Majorization in de Branges spaces II. Banach spaces generated by majorants

Anton Baranov, Harald Woracek

Abstract

This is the second part in a series dealing with subspaces of de Branges spaces of entire functions generated by majorization on subsets of the closed upper half-plane. In this part we investigate certain Banach spaces generated by admissible majorants. We study their interplay with the original de Branges space structure, and their geometry. In particular, we will show that, generically, they will be nonreflexive and nonseparable.

AMS Classification Numbers: 46E15, 46B26, 46E22
Keywords: de Branges subspace, majorant, Banach space

1 Introduction

A de Branges space $\mathcal{H}$ is a Hilbert space whose elements are entire functions, and which has the following properties:

(dB1) For each $w \in \mathbb{C}$ the point evaluation $F \mapsto F(w)$ is a continuous linear functional on $\mathcal{H}$.

(dB2) If $F \in \mathcal{H}$, also $F^{\#}(z) := \overline{F(\overline{z})}$ belongs to $\mathcal{H}$ and $\|F^{\#}\| = \|F\|$.

(dB3) If $w \in \mathbb{C} \setminus \mathbb{R}$ and $F \in \mathcal{H}$, $F(w) = 0$, then

$$\frac{z-\bar{w}}{z-w}F(z) \in \mathcal{H} \quad \text{and} \quad \Big\|\frac{z-\bar{w}}{z-w}F(z)\Big\| = \big\|F\big\| .$$

Alternatively, de Branges spaces can be defined via Hermite–Biehler functions. These are entire functions E which satisfy:

(HB) For all z in the open upper half-plane $\mathbb{C}^+$, we have $|E(\overline{z})| < |E(z)|$.

For a Hermite–Biehler function E define

$$\mathcal{H}(E) := \Big\{F \text{ entire} : \frac{F}{E}, \frac{F^{\#}}{E} \in H^2(\mathbb{C}^+)\Big\} ,$$

$$\|F\|_{\mathcal{H}(E)} := \Big(\int_{\mathbb{R}} \Big|\frac{F(t)}{E(t)}\Big|^2 dt\Big)^{\frac{1}{2}}, \quad F \in \mathcal{H}(E) ,$$

where $H^2(\mathbb{C})$ denotes the Hardy space in the upper half-plane. Then $\mathcal{H}(E)$ is a de Branges space. Conversely, every de Branges space can be obtained in this way, cf. [dB].

In the theory of de Branges spaces, an important role is played by their de Branges subspaces (dB-subspaces, for short). These are those subspaces $\mathcal{L}$ of a de Branges space $\mathcal{H}$ which are themselves de Branges spaces with the norm inherited from $\mathcal{H}$.

In [BW3] we have investigated a general procedure to construct dB-subspaces of a given de Branges space $\mathcal{H}$ by means of majorization. For a function $\mathfrak{m} : D \to [0, \infty)$, defined on some subset D of the closed upper half-plane $\mathbb{C}^+ \cup \mathbb{R}$, we have defined

$$R_{\mathfrak{m}}(\mathcal{H}) := \big\{ F \in \mathcal{H} : \ \exists C > 0 : |F(z)|, |F^{\#}(z)| \leq C\mathfrak{m}(z), \ z \in D \big\},$$

and

$$\mathcal{R}_{\mathfrak{m}}(\mathcal{H}) := \operatorname{clos}_{\mathcal{H}} R_{\mathfrak{m}}(\mathcal{H}) .$$

Provided $R_{\mathfrak{m}}(\mathcal{H}) \neq \{0\}$ and $\mathfrak{m}$ satisifies a mild regularity condition, the space $\mathcal{R}_{\mathfrak{m}}(\mathcal{H})$ is a dB-subspace of $\mathcal{H}$, cf. [BW3, Theorem 3.1]. In this case we say that $\mathfrak{m}$ is an admissible majorant for $\mathcal{H}$; the set of all admissible majorants is denoted by $\operatorname{Adm}\mathcal{H}$ (see Definition A.11). The main task in [BW3] was to investigate which dB-subspaces of $\mathcal{H}$ can be represented as $\mathcal{R}_{\mathfrak{m}}(\mathcal{H})$. Results, of course, depend on the set D where majorization is permitted. We showed that every dB-subspace $\mathcal{L}$ of $\mathcal{H}$ is of the form $\mathcal{R}_{\mathfrak{m}}(\mathcal{H})$ when D is sufficiently large, and obtained a number of results on representability by specific majorants defined on specific (smaller) subsets D.

The starting point for the present paper is the following observation: Those elements of a space $\mathcal{R}_{\mathfrak{m}}(\mathcal{H})$ about which one has explicit information, are the elements of $R_{\mathfrak{m}}(\mathcal{H})$. Hence, a closer investigation of $R_{\mathfrak{m}}(\mathcal{H})$, rather than just of $\mathcal{R}_{\mathfrak{m}}(\mathcal{H})$, is desirable.

On the space $R_{\mathfrak{m}}(\mathcal{H})$ a stronger norm than $\|.\|_{\mathcal{H}}$ can be defined in a natural way, namely as

$$\|F\|_{\mathfrak{m}} := \max\big\{\|F\|_{\mathcal{H}}, \min\{C \geq 0 : \ |F(z)|, |F^{\#}(z)| \leq C\mathfrak{m}(z), z \in D\}\big\}, \qquad F \in R_{\mathfrak{m}}(\mathcal{H}) .$$

It is seen with a routine argument that $R_{\mathfrak{m}}(\mathcal{H})$, if endowed with the norm $\|.\|_{\mathfrak{m}}$, becomes a Banach space. Although quite simple, this fact has interesting consequences and gives rise to some intriguing geometric problems. The reason which makes the structure of $\langle R_{\mathfrak{m}}(\mathcal{H}), \|.\|_{\mathfrak{m}}\rangle$ involved might be explained as follows: On the one hand $R_{\mathfrak{m}}(\mathcal{H})$ is fairly small as a set; it is a subset of the Hilbert space $\mathcal{H}$. On the other hand, to some extent, the norm $\|.\|_{\mathfrak{m}}$ behaves badly; it involves an L^∞-component.

Let us describe the results and organization of the present paper. In Section 2, after providing some basics, we present two instances of the interaction between the de Branges space structure of $\mathcal{H}$ and the Banach space structure of $R_{\mathfrak{m}}(\mathcal{H})$. Namely, we show that the maximal rate of exponential growth of functions in $\mathcal{R}_{\mathfrak{m}}(\mathcal{H})$ is already attained within $R_{\mathfrak{m}}(\mathcal{H})$, and that reflexivity of $\langle R_{\mathfrak{m}}(\mathcal{H}), \|.\|_{\mathfrak{m}}\rangle$ implies its separability, cf. Proposition 2.3 and Proposition 2.4. The proofs of these results are not difficult, but nicely illustrate the interplay of $\|.\|_{\mathcal{H}}$ and $\|.\|_{\mathfrak{m}}$.

Section 3 is the most involved part of the present paper. There we discuss the geometry of the Banach space $\langle R_{\mathfrak{m}}(\mathcal{H}), \|.\|_{\mathfrak{m}}\rangle$ for a particular majorant and two particular domains of majorization. Namely, if $\mathcal{L} = \mathcal{H}(E_1)$ is a dB-subspace of $\mathcal{H}$, we consider the majorant $\mathfrak{m}_{E_1}|_D$ where

$$\mathfrak{m}_{E_1}(z) := \frac{|E_1(z)|}{|z+i|} \quad \text{and} \quad D := i[1,\infty) \text{ or } D := \mathbb{R} .$$

This majorant already has been used and investigated intensively in [BW3]. Although it is probably one of the simplest majorants one can think of, it is already quite hard to obtain knowledge on $R_{\mathfrak{m}}(\mathcal{H})$. It turns out that the geometric structure of $R_{\mathfrak{m}_{E_1}|_D}(\mathcal{H})$ varies from very simple to extremely complicated, and is closely related to the distribution of zeros of E_1 or, more generally, the behaviour of the inner function $E_1^{-1}E_1^{\#}$. In the case $D = i[1,\infty)$, roughly speaking, the zeros of E which are close to $\mathbb{R}$ give "simple" parts of the space, whereas zeros separated from the real axis give "complicated" parts of the space, cf. Theorem 3.1 and Theorem 3.2. A good illustration of this idea is also Corollary 3.9. The case $D = \mathbb{R}$ is different; it turns out that the geometric structure of $R_{\mathfrak{m}}(\mathcal{H})$ will always be complicated, cf. Theorem 3.10.

In the last section of the paper we revisit the question of representability of dB-subspaces by means of majorization, taking up the refined viewpoint of the space $R_{\mathfrak{m}}(\mathcal{H})$. It is a consequence of the Banach space structure of $R_{\mathfrak{m}}(\mathcal{H})$ that, for each given majorant $\mathfrak{m}$, the set of all majorants $\mathfrak{m}_1$ with $R_{\mathfrak{m}_1}(\mathcal{H}) = R_{\mathfrak{m}}(\mathcal{H})$ contains a smallest element. This majorant is fairly smooth and reflects many properties of $\mathcal{R}_{\mathfrak{m}}(\mathcal{H})$, cf. Proposition 4.6. Moreover, it can be used to characterize minimal elements in the set of all admissible majorants; a topic studied for majorization along $\mathbb{R}$ e.g. in [BH] or [HM]. It turns out that minimal majorants correspond to one-dimensional dB-subspaces representable as $\mathcal{R}_{\mathfrak{m}}(\mathcal{H})$. In conjunction with our previous results on representability, this fact yields criteria for existence of minimal majorants, cf. Corollary 4.11.

The notation in the present paper will follow the notation introduced in [BW3]. In order to make the present paper more self-contained, we collected the necessary definitions and some preliminary facts on de Branges spaces in an appendix. Moreover, since in the present context this is no loss in generality, we will assume that all de Branges spaces $\mathcal{H}$ have the following property: Whenever $x \in \mathbb{R}$, there exists an element $F \in \mathcal{H}$ with $F(x) \neq 0$. Also, bounded sets D give only trivial results, hence we will throughout this paper exclude bounded sets from our discussion.

2 The Banach space $R_{\mathfrak{m}}(\mathcal{H})$

The following simple observation is the basis of all considerations made in this paper. For this reason we provide an explicit proof. Let the set $\operatorname{Adm}\mathcal{H}$ be defined as in Definition A.11.

2.1 Proposition. *Let $\mathcal{H}$ be a de Branges space and $\mathfrak{m} \in \operatorname{Adm}\mathcal{H}$. Then $\langle R_{\mathfrak{m}}(\mathcal{H}), \|.\|_{\mathfrak{m}}\rangle$ is a Banach space.*

Proof. Let $(F_n)_{n\in\mathbb{N}}$, $F_n \in R_{\mathfrak{m}}(\mathcal{H})$, be a Cauchy sequence with respect to the norm $\|.\|_{\mathfrak{m}}$. Since $\|.\|_{\mathfrak{m}} \geq \|.\|_{\mathcal{H}}$, it is thus also a Cauchy sequence in $\mathcal{H}$. By completeness of $\mathcal{H}$, $(F_n)_{n\in\mathbb{N}}$ converges with respect to the norm $\|.\|_{\mathcal{H}}$, say, to $F := \lim_{n\to\infty} F_n \in \mathcal{H}$. Set $C := \sup_{n\in\mathbb{N}} \|F_n\|_{\mathfrak{m}} < \infty$. Since convergence in $\mathcal{H}$ implies pointwise convergence, we have

$$|F(z)| = \lim_{n\to\infty} |F_n(z)| \leq C\mathfrak{m}(z), \quad z \in D\,.$$

Similarly, it follows that $|F^{\#}(z)| \leq C\mathfrak{m}(z)$, $z \in D$. Hence $F \in R_{\mathfrak{m}}(\mathcal{H})$.

Let $\epsilon > 0$ be given, and choose $N \in \mathbb{N}$ with $\|F_n - F_m\|_{\mathfrak{m}} \leq \epsilon$, $n, m \geq N$. Then we have

$$|F_n(z) - F_m(z)| \leq \epsilon \mathfrak{m}(z), \quad z \in D,\ n, m \geq N\,.$$

Passing to the limit $m \to \infty$, it follows that $|F_n(z) - F(z)| \leq \epsilon \mathfrak{m}(z)$, $z \in D$, $n \geq N$. Together with convergence in $\mathcal{H}$, this implies that $\lim_{n\to\infty} F_n = F$ with respect to the norm $\|.\|_{\mathfrak{m}}$. ❑

As a first step towards getting acquainted with this Banach space, let us discuss its dual.

2.2 Remark. Let $\mathfrak{m} : D \to [0, \infty)$ belong to $\operatorname{Adm}\mathcal{H}$. Assume that D is a closed subset of $\mathbb{C}^+ \cup \mathbb{R}$, and that

$$\forall\, w \in D:\ (z - w)^{-\mathfrak{d}_{\mathfrak{m}}(w)}\mathfrak{m}(z) \text{ is continuous at } w\,. \tag{2.1}$$

For the definition of the zero-divisor $\mathfrak{d}_{\mathfrak{m}}$ see Definition A.1. As it will become clear later, assuming (2.1) is no essential restriction. Denote by $C(D)$ the Banach space of all continuous and bounded functions on D endowed with the supremum norm $\|.\|_\infty$. Then the map

$$\mathcal{J} : \begin{cases} \langle R_{\mathfrak{m}}(\mathcal{H}), \|.\|_{\mathfrak{m}}\rangle & \to \quad \langle \mathcal{H} \times C(D), \max\{\|.\|_{\mathcal{H}}, \|.\|_\infty\}\rangle \\ \qquad\qquad F & \mapsto \quad \big(F, \frac{F}{\mathfrak{m}}\big) \end{cases}$$

is an isometric isomorphism of $R_{\mathfrak{m}}(\mathcal{H})$ onto a closed subspace of $\mathcal{H} \times C(D)$. Note that the condition (2.1) ensures that $\mathfrak{m}^{-1}F$ is a continuous function whenever $\mathfrak{d}_F \geq \mathfrak{d}_{\mathfrak{m}}$. Since D is locally compact, the dual space $C(D)'$ is isomorphic to the space $\operatorname{rba}(D)$ of all regular bounded finitely additive set functions defined on the σ-algebra of Borel sets on D. It follows that

$$R_{\mathfrak{m}}(\mathcal{H})' \cong \big(\mathcal{H} \times \operatorname{rba}(D)\big)\big/N\,,$$

where $N := \mathcal{J}(R_{\mathfrak{m}}(\mathcal{H}))^\perp$. In fact, every continuous linear functional on $R_{\mathfrak{m}}(\mathcal{H})$ can be written in the form

$$F \mapsto (F, G)_{\mathcal{H}} + \int_D F\, d\mu, \qquad F \in R_{\mathfrak{m}}(\mathcal{H})\,,$$

with some $G \in \mathcal{H}$ and $\mu \in \operatorname{rba}(D)$. We also see that the annihilator N is given as

$$N = \Big\{(G, \mu) \in \mathcal{H} \times \operatorname{rba}(D) :\ \int_D \frac{F}{\mathfrak{m}}\, d\mu = (F, -G)_{\mathcal{H}}, F \in \mathcal{R}_{\mathfrak{m}}(\mathcal{H})\Big\}\,.$$

Interaction between $\|.\|_{\mathfrak{m}}$ and $\|.\|_{\mathcal{H}}$.

The interplay between the Banach space structure of $R_{\mathfrak{m}}(\mathcal{H})$ and the de Branges space structure of $\mathcal{H}$ leads to interesting insight. We give two results of this kind. First we show that the maximal rate of exponential growth transfers from $R_{\mathfrak{m}}(\mathcal{H})$ to $\mathcal{R}_{\mathfrak{m}}(\mathcal{H})$. For $\alpha \leq 0$, set $\mathcal{H}_{(\alpha)} := \{F \in \mathcal{H} :\ \operatorname{mt}_{\mathcal{H}} F, \operatorname{mt}_{\mathcal{H}} F^\# \leq \alpha\}$. Then $\mathcal{H}_{(\alpha)}$ is a closed subspace of $\mathcal{H}$, cf. [KW, Corollary 5.2]. Hence we have

$$\operatorname{mt}_{\mathcal{H}} \mathcal{R}_{\mathfrak{m}}(\mathcal{H}) = \sup_{F \in R_{\mathfrak{m}}(\mathcal{H})} \operatorname{mt}_{\mathcal{H}} F\,, \tag{2.2}$$

for the definition of $\operatorname{mt}_{\mathcal{H}}$ see Definition A.1.

2.3 Proposition. *Let $\mathcal{H}$ be a de Branges space, and let $\mathfrak{m} \in \operatorname{Adm}\mathcal{H}$. Then there exists a function $F \in R_{\mathfrak{m}}(\mathcal{H})$, such that*

$$\operatorname{mt}_{\mathcal{H}} \mathcal{R}_{\mathfrak{m}}(\mathcal{H}) = \operatorname{mt}_{\mathcal{H}} F .$$

Proof. We have $\|.\|_{\mathfrak{m}} \geq \|.\|_{\mathcal{H}}$ on $R_{\mathfrak{m}}(\mathcal{H})$. Hence, for each $\alpha \leq 0$, the subspace $R_{\mathfrak{m}}(\mathcal{H}) \cap \mathcal{H}_{(\alpha)}$ is $\|.\|_{\mathfrak{m}}$-closed. Consider the value $\alpha := \sup_{F \in R_{\mathfrak{m}}(\mathcal{H})} \operatorname{mt}_{\mathcal{H}} F$. Then

$$R_{\mathfrak{m}}(\mathcal{H}) \cap \mathcal{H}_{(\beta)} \subsetneq R_{\mathfrak{m}}(\mathcal{H}), \qquad \beta < \alpha ,$$

and hence each of the spaces $R_{\mathfrak{m}}(\mathcal{H}) \cap \mathcal{H}_{(\beta)}$, $\beta < \alpha$, is nowhere dense in $\langle R_{\mathfrak{m}}(\mathcal{H}), \|.\|_{\mathfrak{m}} \rangle$. By the Baire Category Theorem the set

$$R_{\mathfrak{m}}(\mathcal{H}) \setminus \bigcup_{n \in \mathbb{N}} \Big(R_{\mathfrak{m}}(\mathcal{H}) \cap \mathcal{H}_{(\alpha - \frac{1}{n})} \Big)$$

is dense in $R_{\mathfrak{m}}(\mathcal{H})$. In particular, there exists $F \in R_{\mathfrak{m}}(\mathcal{H})$ with $\operatorname{mt}_{\mathcal{H}} F = \alpha$. ❑

Secondly, we discuss the geometry of $\langle R_{\mathfrak{m}}(\mathcal{H}), \|.\|_{\mathfrak{m}} \rangle$. Several different topologies play a role:

(i) The topology τ_{lu} of locally uniform convergence.

(ii) The weak topology τ_w of $\langle \mathcal{H}, \|.\|_{\mathcal{H}} \rangle$.

(iii) The weak topology $\tau_w^{\mathfrak{m}}$ of $\langle R_{\mathfrak{m}}(\mathcal{H}), \|.\|_{\mathfrak{m}} \rangle$.

Denote by $B_{\mathfrak{m}}(\mathcal{H})$ the unit ball of the Banach space $\langle R_{\mathfrak{m}}(\mathcal{H}), \|.\|_{\mathfrak{m}} \rangle$. Explicitly, this is

$$B_{\mathfrak{m}}(\mathcal{H}) := \left\{ \; F \in \mathcal{H} : \; \begin{array}{l} \|F\|_{\mathcal{H}} \leq 1 \text{ and} \\ |F(z)|, |F^{\#}(z)| \leq \mathfrak{m}(z), z \in D \end{array} \; \right\}.$$

2.4 Proposition. *Let $\mathfrak{m} \in \operatorname{Adm}\mathcal{H}$. Then the following hold:*

(i) *Let $B \subseteq \mathcal{H}$ be bounded with respect to the norm $\|.\|_{\mathcal{H}}$ of $\mathcal{H}$. Then B is a normal family of entire functions. We have $\tau_w|_B = \tau_{lu}|_B$.*

(ii) *The space $\langle R_{\mathfrak{m}}(\mathcal{H}), \|.\|_{\mathfrak{m}} \rangle$ is reflexive if and only if $\tau_w^{\mathfrak{m}}|_{B_{\mathfrak{m}}(\mathcal{H})} = \tau_{lu}|_{B_{\mathfrak{m}}(\mathcal{H})}$.*

(iii) *If $\langle R_{\mathfrak{m}}(\mathcal{H}), \|.\|_{\mathfrak{m}} \rangle$ is reflexive, then it is also separable.*

Proof. Let B be a $\|.\|_{\mathcal{H}}$-bounded subset of $\mathcal{H}$. The weak closure of B is again $\|.\|_{\mathcal{H}}$-bounded. Hence, for the proof of (i), we may assume in addition that B is τ_w-closed, and thus τ_w-compact. Let $\nabla_{\mathcal{H}}$ be defined as

$$\nabla_{\mathcal{H}}(z) := \sup \big\{ |F(z)| : \, \|F\|_{\mathcal{H}} = 1 \big\} = \big(K(z,z) \big)^{1/2},$$

where $K(z,.)$ is the reproducing kernel at the point z (see Appendix A, Subsection II, for details). Since $|F(z)| \leq \|F\|_{\mathcal{H}} \nabla_{\mathcal{H}}(z)$ and $\nabla_{\mathcal{H}}$ is continuous, the family B is locally uniformly bounded, i.e. a normal family. Since $\mathcal{H}$ is a separable Hilbert space, the restriction $\tau_w|_B$ is metrizable, cf. [M, Theorem 2.6.23]. Let $(F_n)_{n \in \mathbb{N}}$ be a sequence of elements of B which converges to $F \in B$ with respect to $\tau_w|_B$. Then it converges pointwise and, by the Vitali Theorem, thus also locally uniformly. We see that $\tau_{lu}|_B \subseteq \tau_w|_B$; note here that τ_{lu} is also metrizable.

Since $\tau_w|_B$ is compact and $\tau_{lu}|_B$ is Hausdorff, it follows that actually equality holds, cf. [Bou, I.9.Corollary 3]. This finishes the proof of (i).

We come to the proof of (ii). The space $\langle R_{\mathfrak{m}}(\mathcal{H}), \|.\|_{\mathfrak{m}}\rangle$ is reflexive, if and only if its unit ball $B_{\mathfrak{m}}(\mathcal{H})$ is $\tau_w^{\mathfrak{m}}$-compact, cf. [M, Theorem 2.8.2]. Since $\|.\|_{\mathfrak{m}} \geq \|.\|_{\mathcal{H}}$, we have $\tau_w|_{R_{\mathfrak{m}}(\mathcal{H})} \subseteq \tau_w^{\mathfrak{m}}$. The unit ball $B_{\mathfrak{m}}(\mathcal{H})$ is a $\|.\|_{\mathcal{H}}$-bounded subset of $\mathcal{H}$. It is $\|.\|_{\mathcal{H}}$-closed and convex, and hence also τ_w-closed. Thus $B_{\mathfrak{m}}(\mathcal{H})$ is τ_w-compact. It follows that $B_{\mathfrak{m}}(\mathcal{H})$ is $\tau_w^{\mathfrak{m}}$-compact if and only if $\tau_w|_{B_{\mathfrak{m}}(\mathcal{H})} = \tau_w^{\mathfrak{m}}|_{B_{\mathfrak{m}}(\mathcal{H})}$. By the already proved item (i), the desired assertion follows.

For the proof of (iii), assume that $\langle R_{\mathfrak{m}}(\mathcal{H}), \|.\|_{\mathfrak{m}}\rangle$ is reflexive. Then, by (ii), the topology $\tau_w^{\mathfrak{m}}|_{B_{\mathfrak{m}}(\mathcal{H})}$ equals the topology of locally uniform convergence and, hence, is metrizable. We conclude from [M, Theorem 2.6.23] that $\langle R_{\mathfrak{m}}(\mathcal{H}), \|.\|_{\mathfrak{m}}\rangle$ is separable. ❑

3 Geometry of $R_{\mathfrak{m}_{E_1}|_D}(\mathcal{H})$

Let $\mathcal{H}$ be a de Branges space and let $\mathcal{L} \in \operatorname{Sub}^* \mathcal{H}$, i.e. let $\mathcal{L}$ be a dB-subspace of $\mathcal{H}$ with $\mathfrak{d}_{\mathcal{L}} = \mathfrak{d}_{\mathcal{H}}$ $(= 0)$, cf. Definition A.2 and the paragraph following it. Write $\mathcal{L} = \mathcal{H}(E_1)$, and denote by $K_1(w, z)$ the reproducing kernel of $\mathcal{L}$. In our previous work two particular majorants were extensively investigated. Namely, those obtained by restriction of the functions

$$\nabla_{\mathcal{L}}(z) := \|K_1(z, .)\|_{\mathcal{H}} \quad \text{and} \quad \mathfrak{m}_{E_1}(z) := \frac{|E_1(z)|}{|z+i|}$$

to the set D under consideration. In [BW1] and [BW3] we showed that:

(i) Let $D := i[1, \infty)$. Then

$$\mathcal{L} = \mathcal{R}_{\nabla_{\mathcal{L}}|_D}(\mathcal{H}) = \mathcal{R}_{\mathfrak{m}_{E_1}|_D}(\mathcal{H}) \,. \tag{3.1}$$

(ii) Let $D := \mathbb{R}$ and assume that $\operatorname{mt}_{\mathcal{H}} \mathcal{L} = 0$. Then $\mathcal{L} = \mathcal{R}_{\mathfrak{m}_{E_1}|_D}(\mathcal{H})$.

In this section we will study $\langle R_{\mathfrak{m}}(\mathcal{H}), \|.\|_{\mathfrak{m}}\rangle$ for these situations. If $\dim \mathcal{H} < \infty$, of course, all questions about the geometry of $R_{\mathfrak{m}}(\mathcal{H})$ are trivial. Hence we will, once and for all, exclude finite dimensional spaces $\mathcal{H}$ from our discussion.

a. Majorization on the imaginary half-line.

Consider $D := i[1, \infty)$. Since $\mathcal{L} \subseteq R_{\nabla_{\mathcal{L}}|_D}(\mathcal{H})$, (3.1) implies that the space $R_{\nabla_{\mathcal{L}}|_D}(\mathcal{H})$ is $\|.\|_{\mathcal{H}}$-closed. It follows that the norms $\|.\|_{\nabla_{\mathcal{L}}|_D}$ and $\|.\|_{\mathcal{H}}$ both turn $R_{\nabla_{\mathcal{L}}|_D}(\mathcal{H})$ into a Banach space. However, $\|.\|_{\nabla_{\mathcal{L}}|_D} \geq \|.\|_{\mathcal{H}}$, and therefore they are equivalent. Being bicontinuously isomorphic to the Hilbert space $\langle \mathcal{L}, \|.\|_{\mathcal{H}}\rangle$, the geometry of the space $\langle R_{\nabla_{\mathcal{L}}|_D}(\mathcal{H}), \|.\|_{\nabla_{\mathcal{L}}|_D}\rangle$ is very simple. For example it is separable, reflexive, has an unconditional basis, etc.

Things change, when turning to the majorant $\mathfrak{m}_{E_1}|_D$. Then, as we will show below, the geometry of the space $\langle R_{\mathfrak{m}_{E_1}|_D}(\mathcal{H}), \|.\|_{\mathfrak{m}_{E_1}|_D}\rangle$ varies from very simple to highly complicated.

For each majorant $\mathfrak{m}$ we have $R_{\mathfrak{m}}(\mathcal{H}) = R_{\mathfrak{m}}(\mathcal{R}_{\mathfrak{m}}(\mathcal{H}))$. Moreover, trivially, the norms $\|.\|_{\mathfrak{m}}$ of $R_{\mathfrak{m}}(\mathcal{H})$ and $R_{\mathfrak{m}}(\mathcal{R}_{\mathfrak{m}}(\mathcal{H}))$ are equal. In particular, using (3.1),

$$R_{\mathfrak{m}_{E_1}|_D}(\mathcal{H}) = R_{\mathfrak{m}_{E_1}|_D}(\mathcal{L})$$

including equality of norms. Hence, we may restrict explicit considerations to the case when $\mathcal{L} = \mathcal{H}$ (and doing so slightly simplifies notation).

Our aim in this subsection is to prove the following two theorems. Denote by Γ_α, $\alpha \in (0, \frac{\pi}{2})$, the Stolz angle

$$\Gamma_\alpha := \big\{ z \in \mathbb{C} : \, \alpha \leq \arg z \leq \pi - \alpha \big\} .$$

3.1 Theorem. *Let $\mathcal{H} = \mathcal{H}(E)$ be a de Branges space, and set $\mathfrak{m} := \mathfrak{m}_E|_{i[1,\infty)}$. Moreover, denote $\Theta := E^{-1}E^{\#}$, and let $(w_n)_{n\in\mathbb{N}}$ be the sequence of zeros of Θ in $\mathbb{C}^+$ listed according to their multiplicities. Then the following are equivalent:*

- (i) *The norms $\|.\|_{\mathfrak{m}}$ and $\|.\|_{\mathcal{H}}$ are equivalent on $R_{\mathfrak{m}}(\mathcal{H})$.*
- (i') *The space $R_{\mathfrak{m}}(\mathcal{H})$ is $\|.\|_{\mathcal{H}}$-closed.*
- (i'') *We have $R_{\mathfrak{m}}(\mathcal{H}) = \mathcal{H}$.*
- (ii) *There exists $\varphi \in \mathbb{R}$, such that*

$$\sup_{z\in\Gamma_\alpha} \Big(|z| \cdot |e^{i\varphi} - \Theta(z)| \Big) < \infty, \quad \alpha \in \Big(0, \frac{\pi}{2}\Big) .$$

- (ii') *There exists $\varphi \in \mathbb{R}$ and $\alpha \in \Big(0, \frac{\pi}{2}\Big)$, such that*

$$\liminf_{\substack{|z|\to\infty \\ z\in\Gamma_\alpha}} \Big(|z| \cdot |e^{i\varphi} - \Theta(z)| \Big) < \infty .$$

- (iii) *We have $\operatorname{mt}\Theta = 0$ and $\sum_{n\in\mathbb{N}} \operatorname{Im} w_n < \infty$.*
- (iii') *The domain of the multiplication operator $S_{\mathcal{H}}$ in $\mathcal{H}$ is not dense.*

3.2 Theorem. *Let $\mathcal{H} = \mathcal{H}(E)$ be a de Branges space, and set $\mathfrak{m} := \mathfrak{m}_E|_{i[1,\infty)}$. Moreover, denote $\Theta := E^{-1}E^{\#}$, and let $(w_n)_{n\in\mathbb{N}}$ be the sequence of zeros of Θ in $\mathbb{C}^+$ listed according to their multiplicities. Then*

$$(i) \implies (ii) \implies (iii),$$

where (i), (ii), and (iii) are the following conditions:

- (i) *We have $\operatorname{mt}\Theta < 0$ or $\limsup_{n\to\infty} \operatorname{Im} w_n > 0$.*
- (ii) *There exists $\delta > 0$ such that*

$$\liminf_{\substack{|z|\to\infty \\ \operatorname{Im} z\geq\delta}} |\Theta(z)| = 0 . \tag{3.2}$$

- (iii) *There exists a bicontinuous embedding of $\langle \ell^\infty, \|.\|_\infty \rangle$ into $\langle R_{\mathfrak{m}}(\mathcal{H}), \|.\|_{\mathfrak{m}} \rangle$.*

Note that the condition Theorem 3.2, (iii), implies that $R_{\mathfrak{m}}(\mathcal{H})$ is neither separable nor reflexive.

3.3 Remark. Comparing the condition (iii) of Theorem 3.1 with (i) of Theorem 3.2, it is apparent that these theorems do not establish a full dichotomy; there is a gap between the described situations. At present it is not clear to us what happens in this gap. In particular, it is an open question whether there exists a de Branges space $\mathcal{H} = \mathcal{H}(E)$, such that for $\mathfrak{m} := \mathfrak{m}_E|_{i[1,\infty)}$ the space $\langle R_{\mathfrak{m}}(\mathcal{H}), \|.\|_{\mathfrak{m}}\rangle$ is separable (or reflexive) although $R_{\mathfrak{m}}(\mathcal{H}) \neq \mathcal{H}$.

We turn to the proof of Theorem 3.1 and Theorem 3.2. For the first theorem we will use the following variant of the Julia-Carathéodory theorem and Ahern and Clark's result on radial limits of K_Θ functions [AC] adjusted to the point at infinity (for details see also [Ba1, Theorem 2, Corollary 2]).

3.4 Lemma. *Let Θ be an inner function in $\mathbb{C}^+$. Then the following are equivalent:*

- (i) *There exists $\varphi \in \mathbb{R}$ such that $\sup_{y\geq 1} y|e^{i\varphi} - \Theta(iy)| < \infty$.*
- (ii) *We have $\sup_{y\geq 1} y(1 - |\Theta(iy)|) < \infty$.*
- (iii) *There exists $\varphi \in \mathbb{R}$ such that $e^{i\varphi} - \Theta \in H^2$.*

Assume that Θ is of the form $\Theta(z) = e^{-iaz}B(z)$ with $a \leq 0$ and B being a Blaschke product. Denote by $(w_n)_{n\in\mathbb{N}}$ the sequence of zeros of B listed according to their multiplicities, then the conditions (i)–(iii) are further equivalent to

- (iv) *$a = 0$ and $\sum_{n\in\mathbb{N}} \operatorname{Im} w_n < \infty$.*

❑

3.5 Corollary. *Let $\mathcal{H} = \mathcal{H}(E)$ be a de Branges space. Set $\Theta := E^{-1}E^{\#}$, and denote by $(w_n)_{n\in\mathbb{N}}$, $w_n \in \mathbb{C}^+$, the sequence of zeros of Θ in $\mathbb{C}^+$ listed according to their multiplicities. Then the multiplication operator in $\mathcal{H}$ is not densely defined, if and only if $\operatorname{mt}\Theta = 0$ and $\sum_{n\in\mathbb{N}} \operatorname{Im} w_n < \infty$.* ❑

Proof (of Theorem 3.1). By the just stated corollary we have $(iii) \Leftrightarrow (iii')$.

The equivalences $(i) \Leftrightarrow (i') \Leftrightarrow (i'')$ are easy to see: If $\|.\|_{\mathfrak{m}}$ and $\|.\|_{\mathcal{H}}|_{R_{\mathfrak{m}}(\mathcal{H})}$ are equivalent, then $R_{\mathfrak{m}}(\mathcal{H})$ is $\|.\|_{\mathcal{H}}$-complete and hence also $\|.\|_{\mathcal{H}}$-closed. Since in any case $\mathcal{R}_{\mathfrak{m}}(\mathcal{H}) = \mathcal{H}$, $\|.\|_{\mathcal{H}}$-closedness of $R_{\mathfrak{m}}(\mathcal{H})$ implies that $R_{\mathfrak{m}}(\mathcal{H}) = \mathcal{H}$. Finally, if $R_{\mathfrak{m}}(\mathcal{H}) = \mathcal{H}$, then $\|.\|_{\mathfrak{m}}$ and $\|.\|_{\mathcal{H}}$ both turn $R_{\mathfrak{m}}(\mathcal{H})$ into a Banach space. Since $\|.\|_{\mathfrak{m}} \geq \|.\|_{\mathcal{H}}$, this implies that they are equivalent.

Trivially, $(ii) \Rightarrow (ii')$ holds. We will finish the proof by showing that $(i) \Leftrightarrow (iii)$, $(iii') \Rightarrow (ii)$, and $(ii') \Rightarrow (iii')$. To this end let us make a preliminary remark. Substituting the explicit expression for $\nabla_{\mathcal{H}}$, cf. (A.5), we obtain

$$\begin{aligned} \frac{\nabla_{\mathcal{H}}(z)}{\mathfrak{m}_E(z)} &= \frac{|z+i|}{|E(z)|}\Big(\frac{|E(z)|^2 - |E(\overline{z})|^2}{4\pi \operatorname{Im} z}\Big)^{\frac{1}{2}} = \frac{|z+i|}{2\sqrt{\pi \operatorname{Im} z}}\big(1 - |\Theta(z)|^2\big)^{\frac{1}{2}} = \\ &= \frac{1}{2\sqrt{\pi}}\frac{|z+i|}{\operatorname{Im} z}\big(1 + |\Theta(z)|\big)^{\frac{1}{2}} \cdot \Big[\operatorname{Im} z \cdot (1 - |\Theta(z)|)\Big]^{\frac{1}{2}}, \quad z \in \mathbb{C}^+ . \end{aligned} \tag{3.3}$$

By (A.6), we have $\inf_{z\in\mathbb{C}^+} \mathfrak{m}_E^{-1}\nabla_{\mathcal{H}} > 0$. Since the first factor of the last expression in (3.3) is bounded above and away from zero on $D = i[1,\infty)$, it follows that (writing $\mathfrak{m}_1 \lesssim \mathfrak{m}_2$ if there exists a positive constant C, such that

$\mathfrak{m}_1(z) \leq C\mathfrak{m}_2(z)$, $z \in D$, and letting $\mathfrak{m}_1 \asymp \mathfrak{m}_2$ stand for "$\mathfrak{m}_1 \lesssim \mathfrak{m}_2$ and $\mathfrak{m}_2 \lesssim \mathfrak{m}_1$")

$$\nabla_{\mathcal{H}}|_D \lesssim \mathfrak{m} \iff \nabla_{\mathcal{H}}|_D \asymp \mathfrak{m} \iff \sup_{y\geq 1} y\big(1-|\Theta(iy)|\big) < \infty\,. \tag{3.4}$$

(i) $\Leftrightarrow$ *(iii):* Let $C>0$ be such that $\|F\|_{\mathfrak{m}} \leq C\|F\|_{\mathcal{H}}$. Then $|F(z)| \leq C\mathfrak{m}(z)$, $z \in D$, $\|F\|_{\mathcal{H}} \leq 1$. Thus $\nabla_{\mathcal{H}}|_D \lesssim \mathfrak{m}$. Using (3.4) and Lemma 3.4, we obtain that $R_{\mathfrak{m}}(\mathcal{H}) = \mathcal{H}$ if and only if the condition *(iii)* holds.

(iii′) $\Rightarrow$ *(ii):* Let $\mathcal{H}(E)$ be such that $\overline{\operatorname{dom}\mathcal{S}_{\mathcal{H}}} \neq \mathcal{H}$, and assume without loss of generality that $B \in \mathcal{H}(E)$. The function $-B^{-1}A$ has nonnegative imaginary part throughout the upper half-plane, and its Herglotz integral representation is of the form

$$-\frac{A}{B} = pz + \sum_{B(t_n)=0} p_n\Big(\frac{1}{t_n - z} - \frac{t_n}{1+t_n^2}\Big)\,,$$

with some nonnegative numbers p and p_n, $n \in \mathbb{N}$. By [dB, Theorem 22] and its proof, the linear term pz in this representation does not vanish. Hence, for each $\alpha \in (0, \frac{\pi}{2})$,

$$0 < p = -\lim_{\substack{|z|\to\infty\\ z\in\Gamma_\alpha}} \frac{1}{z}\cdot\frac{A}{B} = i\lim_{\substack{|z|\to\infty\\ z\in\Gamma_\alpha}} \frac{1}{z}\cdot\frac{1+\Theta(z)}{1-\Theta(z)}\,.$$

Since $1+\Theta(z)$ is bounded, it follows that $z(1-\Theta(z))$ is bounded throughout Γ_α.

(ii′) $\Rightarrow$ *(iii′):* Again it is enough to consider the case that $\phi = 0$. Hence, assume that $\alpha \in (0, \frac{\pi}{2})$, $C > 0$, and a sequence $(z_n)_{n\in\mathbb{N}}$, $z_n \in \Gamma_\alpha$, is given such that $|z_n|\cdot|1-\Theta(z_n)| \leq C$. Then, in particular, $\lim_{n\to\infty}\Theta(z_n) = 1$, and it follows that

$$\liminf_{n\to\infty}\Big|\frac{1}{z_n}\cdot\frac{A(z_n)}{B(z_n)}\Big| = \liminf_{n\to\infty}\Big|\frac{1}{z_n}\cdot\frac{1+\Theta(z_n)}{1-\Theta(z_n)}\Big| \geq \frac{2}{C}\,.$$

If we had $\overline{\operatorname{dom}S_{\mathcal{H}}} = \mathcal{H}$, then by [dB, Theorem 29] and the proof of [dB, Theorem 22] we would have

$$\lim_{\substack{|z|\to\infty\\ z\in\Gamma_\alpha}} \frac{1}{z}\cdot\frac{A(z)}{B(z)} = 0\,,$$

and obtain a contradiction. Thus *(iii′)* must hold. ❑

For the proof of Theorem 3.2, we will employ the following two lemmata.

3.6 Lemma. *Let $\mathcal{H} = \mathcal{H}(E)$ be a de Branges space, and let $(w_k)_{k\in\mathbb{N}}$, $w_k \in \mathbb{C}^+ \cup \mathbb{R}$, be a sequence of points with*

$$\lim_{k\to\infty}\frac{\mathfrak{m}_E(w_k)}{\nabla_{\mathcal{H}}(w_k)} = 0\,. \tag{3.5}$$

Then there exists a subsequence $(w_{k(n)})_{n\in\mathbb{N}}$, such that the sequence $(\tilde{K}(w_{k(n)},.))_{n\in\mathbb{N}}$ of normalized kernels $\tilde{K}(w,.) := \nabla_{\mathcal{H}}(w)^{-1}K_{\mathcal{H}}(w,.)$ is a Riesz sequence, i.e. a Riesz basis in its closed linear span.

In particular, if $(w_k)_{k\in\mathbb{N}}$ satisfies $|w_k| \to \infty$ and $\sup_{k\in\mathbb{N}}|\Theta(w_k)| < 1$, the hypothesis (3.5) *holds true. Here we have again set $\Theta := E^{-1}E^\#$.*

Proof. Let $(e_k)_{k\in\mathbb{N}}$ be a sequence of elements of some Hilbert space with $\|e_k\| = 1$, $k \in \mathbb{N}$. Then, in order that $(e_k)_{k\in\mathbb{N}}$ is a Riesz sequence, it is sufficient that

$$\sum_{\substack{n,m=1\\ n\neq m}}^{\infty} \big|(e_n, e_m)\big|^2 < 1\,,$$

see e.g. [GK, VI.Theorem 2.1]. We compute

$$\big(\tilde{K}(w_m,.), \tilde{K}(w_n,.)\big)_{\mathcal{H}} = \frac{1}{\nabla_{\mathcal{H}}(w_m)\nabla_{\mathcal{H}}(w_n)} \frac{E(w_n)\overline{E(w_m)} - E(\overline{w_m})E^{\#}(w_n)}{2\pi i(\overline{w_m} - w_n)}\,.$$

Hence, by our assumption (3.5), for each fixed $m \in \mathbb{N}$

$$\lim_{n\to\infty} \big(\tilde{K}(w_m,.), \tilde{K}(w_n,.)\big)_{\mathcal{H}} = 0\,.$$

Therefore we can extract a subsequence $(w_{k(n)})_{n\in\mathbb{N}}$ which satisfies

$$\sum_{\substack{n,m=1\\ n\neq m}}^{\infty} \big|\big(\tilde{K}(w_{k(n)},.), \tilde{K}(w_{k(m)},.)\big)_{\mathcal{H}}\big|^2 \leq \frac{1}{2}\,,$$

and hence gives rise to a Riesz sequence $(\tilde{K}(w_{k(n)},.))_{n\in\mathbb{N}}$.

In order to see the last assertion, it is enough to consider the first line of (3.3):

$$\frac{\mathfrak{m}_E(z)}{\nabla_{\mathcal{H}}(z)} = 2\sqrt{\pi}\frac{(\operatorname{Im} z)^{\frac{1}{2}}}{|z+i|} \frac{1}{(1-|\Theta(z)|^2)^{\frac{1}{2}}}\,.$$

Hence, if $|w_n| \to \infty$ and $\sup_{n\in\mathbb{N}} |\Theta(w_n)| < 1$, certainly (3.5) will hold. ❑

Throughout the following we will denote by $\mathcal{U}$ the linear space of all complex sequences $a = (a_k)_{k\in\mathbb{N}}$ with

$$\|a\|_{\mathcal{U}} := \sup_{n\in\mathbb{N}} \Big|\sum_{k=1}^{n} a_k\Big| < \infty\,.$$

Then $\langle \mathcal{U}, \|.\|_{\mathcal{U}}\rangle$ is a Banach space. Actually, it is isometrically isomorphic to ℓ^∞ via the map $(a_n)_{n\in\mathbb{N}} \mapsto (\sum_{k=1}^{n} a_k)_{n\in\mathbb{N}}$.

3.7 Lemma. *Set $n_l := l^3$, $l \in \mathbb{N}_0$, and consider the linear map Λ which assigns to a sequence $d = (d_k)_{k\in\mathbb{N}}$ the sequence $\Lambda(d) = (\Lambda(d)_k)_{k\in\mathbb{N}}$ defined as ($d_0 := 0$)*

$$\Lambda(d)_k := \frac{d_l - d_{l-1}}{n_l - n_{l-1}}, \qquad n_{l-1} < k \leq n_l,\ l \in \mathbb{N}\,.$$

Then, whenever $d \in \ell^\infty$, we have (with $C := 2\sqrt{1+\frac{\pi^2}{18}}$)

$$\|\Lambda(d)\|_2 \leq C\|d\|_\infty, \quad \|\Lambda(d)\|_\infty \leq \|d\|_\infty, \quad \|\Lambda(d)\|_{\mathcal{U}} \leq \|d\|_\infty\,. \tag{3.6}$$

Proof. We compute

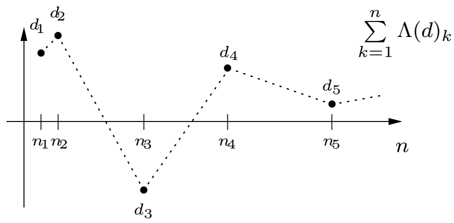
$$\sum_{k=1}^{\infty}|\Lambda(d)_k|^2=\sum_{l=1}^{\infty}(n_l-n_{l-1})\Big|\frac{d_l-d_{l-1}}{n_l-n_{l-1}}\Big|^2\leq 4\|d\|_\infty^2\sum_{l=1}^{\infty}\frac{1}{l^3-(l-1)^3}=$$

$$=4\|d\|_\infty^2\Big(1+\sum_{l=1}^{\infty}\frac{1}{3l^2-3l+1}\Big)\leq 4\big(1+\frac{\pi^2}{18}\big)\,\|d\|_\infty^2\,,$$

and this is the first inequality in (3.6). The second one is obvious since $\Lambda(d)_1=d_1$ and $n_l-n_{l-1}\geq 2$, $l>1$. To see the last inequality, note that

$$\sum_{k=1}^{n_l}\Lambda(d)_k=d_l,\qquad l\in\mathbb{N}\,,$$

and that the number $\sum_{k=1}^{n}\Lambda(d)_k$ lies on the line segment joining $d_{n_{l-1}}$ and d_{n_l} when $l\in\mathbb{N}$ is chosen such that $n_{l-1}\leq n\leq n_l$.



❑

Proof (of Theorem 3.2). The fact that $(i)\Rightarrow(ii)$ is clear. Let us assume that (ii) holds.

Step 1, Extracting a sparse sequence $(v_k)_{k\in\mathbb{N}}$: Let $\delta>0$ be chosen according to (3.2), and let $(w_k)_{k\in\mathbb{N}}$ be a sequence of points with $\operatorname{Im}w_k\geq\delta$, $|w_k|\geq 1$, $|w_k|<|w_{k+1}|$, $|w_k|\to\infty$, and $\lim_{k\to\infty}|\Theta(w_k)|=0$. By Lemma 3.6, we may extract a subsequence $(w_{k(n)})_{n\in\mathbb{N}}$ such that the normalized kernel functions $\tilde{K}(w_{k(n)},.)$, $n\in\mathbb{N}$, form a Riesz sequence in $\mathcal{H}$. Note that therefore also each subsequence of this sequence of functions is a Riesz sequence. Since $\lim_{k\to\infty}|\Theta(w_k)|=0$, we may extract a subsequence $(w_{k(n(l))})_{l\in\mathbb{N}}$ such that $\sum_{l=1}^{\infty}|\Theta(w_{k(n(l))})|\leq\frac{1}{4}$. We thus have found a sequence $(u_k)_{k\in\mathbb{N}}$ with the following properties

- (i) $\operatorname{Im}u_k\geq\delta>0$, $|u_k|\geq 1$, $|u_k|<|u_{k+1}|$, $k\in\mathbb{N}$, $|u_k|\to\infty$;
- (ii) The sequence $(\tilde{K}(u_k,.))_{k\in\mathbb{N}}$ is a Riesz sequence in $\mathcal{H}$;
- (iii) $\sum_{n=1}^{\infty}|\Theta(u_n)|\leq\frac{1}{4}$.

A straightforward induction shows that we may extract yet another subsequence $(v_k)_{k\in\mathbb{N}}$ of $(u_k)_{k\in\mathbb{N}}$ which satisfies

- (iv) $\sum_{n=1}^{k}|v_n|\leq\frac{1}{8}|v_kv_{k+1}|^{\frac{1}{2}}$, $k\in\mathbb{N}$;
- (v) $\sum_{n=l}^{k}\frac{1}{|v_n|}<\frac{1}{8}|v_{l-1}v_l|^{-\frac{1}{2}}$, $l=2,\dots,k$; and hence, letting $k\to\infty$,

$$\sum_{n=l}^{\infty}\frac{1}{|v_n|}\leq\frac{1}{8}|v_{l-1}v_l|^{-\frac{1}{2}},\quad l\geq 2\,.$$

Since each of the properties (i)–(iii) remains valid when passing to subsequences, the sequence $(v_k)_{k\in\mathbb{N}}$ satisfies (i)–(v).

Step 2, Definition of Ψ*:* Since $(\tilde{K}(v_k,.))_{k\in\mathbb{N}}$ is a Riesz sequence in $\mathcal{H}$, the map ρ defined as

$$\rho:(a_k)_{k\in\mathbb{N}}\mapsto\sum_{k=1}^{\infty}a_k\tilde{K}(v_k,.) \tag{3.7}$$

induces a bicontinuous embedding of ℓ^2 into $\mathcal{H}$. Set

$$\mu_k:=i\Big(\pi\frac{1-|\Theta(v_k)|^2}{\operatorname{Im}v_k}\Big)^{1/2}\frac{|E(v_k)|}{\overline{E(v_k)}},\qquad k\in\mathbb{N}.$$

Since $\operatorname{Im}v_k\geq\delta>0$, $k\in\mathbb{N}$, the sequence $(\mu_k)_{k\in\mathbb{N}}$ belongs to ℓ^∞. Thus the multiplication operator

$$\mu:(a_k)_{k\in\mathbb{N}}\mapsto(\mu_k a_k)_{k\in\mathbb{N}} \tag{3.8}$$

induces a bounded operator of ℓ^2 into itself. Finally, let Λ be the operator constructed in Lemma 3.7 considered as an element of $\mathcal{B}(\ell^\infty,\ell^2)$, and set

$$\Psi:=\rho\circ\mu\circ\Lambda.$$

Then $\Psi\in\mathcal{B}(\ell^\infty,\mathcal{H})$.

Step 3, $\Psi\in\mathcal{B}(\ell^\infty,R_{\mathfrak{m}}(\mathcal{H}))$*:* Let $d\in\ell^\infty$. We have, by our choice of μ_k,

$$\begin{aligned}\Psi(d)(z)&=\sum_{n=1}^{\infty}i\Big(\pi\frac{1-|\Theta(v_n)|^2}{\operatorname{Im}v_n}\Big)^{1/2}\frac{|E(v_n)|}{\overline{E(v_n)}}\Lambda(d)_n\cdot\tilde{K}(v_n,z)=\\&=\sum_{n=1}^{\infty}i\Big(\pi\frac{1-|\Theta(v_n)|^2}{\operatorname{Im}v_n}\Big)^{1/2}\frac{|E(v_n)|}{\overline{E(v_n)}}\Lambda(d)_n\cdot\Big[\Big(\frac{|E(v_n)|^2-|E(\overline{v_n})|^2}{4\pi\operatorname{Im}v_n}\Big)^{-1/2}\cdot\\&\qquad\cdot\frac{E(z)\overline{E(v_n)}-E(\overline{v_n})E^\#(z)}{2\pi i(\overline{v_n}-z)}\Big]=E(z)\sum_{n=1}^{\infty}\Lambda(d)_n\frac{1-\overline{\Theta(v_n)}\Theta(z)}{\overline{v_n}-z},\end{aligned} \tag{3.9}$$

and hence

$$\frac{\Psi(d)(z)}{E(z)}=\sum_{n=1}^{\infty}\Lambda(d)_n\frac{1-\overline{\Theta(v_n)}\Theta(z)}{\overline{v_n}-z}=\sum_{n=1}^{\infty}\Big[\frac{\Lambda(d)_n}{\overline{v_n}-z}-\Lambda(d)_n\overline{\Theta(v_n)}\frac{\Theta(z)}{\overline{v_n}-z}\Big].$$

Let $y\in[|v_1|,\infty)$ be given, and choose $k\in\mathbb{N}$ such that $|v_k|\leq y\leq|v_{k+1}|$. Using (3.6) and the properties (iii), (iv), (v) of $(v_k)_{k\in\mathbb{N}}$, we obtain the estimates below. To see these estimates, remember that $v_n\in\mathbb{C}^+$ and hence $|\overline{v_n}-iy|\geq\max\{|v_n|,y\}$.

$$\begin{aligned}\Big|\sum_{n=1}^{k-1}\frac{\Lambda(d)_n}{\overline{v_n}-iy}\Big|&=\Big|\sum_{n=1}^{k-1}\frac{\Lambda(d)_n}{iy}\Big(-1+\frac{\overline{v_n}}{\overline{v_n}-iy}\Big)\Big|\leq\\&\leq\frac{1}{y}\|\Lambda(d)\|_{\mathcal{U}}+\frac{\|\Lambda(d)\|_\infty}{y^2}\underbrace{\sum_{n=1}^{k-1}|v_n|}_{\leq\frac{1}{8}|v_{k-1}v_k|^{\frac{1}{2}}\leq\frac{y}{8}}\leq\frac{9}{8y}\|d\|_\infty,\end{aligned}$$

$$\Big|\frac{\Lambda(d)_k}{\overline{v_k}-iy}+\frac{\Lambda(d)_{k+1}}{\overline{v_{k+1}}-iy}\Big|\leq\frac{2}{y}\|\Lambda(d)\|_\infty\leq\frac{2}{y}\|d\|_\infty\,,$$

$$\Big|\sum_{n=k+2}^\infty\frac{\Lambda(d)_n}{\overline{v_n}-iy}\Big|\leq\|\Lambda(d)\|_\infty\underbrace{\sum_{n=k+2}^\infty\frac{1}{|v_n|}}_{\leq\frac{1}{8}|v_{k+1}v_{k+2}|^{-\frac{1}{2}}\leq\frac{1}{8y}}\leq\frac{1}{8y}\|d\|_\infty\,,$$

$$\Big|\sum_{n=1}^\infty\Lambda(d)_n\overline{\Theta(v_n)}\frac{\Theta(iy)}{\overline{v_n}-iy}\Big|\leq\|\Lambda(d)\|_\infty\frac{1}{y}\sum_{n=1}^\infty|\Theta(v_n)|\leq\frac{1}{4y}\|d\|_\infty\,.$$

Altogether, it follows that

$$y\Big|\frac{\Psi(d)(iy)}{E(iy)}\Big|\leq\frac{7}{2}\|d\|_\infty,\qquad y\in[|v_1|,\infty)\,.$$

Consider the majorant $\tilde{\mathfrak{m}}(z):=\frac{|E(z)|}{|z|}$ defined on the ray $\tilde{D}:=i[|v_1|,\infty)$. The above estimate, together with the already known fact that $\Psi\in\mathcal{B}(\ell^\infty,\mathcal{H})$, implies that $\Psi(\ell^\infty)\subseteq R_{\tilde{\mathfrak{m}}}(\mathcal{H})$ and $\Psi\in\mathcal{B}(\ell^\infty,\langle R_{\tilde{\mathfrak{m}}}(\mathcal{H}),\|.\|_{\tilde{\mathfrak{m}}}\rangle)$.

Since the domains of $\mathfrak{m}$ and $\tilde{\mathfrak{m}}$ differ only by a bounded set, and $\tilde{\mathfrak{m}}\asymp\mathfrak{m}|_{\tilde{D}}$, we have $R_{\mathfrak{m}}(\mathcal{H})=R_{\tilde{\mathfrak{m}}}(\mathcal{H})$ and the norms $\|.\|_{\mathfrak{m}}$ and $\|.\|_{\tilde{\mathfrak{m}}}$ are equivalent. We conclude that

$$\Psi\in\mathcal{B}\big(\ell^\infty,\langle R_{\mathfrak{m}}(\mathcal{H}),\|.\|_{\mathfrak{m}}\rangle\big)\,.$$

Step 4, Ψ is bicontinuous: Let $n_l=l^3$, cf. Lemma 3.7, and consider the values $y_k:=|v_{n_k}v_{n_k+1}|^{\frac{1}{2}}$, $k\in\mathbb{N}$. Then, for each sequence $d\in\ell^\infty$,

$$\frac{\Psi(d)(iy_k)}{E(iy_k)}=-\frac{1}{iy_k}\underbrace{\sum_{n=1}^{n_k}\Lambda(d)_n}_{=d_k}+\sum_{n=1}^{n_k}\Lambda(d)_n\frac{1}{iy_k}\frac{\overline{v_n}}{\overline{v_n}-iy_k}+$$
$$+\sum_{n=n_k+1}^\infty\frac{\Lambda(d)_n}{\overline{v_n}-iy_k}-\sum_{n=1}^\infty\Lambda(d)_n\overline{\Theta(v_n)}\frac{\Theta(iy_k)}{\overline{v_n}-iy_k}\,.$$

However, we estimate

$$\Big|\sum_{n=1}^{n_k}\Lambda(d)_n\frac{1}{iy_k}\frac{\overline{v_n}}{\overline{v_n}-iy_k}\Big|\leq\frac{\|\Lambda(d)\|_\infty}{y_k^2}\underbrace{\sum_{n=1}^{n_k}|v_n|}_{\leq\frac{1}{8}|v_{n_k}v_{n_k+1}|^{\frac{1}{2}}=\frac{y_k}{8}}\leq\frac{1}{8y_k}\|d\|_\infty\,,$$

$$\Big|\sum_{n=n_k+1}^\infty\frac{\Lambda(d)_n}{\overline{v_n}-iy}\Big|\leq\|\Lambda(d)\|_\infty\underbrace{\sum_{n=n_k+1}^\infty\frac{1}{|v_n|}}_{\leq\frac{1}{8}|v_{n_k}v_{n_k+1}|^{-\frac{1}{2}}=\frac{1}{8y_k}}\leq\frac{1}{8y_k}\|d\|_\infty\,,$$

$$\Big|\sum_{n=1}^\infty\Lambda(d)_n\overline{\Theta(v_n)}\frac{\Theta(z)}{\overline{v_n}-z}\Big|\leq\|\Lambda(d)\|_\infty\frac{1}{y_k}\sum_{n=1}^\infty|\Theta(v_n)|\leq\frac{1}{4y_k}\|d\|_\infty\,,$$

and it follows that

$$\|\Psi(d)\|_{\tilde{\mathfrak{m}}} \geq \frac{\Psi(d)(iy_k)}{\tilde{\mathfrak{m}}(iy_k)} \geq |d_k| - \frac{1}{2}\|d\|_\infty .$$

Taking the supremum over $k \in \mathbb{N}$, yields $\|\Psi(d)\|_{\tilde{\mathfrak{m}}} \geq \frac{1}{2}\|d\|_\infty$. ❑

With the help of Lemma 3.6 we can also show that $R_{\mathfrak{m}}(\mathcal{H})$ may contain infinite dimensional closed subspaces $\mathcal{M}$ on which the norms $\|.\|_{\mathfrak{m}}$ and $\|.\|_{\mathcal{H}}$ are equivalent, even if $R_{\mathfrak{m}}(\mathcal{H})$ itself is not $\|.\|_{\mathcal{H}}$-closed.

3.8 Proposition. *Let $\mathcal{H} = \mathcal{H}(E)$ be a de Branges space, and set $\mathfrak{m} := \mathfrak{m}_E|_{i[1,\infty)}$. Moreover, denote $\Theta := E^{-1}E^{\#}$. Assume that there exists a sequence $(w_k)_{k\in\mathbb{N}}$, $w_k \in \mathbb{C}^+$, such that*

$$\liminf_{k\in\mathbb{N}} \operatorname{Im} w_k = 0 \quad \textit{and} \quad \sup_{k\in\mathbb{N}} |\Theta(w_k)| < 1 .$$

Then there exists an infinite dimensional subspace $\mathcal{M}$ of $R_{\mathfrak{m}}(\mathcal{H})$ which is $\|.\|_{\mathcal{H}}$-closed. In particular, the norms $\|.\|_{\mathfrak{m}}|_{\mathcal{M}}$ and $\|.\|_{\mathcal{H}}|_{\mathcal{M}}$ are equivalent.

Proof. The normalized kernel function $\tilde{K}(w,.)$, $w \in \mathbb{C}^+$, is explicitly given as

$$\tilde{K}(w,z) = \Big(\frac{|E(w)|^2 - |E(\overline{w})|^2}{4\pi \operatorname{Im} w}\Big)^{-\frac{1}{2}} \frac{E(z)\overline{E(w)} - E(\overline{w})E^{\#}(z)}{2\pi i(\overline{w} - z)} =$$

$$= E(z)\frac{1-\overline{\Theta(w)}\Theta(z)}{i\sqrt{\pi}(\overline{w}-z)}\Big(\frac{\operatorname{Im} w}{1-|\Theta(w)|^2}\Big)^{\frac{1}{2}} \frac{\overline{E(w)}}{|E(w)|} .$$

We obtain the estimate ($\delta := \sup_{n\in\mathbb{N}} |\Theta(w_n)|$)

$$\Big|\frac{\tilde{K}(w_n,iy)}{\mathfrak{m}(iy)}\Big| \leq \frac{2}{\sqrt{\pi}} \frac{y+1}{y} \frac{\sqrt{\operatorname{Im} w_n}}{1-\delta^2}, \quad y \geq 1 .$$

Hence $\tilde{K}(w_n,.) \in R_{\mathfrak{m}}(\mathcal{H})$ and

$$\Big\|\frac{\tilde{K}(w_n,.)}{\mathfrak{m}}\Big\|_\infty \leq \frac{4\sqrt{\operatorname{Im} w_n}}{\sqrt{\pi}(1-\delta^2)} .$$

Since $\liminf_{n\to\infty} \operatorname{Im} w_n = 0$, we can extract a subsequence $(v_n)_{n\in\mathbb{N}}$ of $(w_n)_{n\in\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \operatorname{Im} v_n < \infty$. Moreover, by Lemma 3.6, the choice of $(v_n)_{n\in\mathbb{N}}$ can be made such that $(\tilde{K}(v_k,.))_{k\in\mathbb{N}}$, is a Riesz sequence. Consider the bicontinuous embedding $\rho : \ell^2 \to \mathcal{H}$ defined as in (3.7). Since, the sequence $(\|\mathfrak{m}^{-1}\tilde{K}(v_n,.)\|_\infty)_{n\in\mathbb{N}}$ is by our choice of $(v_n)_{n\in\mathbb{N}}$ square summable, for each sequence $a = (a_n)_{n\in\mathbb{N}} \in \ell^2$

$$\Big\|\frac{1}{\mathfrak{m}}\sum_{n\in\mathbb{N}} a_n\tilde{K}_n\Big\|_\infty \leq \sum_{n\in\mathbb{N}} |a_n| \cdot \Big\|\frac{\tilde{K}_n}{\mathfrak{m}}\Big\|_\infty \leq \|a\|_2 \cdot \big\|(\|\mathfrak{m}^{-1}\tilde{K}_n\|_\infty)_{n\in\mathbb{N}}\big\|_2 < \infty .$$

Thus every element of $\mathcal{M} := \operatorname{clos}_{\mathcal{H}} \operatorname{span}\{\tilde{K}(v_n,.) : n \in \mathbb{N}\}$ belongs to $R_{\mathfrak{m}}(\mathcal{H})$. Since $\|.\|_{\mathfrak{m}} \geq \|.\|_{\mathcal{H}}$, this implies that $\|.\|_{\mathfrak{m}}|_{\mathcal{M}}$ and $\|.\|_{\mathcal{H}}|_{\mathcal{M}}$ are equivalent. ❑

Let us explicitly state the following observation, which we obtain from Proposition 3.8 in conjunction with Theorem 3.2.

3.9 Corollary. *Let $\mathcal{H} = \mathcal{H}(E)$ be a de Branges space, and set $\mathfrak{m} := \mathfrak{m}_E|_{i[1,\infty)}$. Denote by $(w_k)_{k\in\mathbb{N}}$ the sequence of zeros of $E^\#$, and assume that*

$$\liminf_{k\to\infty} \operatorname{Im} w_k = 0, \quad \limsup_{k\to\infty} \operatorname{Im} w_k > 0 \,.$$

Then the space $\langle R_\mathfrak{m}(\mathcal{H}), \|.\|_\mathfrak{m}\rangle$ contains two closed infinite dimensional subspaces $\mathcal{M}_1$, $\mathcal{M}_2$, such that $\|.\|_\mathfrak{m}|_{\mathcal{M}_1}$ is equivalent to the Hilbert space norm $\|.\|_\mathcal{H}|_{\mathcal{M}_1}$, and $\langle \mathcal{M}_2, \|.\|_\mathfrak{m}|_{\mathcal{M}_2}\rangle$ is bicontinuously isomorphic to $\langle \ell^\infty, \|.\|_\infty\rangle$. ❑

b. Majorization along the real line.

In this subsection we study majorization on $D := \mathbb{R}$. It turns out that in this case the situation is different; the geometry of $R_{\mathfrak{m}_{E_1}|_D}(\mathcal{H})$ is always complicated.

3.10 Theorem. *Let $\mathcal{H}$ be a de Branges space, let $\mathcal{L} = \mathcal{H}(E_1) \in \operatorname{Sub}^* \mathcal{H}$, and set $\mathfrak{m} := \mathfrak{m}_{E_1}|_\mathbb{R}$. Then there exists a bicontinuous embedding of $\langle \ell^\infty, \|.\|_\infty\rangle$ into $\langle R_\mathfrak{m}(\mathcal{H}) \cap \mathcal{L}, \|.\|_\mathfrak{m}\rangle$. In particular, $R_\mathfrak{m}(\mathcal{H})$ is neither separable nor reflexive.*

The basic idea for the proof of this result is similar to the one which led to Theorem 3.2. Still, in some details, one has to argue differently, cf. Remark 3.12 below. We have to consider separately the cases where the zeros w_k of $\Theta := E_1^{-1} E^\#$ approach the real axis or are contained in some Stolz angle.

Proof (of Theorem 3.10. Case 1: zeros approaching the real axis.) Assume that

$$\liminf_{k\to\infty, \operatorname{Re} w_k > 0} \frac{\operatorname{Im} w_k}{|w_k|} = 0.$$

Let $(w_{k(n)})_{n\in\mathbb{N}}$ be a subsequence of $(w_k)_{k\in\mathbb{N}}$ with $\operatorname{Re} w_{k(n)} > 0$ and $\lim_{n\to\infty} |w_n|^{-1} \operatorname{Im} w_n = 0$. From this sequence we can extract yet another subsequence $(v_k)_{k\in\mathbb{N}}$ which satisfies

- (i) $|v_k + i| \leq 2 \operatorname{Re} v_k$, $k \in \mathbb{N}$,
- (ii) $\operatorname{Re} v_{k+1} > 2 \operatorname{Re} v_k$, $k \in \mathbb{N}$,
- (iii) $\sum_{n=1}^\infty \frac{\operatorname{Im} v_n}{|v_n + i|} \leq \frac{1}{24}$.

Denote by $K_1(w, z)$ the reproducing kernel of $\mathcal{L}$, and set $\tilde{K}_1(w, .) := \|K_1(w, .)\|_\mathcal{H}^{-1} K_1(w, .)$. The functions $\tilde{K}_1(v_k, .)$ are normalized and belong to the subspace $\mathcal{L}$. Hence the map ρ defined by (3.7), using $\tilde{K}_1(v_k, .)$ in place of $\tilde{K}(v_k, .)$, maps the space ℓ^1 contractively into $\mathcal{L}$. Set

$$\gamma_k := \frac{\operatorname{Im} v_k}{|v_k + i|}, \qquad k \in \mathbb{N} \,.$$

Then, by (iii), the multiplication operator $\gamma : (a_k)_{k\in\mathbb{N}} \mapsto (\gamma_k a_k)_{k\in\mathbb{N}}$ maps ℓ^∞ boundedly into ℓ^1. Finally, the multiplication operator μ defined in (3.8) maps ℓ^∞ boundedly into itself. Altogether, we have

$$\Psi_1 := \rho \circ \gamma \circ \mu \in \mathcal{B}(\ell^\infty, \mathcal{H}) \,.$$

Note that, as in (3.9),

$$\frac{\Psi_1(c)(z)}{E_1(z)}=\sum_{n=1}^{\infty}\frac{c_n\gamma_n}{\overline{v_n}-z},\qquad c=(c_k)_{k\in\mathbb{N}}\in\ell^\infty .$$

In order to estimate $\mathfrak{m}$-norms let some point $x\in\mathbb{R}\setminus(-\operatorname{Re}v_1,\operatorname{Re}v_1)$ be given. If $x\le-\operatorname{Re}v_1\le 0$, clearly $|E_1^{-1}(x)\Psi_1(c)(x)|\le\frac{\|c\|_\infty}{|x|}\sum_{n=1}^\infty\gamma_n\le\frac{\|c\|_\infty}{24|x|}$. Next, note that

$$\frac{1}{|\overline{v_n}-x|}\le\begin{cases}\frac{1}{\operatorname{Re}v_n-x}\le\frac{1}{x}, & x\le\frac12\operatorname{Re}v_n,\\ \frac{1}{\operatorname{Re}v_k-\operatorname{Re}v_{k-1}}\le\frac{2}{\operatorname{Re}v_k}\le\frac{4}{x}, & n\le k-1,\ \operatorname{Re}v_k\le x\le 2\operatorname{Re}v_k,\\ \frac{1}{x-\operatorname{Re}v_n}\le\frac{2}{x}, & 2\operatorname{Re}v_n\le x,\end{cases}$$

$$\frac{1}{|v_n+i|}\le\frac{1}{\operatorname{Re}v_n}\le\begin{cases}\frac1x, & x\le\operatorname{Re}v_n,\\ \frac2x, & x\le 2\operatorname{Re}v_n.\end{cases}$$

Using these inequalities, it follows that

$$\Big|\sum_{n=1}^{\infty}\frac{\gamma_n}{\overline{v_n}-x}\Big|\le\sum_{n=1}^{k-1}\frac{\gamma_n}{|\overline{v_n}-x|}+\frac{1}{|\overline{v_k}-x|}\frac{\operatorname{Im}v_k}{|v_k+i|}+\frac{1}{|\overline{v_{k+1}}-x|}\frac{\operatorname{Im}v_{k+1}}{|v_{k+1}+i|}$$
$$+\sum_{n=k+2}^{\infty}\frac{\gamma_n}{|\overline{v_n}-x|}\le\frac1x\Big(\frac{4}{24}+2+1+\frac{1}{24}\Big),\qquad\operatorname{Re}v_k\le x\le\operatorname{Re}v_{k+1}\,.$$

It follows that $\Psi_1\in\mathcal{B}(\ell^\infty,R_{\tilde{\mathfrak{m}}}(\mathcal{H}))$ where $\tilde{\mathfrak{m}}$ is the majorant $\tilde{\mathfrak{m}}(x):=|x|^{-1}|E(x)|$ defined on $\tilde{D}:=\mathbb{R}\setminus(-\operatorname{Re}v_1,\operatorname{Re}v_1)$.

We can also obtain an estimate from below. Set $x_k:=\operatorname{Re}v_k$, $k\in\mathbb{N}$, then

$$\Big|\frac{\Psi_1(c)(x_k)}{E_1(x_k)}\Big|\ge\frac{|c_k|}{|\overline{v_k}-x_k|}\frac{\operatorname{Im}v_k}{|v_k+i|}-\|c\|_\infty\sum_{\substack{n=1\\n\ne k}}^{\infty}\frac{1}{|\overline{v_n}-x|}\frac{\operatorname{Im}v_n}{|v_n+i|}\ge$$

$$\ge\frac{|c_k|}{2\operatorname{Re}v_k}-\frac{\|c\|_\infty}{x_k}\Big(\frac{4}{24}+\underbrace{\frac{\operatorname{Im}v_{k+1}}{|v_{k+1}+i|}}_{\le\frac{1}{24}}+\frac{1}{24}\Big)\ge\frac{1}{x_k}\Big(\frac{|c_k|}{2}-\frac{\|c\|_\infty}{4}\Big).$$

Taking the supremum over all $k\in\mathbb{N}$ yields $\|\Psi_1(c)\|_{\tilde{\mathfrak{m}}}\ge\frac14\|c\|_\infty$. Since $R_{\tilde{\mathfrak{m}}}(\mathcal{H})=R_{\mathfrak{m}}(\mathcal{H})$ and the norms $\|.\|_{\tilde{\mathfrak{m}}}$ and $\|.\|_{\mathfrak{m}}$ are equivalent, we obtain that Ψ_1 maps ℓ^∞ bicontinuously into $\langle R_{\mathfrak{m}}(\mathcal{H})\cap\mathcal{L},\|.\|_{\mathfrak{m}}\rangle$. ❑

The case when $\liminf\limits_{k\to\infty,\operatorname{Re}w_k<0}\frac{\operatorname{Im}w_k}{|w_k|}=0$ is treated in a completely similar way, and we will therefore omit explicit proof.

Proof (of Theorem 3.10. Case 2: Zeros in a Stolz angle.) Assume that there exists $\alpha\in\big(0,\frac{\pi}{2}\big)$ such that $w_n\in\Gamma_\alpha$, $n\in\mathbb{N}$. We can extract a subsequence $(v_k)_{k\in\mathbb{N}}$ of $(w_k)_{k\in\mathbb{N}}$ with

(*i*) $\operatorname{Im}v_k\ge 1$, $\operatorname{Im}v_k<\operatorname{Im}v_{k+1}$, $k\in\mathbb{N}$, $\operatorname{Im}v_k\to\infty$,

(*ii*) $\sum_{n=1}^\infty(\operatorname{Im}v_k)^{-\frac12}\le 1$.

As seen by a straightforward induction, we may choose $(v_k)_{k\in\mathbb{N}}$ in such a way that moreover

(*iii*) $\sum_{n=1}^{k}|v_n| \leq \frac{\sin\alpha}{8}(\operatorname{Im} v_k \operatorname{Im} v_{k+1})^{\frac12}$,

(*iv*) $\sum_{n=l}^{k}\frac{1}{\operatorname{Im} v_k} < \frac18(\operatorname{Im} v_{l-1}\operatorname{Im} v_l)^{-\frac12}$, $l=2,\ldots,k$; and hence, letting $k\to\infty$,

$$\sum_{n=l}^{\infty}\frac{1}{\operatorname{Im} v_k} \leq \frac18(\operatorname{Im} v_{l-1}\operatorname{Im} v_l)^{-\frac12}, \quad l\geq 2\,.$$

Again the map ρ defined by (3.7) with $\tilde K_1(v_k,.)$ is a contraction of ℓ^1 into $\mathcal{L}$. Let μ be the multiplication operator defined in (3.8). By (*ii*) we have $(\mu_k)_{k\in\mathbb{N}}\in\ell^1$, and hence $\mu\in\mathcal{B}(\ell^\infty,\ell^1)$. Finally, let Λ be the map defined in Lemma 3.7, then Λ maps ℓ^∞ contractively into itself. Consider the map

$$\Psi := \rho\circ\mu\circ\Lambda\in\mathcal{B}(\ell^\infty,\mathcal{L})\,.$$

We have to estimate $\mathfrak{m}$-norms. Note that always $|\overline{v_n}-x|\geq\max\{|x|\sin\alpha,\operatorname{Im} v_n\}$. Let $x\in\mathbb{R}\setminus(-\operatorname{Im} v_1,\operatorname{Im} v_1)$ be given, and choose $k\in\mathbb{N}$ such that $\operatorname{Im} v_k\leq|x|\leq\operatorname{Im} v_{k+1}$. Then

$$\Big|\sum_{n=1}^{k-1}\frac{\Lambda(d)_n}{\overline{v_n}-x}\Big| = \Big|\sum_{n=1}^{k-1}\frac{\Lambda(d)_n}{x}\Big(-1+\frac{\overline{v_n}}{\overline{v_n}-x}\Big)\Big| \leq$$

$$\leq \frac{1}{|x|}\|\Lambda(d)\|_{\mathcal{U}} + \frac{\|\Lambda(d)\|_\infty}{|x|^2\sin\alpha}\underbrace{\sum_{n=1}^{k-1}|v_n|}_{\leq\frac{\sin\alpha}{8}(\operatorname{Im} v_{k-1}\operatorname{Im} v_k)^{\frac12}\leq\frac{|x|\sin\alpha}{8}} \leq \frac{9}{8|x|}\|d\|_\infty,$$

$$\Big|\frac{\Lambda(d)_k}{\overline{v_k}-x}+\frac{\Lambda(d)_{k+1}}{\overline{v_{k+1}}-x}\Big| \leq \frac{2\|\Lambda(d)\|_\infty}{|x|\sin\alpha} \leq \frac{2}{|x|\sin\alpha}\|d\|_\infty,$$

$$\Big|\sum_{n=k+2}^{\infty}\frac{\Lambda(d)_n}{\overline{v_n}-x}\Big| \leq \|\Lambda(d)\|_\infty\underbrace{\sum_{n=k+2}^{\infty}\frac{1}{\operatorname{Im} v_n}}_{\leq\frac18(\operatorname{Im} v_{k+1}\operatorname{Im} v_{k+2})^{-\frac12}\leq\frac{1}{8|x|}} \leq \frac{1}{8|x|}\|d\|_\infty.$$

Putting together these inequalities, it follows that

$$\Big|\frac{\Psi(d)(x)}{E(x)}\Big| \leq \Big(\frac54+\frac{2}{\sin\alpha}\Big)\frac{1}{|x|}\|d\|_\infty, \quad |x|\geq\operatorname{Im} v_1\,. \tag{3.10}$$

Consider the majorant $\tilde{\mathfrak{m}}(x):=|x|^{-1}|E(x)|$ defined on $\tilde D:=\mathbb{R}\setminus(-\operatorname{Im} v_1,\operatorname{Im} v_1)$. Then (3.10) says that $\Psi\in\mathcal{B}(\ell^\infty,\langle R_{\tilde{\mathfrak{m}}}(\mathcal{H})\cap\mathcal{L},\|.\|_{\tilde{\mathfrak{m}}}\rangle)$. Equivalence of norms implies

$$\Psi\in\mathcal{B}(\ell^\infty,\langle R_{\mathfrak{m}}(\mathcal{H})\cap\mathcal{L},\|.\|_{\mathfrak{m}}\rangle)\,.$$

Let $n_l := l^3$ as in Lemma 3.7, and set $x_k:=(\operatorname{Im} v_{n_k}\operatorname{Im} v_{n_k+1})^{\frac12}$. Then we can estimate

$$\Big|\sum_{n=1}^{n_k}\frac{1}{x_k}\frac{\Lambda(d)_n\overline{v_n}}{\overline{v_n}-x_k}\Big| \leq \frac{\|\Lambda(d)\|_\infty}{|x_k|^2\sin\alpha}\sum_{n=1}^{n_k}|v_n| \leq$$

$$\leq \frac{\|\Lambda(d)\|_\infty}{|x_k|^2\sin\alpha}\frac{\sin\alpha}{8}\big(\operatorname{Im} v_{n_k}\operatorname{Im} v_{n_k+1}\big)^{\frac12} \leq \frac{1}{8|x_k|}\|d\|_\infty$$

$$\Big|\sum_{n=n_k+1}^{\infty}\frac{\Lambda(d)_n}{\overline{v_n}-x}\Big|\leq\|\Lambda(d)\|_\infty\sum_{n=n_k+1}^{\infty}\frac{1}{\operatorname{Im}v_n}\leq\frac{1}{8|x_k|}\|d\|_\infty$$

It follows that

$$\Big|\frac{x_k\Psi(d)(x_k)}{E(x_k)}\Big|\geq\underbrace{\Big|\sum_{n=1}^{n_k}\Lambda(d)_n\Big|}_{=|d_k|}-\frac{\|d\|_\infty}{4},$$

and, taking the supremum over all $k\in\mathbb{N}$,

$$\|\Psi(d)\|_{\tilde{\mathfrak{m}}}\geq\frac{3}{4}\|d\|_\infty .$$

Again by equivalence of norms, this shows that Ψ maps ℓ^∞ bicontinuously into $\langle R_{\mathfrak{m}}(\mathcal{H})\cap\mathcal{L},\|.\|_{\mathfrak{m}}\rangle$. ❑

3.11 Remark. With similar arguments as used in Theorem 3.10 for the case when the zeros of Θ approach $\mathbb{R}$, one can prove the following statement: *Let $\mathcal{H}$ be a de Branges space and let $\mathcal{L}=\mathcal{H}(E_1)\in\operatorname{Sub}^*\mathcal{H}$, $\dim\mathcal{L}=\infty$. Moreover, let $D\subset\mathbb{C}^+\cup\mathbb{R}$ be such that*

$$\limsup_{\substack{|z|\to\infty\\ z\in D}}\frac{\operatorname{Im}z}{|z|}=0 .$$

Assume that there exists a sequence $(w_k)_{k\in\mathbb{N}}$, $w_k\in\mathbb{C}^+$, with

$$\lim_{k\to\infty}\frac{\operatorname{Im}w_k}{|w_k|}=0,\quad\sup_{n\in\mathbb{N}}|\Theta(w_n)|<1 .$$

Then there exists a bicontinuous embedding of ℓ^∞ into $R_{\mathfrak{m}}(\mathcal{H})\cap\mathcal{L}$.

We will not go into details.

3.12 Remark. It is interesting to analyze the construction of the respective embeddings in the proofs of Theorem 3.2, and Theorem 3.10, Case 1 / Case 2:

Theorem 3.2:	$\ell^\infty\xrightarrow{\Lambda}\ell^2\xrightarrow{\mu}\ell^2\xrightarrow{\rho}\mathcal{H}$
Theorem 3.10, Case 2:	$\ell^\infty\xrightarrow{\Lambda}\ell^\infty\xrightarrow{\mu}\ell^1\xrightarrow{\rho}\mathcal{H}$
Theorem 3.10, Case 1:	$\ell^\infty\xrightarrow{\mu}\ell^\infty\xrightarrow{\gamma}\ell^1\xrightarrow{\rho}\mathcal{H}$

In Theorem 3.10, Case 2, the rough argument that $\rho\in\mathcal{B}(\ell^1,\mathcal{H})$ was sufficient, whereas in Theorem 3.2 we had to use the much finer argument that we can extract a Riesz sequence of normalized kernels. This is necessary, since in Theorem 3.2 we only assume that $\operatorname{Im}w_n$ is bounded away from zero, and not that the points w_n are contained in some Stolz angle.

4 The majorant $\mathfrak{m}^\flat$

In this section we investigate the question in how many ways a given dB-subspace can be realized by majorization (provided it can be realized at all). In particular, we ask for "small" majorants which do the job. We will also view representability by different majorants from a refined viewpoint, requiring instead of $\mathcal{R}_{\mathfrak{m}_1}(\mathcal{H}) = \mathcal{R}_{\mathfrak{m}_2}(\mathcal{H})$ that $R_{\mathfrak{m}_1}(\mathcal{H}) = R_{\mathfrak{m}_2}(\mathcal{H})$ or even $B_{\mathfrak{m}_1}(\mathcal{H}) = B_{\mathfrak{m}_2}(\mathcal{H})$.

To start with, let us show by examples that in general there will actually exist many majorants $\mathfrak{m}$ generating the same dB-subspace $\mathcal{R}_{\mathfrak{m}}(\mathcal{H})$.

4.1 Example. Consider a Paley–Wiener space $\mathcal{P}W_a$, $a > 0$, and majorization on $D := \mathbb{R}$. For every $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{R}} \mathcal{P}W_a$ which is separated from zero on each compact interval, we have $\mathcal{R}_{\mathfrak{m}}(\mathcal{P}W_a) = \mathcal{P}W_a$, cf. [BW1, Corollary 3.12]. However, for each $\alpha \in (0,1)$, the function

$$\mathfrak{m}(x) := \exp(-|x|^\alpha), \qquad x \in \mathbb{R},$$

is an admissible majorant, cf. [BW1, Example 2.14].

This example relies on the fact that no proper dB-subspaces of $\mathcal{P}W_a$ can be obtained by majorization on the real line. The following example is of a different nature.

4.2 Example. Let $\mathcal{H}$ be a de Branges space and let $\mathcal{L} \in \operatorname{Sub}^* \mathcal{H}$, $\dim \mathcal{L} = \infty$, be such that $\mathcal{L} = \operatorname{clos}_{\mathcal{H}} \bigcup\{\mathcal{K} \in \operatorname{Sub}^* \mathcal{H} : \mathcal{K} \subsetneq \mathcal{L}\}$. We will consider majorization on $D := i[1,\infty)$, so we know that $\nabla_{\mathcal{L}}|_D$ belongs to $\operatorname{Adm}_D \mathcal{H}$ and $\mathcal{R}_{\nabla_{\mathcal{L}}|_D}(\mathcal{H}) = \mathcal{L}$.

By the Baire Category Theorem we have $\mathcal{L} \neq \bigcup\{\mathcal{K} \in \operatorname{Sub}^* \mathcal{H} : \mathcal{K} \subsetneq \mathcal{L}\}$. Note here that this union is actually equal to some at most countable union. Moreover, each dB-subspace is invariant with respect to $F \mapsto F^\#$. Hence we can choose $F \in \mathcal{L} \setminus \bigcup\{\mathcal{K} \in \operatorname{Sub}^* \mathcal{H} : \mathcal{K} \subsetneq \mathcal{L}\}$, $\|F\| = 1$, with $F = F^\#$.

It follows from a general argument that the function F has infinitely many zeros: Assume on the contrary that $w_1, \ldots, w_n$ are all the zeros of F (listed according to their multiplicities). Then the function

$$\tilde{F}(z) := F(z) \prod_{k=1}^{n} (z - w_k)^{-1}$$

belongs to $\mathcal{H}$, satisfies $\tilde{F}^\# = \tilde{F}$, and has no zeros. Thus $\operatorname{span}\{\tilde{F}\} \in \operatorname{Sub}^* \mathcal{H}$. Hence also each of the spaces

$$\mathcal{L}_m := \big\{p\tilde{F} : p \in \mathbb{C}[z], \deg p \leq m\big\} \cap \mathcal{H}, \qquad m \in \mathbb{N},\ m \leq n,$$

belongs to $\operatorname{Sub}^* \mathcal{H}$. By the construction of $\tilde{F}$, we have $F \in \mathcal{L}_n$. Thus $\mathcal{L}_n$ is not contained in any of the spaces $\mathcal{K} \in \operatorname{Sub}^* \mathcal{H}$, $\mathcal{K} \subsetneq \mathcal{L}$, and, by de Branges' Ordering Theorem, must therefore contain each of them. By our assumption, $\bigcup_{\mathcal{K} \subsetneq \mathcal{L}} \mathcal{K}$ is dense in $\mathcal{L}$, and it follows that also $\mathcal{L} \subseteq \mathcal{L}_n$. This contradicts the assumption that $\dim \mathcal{L} = \infty$.

Denote the sequence of zeros of F which lie in $\mathbb{C}^+ \cup \mathbb{R}$ by $(w_n)_{n\in\mathbb{N}}$, and let $B(z)$ be the Blaschke product build from the zeros w_n with positive imaginary part. Define

$$\mathfrak{m}_k(z) := \frac{|F(z)|}{|z+i|^k |B(z)|}, \qquad z \in D,\ k \in \mathbb{N}_0.$$

Clearly, we have $\mathfrak{m}_k > 0$ and $\mathfrak{m}_0 \geq \mathfrak{m}_1 \geq \mathfrak{m}_2 \geq \ldots$. Moreover, for each $k \in \mathbb{N}_0$,

$$F_k(z) := \frac{F(z)}{\prod_{n=1}^{k}(z - \overline{w}_n) \cdot B(z)} \in R_{\mathfrak{m}_k}(\mathcal{H}) \setminus R_{\mathfrak{m}_{k+1}}(\mathcal{H}).$$

We see that $\mathfrak{m}_k \in \operatorname{Adm}_D \mathcal{H}$. The same argument as used in [BW2, Lemma 3.5, (i)] gives

$$\dim \mathcal{R}_{\mathfrak{m}_k}(\mathcal{H})/\mathcal{R}_{\mathfrak{m}_{k-1}}(\mathcal{H}) \leq 1 .$$

Since each dB-subspace is closed with respect to multiplication by Blaschke products, the function F_0 does not belong to any of the spaces $\mathcal{K} \in \operatorname{Sub}^* \mathcal{H}$, $\mathcal{K} \subsetneq \mathcal{L}$. Hence, $\mathcal{R}_{\mathfrak{m}_0}(\mathcal{H}) \supseteq \mathcal{L}$. On the other hand, $F_0 \in \mathcal{L}$ and thus $\mathfrak{m}_0 \lesssim \nabla_{\mathcal{L}}$. This implies that $\mathcal{R}_{\mathfrak{m}_0}(\mathcal{H}) = \mathcal{L}$, and altogether we obtain $\mathcal{R}_{\mathfrak{m}_k}(\mathcal{H}) = \mathcal{L}$, $k \in \mathbb{N}_0$.

We have obtained a decreasing family $\mathfrak{m}_k$ of majorants with

$$\mathcal{R}_{\mathfrak{m}_k}(\mathcal{H}) = \mathcal{L} \text{ but } R_{\mathfrak{m}_{k+1}}(\mathcal{H}) \subsetneq R_{\mathfrak{m}_k}(\mathcal{H}), \qquad k \in \mathbb{N}_0 .$$

Let us remark moreover that $\lim_{k\to\infty} \mathfrak{m}_k(z) = 0$, $z \in D$, and thus also

$$\bigcap_{k\in\mathbb{N}} B_{\mathfrak{m}_k}(\mathcal{H}) = \{0\} .$$

In particular, these examples show that the space $R_{\mathfrak{m}}(\mathcal{H})$ might be very small, despite the fact that $\mathcal{R}_{\mathfrak{m}}(\mathcal{H})$ is always the same.

Let $\mathfrak{m} \in \operatorname{Adm} \mathcal{H}$ be given. Among all those majorants $\mathfrak{m}_1$ for which even $B_{\mathfrak{m}_1}(\mathcal{H}) = B_{\mathfrak{m}}(\mathcal{H})$ there is a natural "sharp" majorant.

4.3 Definition. Let $\mathcal{H}$ be a de Branges space and $\mathfrak{m} \in \operatorname{Adm} \mathcal{H}$. Define a function $\mathfrak{m}^\flat : \mathbb{C}^+ \cup \mathbb{R} \to [0,\infty)$ by

$$\mathfrak{m}^\flat(z) := \sup\big\{|F(z)| : F \in B_{\mathfrak{m}}(\mathcal{H})\big\}, \quad z \in \mathbb{C}^+ \cup \mathbb{R} .$$

4.4 Lemma. *Let $\mathcal{H}$ be a de Branges space, $D \subseteq \mathbb{C}^+ \cup \mathbb{R}$, and $\mathfrak{m} \in \operatorname{Adm}_D \mathcal{H}$. Then $\mathfrak{m}^\flat \in \operatorname{Adm} \mathcal{H}$ and*

$$B_{\mathfrak{m}^\flat}(\mathcal{H}) = B_{\mathfrak{m}^\flat|_D}(\mathcal{H}) = B_{\mathfrak{m}}(\mathcal{H}) .$$

Moreover, $\mathfrak{m}^\flat|_D \leq \mathfrak{m}$ and $\mathfrak{m}^{\flat\flat} = \mathfrak{m}^\flat$.

Proof. The inclusion $B_{\mathfrak{m}^\flat}(\mathcal{H}) \subseteq B_{\mathfrak{m}^\flat|_D}(\mathcal{H})$ is trivial. By the definition of $\mathfrak{m}^\flat$, we have $\mathfrak{m}^\flat|_D \leq \mathfrak{m}$, and therefore $B_{\mathfrak{m}^\flat|_D}(\mathcal{H}) \subseteq B_{\mathfrak{m}}(\mathcal{H})$. Let $F \in B_{\mathfrak{m}}(\mathcal{H})$ be given. Then $|F(z)| \leq \mathfrak{m}^\flat(z)$, $z \in \mathbb{C}^+ \cup \mathbb{R}$, and hence $F \in B_{\mathfrak{m}^\flat}(\mathcal{H})$. Finally, we have

$$\mathfrak{m}^{\flat\flat}(z) = \sup_{F\in B_{\mathfrak{m}^\flat}(\mathcal{H})} |F(z)| = \sup_{F\in B_{\mathfrak{m}}(\mathcal{H})} |F(z)| = \mathfrak{m}^\flat(z) .$$

❑

4.5 Remark. We see that, given $\mathfrak{m} \in \operatorname{Adm}_D \mathcal{H}$, the majorant $\mathfrak{m}^\flat$ is the smallest among all majorants $\mathfrak{m}_1 \in \operatorname{Adm} \mathcal{H}$ with $B_{\mathfrak{m}_1}(\mathcal{H}) = B_{\mathfrak{m}}(\mathcal{H})$.

Lemma 4.4 can often be used to reduce considerations to majorants of the form $\mathfrak{m}^\flat$. In view of this, it is worth mentioning that the majorant $\mathfrak{m}^\flat$ is always fairly smooth and preserves real zeros as well as exponential growth.

4.6 Proposition. *Let $\mathcal{H}$ be a de Branges space, $D \subseteq \mathbb{C}^+ \cup \mathbb{R}$, and $\mathfrak{m} \in \operatorname{Adm}_D \mathcal{H}$. Then $\mathfrak{m}^\flat$ is continuous on $\mathbb{C}^+ \cup \mathbb{R}$ and $\log \mathfrak{m}^\flat$ is subharmonic on $\mathbb{C}^+$. Moreover, we have $\mathfrak{d}_{\mathfrak{m}^\flat} = \max\{\mathfrak{d}_{\mathfrak{m}}, \mathfrak{d}_{\mathcal{H}}\}$ and $\operatorname{mt}_{\mathcal{H}} \mathfrak{m}^\flat = \operatorname{mt}_{\mathcal{H}} \mathcal{R}_{\mathfrak{m}}(\mathcal{H})$.*

Proof. The unit ball of $\mathcal{H}$ is a locally bounded, and thus normal, family of entire functions. This shows that the subset $B_\mathfrak{m}(\mathcal{H})$ is also a normal family of entire functions, and thus equicontinuous. This implies that $\mathfrak{m}^\flat$ is continuous. The function $\log \mathfrak{m}^\flat$ is the supremum of the subharmonic functions $\log|F(z)|$, $F \in B_\mathfrak{m}(\mathcal{H})$. Since $\log \mathfrak{m}^\flat$ is continuous on $\mathbb{C}^+$, this implies that $\log \mathfrak{m}^\flat$ is subharmonic on $\mathbb{C}^+$.

Let $w \in \mathbb{C}$, $n \in \mathbb{N}_0$, be such that $n < \mathfrak{d}_\mathcal{H}(w)$. Then

$$\Big\{\frac{F(z)}{(z-w)^n} : F \in B_\mathfrak{m}(\mathcal{H})\Big\}$$

is a normal family of entire functions, and hence equicontinuous. Thus, given $\epsilon > 0$, there exists $r > 0$ such that

$$\Big|\frac{F(z)}{(z-w)^n}\Big| \leq \epsilon, \quad |z-w| \leq r,\ F \in B_\mathfrak{m}(\mathcal{H}) .$$

This implies that also $|z-w|^{-n}\mathfrak{m}^\flat(z) \leq \epsilon$ for $0 < |z-w| \leq r$, and hence that $n < \mathfrak{d}_{\mathfrak{m}^\flat}(w)$. We conclude that $\mathfrak{d}_{\mathfrak{m}^\flat}(w) \geq \mathfrak{d}_\mathcal{H}(w)$, $w \in \mathbb{C}$. Now we obtain from (A.8) that

$$\mathfrak{d}_{\mathfrak{m}^\flat} = \max\{\mathfrak{d}_{\mathfrak{m}^\flat}, \mathfrak{d}_\mathcal{H}\} = \mathfrak{d}_{\mathcal{R}_{\mathfrak{m}^\flat}(\mathcal{H})} = \mathfrak{d}_{\mathcal{R}_\mathfrak{m}(\mathcal{H})} = \max\{\mathfrak{d}_\mathfrak{m}, \mathfrak{d}_\mathcal{H}\} .$$

Since $B_\mathfrak{m}(\mathcal{H})$ is contained in the unit ball (with respect to the norm of $\mathcal{H}$) of $\mathcal{R}_\mathfrak{m}(\mathcal{H})$, we have $\mathfrak{m}^\flat \leq \nabla_{\mathcal{R}_\mathfrak{m}(\mathcal{H})}$. Hence, by (A.8), $\operatorname{mt}_\mathcal{H} \mathfrak{m}^\flat \leq \operatorname{mt}_\mathcal{H} \mathcal{R}_\mathfrak{m}(\mathcal{H})$. Conversely, by (2.2),

$$\operatorname{mt}_\mathcal{H} \mathcal{R}_\mathfrak{m}(\mathcal{H}) = \sup_{F \in R_\mathfrak{m}(\mathcal{H})} \operatorname{mt}_\mathcal{H} F = \sup_{F \in B_\mathfrak{m}(\mathcal{H})} \operatorname{mt}_\mathcal{H} F \leq \operatorname{mt}_\mathcal{H} \mathfrak{m}^\flat .$$

Thus $\operatorname{mt}_\mathcal{H} \mathfrak{m}^\flat \geq \operatorname{mt}_\mathcal{H} \mathcal{R}_\mathfrak{m}(\mathcal{H})$. ❑

The next result says that equality of two spaces $R_{\mathfrak{m}_1}(\mathcal{H})$ and $R_{\mathfrak{m}_2}(\mathcal{H})$ can be characterized via $\mathfrak{m}_1^\flat$ and $\mathfrak{m}_2^\flat$. It is again a consequence of the completeness of $R_\mathfrak{m}(\mathcal{H})$.

4.7 Proposition. *Let $\mathcal{H}$ be a de Branges space, $D \subseteq \mathbb{C}^+ \cup \mathbb{R}$, and $\mathfrak{m}_1, \mathfrak{m}_2 \in \operatorname{Adm}_D \mathcal{H}$. Then the following equivalences hold:*

$$\begin{aligned} R_{\mathfrak{m}_1}(\mathcal{H}) \subseteq R_{\mathfrak{m}_2}(\mathcal{H}) &\iff \exists\, \lambda > 0 : B_{\mathfrak{m}_1}(\mathcal{H}) \subseteq \lambda B_{\mathfrak{m}_2}(\mathcal{H}) \\ &\iff \mathfrak{m}_1^\flat \lesssim \mathfrak{m}_2^\flat; \end{aligned} \tag{4.1}$$

$$\begin{aligned} R_{\mathfrak{m}_1}(\mathcal{H}) = R_{\mathfrak{m}_2}(\mathcal{H}) &\iff R_{\mathfrak{m}_1}(\mathcal{H}) = R_{\mathfrak{m}_2}(\mathcal{H}) \text{ and } \|.\|_{\mathfrak{m}_1} \asymp \|.\|_{\mathfrak{m}_2} \\ &\iff \exists\, \lambda_1, \lambda_2 > 0 : \lambda_1 B_{\mathfrak{m}_1}(\mathcal{H}) \subseteq B_{\mathfrak{m}_2}(\mathcal{H}) \subseteq \lambda_2 B_{\mathfrak{m}_1}(\mathcal{H}) \\ &\iff \mathfrak{m}_1^\flat \asymp \mathfrak{m}_2^\flat; \end{aligned} \tag{4.2}$$

$$\begin{aligned} B_{\mathfrak{m}_1}(\mathcal{H}) = B_{\mathfrak{m}_2}(\mathcal{H}) &\iff R_{\mathfrak{m}_1}(\mathcal{H}) = R_{\mathfrak{m}_2}(\mathcal{H}) \text{ and } \|.\|_{\mathfrak{m}_1} = \|.\|_{\mathfrak{m}_2} \\ &\iff \mathfrak{m}_1^\flat = \mathfrak{m}_2^\flat. \end{aligned} \tag{4.3}$$

Proof. Assume that $R_{\mathfrak{m}_1}(\mathcal{H}) \subseteq R_{\mathfrak{m}_2}(\mathcal{H})$. Since $\|.\|_{\mathfrak{m}_j} \geq \|.\|_\mathcal{H}$, point evaluation is continuous with respect to each of the norms $\|.\|_{\mathfrak{m}_j}$. Hence the map id : $\langle R_{\mathfrak{m}_1}(\mathcal{H}), \|.\|_{\mathfrak{m}_1}\rangle \to \langle R_{\mathfrak{m}_2}(\mathcal{H}), \|.\|_{\mathfrak{m}_2}\rangle$ has closed graph. By the Closed Graph

Theorem, it is therefore bounded, i.e. there exists a positive constant λ, such that $B_{\mathfrak{m}_1}(\mathcal{H}) \subseteq \lambda B_{\mathfrak{m}_2}(\mathcal{H})$. Next assume that $B_{\mathfrak{m}_1}(\mathcal{H}) \subseteq \lambda B_{\mathfrak{m}_2}(\mathcal{H})$ with some $\lambda > 0$. Then it follows that

$$\frac{1}{\lambda}\mathfrak{m}_1^\flat(z) = \sup_{F\in B_{\mathfrak{m}_1}(\mathcal{H})} \Big|\frac{1}{\lambda}F(z)\Big| \leq \sup_{G\in B_{\mathfrak{m}_2}(\mathcal{H})} |G(z)| = \mathfrak{m}_2^\flat(z)\,.$$

Assume finally that $\mathfrak{m}_1^\flat \lesssim \mathfrak{m}_2^\flat$. If $F \in B_{\mathfrak{m}_1}(\mathcal{H})$, then $F \in \mathcal{H}$ and

$$|F(z)| \leq \mathfrak{m}_1^\flat(z) \lesssim \mathfrak{m}_2^\flat(z) \leq \mathfrak{m}_2(z), \qquad z \in D\,. \tag{4.4}$$

Hence $F \in R_{\mathfrak{m}_2}(\mathcal{H})$, and we conclude that $R_{\mathfrak{m}_1}(\mathcal{H}) \subseteq R_{\mathfrak{m}_2}(\mathcal{H})$. This finishes the proof of (4.1).

The equivalences in (4.2) are an immediate consequence of (4.1). Those in (4.3) are seen by similar arguments as above, noting that $\mathfrak{m}_1^\flat = \mathfrak{m}_2^\flat$ gives a more accurate estimate in (4.4). ❑

When investigating "small" majorants, naturally the question comes up whether there exist minimal ones among all admissible majorants. First, let us make precise what we understand by the terms "small" or "minimal".

4.8 Definition. On the set of all pairs $(\mathfrak{m}, D)$, where $D \subseteq \mathbb{C}^+ \cup \mathbb{R}$ and $\mathfrak{m} : D \to [0,\infty)$, we define a relation $\preceq$ by

$$(\mathfrak{m}_1, D_1) \preceq (\mathfrak{m}_2, D_2) \iff D_1 \supseteq D_2 \text{ and } \mathfrak{m}_1|_{D_2} \lesssim \mathfrak{m}_2\,.$$

Clearly, the relation $\preceq$ is reflexive and transitive, i.e. $\operatorname{Adm}\mathcal{H}$ is preordered by $\preceq$. Moreover, $(\mathfrak{m}_1, D_1) \preceq (\mathfrak{m}_2, D_2)$ and $(\mathfrak{m}_2, D_2) \preceq (\mathfrak{m}_1, D_1)$ both hold at the same time if and only if

$$D_1 = D_2 \quad \text{and} \quad \mathfrak{m}_1 \asymp \mathfrak{m}_2\,.$$

Whenever we speak of order-theoretic terms in the context of majorization, we refer to the order induced by $\preceq$.

The validity of $\mathfrak{m}_1 \preceq \mathfrak{m}_2$ means that majorization by $\mathfrak{m}_1$ is a stronger requirement than majorization by $\mathfrak{m}_2$. In fact, $(\mathfrak{m}_1, D_1) \preceq (\mathfrak{m}_2, D_2)$ implies

$$R_{\mathfrak{m}_1}(\mathcal{H}) \subseteq R_{\mathfrak{m}_2}(\mathcal{H})\,.$$

Let us note that in general the converse does not hold, even if $D_1 = D_2$.

4.9 Example. Assume that $\mathcal{H} = \mathcal{H}(E)$ contains the set of all polynomials $\mathbb{C}[z]$ as a dense linear subspace. Such de Branges subspaces were studied in [Ba2]; in particular, whenever E is a canonical product whose zeros all lie on the imaginary axis and have genus zero, the space $\mathcal{H}(E)$ will have this property.

Let $n \in \mathbb{N}$, and set $\mathfrak{m}_1(z) = (1+|z|)^{n+1/2}$ and $\mathfrak{m}_2(z) = (1+|z|)^n$, $z \in \mathbb{C}^+ \cup \mathbb{R}$. Then $R_{\mathfrak{m}_1}(\mathcal{H}) = R_{\mathfrak{m}_2}(\mathcal{H})$ equals the set of all polynomials whose degree does not exceed n. Moreover, the norms $\|.\|_{\mathfrak{m}_1}$ and $\|.\|_{\mathfrak{m}_2}$ are equivalent on $R_{\mathfrak{m}_1}(\mathcal{H})$ and so there is $\delta > 0$ such that $B_{\delta\mathfrak{m}_1}(\mathcal{H}) \subseteq B_{\mathfrak{m}_2}(\mathcal{H})$. However, it is not true that $(\delta\mathfrak{m}_1, \mathbb{C}^+ \cup \mathbb{R}) \preceq (\mathfrak{m}_2, \mathbb{C}^+ \cup \mathbb{R})$.

The following result is an extension of [BW1, Theorem 4.2] where the case of majorization along $\mathbb{R}$ was treated. For a de Branges space $\mathcal{H}$ denote by $\mathfrak{r}_{\mathcal{H}} : \operatorname{Adm} \mathcal{H} \to \operatorname{Sub} \mathcal{H}$ the map

$$\mathfrak{r}_{\mathcal{H}}(\mathfrak{m}) := \mathcal{R}_{\mathfrak{m}}(\mathcal{H}),$$

and let $\operatorname{Min} \operatorname{Adm}_D \mathcal{H}$ be the set of all minimal elements of $\operatorname{Adm}_D \mathcal{H}$ modulo $\asymp$.

4.10 Proposition. *Let $\mathcal{H}$ be a de Branges space, and let $D \subseteq \mathbb{C}^+ \cup \mathbb{R}$. Then $\mathfrak{r}_{\mathcal{H}}$ maps $\operatorname{Min} \operatorname{Adm}_D \mathcal{H}$ bijectively onto*

$$\mathfrak{L} := \big\{ \mathcal{L} \in \mathfrak{r}_{\mathcal{H}}(\operatorname{Adm}_D \mathcal{H}) : \dim \mathcal{L} = 1 \big\}.$$

If $\mathcal{L} \in \mathfrak{L}$, then $(\mathfrak{r}_{\mathcal{H}}|_{\operatorname{Min} \operatorname{Adm}_D \mathcal{H}})^{-1}(\mathcal{L}) \asymp \nabla_{\mathcal{L}}|_D$.

Proof. Assume that $\mathfrak{m} \in \operatorname{Adm}_D \mathcal{H}$ is minimal, but $\dim R_{\mathfrak{m}}(\mathcal{H}) > 1$. Then there exists an element $F \in R_{\mathfrak{m}}(\mathcal{H})$ with $F(i) = 0$. It follows that

$$\frac{F(z)}{z - i} \in R_{\tilde{\mathfrak{m}}}(\mathcal{H})$$

where $\tilde{\mathfrak{m}}(z) := (1 + |z|)^{-1} \mathfrak{m}(z)$, $z \in D$. This yields that $\tilde{\mathfrak{m}} \in \operatorname{Adm}_D \mathcal{H}$. However, $\tilde{\mathfrak{m}} \preceq \mathfrak{m}$ but $\mathfrak{m} \not\preceq \tilde{\mathfrak{m}}$, and we have obtained a contradiction. We conclude that $\mathfrak{r}_{\mathcal{H}}$ maps $\operatorname{Min} \operatorname{Adm}_D \mathcal{H}$ into $\mathfrak{L}$.

We proceed with an intermediate remark: Let $\mathcal{L}$ be any one-dimensional dB-subspace of $\mathcal{H}$. Fix $F_0 \in \mathcal{L} \setminus \{0\}$, then $\mathcal{L} = \operatorname{span}\{F_0\}$ and thus $\nabla_{\mathcal{L}}(z) = \|F_0\|_{\mathcal{H}}^{-1} |F_0(z)|$. If $\mathfrak{m} \in \operatorname{Adm}_D \mathcal{H}$ has the property that $\mathcal{L} = \mathcal{R}_{\mathfrak{m}}(\mathcal{H})$, then

$$B_{\mathfrak{m}}(\mathcal{H}) = \big\{ \lambda F_0 : |\lambda| \leq \|F_0\|_{\mathfrak{m}} \big\} \quad \text{and} \quad \mathfrak{m}^\flat(z) = \frac{|F_0(z)|}{\|F_0\|_{\mathfrak{m}}}.$$

It follows that $\mathfrak{m}^\flat \asymp \nabla_{\mathcal{L}}|_D$.

By Lemma 4.4 a minimal majorant $\mathfrak{m}$ satisfies $\mathfrak{m} \asymp \mathfrak{m}^\flat|_D$. Hence, together with the first paragraph of this proof, this remark already shows that $\mathfrak{r}_{\mathcal{H}}|_{\operatorname{Min} \operatorname{Adm}_D \mathcal{H}}$ is injective. To see surjectivity, let $\mathcal{L} \in \mathfrak{L}$ be given. Then the above remark yields $\mathcal{L} = \mathcal{R}_{\nabla_{\mathcal{L}}|_D}(\mathcal{H})$. If $\mathfrak{m} \in \operatorname{Adm}_D \mathcal{H}$ and $\mathfrak{m} \preceq \nabla_{\mathcal{L}}|_D$, then $\{0\} \neq \mathcal{R}_{\mathfrak{m}}(\mathcal{H}) \subseteq \mathcal{R}_{\nabla_{\mathcal{L}}|_D}(\mathcal{H}) = \mathcal{L}$, and hence also $\mathcal{R}_{\mathfrak{m}}(\mathcal{H}) = \mathcal{L}$. It follows that $\mathfrak{m}^\flat|_D \asymp \nabla_{\mathcal{L}}|_D$. Since $\mathfrak{m}^\flat|_D \lesssim \mathfrak{m}$, we obtain that also $\mathfrak{m} \asymp \nabla_{\mathcal{L}}|_D$. Thus $\nabla_{\mathcal{L}}|_D \in \operatorname{Min} \operatorname{Adm}_D \mathcal{H}$. ❑

In conjunction with the representability results shown in [BW3], we obtain the following analogue of [BW1, Theorem 4.9] for majorization on the imaginary half-line.

4.11 Corollary. *Let $\mathcal{H}$ be a de Branges space, and set $D := i[1, \infty)$. Then the set $\operatorname{Adm}_D \mathcal{H}$ contains a minimal element if and only if $\operatorname{Sub}^* \mathcal{H}$ contains a one-dimensional subspace $\mathcal{L}_0$. In this case there exists exactly one minimal element, namely $\nabla_{\mathcal{L}_0}|_D$.*

Proof. By [BW3, Theorem 4.1], we have $\mathfrak{r}_{\mathcal{H}}(\operatorname{Adm}_D \mathcal{H}) = \operatorname{Sub}^* \mathcal{H}$. By de Branges' Ordering Theorem, the set $\operatorname{Sub}^* \mathcal{H}$ can contain at most one one-dimensional subspace. ❑

Appendix A. Notation and preliminaries

I. Mean type and zero divisors

We will use the standard theory of Hardy spaces in the half-plane as presented e.g. in [G] or [RR]. In this place, let us only recall the following notations. We denote by

(i) $\mathcal{N} = \mathcal{N}(\mathbb{C}^+)$ the set of all functions of *bounded type*, that is, of all functions f analytic in $\mathbb{C}^+$, which can be represented as a quotient $f = g^{-1}h$ of two bounded and analytic functions g and h.

(ii) $H^2 = H^2(\mathbb{C}^+)$ the *Hardy space*, that is, the set of all functions f analytic in $\mathbb{C}^+$ which satisfy

$$\sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2 \, dx < \infty \, .$$

If $f \in \mathcal{N}$, the *mean type of* f is defined by the formula

$$\operatorname{mt} f := \limsup_{y \to +\infty} \frac{1}{y} \log |f(iy)| \, .$$

Then $\operatorname{mt} f \in \mathbb{R}$, and the radial growth of f is determined by the number $\operatorname{mt} f$ in the following sense: For every $a \in \mathbb{R}$ and $0 < \alpha < \beta < \pi$, there exists an open set $\Delta_{a,\alpha,\beta} \subseteq (0, \infty)$ with finite logarithmic length, such that

$$\lim_{\substack{r\to\infty \\ r \notin \Delta_{a,\alpha,\beta}}} \frac{1}{r} \log \left| f(a + re^{i\theta}) \right| = \operatorname{mt} f \cdot \sin \theta \, , \tag{A.1}$$

uniformly for $\theta \in [\alpha, \beta]$. If, for some $\epsilon > 0$, the angle $[\alpha - \epsilon, \beta + \epsilon]$ does not contain any zeros of $f(a + z)$, then one can choose $\Delta_{a,\alpha,\beta} = \emptyset$.

Here we understand by the logarithmic length of a subset M of $\mathbb{R}^+$ the value of the integral $\int_M x^{-1} \, dx$. When speaking about logarithmic length of a set M, we always include that M should be measurable.

A.1 Definition. Let $\mathfrak{m} : D \to \mathbb{C}$ be a function defined on some subset D of the complex plane.

(i) By analogy with (A.1) we define the *mean type of* $\mathfrak{m}$ as

$$\operatorname{mt}_{\mathcal{H}} \mathfrak{m} := \inf \left\{ \frac{1}{\sin \theta} \limsup_{\substack{r\to\infty \\ r \in M}} \frac{1}{r} \log |\mathfrak{m}(a + re^{i\theta})| \right\} \in [-\infty, +\infty] \, ,$$

where the infimum is taken over those values $a \in \mathbb{R}$, $\theta \in (0, \pi)$, and those sets $M \subseteq \mathbb{R}^+$ of infinite logarithmic length, for which $\{a + re^{i\theta} : r \in M\} \subseteq D$. Thereby we understand the infimum of the empty set as $+\infty$.

(ii) We associate to $\mathfrak{m}$ its *zero divisor* $\mathfrak{d}_{\mathfrak{m}} : \mathbb{C} \to \mathbb{N}_0 \cup \{\infty\}$. If $w \in \mathbb{C}$, then $\mathfrak{d}_{\mathfrak{m}}(w)$ is defined as the infimum of all numbers $n \in \mathbb{N}_0$, such that there exists a neighbourhood U of w with the property

$$\inf_{\substack{z \in U \cap D \\ |z-w|^n \neq 0}} \frac{|\mathfrak{m}(z)|}{|z-w|^n} > 0 \, .$$

Note that in general $\operatorname{mt}\mathfrak{m}$ may take the values $\pm\infty$. However, the above definition ensures that $\operatorname{mt}\mathfrak{m}$ coincides with the classical notion in case $\mathfrak{m} \in \mathcal{N}$.

A similar remark applies to $\mathfrak{d}_\mathfrak{m}$. If D is open, and $\mathfrak{m}$ is analytic, then $\mathfrak{d}_\mathfrak{m}|_D$ is just the usual zero divisor of $\mathfrak{m}$, i.e. $\mathfrak{d}_\mathfrak{m}(w)$ is the multiplicity of the point w as a zero of $\mathfrak{m}$ whenever $w \in D$. Moreover, note that the definition of $\mathfrak{d}_\mathfrak{m}$ is made in such a way that $\mathfrak{d}_\mathfrak{m}(w) = 0$ whenever $w \notin \overline{D}$.

II. De Branges spaces of entire functions

By the axiom (dB1) a de Branges space $\mathcal{H}$ is a reproducing kernel Hilbert space. We will denote the kernel corresponding to $w \in \mathbb{C}$ by $K(w, \cdot)$ or, if it is necessary to be more specific, by $K_\mathcal{H}(w, \cdot)$. A particular role is played by the norm of reproducing kernel functions. We will denote

$$\nabla_\mathcal{H}(z) := \|K(z, \cdot)\|_\mathcal{H}, \qquad z \in \mathbb{C}\,.$$

This norm can be computed e.g. as

$$\nabla_\mathcal{H}(z) = \sup\big\{|F(z)| :\ \|F\|_\mathcal{H} = 1\big\} = \big(K(z,z)\big)^{1/2}\,.$$

Let us explicitly point out that every element of $\mathcal{H}$ is majorized by $\nabla_\mathcal{H}$: By the Schwarz inequality we have

$$|F(z)| \leq \|F\|\nabla_\mathcal{H}(z), \qquad z \in \mathbb{C}, \quad F \in \mathcal{H}\,. \tag{A.2}$$

A.2 Definition. Let $\mathcal{H}$ be a de Branges space. For a subset $L \subseteq \mathcal{H}$ we define $\mathfrak{d}_L : \mathbb{C} \to \mathbb{N}_0$ as

$$\mathfrak{d}_L(w) := \min_{F \in L} \mathfrak{d}_F(w)\,.$$

Due to the axiom (dB3), we have $\mathfrak{d}_\mathcal{H}(w) = 0$, $w \in \mathbb{C} \setminus \mathbb{R}$. In fact, if $F \in \mathcal{H}$ and w is a nonreal zero of F, then $(z-w)^{-1}F(z) \in \mathcal{H}$. This need not be true for real points w. However, one can show that, if $w \in \mathbb{R}$ and $\mathfrak{d}_F(w) > \mathfrak{d}_\mathcal{H}(w)$, then $(z-w)^{-1}F(z) \in \mathcal{H}$.

A.3 Definition. Let $\mathcal{H}$ be a de Branges space, and let $\mathfrak{m} : D \to \mathbb{C}$ be a function defined on some subset D of the complex plane. We define the *mean type of* $\mathfrak{m}$ *relative to* $\mathcal{H}$ by

$$\operatorname{mt}_\mathcal{H}\mathfrak{m} := \operatorname{mt}\frac{\mathfrak{m}}{\nabla_\mathcal{H}}.$$

If L is a subset of $\mathcal{H}$, the *mean type of* L *relative to* $\mathcal{H}$ is

$$\operatorname{mt}_\mathcal{H} L := \sup_{F \in L} \operatorname{mt}_\mathcal{H} F\,.$$

Note that, by (A.2), we have $\operatorname{mt}_\mathcal{H} L \leq 0$. Moreover, for each $\alpha \leq 0$ the set $\{F \in \mathcal{H} : \operatorname{mt}_\mathcal{H} F \leq \alpha\}$ is closed, cf. [KW]. This implies that always $\operatorname{mt}_\mathcal{H} \operatorname{clos}_\mathcal{H} L = \operatorname{mt}_\mathcal{H} L$.

As we have already remarked in the introduction, a de Branges space $\mathcal{H}$ is completely determined by a single function of Hermite-Biehler class.

A.4 Definition. We say that an entire function E belongs to the *Hermite–Biehler class* $\mathcal{HB}$, if

$$|E^{\#}(z)| < |E(z)|, \qquad z \in \mathbb{C}^+ .$$

If $E \in \mathcal{HB}$, define

$$\mathcal{H}(E) := \Big\{F \text{ entire} : \frac{F}{E}, \frac{F^{\#}}{E} \in H^2(\mathbb{C}^+)\Big\},$$

and

$$(F, G)_E := \int_{\mathbb{R}} \frac{F(t)\overline{G(t)}}{|E(t)|^2}\, dt, \qquad F \in \mathcal{H}(E) .$$

Instead of $E^{-1}F, E^{-1}F^{\#} \in H^2$ one could, equivalently, require that $E^{-1}F$ and $E^{-1}F^{\#}$ are of bounded type and nonpositive mean type in the upper half-plane, and that $\int_{\mathbb{R}} |E^{-1}(t)F(t)|^2\, dt < \infty$. This is, in fact, the original definition in [dB1].

The relation between de Branges spaces and Hermite-Biehler functions is established by the following fact:

A.5. *De Branges spaces via* $\mathcal{HB}$***:*** *For every function* $E \in \mathcal{HB}$*, the space* $\langle \mathcal{H}(E), (\cdot,\cdot)_E\rangle$ *is a de Branges space, and conversely every de Branges space can be obtained in this way.*

The function $E \in \mathcal{HB}$ which realizes a given de Branges space $\langle \mathcal{H}, (\cdot,\cdot)\rangle$ as $\langle \mathcal{H}(E), (\cdot,\cdot)_E\rangle$ is not unique. However, if $E_1, E_2 \in \mathcal{HB}$ and $\langle \mathcal{H}(E_1), (\cdot,\cdot)_{E_1}\rangle = \langle \mathcal{H}(E_2), (\cdot,\cdot)_{E_2}\rangle$, then there exists a constant 2×2-matrix M with real entries and determinant 1, such that

$$(A_2, B_2) = (A_1, B_1)M .$$

Here, and later on, we use the generic decomposition of a function $E \in \mathcal{HB}$ as $E = A - iB$ with

$$A := \frac{E + E^{\#}}{2}, \qquad B := i\frac{E - E^{\#}}{2} . \tag{A.3}$$

For each two function $E_1, E_2 \in \mathcal{HB}$ with $\langle \mathcal{H}(E_1), (\cdot,\cdot)_{E_1}\rangle = \langle \mathcal{H}(E_2), (\cdot,\cdot)_{E_2}\rangle$, there exist constants $c, C > 0$ such that

$$c|E_1(z)| \le |E_2(z)| \le C|E_1(z)|, \qquad z \in \mathbb{C}^+ \cup \mathbb{R} .$$

The notion of a phase function is important in the theory of de Branges spaces. For $E \in \mathcal{HB}$, a *phase function* of E is a continuous, increasing function $\varphi_E : \mathbb{R} \to \mathbb{R}$ with $E(t)\exp(i\varphi_E(t)) \in \mathbb{R}$, $t \in \mathbb{R}$. A phase function φ_E is by this requirement defined uniquely up to an additive constant which belongs to $\pi\mathbb{Z}$. Its derivative is continuous, positive, and can be computed as

$$\varphi'(t) = \pi\frac{K(t,t)}{|E(t)|^2} = a + \sum_n \frac{|\operatorname{Im} z_n|}{|t - z_n|^2} , \tag{A.4}$$

where z_n are zeros of E listed according to their multiplicities, and $a := -\operatorname{mt}(E^{-1}E^{\#})$.

A.6 Remark. Let $\langle\mathcal{H},(\cdot,\cdot)\rangle$ be a de Branges space, and let $E \in \mathcal{H}B$ be such that $\langle\mathcal{H},(\cdot,\cdot)\rangle = \langle\mathcal{H}(E),(\cdot,\cdot)_E\rangle$. Then all information about $\mathcal{H}$ can, theoretically, be extracted from E. In general this is a difficult task, however, for some items it can be done explicitly. For example:

(*i*) The reproducing kernel $K(w,\cdot)$ of $\mathcal{H}$ is given as

$$K(w,z) = \frac{E(z)E^{\#}(\bar{w}) - E(\bar{w})E^{\#}(z)}{2\pi i(\bar{w}-z)}\,.$$

(*ii*) We have $\mathfrak{d}_{\mathcal{H}} = \mathfrak{d}_E$. This equality even holds if we only assume that $\mathcal{H} = \mathcal{H}(E)$ as sets, i.e., without assuming equality of norms.

(*iii*) The function $\nabla_{\mathcal{H}}$ is given as

$$\nabla_{\mathcal{H}}(z) = \begin{cases} \Big(\frac{|E(z)|^2-|E(\overline{z})|^2}{4\pi\,\mathrm{Im}\,z}\Big)^{1/2}, & z \in \mathbb{C}\setminus\mathbb{R}, \\ \pi^{-1/2}|E(z)|(\varphi_E'(z))^{1/2}, & z \in \mathbb{R}. \end{cases} \tag{A.5}$$

In particular, we have $\mathfrak{d}_{\nabla_{\mathcal{H}}} = \mathfrak{d}_{\mathcal{H}}$.

(*iv*) We have

$$\mathrm{mt}_{\mathcal{H}} F = \mathrm{mt}\,\frac{F}{E}, \qquad F \in \mathcal{H}\,.$$

This follows from the estimates (with $w_0 \in \mathbb{C}^+$ fixed)

$$\frac{|E(w_0)|(1-|\frac{E(\overline{w_0})}{E(w_0)}|)}{2\pi\nabla_{\mathcal{H}}(w_0)}\frac{1}{|z-\overline{w}_0|} \leq \frac{\nabla_{\mathcal{H}}(z)}{|E(z)|} \leq \frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{\mathrm{Im}\,z}}, \quad z \in \mathbb{C}^+\,, \tag{A.6}$$

which are deduced from the inequality $|K(w_0,z)| = \big|\big(K(w_0,\cdot),K(z,\cdot)\big)\big| \leq \nabla_{\mathcal{H}}(w_0)\nabla_{\mathcal{H}}(z)$ and (A.5).

III. Structure of dB-subspaces

The, probably, most important notion in the theory of de Branges spaces is the one of de Branges subspaces.

A.7 Definition. A subset $\mathcal{L}$ of a de Branges space $\mathcal{H}$ is called a *dB-subspace* of $\mathcal{H}$, if it is itself, with the norm inherited from $\mathcal{H}$, a de Branges space.

We will denote the set of all dB-subspaces of a given space $\mathcal{H}$ by $\mathrm{Sub}\,\mathcal{H}$. If $\mathfrak{d}:\mathbb{C}\to\mathbb{N}_0$, we set

$$\mathrm{Sub}_{\mathfrak{d}}\,\mathcal{H} := \big\{\mathcal{L}\in\mathrm{Sub}\,\mathcal{H} :\, \mathfrak{d}_{\mathcal{L}} = \mathfrak{d}\big\}\,.$$

Since dB-subspaces with $\mathfrak{d}_{\mathcal{L}} = \mathfrak{d}_{\mathcal{H}}$ appear quite frequently, we introduce the shorthand notation $\mathrm{Sub}^*\,\mathcal{H} := \mathrm{Sub}_{\mathfrak{d}_{\mathcal{H}}}\,\mathcal{H}$.

It is apparent from the axioms (dB1)–(dB3) that a subset $\mathcal{L}$ of $\mathcal{H}$ is a dB-subspace if and only if the following three conditions hold:

(*i*) $\mathcal{L}$ is a closed linear subspace of $\mathcal{H}$;

(*ii*) If $F \in \mathcal{L}$, then also $F^{\#} \in \mathcal{L}$;

(*iii*) If $F \in \mathcal{L}$ and $z_0 \in \mathbb{C} \setminus \mathbb{R}$ is such that $F(z_0) = 0$, then $\frac{F(z)}{z-z_0} \in \mathcal{L}$.

A.8 Example. Some examples of dB-subspaces can be obtained by imposing conditions on real zeros or on mean type.

If $\mathfrak{d} : \mathbb{C} \to \mathbb{N}_0$, $\operatorname{supp} \mathfrak{d} \subseteq \mathbb{R}$, is a function such that $\mathfrak{d}_{F_0} \geq \mathfrak{d}$ for some $F_0 \in \mathcal{H} \setminus \{0\}$, then

$$\mathcal{H}_{\mathfrak{d}} := \big\{ F \in \mathcal{H} : \, \mathfrak{d}_F \geq \mathfrak{d} \big\} \in \operatorname{Sub} \mathcal{H} \, .$$

We have $\mathfrak{d}_{\mathcal{H}_{\mathfrak{d}}} = \max\{\mathfrak{d}, \mathfrak{d}_{\mathcal{H}}\}$.

If $\alpha \leq 0$ is such that $\operatorname{mt}_{\mathcal{H}} F_0, \operatorname{mt}_{\mathcal{H}} F_0^{\#} \leq \alpha$ for some $F_0 \in \mathcal{H} \setminus \{0\}$, then

$$\mathcal{H}_{(\alpha)} := \big\{ F \in \mathcal{H} : \, \operatorname{mt}_{\mathcal{H}} F, \, \operatorname{mt}_{\mathcal{H}} F^{\#} \leq \alpha \big\} \in \operatorname{Sub}^* \mathcal{H} \, ,$$

and we have $\operatorname{mt}_{\mathcal{H}} \mathcal{H}_{(\alpha)} = \alpha$.

Those dB-subspaces which are defined by mean type conditions will in general not exhaust all of $\operatorname{Sub}^* \mathcal{H}$. However, sometimes, this also might be the case.

Trivially, the set $\operatorname{Sub} \mathcal{H}$, and hence also each of the sets $\operatorname{Sub}_{\mathfrak{d}} \mathcal{H}$, is partially ordered with respect to set-theoretic inclusion. One of the most fundamental and deep results in the theory of de Branges spaces is the *Ordering Theorem for subspaces of* $\mathcal{H}$, cf. [dB, Theorem 35] where even a somewhat more general version is proved.

A.9. *De Branges' Ordering Theorem:* *Let* $\mathcal{H}$ *be a de Branges space and let* $\mathfrak{d} : \mathbb{C} \to \mathbb{N}_0$*. Then* $\operatorname{Sub}_{\mathfrak{d}} \mathcal{H}$ *is totally ordered.*

The chains $\operatorname{Sub}_{\mathfrak{d}} \mathcal{H}$ have the following continuity property: For a dB-subspace $\mathcal{L}$ of $\mathcal{H}$, set

$$\check{\mathcal{L}} := \bigcap \big\{ \mathcal{K} \in \operatorname{Sub}_{\mathfrak{d}_{\mathcal{L}}} \mathcal{H} : \, \mathcal{K} \supsetneq \mathcal{L} \big\}, \quad \text{if } \mathcal{L} \neq \mathcal{H} \, , \tag{A.7}$$

$$\tilde{\mathcal{L}} := \operatorname{clos}_{\mathcal{H}} \bigcup \big\{ \mathcal{K} \in \operatorname{Sub}_{\mathfrak{d}_{\mathcal{L}}} \mathcal{H} : \, \mathcal{K} \subsetneq \mathcal{L} \big\}, \quad \text{if } \dim \mathcal{L} > 1 \, .$$

Then

$$\dim \big(\check{\mathcal{L}} / \mathcal{L} \big) \leq 1 \quad \text{and} \quad \dim \big(\mathcal{L} / \tilde{\mathcal{L}} \big) \leq 1 \, .$$

A.10 Example. Let us explicitly mention two examples of de Branges spaces, which show in some sense extreme behaviour.

(*i*) Consider the *Paley–Wiener space* $\mathcal{P}W_a$ where $a > 0$. This space is a de Branges space. It can be obtained as $\mathcal{H}(E)$ with $E(z) = e^{-iaz}$. The chain $\operatorname{Sub}^*(\mathcal{P}W_a)$ is equal to

$$\operatorname{Sub}^* \mathcal{P}W_a = \big\{ \mathcal{P}W_b : \, 0 < b \leq a \big\} \, .$$

Apparently, we have $\mathcal{P}W_b = (\mathcal{P}W_a)_{(b-a)}$, and hence in this example all dB-subspaces are obtained by mean type restrictions.

(ii) In the study of the indeterminate Hamburger moment problem de Branges spaces occur which contain the set of all polynomials $\mathbb{C}[z]$ as a dense linear subspace, see e.g. [Ba2], [BS], [DK, §5.9]

If $\mathcal{H}$ is such that $\mathcal{H} = \operatorname{clos}_{\mathcal{H}} \mathbb{C}[z]$, then the chain $\operatorname{Sub}^* \mathcal{H}$ has order type $\mathbb{N}$. In fact,

$$\operatorname{Sub}^* \mathcal{H} = \big\{ \mathbb{C}[z]_n : n \in \mathbb{N}_0 \big\} \cup \{\mathcal{H}\},$$

where $\mathbb{C}[z]_n$ denotes the set of all polynomials whose degree is at most n.

Examples of de Branges spaces $\mathcal{H}$ for which the chain $\operatorname{Sub}^* \mathcal{H}$ has all different kinds of order types can be constructed using canonical systems of differential equations, see e.g. [dB, Theorems 37,38], [GK], or [HSW].

With help of the estimates (A.6), it is easy to see that

$$\operatorname{mt} \frac{K_{\mathcal{H}}(w,\cdot)}{\nabla_{\mathcal{H}}(z)} = 0 .$$

This implies that for any dB-subspace $\mathcal{L}$ of $\mathcal{H}$ the supremum in the definition of $\operatorname{mt}_{\mathcal{H}} \mathcal{L}$ is attained (e.g. on the reproducing kernel functions $K_{\mathcal{L}}(w,\cdot)$). Moreover, we obtain $\operatorname{mt}_{\mathcal{H}} \mathcal{L} = \operatorname{mt}_{\mathcal{H}} \nabla_{\mathcal{L}}$.

IV. Admissible majorants

We start with the definition of an admissible majorant for a de Branges space $\mathcal{H}$.

A.11 Definition. Let $\mathcal{H}$ be a de Branges space. A function $\mathfrak{m} : D \to [0,\infty)$ where $D \subseteq \mathbb{C}^+ \cup \mathbb{R}$, is called an *admissible majorant for $\mathcal{H}$* if it satisfies the conditions

(Adm1) $\operatorname{supp} \mathfrak{d}_{\mathfrak{m}} \subseteq \mathbb{R}$;

(Adm2) $R_{\mathfrak{m}}(\mathcal{H})$ contains a nonzero element.

The set of all admissible majorants is denoted by $\operatorname{Adm} \mathcal{H}$. For the set of all those admissible majorants which are defined on a fixed set D, we write $\operatorname{Adm}_D \mathcal{H}$.

The significance of this notion is shown by the following fact.

A.12. ***Subspaces generated by majorants:*** *Let $\mathcal{H}$ be a de Branges space, and let $\mathfrak{m} : D \to [0,\infty)$ be a function with $D \subseteq \mathbb{C}^+ \cup \mathbb{R}$. Then $\mathcal{R}_{\mathfrak{m}}(\mathcal{H})$ is a dB-subspace of $\mathcal{H}$ if and only if $\mathfrak{m}$ is an admissible majorant for $\mathcal{H}$. In this case we have*

$$\mathfrak{d}_{\mathcal{R}_{\mathfrak{m}}(\mathcal{H})} = \max\{\mathfrak{d}_{\mathfrak{m}}, \mathfrak{d}_{\mathcal{H}}\} \quad \text{and} \quad \operatorname{mt}_{\mathcal{H}} \mathcal{R}_{\mathfrak{m}}(\mathcal{H}) \leq \operatorname{mt}_{\mathcal{H}} \mathfrak{m} . \tag{A.8}$$

An obvious, but surprisingly important, example of admissible majorants is provided by the functions $\nabla_{\mathcal{L}}|_{\mathbb{C}^+\cup\mathbb{R}}$, $\mathcal{L} \in \operatorname{Sub} \mathcal{H}$. Since always $\mathcal{L} \subseteq R_{\nabla_{\mathcal{L}}|_{\mathbb{C}^+\cup\mathbb{R}}}(\mathcal{H})$, (Adm2) is satisfied. Also, it follows that $\mathfrak{d}_{\nabla_{\mathcal{L}}} \leq \mathfrak{d}_{\mathcal{L}}$ and this yields (Adm1). Thus $\nabla_{\mathcal{L}}|_{\mathbb{C}^+\cup\mathbb{R}} \in \operatorname{Adm} \mathcal{H}$.

References

[AC] P.R. Ahern, D.N. Clark: *Radial limits and invariant subspaces,* Amer. J. Math. **92** (2) (1970), 332–342.

[Ba1] A.D.Baranov: *Isometric embeddings of the spaces K_θ in the upper half-plane,* Journal of Mathematical Sciences **105** (5) (2001), 2319–2329.

[Ba2] A.Baranov: *Polynomials in the de Branges spaces of entire functions,* Ark. Mat. **44** (1) (2006), 16–38.

[BH] A.D.Baranov, V.P.Havin: *Admissible majorants for model subspaces and arguments of inner functions,* Functional. Anal. Appl. **40** (4) (2006), 249–263.

[BW1] A.Baranov, H.Woracek: *Subspaces of de Branges spaces generated by majorants,* Canad. Math. J. **61** (3) (2009), 503–517.

[BW2] A.Baranov, H.Woracek: *Finite dimensional de Branges subspaces generated by majorants,* Oper. Theory Adv. Appl. **188** (2008), 45–56.

[BW3] A.Baranov, H.Woracek: *Majorization in de Branges spaces I. Representability of subspaces,* J.Funct.Anal. (DOI 10.1016/j.jfa.2010.01.016). Preprint available online as ASC Preprint Series 12/2009, http://asc.tuwien.ac.at.

[BS] A. Borichev, M. Sodin: *Weighted polynomial approximation and the Hamburger moment problem,* Complex analysis and differential equations, Proceedings of the Marcus Wallenberg Symposium in Honor of Matts Essén, Uppsala University, 1998.

[Bou] N.Bourbaki: *General Topology,* Hermann, Paris, 1966.

[dB1] L. de Branges: *Some Hilbert spaces of entire functions,* Proc. Amer. Math. Soc. 10 (5) (1959), 840–846.

[dB] L.de Branges: *Hilbert spaces of entire functions,* Prentice-Hall, London, 1968.

[DK] H. Dym, H. McKean: *Gaussian processes, function theory, and the inverse spectral problem,* Academic Press, New York, 1976.

[G] J.B. Garnett: *Bounded analytic functions,* Academic Press, New York, 1981.

[GK] I.Gohberg, M.G.Krein: *Introduction to the theory of linear nonselfadjoint operators,* Izdat. Nauka, Moscow, 1965 (in Russian). English translation: Translations of Mathematical Monographs 18, American Mathematical Society, Rhode Island, 1969.

[GK] I. Gohberg, M.G. Kreĭn: *Theory and applications of Volterra operators in Hilbert space,* Translations of Mathematical Monographs, AMS. Providence, Rhode Island, 1970.

[HSW] S. Hassi, H.S.V. de Snoo, H. Winkler: *Boundary-value problems for two-dimensional canonical systems,* Integr. Equ. Oper. Theory 36 (4) (2000), 445–479.

[HM] V.P.Havin, J.Mashreghi: *Admissible majorants for model subspaces of H^2. Part I: slow winding of the generating inner function,* Can. J. Math. **55** (6) (2003), 1231–1263.

[KW] M.Kaltenbäck, H.Woracek: *De Branges spaces of exponential type: General theory of growth,* Acta Sci. Math (Szeged) **71** (1/2) (2005), 231–284.

[M] R.Megginson: *An Introduction to Banach space theory,* GTM 183, Springer, New York, 1998.

[RR] M. Rosenblum, J. Rovnyak: *Topics in Hardy classes and univalent functions,* Birkhäuser Verlag, Basel, 1994.

A. Baranov
Department of Mathematics and Mechanics
Saint Petersburg State University
28, Universitetski pr.
198504 Petrodvorets
RUSSIA
email: a.baranov@ev13934.spb.edu

H. Woracek
Institut für Analysis und Scientific Computing
Technische Universität Wien
Wiedner Hauptstr. 8–10/101
A–1040 Wien
AUSTRIA
email: harald.woracek@tuwien.ac.at