De Branges' theorem on approximation problems of Bernstein type

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Abstract

The Bernstein approximation problem is to determine whether or not the space of all polynomials is dense in a given weighted C_0-space on the real line. A theorem of L. de Branges characterizes non–density by existence of an entire function of Krein class being related with the weight in a certain way. An analogous result holds true for weighted sup–norm approximation by entire functions of exponential type at most τ and bounded on the real axis ($\tau > 0$ fixed).

We consider approximation in weighted C_0-spaces by functions belonging to a prescribed subspace of entire functions which is solely assumed to be invariant under division of zeros and passing from $F(z)$ to $\overline{F(\overline{z})}$, and establish the precise analogue of de Branges' theorem. For the proof we follow the lines of de Branges' original proof, and employ some results of L. Pitt.

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1 Introduction

Several classical problems revolve around the following general question:

> *Let X be a Banach space of functions, and let $\mathcal{L}$ be a linear subspace of X. When is $\mathcal{L}$ dense in X ?*

As a model example, let us discuss weighted sup-norm approximation by polynomials. Let $W : \mathbb{R} \to (0, \infty)$ be continuous, and assume that $\lim_{|x|\to\infty} \frac{x^n}{W(x)} = 0$, $n \in \mathbb{N}$. Take for X the space $C_0(W)$ of all continuous functions on the real line such that $\frac{f}{W}$ tends to zero at infinity, endowed with the norm $\|f\|_{C_0(W)} := \sup_{x\in\mathbb{R}} \frac{|f(x)|}{W(x)}$. And take for $\mathcal{L}$ the space $\mathbb{C}[z]$ of all polynomials with complex coefficients. Then the above quoted question is known as the Bernstein problem. Several answers were obtained in the 1950's, e.g., in the work of Mergelyan [M], Akhiezer [A1], Pollard [Po], or de Branges [dB2]. A comprehensive exposition can be found in [K, Chapter VI.A–D], let us also refer to the survey article [L] where also quantitative results are reviewed.

Most characterizations of density use in some way the function

$$\mathfrak{m}(z) := \sup\left\{ |P(z)| : \, P \in \mathbb{C}[z], \|P\|_{C_0(W)} \leq 1 \right\},$$

also referred to as the Hall majorant associated with the weight W. The value $\mathfrak{m}(z)$ is of course nothing but the norm of the point evaluation functional $F \mapsto F(z)$ on $\mathbb{C}[z]$ with respect to the norm $\|.\|_{C_0(W)}$, understanding '$\mathfrak{m}(z) = \infty$' as point evaluation being unbounded.

1.1 Theorem. *Let $W : \mathbb{R} \to (0, \infty)$ be a continuous weight, and assume that* $\lim_{|x|\to\infty} x^n W(x)^{-1} = 0$, $n \in \mathbb{N}$. *Then the following are equivalent:*

(*i*) $\mathbb{C}[z]$ *is dense in* $C_0(W)$.

(*ii*) (Mergelyan, 1956) *We have* $\mathfrak{m}(z) = \infty$ *for one* (*equivalently, for all*) $z \in \mathbb{C} \setminus \mathbb{R}$.

(*iii*) (Akhiezer, 1956) *We have*

$$\int_{\mathbb{R}} \frac{\log \mathfrak{m}(x)}{1+x^2}\, dx = \infty\,.$$

(*iv*) (Pollard, 1953) *We have*

$$\sup\Big\{ \int_{\mathbb{R}} \frac{\log |P(x)|}{1+x^2}\, dx : \ P \in \mathbb{C}[z], \|P\|_{C_0(W)} \le 1 \Big\} = \infty\,.$$

The criterion of de Branges is of different nature.

1.2 Theorem (de Branges, 1959)**.** *Let $W : \mathbb{R} \to (0, \infty)$ be a continuous weight, and assume that* $\lim_{|x|\to\infty} \frac{x^n}{W(x)} = 0$, $n \in \mathbb{N}$. *Then the following are equivalent:*

(*i*) $\mathbb{C}[z]$ *is not dense in* $C_0(W)$.

(*ii*) *There exists an entire function B which possesses the properties*

- *B is not a polynomial. We have $B(z) = \overline{B(\overline{z})}$. All zeros of B are real and simple.*
- *B is of finite exponential type, and* $\int_{\mathbb{R}} \frac{\log^+ |B(x)|}{1+x^2}\, dx < \infty$.
- $\displaystyle\sum_{x:\, B(x)=0} \frac{W(x)}{|B'(x)|} < \infty$.

If $\mathbb{C}[z]$ is not dense in $C_0(W)$, the function B in (*ii*) *can be chosen of zero exponential type.*

De Branges' original proof uses mainly basic functional analysis (the Krein–Milman theorem) and complex analysis (bounded type theory). Several other approaches are known; for example [SY] where the result is obtained as a consequence of a deep study of Chebyshev sets, or [P] where some properties of singular (Cauchy-) integral operators are invoked.

Also other instances for X and $\mathcal{L}$ in the question quoted in the very first paragraph of this introduction are classical objects of study.

Working in other spaces X: Density of polynomials in a space $L^2(\mu)$ or $L^1(\mu)$ is closely related with the structure of the solution set of the Hamburger power moment problem generated by the sequence $(\int_{\mathbb{R}} x^n\, d\mu(x))_{n\in\mathbb{N}_0}$. In fact, by theorems of M. Riesz and M.A. Naimark, density is equivalent to extremal properties of μ in this solution set, cf. [A2, §2.3]. Also a characterization in terms of the norms of point evaluation maps was obtained already at a very early stage (in the 1920's) by M. Riesz, see, e.g., [K, Chapter V.D]. By the recent work of A.G. Bakan the results for L^p-spaces and weighted C_0-spaces are closely related, see [B].

Approximation with functions different from polynomials: For example, let us mention approximation with entire functions of finite exponential type. There the space $\mathcal{L}$ is taken, e.g., as the set of all finite linear combinations of exponentials $e^{i\lambda x}$, $|\lambda| \leq a$, (with some fixed $a > 0$) or as the space of all Fourier transforms of C^∞-functions compactly supported in $(-a, a)$. Analogues of the mentioned theorems are again classical, see, e.g., [K, Chapter VI.E–F]. A proof of the 'exponential version' of de Branges' theorem following the method of [SY] is given in [BS].

In the 1980's, L. Pitt proposed a unifying (and generalizing) approach to approximation problems of this kind, cf. [Pi]. He considered quite general instances of X and $\mathcal{L}$: the Banach space X is only assumed to be a so-called regular function space, and the space $\mathcal{L}$ can be any space of entire functions which is closed with respect to forming difference quotients and passing from $F(z)$ to $\overline{F(\overline{z})}$ and which is contained injectively in X. The class of regular function spaces is quite large; for example it includes spaces $C_0(W)$, or $L^p(\mu)$, $p \in [1, \infty)$, or weighted Sobolev spaces. Under some mild regularity conditions Pitt shows analogues of the results mentioned in Theorem 1.1 above, as well as versions of some more detailed results in the same flavour which we did not list above. A general version of Theorem 1.2, however, is not given.

The present contribution.

Our aim in the present paper is to prove a theorem of de Branges type for weighted sup-norm approximation by entire functions of a space $\mathcal{L}$ as considered in the work of Pitt: namely, the below Theorem 1.6. To establish this theorem, we follow de Branges' original method.

Independently of the present work, M. Sodin and P. Yuditskii have generalized the method first used in [SY], and obtained precisely the same result[1].

In order to concisely formulate the presently discussed general version of Theorem 1.2, we introduce some notation. First, for completeness, the class of weighted spaces under consideration.

1.3 Definition. We call a function $W : \mathbb{R} \to (0, \infty]$ a weight, if W is lower semicontinuous and not identically equal to ∞.

We denote by $C_0(W)$ the space of all continuous functions f on the real line such that $\lim_{|x|\to\infty} \frac{|f(x)|}{W(x)} = 0$ (here $\frac{f(x)}{W(x)}$ is understood as 0, if $W(x) = \infty$). This linear space is endowed with the seminorm

$$\|f\|_{C_0(W)} := \sup_{x\in\mathbb{R}} \frac{|f(x)|}{W(x)} .$$

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Clearly, $\|.\|_{C_0(W)}$ induces a locally convex vector topology on $C_0(W)$ and all topological notions are understood with respect to it. Notice that this topology need not be Hausdorff. In fact, the seminorm $\|.\|_{C_0(W)}$ is a norm if and only if the set

$$\Omega := \big\{x \in \mathbb{R} : W(x) \neq \infty\big\}$$

[1]This is work in progress communicated to us by M. Sodin, who also presented it at the 'International Symposium on Orthogonal Polynomials and Special Functions – a Complex Analytic Perspective' (June 11–15, 2012, Copenhagen).

is dense in $\mathbb{R}$.

Next, a terminology for the spaces $\mathcal{L}$ with which we deal.

1.4 Definition. Let $\mathcal{L}$ be a linear space. We call $\mathcal{L}$ an algebraic de Branges space, if

(B1) The elements of $\mathcal{L}$ are entire functions.

(B2) If $F \in \mathcal{L}$ and $w \in \mathbb{C}$ with $F(w) = 0$, then also the function $\frac{F(z)}{z-w}$ belongs to $\mathcal{L}$.

(B3) If $F \in \mathcal{L}$, then also the function $F^{\#}(z) := \overline{F(\overline{z})}$ belongs to $\mathcal{L}$.

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The appropriate weighted analogue of the class of entire functions appearing in de Branges' theorem, which is also referred to as the Krein class, is the following.

1.5 Definition. Let $\mathcal{L}$ be an algebraic de Branges space. Let $W : \mathbb{R} \to (0, \infty]$ be a lower semicontinuous function, and assume that $\mathcal{L} \subseteq C_0(W)$. Then we define the W-weighted Krein class $\mathcal{K}(\mathcal{L}, W)$ with respect to $\mathcal{L}$ as the set of all entire function B which satisfy

(K1) The function B has at least one zero, and all its zeros are real and simple. We have $B = B^{\#}$.

(K2) For each $F \in \mathcal{L}$, the function $\frac{F}{B}$ is of bounded type in both half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$.

(K3) For each $F \in \mathcal{L}$ we have

$$|y| \cdot |F(iy)| = o\big(|B(iy)|\big), \quad y \to \pm\infty .$$

(K4) $\displaystyle\sum_{x:\, B(x)=0} \frac{W(x)}{|B'(x)|} < \infty.$ [2]

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Finally, we denote $\operatorname{mt} h := \limsup_{y \to +\infty} \frac{1}{y} \log |h(iy)| \in [-\infty, \infty]$ whenever h is a complex-valued function defined (at least) on the ray $i\mathbb{R}^+$, and refer to this number as the mean type of h[3].

The statement we are going to prove in this paper can now be formulated as follows.

1.6 Theorem. *Let $\mathcal{L}$ be an algebraic de Branges space. Let $W : \mathbb{R} \to (0, \infty]$ be a weight, i.e. lower semicontinuous and not identically equal to ∞, and assume that $\mathcal{L} \subseteq C_0(W)$. Then the following are equivalent:*

(i) $\mathcal{L}$ is not dense in $C_0(W)$.

(ii) $\mathcal{K}(\mathcal{L}, W)$ is nonempty.

[2] Convergence of this series implicitly includes the requirement that $W(x) < \infty$ whenever x is a zero of B.

[3] Concerning this notation, we do not assume that h is subharmonic or even analytic.

If $\mathcal{L}$ is not dense in $C_0(W)$, then there exists a function $B \in \mathcal{K}(\mathcal{L}, W)$ with

$$\sup\Big\{ \operatorname{mt}\frac{F}{B} : F \in \mathcal{L}\Big\} = 0 . \tag{1.1}$$

Let us observe that, unless the set $\Omega = \{x \in \mathbb{R} : W(x) \neq \infty\}$ has discrete closure, the condition (ii) is the theorem is stable with respect to polynomial changes of growth.

1.7 Remark. Assume that Ω has at least one finite accumulation point. Then the following are equivalent:

(i) We have $\mathcal{K}(\mathcal{L}, W) \neq \emptyset$.

(ii) There exists $d \in \mathbb{R}$ and there exists an entire function B with (K1) and (K2), such that

$$\begin{gathered} \forall\, F \in \mathcal{L}: \quad |y|^d \cdot |F(iy)| = o\big(|B(iy)|\big),\ y \to \pm\infty , \\ \sum_{x:\, B(x)=0} (1+|x|)^d \frac{W(x)}{|B'(x)|} < \infty . \end{gathered} \tag{1.2}$$

(iii) For all $d \in \mathbb{R}$ there exists an entire function B with (K1) and (K2), such that (1.2) holds.

This is seen easily: Assume that a function B satisfies (K1), (K2), and (1.2) with some $d \in \mathbb{R}$. Clearly, then B also satisfies (1.2) with every $d' < d$. Choose a sequence $(x_k)_{k\in\mathbb{N}}$ of pairwise distinct points in Ω with $B(x_k) \neq 0$, $k \in \mathbb{N}$. Such a choice is possible by our assumption that Ω has a finite accumulation point. Then the function $B_N(z) := B(z)\prod_{k=1}^N (z - x_k)$ satisfies (K1), (K2), and (1.2) with $d + N$. //

In sharp contrast, if Ω has no finite accumulation point, the growth condition (K3) is highly sensitive.

1.8 Example. Let Λ be a nonempty subset of $\mathbb{R}$ without finite accumulation point. Choose a real entire function B whose zeros are all simple and located precisely at the points of Λ. Moreover, choose a sequence $(\nu_\lambda)_{\lambda\in\Lambda}$ with $\sum_{\lambda\in\Lambda} \frac{\nu_\lambda}{1+|\lambda|} < \infty$.

From this data, we define entire functions A and E as

$$A(z) := B(z) \cdot \sum_{\lambda\in\Lambda} \frac{\nu_\lambda}{z-\lambda}, \quad E(z) := A(z) - iB(z) .$$

Then the function E generates a de Branges space $\mathcal{H}(E)$. Moreover, E has no real zeros, and hence $\mathcal{H}(E)$ is invariant with respect to dividing zeros also at real points. According to our terminology, thus, $\mathcal{H}(E)$ is an algebraic de Branges space.

The function B clearly satisfies (K1) and (K2). By the definition of A, we have $|A(iy)| = o(|B(iy)|)$, $y \to \pm\infty$. Hence, $\lim_{y\to\pm\infty} \frac{|E(iy)|}{|B(iy)|} = 1$. By the Schwarz inequality and the concrete form of the reproducing kernel in the space $\mathcal{H}(E)$, we have

$$|y|^{\frac{1}{2}} |F(iy)| = O\big(|E(iy)|\big), \quad y \to \pm\infty .$$

Hence, the first relation in (1.2) holds for every $d < \frac{1}{2}$.

Let W be a weight with $\Omega \subseteq \Lambda$, and set $\mathcal{L} := \mathcal{H}(E) \cap C_0(W)$. Then $\mathcal{L}$ (considered as a space of entire functions) is an algebraic de Branges space. For each $\lambda \in \Lambda$ the function $\frac{B(z)}{z-\lambda}$ belongs to $\mathcal{H}(E)$. Hence, the space $\mathcal{L}$ (considered as a subspace of $C_0(W)$) contains all functions with finite support. This immediately implies that $\mathcal{L}$ is dense in $C_0(W)$. By choosing the values $W(x)$ sufficiently small, we can clearly satisfy the second relation in (1.2) for arbitrary d (or any –possibly stronger– condition of a similar form). //

2 The toolbox

Following de Branges' original proof requires several (crucial!) tools:

(1) A description of the topological dual space of $C_0(W)$. Knowledge about duals of weighted C_0-spaces (actually, in a much more general setting than the present) was obtained already in the 1960's by W.H. Summers following the work of L. Nachbin, cf. [S1], [S2], [N].

(2) A weighted version of de Branges' lemma on extremal points μ of the annihilator of $\mathcal{L}$. The original version (without any weights) is [dB1, Lemma][4].

(3) Some of L. Pitt's theorems, applied with the Banach space $L^1(\mu)$. These can be extracted from [Pi].

We start with the description of bounded linear functionals. Let X be a locally compact Hausdorff space, and let $W : X \to (0, \infty]$ be a lower semicontinuous function. Set $\Omega := \{x \in X : W(x) \neq \infty\}$ and

$$V(x) := \begin{cases} \frac{1}{W(x)}, & x \in \Omega, \\ 0, & x \in X \setminus \Omega, \end{cases}, \tag{2.1}$$

and denote by $\mathbb{M}_b(X)$ the space of all complex (bounded) Borel measures on X endowed with the norm $\|\mu\| := |\mu|(X)$, where $|\mu|$ denotes the total variation of the complex measure μ.

Consider the map T which assigns to each measure $\mu \in \mathbb{M}_b(X)$ the linear functional $T\mu$ defined as

$$(T\mu)f := \int_X fV \, d\mu, \quad f \in C_0(W). \tag{2.2}$$

Obviously, T is well-defined and maps $\mathbb{M}_b(X)$ into $C_0(W)'$, in fact

$$\|T\mu\| \leq \|\mu\|, \quad \mu \in \mathbb{M}_b(X).$$

The following statement is an immediate consequence of [S2, Theorems 3.1 and 4.5], just using in addition some standard approximation arguments (like Lusin's Theorem, cf. [R, 2.24]); we will not go into the details.

2.1 Theorem (Summers)**.** *Let $X \neq \emptyset$ be a locally compact Hausdorff space, let $W : X \to (0, \infty]$ be lower semicontinuous, and assume that $\Omega := \{x \in X : W(x) \neq \infty\}$ is dense in X. Then the map T defined by* (2.2) *maps $\mathbb{M}_b(X)$ surjectively onto $C_0(W)'$. Moreover, for each $\mu \in \mathbb{M}_b(X)$, the following hold*[5]*.*

[4]See also [K, Chapter VI.F], where a particular case is elaborated.

[5]We write $h\mu$ for the measure which is absolutely continuous with respect to μ and has Radon–Nikodym derivative h. Moreover, $\mathbb{1}_Y$ denotes the characteristic function of the set Y.

(*i*) *We have* $T\mu = T(\mathbb{1}_\Omega \mu)$ *and* $\|T\mu\| = \|\mathbb{1}_\Omega \mu\|$.

(*ii*) *The functional* $T\mu$ *is real (i.e.* $\forall\, f \in C_0(X), f \geq 0 : \ (T\mu)f \in \mathbb{R}$*), if and only if* $\mathbb{1}_\Omega \mu$ *is a real-valued measure.*

Next, we show the required weighted version of de Branges' lemma. This is done in essence by repeating the proof given in [dB1]; for completeness, we provide the details.

2.2 Lemma (de Branges)**.** *Let* W *be a weight, and let* $\mathcal{L}$ *be a linear subspace of* $C_0(W)$ *which is invariant with respect to complex conjugation. Assume that* $\overline{\mathcal{L}} \neq C_0(W)$. *Then there exists a positive Borel measure on* $\mathbb{R}$, $\mu \neq 0$, *such that the following hold.*

(*i*) $\displaystyle\int_{\mathbb{R}} W \, d\mu < \infty$.

(*ii*) *The annihilator of the space* $\mathcal{L} = \{f : f \in \mathcal{L}\} \subseteq L^1(\mu)$ *with respect to the duality between* $L^1(\mu)$ *and* $L^\infty(\mu)$ *is one-dimensional. It is spanned by a function* $g_0 \in L^\infty(\mu)$ *with* $|g_0(x)| = 1$, μ*-a.a.* $x \in \mathbb{R}$. *This function can be chosen to be real-valued.*

Proof. For each $\sigma \in \mathbb{M}_b(\overline{\Omega})$ we define a positive Borel measure $\tilde{\sigma}$ on $\mathbb{R}$ by (the function V is again defined by (2.1))

$$\tilde{\sigma}(E) := \int_{E\cap\Omega} V \, d|\sigma|, \quad E \text{ is a Borel set}\,.$$

Note that V is upper semicontinuous, and hence bounded on every compact subset of $\overline{\Omega}$. Thus $\tilde{\sigma}(E) < \infty$ whenever $E \subseteq \mathbb{R}$ is compact, and hence $\tilde{\sigma}$ indeed is a positive Borel measure on the real line[6]. Clearly, we have $\tilde{\sigma}(\mathbb{R} \setminus \Omega) = 0$ and $\int_{\mathbb{R}} W \, d\tilde{\sigma} < \infty$, in particular thus $C_0(W) \subseteq L^1(\tilde{\sigma})$.

Step 1: Let $\sigma \in \mathbb{M}_b(\overline{\Omega})$ with $T\sigma \neq 0$ be fixed. For each measurable and bounded function $g : \mathbb{R} \to \mathbb{C}$, we set ($T$ is defined as in (2.2) using the weight $W|_{\overline{\Omega}}$)

$$\Gamma_\sigma g := T(g\sigma)\,.$$

Then

$$(\Gamma_\sigma g)f = \int_{\overline{\Omega}} fV \cdot g \, d\sigma = \int_{\overline{\Omega}} f \cdot g \frac{d\sigma}{d|\sigma|} \, d\tilde{\sigma}, \quad f \in C_0(W|_{\overline{\Omega}})\,. \tag{2.3}$$

Denote by $C_{00}(\overline{\Omega})$ the space of all continuous functions on $\overline{\Omega}$ which have compact support. Then $C_{00}(\overline{\Omega}) \subseteq C_0(W|_{\overline{\Omega}})$, and hence (2.3) implies that $\Gamma_\sigma g = 0$ if and only if $g(x) = 0$ for $\tilde{\sigma}$-a.a. $x \in \overline{\Omega}$. We conclude that Γ_σ induces a well-defined and injective linear operator (again denoted as Γ_σ)

$$\Gamma_\sigma : L^\infty(\tilde{\sigma}) \to C_0(W|_{\overline{\Omega}})'\,.$$

Using the properties of T, we see that

(*a*) $\|\Gamma_\sigma\| \leq \|\sigma\|$;

[6] We always include the requirement that the measure of compact sets is finite into the notion of a Borel measure.

(*b*) if σ is real-valued, then Γ_σ maps real-valued functions to real functionals.

In the remaining part of this proof we work with annihilators of $\mathcal{L}$ buildt with respect to different dualities. To take this into account notationally, we denote the annihilator of $\mathcal{L}$ with respect to the duality between $C_0(W|_{\overline{\Omega}})$ and $\mathbb{M}_b(\overline{\Omega})$ by $\mathcal{L}^{\perp_0}$ and the one with respect to the duality between $L^1(\tilde{\sigma})$ and $L^\infty(\tilde{\sigma})$ by $\mathcal{L}^{\perp_1}$.

Invoking (2.3) we see that

(*c*) $\Gamma_\sigma g \in \mathcal{L}^{\perp_0}$ if and only if $g \in \big(\frac{d\sigma}{d|\sigma|}\big)^{-1}\mathcal{L}^{\perp_1}$.

Step 2: Consider the set

$$\Sigma := \big\{\phi \in C_0(W|_{\overline{\Omega}})' : \|\phi\| \leq 1,\ \phi \text{ real}\big\} \cap \mathcal{L}^{\perp_0}\,.$$

Clearly, Σ is w^*-compact and convex. Since $\overline{\mathcal{L}} \neq C_0(W)$, also $\overline{\mathcal{L}} \neq C_0(W|_{\overline{\Omega}})$. Hence, there exists $\phi \in C_0(W|_{\overline{\Omega}})'$, $\|\phi\| = 1$, with $\phi(f|_{\overline{\Omega}}) = 0$, $f \in \mathcal{L}$. Let $\mu \in \mathbb{M}_b(\overline{\Omega})$, $\|\mu\| = 1$ be such that $T\mu = \phi$. Since $\mathcal{L}$ is invariant with respect to complex conjugation, the functional $T\overline{\mu}$ also annihilates $\mathcal{L}$. Since $T\mu \neq 0$, one of $T(\operatorname{Re}\mu)$ and $T(\operatorname{Im}\mu)$ must be nonzero. We conclude that Σ contains a nonzero element.

Let $\sigma \in \mathbb{M}_b(\overline{\Omega})$ with $|\sigma|(\overline{\Omega}\setminus\Omega) = 0$ and $T\sigma \in \Sigma\setminus\{0\}$, and set

$$\mathcal{M}_\sigma := \Big\{g \in \Big(\frac{d\sigma}{d|\sigma|}\Big)^{-1}\mathcal{L}^{\perp_1} : g \geq 0,\ \int_{\overline{\Omega}} g\, d|\sigma| = \|\sigma\|\Big\}\,.$$

Due to the properties (*a*)–(*c*) from above, we see that

(*A*) $1 \in \mathcal{M}_\sigma$;

(*B*) $\Gamma_\sigma(\mathcal{M}_\sigma) \subseteq \Sigma$.

Claim: If $\dim \mathcal{L}^{\perp_1} > 1$, *then* 1 *is not an extremal point of* $\mathcal{M}_\sigma$.

Once this claim is established, the assertion of the lemma follows immediately: By the Krein–Milman theorem, the set Σ must contain a nonzero extremal point ϕ_0. Let $\sigma_0 \in \mathbb{M}_b(\overline{\Omega})$ be such that $|\sigma_0|(\overline{\Omega}\setminus\Omega) = 0$ and $T\sigma_0 = \phi_0$. Then the function 1 must be an extremal point of $\mathcal{M}_{\sigma_0}$ (otherwise, by the property (B) and linearity of Γ_{σ_0}, the function $\phi_0 = T\sigma_0 = \Gamma_{\sigma_0}1$ will not be an extremal point of Σ). Hence, the measure $\mu := \tilde{\sigma}_0$ has all properties required in the assertion of the lemma, the function g_0 in (*ii*) being $\frac{d\tilde{\sigma}_0}{d|\tilde{\sigma}_0|}$. Since $\mathcal{L}$ is invariant with respect to complex conjugation, also $\mathcal{L}^{\perp_1}$ has this property. Hence, there exists $\theta \in \mathbb{R}$ with $\overline{g_0} = e^{i\theta}g_0$. The function $e^{i\frac{\theta}{2}}g_0$ spans $\mathcal{L}^{\perp_1}$, and is unimodular and real-valued.

Step 3; Proving the claim: The measure σ is real-valued. This implies that $\big(\frac{d\sigma}{d|\sigma|}\big)^{-1}\mathcal{L}^{\perp_1}$ is invariant under complex conjugation, and hence that it equals the linear span of its real-valued elements. If $\dim\mathcal{L}^{\perp_1} > 1$, therefore, there must exist a real-valued element $g \in \big(\frac{d\sigma}{d|\sigma|}\big)^{-1}\mathcal{L}^{\perp_1}$ which is not equal to a constant $\tilde{\sigma}$-a.e. Set

$$h := \big(g + 2\|g\|_\infty\big)\cdot\Big(\int_{\overline{\Omega}} |g + 2\|g\|_\infty|\, d|\sigma|\Big)^{-1}\cdot\|\sigma\|\,.$$

Then $h \in \mathcal{M}_\sigma$, and is not equal to a constant $\tilde\sigma$-a.e.

Let us show that $\|h\|_\infty > 1$. We argue by contradiction. If we had $\|h\|_\infty \le 1$, then $1-h \ge 0$ $\tilde\sigma$-a.e. (and hence also $|\sigma|$-a.e.). This implies

$$\int_\Omega |1-h|\, d|\sigma| = \int_\Omega (1-h)\, d|\sigma| = \|\sigma\| - \int_\Omega h\, d|\sigma| = 0\,,$$

and hence $h = 1$ $|\sigma|$-a.e., a contradiction.

Set $t := \frac{1}{\|h\|_\infty}$, then $t \in (0,1)$. Consider the function

$$\tilde h := \frac{1-th}{1-t}\,.$$

Clearly, $\tilde h \in \big(\frac{d\sigma}{d|\sigma|}\big)^{-1}\mathcal{L}^{\perp_1}$. Since $t<1$, we have $\tilde h \ge 0$. Moreover,

$$\int_\Omega \tilde h\, d|\sigma| = \frac{1}{1-t}\Big(\|\sigma\| - t\int_\Omega h\, d|\sigma|\Big) = \|\sigma\|\,.$$

Together, we see that $\tilde h \in \mathcal{M}_\sigma$. Writing

$$1 = t\cdot h + (1-t)\cdot \frac{1-th}{1-t}$$

shows that 1 is not an extremal point of $\mathcal{M}_\sigma$. ❑

Finally, let us provide the required facts from [Pi]. Again for completeness, we show how they are extracted from this paper.

2.3 Theorem (Pitt). *Let μ be a positive Borel measure on the real line. Let $\mathcal{L}$ be an algebraic de Branges space with*

(i) $\displaystyle\int_\mathbb{R} |F|\, d\mu < \infty, \quad F \in \mathcal{L};$

(ii) If $F \in \mathcal{L}$ and $\int_\mathbb{R} |F|\, d\mu = 0$, then $F = 0$.

Assume that $\mathcal{L}$ is not dense in $L^1(\mu)$.

Then the function $\mathfrak{m} : \mathbb{C} \to [0,\infty]$ defined as

$$\mathfrak{m}(z) := \sup\big\{|F(z)| :\ F \in \mathcal{L}, \|F\|_{L^1(\mu)} \le 1\big\}$$

is everywhere finite and continuous. Each element $f \in \mathrm{Clos}_{L^1(\mu)}\mathcal{L}$ equals μ-a.e. the restriction of an entire function F with

$$|F(z)| \le \mathfrak{m}(z)\|f\|, \qquad z \in \mathbb{C}\,. \tag{2.4}$$

For each two functions $f,g \in \mathrm{Clos}_{L^1(\mu)}\mathcal{L}$, and entire functions F,G with $F|_\mathbb{R} = f$, $G|_\mathbb{R} = g$ μ-a.e., which are subject to (2.4), the quotient $\frac{F}{G}$ is a meromorphic function of bounded type in both half–planes $\mathbb{C}^+$ and $\mathbb{C}^-$.

Proof. Assume that there exists $z \in \mathbb{C}\setminus\mathbb{R}$ with $\mathfrak{m}(z) = \infty$. By symmetry, also $\mathfrak{m}(\overline{z}) = \infty$. We obtain from [Pi, Proposition 2.4,Theorem 3.1] that $\mathcal{L}$ is dense in $L^1(\mu)$, a contradiction. Hence, $\mathfrak{m}$ is finite on $\mathbb{C}\setminus\mathbb{R}$. By [Pi, Theorem 3.2], the function $\mathfrak{m}$ is finite and continuous in the whole plane. The fact that each function $f \in \mathrm{Clos}_{L^1(\mu)}\mathcal{L}$ can be (μ-a.e.) extended to an entire function subject to (2.4) is shown in the same theorem[7]. By [Pi, Proposition 3.4, Theorem A.1], the quotient of each two such functions is of bounded type in $\mathbb{C}^+$ and $\mathbb{C}^-$. ❑

[7] In [Pi, Theorem 3.2] it is claimed that this extension is unique. It seems that this is not true in general: the word ‘unique’ at the end of the third line of this statement should be deleted.

3 Necessity of '$\mathcal{K}(\mathcal{L}, W) \neq \emptyset$'

In this section we show the implication $(i) \Rightarrow (ii)$ in Theorem 1.6. The proof is carried out in five steps. Throughout the discussion, we denote

– by $\mathbb{H}(\mathbb{C})$ the space of all entire functions endowed with the topology of locally uniform convergence;

– by $\rho : \mathbb{H}(\mathbb{C}) \to C(\mathbb{R})$ the restriction map $F \mapsto F|_{\mathbb{R}}$;

– by $\chi_w : \mathbb{H}(\mathbb{C}) \to \mathbb{C}$ the point evaluation map $F \mapsto F(w)$.

Notice that the case that $\mathcal{L} = \{0\}$ in Theorem 1.6 is trivial: we can choose for B any function of the form $B(z) = z - x_0$ with $x_0 \in \mathbb{R}$ such that $W(x_0) \neq 0$. Hence, we may assume throughout that $\mathcal{L} \neq \{0\}$.

In Steps 1 and 2, we do not use the assumption Theorem 1.6, (i). The arguments given in these steps work in general. From Step 3 on, the assumption of the theorem enters in the form of de Branges lemma.

Step 1; The bounded extension operator.

Let μ be a positive Borel measure on the real line, and let $\mathcal{L}$ be an algebraic de Branges space which is injectively contained in $L^1(\mu)$ and is not dense in this space.

By Pitt's theorem each element $f \in \overline{\rho\mathcal{L}}$ can be extended to an entire function. In this step we show, among other things, that one can achieve that this extension process is a linear and continuous map.

Applying Pitt's theorem, we obtain that the function

$$\mathfrak{m}(z) := \sup\big\{|F(z)| : \, f \in \mathcal{L}, \|\rho F\|_{L^1(\mu)} \leq 1\big\}, \quad z \in \mathbb{C},$$

is everywhere finite and continuous. In particular, it is locally bounded, and hence the map

$$\iota := (\rho|_{\mathcal{L}})^{-1} : \rho\mathcal{L} \subseteq L^1(\mu) \longrightarrow \mathcal{L}$$

is continuous in the topology of $\mathbb{H}(\mathbb{C})$. Denote by $\tilde{\iota} : \overline{\rho\mathcal{L}} \subseteq L^1(\mu) \to \mathbb{H}(\mathbb{C})$ its extension by continuity.

It is important to show that $\rho \circ \tilde{\iota} = \mathrm{id}_{\overline{\rho\mathcal{L}}}$. Note that the map ρ is in general not continuous; locally uniform convergence need not imply L^1-convergence. Hence the stated equality does not follow at once, just by 'extension by continuity'. Let $f \in \overline{\rho\mathcal{L}}$ be given. Choose a sequence $(F_n)_{n\in\mathbb{N}}$, $F_n \in \mathcal{L}$, with

$$\lim_{n\to\infty} \rho F_n = f \quad \text{in } L^1(\mu)\,, \tag{3.1}$$

and extract a subsequence $(F_{n_k})_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} (\rho F_{n_k})(x) = f(x), \quad x \in \mathbb{R} \ \mu\text{-a.e.}$$

By continuity of $\tilde{\iota}$, the relation (3.1) implies that $\lim_{n\to\infty} \tilde{\iota}\rho F_n = \tilde{\iota} f$ locally uniformly. In particular,

$$\lim_{k\to\infty} \big(\rho\tilde{\iota}\rho F_{n_k}\big)(x) = (\rho\tilde{\iota} f)(x), \quad x \in \mathbb{R}\,.$$

However, by the definition of $\tilde{\iota}$ we have $\rho \circ \tilde{\iota}|_{\rho\mathcal{L}} = \mathrm{id}_{\rho\mathcal{L}}$, and hence

$$\rho\tilde{\iota}\rho F_{n_k} = \rho F_{n_k}\,.$$

We conclude that $f(x) = (\rho\tilde{\iota}f)(x)$, $x \in \mathbb{R}$ μ-a.e., and this means that $f = \rho\tilde{\iota}f$ in $L^1(\mu)$.

Setting

$$\tilde{\mathcal{L}} := \tilde{\iota}(\overline{\rho\mathcal{L}}), \qquad \tilde{\phi}_w := \chi_w \circ \tilde{\iota}, \quad \phi_w := \chi_w \circ \iota = \tilde{\phi}_w|_{\rho\mathcal{L}}\,,$$

we can summarize in a diagram:

$$\begin{tikzcd}
\begin{matrix}\{F\in\mathbb{H}(\mathbb{C}):\\ \rho F\in L^1(\mu)\}\end{matrix} \arrow[rr, "\rho"] & & L^1(\mu) \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\tilde{\mathcal{L}} \arrow[rr, shift left, "\rho|_{\tilde{\mathcal{L}}}"] \arrow[dddr, bend right=60, "\chi_w|_{\tilde{\mathcal{L}}}"'] & & \overline{\rho\mathcal{L}} \arrow[ll, shift left, "\tilde{\iota}"] \arrow[dddl, bend left=60, "\tilde{\phi}_w"] \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\mathcal{L} \arrow[rr, shift left, "\rho|_{\mathcal{L}}"] \arrow[dr, "\chi_w|_{\mathcal{L}}"'] & & \rho\mathcal{L} \arrow[ll, shift left, "\iota"] \arrow[dl, "\phi_w"] \\
& \mathbb{C} &
\end{tikzcd}$$

Notice that, since $\tilde{\mathcal{L}} = \operatorname{ran}\tilde{\iota}$, we also have $\tilde{\iota} \circ \rho|_{\tilde{\mathcal{L}}} = \mathrm{id}_{\tilde{\mathcal{L}}}$. I.e., the map $\rho|_{\tilde{\mathcal{L}}}$ maps $\tilde{\mathcal{L}}$ bijectively onto $\overline{\rho\mathcal{L}}$ and its inverse equals $\tilde{\iota}$. Moreover, the already noted fact that ρ is in general not continuous reflects in the fact that $\tilde{\mathcal{L}}$ is in general not closed in $\mathbb{H}(\mathbb{C})$. For example, in the case that $\mathcal{L} = \mathbb{C}[z]$, the closure of $\mathcal{L}$ is all of $\mathbb{H}(\mathbb{C})$.

Let us compute the norm of the functionals $\tilde{\phi}_w$. Let $f \in \rho\mathcal{L}$ and set $F := \tilde{\iota}f$. Then

$$|\tilde{\phi}_w f| = |(\chi_w \circ \tilde{\iota})f| = |F(w)| \leq \mathfrak{m}(w)\|f\|\,.$$

By continuity this relation holds for all $f \in \overline{\rho\mathcal{L}}$, and we obtain that

$$\|\tilde{\phi}_w\| \leq \mathfrak{m}(w), \qquad w \in \mathbb{C}\,.$$

In other words, each function $F \in \tilde{\mathcal{L}}$ satisfies

$$|F(w)| \leq \mathfrak{m}(w)\|\rho F\|, \qquad w \in \mathbb{C}\,.$$

Step 2; Showing 'algebraic de Branges space'.

Consider the same setting as in the previous step. We are going to show that $\tilde{\mathcal{L}}$ is an algebraic de Branges space.

The map $.^\# : F \mapsto F^\#$ maps $\mathbb{H}(\mathbb{C})$ continuously into itself and is involutory. Since $\mathcal{L}$ is an algebraic de Branges space, its restriction $.^\#|_{\mathcal{L}}$ maps $\mathcal{L}$ onto $\mathcal{L}$. Complex conjugation $\bar{.} : f \mapsto \overline{f}$ is an involutory homeomorphism of $L^1(\mu)$ onto itself. Clearly,

$$\rho|_{\mathcal{L}} \circ \big(.^\#|_{\mathcal{L}}\big) = \big(\bar{.}|_{\rho\mathcal{L}}\big) \circ \rho|_{\mathcal{L}}\,.$$

First, this relation implies that $\bar{.}$ maps $\rho\mathcal{L}$ onto itself, and hence also $\overline{\rho\mathcal{L}}$ onto itself. Second, it implies that $.^{\#}\circ\iota=\iota\circ(\bar{.}|_{\rho\mathcal{L}})$, and hence, by continuity, that also $.^{\#}\circ\tilde{\iota}=\tilde{\iota}\circ(\bar{.}|_{\overline{\rho\mathcal{L}}})$. Thus, in fact, $.^{\#}$ maps $\tilde{\mathcal{L}}$ into itself.

Let $w\in\mathbb{C}$ be fixed, and consider the difference quotient operator $\mathcal{R}_w:\mathbb{H}(\mathbb{C})\times\mathbb{H}(\mathbb{C})\to\mathbb{H}(\mathbb{C})$, that is

$$\mathcal{R}_w[F,G](z):=\begin{cases}\frac{F(z)G(w)-G(z)F(w)}{z-w}\,, & z\neq w\\ F'(w)G(w)-G'(w)F(w)\,, & z=w\end{cases}\,,\quad F,G\in\mathbb{H}(\mathbb{C})\,.$$

By the Schwarz lemma we have, for each compact set K (denote $B_1(w):=\{z\in\mathbb{C}:\,|z-w|\leq 1\}$),

$$\sup_{z\in K}|\mathcal{R}_w[F,G](z)|\leq 2\sup_{z\in K\cup B_1(w)}|F(z)|\cdot\sup_{z\in K\cup B_1(w)}|G(z)|,\quad F,G\in\mathbb{H}(\mathbb{C})\,,$$

and this implies that $\mathcal{R}_w$ is continuous.

The L^1-counterpart of the difference quotient operator is the map $\mathcal{R}^1_w$ defined on $\overline{\rho\mathcal{L}}\times\overline{\rho\mathcal{L}}$ as (the second alternative occurs of course only if $w\in\mathbb{R}$)

$$\mathcal{R}^1_w[f,g](x):=\begin{cases}\frac{f(x)(\tilde{\iota}g)(w)-g(x)(\tilde{\iota}f)(w)}{x-w}\,, & x\neq w\\ (\tilde{\iota}f)'(w)(\tilde{\iota}g)(w)-(\tilde{\iota}g)'(w)(\tilde{\iota}f)(w)\,, & x=w\end{cases}\,,\quad f,g\in\overline{\rho\mathcal{L}}\,.$$

From this definition we immediately see that

$$\mathcal{R}^1_w=\rho\circ\mathcal{R}_w\circ(\tilde{\iota}\times\tilde{\iota})\,. \tag{3.2}$$

Let us show that indeed $\mathcal{R}^1_w[f,g]\in L^1(\mu)$ and that $\mathcal{R}^1_w$ is continuous. First,

$$\big\|\mathbb{1}_{\mathbb{R}\setminus B_1(w)}\mathcal{R}^1_w[f,g]\big\|_{L^1(\mu)}\leq 2\mathfrak{m}(w)\|f\|_{L^1(\mu)}\|g\|_{L^1(\mu)}\,,$$

and this implies that the map $(f,g)\mapsto\mathbb{1}_{\mathbb{R}\setminus B_1(w)}\mathcal{R}^1_w[f,g]$ maps $\overline{\rho\mathcal{L}}\times\overline{\rho\mathcal{L}}$ continuously into $L^1(\mu)$. The function $\mathbb{1}_{\mathbb{R}\cap B_1(w)}\mathcal{R}^1_w[f,g]$ is μ-a.e. piecewise continuous and has compact support. Hence, it clearly belongs to $L^1(\mu)$. Moreover, due to (3.2), the map $(f,g)\mapsto\mathbb{1}_{\mathbb{R}\cap B_1(w)}\mathcal{R}^1_w[f,g]$ is continuous as a composition of continuous maps. Note here that, although ρ itself is not continuous, for each compact set K the map $F\mapsto\mathbb{1}_K\cdot\rho F$ is.

Since $\mathcal{L}$ is an algebraic de Branges space, we have $\mathcal{R}_w(\mathcal{L}\times\mathcal{L})\subseteq\mathcal{L}$. Due to (3.2), thus also $\mathcal{R}^1_w(\rho\mathcal{L}\times\rho\mathcal{L})\subseteq\rho\mathcal{L}$, and continuity implies

$$\mathcal{R}^1_w\big(\overline{\rho\mathcal{L}}\times\overline{\rho\mathcal{L}}\big)\subseteq\overline{\rho\mathcal{L}}\,.$$

Now we may multiply (3.2) with $\tilde{\iota}$ from the left and $\rho\times\rho$ from the right to obtain

$$\tilde{\iota}\circ\mathcal{R}^1_w\circ(\rho\times\rho)|_{\tilde{\mathcal{L}}\times\tilde{\mathcal{L}}}=\mathcal{R}_w|_{\tilde{\mathcal{L}}\times\tilde{\mathcal{L}}}$$

and conclude that $\mathcal{R}_w(\tilde{\mathcal{L}}\times\tilde{\mathcal{L}})\subseteq\tilde{\mathcal{L}}$. In particular, $\tilde{\mathcal{L}}$ is invariant with respect to division of zeros.

<u>*Step 3; Invoking de Branges' Lemma.*</u>

From now on assume that Theorem 1.6, (i), holds. De Branges' lemma provides us with a positive Borel measure μ, $\mu\neq 0$, and a real-valued function $g_0\in L^\infty(\mu)$ with $|g_0|=1$ μ-a.e., such that

(i) $\int_{\mathbb{R}} W\, d\mu < \infty$, in particular $\rho\mathcal{L} \subseteq L^1(\mu)$;

(ii) $(\rho\mathcal{L})^\perp = \operatorname{span}\{g_0\}$, and hence $\overline{\rho\mathcal{L}} = \{g_0\}^\perp$.

Since $\mathcal{L} \neq \{0\}$, the support of the measure μ must contain at least two points.

The first thing to show is that ρ maps $\mathcal{L}$ injectively into $L^1(\mu)$. Assume that $F \in \mathcal{L}$ and that $F|_{\mathbb{R}} = 0$ μ-a.e. If $\operatorname{supp}\mu$ is not discrete, this implies immediately that $F = 0$. Hence assume that $\operatorname{supp}\mu$ is discrete. Then we must have $F(x) = 0$, $x \in \operatorname{supp}\mu$. Pick $x_0 \in \operatorname{supp}\mu$, denote by l the multiplicity of x_0 as a zero of F, and set $G(z) := (z - x_0)^{-l} F(z)$. Then $G \in \mathcal{L}$, and $G(x_0) \neq 0$ whereas $G(x) = 0$ for all $x \in \operatorname{supp}\mu \setminus \{x_0\}$. Since $g_0(x_0) \neq 0$, this contradicts the fact that $\int_{\mathbb{R}} G g_0\, d\mu = 0$.

Next, we show that the measure μ is discrete. Assume on the contrary that $x_0 \in \mathbb{R}$ is an accumulation point of $\operatorname{supp}\mu$. Choose an interval $[a,b]$, such that $x_0 \notin [a,b]$ and $\operatorname{card}([a,b] \cap \operatorname{supp}\mu) \geq 2$. Then $\dim L^1(\mu|_{[a,b]}) > 1$, and we can choose $f \in L^1(\mu) \setminus \{0\}$ with

$$f(x) = 0,\ x \in \mathbb{R} \setminus [a,b], \qquad \int_{\mathbb{R}} f g_0\, d\mu = 0\,.$$

Then $f \in \overline{\rho\mathcal{L}}$, and hence

$$f(x) = (\rho\tilde{\iota} f)(x), \quad x \in \mathbb{R}\ \mu\text{-a.e.}$$

The function $F := \tilde{\iota} f$ is entire and does not vanish identically. However, since $\rho F = f$ μ-a.e., we must have $F(x) = 0$, $x \in \mathbb{R}\setminus[a,b]$ μ-a.e. The set $(\operatorname{supp}\mu)\setminus[a,b]$ has the accumulation point x_0, and we conclude that $F = 0$, a contradiction.

As a consequence, we can interpret the action of $\tilde{\phi}_x$ for $x \in \operatorname{supp}\mu$ as point evaluation: Let $f \in \overline{\rho\mathcal{L}}$, then $f = \rho\tilde{\iota} f$ in $L^1(\mu)$, i.e.,

$$f(x) = (\tilde{\iota} f)(x), \quad x \in \operatorname{supp}\mu\,.$$

It follows that

$$\tilde{\phi}_x f = (\chi_x \circ \tilde{\iota}) f = (\tilde{\iota} f)(x) = f(x), \quad x \in \operatorname{supp}\mu\,.$$

<u>*Step 4; The functions H_t.*</u>

Fix a point $t_0 \in \operatorname{supp}\mu$. For each $t \in [\operatorname{supp}\mu] \setminus \{t_0\}$ we define a function $h_t : \mathbb{R} \to \mathbb{C}$ as

$$h_t(x) := \begin{cases} -[(t - t_0) g_0(t_0) \mu(\{t_0\})]^{-1}, & x = t_0, \\ [(t - t_0) g_0(t) \mu(\{t\})]^{-1}, & x = t, \\ 0, & \text{otherwise.} \end{cases}$$

Remember here that $\operatorname{supp}\mu$ contains at least two points. We have $h_t \in L^1(\mu)$ and $\int_{\mathbb{R}} h_t g_0\, d\mu = 0$, and hence $h_t \in \overline{\rho\mathcal{L}}$. Moreover, $h_t \in \ker\tilde{\phi}_{t'}$ whenever $t' \in [\operatorname{supp}\mu] \setminus \{t_0, t\}$. Define

$$H_t := \tilde{\iota} h_t\,.$$

We establish the essential properties of the functions H_t in the following three lemmata.

3.1 Lemma. *Let $t, t' \in [\operatorname{supp}\mu] \setminus \{t_0\}$. Then*

$$(z-t)H_t(z) = (z-t')H_{t'}(z)\,. \tag{3.3}$$

Proof. For $t = t'$ this relation is of course trivial. Hence, assume that $t \neq t'$. Choose a function $G_0 \in \mathcal{L}$ with $G_0(t') = 1$, and consider the function

$$f := \big(\operatorname{id} + (t-t')\mathcal{R}^1_{t'}[.,\rho G_0]\big)h_t \in \overline{\rho\mathcal{L}}\,.$$

The value of f at points $x \in \mathbb{R} \setminus \{t'\}$ is computed easily from the definition of $\mathcal{R}^1_{t'}[.,\rho G_0]$:

$$\begin{aligned} f(x) = h_t(x) + (t-t')\frac{h_t(x)}{x-t'} = \frac{x-t}{x-t'}h_t(x) = \\ = \begin{cases} -[(t'-t_0)g_0(t_0)\mu(\{t_0\})]^{-1}, & x = t_0, \\ 0, & x \in \mathbb{R} \setminus \{t_0, t'\}, \end{cases} \end{aligned}$$

Since $f \in \overline{\rho\mathcal{L}}$, we have $\int_{\mathbb{R}} f g_0\, d\mu = 0$, and hence the value of f at t' must be

$$f(t') = [(t'-t_0)g_0(t')\mu(\{t'\})]^{-1}\,.$$

We see that $f = h_{t'}$.

Now we can compute

$$\begin{aligned} H_{t'} = \tilde{\imath}h_{t'} = \tilde{\imath}\big([\operatorname{id} + (t-t')\mathcal{R}^1_{t'}[.,\rho G_0]\big)h_t = \\ = \big(\operatorname{id} + (t'-t)\mathcal{R}_{t'}[.,\rho G_0]\big)\tilde{\imath}h_t = \big(\operatorname{id} + (t'-t)\mathcal{R}_{t'}[.,\rho G_0]\big)H_t\,. \end{aligned}$$

However,

$$\big(\operatorname{id} + (t'-t)\mathcal{R}_{t'}[.,\rho G_0]\big)H_t(z) = \frac{z-t}{z-t'}H_t(z)\,,$$

and the desired relation (3.3) follows. ❑

3.2 Lemma. *We have $H_t = H_t^{\#}$. The function H_t has simple zeros at the points $[\operatorname{supp}\mu] \setminus \{t_0, t\}$, and no zeros elsewhere.*

Proof. Since $h_t = \overline{h_t}$, we have

$$H_t^{\#} = (\tilde{\imath}h_t)^{\#} = \tilde{\imath}(\overline{h_t}) = \tilde{\imath}h_t = H_t\,.$$

Let $t' \in [\operatorname{supp}\mu] \setminus \{t_0, t\}$. Since $H_{t'}(t') = h_{t'}(t') \neq 0$, the relation (3.3) shows that H_t has a simple zero at t'. For $x \in \{t_0, t\}$, we have $H_t(x) = h_t(x) \neq 0$.

Let $w \in \mathbb{C} \setminus \operatorname{supp}\mu$, and assume on the contrary that $H_t(w) = 0$. Then $H_t \in \ker(\chi_w|_{\tilde{\mathcal{L}}})$, and therefore (choose $G_0 \in \mathcal{L}$ with $G_0(w) = 1$)

$$G := \big(\operatorname{id} + (w-t)\mathcal{R}_w[.,G_0]\big)H_t \in \tilde{\mathcal{L}}\,.$$

This implies that $\rho G \in \overline{\rho\mathcal{L}}$.

Clearly, $G(z) = \frac{z-t}{z-w}H_t(z)$, and we can evaluate G at points $x \in \operatorname{supp}\mu$ as

$$G(x) = \begin{cases} [(t_0-w)g_0(t_0)\mu(\{t_0\})]^{-1}, & x = t_0, \\ 0, & x \in [\operatorname{supp}\mu] \setminus \{t_0\}. \end{cases}$$

This shows that $\int_{\mathbb{R}} G g_0\, d\mu \neq 0$, and we have reached a contradiction. ❑

3.3 Lemma. *For each $F \in \tilde{\mathcal{L}}$ we have*

$$F(z) = \sum_{\substack{t \in \operatorname{supp}\mu \\ t \neq t_0}} F(t)\mu(\{t\})g_0(t)(t-t_0)H_t(z)\,,$$

where the series converges locally uniformly.

Proof. Since $|g_0| = 1$ μ-a.e., we have

$$\|h_t\|_{L^1(\mu)} = \int_{\mathbb{R}} |h_t|\,d\mu = \frac{1}{|t-t_0|}\Big(\frac{1}{|g_0(t_0)|} + \frac{1}{|g_0(t)|}\Big) = \frac{2}{|t-t_0|}\,.$$

Hence,

$$\|g_0(t)(t-t_0)h_t\|_{L^1(\mu)} = 2\,,$$

and therefore for each $f \in L^1(\mu)$ the series

$$g := \sum_{\substack{t \in \operatorname{supp}\mu \\ t \neq t_0}} f(t)\mu(\{t\}) \cdot g_0(t)(t-t_0)h_t$$

converges in the norm of $L^1(\mu)$. Since $h_t \in \overline{\rho\mathcal{L}}$, it follows that also $g \in \overline{\rho\mathcal{L}}$, i.e. $\int_{\mathbb{R}} gg_0\,d\mu = 0$.

For $x \in [\operatorname{supp}\mu] \setminus \{t_0\}$, we can evaluate

$$g(x) = \sum_{\substack{t \in \operatorname{supp}\mu \\ t \neq t_0}} f(t)\mu(\{t\}) \cdot g_0(t)(t-t_0)h_t(x) =$$

$$= f(x)\mu(\{x\}) \cdot g_0(x)(x-t_0)h_x(x) = f(x)\,.$$

Hence, the functions g and f differ, up to a μ-zero set, at most at the point t_0. If we know in addition that $f \in \overline{\rho\mathcal{L}}$, then also $\int_{\mathbb{R}} fg_0\,d\mu = 0$, and it follows that $f(t_0) = g(t_0)$, i.e., that $f = g$ in $L^1(\mu)$.

Now let $F \in \tilde{\mathcal{L}}$ be given. Then $\rho F \in \overline{\rho\mathcal{L}}$, and therefore

$$\rho F = \sum_{\substack{t \in \operatorname{supp}\mu \\ t \neq t_0}} F(t)\mu(\{t\}) \cdot g_0(t)(t-t_0)h_t\,.$$

Applying $\tilde{\iota}$, yields the desired representation of F. ❑

Step 5; Construction of $B \in \mathcal{K}(\mathcal{L}, W)$.

Choose $t \in [\operatorname{supp}\mu] \setminus \{t_0\}$, and define

$$B(z) := (z-t_0)(z-t)H_t(z)\,. \tag{3.4}$$

Due to (3.3), this definition does not depend on the choice of t. We are going to show that $B \in \mathcal{K}(\mathcal{L}, W)$. In fact, we will see that (K2) and (K3) hold for all $F \in \tilde{\mathcal{L}}$, i.e., $B \in \mathcal{K}(\tilde{\mathcal{L}}, W)$.

By Lemma 3.3, we have $B = B^\#$, and know that B has simple zeros at the points $\operatorname{supp}\mu$ and no zeros otherwise; this is (K1). By Theorem 2.3, for each $F \in \tilde{\mathcal{L}}$, the function $\frac{F}{H_t}$ is of bounded type in $\mathbb{C}^+$ and $\mathbb{C}^-$. Hence also $\frac{F}{B}$ has this property; and this is (K2).

To show (K3), let $F \in \tilde{\mathcal{L}}$ be given. By Lemma 3.3,

$$\begin{aligned}(z-t_0)\frac{F(z)}{B(z)} =&(z-t_0)\sum_{\substack{t\in\operatorname{supp}\mu\\ t\neq t_0}} F(t)\mu(\{t\})g_0(t)(t-t_0)\frac{H_t(z)}{B(z)} = \\ =& \sum_{\substack{t\in\operatorname{supp}\mu\\ t\neq t_0}} F(t)\mu(\{t\})\cdot g_0(t)\frac{t-t_0}{z-t}\,.\end{aligned}$$

We have

$$\sum_{t\in\operatorname{supp}\mu}|F(t)|\mu(\{t\})<\infty,\quad |g_0(t)|=1,\qquad \sup_{\substack{|y|\geq 1\\ t\in\operatorname{supp}\mu}}\Big|\frac{t-t_0}{iy-t}\Big|<\infty\,,$$

and hence, by bounded convergence,

$$\lim_{y\to\pm\infty}(iy-t_0)\frac{F(iy)}{B(iy)}=0\,.$$

Finally, for (K4), compute ($t\in[\operatorname{supp}\mu]\setminus\{t_0\}$ arbitrary)

$$\begin{aligned}B'(x)=&\begin{cases}(x-t_0)H_x(x), & x\in[\operatorname{supp}\mu]\setminus\{t_0\}\\ (x-t)H_t(x), & x=t_0\end{cases}\\ =&\frac{1}{g_0(x)\mu(\{x\})}\,.\end{aligned}$$

Remembering that $|g_0(x)|=1$ μ-a.e., we conclude that

$$\sum_{x\in\operatorname{supp}\mu}\frac{W(x)}{|B'(x)|}=\int_{\mathbb{R}}W\,d\mu<\infty\,,$$

and this is (K4).

4 Computing mean type

In this section we show the additional statement in Theorem 1.6, that the function B can be chosen such that (1.1) holds. In fact, we show that the function B constructed in the previous section, cf. (3.4), satisfies (1.1).

First, let us observe that it is enough to prove that

$$\sup\Big\{\operatorname{mt}\frac{F}{\mathfrak{m}}:\,F\in\mathcal{L}\Big\}=0\,. \tag{4.1}$$

Indeed, by the definition of B, we have ($t\in[\operatorname{supp}\mu]\setminus\{t_0\}$)

$$|B(iy)|=|iy-t_0|\cdot|iy-t|\cdot|H_t(iy)|\leq(y^2+t_0^2)^{\frac12}(y^2+t^2)^{\frac12}\cdot\mathfrak{m}(iy)\|H_t\|_{L^1(\mu)}\,.$$

Hence, for each $F\in\mathcal{L}$,

$$\Big|\frac{F(iy)}{B(iy)}\Big|\geq\Big[(y^2+t_0^2)^{\frac12}(y^2+t^2)^{\frac12}\cdot\|H_t\|_{L^1(\mu)}\Big]^{-1}\cdot\frac{|F(iy)|}{\mathfrak{m}(iy)}\,,$$

and this implies that $\operatorname{mt}\frac{F}{B} \geq \operatorname{mt}\frac{F}{\mathfrak{m}}$. By (K3) always $\operatorname{mt}\frac{F}{B} \leq 0$, and it follows that (4.1) implies (1.1).

A proof of (4.1) can be obtained from studying of the space

$$\mathcal{K} := \Big\{ F \in \mathbb{H}(\mathbb{C}) : \quad F|_{\mathbb{R}} \in L^1(\mu), \quad |F(iy)| = O\big(\mathfrak{m}(iy)\big) \text{ as } |y| \to \infty, \\ \exists\, F_0 \in \mathcal{L} \setminus \{0\} : \frac{F}{F_0} \text{ is of bounded type in } \mathbb{C}^+ \text{ and } \mathbb{C}^- \Big\},$$

and the group of operators

$$M_\alpha : \left\{ \begin{array}{ccc} \mathbb{H}(\mathbb{C}) & \to & \mathbb{H}(\mathbb{C}) \\ F(z) & \mapsto & e^{i\alpha z}F(z) \end{array} \right. , \qquad \alpha \in \mathbb{R}\,.$$

The following fact is crucial.

4.1 Lemma. *The restriction map $\rho : \mathcal{K} \to L^1(\mu)$ is injective.*

This fact follows immediately from the last sentence in [Pi, p.284, Remarks]. However, in [Pi] no explicit proofs of these remarks are given. Hence, we include a proof, sticking to what is needed for the present purpose[8].

Proof (of Lemma 4.1). Assume on the contrary that there exists a function $G \in \mathcal{K} \setminus \{0\}$ with $G|_{\mathbb{R}} = 0$ μ-a.e. Let $F \in \mathcal{L}$, let $z \in \mathbb{C}^+$ with $G(z) \neq 0$, and consider the function

$$H(z,x) = \frac{F(z)G(x) - G(z)F(x)}{z - x}, \quad x \in \mathbb{R}\,.$$

Then $H(z,.) \in L^1(\mu)$, and

$$H(z,x) = -G(z)\frac{F(x)}{z-x}, \quad x \in \mathbb{R}\ \mu\text{-a.e.}$$

The proof of [Pi, Theorem 3.3], with the modification also used in [Pi, Theorem 3.4], shows that $H(z,.) \in \overline{\rho\mathcal{L}}$. Hence, $\frac{F(x)}{z-x} \in \overline{\rho\mathcal{L}}$ whenever $F \in \mathcal{L}$, and since multiplication with $\frac{1}{z-x}$ is for each fixed $z \in \mathbb{C}\setminus\mathbb{R}$ a bounded operator on $L^1(\mu)$, it follows that

$$\frac{F(x)}{z-x} \in \overline{\rho\mathcal{L}}, \quad F \in \tilde{\mathcal{L}},\ z \in \mathbb{C} \setminus \mathbb{R}, G(z) \neq 0\,.$$

Consider now the function H_t constructed in Step 4 of the previous section. A short computation shows that for no nonreal z the function $\frac{H_t(x)}{z-x}$ is annihilated by g_0. We have reached a contradiction. ❑

4.2 Corollary. *We may define a norm on $\mathcal{K}$ by*

$$\|F\|_{\mathcal{K}} := \|\rho F\|_{L^1(\mu)}, \quad F \in \mathcal{K}\,,$$

and $\mathcal{K}$ is complete with respect to this norm. The point evaluation maps $\chi_w|_{\mathcal{K}} : \mathcal{K} \to \mathbb{C}$ are continuous with respect to $\|.\|_{\mathcal{K}}$.

[8] The general assertion in [Pi, p. 284, Remarks] can be shown with the same argument, only using more of the machinery developed earlier in [Pi].

Proof. Since ρ is injective, the norm $\|.\|_{\mathcal{K}}$ is well-defined. Since $\mathcal{K} \supseteq \tilde{\mathcal{L}}$, and $\dim\big(L^1(\mu)/\overline{\rho\mathcal{L}}\big) = 1$, there are only two possibilities: Either $\rho\mathcal{K} = \overline{\rho\mathcal{L}}$, or $\rho\mathcal{K} = L^1(\mu)$. In both cases, $\mathcal{K}$ is complete.

Since $\tilde{\mathcal{L}}$ is a closed subspace of $\mathcal{K}$ with finite codimension (and hence a complemented subspace), and since the restriction $(\chi_w|_{\mathcal{K}})|_{\tilde{\mathcal{L}}}$ of the point evaluation map $\chi_w|_{\mathcal{K}}$ to $\tilde{\mathcal{L}}$ is continuous, it follows that $\chi_w|_{\mathcal{K}}$ itself is continuous. ❑

Now bring in the family of spaces (parametrized by $\tau_+, \tau_- \leq 0$)

$$\mathcal{K}_{(\tau_+,\tau_-)} := \Big\{ F \in \mathcal{K} : \quad |F(iy)| = O\big(e^{\tau_+ y}\mathfrak{m}(iy)\big) \text{ as } y \to +\infty\,, \\ |F(iy)| = O\big(e^{\tau_- |y|}\mathfrak{m}(iy)\big) \text{ as } y \to -\infty \Big\}\,.$$

It is obvious that

$$M_\alpha\big(\mathcal{K}_{(\tau_+,\tau_-)}\big) \subseteq \mathcal{K}_{(\tau_+-\alpha,\tau_-+\alpha)}, \quad \alpha \in [\tau_+, -\tau_-]\,.$$

Applying this once again with $M_{-\alpha}$ and $\mathcal{K}_{(\tau_+-\alpha,\tau_-+\alpha)}$ in place of M_α and $\mathcal{K}_{(\tau_+,\tau_-)}$, it follows that in fact $M_\alpha|_{\mathcal{K}_{(\tau_+,\tau_-)}}$ is a bijection of $\mathcal{K}_{(\tau_+,\tau_-)}$ onto $\mathcal{K}_{(\tau_+-\alpha,\tau_-+\alpha)}$. Since M_α is isometric, there exists an extension $\hat{M}_{\alpha;\tau_+,\tau_-}$ to an isometric bijection of $\overline{\mathcal{K}_{(\tau_+,\tau_-)}}$ onto $\overline{\mathcal{K}_{(\tau_+-\alpha,\tau_-+\alpha)}}$. Since the point evaluation maps are continuous, we have $\hat{M}_{\alpha;\tau_+,\tau_-} = M_\alpha|_{\overline{\mathcal{K}_{(\tau_+,\tau_-)}}}$.

As a consequence, we obtain that the space $\mathcal{K}_{(\tau_+,\tau_-)}$ is a closed subspace of $\mathcal{K}$. Indeed, let $F \in \overline{\mathcal{K}_{(\tau_+,\tau_-)}}$ be given. Then $M_{\tau_+}F, M_{-\tau_-}F \in \mathcal{K}$. Hence,

$$\big|e^{i\tau_+(iy)}F(iy)\big| = O\big(\mathfrak{m}(iy)\big) \text{ as } y \to +\infty\,,$$
$$\big|e^{-i\tau_-(iy)}F(iy)\big| = O\big(\mathfrak{m}(iy)\big) \text{ as } y \to -\infty\,,$$

and this gives $|F(iy)| = O(e^{\tau_+ y}\mathfrak{m}(iy))$, $y \to +\infty$, and $|F(iy)| = O(e^{\tau_-|y|}\mathfrak{m}(iy))$, $y \to -\infty$.

To finish the proof of (4.1), one more simple observation is needed.

4.3 Remark. Let $F \in \mathcal{K} \setminus \{0\}$, and set $\tau_F := \operatorname{mt}\frac{F}{\mathfrak{m}}$. Then $-\infty < \tau_F \leq 0$, and

$$\begin{aligned} &F \in \mathcal{K}_{(\tau_+,0)}, \quad \tau_+ \in (\tau_F, 0]\,, \\ &F \notin \mathcal{K}_{(\tau_+,0)}, \quad \tau_+ < \tau_F\,. \end{aligned} \tag{4.2}$$

It is obvious that $\operatorname{mt}\frac{F}{\mathfrak{m}} \leq 0$ and that the relations (4.2) hold. Assume that $\operatorname{mt}\frac{F}{\mathfrak{m}} = -\infty$. Then $F \in \bigcap_{\tau_+\leq 0}\mathcal{K}_{(\tau_+,0)}$, and hence $M_\alpha F \in \mathcal{K}$ for all $\alpha \leq 0$. The family $\{M_\alpha F : \alpha \leq 0\}$ is bounded with respect to the norm $\|.\|_{\mathcal{K}}$, and by continuity of point evaluations thus also pointwise bounded. It follows that $F(z) = 0$ for all $z \in \mathbb{C}^+$, and hence everywhere. //

Proof (of (4.1)). Assume on the contrary that

$$\tau := \sup\Big\{ \operatorname{mt}\frac{F}{\mathfrak{m}} : F \in \mathcal{L}\Big\} < 0\,.$$

Then $\mathcal{L} \subseteq \mathcal{K}_{(\tau+\varepsilon,0)}$ and, since $\mathcal{K}_{(\tau+\varepsilon,0)}$ is closed in $\mathcal{K}$, we conclude that $\tilde{\mathcal{L}} \subseteq \mathcal{K}_{(\tau+\varepsilon,0)}$. Fix $F \in \mathcal{L} \setminus \{0\}$. Let $\alpha \in (\tau_F, \tau_F - \tau)$, and choose $\varepsilon > 0$ such that $[\alpha-\varepsilon, \alpha+\varepsilon] \subseteq (\tau_F, \tau_F - \tau)$ and $\tau + \varepsilon \leq 0$.

The function F belongs to $\mathcal{K}_{(\tau_F+\varepsilon,0)}$, and we have $\tau_F+\varepsilon\leq\alpha\leq 0$. Hence, $M_\alpha F\in\mathcal{K}_{(\tau_F+\varepsilon-\alpha,\alpha)}$. Assume that this function would belong to $\tilde{\mathcal{L}}$. Then it would also belong to $\mathcal{K}_{(\tau+\varepsilon,0)}$, and together with what we know, thus $M_\alpha F\in\mathcal{K}_{(\tau+\varepsilon,\alpha)}$. We have $\tau+\varepsilon<\tau_F-\alpha\leq-\alpha$, and hence

$$F=M_{-\alpha}(M_\alpha F)\in\mathcal{K}_{(\tau+\varepsilon+\alpha,0)}\,.$$

However, $\tau+\varepsilon+\alpha<\tau_F$, a contradiction in view of (4.2). We conclude that

$$M_\alpha F\in\mathcal{K}\setminus\tilde{\mathcal{L}},\quad \alpha\in(\tau_F,\tau_F-\tau)\,.$$

From this it immediately follows that

$$\operatorname{span}\big\{M_\alpha F:\,\alpha\in(\tau_F,\tau_F-\tau)\big\}\cap\tilde{\mathcal{L}}=\{0\}\,.$$

Clearly, the dimension of this linear span is infinite. Since ρ is injective, we however have $\dim(\mathcal{K}/\tilde{\mathcal{L}})=\dim(\rho\mathcal{K}/\overline{\rho\mathcal{L}})\leq 1$, and thus arrived at a contradiction. ❑

5 Sufficiency of '$\mathcal{K}(\mathcal{L},W)\neq\emptyset$'

The proof of the implication '$(ii)\Rightarrow(i)$' in Theorem 1.6 is established in the standard way based on a Lagrange-type interpolation formula for functions in $\mathcal{L}$. For the convenience of the reader, we provide the details.

5.1 Lemma. *Let $F\in\mathcal{L}$ and $B\in\mathcal{K}(\mathcal{L},W)$. Then*

$$(z-i)F(z)=\sum_{x:\,B(x)=0}\frac{(x-i)F(x)}{B'(x)}\frac{B(z)}{z-x}\,, \tag{5.1}$$

where the series converges locally uniformly in $\mathbb{C}$.

Proof. Since $F\in C_0(W)$, the series $\sum_{x:\,B(x)=0}\frac{F(x)}{B'(x)}$ converges absolutely. For each $R>0$, we have $\sup\big\{\big|\frac{(x-i)B(z)}{z-x}\big|:|z|\leq R, B(x)=0\big\}<\infty$, and hence the right side of (5.1) converges locally uniformly on $\mathbb{C}$.

The function

$$H(z):=\frac{(z-i)F(z)}{B(z)}-\sum_{x:\,B(x)=0}\frac{(x-i)F(x)}{B'(x)}\frac{1}{z-x}$$

is entire, since the only possible poles (which are the points x with $B(x)=0$) cancel. It is of bounded type in both half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$ (the first summand by assumption and the second since it is the Cauchy integral of a measure with $(1+|x|)^{-1}$ being integrable). Moreover, it tends to 0 along the imaginary axis (the first summand by assumption and the second by bounded convergence). Therefore, H vanishes identically. ❑

Proof (of Theorem 1.6, *'$\Leftarrow$').* Choose $B\in\mathcal{K}(\mathcal{L},W)$, and consider the measure

$$\mu:=\sum_{x:\,B(x)=0}\frac{1}{B'(x)}\delta_x\,,$$

where δ_x denotes the Dirac measure supported on $\{x\}$. Since

$$\int_{\mathbb{R}} W\, d|\mu| = \sum_{x:\, B(x)=0} \frac{W(x)}{|B'(x)|} < \infty\,,$$

the functional $\phi : f \mapsto \int_{\mathbb{R}} f\, d\mu$ belongs to $C_0(W)'$. Since $C_{00}(\mathbb{R}) \subseteq C_0(W)$ and B has at least one zero, ϕ does not vanish identically.

For each $F \in \mathcal{L}$, we apply (5.1) with '$z = i$' and obtain

$$0 = \sum_{x:\, B(x)=0} \frac{(x-i)F(x)}{B'(x)} \frac{B(i)}{i-x} = -B(i) \sum_{x:\, B(x)=0} \frac{F(x)}{B'(x)} = -B(i)\phi(F|_{\mathbb{R}})\,.$$

Thus, $\mathcal{L}$ is annihilated by μ, in particular, $\mathcal{L}$ is not dense in $C_0(W)$. ❑

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