SUMS, COUPLINGS, AND COMPLETIONS OF ALMOST PONTRYAGIN SPACES

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ABSTRACT. An almost Pontryagin space can be written as the direct and orthogonal sum of a Hilbert space, a finite-dimensional anti-Hilbert space, and a finite-dimensional neutral space. In this paper orthogonal sums of almost Pontryagin spaces and completions to almost Pontryagin spaces are studied.

1. INTRODUCTION

The notion of an almost Pontryagin space was introduced in [5] as a generalization of the more familiar notion of a Pontryagin space; cf. [2], [3]. A Pontryagin space is an inner product space which can be written as the direct and orthogonal sum of a Hilbert space and a finite dimensional anti-Hilbert space, whereas an almost Pontryagin space can be written as the direct and orthogonal sum of a Hilbert space, a finite-dimensional anti-Hilbert space, and a finite-dimensional neutral space. Almost Pontryagin spaces appear, sometimes implicitly, in [7], [8], [9], [10]. The introduction of these more general objects was motivated by several classical interpolation and extrapolation problems. Here is an example that may be illuminating; cf. [6]. Let the continuous function $f : [-2a, 2a] \to \mathbb{C}$ be hermitian, in the sense that $f(-t) = \overline{f(t)}$, with κ negative squares, so that the kernel $f(t-s)$, $s, t \in (-a, a)$ has κ negative squares. Then f has exactly one continuous hermitian extension to $\mathbb{R}$ with κ negative squares or it has infinitely many continuous hermitian extensions to $\mathbb{R}$ with κ negative squares. In the latter case f has also infinitely many continuous hermitian extensions to $\mathbb{R}$ with κ_1 negative squares for every $\kappa_1 \geq \kappa$. This result originates from the usual operator theoretic considerations involving the Pontryagin space induced by the problem. However, in the first case of the alternative it turns out that there exists a number $0 < \Delta \leq \infty$ such that f has no continuous hermitian extensions to $\mathbb{R}$ with κ_1 negative squares for $\kappa < \kappa_1 < \kappa + \Delta$, and infinitely many continuous hermitian extensions to $\mathbb{R}$ with κ_1 negative squares for $\kappa_1 \geq \kappa + \Delta$. This addition to the case where f has a unique extension originates from operator theoretic considerations involving an almost Pontryagin space induced by the problem.

The present paper continues the study of almost Pontryagin spaces, begun in [5], providing the tools for studying exit space extensions of isometric operators in almost Pontryagin spaces. This requires the notions of sums and orthogonal couplings of almost Pontryagin spaces. An elementary construction for inner product structures is presented; the discussion becomes more involved when almost Pontryagin spaces are allowed. Furthermore, the present paper contains a discussion

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of almost Pontryagin space completions of inner product spaces. Such completions have been investigated in [5]; with some basic ideas going back to [4]. In these papers the existence of completions was shown, and it was seen that completions are related to linear functionals. The present paper provides a complete treatment of this topic. As a byproduct an alternative proof is provided of the uniqueness part in [5, Proposition 4.4], where a more 'basis dependent' approach was used. Furthermore, a partial ordering on the set of all almost Pontryagin spaces is introduced and its properties are studied. The above topics are treated with specific applications in mind; however it is felt that the general geometric theory is also of independent interest.

The contents of the paper are as follows. In Section 2 some preliminary facts about almost Pontryagin spaces are recalled. Section 3 is concerned with direct (but not necessarily orthogonal) sums of general inner product spaces. The orthogonal coupling of inner product spaces is treated in Section 4. In Section 5 it is shown how to associate a Pontryagin space with a given almost Pontryagin space. Almost Pontryagin space completions are treated in Section 6.

2. Preliminaries on almost Pontryagin spaces

An inner product space is a pair $(\mathcal{L}, [\cdot,\cdot])$ consisting of a linear space $\mathcal{L}$ over $\mathbb{C}$ and an inner product $[.,.]$ on $\mathcal{L}$; the inner product being a sesquilinear form, linear in the first entry and anti-linear in the second entry. Usually the inner product $[.,.]$ is not mentioned explicitly. The negative index of an inner product space $\mathcal{L}$ is defined as

$$\operatorname{ind}_- \mathcal{L} := \sup\big\{ \dim \mathcal{N} : \mathcal{N} \text{ negative subspace of } \mathcal{L} \big\} \in \mathbb{N}_0 \cup \{\infty\},$$

where a subspace $\mathcal{N}$ of $\mathcal{L}$ is called negative, if $[x,x] < 0$, $x \in \mathcal{N}\setminus\{0\}$. Moreover, $\mathcal{L}^\circ$ denotes the isotropic part of $\mathcal{L}$, i.e. $\mathcal{L}^\circ := \mathcal{L} \cap \mathcal{L}^\perp$, and $\operatorname{ind}_0 \mathcal{L} := \dim \mathcal{L}^\circ$ is called the degree of degeneracy of $\mathcal{L}$. The inner product space $\mathcal{L}$ is called nondegenerated if $\operatorname{ind}_- \mathcal{L} = 0$; otherwise $\mathcal{L}$ is called degenerated.

Definition 2.1. An *almost Pontryagin space* is a triple $\langle \mathcal{A}, [.,.], \mathcal{T}\rangle$ consisting of a linear space $\mathcal{A}$, an inner product $[.,.]$ on $\mathcal{A}$, and a topology $\mathcal{T}$ on $\mathcal{A}$, such that

(aPs1) $\mathcal{T}$ is a Banach space topology on $\mathcal{A}$;
(aPs2) $[.,.]$ is $\mathcal{T}$-continuous;
(aPs3) There exists a $\mathcal{T}$-closed linear subspace $\mathcal{M}$ of $\mathcal{A}$ with finite codimension such that $\langle \mathcal{M}, [.,.]\rangle$ is a Hilbert space.

Usually, as with the inner product $[.,.]$, the topology $\mathcal{T}$ is not mentioned explicitly and one speaks of an almost Pontryagin space. Note that the subspace $\mathcal{M}$ in (aPs3) is complemented in the Banach space $\mathcal{A}$. By means of the open mapping theorem one can easily deduce that the topology $\mathcal{T}$ is actually induced by some Hilbert space inner product on $\mathcal{A}$.

Remark 2.2. In order to provide a more concrete picture of almost Pontryagin spaces, recall the following facts [5, Proposition 2.5].

(*i*) Let $\mathcal{A}$ be an almost Pontryagin space. Then there exist closed subspaces $\mathcal{A}_+$ and $\mathcal{A}_-$ of $\mathcal{A}$ such that $\langle \mathcal{A}_+, [.,.]\rangle$ is a Hilbert space, $\langle \mathcal{A}_-, -[.,.]\rangle$ is a negative subspace with $\dim \mathcal{A}_- = \operatorname{ind}_- \mathcal{A} < \infty$, and

$$\mathcal{A} = \mathcal{A}_+ [\dot{+}] \mathcal{A}_- [\dot{+}] \mathcal{A}^\circ,$$

where '$[\dot{+}]$' denotes a direct and orthogonal sum.

(*ii*) Let $\langle \mathcal{A}_+, [.,.]_+ \rangle$ be a Hilbert space, let $\langle \mathcal{A}_-, [.,.]_- \rangle$ be a finite dimensional negative inner product space, and let $\mathcal{A}_0$ be a finite dimensional linear space. Let $\mathcal{A}_0$ be endowed with the euclidean topology and let $\mathcal{A}_+$ and $\mathcal{A}_-$ carry their natural topologies induced by the inner product. Define a linear space $\mathcal{A}$ as

$$\mathcal{A} := \mathcal{A}_+ \times \mathcal{A}_- \times \mathcal{A}^\circ ,$$

with an inner product on $\mathcal{A}$ as

$$\big[(x_+, x_-, x_0), (y_+, y_-, y_0)\big] := [x_+, y_+]_+ + [x_-, y_-], \qquad (x_+, x_-, x_0), (y_+, y_-, y_0) \in \mathcal{A} ,$$

provided with the product topology of $\mathcal{A}_+$, $\mathcal{A}_-$, and $\mathcal{A}_0$. Then $\mathcal{A}$ is an almost Pontryagin space.

Remark 2.3. Pontryagin spaces form a subclass of almost Pontryagin spaces. In fact, if $\langle \mathcal{A}, [.,.], \mathcal{T} \rangle$ is an almost Pontryagin space, then $\langle \mathcal{A}, [.,.] \rangle$ is a Pontryagin space if and only if $\operatorname{ind}_0 \mathcal{A} = 0$. Conversely, let $\langle \mathcal{A}, [.,.] \rangle$ be a Pontryagin space. If $\mathcal{T}$ denotes the natural topology of $\mathcal{A}$, then $\langle \mathcal{A}, [.,.], \mathcal{T} \rangle$ is an almost Pontryagin space. These facts have been shown in [5, Corollary 2.7].

Definition 2.4. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be almost Pontryagin spaces. Then a linear map $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is said to be *isometric* if $[\phi x, \phi x] = [x, x]$ for all $x \in \mathcal{A}_1$ or, equivalently, $[\phi x, \phi y] = [x, y]$ for all $x, y \in \mathcal{A}_1$. Moreover, a map $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is called a *morphism* from $\mathcal{A}_1$ to $\mathcal{A}_2$ if it is linear, isometric, continuous, and maps closed subspaces of $\mathcal{A}_1$ onto closed subspaces of $\mathcal{A}_2$. A morphism $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is said to be an *isomorphism* if there exists a morphism $\psi : \mathcal{A}_2 \to \mathcal{A}_1$, such that $\psi \circ \phi = \mathrm{id}_{\mathcal{A}_1}$ and $\phi \circ \psi = \mathrm{id}_{\mathcal{A}_2}$.

Remark 2.5. The following basic results concerning almost Pontryagin spaces will be needed; see [5, §3].

(*i*) Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces, and let $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ be linear and isometric. Then $\phi^{-1}\big([\operatorname{ran} \phi]^\circ\big) = \mathcal{L}_1^\circ$. In particular, $\ker \phi \subseteq \mathcal{L}_1^\circ$. Hence, if $\mathcal{L}_1$ is nondegenerated, then ϕ is injective.

(*ii*) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be Pontryagin spaces with $\operatorname{ind}_- \mathcal{A}_1 = \operatorname{ind}_- \mathcal{A}_2$, and let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a map. Then ϕ is a morphism if and only if ϕ is linear and isometric.

(*iii*) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be almost Pontryagin space and let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a map. If ϕ is linear, isometric, continuous and surjective, then ϕ is a morphism.

(*iv*) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be almost Pontryagin space and let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a map. Then ϕ is an isomorphism if and only if ϕ is linear, isometric, continuous, and bijective.

(*v*) Let $\mathcal{A}$ be an almost Pontryagin space and let $\mathcal{A}_0$ be a closed subspace of $\mathcal{A}$. Then $\mathcal{A}_0$ is, with the inner product and topology naturally inherited from $\mathcal{A}$, an almost Pontryagin space. The set-theoretic inclusion map $\subseteq : \mathcal{A}_0 \to \mathcal{A}$ is a morphism.

(*vi*) Let $\mathcal{A}$ be an almost Pontryagin space and let $\mathcal{B}$ be a linear subspace of $\mathcal{A}^\circ$. Then $\mathcal{A}/_{\mathcal{B}}$ is an almost Pontryagin space, with the inner product and topology naturally inherited from $\mathcal{A}$. The canonical projection $\pi : \mathcal{A} \to \mathcal{A}/_{\mathcal{B}}$ is a morphism.

(*vii*) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be almost Pontryagin space and let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a morphism. Then there exits a unique isomorphism $\tilde{\phi} : \mathcal{A}_1/_{\ker\phi} \to \operatorname{ran}\phi$, such that

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{\ \phi\ } & \mathcal{A}_2 \\ {\scriptstyle\pi}\downarrow & & \uparrow{\scriptstyle\subseteq} \\ \mathcal{A}_1/_{\ker\phi} & \xrightarrow[\ \tilde{\phi}\ ]{} & \operatorname{ran}\phi \end{array}$$

3. Direct sums of inner product spaces

Consider an inner product space $\mathcal{L}$ and two linear subspaces $\mathcal{L}_1, \mathcal{L}_2$ of $\mathcal{L}$. Then $\mathcal{L}_1$ and $\mathcal{L}_2$ are themselves inner product spaces, namely with the inner product inherited from $\mathcal{L}$. Each element $x_1 \in \mathcal{L}_1$ gives rise to a linear functional on $\mathcal{L}_2$, namely by $[.,x_1]_{\mathcal{L}} : x_2 \mapsto [x_2,x_1]_{\mathcal{L}}$. Moreover, the map

$$c : \begin{cases} \mathcal{L}_1 & \to \quad \mathcal{L}_2^* \\ x_1 & \mapsto \quad [.,x_1]_{\mathcal{L}} \end{cases} \tag{3.1}$$

where $\mathcal{L}_2^*$ denotes the algebraic dual of $\mathcal{L}_2$, is conjugate linear. Clearly, the inner product of arbitrary elements of $\mathcal{L}_1 + \mathcal{L}_2$ can be recovered as

$$\big[x_1+x_2, y_1+y_2\big]_{\mathcal{L}} = [x_1,y_1]_{\mathcal{L}_1} + \overline{c(x_1)y_2} + c(y_1)x_2 + [x_2,y_2]_{\mathcal{L}_2}, \qquad x_1,y_1 \in \mathcal{L}_1,\ x_2,y_2 \in \mathcal{L}_2\,. \tag{3.2}$$

This situation will now be extended.

Definition 3.1. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two inner product spaces whose inner products are denoted by $[.,.]_1$ and $[.,.]_2$, respectively. Moreover, let

$$c : \mathcal{L}_1 \to \mathcal{L}_2^*$$

be a conjugate linear map of $\mathcal{L}_1$ into the algebraic dual space of $\mathcal{L}_2$. Denote by $\mathcal{L}_1 \ltimes_c \mathcal{L}_2$ the product space $\mathcal{L}_1 \times \mathcal{L}_2$ provided with an inner product by

$$\big[(x_1,x_2),(y_1,y_2)\big]_c := [x_1,y_1]_1 + \overline{c(x_1)y_2} + c(y_1)x_2 + [x_2,y_2]_2, \qquad x_1,y_1 \in \mathcal{L}_1,\ x_2,y_2 \in \mathcal{L}_2.$$

The fact that $[.,.]_c$ actually is an inner product follows with a straightforward computation using that c is conjugate linear.

Example 3.2. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces. The zero map $0 : \mathcal{L}_1 \to \mathcal{L}_2^*$, $0(x_1)x_2 := 0$, is conjugate linear, and one has

$$\mathcal{L}_1 \ltimes_0 \mathcal{L}_2 = \mathcal{L}_1[\dotplus]\mathcal{L}_2\,.$$

There are natural embeddings of $\mathcal{L}_j$ into $\mathcal{L}_1 \ltimes_c \mathcal{L}_2$, namely the maps $\iota_{c,j}$ defined as

$$\iota_{c,1}(x) := (x,0),\ x \in \mathcal{L}_1, \quad \iota_{c,2}(x) := (0,x),\ x \in \mathcal{L}_2\,. \tag{3.3}$$

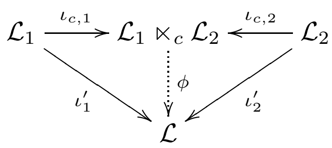
These mappings are injective and isometric, and

$$\mathcal{L}_1 \ltimes_c \mathcal{L}_2 = \operatorname{ran}\iota_{c,1} \dotplus \operatorname{ran}\iota_{c,2}\,,$$

where '$\dotplus$' denotes a direct sum. Hence, $\mathcal{L}_1$ and $\mathcal{L}_2$ may be considered as summands in a direct sum decomposition of $\mathcal{L}_1 \ltimes_c \mathcal{L}_2$. Recall the preliminary computation (3.2); hence, conversely, each decomposition of an inner product space $\mathcal{L}$ into a direct sum gives rise to a representation $\mathcal{L} = \mathcal{L}_1 \ltimes_c \mathcal{L}_2$, where c is as in (3.1). This fact can be formulated in a slightly more general way.

Proposition 3.3. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces and let $\mathcal{L}$ be an inner product space together with isometric maps $\iota'_j : \mathcal{L}_j \to \mathcal{L}$, $j = 1, 2$. Then there exists a unique conjugate linear map $c : \mathcal{L}_1 \to \mathcal{L}_2^*$ such that*

(3.4)



with some isometric linear map $\phi : \mathcal{L}_1 \ltimes_c \mathcal{L}_2 \to \mathcal{L}$. Explicitly, c is given as

$$c : \begin{cases} \mathcal{L}_1 \to \mathcal{L}_2^* \\ x_1 \mapsto \big(x_2 \mapsto [\iota'_2(x_2), \iota'_1(x_1)]_{\mathcal{L}}\big) \end{cases} \tag{3.5}$$

The map ϕ in the diagram (3.4) is uniquely determined. Explicitly, ϕ is given as

$$\phi : \begin{cases} \mathcal{L}_1 \ltimes_c \mathcal{L}_2 \to \mathcal{L} \\ \quad (x_1, x_2) \mapsto \iota'_1(x_1) + \iota'_2(x_2) \end{cases} \tag{3.6}$$

Moreover, one has

$$\ker \phi = \big\{(x_1, x_2) : \; \iota'_1(x_1) = -\iota'_2(x_2)\big\}, \quad \operatorname{ran} \phi = \operatorname{ran} \iota'_1 + \operatorname{ran} \iota'_2 \,. \tag{3.7}$$

Proof. Let c and ϕ be defined by (3.5) and (3.6). A short calculation will show that ϕ is isometric. By the definition of $\iota_{c,j}$, the diagram (3.4) commutes. One has $\phi(x_1, x_2) = 0$ if and only if $\iota'_1(x_1) = -\iota'_2(x_2)$. Hence, the kernel of ϕ has the asserted form. Moreover, clearly, $\operatorname{ran} \phi = \operatorname{ran} \iota'_1 + \operatorname{ran} \iota'_2$.

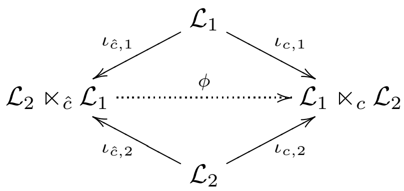
It remains to show uniqueness of c and ϕ. Assume that $c' : \mathcal{L}_1 \to \mathcal{L}_2^*$ is conjugate linear and that there exists an isometric map ϕ' of $\mathcal{L}_1 \ltimes_{c'} \mathcal{L}_2$ into $\mathcal{L}$ which makes the diagram (3.4) commute. Then

$$\begin{aligned} c'(x_1)x_2 = \big[\iota_{c',2}(x_2), \iota_{c',1}(x_1)\big]_{c'} &= \big[\phi'(\iota_{c',2}(x_2)), \phi'(\iota_{c',1}(x_1))\big]_{\mathcal{L}} = \\ &= \big[\iota'_2(x_2), \iota'_1(x_1)\big]_{\mathcal{L}} = c(x_1)x_2, \quad x_1 \in \mathcal{L}_1, x_2 \in \mathcal{L}_2 \,, \end{aligned}$$

i.e. $c' = c$. The map ϕ is uniquely determined by (3.4) since the ranges of $\iota_{c,1}$ and $\iota_{c,2}$ jointly span $\mathcal{L}_1 \ltimes_c \mathcal{L}_2$. $\square$

Corollary 3.4. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces and let $c : \mathcal{L}_1 \to \mathcal{L}_2^*$ be a conjugate linear map. Then there exists a unique conjugate linear map $\hat{c} : \mathcal{L}_2 \to \mathcal{L}_1^*$ such that*

(3.8)



with some isometric linear map ϕ. Explicitly, c is given as

$$\hat{c}(x_2)x_1 = \overline{c(x_1)x_2} \,. \tag{3.9}$$

The map ϕ in the diagrams (3.8) is uniquely determined. Explicitly, ϕ is given as

$$\phi\big((x_2, x_1)\big) = (x_1, x_2) \,. \tag{3.10}$$

The map ϕ is bijective.

Proof. Applying Proposition 3.3 with the spaces $\mathcal{L}_2$ and $\mathcal{L}_1$, and

$$\mathcal{L} := \mathcal{L}_1 \ltimes_c \mathcal{L}_2, \quad \iota_1' := \iota_{c,2},\ \iota_2' := \iota_{c,1}\,,$$

gives the mappings $\hat{c}$ and ϕ as asserted in (3.9) and (3.10). □

The next result gives some of information about the isotropic part of $\mathcal{L}_1 \ltimes_c \mathcal{L}_2$. For a linear space $\mathcal{L}$ and a subset M of $\mathcal{L}^*$, denote by ${}^\perp M$ the left annihilator with respect to the natural duality between $\mathcal{L}$ and $\mathcal{L}^*$, i.e.

$${}^\perp M := \big\{x \in \mathcal{L} :\ f(x) = 0, f \in M\big\}\,.$$

Proposition 3.5. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces and let $c : \mathcal{L}_1 \to \mathcal{L}_2^*$ be a conjugate linear map. Then*

$$\iota_{c,1}(\mathcal{L}_1) \cap \big(\mathcal{L}_1 \ltimes_c \mathcal{L}_2\big)^\circ = \iota_{c,1}\big(\mathcal{L}_1^\circ \cap \ker c\big)\,,$$

$$\iota_{c,2}(\mathcal{L}_2) \cap \big(\mathcal{L}_1 \ltimes_c \mathcal{L}_2\big)^\circ = \iota_{c,2}\big(\mathcal{L}_2^\circ \cap {}^\perp \mathrm{ran}\, c\big)\,.$$

Proof. Let $y_1 \in \mathcal{L}_1$, then

$$\big[(x_1, x_2), (y_1, 0)\big]_c = [x_1, y_1]_1 + c(y_1)x_2, \quad x_1 \in \mathcal{L}_1, x_2 \in \mathcal{L}_2\,.$$

Hence $(y_1, 0) \in (\mathcal{L}_1 \ltimes_c \mathcal{L}_2)^\circ$ if and only if

$$[x_1, y_1]_1 = 0,\ x_1 \in \mathcal{L}_1 \ \text{ and } \ c(y_1) = 0\,.$$

Let $y_2 \in \mathcal{L}_2$, then

$$\big[(x_1, x_2), (0, y_2)\big]_c = \overline{c(x_1)y_2} + [x_2, y_2]_2, \quad x_1 \in \mathcal{L}_1, x_2 \in \mathcal{L}_2\,.$$

Hence $(0, y_2) \in (\mathcal{L}_1 \ltimes_c \mathcal{L}_2)^\circ$ if and only if $c(x_1)y_2 = 0$, $x_1 \in \mathcal{L}_1$, and $[x_2, y_2]_2 = 0$, $x_2 \in \mathcal{L}_2$. □

In general not much information on $\mathcal{L}_1 \ltimes_c \mathcal{L}_2$ can be obtained. Concerning negative indices and degrees of degeneracy there are only the following weak estimates:

$$\mathrm{ind}_-\, \mathcal{L}_1 \ltimes_c \mathcal{L}_2 \geq \max\big\{\, \mathrm{ind}_-\, \mathcal{L}_1, \mathrm{ind}_-\, \mathcal{L}_2\big\}\,,$$

$$\mathrm{ind}_0\, \mathcal{L}_1 \ltimes_c \mathcal{L}_2 \geq \max\big\{\, \dim(\mathcal{L}_1^\circ \cap \ker c), \dim(\mathcal{L}_2^\circ \cap {}^\perp \mathrm{ran}\, c)\big\}\,.$$

It is easy to give examples which show that negative indices or degrees of degeneracy may increase arbitrarily.

Example 3.6. Let $\mathcal{L}$ be a linear space and let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two linear subspaces of $\mathcal{L}$ with the same dimension, such that $\mathcal{L}_1 \cap \mathcal{L}_2 = \{0\}$. Choose bases $\{b_j^1 :\ j \in J\}$ and $\{b_j^2 :\ j \in J\}$ of $\mathcal{L}_1$ and $\mathcal{L}_2$ and let $\mathcal{L}_1 \times \mathcal{L}_2$ be endowed with inner products $[.,.]$ and $[.,.]'$ given by the Gram-matrices

$$G := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad G' := \begin{pmatrix} I & I \\ I & I \end{pmatrix}.$$

Explicitly, this means that

$$\Big[\sum \lambda_i b_i^1 + \sum \mu_i b_i^2, \sum \lambda_j' b_j^1 + \sum \mu_j' b_j^2\Big] = \sum \big(\lambda_i \overline{\mu_i'} + \mu_i \overline{\lambda_i'}\big),$$

while

$$\Big[\sum \lambda_i b_i^1 + \sum \mu_i b_i^2, \sum \lambda_j' b_j^1 + \sum \mu_j' b_j^2\Big]' = \sum \big(\lambda_i \overline{\lambda_i'} + \lambda_i \overline{\mu_i'} + \mu_i \overline{\lambda_i'} + \mu_i \overline{\mu_i'}\big).$$

Define inner products $[.,.]_j$ and $[.,.]_j'$ on $\mathcal{L}_j$ by $[x_1,x_2]_j := 0$, $j=1,2$, and

$$\Big[\sum \lambda_i b_i^j, \sum \mu_i b_i^j\Big]_j' := \sum \lambda_i \overline{\mu_i}, \quad j=1,2\,.$$

Then $\langle \mathcal{L}_1\times\mathcal{L}_2,[.,.]\rangle$ and $\langle \mathcal{L}_1\times\mathcal{L}_2,[.,.]'\rangle$ can be realized as $\langle\mathcal{L}_1,[.,.]_1\rangle \ltimes_c \langle\mathcal{L}_2,[.,.]_2\rangle$ and $\langle\mathcal{L}_1,[.,.]_1'\rangle \ltimes_{c'} \langle\mathcal{L}_2,[.,.]_2'\rangle$, respectively, with some appropriate mappings c and c'. Then

$$\operatorname{ind}_-\langle\mathcal{L}_1,[.,.]_1\rangle = \operatorname{ind}_-\langle\mathcal{L}_2,[.,.]_2\rangle = 0, \quad \operatorname{ind}_-\langle\mathcal{L}_1,[.,.]_1\rangle \ltimes_c \langle\mathcal{L}_2,[.,.]_2\rangle = |J|,$$
$$\operatorname{ind}_0\langle\mathcal{L}_1,[.,.]_1'\rangle = \operatorname{ind}_0\langle\mathcal{L}_2,[.,.]_2'\rangle = 0, \quad \operatorname{ind}_0\langle\mathcal{L}_1,[.,.]_1'\rangle \ltimes_c \langle\mathcal{L}_2,[.,.]_2'\rangle = |J|.$$

If one of the spaces $\mathcal{L}_1$ or $\mathcal{L}_2$ is finite dimensional then at least rough upper estimates can be given for $\operatorname{ind}_-\mathcal{L}_1\ltimes_c\mathcal{L}_2$ and $\operatorname{ind}_0\mathcal{L}_1\ltimes_c\mathcal{L}_2$.

Remark 3.7. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces and let $c:\mathcal{L}_1\to\mathcal{L}_2^*$ be a conjugate linear map.

(i) Assume that $\dim\mathcal{L}_1<\infty$. Since, for each subspace $\mathcal{K}$ of $\mathcal{L}_1\ltimes_c\mathcal{L}_2$ one has $\dim(\mathcal{K}\cap\mathcal{L}_2)\geq\dim\mathcal{K}-\dim\mathcal{L}_1$, it follows that

$$\operatorname{ind}_-\mathcal{L}_1\ltimes_c\mathcal{L}_2 \leq \operatorname{ind}_-\mathcal{L}_2+\dim\mathcal{L}_1, \quad \operatorname{ind}_0\mathcal{L}_1\ltimes_c\mathcal{L}_2 \leq \operatorname{ind}_0\mathcal{L}_2+\dim\mathcal{L}_1$$

(ii) Assume that $\dim\mathcal{L}_2<\infty$. Then it is seen from Corollary 3.4 that analogous inequalities hold.

Now consider the case where the inner product spaces are almost Pontryagin spaces. Assume that $\langle\mathcal{A}_1,[.,.]_1,\mathcal{T}_1\rangle$ and $\langle\mathcal{A}_2,[.,.]_2,\mathcal{T}_2\rangle$ are almost Pontryagin spaces and let $c:\mathcal{A}_1\to\mathcal{A}_2^*$ be conjugate linear. Neither $\operatorname{ind}_-(\mathcal{A}_1\ltimes_c\mathcal{A}_2)$ nor $\operatorname{ind}_0(\mathcal{A}_1\ltimes_c\mathcal{A}_2)$ need to be finite so that $\mathcal{A}_1\ltimes_c\mathcal{A}_2$ will in general be far from an almost Pontryagin space. Also topologically $\mathcal{A}_1\ltimes_c\mathcal{A}_2$ is not that simple. Of course, $\mathcal{A}_1\ltimes_c\mathcal{A}_2$ carries a natural Banach space topology, namely the product topology $\mathcal{T}:=\mathcal{T}_1\times\mathcal{T}_2$. However, the inner product $[.,.]_c$ will in general not be continuous.

Let $\mathcal{A}$ be an almost Pontryagin space. Then $\mathcal{A}'$ denotes its topological dual space and τ_{w^*} denotes the weak-* topology on $\mathcal{A}'$.

Proposition 3.8. *Let $\langle\mathcal{A}_1,[.,.]_1,\mathcal{T}_1\rangle$ and $\langle\mathcal{A}_2,[.,.]_2,\mathcal{T}_2\rangle$ be almost Pontryagin spaces and let $c:\mathcal{A}_1\to\mathcal{A}_2^*$ be a conjugate linear map. Then the inner product*

$$[.,.]_c:(\mathcal{A}_1\ltimes_c\mathcal{A}_2)^2\to\mathbb{C}$$

is $\mathcal{T}$-continuous if and only if $c(\mathcal{A}_1)\subseteq\mathcal{A}_2'$ and c is $\mathcal{T}_1$-to-τ_{w^*}–continuous.

Proof. First assume that c maps $\mathcal{A}_1$ $\mathcal{T}_1$-to-τ_{w^*}–continuously into $\mathcal{A}_2'$. Choose norms $\|.\|_1$, $\|.\|_2$, which induce $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, and put $\|.\|:=\max\{\|.\|_1,\|.\|_2\}$. Let $M_1,M_2>0$ be such that

$$\big|[x_j,y_j]_j\big| \leq M_j\|x_j\|_j\|y_j\|_j, \quad x_j,y_j\in\mathcal{A}_j, j=1,2\,.$$

Since c is $\mathcal{T}_1$-to-τ_{w^*}–continuous, for each fixed $x_2\in\mathcal{A}_2$ there exists $M_{x_2}>0$ such that

$$|c(y_1)x_2|\leq M_{x_2}, \quad y_1\in\mathcal{A}_1, \|y_1\|_1\leq 1\,.$$

The uniform boundedness principle implies

$$M:=\sup\big\{\|c(y_1)\|:\, y_1\in\mathcal{A}_1,\|y_1\|_1\leq1\big\}<\infty\,.$$

For $x_1+x_2, y_1+y_2\in\mathcal{A}_1\ltimes_c\mathcal{A}_2$ with $\|x_1+x_2\|,\|y_1+y_2\|\leq 1$, there is thus the estimate

$$\big|[x_1+x_2,y_1+y_2]_c\big| \leq |[x_1,y_1]_1|+|c(x_1)y_2|+|c(y_1)x_2|+|[x_2,y_2]_2| \leq M_1+2M+M_2\,.$$

This shows that $[.,.]_c$ is $\mathcal{T}$-continuous.

Conversely, assume that $[.,.]_c$ is $\mathcal{T}$-continuous. One has

$$c(y_1)x_2 = [0 + x_2, y_1 + 0]_c, \quad y_1 \in \mathcal{A}_1, x_2 \in \mathcal{A}_2 \,.$$

Keeping y_1 fixed and letting x_2 vary through $\mathcal{A}_2$ shows that the functional $c(y_1)$ belongs to $\mathcal{A}_2'$. Keeping x_2 fixed and letting y_1 vary through $\mathcal{A}_1$ shows that c is $\mathcal{T}_1$-to-τ_{w^*}–continuous. □

Proposition 3.8 does not state that $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ is an almost Pontryagin space. But if $\mathcal{A}_1$ or $\mathcal{A}_2$ is finite dimensional, then $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ is an almost Pontryagin space.

Corollary 3.9. *Let $\langle \mathcal{A}_1, [.,.]_1, \mathcal{T}_1 \rangle$ and $\langle \mathcal{A}_2, [.,.]_2, \mathcal{T}_2 \rangle$ be almost Pontryagin spaces and let $c : \mathcal{A}_1 \to \mathcal{A}_2^*$ be a conjugate linear map.*

- (*i*) *Assume that* $\dim \mathcal{A}_1 < \infty$. *Then $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ is an almost Pontryagin space if and only if $c(\mathcal{A}_1) \subseteq \mathcal{A}_2'$.*
- (*ii*) *Assume that* $\dim \mathcal{A}_2 < \infty$. *Then $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ is an almost Pontryagin space if and only if c is continuous. Here $\mathcal{A}_2' = \mathcal{A}_2^*$ and its topology is just the euclidean topology.*

Proof. (i) Let $\mathcal{A}_1$ be finite dimensional. Assume that $c(\mathcal{A}_1) \subseteq \mathcal{A}_2'$. Since c is conjugate linear, $\dim \mathcal{A}_1 < \infty$ implies that c is $\mathcal{T}_1$-to-τ_{w^*}–continuous. By Proposition 3.8, $[.,.]_c$ is $\mathcal{T}$-continuous. Let $\mathcal{M}$ be a $\mathcal{T}_2$-closed subspace of $\mathcal{A}_2$ which is a Hilbert space and has finite codimension in $\mathcal{A}_2$. Then $\mathcal{M}$ is also $\mathcal{T}$-closed and has finite codimension in $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$. Moreover, $[.,.]_c|_{\mathcal{M}\times\mathcal{M}} = [.,.]_2|_{\mathcal{M}\times\mathcal{M}}$, and hence $\mathcal{M}$ is a Hilbert space with respect to $[.,.]_c$. One sees that $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ is an almost Pontryagin space. Conversely, if $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ is an almost Pontryagin space, then Proposition 3.8 yields $c(\mathcal{A}_1) \subseteq \mathcal{A}_2'$.

(ii) The case that $\mathcal{A}_2$ is finite dimensional is settled in the same manner. □

Remark 3.10. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be almost Pontryagin spaces and let $c : \mathcal{A}_1 \to \mathcal{A}_2^*$ be a conjugate linear map. The embeddings $\iota_{c,j} : \mathcal{A}_j \to \mathcal{A}_1 \ltimes_c \mathcal{A}_2$ are continuous and map closed subsets of $\mathcal{A}_j$ to closed subsets of $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$. Hence, whenever $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ is an almost Pontryagin space, then $\iota_{c,j}$ will be morphisms.

Here are the analogs of Proposition 3.3 and Corollary 3.4 in the setting of an almost Pontryagin space.

Proposition 3.11. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be almost Pontryagin spaces.*

- (*i*) *Let $\mathcal{A}$ be an almost Pontryagin space with morphisms $\iota_j' : \mathcal{A}_j \to \mathcal{A}$, $j = 1, 2$. Let the conjugate linear map $c : \mathcal{A}_1 \to \mathcal{A}_2^*$ and the isometry $\phi : \mathcal{A}_1 \ltimes_c \mathcal{A}_2 \to \mathcal{A}$ be as in Proposition 3.3. Then $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ is an almost Pontryagin space and ϕ is a morphism if and only if*

$$\dim \big(\operatorname{ran} \iota_1' \cap \operatorname{ran} \iota_2'\big) < \infty \quad \text{and} \quad \operatorname{ran} \iota_1' + \operatorname{ran} \iota_2' \text{ closed in } \mathcal{A} \,.$$

- (*ii*) *Let $c : \mathcal{A}_1 \to \mathcal{A}_2^*$ be a conjugate linear map, let $\hat{c} : \mathcal{A}_2 \to \mathcal{A}_1^*$, and let $\phi : \mathcal{A}_2 \ltimes_{\hat{c}} \mathcal{A}_1 \to \mathcal{A}_1 \ltimes_c \mathcal{A}_2$ be as in Corollary 3.4. Then $\mathcal{A}_2 \ltimes_{\hat{c}} \mathcal{A}_1$ is an almost Pontryagin space if and only if $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ is an almost Pontryagin space, and in this case ϕ is an isomorphism between these almost Pontryagin spaces.*

Proof. For the proof of (i) let $\mathcal{A}$ and $\mathcal{A}_j$, ι_j', $j = 1, 2$, be given. Since ι_1' and ι_2' are continuous, the map c is explicitly given by (3.5), it maps $\mathcal{A}_1$ into $\mathcal{A}_2'$, and is $\mathcal{T}_1$-to-τ_{w^*}–continuous. Thus $[.,.]_c$ is continuous. Observe that

$$\dim\ker\phi < \infty \iff \dim\big(\operatorname{ran}\iota_1' \cap \operatorname{ran}\iota_2'\big) < \infty \tag{3.11}$$

To see this, let $\pi_1 : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ denote the projection onto the first component, and consider the map $\mu := \pi_1 \circ (\iota_1' \times (-\iota_2')) : \mathcal{A}_1 \ltimes_c \mathcal{A}_2 \to \mathcal{A}$. By (3.7), $\mu(\ker\phi) = \operatorname{ran}\iota_1' \cap \operatorname{ran}\iota_2'$. Moreover, $\ker(\mu|_{\ker\phi}) = \ker\iota_1' \times \ker\iota_2'$. Since $\ker\iota_j' \subseteq \mathcal{A}_j^\circ$, and hence $\dim\ker(\mu|_{\ker\phi}) < \infty$, (3.11) follows.

Assume that $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ is an almost Pontryagin space and $\phi : \mathcal{A}_1 \ltimes_c \mathcal{A}_2 \to \mathcal{A}$ is a morphism. Then $\operatorname{ran}\iota_1' + \operatorname{ran}\iota_2' = \operatorname{ran}\phi$ is closed in $\mathcal{A}$ since ϕ maps closed subspaces to closed subspaces. Moreover, since $\ker\phi \subseteq (\mathcal{A}_1 \ltimes_c \mathcal{A}_2)^\circ$, one must have $\dim\ker\phi < \infty$, and (3.11) gives $\dim(\operatorname{ran}\iota_1' \cap \operatorname{ran}\iota_2') < \infty$.

Conversely, assume that $\dim(\operatorname{ran}\iota_1' \cap \operatorname{ran}\iota_2') < \infty$ and $\operatorname{ran}\iota_1' + \operatorname{ran}\iota_2'$ is closed in $\mathcal{A}$. Then, by (3.11), $\ker\phi$ is finite dimensional. Moreover, since $[\operatorname{ran}\phi]^\circ$ is a neutral subspace of $\mathcal{A}$, $\dim([\operatorname{ran}\phi]^\circ) \leq \operatorname{ind}_-\mathcal{A} + \operatorname{ind}_0\mathcal{A}$. Since $\phi^{-1}([\operatorname{ran}\phi]^\circ) = (\mathcal{A}_1 \ltimes_c \mathcal{A}_2)^\circ$, it follows that

$$\dim(\mathcal{A}_1 \ltimes_c \mathcal{A}_2)^\circ < \infty\,.$$

The map ϕ is isometric, and hence clearly $\operatorname{ind}_-(\mathcal{A}_1 \ltimes_c \mathcal{A}_2) \leq \operatorname{ind}_-\mathcal{A} < \infty$.

Since $\dim\ker\phi < \infty$, the space $\ker\phi$ is complemented in the Banach space $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$, i.e. one may choose a closed subspace $\mathcal{M}_1$ of $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ with $\mathcal{M}_1 \dot{+} \ker\phi = \mathcal{A}_1 \ltimes_c \mathcal{A}_2$. Then $\phi|_{\mathcal{M}_1}$ is a continuous bijection between the Banach spaces $\mathcal{M}_1$ and $\operatorname{ran}\phi$, and hence a homeomorphism. Let $\mathcal{N}$ be a closed subspace of $\operatorname{ran}\phi$ with finite codimension which is a Hilbert space with respect to the inner product of $\mathcal{A}$. Then $\mathcal{M} := (\phi|_{\mathcal{M}_1})^{-1}(\mathcal{N})$ is a closed subspace of $\mathcal{M}_1$ with finite codimension and, since ϕ is isometric, is a Hilbert space with respect to the inner product of $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$. Since $\mathcal{M}_1$ itself is closed and has finite codimension in $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$, $\mathcal{M}$ is a subspace with the properties required in (aPs3). Let $\mathcal{L}$ be a closed subspace of $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$, then $\phi(\mathcal{L}) = \phi|_{\mathcal{M}_1}(\mathcal{L} \cap \mathcal{M}_1)$, hence is closed in $\operatorname{ran}\phi$ and thus also in $\mathcal{A}$. As a closed subspace of an almost Pontryagin space, the space $\operatorname{ran}\phi$ is itself an almost Pontryagin space.

The second item is immediate, since ϕ is, besides being bijective and isometric, in any case a homeomorphism. □

4. Orthogonal coupling of inner product spaces

Let $\langle\mathcal{L}_1, [.,.]_1\rangle$ and $\langle\mathcal{L}_2, [.,.]_2\rangle$ be inner product spaces. Their direct and orthogonal sum $\mathcal{L}_1[\dot{+}]\mathcal{L}_2$ is defined as the linear space $\mathcal{L}_1 \times \mathcal{L}_2$ with the inner product

$$\big[(x_1, x_2), (y_1, y_2)\big] := [x_1, y_1] + [x_2, y_2], \quad (x_1, y_1), (x_2, y_2) \in \mathcal{L}_1[\dot{+}]\mathcal{L}_2\,.$$

Properties of $\mathcal{L}_1$ and $\mathcal{L}_2$ immediately transfer to $\mathcal{L}_1[\dot{+}]\mathcal{L}_2$; for example

$$\operatorname{ind}_-\mathcal{L}_1[\dot{+}]\mathcal{L}_2 = \operatorname{ind}_-\mathcal{L}_1 + \operatorname{ind}_-\mathcal{L}_2,\ \operatorname{ind}_0\mathcal{L}_1[\dot{+}]\mathcal{L}_2 = \operatorname{ind}_0\mathcal{L}_1 + \operatorname{ind}_0\mathcal{L}_2\,.$$

In fact, $(\mathcal{L}_1[\dot{+}]\mathcal{L}_2)^\circ = \mathcal{L}_1^\circ \times \mathcal{L}_2^\circ$. Recall that (with the notation of the previous section) $\mathcal{L}_1[\dot{+}]\mathcal{L}_2 = \mathcal{L}_1 \ltimes_0 \mathcal{L}_2$, where $0 : \mathcal{L}_1 \to \mathcal{L}_2^*$ denotes the zero map. The following observation is the starting point for the present considerations.

Remark 4.1. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are nondegenerated inner product spaces, then the direct and orthogonal sum $\mathcal{L}_1[\dot{+}]\mathcal{L}_2$ is (up to isomorphisms) the unique inner product space containing $\mathcal{L}_1$ and $\mathcal{L}_2$ isometrically as orthogonal subspaces which together span the whole space. In the degenerated situation uniqueness will fail.

Definition 4.2. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces and let α be a linear subspace of $\mathcal{L}_1^\circ \times \mathcal{L}_2^\circ$. Define

$$\mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2 := \big(\mathcal{L}_1[\dot{+}]\mathcal{L}_2\big)/_\alpha .$$

Then $\mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2$ is called the orthogonal coupling of $\mathcal{L}_1$ and $\mathcal{L}_2$ with overlapping relation α. With ι_j the canonical embedding of $\mathcal{L}_j$ into $\mathcal{L}_1[\dot{+}]\mathcal{L}_2$ and π_α the canonical projection of $\mathcal{L}_1[\dot{+}]\mathcal{L}_2$ onto $(\mathcal{L}_1[\dot{+}]\mathcal{L}_2)/_\alpha$, define $\iota_1^\alpha := \pi_\alpha \circ \iota_1$, $\iota_2^\alpha := \pi_\alpha \circ \iota_2$, that is

$$\begin{array}{ccc} & & \mathcal{L}_1[\dot{+}]\mathcal{L}_2 \\ & \nearrow^{\iota_j} & \downarrow{\scriptstyle \pi_\alpha} \\ \mathcal{L}_j & \xrightarrow[\iota_j^\alpha]{} & \mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2 \end{array}$$

Remark 4.3. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces and let α be a linear subspace of $\mathcal{L}_1^\circ \times \mathcal{L}_2^\circ$. In the following α will be identified with the graph of a linear operator, precisely when $(0, y) \in \alpha$ implies $y = 0$.

(i) Since $\mathcal{L}_1^\circ \times \mathcal{L}_2^\circ = (\mathcal{L}_1[\dot{+}]\mathcal{L}_2)^\circ$, both $\iota_1^\alpha : \mathcal{L}_1 \to \mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2$ and $\iota_2^\alpha : \mathcal{L}_2 \to \mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2$ are isometric. Moreover,

$$\iota_1^\alpha(\mathcal{L}_1) \perp \iota_2^\alpha(\mathcal{L}_2) \quad \text{and} \quad \mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2 = \operatorname{ran} \iota_1^\alpha + \operatorname{ran} \iota_2^\alpha .$$

(ii) The mappings ι_1^α and ι_2^α are both injective if and only if the linear subspace α is the graph of a bijective map $\alpha : \operatorname{dom} \alpha \to \operatorname{ran} \alpha$ between some linear subspaces $\operatorname{dom} \alpha \subseteq \mathcal{L}_1^\circ$ and $\operatorname{ran} \alpha \subseteq \mathcal{L}_2^\circ$. In order to see this, note that

$$(0, x_2) \in \alpha \iff \iota_2^\alpha(x_2) = 0, \quad (x_1, 0) \in \alpha \iff \iota_1^\alpha(x_1) = 0 .$$

Proposition 4.4. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces and let $\mathcal{L}$ be an inner product space with isometric maps $\iota_j' : \mathcal{L}_j \to \mathcal{L}$, $j = 1, 2$, such that $\iota_1'(\mathcal{L}_1) \perp \iota_2'(\mathcal{L}_2)$. Then there exists a unique linear subspace $\alpha \subseteq \mathcal{L}_1^\circ \times \mathcal{L}_2^\circ$, such that*

$$\begin{array}{ccccc} \mathcal{L}_1 & \xrightarrow{\iota_1^\alpha} & \mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2 & \xleftarrow{\iota_2^\alpha} & \mathcal{L}_2 \\ & {\scriptstyle \iota_1'} \searrow & \vdots \, {\scriptstyle \psi} & \swarrow {\scriptstyle \iota_2'} & \\ & & \mathcal{L} & & \end{array} \tag{4.1}$$

with some injective and isometric linear map ψ. Explicitly, α is given as

$$\alpha = \big\{(x_1, x_2) \in \mathcal{L}_1 \times \mathcal{L}_2 : \iota_1'(x_1) = -\iota_2'(x_2)\big\} .$$

The map ψ in the diagram (4.1) *is uniquely determined. Explicitly, ψ is given as*

$$\psi\big((x_1, x_2)/_\alpha\big) = \iota_1'(x_1) + \iota_2'(x_2) .$$

The map ι_j^α is injective if and only if ι_j' has this property, $j = 1, 2$. Moreover, if $\operatorname{ran} \iota_1' + \operatorname{ran} \iota_2' = \mathcal{L}$, *then ψ is bijective.*

Proof. The map $\phi(x) := \iota_1'(x) + \iota_2'(x)$ is an isometry of $\mathcal{L}_1[\dot{+}]\mathcal{L}_2$ into $\mathcal{L}$. It satisfies

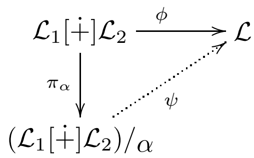
$$\begin{array}{ccccc} \mathcal{L}_1 & \xrightarrow{\iota_1} & \mathcal{L}_1[\dot{+}]\mathcal{L}_2 & \xleftarrow{\iota_2} & \mathcal{L}_2 \\ & {\scriptstyle \iota_1'} \searrow & \downarrow {\scriptstyle \phi} & \swarrow {\scriptstyle \iota_2'} & \\ & & \mathcal{L} & & \end{array} \tag{4.2}$$

and $\ker\phi = \{(x_1,x_2) \in \mathcal{L}_1[\dot{+}]\mathcal{L}_2 : \iota_1'(x) = -\iota_2'(x)\}$. Now it will be shown that $\ker\phi \subseteq (\mathcal{L}_1[\dot{+}]\mathcal{L}_2)^\circ$. To this end, let $(x_1,x_2) \in \ker\phi$ be given. If $y_1 \in \mathcal{L}_1$, then

$$\big[(x_1,x_2),(y_1,0)\big]_{\mathcal{L}_1[\dot{+}]\mathcal{L}_2} = [x_1,y_1]_{\mathcal{L}_1} = \big[\iota_1'(x_1),\iota_1'(y_1)\big] = \big[-\iota_2'(x_2),\iota_1'(y_1)\big] = 0\,.$$

An analogous computation will show that $[(x_1,x_2),(0,y_2)] = 0$ for all $y_2 \in \mathcal{L}_2$. Hence, the linear subspace $\alpha := \ker\phi$ qualifies as being used to define $\mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2$.

Let ψ be the isometry which makes the diagram

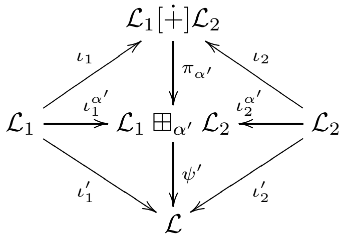


commute. Clearly, ψ is injective and the diagram (4.1) commutes. Moreover,

$$\operatorname{ran}\psi = \operatorname{ran}\phi = \operatorname{ran}\iota_1' + \operatorname{ran}\iota_2'\,.$$

The injectivity of ψ shows that ι_j^α is injective if and only if ι_j' is injective.

In order to show uniqueness, assume that (4.1) holds with some $\alpha' \subseteq \mathcal{L}_1^\circ \times \mathcal{L}_2^\circ$ and $\psi' : \mathcal{L}_1 \boxplus_{\alpha'} \mathcal{L}_2 \to \mathcal{L}$. Then one has



By uniqueness in Proposition 3.3, recall that $\mathcal{L}_1[\dot{+}]\mathcal{L}_2$ can be viewed as $\mathcal{L}_1 \ltimes_0 \mathcal{L}_2$, one must have $\psi' \circ \pi_{\alpha'} = \phi$. Since ψ' is injective, this implies

$$\alpha' = \ker\pi_{\alpha'} = \ker\big(\psi' \circ \pi_{\alpha'}\big) = \ker\phi = \alpha\,.$$

The map ψ is uniquely determined by (4.1), since $\operatorname{ran}\iota_1^\alpha$ and $\operatorname{ran}\iota_2^\alpha$ together span $\mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2$. □

Proposition 4.4 with Remark 4.3, *(ii)* lead to the following corollary.

Corollary 4.5. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces. An inner product space contains isomorphic copies of $\mathcal{L}_1$ and $\mathcal{L}_2$ as orthogonal subspaces which span the whole space if and only if it is isomorphic to $\mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2$ with some bijective map α between subspaces of $\mathcal{L}_1^\circ$ and $\mathcal{L}_2^\circ$.* □

Remark 4.6. G30 Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be almost Pontryagin spaces and let α be a linear subspace of $\mathcal{A}_1^\circ \times \mathcal{A}_2^\circ$. Then also $\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$ is an almost Pontryagin space and

$$\operatorname{ind}_-\big(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2\big) = \operatorname{ind}_-\mathcal{A}_1 + \operatorname{ind}_-\mathcal{A}_2\,,$$

$$\operatorname{ind}_0\big(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2\big) = \operatorname{ind}_0\mathcal{A}_1 + \operatorname{ind}_0\mathcal{A}_2 - \dim\alpha\,.$$

The almost Pontryagin space version of Proposition 4.4 now reads as follows.

Proposition 4.7. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be an almost Pontryagin spaces and let $\mathcal{A}$ be an almost Pontryagin space together with morphisms $\iota_j' : \mathcal{A}_j \to \mathcal{A}$, $j = 1, 2$, such that $\iota_1'(\mathcal{A}_1) \perp \iota_2'(\mathcal{A}_2)$. Then the isometry $\psi : \mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2 \to \mathcal{A}$ in Proposition 4.4 is a morphism.*

Proof. Apply Proposition 3.11, (i), with the presently given data $\mathcal{A}, \mathcal{A}_j, \iota_j'$, $c := 0$, and the map ϕ in (4.2). Note first that $\operatorname{ran} \iota_1' \cap \operatorname{ran} \iota_2'$, being a neutral subspace of $\mathcal{A}$, is finite dimensional. Since $\operatorname{ran} \iota_1'$ and $\operatorname{ran} \iota_2'$, as closed subspaces of the almost Pontryagin space $\mathcal{A}$, are themselves almost Pontryagin spaces, one may choose closed subspaces $\mathcal{M}_j$ of $\operatorname{ran} \iota_j'$, $j = 1, 2$, which are closed, have finite codimension in $\operatorname{ran} \iota_j'$, and are Hilbert spaces with respect to the inner product inherited from $\mathcal{A}$. Clearly, they are orthogonal to each other. This also implies that $\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$. Their sum $\mathcal{M} := \mathcal{M}_1 [\dot{+}] \mathcal{M}_2$ is thus also a Hilbert space in the inner product of $\mathcal{A}$. Moreover $\mathcal{M}$, as the orthogonal sum of two uniformly positive subspaces, is itself uniformly positive. Hence $\mathcal{M}$ is closed in the norm of $\mathcal{A}$. Clearly, $\mathcal{M}$ has finite codimension in $\operatorname{ran} \iota_1' + \operatorname{ran} \iota_2'$, and one may conclude that $\operatorname{ran} \iota_1' + \operatorname{ran} \iota_2'$ is closed in the norm of $\mathcal{A}$.

Remark 3.11 implies that the map ϕ in (4.2) is an almost Pontryagin space-morphism. Hence, also ψ is an almost Pontryagin space-morphism. □

Remark 4.8 (Concrete realization of $\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$)**.** Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be almost Pontryagin spaces and let α be a bijective map between some subspaces $\operatorname{dom} \alpha$ and $\operatorname{ran} \alpha$ of $\mathcal{A}_1^\circ$ and $\mathcal{A}_2^\circ$, respectively. The space $\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$ can also be described explicitly. For this purpose choose closed subspaces $\mathcal{A}_{1,r}$ and $\mathcal{A}_{2,r}$ such that

$$\mathcal{A}_1 = \mathcal{A}_{1,r} [\dot{+}] \mathcal{A}_1^\circ, \ \mathcal{A}_2 = \mathcal{A}_{2,r} [\dot{+}] \mathcal{A}_2^\circ \,,$$

choose D_1 and D_2 such that

$$\mathcal{A}_1^\circ = D_1 \dot{+} \operatorname{dom} \alpha, \ \mathcal{A}_2^\circ = D_2 \dot{+} \operatorname{ran} \alpha \,,$$

and set $D := \operatorname{ran} \alpha$. Consider the almost Pontryagin space

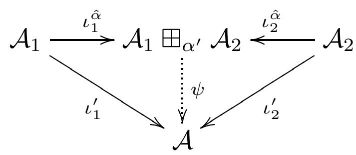
$$\mathcal{A} := \mathcal{A}_{1,r} [\dot{+}] \big(D_1 \dot{+} D \dot{+} D_2 \big) [\dot{+}] \mathcal{A}_{2,r} \tag{4.3}$$

where the inner product and topology on $\mathcal{A}_{1,r}$ and $\mathcal{A}_{2,r}$ are inherited from $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, and where $D_1 \dot{+} D \dot{+} D_2$ is neutral and endowed with the euclidean topology. Moreover, define $\iota_1' : \mathcal{A}_1 \to \mathcal{A}$ by

$$\iota_1'|_{\mathcal{A}_{1,r} \dot{+} D_1} := \mathrm{id}, \ \iota_1'|_{\operatorname{dom} \alpha} := -\alpha \,,$$

and let $\iota_2' : \mathcal{A}_2 \to \mathcal{A}$ be the identity map. Then ι_1' and ι_2' are morphisms. Moreover, it is clear from their definition that $\iota_1'(\mathcal{A}_1) \perp \iota_2'(\mathcal{A}_2)$ and $\iota_1'(\mathcal{A}_1) + \iota_2'(\mathcal{A}_2) = \mathcal{A}$.

By Proposition 4.4 there exist a linear subspace $\hat{\alpha} \subseteq \mathcal{A}_1^\circ \times \mathcal{A}_2^\circ$ and an isomorphism $\psi : \mathcal{A}_1 \boxplus_{\hat{\alpha}} \mathcal{A}_2 \to \mathcal{A}$ with



Thereby the linear subspace $\hat{\alpha}$ is given as $\hat{\alpha} = \{(x_1, x_2) \in \mathcal{A}_1^\circ \times \mathcal{A}_2^\circ : \iota_1'(x_1) = \iota_2'(x_2)\}$. Write $x_1 = a_1 + b_1$ according to the decomposition $\mathcal{A}_1^\circ = D_1 \dot{+} \operatorname{dom} \alpha$, and let $x_2 = a_2 + b_2$ according to $\mathcal{A}_2^\circ = D_2 \dot{+} \operatorname{ran} \alpha$. Then $\iota_1'(x_1) = a_1 - \alpha(b_1)$ and

$\iota_2'(x_2) = a_2 + b_2$. Hence $\iota_1'(x_1) = \iota_2'(x_2)$ if and only if $a_1 = a_2 = 0$ and $b_2 = \alpha(b_1)$. This, in turn, is equivalent to $(x_1, x_2) \in \alpha$.

One sees that $\hat{\alpha} = \alpha$, and hence ψ is actually an isomorphism between $\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$ and $\mathcal{A}$, i.e. $\mathcal{A}$ can be regarded as a concrete realization of $\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$.

5. The canonical Pontryagin space extension of an almost Pontryagin space

There is a natural way to associate with a given almost Pontryagin space $\mathcal{A}$ a Pontryagin space $\mathfrak{P}(\mathcal{A})$ by means of a factorization process $\mathfrak{P}(\mathcal{A}) := \mathcal{A}/\mathcal{A}^\circ$. However, there is also another natural way via an extension process; and this construction has turned out to be important.

Definition 5.1. Let $\mathcal{A}$ be an almost Pontryagin space. A pair $(\iota, \mathcal{P})$ is called a canonical Pontryagin space extension of $\mathcal{A}$, if $\mathcal{P}$ is a Pontryagin space, the extension emdding $\iota : \mathcal{A} \to \mathcal{P}$ is an injective morphism, and

$$\dim \mathcal{P}/\iota(\mathcal{A}) = \operatorname{ind}_0 \mathcal{A}\,.$$

Let $\mathcal{P}$ be a canonical Pontryagin space extension of $\mathcal{A}$, then it follows that $\operatorname{ind}_- \mathcal{P} = \operatorname{ind}_- \mathcal{A} + \operatorname{ind}_0 \mathcal{A}$. Canonical Pontryagin space extensions are in some sense minimal among all Pontryagin spaces which contain $\mathcal{A}$ as an isometric subspace. If $\mathcal{P}$ is a Pontryagin space which contains $\mathcal{A}$ as an isometric subspace, then certainly $\dim \mathcal{P}/\mathcal{A} \geq \operatorname{ind}_0 \mathcal{A}$ and $\operatorname{ind}_- \mathcal{P} \geq \operatorname{ind}_- \mathcal{A} + \operatorname{ind}_0 \mathcal{A}$.

Remark 5.2 (Existence of canonical Pontryagin space extensions)**.** Let $\mathcal{A}$ be an almost Pontryagin space. Choose a closed subspace $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{B}[\dot{+}]\mathcal{A}^\circ$, and let C be a linear space with $\dim C = \dim \mathcal{A}^\circ$. Consider the linear space

$$\mathfrak{P}_{\text{ext}}(\mathcal{A}) := \mathcal{A} \dot{+} C = \mathcal{B} \dot{+} \mathcal{A}^\circ \dot{+} C\,,$$

and define on this linear space an inner product $[.,.]$ by the requirements

$$[.,.]|_{\mathcal{A}\times\mathcal{A}} = [.,.]_{\mathcal{A}}, \quad \mathcal{B} \perp C, \quad \mathcal{A}^\circ \# C\,.$$

The notation $A \# B$ means that A and B are skewly linked, i.e. that A and B are neutral, $\dim A = \dim B$, and $A \dot{+} B$ is nondegenerated, cf. [1, §I.10] or [3, §I.3].

It is easy to see that $\mathfrak{P}_{\text{ext}}(\mathcal{A})$ is a Pontryagin space. Moreover, the set-theoretic inclusion map ι_{ext} of $\mathcal{A}$ into $\mathfrak{P}_{\text{ext}}(\mathcal{A})$ is a morphism. Clearly, ι_{ext} is injective and $\dim \mathcal{P}_{\text{ext}}(\mathcal{A})/\mathcal{A} = \dim \mathcal{A}^\circ$.

In Corollary 5.6 below it will be shown that canonical Pontryagin space extensions are unique up to isomorphisms.

Extension of morphisms. It is important to see how morphisms between almost Pontryagin spaces can be extended to morphisms between canonical Pontryagin space extensions. First concrete extensions as constructed in Remark 5.2 are dealt with.

Proposition 5.3. *Let $\mathcal{A}_1, \mathcal{A}_2$ be almost Pontryagin spaces and let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a morphism. Let spaces $\mathfrak{P}_{\text{ext}}(\mathcal{A}_1/\ker\phi)$ and $\mathfrak{P}_{\text{ext}}(\mathcal{A}_2)$ be constructed as in Remark 5.2 from some choices of subspaces $\mathcal{B}_1 \subseteq \mathcal{A}_1/\ker\phi$ and $\mathcal{B}_2 \subseteq \mathcal{A}_2$, respectively. Then* *there exists a morphism* $\tilde{\phi} : \mathfrak{P}_{\rm ext}(\mathcal{A}_1/\ker\phi) \to \mathfrak{P}_{\rm ext}(\mathcal{A}_2)$*, such that*

$$\begin{array}{ccccc} \mathcal{A}_1 & \xrightarrow{\pi} & \mathcal{A}_1/\ker\phi & \xrightarrow{\iota_{\rm ext}} & \mathfrak{P}_{\rm ext}(\mathcal{A}_1/\ker\phi) \\ {\scriptstyle\phi}\downarrow & & & & \vdots\,{\scriptstyle\tilde{\phi}} \\ \mathcal{A}_2 & & \xrightarrow[\iota_{\rm ext}]{\qquad\qquad} & & \mathfrak{P}_{\rm ext}(\mathcal{A}_2) \end{array} \tag{5.1}$$

Proof. There exists an injective morphism $\phi' : \mathcal{A}_1/\ker\phi \to \mathcal{A}_2$ such that

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{\pi} & \mathcal{A}_1/\ker\phi \\ {\scriptstyle\phi}\downarrow & \swarrow{\scriptstyle\phi'} & \\ \mathcal{A}_2 & & \end{array}$$

cf. Remark 2.5, (vii). Obviously, it is enough to prove the assertion for ϕ'. Hence, assume without loss of generality that ϕ is injective.

The subspace $(\iota_{\rm ext}\circ\phi)(\mathcal{B}_1)$ of $\mathfrak{P}_{\rm ext}(\mathcal{A}_2)$ is closed and nondegenerated. Moreover, $(\iota_{\rm ext}\circ\phi)(\mathcal{A}_1^\circ)$ is a neutral subspace of $(\iota_{\rm ext}\circ\phi)(\mathcal{B}_1)^\perp$. Hence there exists a subspace C' of $(\iota_{\rm ext}\circ\phi)(\mathcal{B}_1)^\perp$, such that $(\iota_{\rm ext}\circ\phi)(\mathcal{A}_1^\circ)\#C'$, cf. [1, §I.10].

The space $\mathfrak{P}_{\rm ext}(\mathcal{A}_1)$ is defined as $\mathcal{B}_1[\dot{+}](\mathcal{A}_1^\circ\dot{+}C)$ with $\mathcal{A}_1^\circ\#C$. Choose a basis $\{\delta_1,\dots,\delta_n\}$ of $\mathcal{A}_1^\circ$ and let $\{\epsilon_1,\dots,\epsilon_n\}$ be a basis of C with

$$[\delta_j,\epsilon_k] = \begin{cases} 0\,, & j\neq k, \\ 1\,, & j=k. \end{cases}$$

Since $\iota_{\rm ext}\circ\phi$ is injective, the set $\{(\iota_{\rm ext}\circ\phi)(\delta_1),\dots,(\iota_{\rm ext}\circ\phi)(\delta_n)\}$ is a basis of $(\iota_{\rm ext}\circ\phi)(\mathcal{A}_1^\circ)$. Hence there exists a basis $\{\epsilon_1',\dots,\epsilon_n'\}$ of C' such that

$$\big[(\iota_{\rm ext}\circ\phi)(\delta_j),\epsilon_k'\big] = \begin{cases} 0\,, & j\neq k, \\ 1\,, & j=k. \end{cases}$$

With these notations define $\tilde{\phi} : \mathfrak{P}_{\rm ext}(\mathcal{A}_1) \to \mathfrak{P}_{\rm ext}(\mathcal{A}_2)$ by

$$\tilde{\phi}|_{\iota_{\rm ext}(\mathcal{A}_1)} := \iota_{\rm ext}\circ\phi\circ\iota_{\rm ext}^{-1}, \quad \tilde{\phi}(\epsilon_j) := \epsilon_j',\ j=1,\dots,n\,.$$

It is straightforward to check that $\tilde{\phi}$ is isometric. Moreover, the commutativity of (5.1) is built into the definition. □

Remark 5.4. The extension $\tilde{\phi}$ in Proposition 5.3 is in general not unique. In fact, whenever $\mathcal{P}$ is a Pontryagin space with

$$(\iota_{\rm ext}\circ\phi)(\mathcal{A}_1) \subseteq \mathcal{P} \subseteq \mathfrak{P}_{\rm ext}(\mathcal{A}_2)\,,$$

the extension $\tilde{\phi}$ can be chosen such that $\operatorname{ran}\tilde{\phi}\subseteq\mathcal{P}$.

Corollary 5.5. *Let $\mathcal{A}$ be an almost Pontryagin space and let $(\iota,\mathcal{P})$ be a canonical Pontryagin space extension of $\mathcal{A}$. Moreover, let $(\iota_{\rm ext},\mathfrak{P}_{\rm ext}(\mathcal{A})$ be the canonical Pontryagin space extension constructed in Remark 5.2 from some subspace $\mathcal{B}$. Then there exists an isomorphism $\lambda : \mathfrak{P}_{\rm ext}(\mathcal{A}) \to \mathcal{P}$ such that*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\iota_{\rm ext}} & \mathfrak{P}_{\rm ext}(\mathcal{A}) \\ {\scriptstyle\iota}\downarrow & \swarrow{\scriptstyle\lambda} & \\ \mathcal{P} & & \end{array}$$

Proof. Since $\mathcal{P}$ is a Pontryagin space, one has $\mathfrak{P}_{\mathrm{ext}}(\mathcal{P}) = \mathcal{P}$ and $\iota_{\mathrm{ext}} = \mathrm{id}$. Proposition 5.3 applied with the map $\iota : \mathcal{A} \to \mathcal{P}$ gives a morphism $\lambda : \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}) \to \mathcal{P}$.

Since a morphism between Pontryagin spaces is injective, one concludes from $\lambda(\iota_{\mathrm{ext}}(\mathcal{A})) = \iota(\mathcal{A})$ and

$$\dim \mathcal{P}/\iota(\mathcal{A}) = \dim \mathcal{A}^\circ = \dim \mathfrak{P}_{\mathrm{ext}}(\mathcal{A})/\iota_{\mathrm{ext}}(\mathcal{A}),$$

that λ is an isomorphism. $\square$

This fact has some immediate, important, consequences.

Corollary 5.6. (*i*) *Let $\mathcal{A}$ be an almost Pontryagin space. If $(\iota_1, \mathcal{P}_1)$ and $(\iota_2, \mathcal{P}_2)$ are canonical Pontryagin space extensions of $\mathcal{A}$, then there exists an isomorphism $\lambda : \mathcal{P}_1 \to \mathcal{P}_2$ with*

$$\begin{array}{ccc} & \mathcal{A} & \\ {\scriptstyle \iota_1}\swarrow & & \searrow{\scriptstyle \iota_2} \\ \mathcal{P}_1 & \xrightarrow[\lambda]{\quad\cdots\quad} & \mathcal{P}_2 \end{array}$$

(*ii*) *Let $\mathcal{A}_1, \mathcal{A}_2$ be almost Pontryagin spaces and let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a morphism. Let $(\iota_1, \mathcal{P}_1)$ and $(\iota_2, \mathcal{P}_2)$ be canonical Pontryagin space extensions of $\mathcal{A}_1/\ker\phi$ and $\mathcal{A}_2$, respectively. Then there exists a morphism $\tilde{\phi} : \mathcal{P}_1 \to \mathcal{P}_2$, such that*

$$\begin{array}{ccccc} \mathcal{A}_1 & \xrightarrow{\pi} & \mathcal{A}_1/\ker\phi & \xrightarrow{\iota_1} & \mathcal{P}_1 \\ {\scriptstyle \phi}\downarrow & & & & \vdots\,{\scriptstyle \tilde{\phi}} \\ \mathcal{A}_2 & & \xrightarrow[\iota_2]{\qquad\qquad} & & \mathcal{P}_2 \end{array}$$

Compatibility with orthogonal coupling. The following fairly simple consequence of Proposition 5.3 turns out to be useful.

Proposition 5.7. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be almost Pontryagin spaces and let α be a bijective function between subspaces of $\mathcal{A}_1^\circ$ and $\mathcal{A}_2^\circ$. Then there exist morphisms $\tilde{\iota}_1^\alpha$ and $\tilde{\iota}_2^\alpha$, such that*

$$\begin{array}{ccccc} \mathcal{A}_1 & \xrightarrow{\iota_1^\alpha} & \mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2 & \xleftarrow{\iota_2^\alpha} & \mathcal{A}_2 \\ {\scriptstyle \iota_{\mathrm{ext}}}\downarrow & & \downarrow{\scriptstyle \iota_{\mathrm{ext}}} & & \downarrow{\scriptstyle \iota_{\mathrm{ext}}} \\ \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_1) & \xrightarrow[\tilde{\iota}_1^\alpha]{\cdots} & \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2) & \xleftarrow[\tilde{\iota}_2^\alpha]{\cdots} & \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_2) \end{array} \tag{5.2}$$

The choice of $\tilde{\iota}_1^\alpha$ and $\tilde{\iota}_2^\alpha$ can be made such that $\operatorname{ran}\tilde{\iota}_1^\alpha \cap \operatorname{ran}\tilde{\iota}_2^\alpha$ is a nondegenerated subspace of $\mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2)$ with dimension $2\dim(\operatorname{dom}\alpha)$ which contains the space $(\iota_{\mathrm{ext}} \circ \iota_1^\alpha)(\operatorname{dom}(\alpha))$.

Proof. By Remark 4.3, (*ii*), the maps ι_1^α and ι_2^α are injective. Hence Proposition 5.3 guarantees the existence of $\tilde{\iota}_1^\alpha$ and $\tilde{\iota}_2^\alpha$ which satisfy (5.2). It will be shown that they can be chosen so as to satisfy the stated additional requirement. To this end use the concrete realization of orthogonal couplings given in Remark 4.8, the concrete form of canonical Pontryagin space extensions given in Remark 5.2, and trace the construction of $\tilde{\iota}_1^\alpha$ and $\tilde{\iota}_2^\alpha$ in the proof of Proposition 5.3.

Choose closed subspaces $\mathcal{A}_{j,r}$ of $\mathcal{A}_j$ with $\mathcal{A}_j = \mathcal{A}_{j,r}[\dot{+}]\mathcal{A}_j^\circ$, $j = 1,2$, choose D_j with $\mathcal{A}_1^\circ = D_1\dot{+}\operatorname{dom}\alpha$ and $\mathcal{A}_2^\circ = D_2\dot{+}\operatorname{ran}\alpha$, and set $D := \operatorname{ran}\alpha$. Then one has

$$\mathcal{A}_1 = \mathcal{A}_{1,r}[\dot{+}](D_1\dot{+}\operatorname{dom}\alpha), \quad \mathcal{A}_2 = \mathcal{A}_{2,r}[\dot{+}](D_2\dot{+}D)\,,$$

and one can identify

$$\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2 \cong \mathcal{A}_{1,r}[\dot{+}](D_1\dot{+}D\dot{+}D_2)[\dot{+}]\mathcal{A}_{2,r}\,.$$

In this identification, the embeddings ι_1^α and ι_2^α act as

$$\iota_1^\alpha(x_r + x_1 + x_d) = x_r + x_1 + \alpha(x_d), \quad x_r \in \mathcal{A}_{1,r},\ x_1 \in D_1,\ x_d \in \operatorname{dom}\alpha\,,$$

$$\iota_2^\alpha(x) = x, \quad x \in \mathcal{A}_2\,,$$

and the isotropic part of $\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$ is given as

$$\big(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2\big)^\circ = D_1\dot{+}D\dot{+}D_2\,,$$

For the construction of $\mathfrak{P}_{\text{ext}}(\mathcal{A}_1)$, $\mathfrak{P}_{\text{ext}}(\mathcal{A}_2)$, and $\mathfrak{P}_{\text{ext}}(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2)$, use the closed nondegenerated subspaces $\mathcal{A}_{1,r}$, $\mathcal{A}_{2,r}$, and $\mathcal{A}_{1,r}[\dot{+}]\mathcal{A}_{2,r}$, respectively. Then one can write (note that $\dim\operatorname{dom}\alpha = \dim\operatorname{ran}\alpha$)

$$\mathfrak{P}_{\text{ext}}(\mathcal{A}_1) = \mathcal{A}_{1,r}\ [\dot{+}]\ \Big((D_1\dot{+}C_1)[\dot{+}](\operatorname{dom}\alpha\dot{+}C)\Big),$$

$$\mathfrak{P}_{\text{ext}}(\mathcal{A}_2) = \mathcal{A}_{2,r}\ [\dot{+}]\ \Big((D_2\dot{+}C_2)[\dot{+}](D\dot{+}C)\Big),$$

$$\mathfrak{P}_{\text{ext}}(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2) = \mathcal{A}_{1,r}\ [\dot{+}]\ \Big((D_1\dot{+}C_1)[\dot{+}](D\dot{+}C)[\dot{+}](D_2\dot{+}C_2)\Big)\ [\dot{+}]\,\mathcal{A}_{2,r},$$

with neutral spaces C_1, C_d, C, C_2 satisfying $C_1\#D_1$, $C\#\operatorname{dom}\alpha$, $C_2\#D_2$, $C\#D$, and the extension embeddings are the respective set-theoretic inclusion maps. The maps constructed in Proposition 5.3 act as

$$\tilde{\iota}_1^\alpha\big(x_r + (x_1 + y_1) + (x_d + y)\big) = x_r + (x_1 + y_1) + (\alpha(x_d) + y),$$
$$x_r \in \mathcal{A}_{1,r}, x_1 \in D_1, y_1 \in C_1, x_d \in \operatorname{dom}\alpha, y \in C\,,$$

and

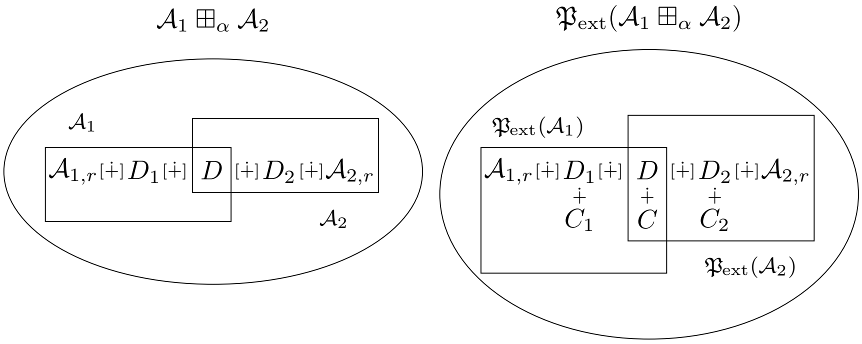
$$\tilde{\iota}_2^\alpha(x) = x, \quad x \in \mathfrak{P}_{\text{ext}}(\mathcal{A}_2)\,.$$

From this one sees that

$$\operatorname{ran}\tilde{\iota}_1^\alpha \cap \operatorname{ran}\tilde{\iota}_2^\alpha = D\dot{+}C\,. \qquad \square$$

Remark 5.8. In Proposition 5.7 the mappings ι_1^α and ι_2^α are injective, all extension embeddings ι_{ext} are by definition injective, and $\tilde{\iota}_1^\alpha, \tilde{\iota}_1^\alpha$ are morphisms with nondegenerated domain and they are also injective. Hence, one can think of $\mathfrak{P}_{\text{ext}}(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2)$ as the biggest of the six spaces in (5.2) which contains the other ones.

If the distinction between the spaces $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$ is suppressed and both are thought of as equal to the space D, then the situation can be pictured as follows:



In the case that $\operatorname{dom}\alpha = \mathcal{A}_1^\circ$ and $\operatorname{ran}\alpha = \mathcal{A}_2^\circ$ in Proposition 5.7, more can be said. Then $D_1 = D_2 = C_1 = C_2 = 0$ and $(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2)^\circ = D = \mathcal{A}_1^\circ = \mathcal{A}_2^\circ$. Denote by P_D, P_C, $P_{\mathcal{A}_{1,r}[\dot{+}]\mathcal{A}_{2,r}}$, and $P_{\mathcal{A}_1\boxplus_\alpha\mathcal{A}_2}$ the projections of the space $\mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_1\boxplus_\alpha\mathcal{A}_2)$ onto the space denoted as index according to the above pictured direct sum decomposition. Thus, e.g., one has $\operatorname{ran} P_D = D$ and $\ker P_D = \mathcal{A}_{1,r}[\dot{+}]C[\dot{+}]\mathcal{A}_{2,r}$.

Lemma 5.9. *Assume that in the situation of Proposition 5.7 one has* $\operatorname{dom}\alpha = \mathcal{A}_1^\circ$ *and* $\operatorname{ran}\alpha = \mathcal{A}_2^\circ$. *Then the following statements hold:*

(*i*) *The projections* $P_{\mathcal{A}_{1,r}[\dot{+}]\mathcal{A}_{2,r}}$ *and* $P_{\mathcal{A}_1\boxplus_\alpha\mathcal{A}_2}$ *satisfy*

$$P_{\mathcal{A}_{1,r}[\dot{+}]\mathcal{A}_{2,r}} + P_D + P_C = I, \quad P_{\mathcal{A}_1\boxplus_\alpha\mathcal{A}_2} + P_C = I,$$

$$P_{\mathcal{A}_{1,r}[\dot{+}]\mathcal{A}_{2,r}}\big(\mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_j)\big) = \mathcal{A}_{j,r},\ P_{\mathcal{A}_1\boxplus_\alpha\mathcal{A}_2}\big(\mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_j)\big) = \mathcal{A}_j, \quad j = 1,2.$$

Let elements $x_1 \in \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_1)$ *and* $x_2 \in \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_2)$ *be given. Then*

(*ii*) $[x_1, x_2] = [P_D x_1, P_C x_2] + [P_C x_1, P_D x_2]$.

(*iii*) $P_C x_1 = P_C x_2$ *if and only if* $[x_1, h] = [x_2, h],\ h \in D$, *in which case*

$$x_1 + P_{\mathcal{A}_1\boxplus_\alpha\mathcal{A}_2} x_2 = P_{\mathcal{A}_1\boxplus_\alpha\mathcal{A}_2} x_1 + x_2.$$

Moreover, let elements $y_1 \in \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_1)$ *and* $y_2 \in \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_2)$ *be given.*

(*iv*) *If* $P_C x_1 = P_C x_2$ *and* $P_C y_1 = P_C y_2$, *then*

$$[x_1 + P_{\mathcal{A}_1\boxplus_\alpha\mathcal{A}_2} x_2, y_1 + P_{\mathcal{A}_1\boxplus_\alpha\mathcal{A}_2} y_2] = [x_1, y_1] + [x_2, y_2]\,.$$

Proof. The formulas in (*i*) are immediate from the definitions of the corresponding projections. In order to see the equality asserted in (*ii*) compute

$$\begin{aligned}[x_1, x_2] &= \big[(P_{\mathcal{A}_{1,r}[\dot{+}]\mathcal{A}_{2,r}} + P_D + P_C)x_1, (P_{\mathcal{A}_{1,r}[\dot{+}]\mathcal{A}_{2,r}} + P_D + P_C)x_2\big] \\ &= \big[P_{\mathcal{A}_{1,r}[\dot{+}]\mathcal{A}_{2,r}} x_1, P_{\mathcal{A}_{1,r}[\dot{+}]\mathcal{A}_{2,r}} x_2\big] + \big[(P_D + P_C)x_1, (P_D + P_C)x_2\big] \\ &= [P_D x_1, P_C x_2] + [P_C x_1, P_D x_2].\end{aligned}$$

As to the proof of (*iii*), observe that for each $h \in D$,

$$[x_1, h] = \big[(P_{\mathcal{A}_1\boxplus_\alpha\mathcal{A}_2} + P_C)x_1, h\big] = [P_C x_1, h],$$

$$[x_2, h] = \big[(P_{\mathcal{A}_1\boxplus_\alpha\mathcal{A}_2} + P_C)x_2, h\big] = [P_C x_2, h].$$

Since $D\#C$ the asserted equivalence follows. Moreover, if $P_C x_1 = P_C x_2$, then

$$\begin{aligned}x_1 + P_{\mathcal{A}_1\boxplus_\alpha\mathcal{A}_2} x_2 &= P_{\mathcal{A}_1\boxplus_\alpha\mathcal{A}_2} x_1 + P_C x_1 + P_{\mathcal{A}_1\boxplus_\alpha\mathcal{A}_2} x_2 \\ &= P_{\mathcal{A}_1\boxplus_\alpha\mathcal{A}_2} x_1 + P_C x_2 + P_{\mathcal{A}_1\boxplus_\alpha\mathcal{A}_2} x_2 = P_{\mathcal{A}_1\boxplus_\alpha\mathcal{A}_2} x_1 + x_2.\end{aligned}$$

The situation in (iv) leads to

$$\begin{aligned}[x_2, y_2] &= \big[(P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} + P_C)x_2, (P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} + P_C)y_2\big] \\ &= [P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2, P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} y_2] + [P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2, P_C y_2] + [P_C x_2, P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} y_2] \\ &= [P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2, P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} y_2] + [P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2, P_C y_1] + [P_C x_1, P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} y_2].\end{aligned}$$

Hence it follows that

$$\begin{aligned}&[x_1 + P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2, y_1 + P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} y_2] \\ &\quad = [x_1, y_1] + [P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2, y_1] + [x_1, P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} y_2] + [P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2, P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} y_2] \\ &\quad = [x_1, y_1] + [P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2, P_C y_1] + [P_C x_1, P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} y_2] + [P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2, P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} y_2] \\ &\quad = [x_1, y_1] + [x_2, y_2].\end{aligned}$$

□

6. Almost Pontryagin space completions

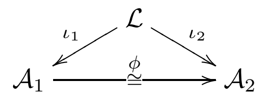
Definition 6.1. Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space. A pair $(\iota, \mathcal{A})$ is called an almost Pontryagin space-completion of $\mathcal{L}$ if $\mathcal{A}$ is an almost Pontryagin space and ι is an isometric map of $\mathcal{L}$ onto a dense subspace of $\mathcal{A}$.

Note that the isometric map ι in Definition 6.1 need not be injective; see Remark 2.5 (i).

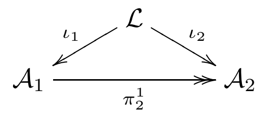
Two almost Pontryagin space completions of an inner product space $\mathcal{L}$ might be 'the same' or one might be 'larger' than the other. This is made precise by the following notions.

Definition 6.2. Let $(\iota_1, \mathcal{A}_1)$ and $(\iota_2, \mathcal{A}_2)$ be two almost Pontryagin space completions of an inner product space $\mathcal{L}$.

(i) The completions $(\iota_1, \mathcal{A}_1)$ and $(\iota_2, \mathcal{A}_2)$ are isomorphic, $(\iota_1, \mathcal{A}_1) \cong (\iota_2, \mathcal{A}_2)$, if there exists an isomorphism ϕ of $\mathcal{A}_1$ onto $\mathcal{A}_2$, such that $\phi \circ \iota_1 = \iota_2$, i.e.



(ii) The completions $(\iota_1, \mathcal{A}_1)$ and $(\iota_2, \mathcal{A}_2)$ satisfy $(\iota_1, \mathcal{A}_1) \succeq (\iota_2, \mathcal{A}_2)$, if there exists a surjective morphism π_2^1 of $\mathcal{A}_1$ onto $\mathcal{A}_2$, such that $\pi_2^1 \circ \iota_1 = \iota_2$, i.e.



Obviously, isomorphism is an equivalence relation on the set of all almost Pontryagin space-completions of $\mathcal{L}$ and the relation $\succeq$ is reflexive and transitive. Moreover, a short argument will show that

$$\Big((\iota_1, \mathcal{A}_1) \succeq (\iota_2, \mathcal{A}_2) \wedge (\iota_2, \mathcal{A}_2) \succeq (\iota_1, \mathcal{A}_1) \Big) \iff (\iota_1, \mathcal{A}_1) \cong (\iota_2, \mathcal{A}_2)$$

The relation $\succeq$ induces a partial order on the set of all almost Pontryagin space-completions of $\mathcal{L}$ modulo isomorphism.

Remark 6.3. If $(\iota_1, \mathcal{A}_1)$ is an almost Pontryagin space-completion of $\mathcal{L}$, $\mathcal{A}_2$ is an almost Pontryagin space, and π is a surjective morphism of $\mathcal{A}_1$ onto $\mathcal{A}_2$, then $(\pi \circ \iota_1, \mathcal{A}_2)$ is an almost Pontryagin space-completion of $\mathcal{L}$ and $(\iota_1, \mathcal{A}_1) \succeq (\pi \circ \iota_1, \mathcal{A}_2)$.

Let $\mathcal{L}$ be an inner product space. If in some almost Pontryagin space-completion $(\iota, \mathcal{A})$ of $\mathcal{L}$ the space $\mathcal{A}$ is nondegenerated, i.e. a Pontryagin space, then $(\iota, \mathcal{A})$ is said to be a Pontryagin space completion of $\mathcal{L}$.

Remark 6.4. The space $\mathcal{L}$ admits a Pontryagin space completion if and only if $\operatorname{ind}_- \mathcal{L} < \infty$; cf. [1, §V.2,§I.11]. Moreover, in this case Pontryagin space completions are isomorphic. Since $\operatorname{ind}_- \mathcal{L} < \infty$ is obviously a necessary condition for the existence of an almost Pontryagin space-completion, one concludes that $\mathcal{L}$ admits an almost Pontryagin space-completion if and only if $\operatorname{ind}_- \mathcal{L} < \infty$.

Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$ and consider the map $\mathfrak{L}$ which assigns to each almost Pontryagin space-completion $(\iota, \mathcal{A})$ of $\mathcal{L}$ the linear subspace

$$\mathfrak{L}(\iota, \mathcal{A}) := \iota^* \mathcal{A}'$$

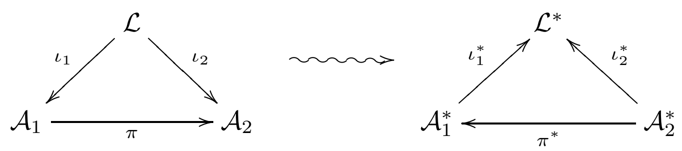
of the algebraic dual $\mathcal{L}^*$ of $\mathcal{L}$. Here ι^* denotes the (algebraic) adjoint of ι, that is $\iota^* : \mathcal{A}^* \to \mathcal{L}^*$ and $\iota^* f = f \circ \iota$.

The next statement already contains a good portion of the description of an almost Pontryagin space-completion.

Lemma 6.5. *Let $\mathcal{L}$ be an inner product space with* $\operatorname{ind}_- \mathcal{L} < \infty$ *and let* $(\iota_1, \mathcal{A}_1)$ *and* $(\iota_2, \mathcal{A}_2)$ *be two almost Pontryagin space-completions of $\mathcal{L}$ with* $(\iota_1, \mathcal{A}_1) \succeq (\iota_2, \mathcal{A}_2)$. *Then*

$$\mathfrak{L}(\iota_1, \mathcal{A}_1) \supseteq \mathfrak{L}(\iota_2, \mathcal{A}_2) \quad \text{and} \quad \dim \left(\mathfrak{L}(\iota_1, \mathcal{A}_1) \big/ \mathfrak{L}(\iota_2, \mathcal{A}_2)\right) = \operatorname{ind}_0 \mathcal{A}_1 - \operatorname{ind}_0 \mathcal{A}_2 \,.$$

Proof. Let $\pi : \mathcal{A}_1 \to \mathcal{A}_2$ be a surjective morphism with $\pi \circ \iota_1 = \iota_2$. Passing to adjoints yields



Since π is continuous one has $\pi^* \mathcal{A}_2' \subseteq \mathcal{A}_1'$. It readily follows that

$$\mathfrak{L}(\iota_2, \mathcal{A}_2) = \iota_2^* \mathcal{A}_2' = \iota_1^* \pi^* \mathcal{A}_2' \subseteq \iota_1^* \mathcal{A}_1' = \mathfrak{L}(\iota_1, \mathcal{A}_1) \,.$$

As $\operatorname{ran} \iota_1$ is dense in $\mathcal{A}_1$, the restriction of ι_1^* to $\mathcal{A}_1'$ is injective. Thus the codimension satisfies

$$\dim \left(\mathfrak{L}(\iota_1, \mathcal{A}_1) \big/ \mathfrak{L}(\iota_2, \mathcal{A}_2)\right) = \dim \left(\iota_1^* \mathcal{A}_1' \big/ \iota_1^* \pi^* \mathcal{A}_2'\right) = \dim \left(\mathcal{A}_1' \big/ \pi^* \mathcal{A}_2'\right) .$$

Since π is surjective, by the closed range theorem, $\pi^* \mathcal{A}_2'$ is a w^*-closed subspace of $\mathcal{A}_1'$. It follows that

$$\pi^* \mathcal{A}_2' = \overline{\pi^* \mathcal{A}_2'}^{w^*} = (\ker \pi)^\perp \,,$$

and hence

$$\dim \left(\mathcal{A}_1' \big/ \pi^* \mathcal{A}_2'\right) = \dim \left(\mathcal{A}_1' \big/ (\ker \pi)^\perp\right) = \dim (\ker \pi)' \,.$$

The mapping π is isometric, so that $\ker \pi \subseteq \mathcal{A}_1^\circ$. In particular, $\ker \pi$ is finite dimensional, and hence

$$\dim (\ker \pi)' = \dim \ker \pi \,.$$

The inclusion $\ker \pi \subseteq \mathcal{A}_1^\circ$ also shows that $\ker \pi = \ker(\pi|_{\mathcal{A}_1^\circ})$. Since π is surjective, one has $\pi^{-1}(\mathcal{A}_2^\circ) = \mathcal{A}_1^\circ$, and hence $\pi|_{\mathcal{A}_1^\circ}$ maps $\mathcal{A}_1^\circ$ onto $\mathcal{A}_2^\circ$. It follows that

$$\dim \ker \pi = \dim \ker(\pi|_{\mathcal{A}_1^\circ}) = \dim \mathcal{A}_1^\circ - \dim \mathcal{A}_2^\circ \,.$$

Putting together these identities, the desired formula follows. □

In particular Lemma 6.5 shows that

$$(\iota_1, \mathcal{A}_1) \cong (\iota_2, \mathcal{A}_2) \implies \mathfrak{L}(\iota_1, \mathcal{A}_1) = \mathfrak{L}(\iota_2, \mathcal{A}_2)\,. \tag{6.1}$$

Since Pontryagin space completions of $\mathcal{L}$ are isomorphic, the following notion is well-defined.

Definition 6.6. Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$. Choose a Pontryagin space completion $(\iota, \mathcal{P})$ of $\mathcal{L}$ and let a linear subspace $\mathcal{L}'$ of $\mathcal{L}^*$ be defined as

$$\mathcal{L}' := \mathfrak{L}(\iota, \mathcal{P}), \quad (\iota, \mathcal{P}) \text{ Pontryagin space completion of } \mathcal{L}\,.$$

Remark 6.7. In the terminology of [1, §IV.6] $\mathfrak{L}(\iota, \mathcal{P})$ is nothing else but the topological dual space of $\mathcal{L}$ with respect to the unique decomposition majorant which $\mathcal{L}$ carries as inner product space with finite negative index. Hence the notation $\mathcal{L}'$.

The map $\mathfrak{L}$ is defined on the set of all almost Pontryagin space-completions of $\mathcal{L}$ and maps an almost Pontryagin space-completion to a linear subspace of the algebraic dual $\mathcal{L}^*$. Due to (6.1) it induces a map, again denoted by $\mathfrak{L}$, from equivalence classes of almost Pontryagin space-completions modulo isomorphisms to linear subspaces of $\mathcal{L}^*$. It acts between two partially ordered sets as an injective order homomorphism.

Theorem 6.8. *Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$. Then $\mathfrak{L}$ induces an order-isomorphism of the set of all almost Pontryagin space-completions of $\mathcal{L}$ modulo isomorphism onto the set of all linear subspaces of $\mathcal{L}^*$ which contain $\mathcal{L}'$ with finite codimension. Thereby,*

$$\dim \big(\mathfrak{L}(\iota, \mathcal{A}) \big/ \mathcal{L}'\big) = \operatorname{ind}_0 \mathcal{A}\,. \tag{6.2}$$

Proof. The proof will be given in a number of steps.

Step 1. Let $(\iota, \mathcal{A})$ be an almost Pontryagin space-completion of $\mathcal{L}$. Denote by $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{A}^\circ$ the canonical projection, then π is a surjective morphism. Hence, $(\pi \circ \iota, \mathcal{A}/\mathcal{A}^\circ)$ is also an almost Pontryagin space-completion and $(\iota, \mathcal{A}) \succeq (\pi \circ \iota, \mathcal{A}/\mathcal{A}^\circ)$, cf. Remark 6.3. However, since $\mathcal{A}/\mathcal{A}^\circ$ is nondegenerated, $(\pi \circ \iota, \mathcal{A}/\mathcal{A}^\circ)$ is actually a Pontryagin space completion of $\mathcal{L}$. Thus $\mathfrak{L}(\pi \circ \iota, \mathcal{A}/\mathcal{A}^\circ) = \mathcal{L}'$, and it follows from Lemma 6.5 that $\mathfrak{L}(\iota, \mathcal{A})$ contains $\mathcal{L}'$ with codimension $\operatorname{ind}_0 \mathcal{A}^\circ$.

Step 2. Next assume that $(\iota_1, \mathcal{A}_1)$ and $(\iota_2, \mathcal{A}_2)$ are almost Pontryagin space-completions of $\mathcal{L}$ such that $\mathfrak{L}(\iota_1, \mathcal{A}_1) \supseteq \mathfrak{L}(\iota_2, \mathcal{A}_2)$. Therefore, for each given $f \in \mathcal{A}_2'$, there exists $\tilde{f} \in \mathcal{A}_1'$ with $\iota_1^* \tilde{f} = \iota_2^* f$. Since $\iota_1^*|_{\mathcal{A}_1'}$ is injective, this element $\tilde{f}$ is uniquely determined. Hence, a map $\Lambda : \mathcal{A}_2' \to \mathcal{A}_1'$ is well-defined by the requirement

$$\iota_1^*(\Lambda f) = \iota_2^* f, \quad f \in \mathcal{A}_2'\,.$$

Clearly, Λ is linear.

Now apply the closed graph theorem. Let a sequence $(f_n)_{n\in\mathbb{N}}$ of functionals $f_n \in \mathcal{A}_2'$ be given, and assume that $f_n \to f$ in $\mathcal{A}_2'$ and $\Lambda f_n \to g$ in $\mathcal{A}_1'$. Since convergence in the norm implies w^*-convergence, one has for each $x \in \mathcal{L}$

$$\begin{array}{l} (\iota_2^* f_n)x = f_n(\iota_2 x) \to f(\iota_2 x) = (\iota_2^* f)x = \iota_1^*(\Lambda f)x \\ \quad \| \\ \iota_1^*(\Lambda f_n)x = (\Lambda f_n)(\iota_1 x) \to g(\iota_1 x) = (\iota_1^* g)x \end{array}$$

Since $\iota_1^*|_{\mathcal{A}_1'}$ is injective, this implies that $\Lambda f = g$. It follows that Λ is bounded.

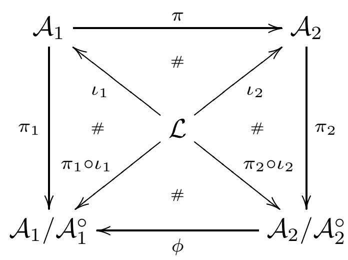
Let $\|.\|_1$ and $\|.\|_2$ be norms on $\mathcal{A}_1$ and $\mathcal{A}_2$ which induce their respective topologies. Moreover, let $\|.\|_1'$ and $\|.\|_2'$ be the corresponding operator norms on $\mathcal{A}_1'$ and $\mathcal{A}_2'$. Note that for $x \in \mathcal{L}$

$$\|\iota_2 x\|_2 = \sup\big\{ \,|\underbrace{f(\iota_2 x)}_{\substack{\| \\ (\iota_2^* f)x = \iota_1^*(\Lambda f)x = (\Lambda f)(\iota_1 x)}}| :\; f \in \mathcal{A}_2', \|f\|_2' \le 1 \big\} =$$

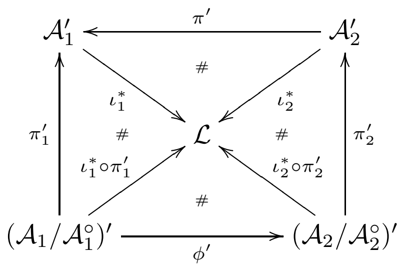
$$= \sup\big\{ |\tilde{f}(\iota_1 x)| :\; \tilde{f} \in \underbrace{\Lambda\big(\{ f \in \mathcal{A}_2' : \|f\|_2' \le 1\}\big)}_{\subseteq \{\tilde{f} \in \mathcal{A}_1' :\, \|\tilde{f}\|_1' \le \|\Lambda\|\}} \big\} \le \|\Lambda\| \cdot \|\iota_1 x\|_1 \,.$$

It follows that $\ker \iota_1 \subseteq \ker \iota_2$, so that $\iota_2 \circ \iota_1^{-1}$ is a well-defined map. Moreover, it follows that $\iota_2 \circ \iota_1^{-1}$ is bounded. Let $\pi : \mathcal{A}_1' \to \mathcal{A}_2'$ be its extension by continuity. Then π is isometric and has dense range in $\mathcal{A}_2'$.

Now let $\pi_j : \mathcal{A}_j \to \mathcal{A}_j/\mathcal{A}_j^\circ$, $j = 1, 2$, denote the canonical projections. Since $(\pi_1 \circ \iota_1, \mathcal{A}_1/\mathcal{A}_1^\circ)$ and $(\pi_2 \circ \iota_2, \mathcal{A}_2/\mathcal{A}_2^\circ)$ are both Pontryagin space completions of $\mathcal{L}$, there exists an isomorphism ϕ of $\mathcal{A}_2/\mathcal{A}_2^\circ$ onto $\mathcal{A}_1/\mathcal{A}_1^\circ$ with $\phi \circ (\pi_2 \circ \iota_2) = \pi_1 \circ \iota_1$. Thus, in the following diagram, each outer triangle commutes.



Passing to adjoints, gives the outer triangles in



Injectivity of $\iota_1^*|_{\mathcal{A}_1'}$ implies $\pi_1' = \pi' \circ \pi_2' \circ \phi'$. In particular, $\operatorname{ran} \pi_1' \subseteq \operatorname{ran} \pi' \subseteq \mathcal{A}_1'$. However, as seen in the proof of Lemma 6.5, $\operatorname{ran} \pi_1'$ is a closed subspace of $\mathcal{A}_1'$ with finite codimension. Hence, also $\operatorname{ran} \pi'$ is closed in $\mathcal{A}_1'$. By the closed range theorem, $\operatorname{ran} \pi$ is closed in $\mathcal{A}_1$, and hence π is surjective. Thus π is a morphism and it has been shown that $(\iota_1, \mathcal{A}_1) \succeq (\iota_2, \mathcal{A}_2)$.

Step 3. So far it is clear that $\mathfrak{L}$ maps almost Pontryagin space-completions into the set of all subspaces of $\mathcal{L}^*$ which contain $\mathcal{L}'$ with finite codimension, that (6.2) holds, and that

$$(\iota_1, \mathcal{A}_1) \succeq (\iota_2, \mathcal{A}_2) \iff \mathfrak{L}(\iota_1, \mathcal{A}_1) \supseteq \mathfrak{L}(\iota_2, \mathcal{A}_2)\,.$$

In particular, $\mathfrak{L}(\iota_1, \mathcal{A}_1) = \mathfrak{L}(\iota_2, \mathcal{A}_2)$ if and only if $(\iota_1, \mathcal{A}_1)$ and $(\iota_2, \mathcal{A}_2)$ are isomorphic.

It remains to show that for each given subspace $\mathcal{M}$ with $\mathcal{L}' \subseteq \mathcal{M}$ and $\dim \mathcal{M}/\mathcal{L}' < \infty$, there exists an almost Pontryagin space-completion $(\iota, \mathcal{A})$ of $\mathcal{L}$ with $\mathfrak{L}(\iota, \mathcal{A}) = \mathcal{M}$. The construction of one such completion goes back to [4] and was formulated and proved in the almost Pontryagin space-context in [5]. This method is now briefly indicated. Put $n := \dim \mathcal{M}/\mathcal{L}'$ and choose $f_1, \dots, f_n \in \mathcal{L}^*$ such that $\mathcal{M} = \operatorname{span}(\mathcal{L}' \cup \{f_1, \dots, f_n\})$. Moreover, let $(\iota_{\mathcal{P}}, \mathcal{P})$ be the Pontryagin space completion of $\mathcal{L}$. Define

- $\rightsquigarrow$ $\mathcal{A} := \mathcal{P}[\dot{+}]\mathbb{C}^n$, and $\mathcal{T}$ the product topology on $\mathcal{A}$,
- $\rightsquigarrow$ $[x + \xi, y + \eta]_{\mathcal{A}} := [x, y]_{\mathcal{P}}$, $x, y \in \mathcal{P}, \xi, \eta \in \mathbb{C}^n$,
- $\rightsquigarrow$ $\iota x := x + (f_1(x), \dots, f_n(x))$, $x \in \mathcal{L}$.

Then one can show that $(\iota, \mathcal{A})$ is an aPs-completion of $\mathcal{L}$ with $\mathfrak{L}(\iota, \mathcal{A}) = \mathcal{M}$. $\square$

Corollary 6.9. *Let $(\iota_1, \mathcal{A}_1)$ and $(\iota_2, \mathcal{A}_2)$ be two almost Pontryagin space-completions of an inner product space $\mathcal{L}$. Then $(\iota_1, \mathcal{A}_1) \succeq (\iota_2, \mathcal{A}_2)$ if and only if* $\ker \iota_1 \subseteq \ker \iota_2$ *and $\iota_2 \circ \iota_1^{-1} : \operatorname{ran} \iota_1 \to \operatorname{ran} \iota_2$ is bounded.*

Proof. If $(\iota_1, \mathcal{A}_1) \succeq (\iota_2, \mathcal{A}_2)$, then the map π_2^1 guaranteed by the definition of $\succeq$ is linear, bounded, and satisfies $\pi_2^1 \circ \iota_1 = \iota_2$. The required properties of ι_1 and ι_2 follow. Conversely, assume that $\ker \iota_1 \subseteq \ker \iota_2$ and $\iota_2 \circ \iota_1^{-1} : \operatorname{ran} \iota_1 \to \operatorname{ran} \iota_2$ is bounded. Let $\pi : \mathcal{A}_1 \to \mathcal{A}_2$ be the extension by continuity of $\iota_2 \circ \iota_1^{-1}$, then $\iota_2^* = \iota_1^* \circ \pi'$ and hence

$$\mathfrak{L}(\iota_2, \mathcal{A}_2) = \iota_2^* \mathcal{A}_2' = \left(\iota_1^* \circ \pi'\right)\mathcal{A}_2' \subseteq \iota_1^* \mathcal{A}_1' = \mathfrak{L}(\iota_1, \mathcal{A}_1)\,. \qquad \square$$

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