm{sym}}_{<\infty}$ with $\mathfrak{P}(E)^{\mathsf{od}} \neq \{0\}$ be given, and set $p := B'(0)$. Moreover, let $q \in \mathbb{R}$ and $l \in \mathbb{R}$ be given, and set*

$$(\tilde{A}, \tilde{B}) := (A, B)\begin{pmatrix} 1 - pl & z\big[(q-p) - pql\big] \\ \frac{l}{z} & 1 + ql \end{pmatrix}$$

Then $\tilde{E} \in \mathcal{H}B^{\mathrm{sym}}_{<\infty}$.

(*i*) *Assume that $p \neq 0$ and $(q-p) - pql = 0$. Then*

$$\mathfrak{P}(\tilde{E}) \stackrel{\mathrm{set}}{=} \operatorname{ran}\mathcal{S}_E \dot{+} \operatorname{span}\Big\{\frac{B(z)}{z}\Big\} \stackrel{\mathrm{set}}{=} \mathfrak{P}(E)\,.$$

The Gram matrix of the inner product $[.,.]_{\tilde{E}}$ with respect to this decomposition is

$$\begin{pmatrix} [.,.]_E & 0 \\ 0 & p(1-pl) \end{pmatrix}$$

(*ii*) *Assume that $p \neq 0$, $A \notin \mathfrak{P}(E)$, and $(q-p) - pql \neq 0$, $1 - pl \neq 0$. Then*

$$\mathfrak{P}(\tilde{E}) \stackrel{\mathrm{set}}{=} \operatorname{ran}\mathcal{S}_E \dot{+} \operatorname{span}\Big\{\frac{B(z)}{z}\Big\} \dot{+} \operatorname{span}\big\{\tilde{A}\big\} \stackrel{\mathrm{set}}{=} \mathfrak{P}(E) \dot{+} \operatorname{span}\big\{A\big\}\,.$$

The Gram matrix of the inner product $[.,.]_{\tilde{E}}$ with respect to this decomposition is

$$\begin{pmatrix} [.,.]_E & 0 & 0 \\ 0 & p(1-pl) & 0 \\ 0 & 0 & \frac{1-pl}{(q-p)-pql} \end{pmatrix}$$

(*iii*) *Assume that $p \neq 0$, $A \notin \mathfrak{P}(E)$, and $(q-p) - pql \neq 0$, $1 - pl = 0$. Then*

$$\mathfrak{P}(\tilde{E}) \stackrel{\mathrm{set}}{=} \operatorname{ran}\mathcal{S}_E \dot{+} \Big(\operatorname{span}\big\{\tilde{A}\big\} \dot{+} \operatorname{span}\Big\{\frac{\tilde{B}(z)}{z}\Big\}\Big) \stackrel{\mathrm{set}}{=} \mathfrak{P}(E) \dot{+} \operatorname{span}\big\{A\big\}\,.$$

The Gram matrix of the inner product $[.,.]_{\tilde{E}}$ with respect to this decomposition is

$$\begin{pmatrix} [.,.]_E & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & q \end{pmatrix}$$

(*iv*) *Assume that $A \in \mathfrak{P}(E)$. Then*

$$\operatorname{ran}\mathcal{S}_E \cap \overline{\operatorname{dom}\mathcal{S}_E} \subseteq \mathfrak{P}(\tilde{E}) \stackrel{\mathrm{set}}{=} \begin{cases} \overline{\operatorname{dom}\mathcal{S}_E} \\ \mathfrak{P}(E) \end{cases}$$

Proof (of Lemma 4.6*; Part 1: Preparation).* Set

$$T(z) := \begin{pmatrix} 1-pl & z\big[(q-p)-pql\big] \\ \frac{l}{z} & 1+ql \end{pmatrix}$$

Since $B(0)=0$, the functions $\tilde{A}$ and $\tilde{B}$ are entire. Clearly, $\tilde{A}$ is even, $\tilde{B}$ is odd, and $\tilde{B}(0)=0$. Moreover, we have

$$\tilde{A}(0) = (1-pl) + lB'(0) = 1\,.$$

Since $\det T(z)=1$, $z\in\mathbb{C}\setminus\{0\}$, the functions $\tilde{A}$ and $\tilde{B}$ have no common zeros in $\mathbb{C}\setminus\{0\}$. Denote by K_T the kernel function (4.2), and by K_E and $K_{\tilde{E}}$ the respective kernel functions (2.1). Then we have

$$K_{\tilde{E}}(w,z) = K_E(w,z) + \underbrace{(A(z),B(z))\cdot K_T(w,z)\cdot (A(w),B(w))^*}_{=:K(w,z)}\,.$$

Since $\mathfrak{P}(E)^{\mathsf{od}} \neq \{0\}$, the functions $A(z)$ and $\frac{B(z)}{z}$ are linearly independent. Thus the map

$$\begin{pmatrix} \alpha \\ \frac{\beta}{z} \end{pmatrix} \mapsto \alpha A(z) + \beta\frac{B(z)}{z}, \quad \begin{pmatrix} \alpha \\ \frac{\beta}{z} \end{pmatrix} \in \mathfrak{K}(T)\,,$$

is an isometric isomorphism of $\mathfrak{K}(T)$ onto the reproducing kernel space $\mathfrak{K}$ generated by the kernel K.

It already follows that $K_{\tilde{E}}$ has a finite number of negative squares, and hence that $\tilde{E}\in\mathcal{H}B^{\mathrm{sym}}_{<\infty}$. ❑

To carry out the required closer analysis, recall how $\mathfrak{P}(\tilde{E})$ can be described considering that the kernel $K_{\tilde{E}}$ is the sum of the two the reproducing kernels K_E and K, see, e.g., [Wor11, Proposition 2.2].

4.7. *Remark.* Denote by $[.,.]_+$ the sum inner product on $\mathfrak{P}(E)\times\mathfrak{K}$, and set

$$\mathcal{D} := \big\{(F,-F):\ F\in\mathfrak{P}(E)\cap\mathfrak{K}\big\}\,.$$

Then the map $\Lambda:(F,G)\mapsto F+G$ is a continuous and surjective isometry of $\langle(\mathfrak{P}(E)\times\mathfrak{K})[-]_+\mathcal{D},[.,.]_+\rangle$ onto $\mathfrak{P}(\tilde{E})$. The spaces $\mathfrak{P}(E)[-]_E(\mathfrak{P}(E)\cap\mathfrak{K})$ and $\mathfrak{K}[-]_{\mathfrak{K}}(\mathfrak{P}(E)\cap\mathfrak{K})$ are isometrically contained in $\mathfrak{P}(\tilde{E})$ as orthogonal subspaces. //

Proof (of Lemma 4.6*; Part 2: Calculations).* If $l=0$ and $q=p$, then $\tilde{E}=E$ and thus the assertions are all clear. Hence, from now on, we exclude this case. Moreover, let us notice that the assumption '$p\neq 0$' in (i)–(iii) implies $\mathfrak{P}(E) = \operatorname{ran}\mathcal{S}_E[\dot{+}]\operatorname{span}\{\frac{B(z)}{z}\}$.

Item (i): In the present situation, we have $(1-pl)(1+ql)=1$. Moreover, if $l=0$ then $p=q$, and this case was excluded. Thus, $l\neq 0$. The space $\mathfrak{K}(T)$ is given as

$$\mathfrak{K}(T) = \operatorname{span}\Big\{\begin{pmatrix} 0 \\ \frac{1}{z}\end{pmatrix}\Big\}, \quad \Big[\begin{pmatrix} 0 \\ \frac{1}{z}\end{pmatrix},\begin{pmatrix} 0 \\ \frac{1}{z}\end{pmatrix}\Big]_{\mathfrak{K}(T)} = \frac{1}{l(1+ql)} = \frac{1}{l} - p\,.$$

The space $\mathfrak{K}$ is thus given as

$$\mathfrak{K} = \operatorname{span}\Big\{\frac{B(z)}{z}\Big\}, \quad \Big[\frac{B(z)}{z}, \frac{B(z)}{z}\Big]_{\mathfrak{K}} = \frac{1}{l} - p\,.$$

We see that $\mathfrak{K} \overset{\text{set}}{\subseteq} \mathfrak{P}(E)$, and that

$$\mathfrak{P}(E) \times \mathfrak{K} = \big(\operatorname{ran}\mathcal{S}_E \times \{0\}\big)[\dot{+}]_+\big(\mathfrak{K}\times\mathfrak{K}\big)\,.$$

Moreover,

$$\mathcal{D} = \operatorname{span}\Big\{\Big(\frac{B(z)}{z}, -\frac{B(z)}{z}\Big)\Big\}, \quad \Big[\Big(\frac{B(z)}{z}, -\frac{B(z)}{z}\Big), \Big(\frac{B(z)}{z}, -\frac{B(z)}{z}\Big)\Big]_+ = \frac{1}{l} \neq 0\,.$$

and

$$\big(\mathfrak{P}(E)\times\mathfrak{K}\big)[-]_+\mathcal{D} = \big(\operatorname{ran}\mathcal{S}_E \times \{0\}\big)[\dot{+}]_+\big((\mathfrak{K}\times\mathfrak{K})[-]_+\mathcal{D}\big)\,.$$

It already follows that

$$\operatorname{ran}\mathcal{S}_E \subseteq \mathfrak{P}(\tilde{E}) \overset{\text{set}}{=} \mathfrak{P}(E)\,,$$

and that

$$\mathfrak{P}(\tilde{E}) = \operatorname{ran}\mathcal{S}_E[\dot{+}]_+\Lambda\big((\mathfrak{K}\times\mathfrak{K})[-]_+\mathcal{D}\big)\,.$$

To compute inner products, consider the element $(\frac{B(z)}{z}, 0) - pl(\frac{B(z)}{z}, -\frac{B(z)}{z}) \in \mathfrak{K}\times\mathfrak{K}$. We have

$$\begin{aligned}
\Big[\Big(\frac{B(z)}{z}, 0\Big) - pl\Big(\frac{B(z)}{z}, -\frac{B(z)}{z}\Big), \Big(\frac{B(z)}{z}, -\frac{B(z)}{z}\Big)\Big]_+ = \\
&= (1-pl)\Big[\frac{B(z)}{z}, \frac{B(z)}{z}\Big]_E - pl\Big[\frac{B(z)}{z}, \frac{B(z)}{z}\Big]_{\mathfrak{K}} = \\
&= (1-pl)p - pl(\frac{1}{l} - p) = 0\,,
\end{aligned} \tag{4.4}$$

i.e. this element belongs to $(\mathfrak{K}\times\mathfrak{K})[-]_+\mathcal{D}$. Its image under Λ equals $\frac{B(z)}{z}$, and hence

$$\begin{aligned}
&\Big[\frac{B(z)}{z}, \frac{B(z)}{z}\Big]_{\tilde{E}} = \\
&\quad = \Big[\Big(\frac{B(z)}{z}, 0\Big) - pl\Big(\frac{B(z)}{z}, -\frac{B(z)}{z}\Big), \Big(\frac{B(z)}{z}, 0\Big) - pl\Big(\frac{B(z)}{z}, -\frac{B(z)}{z}\Big)\Big]_+ = \\
&\quad = (1-pl)^2\Big[\frac{B(z)}{z}, \frac{B(z)}{z}\Big]_E + (pl)^2\Big[\frac{B(z)}{z}, \frac{B(z)}{z}\Big]_{\mathfrak{K}} = p(1-pl)\,.
\end{aligned} \tag{4.5}$$

Item (ii): The space $\mathfrak{K}$ is given as

$$\mathfrak{K} = \operatorname{span}\Big\{A(z), \frac{B(z)}{z}\Big\}, \quad G = \begin{pmatrix} \frac{1+ql}{(q-p)-pql} & -1 \\ -1 & \frac{1}{l} - p \end{pmatrix}\,, \tag{4.6}$$

where G is the Gram matrix of the inner product $[.,.]_{\mathfrak{K}}$ with respect to the written basis.

We see that

$$\mathfrak{P}(E)\cap\mathfrak{K} = \operatorname{span}\Big\{\frac{B(z)}{z}\Big\}, \;\; \mathcal{D} = \operatorname{span}\Big\{\Big(\frac{B(z)}{z}, -\frac{B(z)}{z}\Big)\Big\}\,,$$

Since $p\neq 0$, we have

$$\mathfrak{P}(E)=\operatorname{ran}\mathcal{S}_E[\dot+]_E\operatorname{span}\Big\{\frac{B(z)}{z}\Big\}.$$

Moreover, we have $\tilde A(z)=(1-pl)A(z)+l\frac{B(z)}{z}$, and hence

$$\Big[\tilde A(z),\frac{B(z)}{z}\Big]_{\mathfrak{K}}=(1-pl)\Big[A(z),\frac{B(z)}{z}\Big]_{\mathfrak{K}}+l\Big[\frac{B(z)}{z},\frac{B(z)}{z}\Big]_{\mathfrak{K}}=0.$$

Since $1-pl\neq 0$, thus

$$\mathfrak{K}=\operatorname{span}\Big\{\frac{B(z)}{z}\Big\}[\dot+]_{\mathfrak{K}}\operatorname{span}\{\tilde A\}.$$

The same computation as carried out in (4.4) gives

$$\operatorname{span}\Big\{\frac{B(z)}{z}\Big\}\times\operatorname{span}\Big\{\frac{B(z)}{z}\Big\}=\operatorname{span}\Big\{\Big(\frac{B(z)}{z},0\Big)-pl\Big(\frac{B(z)}{z},-\frac{B(z)}{z}\Big)\Big\}[+]_+\mathcal{D}.$$

Altogether, it follows that $(\mathfrak{P}(E)\times\mathfrak{K})[-]_+\mathcal{D}$ can be written as

$$\begin{aligned}\big(\mathfrak{P}(E)\times\mathfrak{K}\big)[-]_+\mathcal{D}=&\big(\operatorname{ran}\mathcal{S}_E\times\{0\}\big)[\dot+]_+\\&[\dot+]_+\operatorname{span}\Big\{\Big(\frac{B(z)}{z},0\Big)-pl\Big(\frac{B(z)}{z},-\frac{B(z)}{z}\Big)\Big\}[\dot+]_+\\&[\dot+]_+\big(\{0\}\times\operatorname{span}\{\tilde A\}\big).\end{aligned}$$

Applying the isometry Λ, gives that $\operatorname{ran}\mathcal{S}_E\subseteq\mathfrak{P}(\tilde E)$, $\operatorname{span}\{\tilde A\}\subseteq\mathfrak{P}(\tilde E)$, and

$$\mathfrak{P}(\tilde E)=\operatorname{ran}\mathcal{S}_E[\dot+]_{\tilde E}\operatorname{span}\Big\{\frac{B(z)}{z}\Big\}[\dot+]_{\tilde E}\operatorname{span}\{\tilde A\}.$$

The same computation as in (4.5) gives $[\frac{B(z)}{z},\frac{B(z)}{z}]_{\tilde E}=p(1-pl)$. Finally, we compute

$$\begin{aligned}[\tilde A,\tilde A]_{\tilde E}=&[\tilde A,\tilde A]_{\mathfrak{K}}=\Big[(1-pl)A(z)+l\frac{B(z)}{z},(1-pl)A(z)+l\frac{B(z)}{z}\Big]_{\mathfrak{K}}=\\=&(1-pl)^2\frac{1+ql}{(q-p)-pql}-2l(1-pl)+l^2\Big(\frac{1}{l}-p\Big)=\\=&(1-pl)\frac{1+[(q-p)-pql]l}{(q-p)-pql}-l(1-pl)=\frac{1-pl}{(q-p)-pql}.\end{aligned}$$

Item (iii): The space $\mathfrak{K}$ is given by (4.6), and hence we have (remember that $p\neq 0$)

$$\mathfrak{P}(E)\cap\mathfrak{K}=\operatorname{span}\Big\{\frac{B(z)}{z}\Big\},\quad\mathfrak{P}(E)=\operatorname{ran}\mathcal{S}_E[\dot+]_E\operatorname{span}\Big\{\frac{B(z)}{z}\Big\},$$

$$\mathcal{D}=\operatorname{span}\Big\{\Big(\frac{B(z)}{z},-\frac{B(z)}{z}\Big)\Big\},\quad\Big[\Big(\frac{B(z)}{z},-\frac{B(z)}{z}\Big),\Big(\frac{B(z)}{z},-\frac{B(z)}{z}\Big)\Big]=\frac{1}{l}.$$

It follows that

$$\big(\mathfrak{P}(E)\times\mathfrak{K}\big)[-]_+\mathcal{D}=\big(\operatorname{ran}\mathcal{S}_E\times\{0\}\big)[\dot+]_+\Big(\Big(\operatorname{span}\Big\{\Big(\frac{B(z)}{z},-\frac{B(z)}{z}\Big)\times\mathfrak{K}\Big)[-]_+\mathcal{D}\Big),$$

$$\Lambda\Big(\Big(\operatorname{span}\Big\{\Big(\frac{B(z)}{z},-\frac{B(z)}{z}\Big)\times\mathfrak{K}\Big)[-]_+\mathcal{D}\Big)=\mathfrak{K}.$$

This already implies that $\operatorname{ran}\mathcal{S}_E \subseteq \mathfrak{P}(\tilde{E})$ and $\mathfrak{P}(\tilde{E}) = \operatorname{ran}\mathcal{S}_E[\dot{+}]_{\tilde{E}}\mathfrak{K}$. We have

$$\tilde{A}(z) = l\frac{B(z)}{z}, \quad \frac{\tilde{B}(z)}{z} = \big[(q-p) - pql\big]A(z) + (1+ql)\frac{B(z)}{z},$$

and hence $\mathfrak{K} = \operatorname{span}\big\{\tilde{A}(z), \frac{\tilde{B}(z)}{z}\big\}$. Moreover,

$$\begin{aligned}
[\tilde{A},\tilde{A}]_{\tilde{E}} =&[\tilde{A},\tilde{A}]_{\mathfrak{K}} = l^2\Big[\frac{B(z)}{z},\frac{B(z)}{z}\Big]_{\mathfrak{K}} = 0 \\
\Big[\tilde{A}(z),\frac{\tilde{B}(z)}{z}\Big]_{\tilde{E}} =&\tilde{A}(0) = 1 \\
\Big[\frac{\tilde{B}(z)}{z},\frac{\tilde{B}(z)}{z}\Big]_{\tilde{E}} =&\tilde{B}'(0) = \big[(q-p)-pql\big] + (1+ql)p = q
\end{aligned}$$

Item (iv): The space $\mathfrak{K}$ is given by (4.6). Hence $\mathfrak{K} \overset{\text{set}}{\subseteq} \mathfrak{P}(E)$, and

$$\mathcal{D} = \operatorname{span}\Big\{\big(A(z),A(z)\big), \Big(\frac{B(z)}{z},\frac{B(z)}{z}\Big)\Big\}.$$

It follows that

$$\big(\mathfrak{P}(E)\times\mathfrak{K}\big)[-]_+\mathcal{D} = \big((\operatorname{ran}\mathcal{S}_E \cap \overline{\operatorname{dom}\mathcal{S}_E})\times\{0\}\big)[\dot{+}]_+\big((\mathfrak{K}\times\mathfrak{K})[-]_+\mathcal{D}\big).$$

This already shows that

$$(\operatorname{ran}\mathcal{S}_E \cap \overline{\operatorname{dom}\mathcal{S}_E}) \subseteq \mathfrak{P}(\tilde{E}) \overset{\text{set}}{\subseteq} \mathfrak{P}(E).$$

Since $\mathfrak{P}(\tilde{E})$ is itself a dB-space, and is closed as a linear subspace of $\mathfrak{P}(E)$, we must have either $\mathfrak{P}(\tilde{E}) \overset{\text{set}}{=} \mathfrak{P}(E)$ or $\mathfrak{P}(\tilde{E}) \overset{\text{set}}{=} \overline{\operatorname{dom}\mathcal{S}_E}$. ❑

Proof (of Theorem 4.3; Part 1: Reduction). Again our first aim is to show that we may restrict without loss of generality to the case that the additional condition stated in Theorem 4.3 is satisfied.

Choose points $t_1,\dots,t_n > 0$ and weights $\gamma_1,\dots,\gamma_n > 0$, such that the corresponding perturbation $[\![.,.]\!]_{\mathfrak{P}}$ of $[.,.]_{\mathfrak{P}}$ turns $\mathfrak{P}$ into a Hilbert space. Write $\langle\mathfrak{P},[\![.,.]\!]_{\mathfrak{P}}\rangle = \mathfrak{P}(E)$ with $E \in \mathcal{HB}^{\text{sb}}_{<\infty}$, $\operatorname{ind}_- E = 0$, and let the function $E_{0,0} \in \mathcal{HB}^{\text{sym}}_{<\infty}$ be defined by (4.1) with $\gamma = \delta = 0$. Then

$$\langle\mathfrak{P}(E_{0,0}),[.,.]_{E_{0,0}}\rangle^{\mathsf{od}} = \langle\mathfrak{P},[\![.,.]\!]_{\mathfrak{P}}\rangle,$$

cf. Lemma 4.4.

Next, choose points $t_{n+1},\dots,t_m > 0$ and weights $\gamma_{n+1},\dots,\gamma_m > 0$ such that the corresponding perturbation $[\![.,.]\!]^{\mathsf{od}}_{E_{0,0}}$ of $[.,.]_{E_0}$ turns $\mathfrak{P}(E_{0,0})$ into a Hilbert space. Moreover, let $[\![.,.]\!]'_{\mathfrak{P}}$ be the perturbation of $[\![.,.]\!]_{\mathfrak{P}}$ on $\mathfrak{P}$ using $t_{n+1},\dots,t_m$ and $\gamma_{n+1},\dots,\gamma_m$. Then, by Lemma 2.10, (iii),

$$\langle\mathfrak{P}(E_{0,0}),[\![.,.]\!]^{\mathsf{od}}_{E_{0,0}}\rangle^{\mathsf{od}} = \langle\mathfrak{P},[\![.,.]\!]'_{\mathfrak{P}}\rangle.$$

Once it is proved that the totality of all symmetric dB-spaces $\mathfrak{Q}'$ with $(\mathfrak{Q}')^{\mathsf{od}} = \langle\mathfrak{P},[\![.,.]\!]'_{\mathfrak{P}}\rangle$ is described by a family $\mathfrak{L}_{l,q}$, $l \in \mathbb{R}$, $q \in \mathbb{R}\cup\{\infty\}$, we may return to the original inner product on $\mathfrak{P}$ by performing the perturbation with points $t_1,\dots,t_m$ and weights $-\gamma_1,\dots,-\gamma_m$, and in this way obtain the desired description of the family of all symmetric dB-spaces $\mathfrak{Q}$

with $\mathfrak{Q}^{\mathrm{od}} = \langle\mathfrak{P}, [.,.]_{\mathfrak{P}}\rangle$. The space $\langle\mathfrak{P}, [\![.,.]\!]'_{\mathfrak{P}}\rangle$, however, satisfies the additional property stated in Theorem 4.3. ❑

From now on we assume that there exists a dB-Hilbert space $\mathring{\mathfrak{Q}} = \mathfrak{P}(\mathring{E})$, $\mathring{E} \in \mathcal{H}B^{\mathrm{sym}}_{<\infty}$, $\operatorname{ind}_- \mathring{E} = 0$, with $\mathring{\mathfrak{Q}}^{\mathrm{od}} = \mathfrak{P}$. If $\mathring{A} \in \mathfrak{P}(\mathring{E})$, the space $\mathfrak{P}(\hat{E})$ with

$$(\hat{A}, \hat{B}) := (\mathring{A}, \mathring{B}) \begin{pmatrix} 1 & -\frac{1}{[\mathring{A},\mathring{A}]_{\mathring{E}}} z \\ 0 & 1 \end{pmatrix}$$

is again a dB-Hilbert space, in fact, $\mathfrak{P}(\hat{E}) = \overline{\operatorname{dom} \mathcal{S}_{\mathring{E}}}$. It again has the property that $\mathfrak{P}(\hat{E})^{\mathrm{od}} = \mathfrak{P}$. Hence, we may assume without loss of generality that $\mathring{A} \notin \mathring{\mathfrak{Q}}$.

We denote by $\overset{\circ\circ}{\mathfrak{Q}}$ the symmetric dB-Hilbert space $\overset{\circ\circ}{\mathfrak{Q}} := \mathfrak{P}(\overset{\circ\circ}{E})$ with

$$(\overset{\circ\circ}{A}, \overset{\circ\circ}{B}) := (\mathring{A}, \mathring{B}) \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} .$$

Then we have

$$\dim \overset{\circ\circ}{\mathfrak{Q}}/\mathring{\mathfrak{Q}} = 1, \quad \mathring{\mathfrak{Q}} = \overline{\operatorname{dom} \mathcal{S}_{\overset{\circ\circ}{\mathfrak{Q}}}}, \quad \operatorname{ran} \mathcal{S}_{\mathring{\mathfrak{Q}}} = \operatorname{ran} \mathcal{S}_{\overset{\circ\circ}{\mathfrak{Q}}} \cap \overline{\operatorname{dom} \mathcal{S}_{\overset{\circ\circ}{\mathfrak{Q}}}} ,$$

and $\overset{\circ\circ}{A} \in \mathfrak{P}(\overset{\circ\circ}{E})$, and $\overset{\circ\circ}{\mathfrak{Q}}{}^{\mathrm{od}} = \mathfrak{P}$.

We define data as required in Theorem 4.3 as

$$\mathfrak{L} := \operatorname{ran} \mathcal{S}_{\mathring{E}}, \quad p := \mathring{B}'(0), \quad C(z) := -\frac{1}{p} \frac{\mathring{B}(z)}{z}, \; D(z) := \mathring{A}(z) ,$$

and consider the family $\mathfrak{L}_{l,q}$ constructed with this data. Note that, whenever $(l,q) \in M_b$ and $(l',q') \in M_s$, we have

$$\begin{tikzcd}[arrows=dash]
 & \mathfrak{L}_{l,q} \overset{\mathrm{set}}{=} \overset{\circ\circ}{\mathfrak{Q}} \arrow[dl, "D"'] \arrow[dr, "D-pC"] & \\
\mathfrak{L}_{l',q'} \overset{\mathrm{set}}{=} \mathring{\mathfrak{Q}} \arrow[dr, "C"'] & & \operatorname{ran} \mathcal{S}_{\overset{\circ\circ}{\mathfrak{Q}}} \arrow[dl, "C+D"] \\
 & \operatorname{ran} \mathcal{S}_{\mathring{E}} &
\end{tikzcd}$$

Since $\mathring{\mathfrak{Q}}$ is a Hilbert space, also the space $\mathfrak{P}$ is. Hence, it can be written as $\mathfrak{P} = \mathfrak{P}(E)$ with $E \in \mathcal{H}B^{\mathrm{sym}}_{<\infty}$, $\operatorname{ind}_- E = 0$. Lemma 4.4 provides us with two real parameters $\mathring{\gamma}, \mathring{\delta}$, such that $\mathring{E} = E_{\mathring{\gamma},\mathring{\delta}}$. Thereby, in fact, $\mathring{\delta} = p$.

Proof (of Theorem 4.3*; Part 2: Sufficiency).* First we deal with the case that $q \neq \infty$. Our aim is to show that

$$\mathfrak{L}_{l,q} = \mathfrak{P}(E_{\mathring{\gamma}+l,q}), \quad (l,q) \in \mathbb{R} \times \mathbb{R} . \tag{4.7}$$

Once this is known, Lemma 4.4 will imply that $\mathfrak{L}_{l,q} \in \mathcal{D}B^{\mathrm{sym}}$ and $\mathfrak{L}_{l,q}^{\mathrm{od}} = \mathfrak{P}$. In order to prove (4.7), we distinguish three cases. Set $\gamma := \mathring{\gamma} + l$.

Case 1; $(l,q) \in M_s$: The computation Lemma 4.4, (iii), gives

$$(A_{\gamma,q}, B_{\gamma,q}) = (\mathring{A}, \mathring{B}) \begin{pmatrix} 1 - pl & z[(q-p) - pql] \\ \frac{l}{z} & 1 + ql \end{pmatrix} .$$

Since $(l,q) \in M_s$ and $q \neq \infty$, we have $(q-p)-pql = 0$ and $1-pl \neq 0$. Lemma 4.6, (i), implies that

$$\mathfrak{P}(E_{\gamma,q}) \stackrel{\text{set}}{=} \operatorname{ran} \mathcal{S}_{\mathring{E}} \dot{+} \operatorname{span}\{C\},$$

and that the Gram matrix of $[.,.]_{E_{\gamma,q}}$ with respect to this direct sum decomposition is

$$\mathrm{GM}_{[.,.]_{E_{\gamma,q}}} = \begin{pmatrix} [.,.]_{\mathring{E}} & 0 \\ 0 & \frac{1}{p} - l \end{pmatrix}$$

Comparing with the definition of $\mathfrak{L}_{l,q}$, we see that indeed $\mathfrak{L}_{l,q} = \mathfrak{P}(E_{\gamma,q})$.

Case 2; $(l,q) \in M_b$ and $1-pl \neq 0$: Since $(l,q) \in M_b$ and $q \neq \infty$, we have $(q-p)-pql \neq 0$. Thus Lemma 4.6, (ii), can be applied, and we obtain that

$$\mathfrak{P}(E_{\gamma,q}) \stackrel{\text{set}}{=} \operatorname{ran} \mathcal{S}_{\mathring{E}} \dot{+} \operatorname{span}\left\{\frac{\mathring{B}(z)}{z}\right\} \dot{+} \operatorname{span}\left\{A_{\gamma,q}\right\}.$$

We have

$$A_{\gamma,q}(z) = (1-pl)\mathring{A}(z) + l\frac{\mathring{B}(z)}{z} = -lpC(z) + (1-pl)D(z),$$

$$D(z) = -\frac{l}{1-pl}\frac{\mathring{B}(z)}{z} + \frac{1}{1-pl}A_{\gamma,\delta}(z),$$

and hence

$$\operatorname{ran} \mathcal{S}_{\mathring{E}} \dot{+} \operatorname{span}\left\{\frac{\mathring{B}(z)}{z}\right\} \dot{+} \operatorname{span}\left\{A_{\gamma,q}\right\} \stackrel{\text{set}}{=} \operatorname{ran} \mathcal{S}_{\mathring{E}} \dot{+} \operatorname{span}\{C\} \dot{+} \operatorname{span}\{D\}.$$

We compute

$$\begin{aligned}
[C,C]_{E_{\gamma,\delta}} &= \Big[-\frac{1}{p}\frac{\mathring{B}(z)}{z}, -\frac{1}{p}\frac{\mathring{B}(z)}{z}\Big]_{E_{\gamma,q}} = \frac{1}{p^2}\cdot p(1-pl) = \frac{1}{p} - l, \\
[C,D]_{E_{\gamma,\delta}} &= \Big[-\frac{1}{p}\frac{\mathring{B}(z)}{z}, \frac{-l}{1-pl}\frac{\mathring{B}(z)}{z} + \frac{1}{1-pl}A_{E_{\gamma,q}}(z)\Big]_{E_{\gamma,q}} = \\
&= \frac{l}{p(1-pl)}\cdot p(1-pl) = l, \\
[D,D]_{E_{\gamma,\delta}} &= \Big[\frac{-l}{1-pl}\frac{\mathring{B}(z)}{z} + \frac{1}{1-pl}A_{\gamma,\delta}(z), \frac{-l}{1-pl}\frac{\mathring{B}(z)}{z} + \frac{1}{1-pl}A_{E_{\gamma,\delta}}(z)\Big]_{E_{\gamma,q}} = \\
&= \frac{l^2}{(1-pl)^2}\cdot p(1-pl) + \frac{1}{(1-pl)^2}\cdot\frac{1-pl}{(q-p)-pql} = \\
&= \frac{l^2 p[(q-p)-pql]+1}{(1-pl)[(q-p)-pql]} = \frac{q(l^2p)+(1+pl)}{(q-p)-pql}.
\end{aligned}$$

Thus, again, $\mathfrak{L}_{l,q} = \mathfrak{P}(E_{\gamma,q})$.

Case 3; $(l,q) \in M_b$ and $1-pl = 0$: Clearly, $(q-p)-pql \neq 0$, and hence we can apply Lemma 4.6, (iii). This gives

$$\mathfrak{P}(E_{\gamma,q}) \stackrel{\text{set}}{=} \operatorname{ran} \mathcal{S}_{\mathring{E}} \dot{+} \operatorname{span}\left\{A_{\gamma,q}\right\} \dot{+} \operatorname{span}\left\{\frac{B_{\gamma,q}(z)}{z}\right\}.$$

We have

$$C(z)=-\frac{1}{p}\frac{\mathring{B}(z)}{z}=-A_{\gamma,q}(z),\ \ \frac{B_{\gamma,q}(z)}{z}=-p\mathring{A}(z)+(1+ql)\frac{\mathring{B}(z)}{z}\,,$$

$$D(z)=\mathring{A}(z)=(1+ql)A_{\gamma,q}(z)-l\frac{B_{\gamma,q}(z)}{z}\,.$$

Hence

$$\operatorname{ran}\mathcal{S}_{\mathring{E}}\dot{+}\operatorname{span}\big\{A_{\gamma,q}\big\}\dot{+}\operatorname{span}\Big\{\frac{B_{\gamma,q}(z)}{z}\Big\}\overset{\text{set}}{=}\operatorname{ran}\mathcal{S}_{\mathring{E}}\dot{+}\operatorname{span}\{C\}\dot{+}\operatorname{span}\{D\}\,,$$

and we can compute

$$\begin{aligned}
[C,C]_{E_{\gamma,q}} =&[-A_{\gamma,q},A_{\gamma,q}]_{E_{\gamma,q}}=0\\
[C,D]_{E_{\gamma,q}} =&\Big[-A_{\gamma,q},(1+ql)A_{\gamma,q}(z)-l\frac{B_{\gamma,q}(z)}{z}\Big]_{E_{\gamma,q}}=l\\
[D,D]_{E_{\gamma,q}} =&\Big[(1+ql)A_{\gamma,q}(z)-l\frac{B_{\gamma,q}(z)}{z},(1+ql)A_{\gamma,q}(z)-l\frac{B_{\gamma,q}(z)}{z}\Big]_{E_{\gamma,q}}=\\
=&-2l(1+ql)+l^2q=-2l-l^2q\,.
\end{aligned}$$

On the other hand, we have

$$\frac{q(l^2p)+(1+pl)}{q(1-pl)-p}=\frac{ql+2}{-p}=-ql^2-2l\,.$$

Thus also in this case $\mathfrak{L}_{l,q}=\mathfrak{P}(E_{\gamma,q})$.

The proof of (4.7) is finished, and hence the case that $q\neq\infty$ is settled. The case that $q=\infty$ is treated with a limit argument.

If $1-pl\neq 0$, then we have for all sufficiently large values of $q'\in\mathbb{R}$ that $(l,q')\in M_b$. Moreover, for such values of q',

$$\mathfrak{L}_{l,\infty}\overset{\text{set}}{=}\mathfrak{L}_{l,q'},\quad [F,G]_{\mathfrak{L}_{l,\infty}}=\lim_{q'\to\infty}[F,G]_{\mathfrak{L}_{l,q'}},\ F,G\in\mathfrak{L}_{l,\infty}\,.$$

It follows that $\mathfrak{L}_{l,\infty}\in\mathcal{DB}^{\text{sym}}$ and that $\mathfrak{L}_{l,\infty}^{\mathsf{od}}=\mathfrak{P}$.

If $1-pl=0$, we have $(l,\infty)\in M_s$. For $l'\in\mathbb{R}\setminus\{l\}$, consider the value $\mathring{q}(l'):=\frac{p}{1-pl'}$, so that $\mathring{q}(l')\neq\infty$ and $(l',\mathring{q}(l'))\in M_s$. We have

$$\mathfrak{L}_{l,\infty}\overset{\text{set}}{=}\mathfrak{L}_{l',\mathring{q}(l')},\quad [F,G]_{\mathfrak{L}_{l,\infty}}=\lim_{l'\to l}[F,G]_{\mathfrak{L}_{l',\mathring{q}(l')}},\ F,G\in\mathfrak{L}_{l,\infty}\,,$$

and again conclude that $\mathfrak{L}_{l,\infty}\in\mathcal{DB}^{\text{sym}}$, $\mathfrak{L}_{l,\infty}^{\mathsf{od}}=\mathfrak{P}$. ❑

Proof (of Theorem 4.3*; Part 3: Necessity).* Let a symmetric dB-space $\mathfrak{Q}$ with $\mathfrak{Q}^{\mathsf{od}}=\mathfrak{P}$ be given. The case that $\mathfrak{Q}$ is nondegenerated is simple: By Lemma 4.4 there exist parameters $\gamma,\delta\in\mathbb{R}$ with $\mathfrak{Q}=\mathfrak{P}(E_{\gamma,\delta})$, and by (4.7) thus

$$\mathfrak{P}(E_{\gamma,\delta})=\mathfrak{L}_{\gamma-\mathring{\gamma},\delta}\,.$$

From now on assume that $\mathfrak{Q}$ is degenerated. Again, this situation requires more effort.

Case '$\mathfrak{Q}$ degenerated'; $\mathfrak{Q}$ as a set: Choose points $s_1,\dots,s_n>0$ and weights $\delta_1,\dots,\delta_n>0$, such that the correspondingly perturbed inner product $[\![.,.]\!]_{\mathfrak{Q}}^{\mathsf{od}}$

turns $\mathfrak{Q}$ into a Pontryagin space. Since all weights δ_i are positive, $\langle\overset{\circ\circ}{\mathfrak{Q}},[\![.,.]\!]^{\mathsf{od}}_{\overset{\circ\circ}{\mathfrak{Q}}}\rangle$ is also a Hilbert space. Let $[\![.,.]\!]_{\mathfrak{P}}$ be the corresponding perturbation of $[.,.]_{\mathfrak{P}}$, then we can write

$$\begin{aligned}
\langle\mathfrak{P},[\![.,.]\!]_{\mathfrak{P}}\rangle &= \mathfrak{P}(E^+) && \text{with } E^+\in\mathcal{HB}^{\mathrm{sb}}_{<\infty}, \operatorname{ind}_- E^+=0\,,\\
\langle\mathfrak{Q},[\![.,.]\!]^{\mathsf{od}}_{\mathfrak{Q}}\rangle &= \mathfrak{P}(\tilde E^+) && \text{with } \tilde E^+\in\mathcal{HB}^{\mathrm{sym}}_{<\infty},\\
\langle\overset{\circ\circ}{\mathfrak{Q}},[\![.,.]\!]^{\mathsf{od}}_{\overset{\circ\circ}{\mathfrak{Q}}}\rangle &= \mathfrak{P}(\overset{\circ\circ}{E}{}^+) && \text{with } \overset{\circ\circ}{E}{}^+\in\mathcal{HB}^{\mathrm{sym}}_{<\infty}, \operatorname{ind}_- \overset{\circ\circ}{E}{}^+=0\,.
\end{aligned}$$

By Lemma 2.10, (iii),

$$\mathfrak{P}(\tilde E^+)^{\mathsf{od}}=\mathfrak{P}(\overset{\circ\circ}{E}{}^+)^{\mathsf{od}}=\mathfrak{P}(E^+)\,.$$

From Lemma 4.4 we obtain parameters $\tilde\gamma,\tilde\delta\in\mathbb{R}$ and $\overset{\circ\circ}{\gamma},\overset{\circ\circ}{\delta}\in\mathbb{R}$, such that $\tilde E^+=E^+_{\tilde\gamma,\tilde\delta}$ and $\overset{\circ\circ}{E}{}^+=E^+_{\overset{\circ\circ}{\gamma},\overset{\circ\circ}{\delta}}$, and hence

$$(\tilde A^+,\tilde B^+)=(\overset{\circ\circ}{A}{}^+,\overset{\circ\circ}{B}{}^+)\begin{pmatrix}1-pl & z[(q-p)-pql]\\ \frac{l}{z} & 1+ql\end{pmatrix}$$

with

$$p:=\overset{\circ\circ}{\delta},\quad q:=\tilde\delta,\qquad l:=\tilde\gamma-\overset{\circ\circ}{\delta}\,.$$

By Corollary 2.12, we have $\overset{\circ\circ}{A}{}^+\in\mathfrak{P}(\overset{\circ\circ}{E}{}^+)$, and hence we may apply Lemma 4.6, (iv). It follows that

$$\operatorname{ran}\mathcal{S}_{\overset{\circ\circ}{E}{}^+}\cap\overline{\operatorname{dom}\mathcal{S}_{\overset{\circ\circ}{E}{}^+}}\subseteq\mathfrak{P}(\tilde E^+)\overset{\mathrm{set}}{=}\begin{cases}\overline{\operatorname{dom}\mathcal{S}_{\overset{\circ\circ}{E}{}^+}}\\ \mathfrak{P}(\overset{\circ\circ}{E}{}^+)\end{cases}.$$

Returning to the unperturbed inner products, yields that

$$\operatorname{ran}\mathcal{S}_{\mathring{\mathfrak{Q}}}=\operatorname{ran}\mathcal{S}_{\overset{\circ\circ}{\mathfrak{Q}}}\cap\overline{\operatorname{dom}\mathcal{S}_{\overset{\circ\circ}{\mathfrak{Q}}}}\subseteq\mathfrak{Q}\overset{\mathrm{set}}{=}\begin{cases}\mathring{\mathfrak{Q}}\\ \overset{\circ\circ}{\mathfrak{Q}}\end{cases}.$$

Case '$\mathfrak{Q}$ degenerated'; We show that $\mathfrak{Q}=\operatorname{ran}\mathcal{S}_{\mathfrak{Q}}\dot+\mathfrak{Q}^\circ$: Since $\mathfrak{Q}^{\mathsf{od}}=\mathfrak{P}$ is nondegenerated, we have $\mathfrak{Q}^\circ\subseteq\{F\in\mathfrak{Q}: F \text{ even}\}$ and $\dim\mathfrak{Q}^\circ=1$, cf. [KWW06a, Lemma 2.4]. It follows that the equality $\mathfrak{Q}=\operatorname{ran}\mathcal{S}_{\mathfrak{Q}}\dot+\mathfrak{Q}^\circ$ is equivalent to $\mathfrak{Q}^\circ\not\subseteq\operatorname{ran}\mathcal{S}_{\mathfrak{Q}}$ and further equivalent to $\mathfrak{Q}^\circ\cap\operatorname{ran}\mathcal{S}_{\mathfrak{Q}}=\{0\}$.

Assume first that $\mathfrak{Q}\overset{\mathrm{set}}{=}\mathring{\mathfrak{Q}}$. Since $\operatorname{ran}\mathcal{S}_{\mathring{\mathfrak{Q}}}$ is positive definite and $\operatorname{ran}\mathcal{S}_{\mathring{\mathfrak{Q}}}\subseteq\mathfrak{Q}$, we have $\mathfrak{Q}^\circ\cap\operatorname{ran}\mathcal{S}_{\mathring{\mathfrak{Q}}}=\{0\}$. However, in the present case $\operatorname{ran}\mathcal{S}_{\mathring{\mathfrak{Q}}}=\operatorname{ran}\mathcal{S}_{\mathfrak{Q}}$.

Next, assume that $\mathfrak{Q}\overset{\mathrm{set}}{=}\overset{\circ\circ}{\mathfrak{Q}}$ and $\mathfrak{Q}^\circ\overset{\mathrm{set}}{\subseteq}\mathring{\mathfrak{Q}}$. We have $\langle\mathring{\mathfrak{Q}},[.,.]_{\mathfrak{Q}}\rangle^{\mathsf{od}}=\mathfrak{P}$ and $\langle\mathring{\mathfrak{Q}},[.,.]_{\mathfrak{Q}}\rangle^\circ\supseteq\mathfrak{Q}^\circ$. By what we showed in the above paragraph, thus $\mathfrak{Q}^\circ\cap\operatorname{ran}\mathcal{S}_{\mathring{\mathfrak{Q}}}=\{0\}$. Since $\mathfrak{Q}^\circ\subseteq\mathring{\mathfrak{Q}}$, $\operatorname{ran}\mathcal{S}_{\overset{\circ\circ}{\mathfrak{Q}}}\cap\mathring{\mathfrak{Q}}=\operatorname{ran}\mathcal{S}_{\mathring{\mathfrak{Q}}}$, and $\operatorname{ran}\mathcal{S}_{\overset{\circ\circ}{\mathfrak{Q}}}=\operatorname{ran}\mathcal{S}_{\mathfrak{Q}}$, this implies that $\mathfrak{Q}^\circ\cap\operatorname{ran}\mathcal{S}_{\mathfrak{Q}}=\{0\}$.

Finally, consider the case that $\mathfrak{Q}\overset{\mathrm{set}}{=}\overset{\circ\circ}{\mathfrak{Q}}$ and $\mathfrak{Q}^\circ\overset{\mathrm{set}}{\not\subseteq}\mathring{\mathfrak{Q}}$. Then we have $\mathfrak{Q}\overset{\mathrm{set}}{=}\mathring{\mathfrak{Q}}\dot+\mathfrak{Q}^\circ$. Since $\dim\mathfrak{Q}^\circ=1$, thus $\overline{\operatorname{dom}\mathcal{S}_{\mathfrak{Q}}}=\langle\mathring{\mathfrak{Q}},[.,.]_{\mathfrak{Q}}\rangle$ is nondegenerated. Write $\langle\mathring{\mathfrak{Q}},[.,.]_{\mathfrak{Q}}\rangle=\mathfrak{P}(\hat E)$ with some $\hat E\in\mathcal{HB}^{\mathrm{sym}}_{<\infty}$. Then, by Lemma 2.14, $\mathfrak{Q}^\circ$

either equals $\operatorname{span}\{\hat{A}\}$ or $\operatorname{span}\{\hat{B}\}$. Since $\mathfrak{Q}^\circ$ contains only even functions, the second possibility is ruled out. We see that again $\mathfrak{Q}^\circ \cap \operatorname{ran}\mathcal{S}_{\mathfrak{Q}} = \{0\}$.

Let us note explicitly that the relation $\mathfrak{Q} = \operatorname{ran}\mathcal{S}_{\mathfrak{Q}} \dot{+} \mathfrak{Q}^\circ$ implies that $\operatorname{ran}\mathcal{S}_{\mathfrak{Q}}$ is nondegenerated.

Case '$\mathfrak{Q}$ degenerated'; Finish of proof: For each parameter $\varepsilon \in \mathbb{R}\setminus\{0\}$ consider the perturbed inner product $[\![.,.]\!]^\varepsilon_{\mathfrak{Q}}$ on $\mathfrak{Q}$, cf. (2.5). By Lemma 2.10,

$$\langle \mathfrak{Q}, [\![.,.]\!]^\varepsilon_{\mathfrak{Q}} \rangle \in \mathcal{D}B^{\mathrm{sym}}, \quad \langle \mathfrak{Q}, [\![.,.]\!]^\varepsilon_{\mathfrak{Q}} \rangle^{\mathsf{od}} = \mathfrak{P}.$$

Let us show that $[\![.,.]\!]^\varepsilon_{\mathfrak{Q}}$ is nondegenerated: To this end write $\mathfrak{Q}^\circ = \operatorname{span}\{L\}$, where L is an even function with $L(0) = 1$. Let $G \in \langle \mathfrak{Q}, [\![.,.]\!]^\varepsilon_{\mathfrak{Q}} \rangle^\circ$ be given, and write $G = G_0 + \lambda L$ with $G_0 \in \operatorname{ran}\mathcal{S}_{\mathfrak{Q}}$ and $\lambda \in \mathbb{C}$. For each $F \in \operatorname{ran}\mathcal{S}_{\mathfrak{Q}}$ we have

$$[G_0, F]_{\mathfrak{Q}} = [G, F]_{\mathfrak{Q}} = [\![G, F]\!]^\varepsilon_{\mathfrak{Q}} = 0\,.$$

Since $\operatorname{ran}\mathcal{S}_{\mathfrak{Q}}$ is nondegenerated, thus $G_0 = 0$, i.e. $G = \lambda L$. Next,

$$0 = [\![G, G]\!]^\varepsilon_{\mathfrak{Q}} = [\![\lambda L, \lambda L]\!]^\varepsilon_{\mathfrak{Q}} = |\lambda|^2 \cdot \varepsilon\,,$$

and hence $\lambda = 0$. It follows that $\langle \mathfrak{Q}, [\![.,.]\!]^\varepsilon_{\mathfrak{Q}} \rangle^\circ = \{0\}$.

By Lemma 4.4 and (4.7), there exist $\gamma(\varepsilon), \delta(\varepsilon) \in \mathbb{R}$ with

$$\langle \mathfrak{Q}, [\![.,.]\!]^\varepsilon_{\mathfrak{Q}} \rangle = \mathfrak{P}(E_{\gamma(\varepsilon),\delta(\varepsilon)}) = \mathfrak{L}_{\gamma(\varepsilon)-\mathring{\gamma},\delta(\varepsilon)}, \quad \varepsilon \in \mathbb{R}\setminus\{0\}\,.$$

Thus

$$\mathfrak{Q} \overset{\text{set}}{=} \begin{cases} \operatorname{ran}\mathcal{S}_{\mathring{E}} \dot{+} \operatorname{span}\{C\} & , \quad (\gamma(\varepsilon), \delta(\varepsilon)) \in M_s \\ \operatorname{ran}\mathcal{S}_{\mathring{E}} \dot{+} \operatorname{span}\{C\} \dot{+} \operatorname{span}\{D\}, & (\gamma(\varepsilon), \delta(\varepsilon)) \in M_b \end{cases},$$

and the Gram matrix of the inner product $[\![.,.]\!]^\varepsilon_{\mathfrak{Q}}$ with respect to this decomposition is given as

$$\mathrm{GM}_{[\![.,.]\!]^\varepsilon_{\mathfrak{Q}}} = \begin{pmatrix} [.,.]_{\mathring{E}} & 0 & 0 \\ 0 & \frac{1}{p} - (\gamma(\varepsilon) - \mathring{\gamma}) & \gamma(\varepsilon) - \mathring{\gamma} \\ 0 & \gamma(\varepsilon) - \mathring{\gamma} & \frac{\delta(\varepsilon)\big((\gamma(\varepsilon)-\mathring{\gamma})^2 p\big) + \big(1 + p(\gamma(\varepsilon)-\mathring{\gamma})\big)}{q\big(1 - p(\gamma(\varepsilon)-\mathring{\gamma})\big) - p} \end{pmatrix},$$

where the last row and column is present only if $(\gamma(\varepsilon), \delta(\varepsilon)) \in M_b$.

We pass to the limit '$\varepsilon \to 0$'. The fact that $\lim_{\varepsilon\to 0} [\![.,.]\!]^\varepsilon_{\mathfrak{Q}} = [.,.]_{\mathfrak{Q}}$, says nothing else but

$$\mathrm{GM}_{[.,.]_{\mathfrak{Q}}} = \lim_{\varepsilon \to 0} \mathrm{GM}_{[\![.,.]\!]^\varepsilon_{\mathfrak{Q}}}\ .$$

In particular, the limit $\gamma := \lim_{\varepsilon\to 0} \gamma(\varepsilon)$ exists.

If $\mathfrak{Q} \overset{\text{set}}{=} \mathring{\mathfrak{Q}}$, so that the third row and column is not present, we see that γ must equal $\mathring{\gamma} + \frac{1}{p}$, since otherwise $[.,.]_{\mathfrak{Q}}$ would be nondegenerated. This shows that $\langle \mathfrak{Q}, [.,.]_{\mathfrak{Q}} \rangle = \mathfrak{L}_{\gamma,\infty}$.

Consider the case that $\mathfrak{Q} \overset{\text{set}}{=} \overset{\infty}{\mathfrak{Q}}$. Since $[.,.]_{\mathfrak{Q}}$ is degenerated, the determinant of the right lower 2×2-block of $\mathrm{GM}_{[.,.]_{\mathfrak{Q}}}$ must be equal to zero. This rules out the possibility that $\gamma = \mathring{\gamma} + \frac{1}{p}$, since in this case the mentioned determinant were $\frac{1}{p^2}$. Moreover, it follows that

$$\lim_{\varepsilon\to 0} \frac{\delta(\varepsilon)\big((\gamma(\varepsilon)-\mathring{\gamma})^2 p\big) + \big(1+p(\gamma(\varepsilon)-\mathring{\gamma})\big)}{q\big(1-p(\gamma(\varepsilon)-\mathring{\gamma})\big)-p} = \frac{(\gamma-\mathring{\gamma})^2}{\frac{1}{p}-(\gamma-\mathring{\gamma})}\,.$$

We see that $\langle\mathfrak{Q}, [.,.]_{\mathfrak{Q}}\rangle = \mathfrak{L}_{\gamma_0,\infty}$. ❑

The proof of Theorem 4.3 is complete.

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Henrik Winkler
Institut für Mathematik
Technische Universität Illmenau
Curiebau, Weimarer Straße 25
D–98693 Illmenau
GERMANY
e-mail: henrik.winkler@tu-ilmenau.de

Harald Woracek
Institut for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße. 8–10/101
1040 Wien
AUSTRIA
e-mail: harald.woracek@tuwien.ac.at