Eigenvalue Asymptotics for a Star-Graph Damped Vibrations Problem

Vyacheslav Pivovarchik, Harald Woracek

Abstract

We consider a boundary value problem generated by Sturm-Liouville equations on the edges of a star-shaped graph. Thereby a continuity condition and a condition depending on the spectral parameter is imposed at the interior vertex, corresponding to the case of damping in the problem of small transversal vibrations of a star graph of smooth inhomogeneous strings. At the pendant vertices Dirichlet boundary conditions are imposed. We describe the eigenvalue asymptotics of the problem under consideration.

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1 Introduction

Consider S-wave quantum scattering described by the Sturm-Liouville equation and the boundary condition

$$\begin{cases} -y''(x) + q(x)y(x) = \lambda^2 y(x), \ x \in (0, \infty) \\ y(0) = 0\,. \end{cases} \tag{1.1}$$

In [R1, R2] T.Regge proposed this as a simple model of particle interaction with real finite potential, i.e. considering the case that for some $a > 0$ we have $q(x) \overset{\text{a.e}}{=} 0$, $x > a$, while $q|_{(0,a)} \in L_2(0,a)$ and is a real-valued. In this case the Jost function is an entire function of exponential type at most a. Its zeros are located symmetrically with respect to the imaginary axis and are contained in the open upper half-plane except, possibly, a finite number of zeros located on the non-positive imaginary half-axis. The zeros on the negative imaginary half-axis correspond to the bound states, and the zeros in the open upper half-plane are resonances describing energies and time of life of unstable states. The only possible real zero can occur at the origin.

The scattering problem (1.1) is associated with the spectral problem

$$\begin{cases} -y''(x) + q(x)y(x) = \lambda^2 y(x), \ x \in (0, a) \\ y(0) = 0, \quad y'(a) + i\lambda y(a) = 0\,, \end{cases} \tag{1.2}$$

usually called the Regge problem. The set of zeros of the Jost function of (1.1) is nothing but the set of zeros of the characteristic function of (1.2).

Information on the location of the spectrum of (1.2) was obtained, e.g., in [Si, K, P3]. It was shown that the eigenvalues in the open lower half-plane are all simple, and, if one denotes them as $-i\tau_1, -i\tau_2, \ldots, -i\tau_\kappa$ with $0 < \tau_1 < \tau_2 < \ldots < \tau_\kappa$, then each point $i\tau_k$ does not belong to the spectrum and each interval $(i\tau_k, i\tau_{k+1})$ contains an odd number of eigenvalues (counted according to multiplicities). Asymptotics of the eigenvalues of the Regge problem were investigated, e.g., in [Kr, Ko, Se]. Knowledge on the asymptotics, however, could only be obtained under certain conditions: For example the potential was assumed to be continuous with $q(a) \neq 0$, or to be n-times continuously differentiable with $q(a) = \ldots = q^{(n-1)}(a) = 0$ but $q^{(n)}(a) \neq 0$. Inverse spectral problems have been investigated, e.g., in [K, BKW].

The generalized Regge problem is the boundary value problem

$$\begin{cases} -y''(x) + q(x)y(x) = \lambda^2 y(x), \ x \in (0, a) \\ \ y(0) = 0, \quad y'(a) + (i\alpha\lambda + \beta)y(a) = 0\,, \end{cases} \tag{1.3}$$

where $q \in L_2(0, a)$ is real-valued, and $\alpha \in (0, \infty)$, $\beta \in \mathbb{R}$, are parameters. It appears for example when applying the Liouville transformation to investigate small transversal vibrations of a smooth inhomogeneous string subject to damping. This problem has a long history, let us mention, e.g., [CH, A, KN1, KN2, Sh, P1, P2, MP].

The location of the spectrum of (1.3) was investigated, e.g., in [PvM]. There it was shown that the eigenvalues are located according to the same rules as in the case of the classical Regge problem. Later on, different and more general approaches were obtained, e.g., in [PW1, PW2]. Asymptotics of the eigenvalues and inverse problems were investigated, e.g., in [GP] (for the case that $\alpha \neq 1$).

Considering small transversal vibrations of a star-shaped graph consisting of smooth nonhomogeneous strings which is subject to damping at the interior vertex, one arrives at the following boundary value problem, cf. [P4, YY]:

$$\begin{cases} -y_j''(x) + q_j(x)y_j(x) = \lambda^2 y_j(x), \ x \in (0, a), \quad j = 1, \ldots, n \\ \ y_j(0) = 0, \ j = 1, \ldots, n, \quad y_1(a) = y_2(a) = \ldots = y_n(a) \\ \ y_1'(a) + \ldots + y_n'(a) + (i\alpha\lambda + \beta)y_1(a) = 0\,, \end{cases} \tag{1.4}$$

where $q_j \in L_2(0, a)$, $j = 1, \ldots, n$, are real-valued, and $\alpha \in (0, \infty)$, $\beta \in \mathbb{R}$, are parameters. Thereby the boundary conditions in the first line mean that the string is tied at the outer vertices and continuous at the inner vertex, the condition in the second line describes the damping at the central vertex and is known as the Kirchhoff condition.

The location of the eigenvalues of this problem was discussed in [PW3]. It turned out that the picture doesn't change much in comparison to the problem of one single string.

Our aim in the present paper is to discuss the asymptotics of the eigenvalues of the problem (1.4) when $\alpha < n$. We show that the sequence of eigenvalues can be split into n subsequences, with $n - 1$ behaving like in the 'one-string-problem', cf. Theorem 2.7. We do not touch upon inverse spectral theorems; this will be subject of forthcoming work.

2 Eigenvalue asymptotics

For the rest of this paper let $n \in \mathbb{N}$, $a > 0$, real-valued potentials $q_j \in L_2(0, a)$, $j = 1, \ldots, n$, and parameters $\alpha > 0$, $\beta \in \mathbb{R}$, be fixed. Moreover, always denote by $s_j(\lambda, x)$ the unique solution of the differential equation in (1.4) which satisfies the boundary conditions

$$s_j(\lambda, 0) = 0,\ s_j'(\lambda, 0) = 1\,.$$

The function $s_j(\lambda, x)$ can be written in the form

$$s_j(\lambda, x) = \frac{\sin \lambda x}{\lambda} + \int_0^x K_j(x,t) \frac{\sin \lambda t}{\lambda}\, dt\,, \tag{2.1}$$

with some kernel function $K_j(x,t)$, see, e.g., [M, Corollary to Theorem 1.2.1].

For further use let use recall some properties of this kernel, see, e.g., [M]. The function $K_j(x,t)$ possesses partial derivatives of first order and these derivatives belong to $L_2(0,a)$ as functions of each variable when the other variable is fixed. Moreover, $K_j(a,0) = 0$ and $K_j(a,a) = \frac{1}{2}\int_0^a q_j(x)\,dx$.

2.1. Some preparatory formulae: Let us provide some formulas which will be used to obtain representations of the characteristic function $\Xi(\lambda)$. To shorten notation, we set

$$K_{j,t}(x,t) := \frac{\partial K_j(x,t)}{\partial t},\ K_{j,x}(x,t) := \frac{\partial K_j(x,t)}{\partial x}\,,$$

and

$$B_j := K_j(a,a) \quad \Big(= \frac{1}{2}\int_0^a q_j(x)\,dx \Big)\,.$$

Integrating by parts in (2.1) gives

$$s_j(\lambda, x) = \frac{\sin \lambda x}{\lambda} - K_j(x,x)\frac{\cos \lambda x}{\lambda^2} + K_j(x,0)\frac{1}{\lambda^2} + \int_0^x K_{j,t}(x,t)\frac{\cos \lambda t}{\lambda^2}\,dt\,.$$

For $x = a$, thus

$$s_j(\lambda, a) = \frac{\sin \lambda a}{\lambda} - B_j\frac{\cos \lambda a}{\lambda^2} + \int_0^a K_{j,t}(a,t)\frac{\cos \lambda t}{\lambda^2}\,dt\,. \tag{2.2}$$

Differentiating (2.1) with respect to x, we obtain

$$s_j'(\lambda, x) = \cos \lambda x + K_j(x,x)\frac{\sin \lambda x}{\lambda} + \int_0^x K_{j,x}(x,t)\frac{\sin \lambda t}{\lambda}\,dt\,.$$

For $x = a$, thus

$$s_j'(\lambda, a) = \cos \lambda a + B_j\frac{\sin \lambda a}{\lambda} + \int_0^a K_{j,x}(a,t)\frac{\sin \lambda t}{\lambda}\,dt\,.$$

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2.2. Notational conventions: Throughout the following we denote, for each $b > 0$, by $\mathcal{L}^b$ the Paley-Wiener space consisting of all entire functions of exponential type at most b which are square integrable along the real axis. By the Theorem of Paley-Wiener, $\mathcal{L}^b$ is the Fourier image of all square summable functions supported on $[-b, b]$.

For $n, l \in \mathbb{N}_0$, we denote by $\sigma_{n,l}$ the elementary symmetric polynomial in n variables with degree l. Explicitly, this is

$$\sigma_{n,l}(x_1, \dots, x_n) := \sum_{\substack{j_1,\dots,j_l=1 \\ j_1<\dots<j_n}}^{n} x_{j_1} \cdot \dots \cdot x_{j_l} \,.$$

To shorten notation, we write

$$\sigma_{n,l}(x_p) := \sigma_{n,l}(x_1, \dots, x_n)$$

whenever $x_1, \dots, x_n$ are given. Moreover, if $x_1, \dots, x_n$ and $i \in \{1, \dots, n\}$ are given, we write

$$\sigma_{n-1,l}(\neg\, x_i) := \sigma_{n-1,l}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \,.$$

//

For later use, recall that $\sigma_{n,l}$ is homogeneous of degree l, i.e.

$$\sigma_{n,l}(tx_p) = t^l \sigma_{n,l}(x_p) \,,$$

and that

$$\sum_{i=1}^{n} \sigma_{n-1,l}(\neg\, x_i) = (n-l)\sigma_{n,l}(x_p), \quad l = 0, \dots, n-1 \,,$$

$$\sum_{i=1}^{n} x_i \sigma_{n-1,l}(\neg\, x_i) = (l+1)\sigma_{n,l+1}(x_p), \quad l = 0, \dots, n-1 \,.$$

If $p(z) = \prod_{j=1}^{n}(z + x_j)$ and $p'(z) = n\prod_{j=1}^{n-1}(x + y_j)$, then

$$\sigma_{n-1,l}(y_p) = \frac{n-l}{n}\sigma_{n,l}(x_p), \quad l = 0, \dots, n-1 \,.$$

Let a problem (1.4) be given, and consider the functions defined as

$$\phi(\lambda) := \sum_{j=1}^{n} s_j'(\lambda, a) \prod_{\substack{p=1 \\ p\neq j}}^{n} s_p(\lambda, a), \quad \chi(\lambda) := \prod_{p=1}^{n} s_p(\lambda, a)$$

$$\Xi(\lambda) := \phi(\lambda) + (i\alpha\lambda + \beta)\chi(\lambda)$$

It was shown in [P4, Section 3] that Ξ is the characteristic function of the problem (1.4).

We start with some asymptotic formulas for χ, ϕ, and Ξ. These are based on the following general lemma.

2.3 Lemma. *Let $m \in \mathbb{N}$, $a > 0$, and $\alpha_p \in \mathbb{C}$, $f_p \in \mathcal{L}^a$, $p = 1, \ldots, m$. Set*

$$g_p(\lambda) := \frac{\sin \lambda a}{\lambda} + \alpha_p \frac{\cos \lambda a}{\lambda^2} + \frac{f_p(\lambda)}{\lambda^2},$$

then

$$\prod_{p=1}^{m} g_p(\lambda) = \sum_{j=0}^{m} \sigma_{m,m-j}(\alpha_p) \cos^{m-j} \lambda a \frac{\sin^j \lambda a}{\lambda^{2m-j}} + \Omega(\lambda)$$

where the remainder term $\Omega(\lambda)$ is of the form

$$\Omega(\lambda) = \sum_{j=0}^{m-1} \omega_j(\lambda) \frac{\sin^j \lambda a}{\lambda^{2m-j}}$$

with some functions $\omega_j \in \mathcal{L}^{(m-j)a}$, $j = 0, \ldots, m-1$.

Proof. We have

$$\prod_{p=1}^{m} g_p(\lambda) = \prod_{p=1}^{m} \left[\frac{\sin \lambda a}{\lambda} + \Big(\alpha_p \frac{\cos \lambda a}{\lambda^2} + \frac{f_p(\lambda)}{\lambda^2} \Big) \right] =$$

$$= \sum_{j=0}^{m} \Big(\frac{\sin \lambda a}{\lambda} \Big)^j \sigma_{m,m-j} \Big(\alpha_p \frac{\cos \lambda a}{\lambda^2} + \frac{f_p(\lambda)}{\lambda^2} \Big) =$$

$$= \sum_{j=0}^{m} \sigma_{m,m-j} \big(\alpha_p \cos \lambda a + f_p(\lambda) \big) \frac{\sin^j \lambda a}{\lambda^{2m-j}}.$$

Set

$$\omega_j(\lambda) := \sigma_{m,m-j} \Big(\alpha_p \cos \lambda a + f_p(\lambda) \Big) - \sigma_{m,m-j}(\alpha_p) \cos^{m-j} \lambda a.$$

Then the desired representation of $\prod_{p=1}^{m} g_p(\lambda)$ holds. Note here that $\omega_m = 0$. To show that ω_j belongs to $\mathcal{L}^{(m-j)a}$, consider generally an expression of the form

$$\sigma_{m,l}(x_p + y_p) - \sigma_{m,l}(x_p) = \sum_{\substack{j_1, \ldots, j_l = 1 \\ j_1 < \cdots < j_l}}^{m} \Big[(x_{j_1} + y_{j_1}) \cdot \ldots \cdot (x_{j_l} + y_{j_l}) - x_{j_1} \cdot \ldots \cdot x_{j_l} \Big].$$

Each summand on the right hand side is a sum of products of l factors, each of them having at least one factor y_j. Applying this with $x_p := \alpha_p \cos \lambda a$ and $y_p := f_p(\lambda)$, we see that ω_j is a sum of products, each product having $m - j$ factors which are all entire functions of exponential type at most a and are bounded on each strip around the real line. Moreover, in each product, at least one factor belongs to $\mathcal{L}^a$. Altogether, it follows that $\omega_j \in \mathcal{L}^{(m-j)a}$. ❑

2.4 Corollary. *The function $\chi(\lambda)$ can be written as*

$$\chi(\lambda) = \sum_{j=0}^{n} \sigma_{n,n-j}(-B_p) \cos^{n-j} \lambda a \frac{\sin^j \lambda a}{\lambda^{2n-j}} + \Omega_1(\lambda),$$

where the remainder term $\Omega_1(\lambda)$ is of the form

$$\Omega_1(\lambda) = \sum_{j=0}^{n-1} \omega_{1,j}(\lambda) \frac{\sin^j \lambda a}{\lambda^{2n-j}}$$

with some functions $\omega_{1,j} \in \mathcal{L}^{(n-j)a}$, $j = 1, \ldots, n$.

Proof. The integral term $\int_0^a K_{j,t}(a,t)\cos\lambda t\,dt$ occuring in (2.2) belongs to $\mathcal{L}^a$. Hence, we can apply Lemma 2.3 with the functions $s_p(\lambda,a)$, $p=1,\dots,n$. ❑

We also can deduce a representation of $\phi(\lambda)$.

2.5 Lemma. *The function $\phi(\lambda)$ can be written as*

$$\phi(\lambda) = \sum_{j=0}^{n-1}(j+1)\sigma_{n,n-1-j}(-B_p)\cos^{n-j}\lambda a\frac{\sin^j \lambda a}{\lambda^{2n-j-2}} - \\ - \sum_{j=1}^{n}(n+1-j)\sigma_{n,n+1-j}(-B_p)\cos^{n-j}\lambda a\frac{\sin^j \lambda a}{\lambda^{2n-j}} + \Omega_2(\lambda)$$

where the remainder term $\Omega_2(\lambda)$ is of the form

$$\Omega_2(\lambda) = \sum_{j=0}^{n-1}\omega_{2,j}(\lambda)\frac{\sin^j \lambda a}{\lambda^{2n-j-2}}$$

with some functions $\omega_{2,j}\in\mathcal{L}^{(n-j)a}$, $j=1,\dots,n-1$. The function $\lambda\omega_{2,n-1}(\lambda)$ belongs to $\mathcal{L}^a$.

Proof. Let $i\in\{1,\dots,n\}$. Applying Lemma 2.3 with the functions

$$s_1(\lambda,a),\dots,s_{i-1}(\lambda,a),s_{i+1}(\lambda,a),\dots,s_n(\lambda,a)$$

gives

$$\prod_{\substack{p=1\\p\neq i}}^{n} s_p(\lambda,a) = \sum_{j=0}^{n-1}(-1)^{n-1-j}\sigma_{n-1,n-1-j}(\neg B_i)\cos^{n-1-j}\lambda a\frac{\sin^j \lambda a}{\lambda^{2n-j-2}} + \Omega_{(i)}(\lambda)$$

where

$$\Omega_{(i)}(\lambda) = \sum_{j=0}^{n-2}\omega_{ij}(\lambda)\frac{\sin^j \lambda a}{\lambda^{2n-j-2}},\quad \omega_{ij}\in\mathcal{L}^{(n-1-j)a}\,.$$

Set $\tilde f_i(\lambda) := \int_0^a K_{i,x}(a,t)\sin\lambda t\,dt$, then $\tilde f_i\in\mathcal{L}^a$ and $s_i'(\lambda,a)=\cos\lambda a + B_i\frac{\sin\lambda a}{\lambda} + \frac{1}{\lambda}\tilde f_i(\lambda)$. Moreover, since $s_i'(\lambda,a)$ is entire, also $\frac{1}{\lambda}\tilde f_i(\lambda)$ is entire and thus also belongs to $\mathcal{L}^a$. Note that, in particular, $s_i'(\lambda,a)$ is bounded along the real line.

We compute

$$
\begin{aligned}
s_i'(\lambda,a)\prod_{\substack{p=1\\p\neq i}}^{n} s_p(\lambda,a) = \Big(\cos\lambda a + B_i\frac{\sin\lambda a}{\lambda} + \frac{\tilde f_i(\lambda)}{\lambda}\Big)\cdot\\
\cdot\sum_{j=0}^{n-1}(-1)^{n-1-j}\sigma_{n-1,n-1-j}(\neg\, B_i)\cos^{n-1-j}\lambda a\frac{\sin^j\lambda a}{\lambda^{2n-j-2}}+\\
+ s_i'(\lambda,a)\Omega_{(i)}(\lambda) =\\
= \sum_{j=0}^{n-1}(-1)^{n-1-j}\sigma_{n-1,n-1-j}(\neg\, B_i)\cos^{n-j}\lambda a\frac{\sin^j\lambda a}{\lambda^{2n-j-2}}+\\
+\sum_{j=0}^{n-1}(-1)^{n-1-j}B_i\sigma_{n-1,n-1-j}(\neg\, B_i)\cos^{n-1-j}\lambda a\frac{\sin^{j+1}\lambda a}{\lambda^{2n-j-1}}+\\
+\sum_{j=0}^{n-1}\underbrace{(-1)^{n-1-j}\sigma_{n-1,n-1-j}(\neg\, B_i)\cos^{n-1-j}\lambda a\frac{\tilde f_i(\lambda)}{\lambda}}_{\in\mathcal{L}^{(n-j)a}}\frac{\sin^j\lambda a}{\lambda^{2n-j-2}}+\\
+\sum_{j=0}^{n-2}\underbrace{s_i'(\lambda,a)\omega_{ij}(\lambda)}_{\in\mathcal{L}^{(n-j)a}}\frac{\sin^j\lambda a}{\lambda^{2n-j-2}}
\end{aligned}
$$

Summing over $i\in\{1,\dots,n\}$ gives

$$
\begin{aligned}
\phi(\lambda) = &\sum_{j=0}^{n-1}(-1)^{n-1-j}\underbrace{\Big[\sum_{i=1}^{n}\sigma_{n-1,n-1-j}(\neg\, B_i)\Big]}_{=(j+1)\sigma_{n,n-1-j}(B_p)}\cos^{n-j}\lambda a\frac{\sin^j\lambda a}{\lambda^{2n-j-2}}+\\
&+\sum_{j=0}^{n-1}(-1)^{n-1-j}\underbrace{\Big[\sum_{i=1}^{n}B_i\sigma_{n-1,n-1-j}(\neg\, B_i)\Big]}_{=(n-j)\sigma_{n,n-j}(B_p)}\cos^{n-1-j}\lambda a\frac{\sin^{j+1}\lambda a}{\lambda^{2n-j-1}}+\\
&+\Omega_2(\lambda)
\end{aligned}
$$

and this is the desired representation. ❑

Putting together the representations of χ and ϕ, we obtain a representation of the function Ξ.

2.6 Lemma. *The function $\Xi(\lambda)$ can be written as*

$$
\Xi(\lambda) = \sum_{j=0}^{n-1}(j+1)\sigma_{n,n-1-j}(-B_p)\cos^{n-j}\lambda a\frac{\sin^j\lambda a}{\lambda^{2n-j-2}}- \tag{2.3}
$$

$$
-\sum_{j=1}^{n}(n+1-j)\sigma_{n,n+1-j}(-B_p)\cos^{n-j}\lambda a\frac{\sin^j\lambda a}{\lambda^{2n-j}}+ \tag{2.4}
$$

$$
+\sum_{j=1}^{n}\beta\sigma_{n,n-j}(-B_p)\cos^{n-j}\lambda a\frac{\sin^j\lambda a}{\lambda^{2n-j}}+ \tag{2.5}
$$

$$
+ i\alpha\sum_{j=1}^{n}\sigma_{n,n-j}(-B_p)\cos^{n-j}\lambda a\frac{\sin^j\lambda a}{\lambda^{2n-j-1}}+\Omega_3(\lambda) \tag{2.6}
$$

where the remainder term $\Omega_3(\lambda)$ is of the form

$$\Omega_3(\lambda) = \sum_{j=0}^{n-1} \omega_{3,j}(\lambda) \frac{\sin^j \lambda a}{\lambda^{2n-j-2}}$$

with some functions $\omega_{3,j} \in \mathcal{L}^{(n-j)a}$. The function $\lambda\omega_{3,n-1}(\lambda)$ belongs to $\mathcal{L}^a$.

Proof. The asserted formula is immediate from the representations of χ and ϕ with the remainder term Ω_3 being equal to

$$\Omega_3(\lambda) = (i\alpha\lambda + \beta)\sigma_{n,n}(-B_p)\cos^n \lambda a \frac{1}{\lambda^{2n}} + \Omega_2(\lambda) + (i\alpha\lambda + \beta)\Omega_1(\lambda) =$$
$$= (i\alpha\lambda + \beta)\sigma_{n,n}(-B_p)\cos^n \lambda a \frac{1}{\lambda^{2n}} + \sum_{j=0}^{n-1} \omega_{2,j}(\lambda) \frac{\sin^j \lambda a}{\lambda^{2n-j-2}} +$$
$$+ i\alpha \sum_{j=0}^{n-1} \omega_{1,j}(\lambda) \frac{\sin^j \lambda a}{\lambda^{2n-j-1}} + \beta \sum_{j=0}^{n-1} \omega_{1,j}(\lambda) \frac{\sin^j \lambda a}{\lambda^{2n-j}} .$$

In order to rewrite Ω_3 appropriately, note that

$$\frac{\sin^j \lambda a}{\lambda^{2n-j-1}} = \sin \lambda a \frac{\sin^{j-1} \lambda a}{\lambda^{2n-(j-1)-2}}, \quad \frac{\sin^j \lambda a}{\lambda^{2n-j}} = \frac{\sin \lambda a}{\lambda} \frac{\sin^{j-1} \lambda a}{\lambda^{2n-(j-1)-2}}, \quad j \geq 1 .$$

Hence, setting

$$\omega_{3,j}(\lambda) := \begin{cases} \omega_{2,n-1}(\lambda) & , \quad j = n-1 \\ \omega_{2,j}(\lambda) + i\alpha\omega_{1,j+1}(\lambda) \sin \lambda a + \beta\omega_{1,j+1}(\lambda) \frac{\sin \lambda a}{\lambda} , & 1 \leq j \leq n-2 \end{cases}$$

$$\omega_{3,0}(\lambda) := \Big(\omega_{2,0}(\lambda) + i\alpha\omega_{1,1}(\lambda) \sin \lambda a + \beta\omega_{1,1}(\lambda) \frac{\sin \lambda a}{\lambda}\Big) + \Big(i\alpha\omega_{1,0}(\lambda) \frac{1}{\lambda} +$$
$$+ \beta\omega_{1,0}(\lambda) \frac{1}{\lambda^2}\Big) + \Big(i\alpha\sigma_{n,n}(-B_p) \cos^n \lambda a \frac{1}{\lambda} + \beta\sigma_{n,n}(-B_p) \cos^n \lambda a \frac{1}{\lambda^2}\Big)$$

we can write $\Omega_3(\lambda) = \sum_{j=0}^{n-1} \omega_{3,j}(\lambda) \frac{\sin^j \lambda a}{\lambda^{2n-j-2}}$. For $j \geq 1$ the functions $\omega_{3,j}$ obviously belong to $\mathcal{L}^{(n-j)a}$. Once we know that $\omega_{3,0}$ is entire, it will be clear that this function belongs to $\mathcal{L}^{na}$. However, multiplying the equation for $\Xi(\lambda)$ by λ^{2n-2} shows that $\omega_{3,0}$ is indeed entire. ❑

The asymptotics of the zeros of Ξ depends on the numbers B_j in a somewhat involved way. There appear not only the constants B_j, but also the following others: Let p be the polynomial $p(z) := \prod_{j=1}^{n}(z - \pi^{-1}B_j)$, and let $M_1, \ldots, M_{n-1}$ be the zeros of its derivative listed according to their multiplicities, i.e.

$$p'(z) = n \prod_{l=1}^{n-1} (z - M_l) .$$

Since p has only real zeros, namely the values $\pi^{-1}B_j$, also all numbers M_j are real.

2.7 Theorem. *Let a problem* (1.4) *with* $n > 1$ *be given, and assume that* $\alpha < n$. *Then, outside of some sufficiently large disk, the zeros of the characteristic function* Ξ *(including multiplicities) which are located in the right half-plane can be arranged in* n *sequences, namely*

$$\rho_{j;k} = \frac{\pi k}{a} + \frac{M_j}{k} + \frac{\beta_{j;k}}{k}, \quad k \geq k_0, \qquad j = 1, \ldots, n-1\,,$$
$$\rho_{n;k} := \frac{\pi(k-\frac{1}{2})}{a} + \frac{i}{2a}\log\frac{n+\alpha}{n-\alpha} + \frac{\tilde{B}}{\pi k} + \frac{\beta_{n;k}}{k}, \quad k \geq k_0\,,$$

where

$$\tilde{B} := \frac{1}{n}\sum_{j=1}^{n} B_j + \frac{\beta n}{n^2 - \alpha^2}\,,$$

and

$$\Bigg(\prod_{\substack{j' \in \{1,\ldots,n-1\} \\ s.t.\ M_{j'} = M_j}} \beta_{j';k}\Bigg)_{k\in\mathbb{N}} \in \ell^2,\ j = 1, \ldots, n-1, \qquad (\beta_{n;k})_{k\in\mathbb{N}} \in \ell^2\,.$$

Proof. The proof of this theorem proceeds in five steps. In Step 1 we define a comparison function. Then, in Steps 2–4, we establish some estimates which allow us to invoke Rouche's Theorem. This is then carried out in the final Step 5.

Step 1; A comparison function: Consider the sequences

$$\mathring{\rho}_{j;k} := \frac{\pi k}{a} + \frac{M_j}{k}, \quad k \in \mathbb{N}, \qquad j = 1, \ldots, n-1\,,$$
$$\mathring{\rho}_{n;k} := \frac{\pi(k-\frac{1}{2})}{a} + \frac{i}{2a}\log\frac{n+\alpha}{n-\alpha} + \frac{\tilde{B}}{\pi k}, \quad k \in \mathbb{N}\,.$$

The products

$$\mathring{s}_j(\lambda) := a\prod_{k=1}^{\infty}\frac{a^2(\mathring{\rho}_{j;k}^2 - \lambda^2)}{\pi^2 k^2}, \quad j = 1, \ldots, n-1\,,$$
$$\mathring{s}_n(\lambda) := \prod_{k=1}^{\infty}\frac{a^2(\overline{\mathring{\rho}_{n;k}} + \lambda)(\mathring{\rho}_{n;k} - \lambda)}{\pi^2(k-1/2)^2}\,,$$

converge locally uniformly on $\mathbb{C}$ and represent entire functions of exponential type a.

Employing suitable linear transformations, it is easy to deduce the following asymptotic formulas for $\mathring{s}_j$ from [M, Lemma 3.4.2]:

$$\mathring{s}_j(\lambda) = \frac{\sin\lambda a}{\lambda} - \pi M_j\frac{\cos\lambda a}{\lambda^2} + \frac{f_j(\lambda)}{\lambda^2}, \quad j = 1, \ldots, n-1\,, \tag{2.7}$$

$$\begin{aligned}\mathring{s}_n(\lambda) = {} & \cos\Big(\lambda a - \frac{i}{2}\log\frac{n+\alpha}{n-\alpha}\Big) + \tilde{B}\frac{\sin\big(\lambda a - \frac{i}{2}\log\frac{n+\alpha}{n-\alpha}\big)}{\lambda} + \frac{f_n(\lambda)}{\lambda} = \\ = {} & \frac{n}{\sqrt{n^2-\alpha^2}}\cos\lambda a + \frac{i\alpha}{\sqrt{n^2-\alpha^2}}\sin\lambda a + \\ & + \frac{n}{\sqrt{n^2-\alpha^2}}\tilde{B}\frac{\sin\lambda a}{\lambda} - \frac{i\alpha}{\sqrt{n^2-\alpha^2}}\tilde{B}\frac{\cos\lambda a}{\lambda} + \frac{f_n(\lambda)}{\lambda}\end{aligned} \tag{2.8}$$

with some functions $f_j \in \mathcal{L}^a$, $j = 1, \dots, n$. Note that, in particular, $\mathring{s}_j$ is bounded in each strip around the real line.

Let us now consider the function

$$\mathring{\Xi}(\lambda) := \sqrt{n^2 - \alpha^2} \prod_{k=1}^{n} \mathring{s}_j(\lambda) \,.$$

Using Lemma 2.3 with the functions $\mathring{s}_1, \dots, \mathring{s}_{n-1}$, it follows that, with some functions $\mathring{\omega}_j \in \mathcal{L}^{(n-1-j)a}$,

$$\mathring{\Xi}(\lambda) = \Big[\sum_{j=0}^{n-1} \sigma_{n-1,n-1-j}(-\pi M_p) \cos^{n-1-j} \lambda a \frac{\sin^j \lambda a}{\lambda^{2n-j-2}} + \sum_{j=0}^{n-2} \mathring{\omega}_j(\lambda) \frac{\sin^j \lambda a}{\lambda^{2n-j-2}} \Big] \cdot$$

$$\cdot \Big[n \cos \lambda a + i\alpha \sin \lambda a + n\tilde{B} \frac{\sin \lambda a}{\lambda} - i\alpha \tilde{B} \frac{\cos \lambda a}{\lambda} + \frac{f_n(\lambda)}{\lambda} \Big] =$$

Multiplying out, and remembering that

$$\sigma_{n-1,n-1-j}(-\pi M_p) = \pi^{n-1-j} \sigma_{n-1,n-1-j}(-M_p) = \\ = \pi^{n-1-j} \frac{j+1}{n} \sigma_{n,n-1-j}\Big(-\frac{B_p}{\pi}\Big) = \frac{j+1}{n} \sigma_{n,n-1-j}(-B_p) \,,$$

gives

$$\mathring{\Xi}(\lambda) = \sum_{j=0}^{n-1} (j+1) \sigma_{n,n-1-j}(-B_p) \cos^{n-j} \lambda a \frac{\sin^j \lambda a}{\lambda^{2n-j-2}} + \tag{2.9}$$

$$+ \sum_{j=0}^{n-1} \frac{j+1}{n} \sigma_{n,n-1-j}(-B_p) i\alpha \cos^{n-j-1} \lambda a \frac{\sin^{j+1} \lambda a}{\lambda^{2n-j-2}} + \tag{2.10}$$

$$+ \sum_{j=0}^{n-1} (j+1) \sigma_{n,n-1-j}(-B_p) \tilde{B} \cos^{n-j-1} \lambda a \frac{\sin^{j+1} \lambda a}{\lambda^{2n-j-1}} - \tag{2.11}$$

$$- \sum_{j=0}^{n-1} \frac{j+1}{n} \sigma_{n,n-1-j}(-B_p) i\alpha \tilde{B} \cos^{n-j} \lambda a \frac{\sin^j \lambda a}{\lambda^{2n-j-1}} + \tag{2.12}$$

$$+ \sum_{j=0}^{n-1} \underbrace{\frac{j+1}{n} \sigma_{n,n-1-j}(-B_p) \cos^{n-j-1} \lambda a f_n(\lambda)}_{\in \mathcal{L}^{(n-j)a}} \frac{\sin^j \lambda a}{\lambda^{2n-j-1}} + \tag{2.13}$$

$$+ \sum_{j=0}^{n-2} \underbrace{\sqrt{n^2 - \alpha^2} \mathring{s}_n(\lambda) \mathring{\omega}_j(\lambda)}_{\in \mathcal{L}^{(n-j)a}} \frac{\sin^j \lambda a}{\lambda^{2n-j-2}} \tag{2.14}$$

Step 2; Preliminary estimates: Let $\varepsilon \in (0,1]$ be fixed, set $r_k := \frac{\varepsilon}{k}$, and denote by $D^\varepsilon_{j;k}$, $j = 1, \dots, n$, $k \in \mathbb{N}$, the closed disk centered at $\mathring{\rho}_{j;k}$ with radius r_k. We make two observations. First, set

$$K_1 := \max \Big\{ \Big| \frac{\sin \lambda}{\lambda} \Big| : |z| \le a(1 + \max_{1 \le j \le n-1} |M_j|) \Big\} \,,$$

then a short computation shows that

$$|\lambda \sin \lambda a| \le K_1 \big[\pi(1 + \max_{1 \le j \le n-1} |M_j|) + a(1 + \max_{1 \le j \le n-1} |M_j|)^2 \big], \\ \lambda \in D^\varepsilon_{j;k}, \; k \in \mathbb{N}, \; j = 0, \dots, n-1 \,. \tag{2.15}$$

Second, let $l \in \mathbb{N}$ and $f \in \mathcal{L}^{la}$. Choose $a_{j;k}$ with $|a_{j;k}| = r_k$ such that

$$\left|f\left(\mathring{\rho}_{j;k} + a_{j;k}\right)\right| = \max_{z \in D_{j;k}^{\varepsilon}} |f(z)| .$$

Then we can deduce from [M, Lemma 1.4.3] by using the obvious linear transformation that $(|f(\mathring{\rho}_{j;k} + a_{j;k})|)_{k\in\mathbb{N}} \in \ell^2$, i.e.

$$\left(\max_{z \in D_{j;k}^{\varepsilon}} |f(z)| \right)_{k\in\mathbb{N}} \in \ell^2 . \tag{2.16}$$

Step 3; Ξ as a perturbation of $\mathring{\Xi}$: Looking at the difference $\Xi - \mathring{\Xi}$, we see that the sums (2.3) and (2.9) cancel. In order to estimate the other summands we proceed differently, depending whether $j \neq n$ or $j = n$.

Consider first the case that $j \in \{1, \dots, n-1\}$. Using (2.15), and assuming that k is sufficiently large so to ensure that $\mathring{\rho}_{j;k} - r_k > 0$, we can estimate the sums in (2.4)–(2.6) and (2.10)–(2.13) as

$$|\text{sum in } (2.6)|, |(2.10)|, |(2.12)|, |(2.13)| \leq \frac{c_1}{(\mathring{\rho}_{j;k} - r_k)^{2n-1}}, \quad \lambda \in D_{j;k}^{\varepsilon} ,$$

$$|(2.4)|, |(2.5)|, |(2.11)| \leq \frac{c_2}{(\mathring{\rho}_{j;k} - r_k)^{2n}}, \quad \lambda \in D_{j;k}^{\varepsilon} ,$$

with some constants $c_1, c_2 > 0$. Using (2.16), we obtain

$$|\Omega_3 \text{ in } (2.6)|, |(2.14)| \leq \frac{c_{3;k}}{(\mathring{\rho}_{j;k} - r_k)^{2n-2}}, \quad \lambda \in D_{j;k}^{\varepsilon} ,$$

with some sequence $(c_{3;k})_{k\in\mathbb{N}} \in \ell^2$. Together it follows that

$$|\Xi(\lambda) - \mathring{\Xi}(\lambda)| \leq \frac{\beta_{j;k}}{k^{2n-2}}, \ \lambda \in D_{j;k}^{\varepsilon}, k \in \mathbb{N}, \quad j = 1, \dots, n-1 , \tag{2.17}$$

with some sequences $(\beta_{j;k})_{k\in\mathbb{N}} \in \ell^2$.

We turn to the case that $j = n$. The difference $\Xi - \mathring{\Xi}$ can be written as

$$\begin{aligned}
\Xi(\lambda) - \mathring{\Xi}(\lambda) = {} & \frac{\sin^n \lambda a}{\lambda^n} \big[\beta - \sigma_{n,1}(-B_p) - n\tilde{B}\big] + \\
& + i\alpha \cos \lambda a \frac{\sin^{n-1} \lambda a}{\lambda^n} \Big[\tilde{B} + \sigma_{n,1}(-B_p)\frac{1}{n}\Big] + \\
& + \omega_{3,n-1}(\lambda) \frac{\sin^{n-1} \lambda a}{\lambda^{n-1}} + \sum_{j=0}^{n} \frac{\omega_{4,j}}{\lambda^{2n-j}} \\
= {} & \frac{\sin^{n-1} \lambda a}{\lambda^n} \sqrt{n^2 - \alpha^2} \cdot \cos\big(\lambda a - \frac{i}{2} \log \frac{n+\alpha}{n-\alpha}\big) \cdot \Big[\tilde{B} + \sigma_{n,1}(-B_p)\frac{1}{n}\Big] + \\
& + \frac{\sin^n \lambda a}{\lambda^n} \underbrace{\Big[\frac{\alpha^2}{n}\Big(\tilde{B} + \sigma_{n,1}(-B_p)\frac{1}{n}\Big) + \big(\beta - \sigma_{n,1}(-B_p) - n\tilde{B}\big)\Big]}_{=0} + \\
& + \omega_{3,n-1}(\lambda) \frac{\sin^{n-1} \lambda a}{\lambda^{n-1}} + \sum_{j=0}^{n} \frac{\omega_{4,j}(\lambda)}{\lambda^{2n-j}}
\end{aligned}$$

where $\omega_{4,j}$ are entire functions which are bounded in each strip around the real line, and $\omega_{4,n} \in \mathcal{L}^{na}$. Remembering that $\lambda\omega_{3,n-1}(\lambda) \in \mathcal{L}^a$, it follows that

$$|\Xi(\lambda) - \mathring{\Xi}(\lambda)| \leq \frac{\beta_{n;k}}{k^n}, \ \lambda \in D^\varepsilon_{n;k}, k \in \mathbb{N}\,, \tag{2.18}$$

with some sequence $(\beta_{n;k})_{k\in\mathbb{N}} \in \ell^2$.

Step 4; Estimating $\mathring{\Xi}$ from below: In this step we provide a below estimate for $\mathring{\Xi}$ on the circles $C^\varepsilon_{j;k} := \partial D^\varepsilon_{j;k}$. Thereby, we assume that $\varepsilon \in (0,1]$ is so small that

$$\varepsilon < \min\Big(\big\{|M_j - M_{j'}| :\, j,j' \in \{1,\ldots,n-1\} \text{ with } M_j \neq M_{j'}\big\} \cup \big\{\frac{1}{2a}\log\frac{n+\alpha}{n-\alpha}\big\}\Big)\,.$$

Then $\mathring{s}_{j'}$, $j' = 1,\ldots,n$, does not vanish on any punctured disk $D^\varepsilon_{j;k} \setminus \{\rho_{j;k}\}$, $j = 1,\ldots,n$, $k \in \mathbb{N}$. We again distinguish the cases that $j \neq n$ or $j = n$.

Assume first that $j \in \{1,\ldots,n-1\}$. Let $\lambda \in D^\varepsilon_{j;k}$, and write $\lambda = \mathring{\rho}_{j;k} + \tau$ with $|\tau| \leq \frac{\varepsilon}{k}$. By (2.7), we have for $j' \in \{1,\ldots,n-1\}$

$$\begin{aligned}
\mathring{s}_{j'}(\lambda) &= \frac{\sin\lambda a}{\lambda} - \pi M_{j'}\frac{\cos\lambda a}{\lambda^2} + \frac{f_{j'}(\lambda)}{\lambda^2} = \\
&= \frac{1}{(\frac{\pi k}{a} + \frac{M_j}{k} + \tau)^2}\Big[\sin\Big(\pi k + \frac{aM_j}{k} + a\tau\Big)\Big(\frac{\pi k}{a} + \frac{M_j}{k} + \tau\Big) - \\
&\quad - \pi M_{j'}\cos\Big(\pi k + \frac{aM_j}{k} + a\tau\Big) + f_j\Big(\frac{\pi k}{a} + \frac{M_j}{k} + \tau\Big)\Big] = \\
&= \frac{(-1)^k\pi}{(\frac{\pi k}{a} + \frac{M_j}{k} + \tau)^2}\Big[(M_j - M_{j'} + k\tau + \gamma_{j,k}) + O(k^{-2})\Big]
\end{aligned} \tag{2.19}$$

where $\gamma_{j,k} := \frac{(-1)^k}{\pi} f_j(\frac{\pi k}{a} + \frac{M_{j'}}{k} + \tau)$. Assume now that $\lambda \in C^\varepsilon_{j;k}$. Then, by our choice of ε and since $(\gamma_{j,k})_{k\in\mathbb{N}} \in \ell^2$, the number $M_{j'} - M_j + k\tau + \gamma_{j,k}$ is bounded away from zero for k sufficiently large.

We conclude that there exist $\gamma^\varepsilon_{j,j'} > 0$ and $k^\varepsilon_{j,j'} \in \mathbb{N}$ such that

$$|\mathring{s}_{j'}(\lambda)| \geq \frac{\gamma^\varepsilon_{j,j'}}{k^2}, \quad \lambda \in C^\varepsilon_{j;k},\ k \geq k^\varepsilon_{j,j'},\ j,j' = 1,\ldots,n-1\,.$$

By (2.8), we have

$$\lim_{k\to\infty} |\mathring{s}_n(z)| = \frac{n}{\sqrt{n^2 - \alpha^2}} \quad \text{uniformly for } z \in D^\varepsilon_{j;k}\,.$$

In particular, $|\mathring{s}_n(\lambda)| \geq \gamma^\varepsilon_{j,n}$, $\lambda \in D^\varepsilon_{j;k}$, $k \geq k^\varepsilon_{j,n}$, $j, = 1,\ldots,n-1$, with some $\gamma^\varepsilon_{j,n} > 0$ and $k^\varepsilon_{j,n} \in \mathbb{N}$. Together, this gives

$$\big|\mathring{\Xi}(\lambda)\big| \geq \frac{\gamma^\varepsilon_j}{k^{2n-2}}, \quad \lambda \in C^\varepsilon_{j;k},\ k \geq k^\varepsilon_j,\ j = 1,\ldots,n-1\,,$$

with some $\gamma_j > 0$ and $k^\varepsilon_j \in \mathbb{N}$.

We turn to the case that $j = n$. Let $\lambda \in D^\varepsilon_{n;k}$, and again write $\lambda = \mathring{\rho}_{j;k} + \tau$ with $|\tau| \leq \frac{\varepsilon}{k}$. For $j' = 1,\ldots,n-1$ we have

$$\lim_{k\to\infty} |\sin za| = \frac{n}{\sqrt{n^2 - \alpha^2}} \quad \text{uniformly for } z \in D^\varepsilon_{j;k}\,,$$

and hence $|\mathring{s}_{j'}(\lambda)| \geq \frac{\gamma^\varepsilon_{n,j'}}{k}$, $k \geq k^\varepsilon_{n,j'}$, $j' = 1, \ldots, n-1$, with some $\gamma^\varepsilon_{n,j'} > 0$ and $k^\varepsilon_{n,j'} \in \mathbb{N}$. Finally, consider the function $\mathring{s}_n$ on the disks $D^\varepsilon_{n,k}$. We have

$$\begin{aligned}
\mathring{s}_n(\lambda) =& \cos\Big(\lambda a - \frac{i}{2}\log\frac{n+\alpha}{n-\alpha}\Big) + \tilde{B}\frac{\sin\big(\lambda a - \frac{i}{2}\log\frac{n+\alpha}{n-\alpha}\big)}{\lambda} + \frac{f_n(\lambda)}{\lambda} = \\
=& \frac{(-1)^k}{\lambda}\Big[\cos\Big(\frac{\pi}{2} + \frac{a\tilde{B}}{\pi k} + \tau a\Big)\Big(\frac{\pi(k-\frac{1}{2})}{a} + \frac{i}{2a}\log\frac{n+\alpha}{n-\alpha} + \frac{\tilde{B}}{\pi k} + \tau\Big) + \\
& + \tilde{B}\sin\Big(\frac{\pi}{2} + \frac{a\tilde{B}}{\pi k} + \tau a\Big) + (-1)^k f_n(\lambda)\Big] = \\
=& \frac{(-1)^k}{\lambda}\Big[-\pi\tau k + O(k^{-1})\Big] \qquad (2.20)
\end{aligned}$$

Together, this gives

$$\big|\mathring{\Xi}(\lambda)\big| \geq \frac{\gamma^\varepsilon_n}{k^n}, \quad \lambda \in C^\varepsilon_{n;k},\ k \geq k^\varepsilon_n,\ ,$$

with some $\gamma^\varepsilon_n > 0$ and $k^\varepsilon_n \in \mathbb{N}$.

Step 5; Application of Rouche's Theorem: If ε is given as in Step 4, we can combine the estimates established in Steps 3 and 4, to obtain

$$\big|\Xi(\lambda) - \mathring{\Xi}(\lambda)\big| < \big|\mathring{\Xi}(\lambda)\big|, \quad \lambda \in C^\varepsilon_{j;k},\ k \geq k_\varepsilon,\ j = 1, \ldots, n\,,$$

with some $k_\varepsilon \in \mathbb{N}$. By Rouche's Theorem the functions Ξ and $\mathring{\Xi}$ have the same number of zeros (including multiplicities) inside each disk $D^\varepsilon_{j;k}$, $k \geq k_\varepsilon$, $j = 1, \ldots, n-1$. However, the function $\mathring{\Xi}$ has only one zero in $D^\varepsilon_{j;k}$, namely $\mathring{\rho}_{j;k}$, and the multiplicity of this zero is equal to

$$m(j) := \begin{cases} \#\{j' \in \{1, \ldots, n-1\} : M_{j'} = M_j\}, & j \in \{1, \ldots, n-1\} \\ 1 & , \quad j = n \end{cases}.$$

Fix ε_0, and choose $\Delta^l_{j;k} \in \mathbb{C}$, $|\Delta^l_{j;k}| \leq \varepsilon_0$, $l = 1, \ldots, m(j)$, such that

$$\rho^l_{j;k} := \mathring{\rho}_{j;k} + \frac{\Delta^l_{j;k}}{k}, \quad l = 1, \ldots, m(j),\ k \geq k_\varepsilon,\ j = 1, \ldots, n\,,$$

are all zeros of Ξ in $D^{\varepsilon_0}_{j;k}$ listed according to their multiplicities.

For each given $\varepsilon > 0$, we know that the point $\rho^l_{j;k}$ lies in $D^\varepsilon_{j;k}$ when k is sufficiently large. This means that

$$\lim_{k\to\infty} \Delta^l_{j;k} = 0, \quad l = 1, \ldots, m(j),\ j = 1, \ldots, n\,.$$

Let $j \in \{1, \ldots, n-1\}$. By (2.19) we have for $j' \in \{1, \ldots, n-1\}$

$$\mathring{s}_{j'}(\rho^l_{j;k}) = \frac{(-1)^k\pi}{(\frac{\pi k}{a} + \frac{M_j}{k} + \frac{\Delta^l_{j;k}}{k}\tau)^2}\Big[(M_{j'} - M_j + \Delta^l_{j;k} + \gamma^l_{j,k})\pi + O(k^{-2})\Big]$$

and hence

$$\begin{aligned}
\mathring{\Xi}(\rho^l_{j;k}) = \frac{\gamma_k}{k^{2n-2}} &\cdot \prod_{\substack{j'=1 \\ M_{j'} \neq M_j}}^{n-1} \Big[(M_{j'} - M_j + \Delta^l_{j;k} + \gamma^l_{j,k})\pi + O(k^{-2})\Big] \cdot \\
&\cdot \prod_{l=1}^{m(j)} \Big[(\Delta^l_{j;k} + \gamma^l_{j,k})\pi + O(k^{-2})\Big],
\end{aligned}$$

with $(\gamma_k)_{k\in\mathbb{N}}$ having a nonzero limit. However, by (2.17),

$$|\mathring{\Xi}(\rho^l_{j;k})| = |\Xi(\rho^l_{j;k}) - \mathring{\Xi}(\rho^l_{j;k})| \leq \frac{\beta_{j;k}}{k^{2n-2}}$$

with $(\beta_{j;k})_{k\in\mathbb{N}} \in \ell^2$, and we conclude that

$$\Big(\prod_{l=1}^{m(j)} \Delta^l_{j;k}\Big)_{k\in\mathbb{N}} \in \ell^2 .$$

Using (2.20), we obtain that $|\mathring{\Xi}(\rho^1_{n;k})| = \frac{\gamma'_k}{k^n}\Delta^1_{n;k}$ with $(\gamma'_k)_{k\in\mathbb{N}}$ again having a nonzero limit. Now (2.18) implies that

$$\big(\Delta^1_{n;k}\big)_{k\in\mathbb{N}} \in \ell^2 .$$

❑

2.8 Remark. In case of $\alpha > n$ we have in Theorem 2.7

$$\rho_{n;k} = \frac{\pi k}{a} + \frac{i}{2a}\log\frac{\alpha+n}{\alpha-n} + \frac{\tilde{B}}{\pi k} + \frac{\beta_{n;k}}{k}.$$

If $\alpha = n$, the situation is more complicated. In fact, then $\mathrm{Im}\rho_{n;k} \to \infty$. We will return to this case in forthcoming work. //

2.9 Remark. If $q_1(x) \equiv q_2(x) \equiv ... \equiv q_n(x)$ we have

$$\Xi(\lambda) = (ns'_1(\lambda, a) + (i\alpha\lambda + \beta)s_1(\lambda, a))s_1^{n-1}(\lambda, a)$$

and, therefore, the spectrum consists of

(1) $(n-1)$–multiplicity real eigenvalues of the Dirichlet problem on an edge which are independent of α and of β, and

(2) geometrically simple eigenvalues of the Regge-type problem which depend of α and of β.

This fact has been seen earlier, cf. [Pa, EL]. //

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V. Pivovarchik
Department of Applied Mathematics and Informatics
South-Ukrainian State Pedagogical University
Staroportofrankovskaya str. 26
UA–65020 Odessa
UKRAINE
email: v.pivovarchik@paco.net, vnp@dtp.odessa.ua

H. Woracek
Institut for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstr. 8–10/101
A–1040 Vienna
AUSTRIA
email: harald.woracek@tuwien.ac.at