Reparametrizations of non trace-normed Hamiltonians

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Abstract. We consider a Hamiltonian system of the form $y'(x) = JH(x)y(x)$, with a locally integrable and nonnegative 2×2-matrix valued Hamiltonian $H(x)$. In the literature dealing with the operator theory of such equations, it is often required in addition that the Hamiltonian H is trace–normed, i.e. satisfies $\operatorname{tr} H(x) \equiv 1$. However, in many examples this property does not hold. The general idea is that one can reduce to the trace–normed case by applying a suitable change of scale (reparametrization). In this paper we justify this idea and work out the notion of reparametrization in detail.

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1. Introduction

Consider a Hamiltonian system of the form

$$y'(x) = zJH(x)y(x), \quad x \in I\,, \tag{1.1}$$

where I is a (finite or infinite) open interval on the real line, $z \in \mathbb{C}$, $J := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and $H : I \to \mathbb{R}^{2\times 2}$ is a function which does not vanish identically on I a.e., and has the following properties:

(Ham1) Each entry of H is (Lebesgue-to-Borel) measurable and locally integrable on I.

(Ham2) We have $H(x) \geq 0$ almost everywhere on I.

We call a function H satisfying (Ham1) and (Ham2) a *Hamiltonian*.

In the literature dealing with systems of the form (1.1), their operator theory, and their spectral properties, it is often assumed that H is *trace–normed*, i.e. that

(Ham3) We have $\operatorname{tr} H(x) = 1$ almost everywhere on I.

For example, in [HSW], where the operator model associated with (1.1) is introduced from an up-to-date viewpoint, the property (Ham3) is required from the start, in [K] trace–normed Hamiltonians are considered, and also in [GK] it is very soon required that the Hamiltonian under consideration satisfies (Ham3). Contrasting this, in [dB] no normalization conditions are required. However, that work does not deal with the operator theoretic viewpoint on the equation (1.1). In [KW/IV] boundary triples were studied which arise from Hamiltonian functions H which are only assumed to be *non–vanishing*, i.e. have the property that

(Ham3') The function H does not vanish on any set of positive measure.

Let us now list some examples of Hamiltonian systems, where the Hamiltonian is not necessarily trace–normed, or not even non–vanishing, and which have motivated our present work.

1°. When investigating the inverse spectral problem for semibounded spectral measures μ, equations (1.1) with H being of the form

$$H(x) = \begin{pmatrix} v(x)^2 & v(x) \\ v(x) & 1 \end{pmatrix}$$

appear naturally, cf. [W2]. Clearly, Hamiltonians of this kind are non–vanishing but not trace–normed. The function v has intrinsic meaning. For example, when μ is associated with a Kreĭn string $S[L, \mathfrak{m}]$, the function v is the mass function of the dual string of $S[L, \mathfrak{m}]$, cf. [KWW2, §4]. ∥

2°. When identifying a Sturm–Liouville equation without potential term as a Hamiltonian system, one obtains an equation (1.1) with H being of the form

$$H(x) = \begin{pmatrix} p(x) & 0 \\ 0 & \rho(x) \end{pmatrix} .$$

Often the functions p and ρ have physical meaning. For example, consider the propagation of waves in an elastic medium, and assume that the equations of isotropic elasticity hold and that the density of the medium depends only on the depth measured from the surface. Then one arrives at a hyperbolic system whose associated linear spectral problem is of the form

$$-\big(p(x)y'(x)\big)' = \omega^2 \rho(x) y(x), \quad x \geq 0\,,$$

where x measures the depth from the surface, $\rho(x)$ is the density of the media, and $p(x) = \lambda(x) + 2\mu(x)$ with the Lamé parameters λ, μ, cf. [BB], [McL]. Apparently, Hamiltonians of this kind are in general not trace–normed. When the medium under consideration contains layers of vacuum, they will not even be non–vanishing. ∥

3°. Dropping normalization assumptions often leads to significant simplification. For example, transformation of Hamiltonians and their corresponding Weyl–coefficients, like those given in [W1], can be treated with much more ease when

the requirement that all Hamiltonians are trace–normed is dropped. Also, the natural action of such transformations on the associated chain of de Branges spaces becomes much more apparent.

For example, in our recent investigation of symmetry in the class of Hamiltonians, cf. [WW], it is much more suitable to work with Hamiltonians which may vanish on sets of positive measure. When working with transformation formulas like those introduced in [KWW1], dropping the requirement that Hamiltonians are non–vanishing is very helpful. //

One obvious reason why a Hamiltonian H may fail to satisfy (Ham3'), is that there exist whole intervals (α, β) with $H|_{(\alpha,\beta)} = 0$ a.e.; remember the situations described in 2° or 5°. Of course such intervals are somewhat trivial pieces of H. Hence, it is interesting to note that (Ham3') may also fail for a more subtle reason.

1.1. *Example.* Choose a compact subset K of the unit interval $[0, 1]$ whose Lebesgue measure m is positive and less than 1, and which does not contain any open intervals. A typical example of such a set is the Smith-Volterra-Cantor set being obtained by the usual construction of the Cantor set but removing intervals of length $\frac{1}{4^n}$ instead of $\frac{1}{3^n}$ at the n-th step of the process. For more details, see, e.g., [AB, p140 f.]. Set $I := (0, 1)$ and

$$H(x) := \begin{cases} \mathrm{id}_{2\times 2}\,, & x \in I \setminus K \\ 0 \quad\;\;, & x \in K \cap I \end{cases},$$

then H is a Hamiltonian. It vanishes on a set of positive measure, namely on K. However, if J is any open interval, then $J \setminus K$ is open and nonempty. Hence, there exists no interval where H vanishes almost everywhere. //

When dealing with Hamiltonian functions which are not normalized by (Ham3), the notion of reparametrization is (and has always been) present. The idea is:

> *If two Hamiltonian functions differ only by a change of scale, they will share their operator theoretic properties.*

Reparametrizations for non–vanishing Hamiltonians were investigated in [KW/IV, §2.1.f], in the context of generalized strings reparameterizations appeared in [LW].

Our aim in this paper is to provide a rigorous fundament for the theory of (not necessarily non–vanishing) Hamiltonian, the notion of a reparametrization, and the above quoted intuitive statement. We set up the proper environment to deal with Hamiltonians without further normalization or restriction, and provide the practical tool of reparametrization in this general setting. The definition of the associated boundary triple is in essence the same as known from the trace–normed case. The main effort is to thoroughly understand the notion of a reparametrization. As one can guess already from the above Example 1.1, the difficulties which have to be overcome are of measure theoretic nature.

To close this introduction, let us briefly describe the content of the present paper. We define a boundary triple associated with a Hamiltonian in a way which is convenient for the general situation (Section 2); we define and discuss absolutely

continuous reparametrizations (Section 3); we show that for a given Hamiltonian H there always exist reparametrizations which relate H with a trace–normed Hamiltonian, and that the presently defined notion of reparametrization coincides with the previously introduced one in the case of non–vanishing Hamiltonians (Section 4).

2. Hamiltonians and their operator models

Throughout this paper measure theoretic notions like 'integrability', 'almost everywhere', 'measurable set', 'zero set', are understood with respect to the Lebesgue measure unless explicitly stated differently.

Intervals where the Hamiltonian is of a particularly simple form play a special role.

2.1. Definition. Let H be a Hamiltonian on I, and let $(\alpha, \beta) \subseteq I$ be a nonempty open interval.

(*i*) We call (α, β) H-immaterial, if $H(x) = 0$, $x \in (\alpha, \beta)$ a.e.

(*ii*) For $\phi \in \mathbb{R}$ set $\xi_\phi := (\cos\phi, \sin\phi)^T$. We call (α, β) H-indivisible of type $\phi \in \mathbb{R}$, if $H|_{(\alpha,\beta)}$ is of the form

$$H(x) = h(x)\xi_\phi\xi_\phi^T, \ x \in (\alpha, \beta) \text{ a.e.}, \tag{2.1}$$

with some scalar function h, and if no interval (α, γ) or (γ, β) with $\gamma \in (\alpha, \beta)$ is H-immaterial.

(*iii*) We denote by I_{ind} the union of all H-indivisible and H-immaterial intervals.

(*iv*) We say that H has a heavy left endpoint, if it does not start with an immaterial interval. Analogously, H has a heavy right endpoint, if it does not end with an immaterial interval. If both endpoints of H are heavy, we just say that H has heavy endpoints.

If no confusion is possible, we will drop the prefix 'H-' in these notations. //

Note that the type of an indivisible interval is uniquely determined up to multiples of π, and that the function h in (2.1) coincides a.e. with $\operatorname{tr} H$.

For later use, let us list some simple properties of immaterial and indivisible intervals.

2.2. *Remark.*

(*i*) Let (α, β) and (α', β') be immaterial. If the closures of these intervals have nonempty intersection, then the interior of the union of their closures is immaterial.

(*ii*) Each immaterial interval is contained in a maximal immaterial interval.

Let (α, β) be maximal immaterial and let (α', β') be immaterial, then either $(\alpha', \beta') \subseteq (\alpha, \beta)$ or $[\alpha', \beta'] \cap [\alpha, \beta] = \emptyset$.

There exist at most countably many maximal immaterial intervals.

(*iii*) Let (α, β) be indivisible of type ϕ, and let (α', β') be an interval which has nonempty intersection with (α, β). If (α', β') is immaterial, then $[\alpha', \beta'] \subseteq (\alpha, \beta)$. If (α', β') is indivisible of type ϕ', then $\phi = \phi' \bmod \pi$ and the union $(\alpha, \beta) \cup (\alpha', \beta')$ is indivisible of type ϕ.

(*iv*) Each indivisible interval of type ϕ is contained in a maximal indivisible interval of type ϕ.

Let (α, β) be maximal indivisible of type ϕ and let (α', β') be indivisible of type ϕ'. Then either $\phi = \phi' \bmod \pi$ and $(\alpha', \beta') \subseteq (\alpha, \beta)$, or $(\alpha', \beta') \cap (\alpha, \beta) = \emptyset$.

There exist at most countably many maximal indivisible intervals.

(*v*) The set I_{ind} is the disjont union of all maximal indivisible intervals, and all maximal immaterial intervals which are not contained in an indivisible interval.

(*vi*) The following statements are equivalent:

- The interval (α, β) is indivisible of type ϕ.
- We have $\xi_{\phi+\frac{\pi}{2}} \in \ker H(x)$, $x \in (\alpha, \beta)$ a.e. Neither H vanishes a.e. on an interval of the form (α, γ) with $\gamma \in (\alpha, \beta)$, nor on an interval of the form (γ, β).
- We have
$$\int_{(\alpha,\beta)} \xi^*_{\phi+\frac{\pi}{2}} H(x) \xi_{\phi+\frac{\pi}{2}} \, dx = 0 \,.$$
Neither H vanishes a.e. on an interval of the form (α, γ) with $\gamma \in (\alpha, \beta)$, nor on an interval of the form (γ, β).

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The first step towards the definition of the operator model associated with a Hamiltonian is to define the space of H-measurable functions.

2.3. Definition. Let H be a Hamiltonian defined on I. Then we denote by $\mathcal{M}(H)$ the set of all $\mathbb{C}^2$-valued functions f on I, such that:

(*i*) The function $Hf : I \to \mathbb{C}^2$ is (Lebesgue-to-Borel) measurable.

(*ii*) If $(\alpha, \beta) \subseteq I$ is immaterial, then f is constant on $[\alpha, \beta] \cap I$.

(*iii*) If $(\alpha, \beta) \subseteq I$ is indivisible of type ϕ, then $\xi^T_\phi f$ is constant on (α, β).

We define a relation '$=_H$' on $\mathcal{M}(H)$ by

$$f =_H g \qquad \text{if} \qquad H(f - g) = 0 \text{ a.e. on } I$$

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Let us point out explicitly that in the conditions (*ii*) and (*iii*) the respective functions are required to be constant, and not only constant almost everywhere. Apparently, (*ii*) and (*iii*) are a restriction only on the closure of I_{ind}. For example, each measurable function whose support does not intersect this closure certainly belongs to $\mathcal{M}(H)$. Also, note that the set $\mathcal{M}(H)$ does not change when H is changed on a set of measure zero, and that $=_H$ is an equivalence relation.

Usually, in the literature, only measurable functions f are considered. However, it turns out practical to weaken this requirement to (*i*) of Definition 2.3.

The next statement says that each equivalence class modulo $=_H$ in fact contains measurable functions. In particular, this implies that when factorizing modulo '$=_H$' it makes no difference whether we require Hf or f to be measurable.

2.4. Lemma. *Let H be a Hamiltonian defined on I, and let $f \in \mathcal{M}(H)$. Then there exists a measurable function $g \in \mathcal{M}(H)$, such that $f =_H g$.*

Proof. Write $H := \left(\begin{smallmatrix} h_1 & h_3 \\ h_3 & h_2 \end{smallmatrix}\right)$. We divide the interval I into six disjoint parts, namely

$$\begin{aligned}
J_1 &:= \bigcup \big\{ L : \, L \text{ maximal indivisible} \big\} \\
J_2 &:= \bigcup \big\{ I \cap \overline{L} : \, L \text{ maximal immaterial}, L \cap J_1 = \emptyset \big\} \\
J_3 &:= \big\{ x \in I : \, H(x) = 0 \big\} \setminus (J_1 \cup J_2) \\
J_4 &:= \big\{ x \in I : \, H(x) \neq 0, \det H(x) = 0, \ h_2(x) = 0 \big\} \setminus (J_1 \cup J_2) \\
J_5 &:= \big\{ x \in I : \, H(x) \neq 0, \det H(x) = 0, \ h_2(x) \neq 0 \big\} \setminus (J_1 \cup J_2) \\
J_6 &:= \big\{ x \in I : \, \det H(x) \neq 0 \big\} \setminus (J_1 \cup J_2)
\end{aligned}$$

Since J_1 is open, and J_2 is a countable union of (relatively) closed sets, both are measurable. Since each entry of H is measurable, each of the subsets $J_3, \dots, J_6$ is measurable. If two open intervals L_1 and L_2 have empty intersection, also $L_1 \cap \overline{L}_2 = \emptyset$. Thus, $J_1 \cap J_2 = \emptyset$. The other sets $J_3, \dots, J_6$ are trivially pairwise disjoint and disjoint from $J_1$and J_2. We are going to define the required function g on each of the sets J_i, $i = 1, \dots, 6$, separately.

Definition on J_1: Let L be a maximal indivisible interval, say of type ϕ. Then $\xi_\phi^T f(x)$ is constant on L. We set

$$g(x) := \big[\xi_\phi^T f(x)\big] \cdot \xi_\phi, \quad x \in L \,,$$

then

$$\begin{aligned}
H(x)\big(f(x) - g(x)\big) &= h(x) \cdot \xi_\phi \xi_\phi^T \big(f(x) - g(x)\big) = \\
&= h(x) \cdot \xi_\phi \Big[\xi_\phi^T f(x) - \xi_\phi^T \big[\xi_\phi^T f(x)\big] \xi_\phi\Big] = 0, \quad x \in L \text{ a.e.}
\end{aligned}$$

The function $g|_L$ itself, in particular also $\xi_\phi^T g|_L$, is constant. Hence, no matter how we define g on the remaining parts $J_2, \dots, J_6$, the condition (iii) of Definition 2.3 will be satisfied for g.

By the above procedure, g is defined on all of J_1. Since J_1 is a countable union of disjoint open sets where g is constant, $g|_{J_1}$ is measurable.

Definition on J_2: Let L be a maximal immaterial interval which does not intersect any indivisible interval. Then f is constant on $I \cap \overline{L}$. We set

$$g(x) := f(x), \quad x \in I \cap \overline{L} \,,$$

then

$$H(x)\big(f(x) - g(x)\big) = 0, \quad x \in I \cap \overline{L} \,.$$

No matter how we define g on the remaining parts $J_3, \dots, J_6$, the condition (ii) of Definition 2.3 will hold true for g: Assume that (α, β) is immaterial. Then $[\alpha, \beta] \cap I$ is either contained in some maximal indivisible interval or in some maximal immaterial interval which does not intersect any indivisible interval. In both cases, the function g is constant on $[\alpha, \beta] \cap I$. Since J_2 is a countable disjoint union of closed sets where g is constant, $g|_{J_2}$ is measurable.

Definition on J_3: We set $g(x) := 0$, $x \in J_3$, then $g|_{J_3}$ is measurable and $H(x)(f(x) - g(x)) = 0$, $x \in J_3$.

Definition on J_4: For $x \in J_4$ we have $H(x) = h(x)\xi_0\xi_0^T$ with the measurable and positive function $h(x) := \operatorname{tr} H(x)$. Write f as $f(x) = f_1(x)\xi_0 + f_2(x)\xi_{\frac{\pi}{2}}$, then

$$H(x)f(x) = h(x)f_1(x)\xi_0, \quad x \in J_4\,.$$

Since $h(x)$ is positive, it follows that the function

$$g(x) := f_1(x)\xi_0, \quad x \in J_4\,,$$

is measurable. Also, it satisfies

$$H(x)\big(f(x) - g(x)\big) = h(x)\xi_0\xi_0^T \cdot f_2(x)\xi_{\frac{\pi}{2}} = 0, \quad x \in J_4\,.$$

Definition on J_5: We argue similar as for J_4. For $x \in J_5$ we have $H(x) = h(x)\xi_{\phi(x)}\xi_{\phi(x)}^T$ with the measurable and positive function $h(x) := \operatorname{tr} H(x)$ and the measurable function $\phi(x) := \operatorname{Arccot} \frac{h_3(x)}{h_2(x)}$. Write f as $f(x) = f_1(x)\xi_{\phi(x)} + f_2(x)\xi_{\phi(x)+\frac{\pi}{2}}$, $x \in J_5$, then

$$H(x)f(x) = h(x)f_1(x)\xi_{\phi(x)}, \quad x \in J_5\,.$$

Since $h(x)$ is positive, the function $g(x) := f_1(x)\xi_{\phi(x)}$, $x \in J_5$, is measurable. It satisfies, $H(x)(f(x) - g(x)) = 0$, $x \in J_5$.

Definition on J_6: If $\det H(x) \neq 0$, we can write $f(x) = H(x)^{-1} \cdot H(x)f(x)$, and hence $f|_{J_6}$ is measurable. Set $g(x) := f(x)$, $x \in J_6$, then $H(x)(f(x) - g(x)) = 0$, $x \in J_6$. ❑

2.5. Corollary. *Let H be a Hamiltonian defined on I. If $f_1, f_2 \in \mathcal{M}(H)$, then the function $f_2^* H f_1$ is measurable. For each $f \in \mathcal{M}(H)$ the function $f^* H f$ is measurable and almost everywhere nonnegative.*

Proof. Choose measurable functions $g_1, g_2 \in \mathcal{M}(H)$ according to Lemma 2.4. Then

$$f_2^* H f_1 = g_2^* H g_1 + (f_2 - g_2)^* H g_1 + f_2^* H (f_1 - g_1) = g_2^* H g_1 \quad \text{a.e.}\,.$$

Since $H(x)$ is a.e. a nonnegative matrix, each function $f^* H f$, $f \in \mathcal{M}(H)$, is a.e. nonnegative. ❑

Now we can write down the definition of the operator model associated with a Hamiltonian. It reads almost the same as in the trace-normed case.

Denote by $\mathrm{Ac}(H)$ the subset of $\mathcal{M}(H)$, which consists of all locally absolutely continuous functions in $\mathcal{M}(H)$. Moreover, call H regular at the endpoint $s_- := \inf I$, if for one (and hence for all) $s \in I$

$$\int_{s_-}^{s} \operatorname{tr} H(x)\,dx < \infty\,.$$

If this integral is infinite, call H singular at s_-. The terms regular/singular at the endpoint $s_+ := \sup I$ are defined analogously[†].

2.6. Definition. Let H be a Hamiltonian defined on $I = (s_-, s_+)$. Set

$$\begin{aligned} \sigma_- &:= \sup\big\{x \in I :\ (s_-, x) \text{ immaterial}\big\}\,,\\ \sigma_+ &:= \inf\big\{x \in I :\ (x, s_+) \text{ immaterial}\big\}\,,\\ \tilde{I} &:= (\sigma_-, \sigma_+), \quad \tilde{H} := H|_{\tilde{I}} \end{aligned} \tag{2.2}$$

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2.7. Definition. Let H be a Hamiltonian defined on I.

(i) We define the model space $L^2(\tilde{H}) \subseteq \mathcal{M}(\tilde{H})/_{=_{\tilde{H}}}$ as

$$L^2(H) := \Big\{\hat{f}/_{=_{\tilde{H}}} :\ \hat{f} \in \mathcal{M}(\tilde{H}),\ \int_{\tilde{I}} \hat{f}(x)^* H(x) \hat{f}(x)\,dx < \infty\Big\}\,.$$

For $f_1, f_2 \in L^2(H)$ we define an inner product as ($f_1 = \hat{f}_1/_{=_H}$, $f_2 = \hat{f}_2/_{=_H}$)

$$(f_1, f_2)_H := \int_{\tilde{I}} \hat{f}_2(x)^* H(x) \hat{f}_1(x)\,dx\,.$$

(ii) We define the model relation $T_{\max}(H) \subseteq L^2(H) \times L^2(H)$ as

$$\begin{aligned} T_{\max}(H) := \big\{(f;g) \in L^2(H) \times L^2(H) :\ \exists\, \hat{f} \in \mathrm{Ac}(\tilde{H}), \hat{g} \in \mathcal{M}(\tilde{H}) \text{ with}\\ f = \hat{f}/_{=_{\tilde{H}}}, g = \hat{g}/_{=_{\tilde{H}}} \text{ and } \hat{f}' = J\tilde{H}\hat{g} \text{ a.e.}\big\}\,. \end{aligned}$$

(iii) We define the model boundary relation $\Gamma(H) \subseteq T_{\max}(H) \times (\mathbb{C}^2 \times \mathbb{C}^2)$ as the set of all elements $((f;g);(a;b))$ such that there exist representants $\hat{f} \in \mathrm{Ac}(H)$ of f and $\hat{g} \in \mathcal{M}(H)$ of g with $\hat{f}' = JH\hat{g}$ and ($s_- := \inf I$, $s_+ := \sup I$)

$$a = \begin{cases} \lim_{t \searrow s_-} \hat{f}(t)\,, & \text{regular at } s_-\\ 0 \qquad\qquad\ , & \text{singular at } s_- \end{cases}$$

$$b = \begin{cases} \lim_{t \nearrow s_+} \hat{f}(t)\,, & \text{regular at } s_+\\ 0 \qquad\qquad\ , & \text{singular at } s_+ \end{cases}$$

Unless it is necessary, the equivalence relation '$=_H$' will not be mentioned explicitly and equivalence classes and their representants will not be distinguished explicitly.

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[†]Instead of regular and singular, one also speaks of Weyl's limit circle case or Weyl's limit point case.

The operator theoretic properties of these objects, for example the fact that $(L^2(H), T_{\max}(H), \Gamma(H))$ is a Hilbert space boundary triple, could be proved by following the known path. This, however, would be unnecessary labour. As we will see later, it is always possible to reduce to the trace-normed case by means of a reparametrization, cf. Corollary 4.4.

For later reference let us explicitly state the obvious fact that a pair $(f; g)$ belongs to $T_{\max}(H)$ if and only if there exist representants $\hat{f}$ and $\hat{g}$ of f and g, respectively, with

$$\hat{f}(y) = \hat{f}(x) + \int_x^y JH\hat{g}, \quad x, y \in I\,. \tag{2.3}$$

3. Absolutely continuous reparametrizations

Let us define rigorously what we understand by a reparametrization (i.e. a 'change of scale').

3.1. Definition. Let H_1 and H_2 be Hamiltonians defined on intervals I_1 and I_2, respectively.

(*i*) We say that H_2 is a basic reparametrization of H_1, and write $H_1 \rightsquigarrow H_2$, if there exists a nondecreasing, locally absolutely continuous, and surjective map λ of I_1 onto I_2, such that

$$H_1(x) = H_2(\lambda(x)) \cdot \lambda'(x), \quad x \in I_1 \text{ a.e.} \tag{3.1}$$

Here λ' denotes a nonnegative function which coincides a.e. with the derivative of λ.

(*ii*) Let numbers σ_1^-, σ_1^+ and $\sigma_{2,-}, \sigma_{2,+}$ be defined by (2.2) for H_1 and H_2, respectively. Then we write $H_1 \succcurlyeq H_2$, if

$$H_1|_{(\sigma_1^-, \sigma_1^+)} = H_2|_{(\sigma_{2,-}, \sigma_{2,+})}\,.$$

(*iii*) We denote by '$\sim$' the smallest equivalence relation containing both relations '$\rightsquigarrow$' and '$\succcurlyeq$'. If $H \sim \tilde{H}$, we say that H and $\tilde{H}$ are reparametrizations of each other.

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First of all note that '$\succcurlyeq$' is an equivalence relation, and that '$\rightsquigarrow$' is reflexive and transitive; for transitivity apply the chain rule. However, '$\rightsquigarrow$' fails to be symmetric, see the below Example 3.2. This properties of '$\rightsquigarrow$' imply that $H \sim \tilde{H}$ if and only if there exist finitely many Hamiltonians $L_0, \dots, L_m$, such that

$$H = L_0 \approx_1 L_1 \approx_2 L_2 \approx_3 \cdots \approx_{m-1} L_{m-1} \approx_m L_m = \tilde{H} \tag{3.2}$$

where $\approx_i \in \{\succcurlyeq, \rightsquigarrow, \rightsquigarrow^{-1}\}$, $i = 1, \dots, m$.

3.2. *Example.* Let us show by an example that '$\rightsquigarrow$' is not symmetric. One obvious obstacle for symmetry is that a function λ establishing a basic reparametrization by means of (3.1) need not be injective. However, if $\lambda(x_1) = \lambda(x_2)$ for some

$x_1 < x_2$, then λ is constant on the interval (x_1, x_2), and hence $\lambda' = 0$ a.e. on (x_1, x_2). Thus (x_1, x_2) must be a H_1-immaterial interval; a somewhat trivial piece of the Hamiltonian.

A more subtle example is obtained from the Hamiltonian H introduced in Example 1.1. Using the notation from this example, set

$$\tilde{I} := (0, 1-m), \quad \tilde{H}(y) := \mathrm{id}_{2\times 2},\ y \in \tilde{I}\,,$$

and consider the map

$$\lambda(x) := \int_0^x \chi_{I\setminus K}(t)\,dt, \quad x \in I\,.$$

Then, λ is nondecreasing, absolutely continuous, and $\lambda' = \chi_{I\setminus K}$ a.e. Since K does not contain any open interval, λ is in fact an increasing bijection of I onto $\tilde{I}$. Let us show that

$$H(x) = \tilde{H}(\lambda(x))\lambda'(x), \quad x \in I \text{ a.e.}$$

If $x \in I \setminus K$ and $\lambda'(x) = \chi_{I\setminus K}(x)$, both sides equal $\mathrm{id}_{2\times 2}$. If $x \in K$ and $\lambda'(x) = \chi_{I\setminus K}(x)$, both sides equal 0. We see that $H \rightsquigarrow \tilde{H}$ via λ.

Assume on the contrary that $\tilde{H} \rightsquigarrow H$ via some nondecreasing, locally absolutely continuous, and surjective map τ of $\tilde{I}$ onto I, so that

$$\tilde{H}(y) = H(\tau(y))\tau'(y), \quad y \in \tilde{I} \text{ a.e.}$$

For $y \in \tau^{-1}(K)$, the left side of this relation equals $\mathrm{id}_{2\times 2}$ and right side equals 0. Thus $\tau^{-1}(K)$ must be a zero set. Since τ is locally absolutely continuous and surjective, this implies that $K = \tau(\tau^{-1}(K))$ is a zero set. We have reached a contradiction, and conclude that $\tilde{H} \not\rightsquigarrow H$. ∥

Our aim in this section is to show that Hamiltonians which are reparametrizations of each other give rise to isomorphic operator models, for the precise formulation see Theorem 3.8 below. The main effort is to understand basic reparametrizations; and this is our task in the next couple of statements.

3.3. *Remark.* Let I_1 and I_2 be nonempty open intervals on the real line, and let $\lambda : I_1 \to I_2$ be a nondecreasing, locally absolutely continuous, and surjective map.

(*i*) The function λ cannot be constant on any interval of the form $(\inf I, \gamma)$ or $(\gamma, \sup I)$ with $\gamma \in I$. This is immediate from the fact that the image of λ is an open interval.

(*ii*) There exists a nonnegative function λ' which coincides almost everywhere with the derivative of λ, and which has the following property:

$$\text{For each nonempty interval } (\alpha, \beta) \subseteq I \text{ such that } \lambda|_{(\alpha,\beta)} \text{ is constant, we have } \lambda'|_{[\alpha,\beta]} = 0. \tag{3.3}$$

Note here that, due to (*i*), always $[\alpha, \beta] \subseteq I$.

Let us show that λ' can indeed be assumed to satisfy (3.3). Each interval (α, β) where λ is constant is contained in a maximal interval having this property. Each two maximal intervals where λ is constant are either equal or

disjoint. Hence, there can exist at most countably many such. Let (α, β) be one of them. Then the derivative of λ exists and is equal to zero on all of (α, β). Choose any function λ' which coincides almost everywhere with the derivative of λ. By redefining this function on a set of measure zero, we can thus achieve that $\lambda'(x) = 0$, $x \in [\alpha, \beta]$.

//

We will, throughout the following, always assume that the function λ' in Definition 3.1, (i), has the additional property (3.3). By the just said, this is no loss in generality.

3.4. Proposition. *Let H_1 and H_2 be Hamiltonians defined on intervals I_1 and I_2, respectively. Assume that H_2 is a basic reparametrization of H_1, and let λ be a map which establishes this reparametrization. Moreover, let $\tilde{\lambda}$ be a right inverse of λ*[†]*.*

Then the maps $\circ\tilde{\lambda} : f_1 \mapsto f_1 \circ \tilde{\lambda}$ and $\circ\lambda : f_2 \mapsto f_2 \circ \lambda$ induce mutually inverse linear bijections between $\mathcal{M}(H_1)$ and $\mathcal{M}(H_2)$.

$$I_1 \underset{\tilde{\lambda}}{\overset{\lambda}{\rightleftarrows}} I_2 \qquad \lambda \circ \tilde{\lambda} = \mathrm{id} \qquad\qquad \mathcal{M}(H_1) \underset{\circ\tilde{\lambda}}{\overset{\circ\lambda}{\rightleftarrows}} \mathcal{M}(H_2)$$

They respect the equivalence relations $=_{H_1}$ and $=_{H_2}$ in the sense that, for each two elements $f_2, g_2 \in \mathcal{M}(H_2)$,

$$f_2 =_{H_2} g_2 \quad \Longleftrightarrow \quad (f_2 \circ \lambda) =_{H_1} (g_2 \circ \lambda) \tag{3.4}$$

and for each two elements $f_1, g_1 \in \mathcal{M}(H_1)$,

$$f_1 =_{H_1} g_1 \quad \Longleftrightarrow \quad (f_1 \circ \tilde{\lambda}) =_{H_2} (g_1 \circ \tilde{\lambda})$$

In the proof of this proposition there arise some difficulties of measure theoretic nature. Let us state the necessary facts separately.

3.5. Lemma. *Let I_1 and I_2 be nonempty open intervals on the real line, let $\lambda : I_1 \to I_2$ be a nondecreasing, locally absolutely continuous, and surjective map, and let $\tilde{\lambda}$ be a right inverse of λ. Moreover, assume that λ' is a function which coincides almost everywhere with the derivative of λ (and has the property* (3.3)*), and set*

$$L_0 := \left\{ x \in I : \lambda'(x) = 0 \right\}.$$

Then the following hold:

(i) *If $E \subseteq I_1$ is a zero set, so is $\tilde{\lambda}^{-1}(E)$.*
(ii) *The function $\tilde{\lambda}$ is Lebesgue-to-Lebesgue measurable.*
(iii) *The set $\lambda(L_0)$ is measurable and has measure zero.*

[†]For example, one could choose $\tilde{\lambda}(y) := \min\{x \in I_1 : \lambda(x) = y\}$. Due to continuity of λ and Remark 3.3, (i), this minimum exists and belongs to I_1

(*iv*) *The function* $\lambda' \circ \tilde{\lambda}$ *is almost everywhere positive. In fact,*

$$\big\{y \in I_2 : (\lambda' \circ \tilde{\lambda})(y) = 0\big\} = \lambda(L_0)\,.$$

(*v*) *If* $E \subseteq I_2$ *is a measurable set, so is* $\lambda^{-1}(E) \setminus L_0 \subseteq I_1$*. If* E *is a zero set, also* $\lambda^{-1}(E) \setminus L_0$ *has measure zero.*

Proof.
Item (*i*)*:* Since $\tilde{\lambda}$ is a right inverse of λ, we have $\tilde{\lambda}^{-1}(E) \subseteq \lambda(E)$. Since λ is locally absolutely continuous, E being a zero set implies that $\lambda(E)$ is a zero set. Thus $\tilde{\lambda}^{-1}(E)$ is measurable and has measure zero.

Item (*ii*)*:* The function $\tilde{\lambda}$ is nondecreasing, and hence Borel-to-Borel measurable. Let a Lebesgue measurable set $M \subseteq I_1$ be given, and choose Borel sets A, B with $A \subseteq M \subseteq B$ such that the Lebesgue measure of $B \setminus A$ equals zero. Then $\tilde{\lambda}^{-1}(A)$ and $\tilde{\lambda}^{-1}(B)$ are Borel sets,

$$\tilde{\lambda}^{-1}(A) \subseteq \tilde{\lambda}^{-1}(M) \subseteq \tilde{\lambda}^{-1}(B), \quad \tilde{\lambda}^{-1}(B) \setminus \tilde{\lambda}^{-1}(A) = \tilde{\lambda}^{-1}(B \setminus A)\,.$$

However, by (*i*), $\tilde{\lambda}^{-1}(B \setminus A)$ has measure zero, and it follows that $\tilde{\lambda}^{-1}(M)$ is Lebesgue measurable.

Item (*iii*)*:* The crucial observation is the following: If two points $x, y \in I$, $x < y$, have the same image under λ, then λ is constant on $[x, y]$, and by (3.3) thus $x, y \in L_0$. In particular, the set L_0 is saturated with respect to the equivalence relation $\ker \lambda$[‡]. This implies that

$$\lambda(L_0) = \tilde{\lambda}^{-1}(L_0), \quad L_0 = \lambda^{-1}(\lambda(L_0))\,. \tag{3.5}$$

By (*ii*), the first equality already shows that $\lambda(L_0)$ is measurable. To compute the measure of $\lambda(L_0)$, we use the second equality and evaluate

$$\int_{I_2} \chi_{\lambda(L_0)}(y)\,dy = \int_{I_1} \big(\underbrace{\chi_{\lambda(L_0)} \circ \lambda}_{\substack{\| \\ \chi_{\lambda^{-1}(\lambda(L_0))} = \chi_{L_0}}}\big)(x) \cdot \lambda'(x)\,dx = 0\,.$$

Item (*iv*)*:* Consider the function $\lambda' \circ \tilde{\lambda}$. Clearly, it is nonnegative. Let $y \in I_2$ be given. Then $(\lambda' \circ \tilde{\lambda})(y) = 0$ if and only if $\tilde{\lambda}(y) \in L_0$, and in turn, by (3.5), if and only if $y \in \lambda(L_0)$.

Item (*v*)*:* The function $(\chi_E \circ \lambda) \cdot \lambda'$ is measurable, and hence

$$\lambda^{-1}(E) \setminus L_0 = \big\{x \in I_1 : [(\chi_E \circ \lambda) \cdot \lambda'](x) \neq 0\big\}$$

[‡]Here we understand by $\ker \lambda$ the equivalence relation

$$\ker \lambda := \{(x_1; x_2) \in I_1 \times I_1 : \lambda(x_1) = \lambda(x_2)\}\,,$$

and call a subset of I_1 saturated with respect to this equivalence relation, if it is a union of equivalence classes.

is measurable. Moreover, if E is a zero set,

$$0 = \int_{I_2} \chi_E(x)\,dx = \int_{I_1} (\chi_E \circ \lambda)(x) \cdot \lambda'(x)\,dx\,,$$

and hence the (nonnegative) function $(\chi_E \circ \lambda) \cdot \lambda'$ must vanish almost everywhere. ❑

Next, we have to make clear how immaterial and indivisible intervals behave when performing the transformation λ.

3.6. Lemma. *Consider the situation described in Proposition* 3.4*.*

(*i*) *If $(\alpha,\beta) \subseteq I_1$ and λ is constant on this interval, then (α,β) is H_1-immaterial.*
(*ii*) *If $(\alpha,\beta) \subseteq I_1$ is H_1-immaterial, then the set of inner points of the interval $\lambda\big([\alpha,\beta] \cap I_1\big)$ is either empty or H_2-immaterial.*
(*iii*) *If $(\alpha,\beta) \subseteq I_2$ is H_2-immaterial, then the set of inner points of the interval $\lambda^{-1}\big([\alpha,\beta] \cap I_2\big)$ is H_1-immaterial.*
(*iv*) *If $(\alpha,\beta) \subseteq I_1$ is H_1-indivisible of type ϕ, then the interval $\lambda\big((\alpha,\beta)\big)$ is H_2-indivisible of type ϕ.*
(*v*) *If $(\alpha,\beta) \subseteq I_2$ is H_2-indivisible of type ϕ, then the interval $\lambda^{-1}\big((\alpha,\beta)\big)$ is H_1-indivisible of type ϕ.*

Proof.
Item (*i*)*:* This has already been noted in the first paragraph of Example 3.2.

Item (*ii*)*:* If the set of inner points of the interval $\lambda\big([\alpha,\beta] \cap I_1\big)$ is empty, there is nothing to prove. Hence, assume that it is nonempty.

Consider first the case that $[\alpha,\beta] \cap I_1$ is saturated with respect to the equivalence relation $\ker\lambda$. Choose a zero set $E \subseteq I_1$, such that $H_1(x) = 0$, $x \in \big([\alpha,\beta] \cap I_1\big) \setminus E$. Since $H_1(\tilde\lambda(y)) = H_2(y) \cdot (\lambda' \circ \tilde\lambda)(y)$ a.e., we obtain

$$H_2(y) = 0, \quad y \in \tilde\lambda^{-1}\big(\big([\alpha,\beta] \cap I_1\big) \setminus E\big) \setminus \lambda(L_0) \text{ a.e.}$$

Since $[\alpha,\beta] \cap I_1$ is saturated with respect to $\ker\lambda$, we have $\tilde\lambda^{-1}\big([\alpha,\beta] \cap I_1\big) = \lambda\big([\alpha,\beta] \cap I_1\big)$, and it follows that

$$\tilde\lambda^{-1}\big(\big([\alpha,\beta] \cap I_1\big) \setminus E\big) \setminus \lambda(L_0) = \lambda\big([\alpha,\beta] \cap I_1\big) \setminus \big(\tilde\lambda^{-1}(E) \cup \lambda(L_0)\big)\,.$$

In particular, H_2 vanishes almost everywhere on the set of inner points of the interval $\lambda\big([\alpha,\beta] \cap I_1\big)$.

Assume next that (α,β) is an arbitrary H_1-immaterial interval. The union of all equivalence classes of elements $x \in (\alpha,\beta)$ modulo $\ker\lambda$ is a (relatively) closed interval, say $[\alpha_0,\beta_0] \cap I_1$. Since

$$(\alpha_0,\beta_0) = (\alpha_0,\alpha] \cup (\alpha,\beta) \cup [\beta,\beta_0)\,,$$

and λ is certainly constant on $(\alpha_0,\alpha]$ and $[\beta,\beta_0)$, it follows that (α_0,β_0) is H_1-immaterial. Moreover, $[\alpha_0,\beta_0] \cap I_1$ is saturated with respect to $\ker\lambda$. Applying what we have proved in the above paragraph, gives that the set of inner points

of $\lambda\big([\alpha_0,\beta_0]\cap I_1\big)$ is H_2-immaterial. Since $[\alpha,\beta]\cap I_1\subseteq[\alpha_0,\beta_0]\cap I_1$, the required assertion follows.

Item (iii): Choose a zero set $E\subseteq I_2$, such that $H_2(y)=0$, $y\in\big([\alpha,\beta]\cap I_2\big)\setminus E$. Since $H_1(x)=H_2(\lambda(x))\lambda'(x)$ a.e., it follows that

$$H_1(x)=0,\quad x\in\lambda^{-1}\big(\big([\alpha,\beta]\cap I_2\big)\setminus E\big)\cup L_0 \text{ a.e.}$$

However,

$$\lambda^{-1}\big([\alpha,\beta]\cap I_2\big)\setminus\big(\lambda^{-1}(E)\setminus L_0\big)\subseteq\big[\big(\lambda^{-1}\big([\alpha,\beta]\cap I_2\big)\big)\setminus\lambda^{-1}(E)\big]\cup L_0=$$
$$=\lambda^{-1}\big(([\alpha,\beta]\cap I_2)\setminus E\big)\cup L_0$$

and we conclude that H_1 vanishes on $\lambda^{-1}\big([\alpha,\beta]\cap I_2\big)$ with possible exception of a zero set.

Item (iv): The function λ is not constant on any interval of the form $(\alpha,\alpha+\varepsilon)$ or $(\beta-\varepsilon,\beta)$. Hence, the interval (α,β) is saturated with respect to $\ker\lambda$, and $\lambda\big((\alpha,\beta)\big)$ is open.

Choose a zero set $E\subseteq I_1$, such that $H_1(x)=h_1(x)\cdot\xi_\phi\xi_\phi^T$, $x\in(\alpha,\beta)\setminus E$. Then

$$H_2(y)=\frac{h_1(\tilde\lambda(y))}{(\lambda'\circ\tilde\lambda)(y)}\cdot\xi_\phi\xi_\phi^T,\quad y\in\tilde\lambda^{-1}\big((\alpha,\beta)\setminus E\big)\setminus\lambda(L_0)\text{ a.e.}$$

However,

$$\tilde\lambda^{-1}\big((\alpha,\beta)\setminus E\big)\setminus\lambda(L_0)=\lambda\big((\alpha,\beta)\big)\setminus\big(\tilde\lambda^{-1}(E)\cup\lambda(L_0)\big)\,.$$

Hence, H_2 has the required form.

Set $(\alpha',\beta'):=\lambda\big((\alpha,\beta)\big)$, and assume that for some $\gamma'>\alpha'$ the interval (α',γ') is H_2-immaterial. Then the interval $\lambda^{-1}\big((\alpha',\gamma')\big)$ is H_1-immaterial. Since λ is continuous and (α,β) is saturated with respect to $\ker\lambda$, we have $\lambda^{-1}\big((\alpha',\gamma')\big)=(\alpha,\gamma)$ with some $\gamma\in(\alpha,\beta)$. We have reached a contradiction. The same argument shows that no interval of the form (γ',β') can be H_2-immaterial.

Item (v): Choose a zero set $E\subseteq I_2$, such that

$$H_2(y)=h_2(y)\cdot\xi_\phi\xi_\phi^T,\quad y\in(\alpha,\beta)\setminus E\,.$$

Moreover, set $\lambda^{-1}\big((\alpha,\beta)\big)=:(\alpha',\beta')\subseteq I_1$.

First, we have

$$H_1(x)=h_2(\lambda(x))\lambda'(x)\cdot\xi_\phi\xi_\phi^T,\quad x\in\lambda^{-1}\big((\alpha,\beta)\setminus E\big)\text{ a.e.}$$

On the set L_0 this equality trivially remains true a.e. We conclude that $H_1(x)$ is of the form $h_1(x)\cdot\xi_\phi\xi_\phi^T$ for all $x\in\lambda^{-1}\big((\alpha,\beta)\big)\setminus\big(\lambda^{-1}(E)\setminus L_0\big)$ a.e.

Second, assume that for some $\gamma'\in(\alpha',\beta')$ the interval (α',γ') is H_1-immaterial. Then the set of inner points of $\lambda\big((\alpha',\gamma')\big)$ is H_2-immaterial. However, since $\lambda\big((\alpha',\beta')\big)=(\alpha,\beta)$, the function λ cannot be constant on any interval $(\alpha',\alpha'+\varepsilon)$, and hence $\lambda\big((\alpha',\gamma')\big)\supseteq(\alpha,\gamma)$ for some $\gamma>\alpha$. We have reached a

contradiction, and conclude that (α', β') cannot start with an immaterial interval. The fact that it cannot end with such an interval is seen in the same way. ❑

After these preparations, we turn to the proof of Proposition 3.4.

Proof (of Proposition 3.4*).*
Step 1: Let $f_2 \in \mathcal{M}(H_2)$ be given, and consider the function $f_1 := f_2 \circ \lambda$. We have

$$H_1 f_1 = H_1(f_2 \circ \lambda) = (H_2 \circ \lambda)\lambda' \cdot (f_2 \circ \lambda) = \big[(H_2 f_2) \circ \lambda\big] \cdot \lambda' \text{ a.e.}, \tag{3.6}$$

and hence $H_1 f_1$ is measurable.

Let $(\alpha, \beta) \subseteq I_1$ be an immaterial interval. Then the set of inner points of $\lambda\big([\alpha, \beta] \cap I_1\big)$ is either empty or H_2-immaterial. In the first case, λ is constant on $[\alpha, \beta] \cap I_1$, and hence also f_1 is constant on this interval. In the second case, f_2 is constant on $\lambda\big([\alpha, \beta] \cap I_1\big)$, and it follows that f_1 is constant on $[\alpha, \beta] \cap I_1$.

If (α, β) is H_1-indivisible of type ϕ, then $\lambda\big((\alpha, \beta)\big)$ is H_2-indivisible of type ϕ. Hence $\xi_\phi^T f_2$ is constant on $\lambda\big((\alpha, \beta)\big)$, and thus $\xi_\phi^T f_1$ is constant on (α, β). It follows that $f_1 \in \mathcal{M}(H_1)$, and we have shown that $\circ\lambda$ maps $\mathcal{M}(H_2)$ into $\mathcal{M}(H_1)$.

Step 2: Let $f_1 \in \mathcal{M}(H_1)$ be given, and set $f_2 := f_1 \circ \tilde{\lambda}$. First note that $(x_1; x_2) \in \ker \lambda$, $x_1 < x_2$, implies that the interval (x_1, x_2) is H_1-immaterial, and hence that $f_1(x_1) = f_1(x_2)$. Using this fact, it follows that

$$f_2 \circ \lambda = (f_1 \circ \tilde{\lambda}) \circ \lambda = f_1 \,. \tag{3.7}$$

Next, we compute (a.e.)

$$(H_1 f_1) \circ \tilde{\lambda} = \big[H_1 \cdot (f_2 \circ \lambda)\big] \circ \tilde{\lambda} = \big[(H_2 \circ \lambda)\lambda' \cdot (f_2 \circ \lambda)\big] \circ \tilde{\lambda} = (H_2 f_2) \cdot (\lambda' \circ \tilde{\lambda}) \,. \tag{3.8}$$

Since $\tilde{\lambda}$ is Lebesgue-to-Lebesgue measurable, the function $(H_1 f_1) \circ \tilde{\lambda}$ is measurable. Since $\lambda' \circ \tilde{\lambda}$ is almost everywhere positive, this implies that $H_2 f_2$ is measurable.

Let $(\alpha, \beta) \subseteq I_2$ be H_2-immaterial, then f_1 is constant on $\lambda^{-1}\big([\alpha, \beta] \cap I_2\big)$. Since $\tilde{\lambda}\big([\alpha, \beta] \cap I_2\big) \subseteq \lambda^{-1}\big([\alpha, \beta] \cap I_2\big)$, it follows that $f_1 \circ \tilde{\lambda}$ is constant on $[\alpha, \beta] \cap I_2$.

If $(\alpha, \beta) \subseteq I_2$ is H_2-indivisible of type ϕ, then $\xi_\phi^T f_1$ is constant on $\lambda^{-1}((\alpha, \beta))$, and in turn $\xi_\phi^T f_2$ is constant on (α, β). It follows that $f_2 \in \mathcal{M}(H_2)$, and we have shown that $\circ\tilde{\lambda}$ maps $\mathcal{M}(H_1)$ into $\mathcal{M}(H_2)$.

Step 3: Since $\tilde{\lambda}$ is a right inverse of λ, we have $(f_2 \circ \lambda) \circ \tilde{\lambda} = f_2$ for any function defined on I_2. The fact that $(f_1 \circ \tilde{\lambda}) \circ \lambda = f_1$ whenever $f_1 \in \mathcal{M}(H_1)$, was shown in (3.7). We conclude that the maps $\circ\lambda$ and $\circ\tilde{\lambda}$ are mutually inverse bijections between $\mathcal{M}(H_1)$ and $\mathcal{M}(H_2)$.

Step 4: To show (3.4), it is clearly enough to consider the case that $g_2 = 0$. Let $f_2 \in \mathcal{M}(H_2)$ be given. Assume first that there exists a set $E \subseteq I_1$ of measure zero, such that $H_1(x)(f_2 \circ \lambda)(x) = 0$, $x \in I_1 \setminus E$. Then, by (3.8) and the fact that $H_1(f_2 \circ \lambda) = [(H_2 f_2) \circ \lambda] \cdot \lambda'$, we have

$$(H_2 f_2)(y) \cdot (\lambda' \circ \tilde{\lambda})(y) = 0, \quad y \in I_2 \setminus \tilde{\lambda}^{-1}(E) \,.$$

Since $\tilde{\lambda}^{-1}(E)$ is a zero set, and $(\lambda' \circ \tilde{\lambda})$ is positive a.e., this implies that $H_2 f_2 = 0$ a.e. on I_2. Conversely, assume that $H_2(y)f_2(y) = 0$, $y \in I_2 \setminus E$, with some set $E \subseteq I_2$ of measure zero. Then, by (3.6), we have

$$H_1(x)(f_2 \circ \lambda)(x) = 0, \quad x \in \big(I_1 \setminus \lambda^{-1}(E)\big) \cup L_0 = I_1 \setminus (\lambda^{-1}(E) \setminus L_0)\,.$$

However, we know that $\lambda^{-1}(E) \setminus L_0$ is a zero set.

Since we already know that $\circ\tilde{\lambda}$ is the inverse of $\circ\lambda$, the last equivalence follows from (3.4). ❑

Continuing the argument, we obtain that the model boundary triples of H_1 and H_2 are isomorphic.

3.7. Proposition. *Consider the situation described in Proposition* 3.4. *Then the maps* $\circ\lambda$ *and* $\circ\tilde{\lambda}$ *induce mutually inverse isometric isomorphisms between* $L^2(H_1)$ *and* $L^2(H_2)$*,*

$$L^2(H_1) \underset{\circ\tilde{\lambda}}{\overset{\circ\lambda}{\rightleftarrows}} L^2(H_2)$$

which satisfy

$$[\circ\lambda \times \circ\lambda]\big(T_{\max}(H_2)\big) = T_{\max}(H_1), \quad \Gamma(H_1) \circ [\circ\lambda \times \circ\lambda] = \Gamma(H_2)\,.$$

Proof.
Step 1; Mapping L^2*:* Let $f_2 \in \mathcal{M}(H_2)$. Then

$$\int_{I_2} f_2^* H_2 f_2 = \int_{I_1} ([f_2^* H_2 f_2] \circ \lambda) \cdot \lambda' = \int_{I_1} (f_2 \circ \lambda)^* \cdot (H_2 \circ \lambda)\lambda' \cdot (f_2 \circ \lambda)\,.$$

Remembering that $\circ\lambda$ maps $\mathcal{M}(H_2)$ bijectively onto $\mathcal{M}(H_1)$ and respects the equivalence relations $=_{H_1}$ and $=_{H_2}$, this relation implies that $\circ\lambda$ induces an isometric isomorphism of $L^2(H_2)$ onto $L^2(H_1)$.

Step 2; Image of $T_{\max}$*:* Let $f_2, g_2 \in L^2(H_2)$, and let $\hat{f}_2, \hat{g}_2$ be some respective representants. Then we have

$$\hat{f}_2(\lambda(x)) + \int_{\lambda(x)}^{\lambda(y)} J H_2 \hat{g}_2 = (\hat{f}_2 \circ \lambda)(x) + \int_x^y J H_1 (\hat{g}_2 \circ \lambda), \quad x, y \in I_1\,. \tag{3.9}$$

If $(f_2; g_2) \in T_{\max}(H_2)$, choose representants $\hat{f}_2, \hat{g}_2$ as in (2.3). If $x, y \in I_1$, then the left side of (3.9) is equal to $\hat{f}_2(\lambda(y))$. Hence also the right side takes this value. We see that $\hat{f}_2 \circ \lambda$ and $\hat{g}_2 \circ \lambda$ are representants as required in (2.3) to conclude that $(f_2 \circ \lambda, g_2 \circ \lambda) \in T_{\max}(H_1)$.

Conversely, assume that $f_2, g_2 \in L^2(H_2)$ with $(f_1; g_1) := (f_2 \circ \lambda; g_2 \circ \lambda) \in T_{\max}(H_1)$, let $\hat{f}_1, \hat{g}_1$ be representants as in (2.3), and set $\hat{f}_2 := \hat{f}_1 \circ \tilde{\lambda}$ and $\hat{g}_2 := \hat{g}_1 \circ \tilde{\lambda}$. First of all notice that $\hat{f}_2$ and $\hat{g}_2$ are representants of f_2 and g_2, respectively, and remember that $\hat{f}_2 \circ \lambda = \hat{f}_1$ and $\hat{g}_2 \circ \lambda = \hat{g}_1$, cf. (3.7). The right hand side of (3.9),

and thus also the left hand side, is equal to $\hat{f}_1(y) = (\hat{f}_2 \circ \lambda)(y)$. Since λ is surjective, it follows that

$$\hat{f}_2(\tilde{y}) = \hat{f}_2(\tilde{x}) + \int_{\tilde{x}}^{\tilde{y}} JH_2\hat{g}_2, \quad \tilde{x}, \tilde{y} \in I_2\,.$$

It follows that $\hat{f}_2$ is absolutely continuous, and satisfies the relation required in (2.3) to conclude that $(f_2; g_2) \in T_{\max}(H_2)$.

Step 3; Boundary values: As we have seen in the previous part of this proof, the map $\circ\lambda \times \circ\lambda$ is not only a bijection of $T_{\max}(H)$ onto $T_{\max}(H)$, but actually between the sets of all possible representants which can be used in (2.3). This implies that also $\Gamma(H_1) \circ [\circ\lambda \times \circ\lambda] = \Gamma(H_2)$. ❑

Now it is easy to reach our aim, and treat arbitrary reparametrizations.

3.8. Theorem. *Let H and $\tilde{H}$ be Hamiltonians which are reparametrizations of each other. Then there exists a linear and isometric bijection Φ of $L^2(H)$ onto $L^2(\tilde{H})$ such that*

$$(\Phi \times \Phi)\big(T_{\max}(H)\big) = T_{\max}(\tilde{H}), \quad \Gamma(\tilde{H}) \circ (\Phi \times \Phi) = \Gamma(H)\,.$$

Proof. Assume that $H \sim \tilde{H}$, and choose $L_0, \ldots, L_m$ as in (3.2). Then there exist isometric isomorphisms $\Phi_i : L^2(L_i) \to L^2(L_{i+1})$, $i = 0, \ldots, m-1$, with $(\Phi_i \times \Phi_i)(T_{\max}(H_i)) = T_{\max}(H_{i+1})$ and $\Gamma(H_{i+1}) \circ (\Phi_i \times \Phi_i) = \Gamma(H_i)$. The composition

$$\Phi := \Phi_{m-1} \circ \cdots \circ \Phi_0$$

hence does the job. ❑

4. Trace-normed and non-vanishing Hamiltonians

In this section we show that indeed it is often no loss in generality to work with trace–normed Hamiltonians. Moreover, we show that the presently introduced notion of reparametrization is consistent with what was used previously.

a. Existence of trace-norming reparametrizations.
The fact that each equivalence class of Hamiltonians modulo reparametrization contains trace-normed elements, is a consequence of the following lemma.

4.1. Lemma. *Let I_1 and I_2 be nonempty open intervals on the real line, and let $\lambda : I_1 \to I_2$ be a nondecreasing, locally absolutely continuous, and surjective map. Moreover, let H_1 be a Hamiltonian on I_1. Then there exists a Hamiltonian H_2 on I_2, such that $H_1 \rightsquigarrow H_2$ via the map λ.*

Proof. Choose a right inverse $\tilde{\lambda}$ of λ, and a function λ' which coincides almost everywhere with the derivative of λ (and satisfies (3.3)). Moreover, set again $L_0 := \{x \in I_1 : \lambda'(x) = 0\}$.

Then we define

$$H_2(y) := \begin{cases} \frac{1}{(\lambda' \circ \tilde{\lambda})(y)} (H_1 \circ \tilde{\lambda})(y)\,, & y \in I_2 \setminus \lambda(L_0) \\ 0 \qquad\qquad\qquad\quad , & y \in \lambda(L_0) \end{cases}$$

Then H_2 is a measurable function, and $H_2(y) \geq 0$ a.e. If $x_1, x_2 \in I_1$, $x_1 < x_2$, and $(x_1; x_2) \in \ker \lambda$, then $x_1, x_2 \in L_0$. Hence,

$$(\tilde{\lambda} \circ \lambda)(x) = x, \quad x \in I_1 \setminus L_0 \,.$$

Thus H_1 and H_2 are related by (3.1).

Let $\alpha, \beta \in I_1$, $\alpha < \beta$. Then

$$\int_{\lambda(\alpha)}^{\lambda(\beta)} \operatorname{tr} H_2 = \int_\alpha^\beta \big([\operatorname{tr} H_2] \circ \lambda\big) \cdot \lambda' = \int_\alpha^\beta \operatorname{tr} H_1 < \infty \,.$$

Whenever K is a compact subset of I_2, we can choose α, β such that $K \subseteq \lambda\big((\alpha, \beta)\big)$. Thus $\operatorname{tr} H_2$, and hence also each entry of H_2, is locally integrable. ❑

4.2. Proposition. *Let H be a Hamiltonian, then there exists a trace-normed reparametrization of H.*

Proof. Since we are only interested in the equivalence class modulo reparametrization which contains H as a representant, we may assume without loss of generality that H has heavy endpoints.

Write the domain of H as $I = (s_-, s_+)$, fix $s \in (s_-, s_+)$, and set

$$\mathfrak{t}(x) := \int_s^x \operatorname{tr} H(t)\, dt, \quad x \in I \,, \tag{4.1}$$

$$\sigma_- := \lim_{x \searrow s_-} \mathfrak{t}(x), \quad \sigma_+ := \lim_{x \nearrow s_+} \mathfrak{t}(x) \,.$$

Then $\mathfrak{t}$ is an absolutely continuous and nondecreasing function which maps I surjectively onto the open interval $\tilde{I} := (\sigma_-, \sigma_+)$. By Lemma 4.1, there exists a basic reparametrization $\tilde{H}$ of H via the map $\mathfrak{t}$.

It remains to compute ($\tilde{\mathfrak{t}}$, $\mathfrak{t}'$, and L_0, are as in Lemma 4.1 for $\lambda := \mathfrak{t}$)

$$\operatorname{tr} \tilde{H}(y) = \frac{1}{(\operatorname{tr} H \circ \tilde{\mathfrak{t}})(y)} \operatorname{tr}(H \circ \tilde{\mathfrak{t}})(y) = 1, \quad y \in \tilde{I} \setminus \mathfrak{t}(L_0) \,,$$

and to remember that $\mathfrak{t}(L_0)$ is a zero set. ❑

4.3. *Remark.* We would like to note that the reparameterization used in Proposition 4.2 could also be obtained as a three–step result: First, we may assume that I is a finite interval and that H has heavy endpoints. This can be achived by an affine reparameterization, and applying the 'scissors'-operation, respectively.

In the second step apply Lemma 4.1 with the map

$$\lambda(t) := \int_{\inf I}^t \chi_J(t)\, dt$$

where

$$J := \{t \in I : \, H(t) \neq 0\} \,.$$

This yields a reparameterization to a non-vanishing Hamiltonian. Note here that, since H has heavy endpoints, the image of λ is an open interval.

Finally, apply the reparameterization via the map (4.1). For this step we need not anymore use Lemma 4.1, but can refer to the classical theory (or Lemma 4.7 below). //

Now we obtain without any further effort that the operator model defined in Section 2 indeed has all the properties known from the trace-normed case. For example:

4.4. Corollary. *Let H be a Hamiltonian. Then $(L^2(H), T_{\max}(H), \Gamma(H))$ is a boundary triple with defect 1 or 2 in the sense of* [KW/IV, §2.2.a]. ❑

b. Description of '$\sim$' for non-vanishing Hamiltonians.
Our last aim in this paper is to show that the restriction of the relation '$\sim$' to the subclass of non-vanishing Hamiltonians can be described in a simple way, namely in exactly the way 'reparametrizations' were defined in [KW/IV], compare Proposition 4.9 below with [KW/IV, §2.1.f]. In particular, this tells us that the present notion of reparametrization is consistent with the one introduced earlier.

To achieve this aim, we provide some lemmata.

4.5. Lemma. *Let H_1 and H_2 be Hamiltonians defined on $I_1 = (s_{1,-}, s_{1,+})$ and $I_2 = (s_{2,-}, s_{2,+})$, respectively, and let $H_i' := H_i|_{(\sigma_{i,-}, \sigma_{i,+})}$ where $\sigma_i^{\pm}$ is defined as in* (2.2)*. Then the following are equivalent:*

(*i*) *We have $H_1 \rightsquigarrow H_2$.*
(*ii*) *We have $H_1' \rightsquigarrow H_2'$. Moreover, H_1 and H_2 together do or do not have a heavy left endpoint, and together do or do not have a heavy right endpoint.*

Proof. Assume that $H_1 \rightsquigarrow H_2$, and let λ be a nondecreasing, locally absolutely continuous surjection of I_1 onto I_2 which establishes this basic reparametrization. First we show that H_1 does not have a heavy left endpoint if and only if H_2 does not have a heavy left endpoint, and that, in this case,

$$\lambda(\sigma_1^-) = \sigma_{2,-}\,. \tag{4.2}$$

Assume that $s_{1,-} < \sigma_1^-$. Then, by Lemma 3.6, the set of inner points of the interval $\lambda\big((s_{1,-}, \sigma_1^-]\big)$ is either empty or H_2-immaterial. However, this set is nothing but the open interval $(s_{1,-}, \lambda(\sigma_1^-))$. We conclude that $\lambda(\sigma_1^-) \leq \sigma_{2,-}$, in particular, $s_{2,-} < \sigma_{2,-}$. For the converse, assume that $s_{2,-} < \sigma_{2,-}$. Then, again by Lemma 3.6, the set of inner points of $\lambda^{-1}\big((s_{2,-}, \sigma_{2,-}]\big)$ is H_1-immaterial. This set is an open interval of the form $(s_{1,-}, x_0)$ with some $x_0 \in I_1$. It already follows that $s_{1,-} < \sigma_1^-$. Assume that $\lambda(\sigma_1^-) < \sigma_{2,-}$. Then there exists a point $x \in (s_{1,-}, x_0)$ with $\lambda(\sigma_1^-) < \lambda(x)$. This implies that $\sigma_1^- < x$, and we have reached a contradiction. Thus the equality (4.2) must hold.

The fact that H_1 and H_2 together do or do not have a heavy right endpoint is seen in exactly the same way. Moreover, we also obtain that $\lambda(\sigma_1^+) = \sigma_{2,+}$, in case $\sigma_{2,-} < s_{2,-}$.

Consider the restriction $\Lambda := \lambda|_{(\sigma_1^-, \sigma_1^+)}$. Then Λ is a nondecreasing and locally absolutely continuous map. Since λ cannot be constant on any interval having σ_1^-

as its left endpoint or σ_1^+ as its right endpoint, we have

$$\Lambda\big((\sigma_1^-, \sigma_1^+)\big) = (\sigma_{2,-}, \sigma_{2,+})\,.$$

Hence Λ establishes a basic reparametrization of H_1' to H_2'. We have shown that $H_1 \rightsquigarrow H_2$ implies that the stated conditions hold true.

For the converse implication, assume that the stated conditions are satisfied, and let Λ be a nondecreasing, locally absolutely continuous surjection of (σ_1^-, σ_1^+) onto $(\sigma_{2,-}, \sigma_{2,+})$ which establishes the basic reparametrization of H_1' to H_2'. If $s_{1,-} < \sigma_1^-$, then also $s_{2,-} < \sigma_{2,-}$, and hence we can choose a linear and increasing bijection Λ_- of $[s_{1,-}, \sigma_1^-]$ onto $[s_{2,-}, \sigma_{2,-}]$. If $s_{1,+} < \sigma_1^+$ choose analogously a linear and increasing bijection Λ_+ of $[\sigma_1^+, s_{1,+}]$ onto $[\sigma_{2,+}, s_{2,+}]$. Then the map $\lambda : I_1 \to I_2$ defined as

$$\lambda(x) := \begin{cases} \Lambda_-(x)\,, & x \in (s_{1,-}, \sigma_1^-] \text{ if } s_{1,-} < \sigma_1^- \\ \Lambda(x) \quad , & x \in (\sigma_1^-, \sigma_1^+) \\ \Lambda_+(x)\,, & x \in [\sigma_1^+, s_{1,+}) \text{ if } \sigma_1^+ < s_{1,+} \end{cases}$$

establishes a basic reparametrization of H_1 to H_2. ❑

4.6. Lemma. *Let H and $\tilde{H}$ be Hamiltonians. Then $H \sim \tilde{H}$ if and only if there exist finitely many Hamiltonians $H_1, \dots, H_n$ with heavy endpoints, such that*

$$H \asymp H_1 \rightsquigarrow H_2 \rightsquigarrow^{-1} H_3 \rightsquigarrow \cdots \rightsquigarrow H_{n-1} \rightsquigarrow^{-1} H_n \asymp \tilde{H}$$

Proof. First we show that

$$\asymp \circ \rightsquigarrow \;=\; \rightsquigarrow \circ \asymp \tag{4.3}$$

Assume that $(H_1; H_2) \in \asymp \circ \rightsquigarrow$. Then there exists a Hamiltonian L, such that

$$H_1 \asymp L \rightsquigarrow H_2$$

Let H_1', H_2', L' be the Hamiltonians with heavy endpoints, such that

$$H_1' \asymp H_1,\ H_2' \asymp H_2,\ L' \asymp L$$

By Lemma 4.5, we have $H_1' = L' \rightsquigarrow H_2'$. Define a Hamiltonian L'' by appending (if necessary) immaterial intervals to L' in such a way that $L'' \asymp L'$, and L'' and H_1 together do or do not heavy left or right endpoints. Analogously, define H_2'', such that $H_2'' \asymp H_2'$, and H_2'' and H_1 together do or do not have heavy left or right endpoints. Then $H_2'' \asymp H_2$ and, by Lemma 4.5,

$$H_1 \rightsquigarrow L'' \rightsquigarrow H_2''$$

Altogether it follows that

$$H_1 \rightsquigarrow H_2'' \asymp H_2$$

i.e. $(H_1; H_2) \in \rightsquigarrow \circ \asymp$. We have established the inclusion '$\subseteq$' in (4.3). The reverse inclusion is seen in the same way.

Assume now that $H \sim \tilde{H}$, and let $L_0, \dots, L_m$ be as in (3.2). By (4.3), reflexivity, and transitivity, there exist Hamiltonians $L_0', \dots, L_n'$ with

$$H = L_0' \asymp L_1' \rightsquigarrow L_2' \rightsquigarrow^{-1} \cdots \rightsquigarrow L_{n-1}' \rightsquigarrow^{-1} L_n' = \tilde{H}$$

Let H_i, $i = 0, \dots, n$, be the Hamiltonians with heavy endpoints and $H_i \rightarrowtail\!\!\!\!\circ\, L_i'$. Then, by Lemma 4.5,

$$H \rightarrowtail\!\!\!\!\circ\, H_0 = H_1 \rightsquigarrow H_2 \rightsquigarrow^{-1} \cdots \rightsquigarrow H_{n-1} \rightsquigarrow^{-1} H_n \rightarrowtail\!\!\!\!\circ\, \tilde{H}$$

❑

4.7. Lemma. *Let H_1 and H_2 be Hamiltonians defined on intervals I_1 and I_2, respectively. Assume that $H_1 \rightsquigarrow H_2$, and let $\lambda : I_1 \to I_2$ be a nondecreasing, locally absolutely continuous, and surjective map such that* (3.1) *holds. If H_1 is non-vanishing, then λ is bijective, λ' is almost everywhere positive, λ^{-1} is locally absolutely continuous, and H_2 is non-vanishing.*

Proof. Assume that H_1 is non-vanishing. Then the function λ' cannot vanish on any set of positive measure, i.e. it is almost everywhere positive. In particular, λ cannot be constant on any nonempty interval. Hence, λ is strictly increasing, and thus also bijective.

Let $E \subseteq I_2$ be a zero set, then

$$0 = \int_{I_2} \chi_E(y)\, dy = \int_{I_1} (\chi_E \circ \lambda)(x)\lambda'(x)\, dx\,.$$

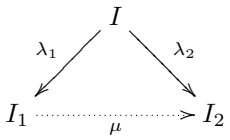
This implies that the (nonnegative) function $\chi_E \circ \lambda$ must vanish almost everywhere. However, $\chi_E \circ \lambda = \chi_{\lambda^{-1}(E)}$, i.e. $\lambda^{-1}(E)$ is a zero set.

It remains to show that H_2 is non-vanishing. Let $E \subseteq I_2$ be measurable. Then

$$\int_E \operatorname{tr} H_2 = \int_{\lambda^{-1}(E)} (\operatorname{tr} H_2 \circ \lambda)\lambda' = \int_{\lambda^{-1}(E)} \operatorname{tr} H_1\,.$$

If $\operatorname{tr} H_2$ vanishes on E, then $\operatorname{tr} H_1$ must vanish on $\lambda^{-1}(E)$. Hence, $\lambda^{-1}(E)$ is a zero set, and thus also E is a zero set. ❑

4.8. Lemma. *Let H, H_1, H_2 be Hamiltonians with heavy endpoints, being defined on respective intervals I, I_1, I_2. Assume that H_1 and H_2 are non–vanishing, and that $H \rightsquigarrow H_1$ and $H \rightsquigarrow H_2$ via maps $\lambda_1 : I \to I_1$ and $\lambda_2 : I \to I_2$. Then there exists a bijective increasing map $\mu : I_1 \to I_2$ such that μ and μ^{-1} are locally absolutely continuous and*



Proof.
Step 1: We start with a preliminary remark. Denote

$$L_0^j := \big\{x \in I :\, \lambda_j'(x) = 0\big\}, \quad j = 1, 2\,.$$

If $x \in L_0^1 \setminus L_0^2$, i.e. $\lambda_1'(x) = 0$ but $\lambda_2'(x) \neq 0$, then

$$H_2(\lambda_2(x)) = \frac{1}{\lambda_2'(x)} H(x) = \frac{1}{\lambda_2'(x)} \cdot H_1(\lambda_1(x))\lambda_1'(x) = 0\,.$$

Since H_2 is non–vanishing, it follows that $\lambda_2(L_0^1 \setminus L_0^2)$ is a zero set. This implies that also

$$\lambda_2^{-1}\big(\lambda_2(L_0^1 \setminus L_0^2)\big) \setminus L_0^2$$

has measure zero. However,

$$L_0^1 \setminus L_0^2 = \big(L_0^1 \setminus L_0^2\big) \setminus L_0^2 \subseteq \lambda_2^{-1}\big(\lambda_2(L_0^1 \setminus L_0^2)\big) \setminus L_0^2\,,$$

and hence also $L_0^1 \setminus L_0^2$ is a zero set. In the same way it follows that $L_0^2 \setminus L_0^1$ is a zero set.

Step 2: We turn to the proof of the lemma. Let $\tilde{\lambda}_1$ be a right inverse of λ_1, and set

$$\mu := \lambda_2 \circ \tilde{\lambda}_1\,.$$

Then μ is a nondecreasing map of I_1 onto I_2.

First, we show that μ is surjective. Let $y \in I_2$ be given, and set $x := \lambda_1(\tilde{\lambda}_2(y))$ where $\tilde{\lambda}_2$ is a right inverse of λ_2. If $\tilde{\lambda}_1(x) = \tilde{\lambda}_2(y)$, we have

$$\mu(x) = \lambda_2(\tilde{\lambda}_1(x)) = \lambda_2(\tilde{\lambda}_2(y)) = y\,.$$

Assume that $\tilde{\lambda}_1(x) < \tilde{\lambda}_2(y)$. We have

$$\lambda_1\big(\tilde{\lambda}_1(x)\big) = x = \lambda_1\big(\tilde{\lambda}_2(y)\big)\,,$$

and hence the interval $(\tilde{\lambda}_1(x), \tilde{\lambda}_2(y))$ is H-immaterial. Thus the set of inner points of $\lambda_2\big([\tilde{\lambda}_1(x), \tilde{\lambda}_2(y)] \cap I\big)$ is either empty or H_2-immaterial. Since H_2 is non–vanishing, the second possibility cannot occur. We conclude that $\lambda_2(\tilde{\lambda}_1(x)) = \lambda_2(\tilde{\lambda}_2(y))$, and hence again $\mu(x) = y$. The case that $\tilde{\lambda}_1(x) > \tilde{\lambda}_2(y)$ is treated in the same way. In any case, the given point y belongs to the image of μ.

Since μ is nondecreasing and surjective, μ must be continuous. To show that μ is locally absolutely continuous, let a set $E \subseteq I_1$ with measure zero be given. Denote by A the union of all equivalence classes modulo $\ker \lambda_2$ which intersect $\lambda_1^{-1}(E)$. Then we have

$$\mu(E) = \lambda_2\big(\tilde{\lambda}_1(E)\big) \subseteq \lambda_2\big(\lambda_1^{-1}(E)\big) = \lambda_2(A)\,.$$

Hence, it suffices to show that $\lambda_2(A)$ has measure zero.

We know that the set $\lambda_1^{-1}(E) \setminus L_0^1$ has measure zero. By what we showed in Step 1, thus also $\lambda_1^{-1}(E) \setminus L_0^2$ has this property. Since λ_2 is absolutely continuous, it follows that also the set

$$\lambda_2\big(\lambda_1^{-1}(E) \setminus L_0^2\big) = \lambda_2\big(A \setminus L_0^2\big)$$

has measure zero. We can rewrite

$$\lambda_2(A) \setminus \lambda_2(L_0^2) = \lambda_2\big[\lambda_2^{-1}\big(\lambda_2(A) \setminus \lambda_2(L_0^2)\big)\big] = \lambda_2\big[\underbrace{\lambda_2^{-1}(\lambda_2(A)) \setminus \lambda_2^{-1}(\lambda_2(L_0^2))}_{=A \setminus L_0^2}\big]\,,$$

and conclude that

$$\lambda_2(A) \subseteq \lambda_2(A \setminus L_0^2) \cup \lambda_2(L_0^2)\,.$$

Thus $\lambda_2(A)$ is a zero set.

We conclude that $H_1 \rightsquigarrow H_2$ via μ. The proof of the lemma is completed by applying Lemma 4.7.

❑

Now we are ready for the proof of the following simple description of '$\sim$' for non-vanishing Hamiltonians.

4.9. Proposition. *Let H and $\tilde{H}$ be non-vanishing Hamiltonians defined on intervals I and $\tilde{I}$, respectively. Then we have $H \sim \tilde{H}$ if and only if there exists an increasing bijection λ of I onto $\tilde{I}$, such that λ and λ^{-1} are both locally absolutely continuous, and*

$$H(x) = \tilde{H}(\lambda(x))\lambda'(x), \quad x \in I_1 \ \text{a.e.}$$

Proof. Let $H_1, \dots, H_n$ be Hamiltonians with heavy endpoints as in Lemma 4.6. Since H and $\tilde{H}$ are non-vanishing, they certainly have heavy endpoints. Thus $H = H_1$ and $\tilde{H} = H_n$.

Let λ_i, $i = 1, \dots, n-1$, be maps which establish the basic reparametrizations

$$\begin{cases} H_i \rightsquigarrow H_{i+1}\,, & i = 1, 3, \dots, n-2 \\ H_{i+1} \rightsquigarrow H_i\,, & i = 2, 4, \dots, n-1 \end{cases}$$

Lemma 4.8 furnishes us with maps μ_i, $i = 1, \dots, n-1$, which establish basic reparametrizations

$$\begin{cases} H_i' \rightsquigarrow H_{i+1}'\,, & i = 1, 3, \dots, n-2 \\ H_{i+1}' \rightsquigarrow H_i'\,, & i = 2, 4, \dots, n-1 \end{cases}$$

where H_i' is trace–normed basic reparametrizations of H_i, e.g. $H_i \rightsquigarrow H_i'$ via the map $\mathfrak{t}_i = \operatorname{tr} H_i$ as in Proposition 4.2:

i odd:
$$\begin{array}{ccc} I_i' & \overset{\mu_i}{\dashrightarrow} & I_{i+1}' \\ {\scriptstyle \mathfrak{t}_i}\uparrow & & \uparrow{\scriptstyle \mathfrak{t}_{i+1}} \\ I_i & \xrightarrow[\lambda_i]{} & I_{i+1} \end{array}$$

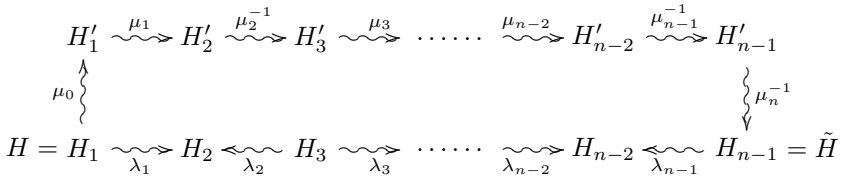
i even:
$$\begin{array}{ccc} I_i' & \overset{\mu_i}{\dashleftarrow} & I_{i+1}' \\ {\scriptstyle \mathfrak{t}_i}\uparrow & & \uparrow{\scriptstyle \mathfrak{t}_{i+1}} \\ I_i & \xleftarrow[\lambda_i]{} & I_{i+1} \end{array}$$

The maps μ_i are bijective and have the property that μ_i^{-1} is locally absolutely continuous. Set $\mu_0 := \mathfrak{t}_1$ and $\mu_n := \mathfrak{t}_n$. Since $H_1 = H$ and $H_n = \tilde{H}$ are non–vanishing, by Lemma 4.7, also μ_0 and μ_n are bijective, and their inverses are locally absolutely continuous.

We see that the composition

$$\lambda := \mu_n^{-1} \circ \mu_{n-1}^{-1} \circ \mu_{n-2} \circ \cdots \circ \mu_3 \circ \mu_2^{-1} \circ \mu_1 \circ \mu_0$$

has the required properties:



❑

4.10. *Remark.* As an immediate corollary of Proposition 4.9, we obtain that the relation '$\rightsquigarrow$' restricted to the set of all non–vanishing Hamiltonians is an equivalence relation. Let us note that the proof of this fact does not require the full strength of Proposition 4.9; it follow immediately from Lemma 4.6 and the chain rule for differentiation. //

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