Pontryagin spaces of entire functions V

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Abstract

The spectral theory of a two-dimensional canonical (or 'Hamiltonian') system is closely related with two notions, depending whether Weyl's limit circle or limit point case prevails. Namely, with its monodromy matrix or its Weyl coefficient, respectively. A Fourier transform exists which relates the differential operator induced by the canonical system to the operator of multiplication by the independent variable in a reproducing kernel space of entire 2-vector valued functions or in a weighted L^2-space of scalar valued functions, respectively.

Motivated from the study of canonical systems or Sturm-Liouville equations with a singular potential and from other developments in Pontryagin space theory, we have suggested a generalization of canonical systems to an indefinite setting which includes a finite number of inner singularities. We have constructed an operator model for such 'indefinite canonical systems'. The present paper is devoted to the construction of the corresponding monodromy matrix or Weyl coefficient, respectively, and of the Fourier transform.

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1 Introduction

A two-dimensional canonical (or Hamiltonian) system is an initial value problem of the form

$$y'(t) = zJH(t)y(t), \quad t \in [s_-, s_+),\ y(s_-) = y_0\,, \tag{1.1}$$

where z is a complex parameter, J denotes the symplectic matrix

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and H is a 2×2-matrix valued function with $H(t) \geq 0$ for $t \in (s_-, s_+)$ a.e., which is locally integrable and does not vanish identically on any set of positive measure. The function H is called the Hamiltonian of the system (1.1). A Hamiltonian H is called regular, if $\int_{s_-}^{s_+} \operatorname{tr} H(t)\, dt < \infty$, and singular otherwise. One also speaks of Weyl's limit circle or limit point case instead of regular or singular, respectively.

The interpretation of (1.1) as a differential operator takes place in a certain L^2-space of 2-vector valued functions. In fact, in order to investigate the spectral theory of a canonical system, one constructs a boundary triplet $\mathfrak{B}(H) = (L^2(H), T_{\max}(H), \Gamma(H))$. This operator theoretic viewpoint goes back to [K], for a more recent compilation see [HSW].

Depending whether limit circle or limit point case prevails, the system (1.1) shows significantly different behaviour. In the following we denote by $W(t,z) = (w_{ij}(t,z))_{i,j=1,2}$ the unique solution of the initial value problem

$$\frac{\partial}{\partial t} W(t,z)J = zW(t,z)H(t),\ t \in [s_-, s_+), \quad W(s_-,z) = I\,. \tag{1.2}$$

Limit circle case: The function $W(t,z)$ admits a continuous extension to s_+. The matrix function $W(s_+,z)$, sometimes also called the monodromy matrix of H, belongs to the class $\mathcal{M}_0$, i.e. the entries of $W(s_+,z)$ are entire functions which are real for real z, $\det W(s_+,z) = 1$, and

$$\frac{W(s_+,z)JW(s_+,z)^* - J}{z - \overline{z}} \geq 0, \quad \operatorname{Im} z > 0\,. \tag{1.3}$$

The family $\omega_H := (W(t,z))_{t\in[s_-,s_+]}$ is, in the language of §3.b below, a finite maximal chain of matrices going downwards from the monodromy matrix $W(s_+,z)$.

The symmetric operator $T_{\min}(H) := T_{\max}(H)^*$ has defect index $(2,2)$, is completely nonselfadjoint, and the set of its points of regular type satisfies $r(T_{\min}(H)) = \mathbb{C}$. The selfadjoint extensions of $T_{\min}(H)$ have compact resolvents, in particular, their spectrum is discrete. The monodromy matrix is a (regularized) u-resolvent matrix of a certain symmetric extension of $T_{\min}(H)$ with defect 1. A Fourier transform exists which maps $L^2(H)$ isometrically onto the reproducing kernel Hilbert space generated by $W(s_+,z)$ via the kernel (1.3). The elements of this reproducing kernel space are entire $\mathbb{C}^2$-valued functions, and the operator $T_{\min}(H)$ corresponds to the operator of multiplication by the independent variable z.

Limit point case: We have $\lim_{t\nearrow s_+} \operatorname{tr}(W(t,0)'J) = +\infty$. The family $\omega_H := (W(t,z))_{t\in[s_-,s_+)}$ is, in the language of [KW/III], a maximal chain of matrices.

Write $W(t,z) = (w_{ij}(t,z))_{i,j=1,2}$. Then, for each $\tau \in \mathbb{R}\cup\{\infty\}$, the limit

$$q_H(z) := \lim_{t\nearrow s_+} \frac{w_{11}(t,z)\tau + w_{12}(t,z)}{w_{21}(t,z)\tau + w_{22}(t,z)} \tag{1.4}$$

exists locally uniformly on $\mathbb{C}\setminus\mathbb{R}$ and does not depend on τ. The function q_H is called the Titchmarsh-Weyl coefficient associated to the Hamiltonian H. It belongs to the Nevanlinna class $\mathcal{N}_0$, i.e. is analytic on $\mathbb{C}\setminus\mathbb{R}$, satisfies $q_H(\overline{z}) = \overline{q_H(z)}$, $z \in \mathbb{C}\setminus\mathbb{R}$, and

$$\operatorname{Im} q_H(z) \geq 0, \quad \operatorname{Im} z > 0\,. \tag{1.5}$$

The symmetric operator $T_{\min}(H) := T_{\max}(H)^*$ has defect index $(1,1)$ and is completely nonselfadjoint. However, the selfadjoint extensions of $T_{\min}(H)$ may have continuous spectrum. The function q_H can be viewed as a Q-function of $T_{\min}(H)$. A Fourier transform exists which maps $L^2(H)$ isometrically onto the space $L^2(\sigma)$ where σ is the measure in the Herglotz–integral representation of q_H (appropriately including a possible point mass at ∞). Thereby, the operator $T_{\min}(H)$ corresponds to a restriction of the operator of multiplication by the independent variable t.

In [KW/IV] we have, as a generalization of the notion of a Hamiltonian function H to an indefinite setting, introduced the notion of general Hamiltonians

$\mathfrak{h}$ which involves a finite number of singularities, cf. Definition IV.8.1 or §3.e below. For each general Hamiltonian we have constructed a Pontryagin space boundary triplet $\mathfrak{B}(\mathfrak{h}) = (\mathcal{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h}))$, which is an indefinite analogue of the boundary triplet $\mathfrak{B}(H)$, and showed that it shares the most important operator theoretic properties of $\mathfrak{B}(H)$. Our aim in the present paper is to establish the indefinite analogoues of the above mentioned items related to the chain ω_H and to the Weyl coefficient q_H.

The main difficulty is to actually construct a (finite) maximal chain $\omega_\mathfrak{h}$ for a given general Hamiltonian $\mathfrak{h}$. Most of this paper is devoted to the construction of $\omega_\mathfrak{h}$ and to the development of the machinery needed for it. Unlike in the positive definite situation, where ω_H is simply obtained as the solution of (1.2), in the indefinite case it is not at all clear how to define $\omega_\mathfrak{h}$. Of course, in between each two singularities we should have a solution of the differential equation in (1.2), the problem is to understand how to 'jump over an inner singularity'. In order to construct $\omega_\mathfrak{h}$, we will combine indefinite analogues of the classical differential equation oriented approach interpreting $W(t,z)$ as boundary values of defect elements, and the more operator theoretic approach via the (generalized) u-resolvent matrix of a certain selfadjoint extension of the minimal operator.

Besides constructing $\omega_\mathfrak{h}$ and proving that it indeed is a (finite) maximal chain (and thereby in particular associating a monodromy matrix to a regular general Hamiltonian), we will also prove in this paper existence of a Fourier transform in both cases, regular and singular. Moreover, we will show that in the singular case the Weyl-coefficient $q_\mathfrak{h}$ can be identified as a Q-function of the minimal operator.

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Let us outline the content of these sections. In Section 2 we consider entire matrix functions of the class $\mathcal{M}_{<\infty}$. We construct, for each $W \in \mathcal{M}_{<\infty}$, a boundary triplet $\mathfrak{B}(W)$ and investigate its properties. Moreover, we recall some relations of $\mathcal{M}_{<\infty}$ to other classes of functions. In Section 3 we deal with maximal chains of matrices and general Hamiltonians. We prove some results which supplement [KW/III] and [KW/IV]. In Section 4 we associate to each boundary triplet $\mathfrak{B}$ an entire matrix function $\omega(\mathfrak{B})$ by means of boundary values, and make the connection with another line of approach by showing that it is a (generalized) u-resolvent matrix in the sense of [KW/0], cf. Theorem 4.20. The matrix $\omega(\mathfrak{B})$ and its properties is vital for our purposes. We will pay particular attention to the situation that $\mathfrak{B}$ is of the form $\mathfrak{B}(\mathfrak{h})$ with a general Hamiltonian $\mathfrak{h}$. Moreover, it is a noteworthy fact that for boundary triplets of the form $\mathfrak{B}(W)$, the construction of $\omega(\mathfrak{B})$ is converse to the construction introduced in §2. In fact, $\omega(\mathfrak{B}(W)) = W$ whenever $W \in \mathcal{M}_{<\infty}$. In Section 5 we give the definition of $\omega_\mathfrak{h}$ for a general Hamiltonian $\mathfrak{h}$, and prove that $\omega_\mathfrak{h}$ actually is a (finite) maximal chain, cf. Theorem 5.1 which can be regarded as the main result of this paper.

Moreover, in the singular case, we give a representation of $q_{\mathfrak{h}}$ as a Q-function associated with a certain selfadjoint extension of the minimal operator. Finally, in Section 6, we construct the Fourier transforms for both cases, regular or singular. In the singular case, thereby the space $L^2(\sigma)$ is substituted by a Pontryagin space induced by a distribution which represents $q_{\mathfrak{h}}$ by application to Poisson-kernels.

Through all levels of development, some operations can be defined; rotation, reversing, and the splitting-and pasting method, cf. §2.b,c. §3.c,e, §4.a,e, §5.d. These operations are a somewhat technical, but essential, tool throughout our considerations.

In its flavour, this paper is operator theoretic. Our methods are based on the theory of very special Pontryagin space boundary triplets which possess the specific properties as introduced in [KW/IV]. There is a vast literature on boundary triplets leading in its generality far beyond the special case treated in detail here. For the Hilbert space case we refer the interested reader for example to [DHMS/I], [DHMS/II] and [DHMS/III]. The algebraic and geometric properties of general boundary triplets and related objects in operator theory remain more or less the same if they are studied not only in Hilbert spaces, but also in the indefinite situation of Pontryagin or even Krein spaces. Regarding this we refer the reader for example to [D/I] and [B].

Let us close this introduction with a technical notice: References to [KW/0]–[KW/IV] will be given as the following examples indicate. E.g. Lemma 0.2.1 refers to Lemma 2.1 of [KW/0], (I.2.1) refers to the equation (2.1) of [KW/I], or Theorem IV.8.6 to Theorem 8.6 of [KW/IV].

2 Matrices of the class $\mathcal{M}_{<\infty}$

In this section we discuss entire 2×2-matrix functions for which a certain kernel function has a finite number of negative squares. After having recalled the definition of $\mathcal{M}_{<\infty}$, this section is divided into five subsections:

a. We show that each nonconstant function $W \in \mathcal{M}_{<\infty}$ generates a boundary triplet $\mathfrak{B}(W)$ in the sense of Definition IV.2.7.
b. We introduce two operations, namely $\circlearrowleft_\alpha$ and rev, with functions from $\mathcal{M}_{<\infty}$ as well as with boundary triplets, and provide some of their properties. Though elementary, these operations will be a very practical tool throughout the whole paper. See [DHMS/I] and [DHMS/III] for related operations in a more general setting.
c. Matrix polynomials do belong to $\mathcal{M}_{<\infty}$. Here we discuss the boundary triplet generated by matrix polynomials of specific form.
d. The class $\mathcal{M}_{<\infty}$ is closed with respect to products. In this subsection we make the relation between the boundary triplets $\mathfrak{B}(W_1)$, $\mathfrak{B}(W_2)$, and $\mathfrak{B}(W_1 \cdot W_2)$ explicit.
e. The class $\mathcal{M}_{<\infty}$ is closely related to other classes of functions. We recall some results on the relationship with indefinite Hermite-Biehler functions and with generalized Nevanlinna functions.

Let us come to the definition of $\mathcal{M}_{<\infty}$. For a complex valued function f defined on some subset D of the complex plane, we denote by $f^\#$ the function $f^\#(z) := \overline{f(\overline{z})}$, which is defined on the set $D^\# := \{z \in \mathbb{C} : \overline{z} \in D\}$. We call f real, if

$D^{\#} = D$ and $f^{\#}(z) = f(z)$, $z \in D$. If W is an analytic 2×2-matrix valued function defined on some open subset D of the complex plane, and satisfies $W(z)JW(\overline{z})^* = J$, $z, \overline{z} \in D$, we consider the kernel H_W defined as

$$H_W(w,z) := \frac{W(z)JW(w)^* - J}{z - \overline{w}}, \ z, w \in D .$$

For $z = \overline{w}$ this formula has to be interpreted appropriately as a derivative, which is possible by analyticity.

2.1 Definition. Let $W = (w_{ij})_{i,j=1,2}$ be a 2×2-matrix valued function defined on $\mathbb{C}$, and let $\kappa \in \mathbb{N}_0$. We write $W \in \mathcal{M}_\kappa$, if

(M1) The entries w_{ij} of W are real and entire functions.

(M2) We have $\det W(z) = 1$, $z \in \mathbb{C}$, and $W(0) = I$.

(M3) The kernel H_W has κ negative squares on $\mathbb{C}$.

Note in this place that the conditions (M1) and (M2) together imply that $W(z)JW(\overline{z})^* = J$.

We will moreover use the notation

$$\mathcal{M}_{<\infty} := \bigcup_{\nu \in \mathbb{N} \cup \{0\}} \mathcal{M}_\nu ,$$

and write $\operatorname{ind}_- W = \kappa$ to express the fact that a matrix function $W \in \mathcal{M}_{<\infty}$ belongs to $\mathcal{M}_\kappa$. For $W \in \mathcal{M}_{<\infty}$, set

$$\mathfrak{t}(W) := \operatorname{tr}(W'(0)J) . \tag{2.0}$$

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a. The boundary triplet associated with $W \in \mathcal{M}_{<\infty}$.

A matrix function $W \in \mathcal{M}_{<\infty}$ generates via the kernel H_W in a standard way a reproducing kernel Pontryagin space $\mathfrak{K}(W)$, cf. [ADSR]. In fact, $\mathfrak{K}(W)$ is the Pontryagin space completion of the inner product space defined by

$$\mathcal{L}(W) := \operatorname{span}\left\{ H_W(w,.)v : \ w \in \mathbb{C}, v \in \mathbb{C}^2 \right\} ,$$

$$\left[H_W(w_1,.)v_1, H_W(w_2,.)v_2 \right] := v_2^* H_W(w_1,w_2)v_1 .$$

The elements of this space are entire 2-vector valued functions. Besides its Pontryagin space structure, $\mathfrak{K}(W)$ carries a conjugate linear and anti-isometric involution: Consider the map $.^{\#}$ defined on the set of all entire 2-vector valued functions by

$$\begin{pmatrix} F \\ G \end{pmatrix}^{\#} := \begin{pmatrix} F^{\#} \\ G^{\#} \end{pmatrix} .$$

Since the entries of $W(z)$ are real, we have

$$\begin{aligned} \big(H_W(w,z)v\big)^{\#} &= \Big(\frac{W(z)JW(w)^* - J}{z - \overline{w}} v\Big)^{\#} = \\ &= \frac{W(z)JW(\overline{w})^* - J}{z - w} \overline{v} = H_W(\overline{w}, z)\overline{v} , \end{aligned} \tag{2.1}$$

and

$$\big[H_W(\overline{w_2},z)\overline{v_2},H_W(\overline{w_1},z)\overline{v_1}\big]=\overline{v_1}^*H_W(\overline{w_2},\overline{w_1})\overline{v_2}=\big(\overline{v_1}^*H_W(\overline{w_2},\overline{w_1})\overline{v_2}\big)^T=$$

$$=v_2^*\Big(\frac{W(\overline{w_1})JW(\overline{w_2})^*-J}{\overline{w_1}-w_2}\Big)^Tv_1=v_2^*\frac{-W(w_2)JW(w_1)^*+J}{\overline{w_1}-w_2}v_1=$$

$$=v_2^*H_W(w_1,w_2)v_1=\big[H_W(w_1,z)v_1,H_W(w_2,z)v_2\big]\,.$$

Hence $.^\#$ maps $\mathcal{L}(W)$ conjugate linearly and anti-isometrically onto itself. With a standard continuity argument, it follows that $.^\#$ maps $\mathfrak{K}(W)$ conjugate linearly and anti-isometrically onto itself.

It is a deeper result that $\mathfrak{K}(W)$ is closed with respect to difference quotients, cf. Proposition I.8.3: Denote by $\mathcal{R}(w)$ the operator

$$\big(\mathcal{R}(w)F\big)(z):=\frac{F(z)-F(w)}{z-w}\,.$$

Then, for each $w\in\mathbb{C}$, we have $\mathcal{R}(w)\mathfrak{K}(W)\subseteq\mathfrak{K}(W)$, and

$$G(u)^*JF(w)=[F,\mathcal{R}(u)G]-[\mathcal{R}(w)F,G]+(w-\overline{u})[\mathcal{R}(w)F,\mathcal{R}(u)G]\,, \qquad (2.2)$$

whenever $w,u\in\mathbb{C}$ and $F,G\in\mathfrak{K}(W)$.

2.2 Definition. Let $W\in\mathcal{M}_{<\infty}$, $W\neq I$, be given. Define $T(W)\subseteq\mathfrak{K}(W)^2$ as

$$T(W):=\operatorname{cls}\big\{(H_W(w,.)v;\overline{w}H_W(w,.)v):\,w\in\mathbb{C},v\in\mathbb{C}^2\big\}\,,$$

and $\Gamma(W)\subseteq T\times(\mathbb{C}^2\times\mathbb{C}^2)$ as

$$\Gamma(W):=\operatorname{cls}\big\{\big((H_W(w,.)v;\overline{w}H_W(w,.)v);(v;W(w)^*v)\big):\,w\in\mathbb{C},v\in\mathbb{C}^2\big\}\,.$$

Moreover, set $S(W):=T(W)^*$. //

For similar constructions involving Nevanlinna pairs in a broader setting see [BHS] and [D/II].

2.3 Proposition. *Let $W\in\mathcal{M}_{<\infty}$, $W\neq I$, be given. Then $\mathfrak{B}(W):=(\mathfrak{K}(W),T(W),\Gamma(W))$ is a boundary triplet which has defect 2 and satisfies (E), cf. Definition IV.2.8, Definition IV.2.16. The symmetry $S(W)$ is completely nonselfadjoint and the set of its points of regular type satisfies $r(S(W))=\mathbb{C}$.*

Proof.
Step 1: First we show that $T(W)$ and $\Gamma(W)$ respect the involution $.^\#$ on $\mathfrak{K}(W)$, and that the abstract Green's identity holds.

From the computation (2.1) we obtain

$$\big((H_W(w,z)v)^\#;(\overline{w}H_W(w,z)v)^\#\big)=\big(H_W(\overline{w},z)\overline{v};wH_W(\overline{w},z)\overline{v}\big)\in T(W)\,,$$

$$\Big(\big((H_W(w,z)v)^\#;(\overline{w}H_W(w,z)v)^\#\big);\big(\overline{v};\overline{W(w)^*v}\big)\Big)=$$
$$=\Big(\big(H_W(\overline{w},z)\overline{v};wH_W(\overline{w},z)\overline{v}\big);\big(\overline{v},W(\overline{w})^*\overline{v}\big)\Big)\in\Gamma(W)\,.$$

It follows by continuity that

$$(f;g)\in T(W)\iff(f^\#;g^\#)\in T(W)$$

$$\big((f;g);(a;b)\big)\in\Gamma(W) \iff \big((f^\#;g^\#);(\overline{a};\overline{b})\big)\in\Gamma(W)$$

The abstract Green's identity (2.2) follows with the help of linearity and continuity from

$$\big[\overline{w_1}H_W(w_1,z)v_1, H_W(w_2,z)v_2\big]-\big[H_W(w_1,z)v_1,\overline{w_2}H_W(w_2,z)v_2\big]=$$
$$=(\overline{w_1}-w_2)v_2^*H_W(w_1,w_2)v_1=-v_2^*W(w_2)JW(w_1)^*v_1+v_2^*Jv_1\,.$$

Step 2: We turn to a closer inspection of $S(W)$. It is apparent from the definition of $T(W)$ that

$$S(W)=\big\{(f(z);zf(z)):\, f(z),zf(z)\in\mathfrak{K}(W)\big\}\,.$$

Since $\mathcal{R}(\eta)$ maps $\mathfrak{K}(W)$ into itself, we obtain

$$\operatorname{ran}\big(S(W)-\eta\big)=\big\{f\in\mathfrak{K}(W):\, f(\eta)=0\big\},\ \eta\in\mathbb{C}\,.$$

In particular, $\operatorname{ran}(S(W)-\eta)$ is closed and

$$\operatorname{ran}\big(S(W)-\eta\big)^\perp=\big\{H_W(\eta,.)v:\, v\in\mathbb{C}^2\big\}\,. \tag{2.3}$$

Obviously, $\ker(S(W)-\eta)=\{0\}$ for all $\eta\in\mathbb{C}$. Hence $r(S(W))=\mathbb{C}$. Moreover, we see that $S(W)$ is completely nonselfadjoint. Using (I.8.4), we obtain

$$S(W)\subseteq\big\{(\mathcal{R}_1(0)g;g):\, g\in\mathfrak{K}(W)\big\}\subseteq S(W)^*=T(W)\,,$$

i.e. $S(W)$ is symmetric. The defect index of $S(W)$ is, by (2.3), given as

$$\begin{cases}(2,2) & ,\ \{H_W(0,.)\binom{1}{0},H_W(0,.)\binom{0}{1}\}\text{ linearly independent}\\ (1,1) & ,\ \{H_W(0,.)\binom{1}{0},H_W(0,.)\binom{0}{1}\}\text{ linearly dependent}, H_W(0,.)\neq 0\\ (0,0) & ,\ H_W(0,.)=0\end{cases}$$

The case of defect $(0,0)$, however, cannot occur, since W is not constant.

By (2.3) and $r(S(W))=\mathbb{C}$, the dimension of $\ker H_W(\eta,.)$ does not depend on $\eta\in\mathbb{C}$. Let us show that actually $\ker H_W(\eta,.)$ is independent of $\eta\in\mathbb{C}$. To this end assume that $m\in\ker H_W(0,.)$, i.e.

$$\big(W(z)J-J\big)m=0,\ z\in\mathbb{C}\,. \tag{2.4}$$

Using $W(z)^{-1}J=JW(\overline{z})^*$, we obtain

$$m=W(\overline{z})^*m,\ z\in\mathbb{C}\,, \tag{2.5}$$

and hence

$$(z-\overline{\eta})H_W(\eta,z)m=\big(W(z)JW(\eta)^*-J\big)m=0,\ z,\eta\in\mathbb{C}\,.$$

Step 3: Next we establish the required properties of $\Gamma(W)$. To start with note the following consequences of the Green's identity: If $\mathfrak{J}$ denotes the Gram-matrix $\mathfrak{J}:=\begin{pmatrix}J&0\\0&-J\end{pmatrix}$ on $\mathbb{C}^2\times\mathbb{C}^2$, then

(i) $\operatorname{mul}\Gamma(W)$ is $\mathfrak{J}$-neutral and $\operatorname{ran}\Gamma(W) \subseteq \operatorname{mul}\Gamma(W)^{\perp_{\mathfrak{J}}}$;

(ii) $\ker\Gamma(W) \subseteq T(W)^* = S(W)$.

By (i) we are left with the possibilities

$$\dim \operatorname{mul}\Gamma(W) = 0, 1, 2$$

and, correspondingly,

$$\dim \operatorname{ran}\Gamma(W) \leq \begin{cases} 4 & , \dim \operatorname{mul}\Gamma(W) = 0 \\ 3 & , \dim \operatorname{mul}\Gamma(W) = 1 \\ 2 & , \dim \operatorname{mul}\Gamma(W) = 2 \end{cases} .$$

Since, for any linear relation $G \subseteq X \times Y$ with $\dim Y < \infty$ and $\operatorname{dom} G = X$, the inequality

$$\dim(X/\ker G) \leq \dim \operatorname{ran} G - \dim \operatorname{mul} G$$

holds, it follows that

$$\dim(T(W)/\ker\Gamma(W)) \leq \begin{cases} 4 & , \dim \operatorname{mul}\Gamma(W) = 0 \\ 2 & , \dim \operatorname{mul}\Gamma(W) = 1 \\ 0 & , \dim \operatorname{mul}\Gamma(W) = 2 \end{cases} .$$

Note here that by (i) in particular $\operatorname{mul}\Gamma(W) \subseteq \operatorname{mul}\Gamma(W)^{\perp_{\mathfrak{J}}}$. It follows from ($ii$) that the case $\ker\Gamma(W) = T(W)$ cannot occur, and that

$$\dim(T(W)/S(W)) = 4 \quad \Longrightarrow \quad \operatorname{mul}\Gamma(W) = 0, \ker\Gamma(W) = S(W)$$

Assume that $\dim T(W)/S(W) = 2$. Then there exists $m \in \mathbb{C}^2 \setminus \{0\}$ such that $H_W(0,.)m = 0$. Hence we obtain

$$\big((0;0);(m;m)\big) = \big((H_W(0,z)m;0);(m;m)\big) \in \Gamma(W)\,,$$

i.e. $(m;m) \in \operatorname{mul}\Gamma(W)$. It follows that $\dim \operatorname{mul}\Gamma(W) = 1$ and thus also that $\dim T(W)/\ker\Gamma(W) \leq 2$. Combining this with ($ii$) yields $\ker\Gamma(W) = S(W)$, and $\operatorname{mul}\Gamma(W) = \operatorname{span}\{(m;m)\}$.

Let us state explictily that, in any case,

$$\operatorname{mul}\Gamma(W) = \operatorname{span}\{(m;m)\} \text{ where } \operatorname{span}\{m\} = \ker H_W(z,0)\,. \tag{2.6}$$

Step 4: So far we have shown that $\mathfrak{B}(W)$ is a boundary triplet with defect 2, that $S(W)$ is completely nonselfadjoint, and that $r(S(W)) = \mathbb{C}$. It remains to show that (E) holds.

We have

$$\ker\big(T(W) - \eta\big) = \operatorname{ran}\big(S(W) - \overline{\eta}\big)^{\perp} = \big\{H_W(\overline{\eta},.)v : v \in \mathbb{C}^2\big\}\,.$$

Let $v \in \mathbb{C}^2$ be such that $H_W(\overline{\eta},.)v \neq 0$, and let $a, b \in \mathbb{C}^2$ be such that

$$\big((H_W(\overline{\eta},z)v;\eta H_W(\overline{\eta},z)v);(a;b)\big) \in \Gamma(W)\,.$$

If $\operatorname{mul}\Gamma(W)=\{0\}$, it follows that

$$a=v,\ b=W(\overline{\eta})^*v\,.$$

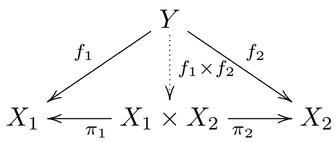
Since $H_W(\overline{\eta},.)v\neq 0$, certainly $v\neq 0$, i.e. $a\neq 0$. Since $\det W(\overline{\eta})=1$ this also implies that $b\neq 0$. Consider the case that $\operatorname{mul}\Gamma(W)\neq\{0\}$, and write $\operatorname{mul}\Gamma(W)=\operatorname{span}\{(m;m)\}$ with $\operatorname{span}\{m\}=\ker H_W(\overline{\eta},.)$. It follows that, for some $\mu\in\mathbb{C}$,

$$a=v+\mu m,\ b=W(\overline{\eta})^*v+\mu m\,.$$

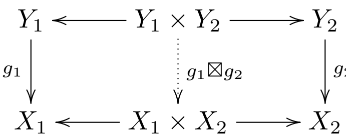
By (2.5), actually $b=W(\overline{\eta})^*(v+\mu m)$. Since $H_W(\overline{\eta},.)v\neq 0$ but $H_W(\overline{\eta},.)m=0$, the elements v and m are linearly independent. In particular, $a\neq 0$. Since $\det W(\overline{\eta})=1$, it follows that also $b\neq 0$. $\square$

The relation (2.6) in the previous proof corresponds to conditions appearing in Lemma 4.1 of [DHMS/II] or Proposition 3.8 of [DHMS/III].

b. Operations with matrix functions and boundary triplets.
We will frequently employ two elementary operations on $\mathcal{M}_{<\infty}$. Before we come to the definition of these operations, let us fix the following notation. Throughout our discussions, products and pairings of maps will appear. We shall be accurate and distinguish these concepts also notationwise: Let $f_1: Y\to X_1$ and $f_2: Y\to X_2$, then $f_1\times f_2: Y\to X_1\times X_2$ denotes the direct product of the maps f_1,f_2. That is the unique map with



Let $g_1: Y_1\to X_1$ and $g_2: Y_2\to X_2$, then $g_1\boxtimes g_2: Y_1\times Y_2\to X_1\times X_2$ denotes the pairing of the maps g_1,g_2. That is the unique map with



The first operation $\circlearrowleft_\alpha\colon\mathcal{M}_{<\infty}\to\mathcal{M}_{<\infty}$ which we will introduce can be thought of as a rotation of a matrix by the angle α. For a related concept for boundary triplets in a more general setting see Proposition 3.18 in [DHMS/III].

2.4 Definition. Denote

$$N_\alpha:=\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix},\quad \alpha\in\mathbb{R}\,, \tag{2.7}$$

and define for each 2×2-matrix W

$$\circlearrowleft_\alpha W:=N_\alpha W N_\alpha^{-1}\,. \tag{2.8}$$

For boundary triplet $\mathfrak{B}=\big(\mathcal{P},T,\Gamma\big)$ and $\alpha\in\mathbb{R}$ we define a rotated boundary triplet $\circlearrowleft_\alpha\mathfrak{B}$: Denote by $\nu_\alpha:\mathbb{C}^2\to\mathbb{C}^2$ the isomorphism $\nu_\alpha(x):=N_\alpha x$ and set

$$\circlearrowleft_\alpha\mathfrak{B}:=\big(\mathcal{P},T,(\nu_\alpha\boxtimes\nu_\alpha)\circ\Gamma\big)\,. \tag{2.9}$$

//

The matrix N_α is unitary and J-unitary, i.e. $N_\alpha^{-1} = N_\alpha^* = N_\alpha^T$ and $N_\alpha J N_\alpha^* = J$, and we have $N_\alpha^{-1} = N_{-\alpha}$. Clearly, $N_{-\frac{\pi}{2}} = J$. Moreover, let us note the following simple properties of $\circlearrowleft_\alpha$ which are seen by straightforward computations (see (2.0)):

$$\circlearrowleft_0 = \mathrm{id}, \quad \circlearrowleft_\alpha \circ \circlearrowleft_\beta = \circlearrowleft_{\alpha+\beta}, \qquad \mathfrak{t}(\circlearrowleft_\alpha W) = \mathfrak{t}(W)\,.$$

$$\circlearrowleft_\alpha = \circlearrowleft_\beta \quad \Longleftrightarrow \quad \alpha \equiv \beta \mod \pi\,.$$

$$\circlearrowleft_\alpha (W_1 \cdot W_2) = \circlearrowleft_\alpha W_1 \cdot \circlearrowleft_\alpha W_2\,, \tag{2.10}$$

For boundary triplet $\mathfrak{B}$ the rotated boundary triplet $\circlearrowleft_\alpha \mathfrak{B}$ is in fact a boundary triplet and $(\mathrm{id}_{\mathcal{P}}, \nu_\alpha \boxtimes \nu_\alpha)$ is an isomorphism from $\mathfrak{B}$ to $\circlearrowleft_\alpha \mathfrak{B}$, cf. Remark IV.2.14.

2.5 Lemma. *Let $W \in \mathcal{M}_{<\infty}$ and $\alpha \in \mathbb{R}$. Then also $\circlearrowleft_\alpha W \in \mathcal{M}_{<\infty}$, and*

$$\mathrm{ind}_- \circlearrowleft_\alpha W = \mathrm{ind}_- W\,.$$

Denote by $\nu_\alpha : \mathbb{C}^2 \to \mathbb{C}^2$ the map $\nu_\alpha x := N_\alpha x$, and let $\varpi f := \nu_\alpha \circ f$ for $\mathbb{C}^2$-valued functions f. Then $(\varpi, \nu_\alpha \boxtimes \nu_\alpha)$ is an isomorphism of the boundary triplets $\mathfrak{B}(W)$ and $\mathfrak{B}(\circlearrowleft_\alpha W)$.

Proof. The kernel relation

$$H_{\circlearrowleft_\alpha W}(w,z) = N_\alpha H_W(w,z) N_\alpha^* \tag{2.11}$$

shows that $\circlearrowleft_\alpha W \in \mathcal{M}_{<\infty}$ if and only if $W \in \mathcal{M}_{<\infty}$ and that $\mathrm{ind}_- \circlearrowleft_\alpha W = \mathrm{ind}_- W$. Moreover, the map $\varpi : f \mapsto N_\alpha f$ is an isometric isomorphism of $\mathfrak{K}(W)$ onto $\mathfrak{K}(\circlearrowleft_\alpha W)$, cf. [ADSR]. Since the entries of N_α are real, ϖ is compatible with the respective involutions.

Again by (2.11), we have

$$H_{\circlearrowleft_\alpha W}(w,.)\nu_\alpha v = \varpi H_W(w,.)v, \ w \in \mathbb{C}, v \in \mathbb{C}^2\,.$$

Hence, the sets of elements written explicitly on the right side of the definition of $T(W)$ and $\Gamma(W)$ will be mapped to the respective sets for $T(\circlearrowleft_\alpha W)$ and $\Gamma(\circlearrowleft_\alpha W)$ when applying $\varpi \boxtimes \varpi$ and $(\varpi \boxtimes \varpi) \boxtimes (\nu_\alpha \boxtimes \nu_\alpha)$, respectively. By continuity, it follows that

$$(\varpi \boxtimes \varpi)T(W) = T(\circlearrowleft_\alpha W), \quad \big[(\varpi \boxtimes \varpi) \boxtimes (\nu_\alpha \boxtimes \nu_\alpha)\big]\Gamma(W) = \Gamma(\circlearrowleft_\alpha W)\,.$$

❑

We will next introduce another operation $\mathrm{rev} : \mathcal{M}_{<\infty} \to \mathcal{M}_{<\infty}$. The meaning of this operation will become clear later, when it will be applied to chains of matrices and Hamiltonians rather than to single matrices, cf. §3.c-e.

2.6 Definition. Denote

$$V := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and define for each 2×2-matrix W

$$\mathrm{rev}\, W := V W^{-1} V\,.$$

Denote by $\phi : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}^2$ the map

$$\phi(a; b) := (Vb; Va)\,.$$

Let $\mathfrak{B} = (\mathcal{P}, T, \Gamma)$ be a boundary triplet. We define the reversed boundary triplet $\operatorname{rev}\mathfrak{B}$ as

$$\operatorname{rev}\mathfrak{B} := (\mathcal{P}, T, \phi \circ \Gamma)\,.$$

//

The matrix V satisfies $V = V^{-1} = V^*$ and $VJV = -J$. Moreover, the following relations are checked by simple computation (see (2.0)):

$$\operatorname{rev}(\operatorname{rev} W) = W, \quad \mathfrak{t}(\operatorname{rev} W) = \mathfrak{t}(W)\,,$$

$$\operatorname{rev}(W_1 \cdot W_2) = \operatorname{rev} W_2 \cdot \operatorname{rev} W_1, \quad \operatorname{rev} \circlearrowleft_\alpha W = \circlearrowleft_{-\alpha} \operatorname{rev} W\,.$$

Clearly, ϕ is an isometric isomorphism of $(\mathbb{C}^2 \times \mathbb{C}^2, (\mathfrak{J}.,.))$ onto itself such that $\phi \circ \phi = \mathrm{id}$. Therefore, by Remark IV.2.14, $\operatorname{rev}\mathfrak{B}$ is a boundary triplet and $(\mathrm{id}; \phi)$ is an isomorphism between $\mathfrak{B}$ and $\operatorname{rev}\mathfrak{B}$. Moreover, it is easy to check that

$$\operatorname{rev} \circlearrowleft_\alpha \mathfrak{B} = \circlearrowleft_{-\alpha} \operatorname{rev} \mathfrak{B}$$

2.7 Lemma. *Let $W \in \mathcal{M}_{<\infty}$. Then also $\operatorname{rev} W \in \mathcal{M}_{<\infty}$, and*

$$\operatorname{ind}_- \operatorname{rev} W = \operatorname{ind}_- W\,.$$

Let $\varpi f := VW^{-1} f$ for $\mathbb{C}^2$-valued functions f, and set $\phi(a; b) := (Vb; Va)$ for $(a; b) \in \mathbb{C}^2 \times \mathbb{C}^2$. Then (ϖ, ϕ) is an isomorphism of the boundary triplets $\mathfrak{B}(W)$ and $\mathfrak{B}(\operatorname{rev} W)$.

Proof. The kernel relation

$$H_{\operatorname{rev} W}(w, z) = \left[VW(z)^{-1}\right] H_W(w, z) \left[VW(w)^{-1}\right]^* \tag{2.12}$$

is verified by a simple computation. Thus $\operatorname{rev} W \in \mathcal{M}_{<\infty}$ if and only if $W \in \mathcal{M}_{<\infty}$. Moreover, in this case, $\operatorname{ind}_- \operatorname{rev} W = \operatorname{ind}_- W$. It also follows that ϖ is an isometric isomorphism of $\mathfrak{K}(W)$ onto $\mathfrak{K}(\operatorname{rev} W)$. Since the entries of VW^{-1} are real, ϖ is compatible with the respective involutions.

The relation (2.12) can be written as

$$H_{\operatorname{rev} W}(w, .)[VW(w)^*]v = \varpi H_W(w, .)v, \;\; w \in \mathbb{C}, v \in \mathbb{C}^2\,,$$

and hence $(\varpi \boxtimes \varpi)T(W) = T(\operatorname{rev} W)$. Finally, a straightforward computation will show that ϕ actually is an isometric isomorphism of $(\mathbb{C}^2 \times \mathbb{C}^2, \mathfrak{J})$ onto itself, cf. Definition IV.2.12, and that $((\varpi \boxtimes \varpi) \boxtimes \phi)\Gamma(W) = \Gamma(\operatorname{rev} W)$. ❑

c. Polynomial matrices.

If $W = (w_{ij})_{i,j=1}^2$ is a real polynomial matrix with $W(0) = I$ and $\det W = 1$, then the space $\mathcal{L}(W)$ is finite-dimensional. Actually,

$$\mathcal{L}(W) \subseteq \Big\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathbb{C}[z]^2 : \max_{i=1,2} \deg f_i < \max_{i,j=1,2} \deg w_{ij} \Big\}\,.$$

In particular, it follows that $W \in \mathcal{M}_{<\infty}$ and $\operatorname{ind}_- W \leq 2 \max_{i,j=1,2} \deg w_{ij}$.

Frequently it will be necessary to have a detailed description of $\mathfrak{K}(W)$ for polynomial matrices W of a specific form at hand. This result is implicitly contained in several earlier works, e.g. in [dB]. However, in view of our later needs we give an explicit proof.

2.8 Proposition. *Let $p \in \mathbb{R}[z]$, write $p(z) := a_1 z + \ldots + a_n z^n$ with $a_n \neq 0$, and consider the matrix function*

$$W(z) := \begin{pmatrix} 1 & 0 \\ -p(z) & 1 \end{pmatrix} .$$

Then $W \in \mathcal{M}_{<\infty}$ and

$$\operatorname{ind}_- W = [\frac{n}{2}] + \begin{cases} 1 & , \ n \text{ odd}, a_n < 0 \\ 0 & , \ \text{otherwise} \end{cases} \tag{2.13}$$

The space $\mathfrak{K}(W)$ is spanned by the functions

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, z\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ldots, z^{n-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and the Gram-matrix of $\mathfrak{K}(W)$ with respect to this basis is of Hankel type, i.e. of the form $(\gamma_{k+l})_{k,l=0}^{n-1}$. Thereby

$$\gamma_0 = \ldots = \gamma_{n-2} = 0 ,$$

and $\gamma_{n-1}, \ldots, \gamma_{2n-2}$ are the unique real numbers which satisfy

$$(a_1, \ldots, a_n) \begin{pmatrix} 0 & \cdots & \gamma_{n-1} \\ \vdots & ⋰ & \vdots \\ \gamma_{n-1} & \cdots & \gamma_{2n-2} \end{pmatrix} = (1, 0, \ldots, 0) . \tag{2.14}$$

Proof. A computation shows that

$$\frac{W(z)JW(w)^* - J}{z - \overline{w}} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{p(z)-p(\overline{w})}{z-\overline{w}} \end{pmatrix} ,$$

cf. (I.8.2). Hence, the reproducing kernel functions of $\mathfrak{K}(W)$ are

$$H_W(w,z)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \beta \frac{p(z) - p(\overline{w})}{z - \overline{w}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

It follows that $\mathfrak{K}(W)$ is contained in the set $\{q(z)\binom{0}{1} : q \in \mathbb{C}[z], \deg q \leq n-1\}$ of polynomials. Moreover,

$$H_W(0,z)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (a_1 + a_2 z + \ldots + a_n z^{n-1}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathfrak{K}(W) . \tag{2.15}$$

Hence, there exists a polynomial of degree $n-1$ in the space $\mathfrak{K}(W)$.

Since $\mathfrak{K}(W)$ is invariant under $\mathcal{R}(0)$ and contains a polynomial of degree $n-1$, it contains a polynomial of each degree $\leq n-1$, i.e.

$$\mathfrak{K}(W) = \{q(z)\begin{pmatrix} 0 \\ 1 \end{pmatrix} : q \in \mathbb{C}[z], \deg q \leq n-1\} =$$
$$= \operatorname{span}\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}, z\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ldots, z^{n-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\} .$$

Moreover, by (2.2) we have

$$\big[z^k\binom{0}{1}, z^l\binom{0}{1}\big] = \big[z^{k+1}\binom{0}{1}, z^{l-1}\binom{0}{1}\big],\ k=0,\dots,n-2, l=1,\dots,n-1\,.$$

Hence, the Gram-matrix of $\mathfrak{K}(W)$ with respect to the basis $\binom{0}{1}, z\binom{0}{1},\dots,z^{n-1}\binom{0}{1}$ is of Hankel type $(\gamma_{k+l})_{k,l=0}^{n-1}$. Again by (2.2) we have

$$\big[z^k\binom{0}{1}, \binom{0}{1}\big] = \big[z^{k+1}\binom{0}{1}, 0\big] = 0,\ k=0,\dots,n-2\,,$$

and hence $\gamma_0=\ldots=\gamma_{n-2}=0$. We have

$$\big[z^k\binom{0}{1}, H_W(0,z)\binom{0}{1}\big] = \begin{cases} 1 & ,\ k=0 \\ 0 & ,\ k=1,\dots,n-1 \end{cases}$$

Since, by (2.15),

$$\big[z^k\binom{0}{1}, H_W(0,z)\binom{0}{1}\big] = \big[z^k\binom{0}{1}, \sum_{l=0}^{n-1} a_{l+1}z^l\binom{0}{1}\big] = \sum_{l=0}^{n-1} a_{l+1}\gamma_{k+l}\,,$$

we conclude that (2.14) holds true. Formula (2.13) for the negative index of $\mathfrak{K}(W)$ follows from the already established form of the Gram-matrix. ❑

Proposition 2.8 can be used to characterize the occurance of a nontrivial multivalued part of $\Gamma(W)$ explicitly in terms of W. For $\alpha\in\mathbb{R}$ denote by ξ_α the vector

$$\xi_\alpha := \begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}. \tag{2.16}$$

It is useful to collect the following elementary relations:

$$N_\phi\xi_\alpha = \xi_{\alpha-\phi},$$

$$N_\alpha\xi_\alpha = \begin{pmatrix}1\\0\end{pmatrix},\ N_\alpha J\xi_\alpha = \begin{pmatrix}0\\1\end{pmatrix},\quad N_\alpha\begin{pmatrix}1\\0\end{pmatrix} = \xi_{-\alpha},\ N_\alpha\begin{pmatrix}0\\1\end{pmatrix} = \xi_{\frac{\pi}{2}-\alpha}\,,$$

$$J\xi_\alpha = \xi_{\frac{\pi}{2}+\alpha},\quad \xi_{\alpha+\pi} = -\xi_\alpha\,.$$

$$\xi_\alpha,\xi_\beta \text{ linearly dependent} \quad\Longleftrightarrow\quad \alpha\equiv\beta \mod \pi$$

2.9 Corollary. *Let $W\in\mathcal{M}_{<\infty}$, $W\neq I$. Then* $\operatorname{mul}\Gamma(W)\neq\{0\}$ *if and only if W is of the form*

$$W(z) = \circlearrowleft_\alpha \begin{pmatrix}1 & 0\\ -p(z) & 1\end{pmatrix} \tag{2.17}$$

with some $\alpha\in\mathbb{R}$. In this case

$$\operatorname{mul}\Gamma(W) = \operatorname{span}\big\{(\xi_{-\alpha};\xi_{-\alpha})\big\}\,, \tag{2.18}$$

and $\mathfrak{K}(W) = \operatorname{span}\{\xi_{\frac{\pi}{2}-\alpha},\dots,z^{\deg p-1}\xi_{\frac{\pi}{2}-\alpha}\}$.

Proof. Assume first that W is of the form (2.17). Then we have

$$H_W(w,z) = \circlearrowleft_\alpha H_{\left(\begin{smallmatrix} 1 & 0 \\ -p & 1\end{smallmatrix}\right)}(w,z) = N_\alpha \begin{pmatrix} 0 & 0 \\ 0 & \frac{p(z)-\overline{p(w)}}{z-\overline{w}} \end{pmatrix} N_{-\alpha} \,.$$

Thus

$$H_W(0,z)\xi_{-\alpha} = N_\alpha \begin{pmatrix} 0 & 0 \\ 0 & \frac{p(z)-\overline{p(w)}}{z-\overline{w}} \end{pmatrix} \underbrace{N_{-\alpha}\xi_{-\alpha}}_{=\binom{1}{0}} = 0 \,,$$

and we conclude from (2.6) that $\operatorname{mul}\Gamma(W) \neq \{0\}$ and, actually, that (2.18) holds.

Conversely, assume that $\operatorname{mul}\Gamma(W) = \operatorname{span}\{(m;m)\} \neq \{0\}$. Clearly, we can choose m such that $m = \xi_{-\alpha}$ with some appropriate $\alpha \in \mathbb{R}$. Remembering (2.4) it follows that

$$\left(\circlearrowleft_{-\alpha} W(z)\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = N_{-\alpha} W(z) N_\alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} = N_{-\alpha} W(z) \xi_{\frac{\pi}{2}-\alpha} =$$

$$= N_{-\alpha} W(z) J \xi_{-\alpha} = N_{-\alpha} J \xi_{-\alpha} = N_{-\alpha} \xi_{\frac{\pi}{2}-\alpha} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

As $\det(\circlearrowleft_{-\alpha} W(z)) = 1$ we also have $(1,0) \circlearrowleft_{-\alpha} W(z) = (1,0)$, and by Lemma II.5.6

$$\circlearrowleft_{-\alpha} W(z) = \begin{pmatrix} 1 & 0 \\ -p(z) & 1 \end{pmatrix}$$

with some polynomial p.

By Lemma 2.5, the map $f \mapsto N_\alpha f$ is an isomorphism of $\mathfrak{K}(\left(\begin{smallmatrix} 1 & 0 \\ -p & 1\end{smallmatrix}\right))$ onto $\mathfrak{K}(W)$. Proposition 2.8 implies that $\mathfrak{K}(W)$ is of the desired form. ❑

d. Products of $\mathcal{M}_{<\infty}$-functions.

It is an immediate consequence of the kernel relation

$$H_{W_1W_2}(w,z) = H_{W_1}(w,z) + W_1(z) H_{W_2}(w,z) W_1(w)^* \tag{2.19}$$

that the class $\mathcal{M}_{<\infty}$ is closed with respect to products:

2.10 Lemma. *Let $W_1, W_2 \in \mathcal{M}_{<\infty}$. Then also $W_1 \cdot W_2 \in \mathcal{M}_{<\infty}$, and*

$$\operatorname{ind}_-(W_1W_2) \leq \operatorname{ind}_- W_1 + \operatorname{ind}_- W_2 \,.$$

❑

It is a more involved task to figure out how, for given $W_1, W_2 \in \mathcal{M}_{<\infty}$, the boundary triplet $\mathfrak{B}(W_1 \cdot W_2)$ is related with $\mathfrak{B}(W_1)$ and $\mathfrak{B}(W_2)$. If one of W_i is equal to I, the matters are trivial. Hence, assume that $W_1, W_2 \neq I$. Let λ be the mapping

$$\lambda : \begin{cases} \mathfrak{K}(W_2) & \rightarrow \quad W_1 \cdot \mathfrak{K}(W_2) \\ \quad f(z) & \mapsto \quad W_1(z) f(z) \end{cases}$$

and define an inner product on $W_1 \cdot \mathfrak{K}(W_2)$ so that λ is isometric. Note here that λ is injective since $\det W_1(z) \equiv 1$. On $W_1 \cdot \mathfrak{K}(W_2)$ we have the conjugate linear involution $W_1 f \mapsto (W_1 f)^\# = W_1(f^\#)$. Clearly, λ respects the respective involutions, and the involution on $W_1 \cdot \mathfrak{K}(W_2)$ is anti-isometric.

Denote by $\mathfrak{B}_{W_1}(W_2)$ the boundary triplet defined on $W_1 \cdot \mathfrak{K}(W_2)$ by the requirement that $(\lambda, \operatorname{id}_{\mathbb{C}^4})$ is an isomorphism between $\mathfrak{B}(W_2)$ and $\mathfrak{B}_{W_1}(W_2)$, cf. Remark IV.2.14. Then $\mathfrak{B}_{W_1}(W_2)$ has defect 2 and satisfies (E).

2.11 Proposition. *Let $W_1, W_2 \in \mathcal{M}_{<\infty}$, $W_1, W_2 \neq I$. Assume that there exists no nonzero constant u with $u \in \mathfrak{K}(W_2)$ and $W_1(z)u \in \mathfrak{K}(W_1)$. Then*

$$\mathfrak{B}(W_1) \uplus \mathfrak{B}_{W_1}(W_2) = \mathfrak{B}(W_1 \cdot W_2)\,.$$

In the proof of this result we employ the following general statement which can also be shown by means of Section 3 of [DHMS/II]. We give a more straight forward proof.

2.12 Lemma. *Let $\mathfrak{B}_1 = (\mathcal{P}, T_1, \Gamma_1)$ and $\mathfrak{B}_2 = (\mathcal{P}, T_2, \Gamma_2)$ be boundary triplets defined on the same space $\mathcal{P}$. Assume that either both have defect 2 or both have defect 1. If $\Gamma_1 \subseteq \Gamma_2$, then already $\mathfrak{B}_1 = \mathfrak{B}_2$.*

Proof. The hypothesis $\Gamma_1 \subseteq \Gamma_2$ implies

$$T_1 = \operatorname{dom}\Gamma_1 \subseteq \operatorname{dom}\Gamma_2 = T_2\,,$$

$$T_1^* = \ker\Gamma_1 \subseteq \ker\Gamma_2 = T_2^*\,.$$

Thus $T_1 = T_2$, and hence in particular $\dim T_1/T_1^* = \dim T_2/T_2^*$. Since $\mathfrak{B}_1$ and $\mathfrak{B}_2$ have the same defect, it follows that

$$\operatorname{mul}\Gamma_1 \neq \{0\} \iff \operatorname{mul}\Gamma_2 \neq \{0\}$$

If $\operatorname{mul}\Gamma_1 = \operatorname{mul}\Gamma_2 = \{0\}$, then $\Gamma_1 = \Gamma_2$ since their domains coincide. Otherwise, we obtain from $\dim\operatorname{mul}\Gamma_1 = \dim\operatorname{mul}\Gamma_2 = 1$ and $\operatorname{mul}\Gamma_1 \subseteq \operatorname{mul}\Gamma_2$ that actually $\operatorname{mul}\Gamma_1 = \operatorname{mul}\Gamma_2$. Again we end up with $\Gamma_1 = \Gamma_2$. ❑

Proof (of Proposition 2.11). Our first task is to show that $\mathfrak{B}(W_1)$ and $\mathfrak{B}_{W_1}(W_2)$ satisfy the condition (LI), cf. Proposition IV.6.2. Assuming the contrary yields

$$\operatorname{mul}\Gamma(W_1) = \operatorname{mul}\Gamma(\mathfrak{B}_{W_1}(W_2)) = \operatorname{span}\{(m; m)\}$$

with some $m \in \mathbb{C}^2 \setminus \{0\}$. However, $\operatorname{mul}\Gamma(\mathfrak{B}_{W_1}(W_2)) = \operatorname{mul}\Gamma(W_2)$, and we conclude from (2.4) and Corollary 2.9 that

$$W_1(z)Jm = W_2(z)Jm = Jm \in \mathfrak{K}(W_1) \cap \mathfrak{K}(W_2)\,.$$

This contradicts the assumption of the present proposition. Thus (LI) holds, and by Proposition IV.6.2 $\mathfrak{B}(W_1) \uplus \mathfrak{B}_{W_1}(W_2)$ is a well-defined boundary triplet, has defect 2, and satisfies (E), cf. Lemma IV.6.7.

The boundary triplet $\mathfrak{B}(W_1) \uplus \mathfrak{B}_{W_1}(W_2)$ acts in the space

$$\mathfrak{K}(W_1) \oplus W_1 \cdot \mathfrak{K}(W_2)\,.$$

However, by the present assumption, this space is isometrically equal to $\mathfrak{K}(W_1W_2)$, cf. [ADSR, §1.5]. In order to prove that $\mathfrak{B}(W_1) \uplus \mathfrak{B}_{W_1}(W_2) = \mathfrak{B}(W_1W_2)$ it is, by Lemma 2.12, enough to show that

$$\Gamma(W_1W_2) \subseteq \Gamma(W_1) \uplus \Gamma(\mathfrak{B}_{W_1}(W_2))\,. \tag{2.20}$$

The relation $\Gamma(W_1W_2)$ is the closed linear span of the elements

$$\big((H_{W_1W_2}(w,z)v; \overline{w}H_{W_1W_2}(w,z)v); (v; W_2(w)^*W_1(w)^*v)\big),\ v \in \mathbb{C}^2, w \in \mathbb{C}\,. \tag{2.21}$$

However, we have

$$H_{W_1 W_2}(w,z) = H_{W_1}(w,z) + W_1(z) H_{W_2}(w,z) W_1(w)^* \,,$$

and

$$\big((H_{W_1}(w,z)v; \overline{w} H_{W_1}(w,z)v); (v; W_1(w)^* v)\big) \in \Gamma(W_1)\,,$$

$$\big((W_1(z) \cdot H_{W_2}(w,z) \cdot W_1(w)^* v; W_1(z) \cdot \overline{w} H_{W_2}(w,z) \cdot W_1(w)^* v);$$
$$(W_1(w)^* v; W_2(w)^* \cdot W_1(w)^* v)\big) \in \Gamma(\mathfrak{B}_{W_1}(W_2))\,.$$

We see that each element of the form (2.21) belongs to $\Gamma(W_1) \uplus \Gamma(\mathfrak{B}_{W_1}(W_2))$, cf. Definition IV.6.1. Thus (2.20) holds. ❑

It is sometimes practical to note that pasting is compatible with operation rev.

2.13 Lemma. *Let $W_1, W_2 \in \mathcal{M}_{<\infty}$, $W_1, W_2 \neq I$, be given. Then W_1 and W_2 satisfy the hypothesis of Proposition 2.11 if and only if $\operatorname{rev} W_2$ and $\operatorname{rev} W_1$ do so.*

Assume that W_1 and W_2 do satisfy this hypothesis, set $W := W_1 W_2$, and let

$$\varpi_W : \mathfrak{K}(W) \to \mathfrak{K}(\operatorname{rev} W)\,,$$

$$\varpi_{W_j} : \mathfrak{K}(W_j) \to \mathfrak{K}(\operatorname{rev} W_j), \quad j = 1, 2\,,$$

be the respective isomorphisms constructed in Lemma 2.7. Then

$$\varpi_W\big(\mathfrak{K}(W_1)\big) = \operatorname{rev} W_2 \, \mathfrak{K}(\operatorname{rev} W_1), \quad \varpi_W\big(W_1 \mathfrak{K}(W_2)\big) = \mathfrak{K}(\operatorname{rev} W_2)\,,$$

i.e. we are in the situation

$$\begin{array}{ccccc} \mathfrak{K}(W) & = & \mathfrak{K}(W_1) & [\dot{+}] & W_1 \mathfrak{K}(W_2) \\ \downarrow{\scriptstyle \varpi_W} & & & \times & \\ \mathfrak{K}(\operatorname{rev} W) & = & \mathfrak{K}(\operatorname{rev} W_2) & [\dot{+}] & \operatorname{rev} W_2 \, \mathfrak{K}(\operatorname{rev} W_1) \end{array}$$

Proof. Assume that u is constant with $u \in \mathfrak{K}(W_2)$ and $W_1 u \in \mathfrak{K}(W_1)$. Then

$$Vu = V W_1^{-1} W_1 u = \varpi_W(W_1 u) \in \mathfrak{K}(\operatorname{rev} W_1)\,,$$

$$\operatorname{rev} W_2 \cdot Vu = V W_2^{-1} V \cdot Vu = \varpi_{W_2}(u) \in \mathfrak{K}(\operatorname{rev} W_2)\,.$$

Since rev is involutory, the first assertion follows.

In order to show the remaining part of the lemma, it suffices to compute

$$\varpi_W f = V W_2^{-1} W_1^{-1} \cdot f = V W_2^{-1} V \cdot V W_1^{-1} f = \operatorname{rev} W_2 \cdot \varpi_{W_1} f, \quad f \in \mathfrak{K}(W_1)\,,$$

$$\varpi_W(W_1 f) = V W_2^{-1} W_1^{-1} \cdot W_1 f = V W_2^{-1} f = \varpi_{W_2} f, \quad f \in \mathfrak{K}(W_2)\,.$$

❑

e. Relation with other classes of functions.

If $E : D \to \mathbb{C}$ is an analytic function defined on some open subset D of the complex plane, we define a kernel K_E as

$$K_E(w,z) := \frac{i}{2}\frac{E(z)\overline{E(w)} - E^{\#}(z)\overline{E^{\#}(w)}}{z-\overline{w}},\ z,w \in D\,.$$

Again, for $z = \overline{w}$, this formula has to be interpreted appropriately.

2.14 Definition. Let E be a complex-valued function, and let $\kappa \in \mathbb{N}_0$. We write $E \in \mathcal{H}B_\kappa$, if

(HB1) E is entire;

(HB2) E and $E^{\#}$ have no common nonreal zeros;

(HB3) The kernel K_E has κ negative squares on $\mathbb{C}$.

We use the notation

$$\mathcal{H}B_{<\infty} := \bigcup_{\kappa \in \mathbb{N}\cup\{0\}} \mathcal{H}B_\kappa\,,$$

and write $\operatorname{ind}_- E = \kappa$ to express that a function $E \in \mathcal{H}B_{<\infty}$ belongs to $\mathcal{H}B_\kappa$. The class $\mathcal{H}B_{<\infty}$ is called the indefinite Hermite-Biehler class. //

Let us recall some facts which can be found in [KW/I]. By means of the reproducing kernel K_E, each function $E \in \mathcal{H}B_{<\infty}$ generates a Pontryagin space $\mathfrak{P}(E)$ which consists of entire functions. This space is referred to as the de Branges Pontryagin space generated by E. In the space $\mathfrak{P}(E)$ we can consider the operator $\mathcal{S}(E)$ of multiplication by the independent variable. This operator is closed, symmetric, has defect index $(1,1)$, and the set $r(\mathcal{S}(E))$ of its points of regular type equals the whole plane. Moreover, we have $\dim(\mathfrak{P}(E)/\overline{\operatorname{dom}\mathcal{S}(E)}) \leq 1$.

The relations $\mathcal{A} \subseteq \mathfrak{P}(E)^2$ which extend $\mathcal{S}(E)$ and have nonempty resolvent set can be described in terms of the set

$$\operatorname{Assoc}\mathfrak{P}(E) := \left\{S(z) : \exists\, F,G \in \mathfrak{P}(E),\ S(z) = F(z) + zG(z)\right\}.$$

This correspondence is given by the formula

$$(\mathcal{A}_S - w)^{-1}F(z) = \frac{F(z) - \frac{S(z)}{S(w)}F(w)}{z-w},\ w \in \rho(\mathcal{A}_S), F \in \mathfrak{P}(E)\,,$$

$$\rho(\mathcal{A}_S) = \{w \in \mathbb{C} :\ S(w) \neq 0\}\,,$$

where $S \in \operatorname{Assoc}\mathfrak{P}(E)$. The relation $\mathcal{A}_S$ has a nontrivial multivalued part if and only if $S \in \mathfrak{P}(E)$ and, in this case, $\operatorname{mul}\mathcal{A}_S = \operatorname{span}\{S\}$. The relation $\mathcal{A}_S$ is selfadjoint if and only if $S = S_\psi$ for some $\psi \in [0,\pi)$, where

$$S_\psi := e^{i\psi}E + e^{-i\psi}E^{\#},\ \psi \in [0,\pi)\,.$$

If $S \in \operatorname{Assoc}\mathfrak{P}(E)$ and $S(0) = 1$, then $\mathcal{A}_S^{-1}$ is a bounded operator which extends $\mathcal{S}(E)^{-1}$. It is clear that there exists a one-dimensional perturbation which turns $\mathcal{A}_S^{-1}$ into a selfadjoint operator. We will in our later discussions need this fact in an explicit form for functions $S \in \mathfrak{P}(E)$.

2.15 Lemma. *Let $E \in \mathcal{H}B_{<\infty}$, $E(0) = -i$, and let $S \in \mathfrak{P}(E)$, $S(0) = 1$. Define $\mathcal{B}_S : \mathfrak{P}(E) \to \mathfrak{P}(E)$ as*

$$\mathcal{B}_S F := \mathcal{A}_S^{-1} F - [\mathcal{A}_S^{-1} F, S] K_E(0,.),\ F \in \mathfrak{P}(E)\,.$$

Then $\mathcal{B}_S$ is a bounded selfadjoint operator in $\mathfrak{P}(E)$. We have

$$\mathcal{B}_S \circ \mathcal{S}(E)|_{\operatorname{dom} \mathcal{S}(E) \cap \operatorname{span}\{S\}^\perp} = \mathrm{id}\,,$$

$$\big(\mathcal{B}_S F; F - F(0)S\big) \in \mathcal{A}_{S_0},\ F \in \mathfrak{P}(E)\,.$$

Proof. The fact that $\mathcal{B}_S$ is bounded is clear. We only need to check symmetry. To this end, note that $K_E(0,.) \in \ker \mathcal{A}_{S_0}$, and hence

$$(\mathcal{B}_S F; F - F(0)S) = \underbrace{\Big(\frac{F(z) - F(0)S(z)}{z}; F(z) - F(0)S(z)\Big)}_{\in \mathcal{S}(E)} -$$

$$-[\mathcal{A}_S F, S](K_E(0,.); 0) \in \mathcal{A}_{S_0}\,.$$

Moreover, we have $S \perp \operatorname{ran} \mathcal{B}_S$. Thus we can compute

$$[\mathcal{B}_S F, G] = [\mathcal{B}_S F, G - G(0)S] = [F - F(0)S, \mathcal{B}_S G] = [F, \mathcal{B}_S G]\,.$$

❑

Matrices of the class $\mathcal{M}_{<\infty}$ give rise to indefinite Hermite-Biehler functions as follows: If $W = (w_{ij})_{i,j=1,2} \in \mathcal{M}_{<\infty}$, define

$$E_W := w_{21} - i w_{22}\,.$$

The kernel relation

$$K_{E_W}(w,z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* H_W(w,z) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.22}$$

shows that $E_W \in \mathcal{H}B_{<\infty}$ and $\operatorname{ind}_- E_W \leq \operatorname{ind}_- W$. Moreover, the projection $\pi_2 : \binom{f_1}{f_2} \mapsto f_2$ onto the second component induces an isometric isomorphism

$$\pi_2 : \operatorname{cls}\Big\{H_W(w,.)\begin{pmatrix} 0 \\ 1 \end{pmatrix} : w \in \mathbb{C}\Big\} \Big/_{\operatorname{cls}\{H_W(w,.)\binom{0}{1} : w \in \mathbb{C}\}^\circ} \to \mathfrak{P}(E_W)\,,$$

cf. Lemma I.8.6. In particular, if

$$\mathfrak{K}(W) = \operatorname{clos}_{\mathfrak{K}(W)} \operatorname{span}\Big\{H_W(w,z)\begin{pmatrix} 0 \\ 1 \end{pmatrix} : w \in \mathbb{C}\Big\}\,, \tag{2.23}$$

then π_2 is an isometric isomorphism of $\mathfrak{K}(W)$ onto $\mathfrak{P}(E_W)$.

Let $W \in \mathcal{M}_{<\infty}$, and assume that (2.23) is satisfied, so that we can identify $\mathfrak{K}(W)$ via π_2 with $\mathfrak{P}(E_W)$. Let us record that $T(W)$ and $\mathcal{S}(E_W)$ are related. To this end denote by $\pi_{l,1} : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$, $\pi_r : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$ the projections

$$\pi_{l,1}\Big(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\Big) := a_1,\ \pi_r(a,b) := b\,,$$

and set

$$T_1(W) := \ker\big(\pi_{l,1} \circ \Gamma(W)\big), \quad S_1(W) := T_1(W)^*\,.$$

Then $S_1(W) = \ker((\pi_{l,1} \times \pi_r) \circ \Gamma(W))$, and $S_1(W)$ is symmetric with defect index $(1,1)$.

2.16 Lemma. *Assume that $W \in \mathcal{M}_{<\infty}$ satisfies* (2.23)*, then*

$$(\pi_2 \boxtimes \pi_2) S_1(W) = S(E_W)\,.$$

We are thus in the situation

$$\begin{array}{ccc}
\mathfrak{K}(W) & \qquad & T(W) \supseteq \underbrace{T_1(W) \supseteq S_1(W)} \supseteq S(W) \\
\Big\downarrow{\scriptstyle \pi_2} & & \Big\downarrow{\scriptstyle \pi_2 \boxtimes \pi_2} \\
\mathfrak{P}(E_W) & & S(E_W)^* \supseteq S(E_W)
\end{array}$$

Proof. The relation $T_1(W)$ contains all pairs $(H_W(w,.)\binom{0}{1}; \overline{w} H_W(w,.)\binom{0}{1})$, $w \in \mathbb{C}$. Thus, cf. (2.22),

$$\big(K_{E_W}(w,.); \overline{w} K_{E_W}(w,.)\big) \in (\pi_2 \boxtimes \pi_2) T_1(W),\ w \in \mathbb{C}\,.$$

We conclude that $[(\pi_2 \boxtimes \pi_2) T_1(W)]^* \subseteq S(E_W)$, and hence

$$(\pi_2 \boxtimes \pi_2) S_1(W) \subseteq S(E_W) \subseteq S(E_W)^* \subseteq (\pi_2 \boxtimes \pi_2) T_1(W)\,.$$

Since $\dim(S(E_W)^*/S(E_W)) = \dim((\pi_2 \boxtimes \pi_2) T_1(W)/(\pi_2 \boxtimes \pi_2) S_1(W)) = 1$, the assertion follows. ❑

If $q : D \to \mathbb{C}$ is an analytic function defined on some open subset D of the complex plane, we define a kernel N_q as

$$N_q(w,z) := \frac{q(z) - \overline{q(w)}}{z - \overline{w}},\ z, w \in D\,.$$

Again, for $z = \overline{w}$, this formula has to be interpreted appropriately.

2.17 Definition. Let q be a complex-valued function, and let $\kappa \in \mathbb{N}_0$. We write $q \in \mathcal{N}_\kappa$, if

- **(N1)** q is real and meromorphic on $\mathbb{C} \setminus \mathbb{R}$;
- **(N2)** The kernel N_q has κ negative squares on the domain of holomorphy of q.

Once more, we set $\mathcal{N}_{<\infty} := \bigcup_{\kappa \in \mathbb{N} \cup \{0\}} \mathcal{N}_\kappa$, and write $\operatorname{ind}_- q = \kappa$ to express that $q \in \mathcal{N}_{<\infty}$ belongs to $\mathcal{N}_\kappa$. //

Matrices of the class $\mathcal{M}_{<\infty}$ give rise to generalized Nevanlinna functions as follows: For a 2×2-matrix valued function $W(z) = (w_{ij}(z))_{i,j=1}^2$ and a scalar function $\tau(z)$, we denote by $W \star \tau$ the scalar function

$$(W \star \tau)(z) := \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_{21}(z)\tau(z) + w_{22}(z)}\,,$$

wherever this expression is defined. For the parameter $\tau = \infty$, we set $W \star \tau := w_{21}^{-1} w_{11}$. A straightforward computation shows that

$$(W_1 W_2) \star \tau = W_1 \star (W_2 \star \tau)\,.$$

The kernel relation

$$(w_{21}(z)\tau(z)+w_{22}(z))\frac{(W\star\tau)(z)-(W\star\tau)(\overline{w})}{z-\overline{w}}(w_{21}(\overline{w})\tau(\overline{w})+w_{22}(\overline{w}))=$$

$$=\begin{pmatrix}-\tau(z)\\1\end{pmatrix}^T\frac{(VW^{-1}V)(z)J(VW^{-1}V)^*(w)-J}{z-\overline{w}}\begin{pmatrix}-\tau(\overline{w})\\1\end{pmatrix}+\frac{\tau(z)-\tau(\overline{w})}{z-\overline{w}}$$

shows that $W\star\tau\in\mathcal{N}_{<\infty}$ provided that $W\in\mathcal{M}_{<\infty}$ and $\tau\in\mathcal{N}_{<\infty}$. Actually, we have $\operatorname{ind}_- W\star\tau\leq\operatorname{ind}_- W+\operatorname{ind}_-\tau$.

Indefinite Hermite-Biehler functions give rise to generalized Nevanlinna functions as follows: If E is an entire function, write $E=A-iB$ with $A:=\frac{1}{2}(E+E^\#)$, $B:=\frac{i}{2}(E-E^\#)$. Assume that E and $E^\#$ have no common nonreal zeros. The kernel relation

$$K_E(w,z)=A(z)N_{\frac{B}{A}}(w,z)\overline{A(w)},\quad z,w\in\mathbb{C}\ A(z),A(w)\neq 0\,,$$

implies that $E\in\mathcal{H}B_{<\infty}$ if and only if $\frac{B}{A}\in\mathcal{N}_{<\infty}$, and that in this case $\operatorname{ind}_- E=\operatorname{ind}_-\frac{B}{A}$.

On the set $\mathcal{N}_{<\infty}$ we can also introduce an operation $\circlearrowleft_\alpha$, namely as

$$\circlearrowleft_\alpha q:=N_\alpha\star q,\ \alpha\in\mathbb{R},q\in\mathcal{N}_{<\infty}\,.$$

From the above kernel relation it follows immediately that $\circlearrowleft_\alpha q\in\mathcal{N}_{<\infty}$ and $\operatorname{ind}_-\circlearrowleft_\alpha q=\operatorname{ind}_- q$. Clearly, we have

$$\circlearrowleft_\beta(\circlearrowleft_\alpha q)=\circlearrowleft_{(\alpha+\beta)}q,\ \circlearrowleft_\alpha\circ\circlearrowleft_{-\alpha}=\operatorname{id},\ \circlearrowleft_\alpha(q_1+q_2)=\circlearrowleft_\alpha q_1+\circlearrowleft_\alpha q_2\,.$$

Moreover, a simple computation shows that

$$(\circlearrowleft_\alpha W)\star\tau=\circlearrowleft_\alpha\big(W\star(\circlearrowleft_{-\alpha}\star\tau)\big)\,.\tag{2.24}$$

3 Maximal chains and general Hamiltonians

In this section we deal with chains of matrices and with positive definite and general Hamiltonians. We set up the necessary notation, give some supplements to earlier results, and provide some tools which are essential for the present work. The content of this section is arranged in five subsections:

a. We recall definition and properties of a maximal chain of matrices, which is the indefinite analogue of the fundamental matrix solution of a canonical system in the limit point case. Moreover, we recall the notion of its Weyl coefficient, and the corresponding variant of the Inverse Spectral Theorem.
b. Finite maximal chains are the indefinite analogue of the fundamental matrix solution of a canonical system in the limit circle case. Besides recalling the definition and the corresponding variant of an Existence/Uniqueness Theorem for finite maximal chains, we give a condition for a function to be a finite maximal chain which will be used later.
c. We formalize the idea of splitting-and-pasting for (finite) maximal chains. This procedure is a technical tool whose use is, at the present stage of development, inevitable. Moreover, we introduce operations $\circlearrowleft_\alpha$ and rev on

(finite) maximal chains analogous to those introduced in the previous section on $\mathcal{M}_{<\infty}$.
d. We revisit the positive definite situation, and recall some results concerning positive definite Hamiltonians. We discuss their relationship with (positive definite) maximal chains, and provide the analogues of the previously defined operations on maximal chains.
e. Here we recall the definition of a general Hamiltonian, investigate the splitting-and-pasting procedure for such, and introduce corresponding operations $\circlearrowleft_\alpha$ and rev.

a. Maximal chains of matrices and their Weyl coefficients.

Let us recall the definition of a maximal chain of matrices, cf. [KW/III].

3.1 Definition. A mapping $\omega : I \to \mathcal{M}_{<\infty}$ is called a maximal chain of matrices if the following axioms are satisfied:

- **(W1)** The set I is of the form $\bigcup_{i=0}^{n}(\sigma_i, \sigma_{i+1})$ for some numbers $n \in \mathbb{N}\cup\{0\}$ and $\sigma_0, \ldots, \sigma_{n+1} \in \mathbb{R} \cup \{\pm\infty\}$ with $\sigma_0 < \sigma_1 < \ldots < \sigma_{n+1}$.
- **(W2)** The function ω is not constant on any interval contained in I.
- **(W3)** For all $s, t \in I$, $s \leq t$, the matrix $\omega(s)^{-1}\omega(t)$ belongs to $\mathcal{M}_{<\infty}$, and
$$\operatorname{ind}_- \omega(t) = \operatorname{ind}_- \omega(s) + \operatorname{ind}_- \omega(s)^{-1}\omega(t)\,.$$
We will refer to $\omega(s)^{-1}\omega(t)$ as the transfer-matrix from s to t.
- **(W4)** Let $t \in I$ and $W \in \mathcal{M}_{<\infty}$, $W \neq I$. If $W^{-1}\omega(t) \in \mathcal{M}_{<\infty}$ and $\operatorname{ind}_- \omega(t) = \operatorname{ind}_- W + \operatorname{ind}_- W^{-1}\omega(t)$, then there exists a number $s \in I$ such that $W = \omega(s)$.
- **(W5)** We have $\lim_{t\nearrow\sigma_{n+1}} \mathfrak{t}(\omega(t)) = +\infty$. If I is not connected, i.e. $n > 0$, there exist numbers $s, t \in (\sigma_n, \sigma_{n+1})$ such that $\omega(s)^{-1}\omega(t)$ is not a linear polynomial.

The points $\sigma_1, \ldots, \sigma_n$ are called the singularities of ω. //

It is apparent from the axiom (W5) that points $s, t \in I$ where the transfer matrix $\omega(s)^{-1}\omega(t)$ is a linear polynomial play a special role. For $l, \phi \in \mathbb{R}$, we set
$$W_{(l,\phi)}(z) := \begin{pmatrix} 1 - lz\sin\phi\cos\phi & lz\cos^2\phi \\ -lz\sin^2\phi & 1 + lz\sin\phi\cos\phi \end{pmatrix}$$
Note that (see (2.8))
$$\circlearrowleft_\alpha W_{(l,\phi)}(z) = W_{(l,\phi-\alpha)}(z) \tag{3.1}$$
A short argument shows that a linear polynomial W belongs to $\mathcal{M}_{<\infty}$ if and only if $W = W_{(l,\phi)}$ for some $l \in \mathbb{R}$ and $\phi \in [0, \pi)$. In this case we have
$$\operatorname{ind}_- W = \operatorname{ind}_- W_{(l,\phi)} = \begin{cases} 0 & , \ l \geq 0 \\ 1 & , \ l < 0 \end{cases}$$
A nonvoid interval $(s,t) \subseteq I$ is called indivisible of type $\phi \in [0, \pi)$, if for all $s', t' \in (s,t)$ there is an $l(s',t') \in \mathbb{R}$ such that
$$\omega(s')^{-1}\omega(t') = W_{(l(s',t'),\phi)}\,.$$

For $s' < t'$ we obtain from Definition 3.1 that $l(s',t') > 0$. Hence, $\sup\{l(s',t') : s' < t', s',t' \in (s,t)\}$ is positive, and we call it the length of the indivisible interval (s,t).

If (s_1,t_1) and (s_2,t_2) are indivisible intervals of types ϕ_1 and ϕ_2, respectively, which have nonempty intersection, then $\phi_1 = \phi_2$ and their union is again an indivisible interval of the same type. Hence every indivisible interval is contained in a maximal indivisible interval. We set

$$I_{\text{sing}} := \bigcup_{(s,t)\ \text{indivisible interval}} (s,t), \quad \text{and} \quad I_{\text{reg}} := I \setminus I_{\text{sing}}.$$

A singularity σ of a maximal chain ω is called of polynomial type, if for some $s_-, s_+ \in I$ the intervals (s_-,σ) and (σ,s_+) are both indivisible.

Let us recall some basic properties of maximal chains which were proved in [KW/III]. For (i) and (ii) of the following statement see Lemma III.3.5, the assertion (iii) is Proposition III.3.16, and (iv) follows from the construction in Theorem II.7.1. For (v) see below.

3.2 Proposition ([KW/III]). *Let ω be a maximal chain of matrices.*

- (i) *The function* $\operatorname{ind}_- \omega(t)$ *is nondecreasing, constant on each connected component of I, and takes different values on different components. In particular, it is bounded and attains its maximum on (σ_n,σ_{n+1}). Moreover,* $\operatorname{ind}_- \omega(t) = 0$ *for $t \in (\sigma_0,\sigma_1)$.*

- (ii) *The function $\mathfrak{t}(\omega(t))$ (see* (2.0)*) is continuous and strictly increasing on each interval (σ_i,σ_{i+1}). We have*

$$\lim_{t\nearrow\sigma_i} \mathfrak{t}(\omega(t)) = +\infty,\ i = 1,\dots,n+1\,,$$

$$\lim_{t\searrow\sigma_i} \mathfrak{t}(\omega(t)) = -\infty,\ i = 1,\dots,n, \quad \lim_{t\searrow\sigma_0} \omega(t) = I\,. \tag{3.2}$$

- (iii) *The condition required in* (W5) *for the interval (σ_n,σ_{n+1}) holds automatically for intervals between two singularities, i.e. none of the intervals (σ_i,σ_{i+1}), $i = 1,\dots,n-1$, is indivisible. The interval (σ_0,σ_1), however, might be indivisible.*

- (iv) *Let $t \in I_{\text{reg}}$ and $s \in I$, $s \geq t$. Then $\mathfrak{K}(\omega(t)) \subseteq \mathfrak{K}(\omega(s))$ and the inclusion map is isometric. The map $f \mapsto \omega(t)f$ is an isometric isomorphism of $\mathfrak{K}(\omega(t)^{-1}\omega(s))$ onto $\mathfrak{K}(\omega(s)) \ominus \mathfrak{K}(\omega(t))$.*

- (v) *Let $i \in \{1,\dots,n\}$. Then there exists a unique angle $\phi(\sigma_i) \in [0,\pi)$ such that*

$$\lim_{t\nearrow\sigma_i} \left[\ \circlearrowleft_{\phi(\sigma_i)} \omega(t)\right]_{12}'(0) < \infty\,.$$

We have

$$\lim_{t\searrow\sigma_i} \left[\ \circlearrowleft_{\phi(\sigma_i)} \omega(t)\right]_{12}'(0) = \lim_{t\nearrow\sigma_i} \left[\ \circlearrowleft_{\phi(\sigma_i)} \omega(t)\right]_{12}'(0)\,.$$

❑

For the verification of (v) note that due to Remark III.5.7 the intermediate Weyl coefficient q of ω at σ_i as introduced in the beginning of Section III.5 is a meromorphic generalized Nevanlinna function including the possibility that it identically equals to ∞. In particular $q(0) \in \mathbb{R} \cup \{\infty\}$.

If q has a singularity at 0 set $\phi(\sigma_i) = 0$. Otherwise chose the unique angle $\phi(\sigma_i) \in (0, \pi)$ so that $q(0) = \frac{\cos \phi(\sigma_i)}{\sin \phi(\sigma_i)}$. In any case there is a unique angle $\phi(\sigma_i) \in [0, \pi)$ such that $\circlearrowleft_{\phi(\sigma_i)} q$ has a singularity at 0.

As $\circlearrowleft_{\phi(\sigma_i)} \omega$ is also a maximal chain of matrix functions and as $\circlearrowleft_{\phi(\sigma_i)} q$ is its intermediate Weyl coefficient at σ_i (see Lemma 3.13) we can apply Lemma III.5.9 and Proposition III.5.8 in order to obtain the properties mention in (v). The uniqueness follows by reversing the mentioned steps of the present arguments.

3.3 Remark. Let us explicitly state the following consequences of the above item Proposition 3.2, (i):

(i) The notation $\operatorname{ind}_- \omega := \max_{t \in I} \operatorname{ind}_- \omega(t)$ is meaningful. The set of all maximal chains ω with $\operatorname{ind}_- \omega = \kappa$ will be denoted by $\mathfrak{M}_\kappa$. As usual we will use the notation

$$\mathfrak{M}_{<\infty} := \bigcup_{\nu \in \mathbb{N} \cup \{0\}} \mathfrak{M}_\nu \,,$$

and write $\operatorname{ind}_- \omega = \kappa$ to express that a chain $\omega \in \mathfrak{M}_{<\infty}$ belongs to $\mathfrak{M}_\kappa$.

(ii) If $s, t \in I$, $s < t$, are such that (s, t) is an indivisible interval, then $\operatorname{ind}_- \omega(t) = \operatorname{ind}_- \omega(s)$. Hence the number l in $\omega(s)^{-1}\omega(t) = W_{(l,\phi)}$ is positive. In particular, the length of an indivisible interval is a positive number or equal to $+\infty$.

It might happen that for some $s, t \in I$, $s < t$, we have $\omega(s)^{-1}\omega(t) = W_{(l,\phi)}$ with some $l < 0$. In this case (s, t) cannot be contained in I. Nevertheless, we shall speak of (s, t) as an indivisible interval of negative length l.

//

Chains which can be obtained out of each other by a change of variable will share their important properties. This is formalized by the notion of reparameterization.

3.4 Definition. Let $J_1, J_2 \subseteq \mathbb{R}$ and let $\omega_i : J_i \to \mathcal{M}_{<\infty}$, $i = 1, 2$, be functions. Then we say that ω_2 is a reparameterization of ω_1 if there exists an increasing and bijective map $\alpha : J_2 \to J_1$ such that $\omega_2 = \omega_1 \circ \alpha$. In this case we write $\omega_2 \leftrightsquigarrow \omega_1$. //

Clearly, the relation $\leftrightsquigarrow$ induces an equivalence relation on the set $\mathfrak{M}_{<\infty}$, and thereby $\omega_1 \leftrightsquigarrow \omega_2$ implies $\operatorname{ind}_- \omega_1 = \operatorname{ind}_- \omega_2$.

A central role in the theory of maximal chains of matrices is played by the Weyl coefficient associated to a maximal chain. Let $\omega : I \to \mathcal{M}_{<\infty}$ be a maximal chain of matrices. Due to the fact that $\lim_{t \nearrow \sup I} \mathfrak{t}(\omega(t)) = +\infty$, for each function $\tau \in \mathcal{N}_0$ the limit

$$q_\infty(\omega)(z) := \lim_{t \nearrow \sup I} (\omega(t) \star \tau)(z) \tag{3.3}$$

exists locally uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{R}$ with respect to the chordal metric, cf. [KW/II]. Moreover, it does not depend on τ. Obviously, if $\omega_1 \leftrightsquigarrow \omega_2$, then $q_\infty(\omega_1) = q_\infty(\omega_2)$.

3.5 Definition. If $\omega \in \mathfrak{M}_{<\infty}$, the function $q_\infty(\omega)$ is called the Weyl coefficient of ω. //

The main result in connection with this notion is the Inverse Spectral Theorem for matrix chains which is obtained by combining Theorem II.8.7 with Theorem II.7.1.

3.6. Inverse Spectral Theorem; chain version ([KW/II])***:*** *For each $\kappa \in \mathbb{N}\cup\{0\}$ the assignment $\omega \mapsto q_\infty(\omega)$ establishes a bijective correspondence between the sets $\mathfrak{M}_\kappa/_{\leftrightsquigarrow}$ and $\mathcal{N}_\kappa$.*

b. Finite maximal chains

Finite maximal chains are bounded analogues of maximal chains.

3.7 Definition. A mapping $\omega : I \to \mathcal{M}_{<\infty}$ is called a finite maximal chain of matrices if

(W1$_f$) the set I is of the form $I = [\sigma_0, \sigma_{n+1}] \setminus \{\sigma_1, \ldots, \sigma_n\}$ for some numbers $n \in \mathbb{N} \cup \{0\}$ and $\sigma_0, \ldots, \sigma_{n+1} \in \mathbb{R}$ with $\sigma_0 < \sigma_1 < \ldots < \sigma_n < \sigma_{n+1}$,

and ω satisfies the axioms (W2), (W3) and (W4). Again, $\sigma_1, \ldots, \sigma_n$ are called the singularities of the chain ω. //

Again we will denote by I_{sing} the union of all indivisible intervals, and set $I_{\text{reg}} := I \setminus I_{\text{sing}}$. With the obvious modifications the statements of Proposition 3.2 remain true.

3.8 Proposition ([KW/III])**.**

(*i*) *The function* $\operatorname{ind}_- \omega(t)$ *is nondecreasing, constant on each connected component of I, and takes different values on different components. We have* $\operatorname{ind}_- \omega(t) = 0$ *for $t \in [\sigma_0, \sigma_1)$.*

(*ii*) *The function* $\mathfrak{t}(\omega(t))$ *is continuous and strictly increasing on each component of I. We have*

$$\lim_{t \nearrow \sigma_i} \mathfrak{t}(\omega(t)) = +\infty, \ \lim_{t \searrow \sigma_i} \mathfrak{t}(\omega(t)) = -\infty, \quad i = 1, \ldots, n,$$

and $\omega(\sigma_0) = I$.

(*iii*) *None of the intervals (σ_i, σ_{i+1}), $i = 1, \ldots, n-1$, is indivisible. The intervals $[\sigma_0, \sigma_1)$ and $(\sigma_n, \sigma_{n+1}]$, however, might be indivisible.*

(*iv*) *Let $t \in I_{\text{reg}}$ and $s \in I$, $s \geq t$. Then $\mathfrak{K}(\omega(t)) \subseteq \mathfrak{K}(\omega(s))$ isometrically, and the map $f \mapsto \omega(t)f$ is an isometric isomorphism of $\mathfrak{K}(\omega(t)^{-1}\omega(s))$ onto $\mathfrak{K}(\omega(s)) \ominus \mathfrak{K}(\omega(t))$.*

Note that this includes, as trivial cases, the points $t = \sigma_0$ and $t = \sigma_{n+1}$.

(*v*) *Let $i \in \{1, \ldots, n\}$. Then there exists a unique angle $\phi(\sigma_i) \in [0, \pi)$ such that*

$$\lim_{t \nearrow \sigma_i} \left[\circlearrowleft_{\phi(\sigma_i)} \omega(t)\right]'_{12}(0) < \infty .$$

We have

$$\lim_{t \searrow \sigma_i} \left[\circlearrowleft_{\phi(\sigma_i)} \omega(t)\right]'_{12}(0) = \lim_{t \nearrow \sigma_i} \left[\circlearrowleft_{\phi(\sigma_i)} \omega(t)\right]'_{12}(0) .$$

❑

Again we will write $\operatorname{ind}_- \omega := \max_{t\in I} \operatorname{ind}_- \omega(t) = \operatorname{ind}_- \omega(\sigma_{n+1})$, denote the set of all finite maximal chains ω with $\operatorname{ind}_- \omega = \kappa$ by $\mathfrak{M}^f_\kappa$, and set

$$\mathfrak{M}^f_{<\infty} := \bigcup_{\nu\in\mathbb{N}\cup\{0\}} \mathfrak{M}^f_\nu \,.$$

Clearly, if $\omega_1 \in \mathfrak{M}^f_{<\infty}$ and $\omega_2 \leftrightsquigarrow \omega_1$, then also $\omega_2 \in \mathfrak{M}^f_{<\infty}$ and $\operatorname{ind}_- \omega_1 = \operatorname{ind}_- \omega_2$.

If $W \in \mathcal{M}_\kappa$ and $\omega : I \to \mathcal{M}_{<\infty}$ is a finite maximal chain with $\omega(\max I) = W$, we speak of ω as a finite maximal chain going downwards from W. Let us recall the following fundamental result, cf. Theorem II.7.1.

3.9. *Existence/Uniqueness of finite maximal chains* ([KW/II])***:*** *Let $W \in \mathcal{M}_{<\infty}$ be given. Then there exists an, up to reparameterizations, unique finite maximal chain ω going downwards from W.*

Note in this place that, clearly, $\omega_1 \leftrightsquigarrow \omega_2$ implies that $\omega_1(\max I_1) = \omega_2(\max I_2)$. Hence, if $W \in \mathcal{M}_{<\infty}$ is given, the set of all finite maximal chains going downwards from W equals exactly one equivalence class of $\mathfrak{M}^f_{\operatorname{ind}_- W}$ modulo $\leftrightsquigarrow$.

Next we give an easy-to-check set of conditions for maximality of a given chain. This result will be of good use later on.

3.10 Proposition. *Let $\omega : I \to \mathcal{M}_{<\infty}$ be given, where the set I is of the form $I = [\sigma_0, \sigma_{n+1}] \setminus \{\sigma_1, \ldots, \sigma_n\}$ with $\sigma_0 < \sigma_1 < \sigma_2 < \ldots < \sigma_n < \sigma_{n+1}$, and let $W \in \mathcal{M}_{<\infty}$. Then ω is a finite maximal chain going downwards from W if and only if the following conditions hold:*

(*i*) *We have $\omega(\sigma_0) = I$ and $\omega(\sigma_{n+1}) = W$.*

(*ii*) *The function $\operatorname{ind}_- \omega(t)$ is nondecreasing on I, constant on each component of I, and takes different values on different components. For each $t \in I$ we have $\omega(t)^{-1}W \in \mathcal{M}_{<\infty}$ and $\operatorname{ind}_- \omega(t)^{-1}W = \operatorname{ind}_- W - \operatorname{ind}_- \omega(t)$.*

(*iii*) *The function $\mathfrak{t}(\omega(t))$ is continuous and strictly increasing on each interval contained in I. Moreover,*

$$\lim_{t\searrow\sigma_i} \mathfrak{t}(\omega(t)) = -\infty,\ \lim_{t\nearrow\sigma_i} \mathfrak{t}(\omega(t)) = +\infty, \quad i = 1, \ldots, n\,.$$

(*iv*) *For each $i = 1, \ldots, n$ there exist functions $\tau_+, \tau_- \in \mathcal{N}_0$ such that*

$$\lim_{t\searrow\sigma_i} \omega(t) \star \tau_+ = \lim_{t\nearrow\sigma_i} \omega(t) \star \tau_- \,.$$

Proof. Necessity of the conditions (*iv*) is clear from what we have recalled in Proposition 3.8 and Theorem III.5.6.

We need to establish sufficiency. Thus let ω be given, and assume that conditions (*i*)–(*iv*) hold true. Let $\hat\omega : \hat I \to \mathcal{M}_{<\infty}$, $\hat I = [\hat\sigma_0, \hat\sigma_{\hat n+1}] \setminus \{\hat\sigma_1, \ldots, \hat\sigma_{\hat n}\}$, be a finite maximal chain going downwards from W. We shall show that ω is a reparameterization of $\hat\omega$.

By our assumption (ii) on the factorization of W and by the maximality (W4) of $\hat{\omega}$, for each $t \in I$ there exists a number $s \in \hat{I}$ such that $\omega(t) = \hat{\omega}(s)$, i.e. $\operatorname{ran} \omega \subseteq \operatorname{ran} \hat{\omega}$. Since $\hat{\omega}$ is injective, we can define $\vartheta := \hat{\omega}^{-1} \circ \omega$, then ϑ maps I into $\hat{I}$.

Write $I = \{\sigma_0\} \cup I_0 \cup \ldots \cup I_n \cup \{\sigma_{n+1}\}$ where $I_j := (\sigma_j, \sigma_{j+1})$. Similarly, let $\hat{I} = \{\hat{\sigma}_0\} \cup \hat{I}_0 \cup \ldots \cup \hat{I}_{\hat{n}} \cup \{\hat{\sigma}_{\hat{n}+1}\}$ with $\hat{I}_j := (\hat{\sigma}_j, \hat{\sigma}_{j+1})$. Let $i \in \{0, \ldots, n\}$ and assume that for some $l \in \{0, \ldots, \hat{n}\}$ we have $\vartheta(I_i) \cap \hat{I}_l \neq \emptyset$. Then, by our assumption (ii) on negative indices, and the property Proposition 3.8, (i), of $\hat{\omega}$, we have $\vartheta(I_i) \subseteq \hat{I}_l$. Hence, there exists a map $l : \{0, \ldots, n\} \to \{0, \ldots, \hat{n}\}$ such that $\vartheta(I_i) \subseteq \hat{I}_{l(i)}$. Moreover, the map l is strictly increasing.

By our assumption (iii), $\mathfrak{t} \circ \omega|_{I_i}$ is continuous and strictly increasing. Moreover, for $i = 1, \ldots, n$, it maps I_i bijectively onto $\mathbb{R}$. In the case that $i = 0$ or $i = n+1$, we obtain from (iii) and (i) that $\mathfrak{t} \circ \omega|_{I_i}$ is a continuous and increasing bijection of I_0 onto $(0, \infty)$ or of I_n onto $(-\infty, \mathfrak{t}(W))$, respectively. The map $\mathfrak{t} \circ \hat{\omega}$ has the same properties. Since $\mathfrak{t} \circ \omega = \mathfrak{t} \circ \hat{\omega} \circ \vartheta$, we have $\vartheta = (\mathfrak{t} \circ \hat{\omega}|_{\hat{I}_l})^{-1} \circ (\mathfrak{t} \circ \omega|_{I_i})$, and conclude that $\vartheta|_{I_i}$ is an increasing bijection of I_i onto $\hat{I}_{l(i)}$.

By (i) and (ii), we have $\operatorname{ind}_- \omega(t) = \operatorname{ind}_- \omega(\sigma_0) = 0$, $t \in I_0$. Since also $\operatorname{ind}_- \hat{\omega}(s) = 0$, $s \in \hat{I}_0$, it follows that $l(0) = 0$. Let $i \in \{0, \ldots, n-1\}$ be given. By our assumption (iv), Proposition III.5.1 and Theorem III.5.6, we have

$$\operatorname{ind}_- \hat{\omega}(\hat{I}_{l(i)}) = \operatorname{ind}_- \omega(I_i) = \operatorname{ind}_- \lim_{t \searrow \sigma_{i+1}} \omega(t) \star \infty =$$

$$= \operatorname{ind}_- \lim_{t \searrow \sigma_{i+1}} (\hat{\omega} \circ \vartheta)(t) \star \infty = \operatorname{ind}_- \lim_{s \searrow \hat{\sigma}_{l(i+1)}} \hat{\omega}(s) \star \infty =$$

$$= \operatorname{ind}_- \hat{\omega}(\hat{I}_{l(i+1)-1}) .$$

It follows that $l(i+1) - 1 = l(i)$. Recursively, we obtain $l(i) = i$ for all $i \in \{0, \ldots, n\}$. Since, by (i), $\operatorname{ind}_- \omega(t) = \operatorname{ind}_- \omega(\sigma_{n+1}) = \operatorname{ind}_- W$, $t \in I_n$, and correspondingly $\operatorname{ind}_- \hat{\omega}(t) = \operatorname{ind}_- \hat{\omega}(\hat{\sigma}_{\hat{n}+1}) = \operatorname{ind}_- W$, $t \in \hat{I}_{\hat{n}}$, we obtain that $l(n) = \hat{n}$. This shows that $n = \hat{n}$ and that l is an increasing bijection.

Altogether it follows that ϑ is an increasing and bijective map of I onto $\hat{I}$. ❑

Since each matrix $W \in \mathcal{M}_{<\infty}$ induces a de Branges Pontryagin space, a (finite) maximal chain of matrices induces a whole family of de Branges spaces. In essence, this is actually a chain of spaces.

3.11 Proposition ([KW/II]). *Let $\omega \in \mathfrak{M}^f_{<\infty}$, and assume that $\omega(t)$ satisfies* (2.23). *Denote $E_t := E_{\omega(t)}$ for $t \in I$. Then the set of all nondegenerated dB-subspaces of $\mathfrak{P}(E_{s_+})$ is equal to*

$$\big\{ \mathfrak{P}(E_t) : t \notin I_{\text{sing}} \big\} .$$

If $t \in I_{\text{sing}}$, and (t_-, t_+) is the maximal indivisible interval which contains t, then $\mathfrak{P}(E_t)$ contains each space $\mathfrak{P}(E_s)$, $s \leq t_-$, isometrically, and is contained in each $\mathfrak{P}(E_s)$, $s \geq t_+$, as a set but not isometrically. For $s \in (t_-, t_+) \setminus \{t\}$ we have $\mathfrak{P}(E_s) = \mathfrak{P}(E_t)$ as sets, but not isometrically. ❑

c. Splitting-and-pasting, and other operations.

The operations $\circlearrowleft_\alpha$ and rev, which were introduced previously for matrices $W \in \mathcal{M}_{<\infty}$, can be applied to chains of matrices. One can think of $\circlearrowleft_\alpha$ as rotation of the whole chain by the angle α, and of rev as reading the chain backwards, i.e. reversing the order in which the chain is run through.

3.12 Definition. Let $\omega \in \mathfrak{M}^f_{<\infty} \cup \mathfrak{M}_{<\infty}$, and let $\alpha \in \mathbb{R}$. Then we define

$$\circlearrowleft_\alpha \omega : \begin{cases} I & \to & \mathcal{M}_{<\infty} \\ t & \mapsto & \circlearrowleft_\alpha \omega(t) \end{cases}$$

If $\omega \in \mathfrak{M}^f_{<\infty}$, define

$$\operatorname{rev} \omega : \begin{cases} -I & \to & \mathcal{M}_{<\infty} \\ t & \mapsto & \operatorname{rev}\big(\omega(-t)^{-1}\omega(\sigma_{n+1})\big) \end{cases}$$

//

3.13 Lemma.

(*i*) *Let* $\omega \in \mathfrak{M}^f_\kappa$, *and let* $\alpha \in \mathbb{R}$. *Then* $\circlearrowleft_\alpha \omega \in \mathfrak{M}^f_\kappa$.

(*ii*) *Let* $\omega \in \mathfrak{M}_\kappa$, *and let* $\alpha \in \mathbb{R}$. *Then* $\circlearrowleft_\alpha \omega \in \mathfrak{M}_\kappa$, *and* $q_\infty(\circlearrowleft_\alpha \omega) = N_\alpha \star q_\infty(\omega)$.

(*iii*) *Let* $\omega \in \mathfrak{M}^f_\kappa$, *then also* $\operatorname{rev}\omega \in \mathfrak{M}^f_\kappa$.

Proof.

(*i*) As $\circlearrowleft_\alpha (\omega(s)^{-1}\omega(t)) = (\circlearrowleft_\alpha \omega(s))^{-1}(\circlearrowleft_\alpha \omega(t))$ for $s \le t$ the assertion follows from Lemma 2.5.

(*ii*) Since $\circlearrowleft_\alpha W$ is a linear polynomial if and only if W is, the first assertion follows from Lemma 2.5 in the same manner. The second is immediate from (3.3) and (2.24).

(*iii*) One easily verifies that $(\operatorname{rev}\omega)(s)^{-1}(\operatorname{rev}\omega)(t) = \operatorname{rev}(\omega(-t)^{-1}\omega(-s))$ for $s \le t$. Hence, the assertion is an immediate consequence of Lemma 2.7.

❑

Clearly,

$$\circlearrowleft_\beta (\circlearrowleft_\alpha \omega) = \circlearrowleft_{\alpha+\beta} \omega,\ \circlearrowleft_0 \omega = \omega \ \text{ and } \ \operatorname{rev}(\operatorname{rev}\omega) = \omega\,.$$

For the following keep in mind that I contains σ_0 and σ_{n+1} if ω is a finite maximal chain.

3.14 Definition. Let $\omega : I \to \mathcal{M}_{<\infty}$ be a maximal or a finite maximal chain. Let $r \in I \cup \{\sigma_0\}, s \in \overline{I}, r < s$. Then we define

$$\omega_{r\leftrightarrow s}(t) := \omega(r)^{-1}\omega(t),\ t \in I \cap [r, s]\,.$$

In parallel we will also use the notations $\omega_{\urcorner s} := \omega_{\sigma_0 \leftrightarrow s}$, and $\omega_{r\ulcorner} := \omega_{r\leftrightarrow\sigma_{n+1}}$.
//

Because of $\mathfrak{t}(WV) = \mathfrak{t}(W) + \mathfrak{t}(V)$ (see (2.0)), it is easy to show that $\omega_{r\leftrightarrow s}$ is a finite maximal chain or a maximal chain, depending whether $s \in I$ or $s \notin I$. Moreover, we have

$$(\omega_{r\leftrightarrow s})_{r'\leftrightarrow s'} = \omega_{r'\leftrightarrow s'}, \quad r, r' \in I \cup \{\sigma_0\}, s, s' \in \overline{I},\ r \leq r' < s' \leq s\,.$$

The following remark shows that one can often reduce statements about maximal chains to corresponding statements about finite maximal chains. Its proof is again obvious from the respective definitions.

3.15 Remark. Let $\omega : I \to \mathcal{M}_{<\infty}$ be a function which satisfies (W1) and (W5). Then the following are equivalent:

(i) ω is a maximal chain.

(ii) For each $s \in I$, the function $\omega_{\urcorner s}$ is a finite maximal chain.

(iii) There exists a sequence $(s_n)_{n\in\mathbb{N}}$, $s_n \in I$, with $s_n \nearrow \sigma_{n+1}$, such that for each $n \in \mathbb{N}$ the function $\omega_{\urcorner s_n}$ is a finite maximal chain.

//

Although the definition of pasting two chains of matrices is most natural, the properties of this operation are more involved.

3.16 Definition. Let $J_1, J_2 \subseteq \mathbb{R}$, and let $\omega_1 : J_1 \to \mathcal{M}_{<\infty}$, $\omega_2 : J_2 \to \mathcal{M}_{<\infty}$. Assume that $\inf J_2 = \sup J_1 \in J_1$, and that we have

$$\begin{cases} \omega_2(\inf J_2) = I & , \ \inf J_2 \in J_2 \\ \lim_{t\searrow \inf J_2} \omega_2(t) = I & , \ \inf J_2 \notin J_2 \end{cases}$$

Then we define $\omega_1 \uplus \omega_2 : J_1 \cup J_2 \to \mathcal{M}_{<\infty}$ by

$$(\omega_1 \uplus \omega_2)(t) := \begin{cases} \omega_1(t) & , \ t \in J_1 \\ \omega_1(s_1)\omega_2(t) & , \ t \in J_2 \end{cases}$$

where $s_1 := \sup J_1$. //

Note that $\uplus$ is associative whenever all operations are defined. From the considerations in [KW/II, §7] we obtain the following statement.

3.17 Proposition ([KW/II]). *Let $\omega_1 : I_1 \to \mathcal{M}_{<\infty}$ belong to $\mathfrak{M}^f_{\kappa_1}$, $\omega_2 : I_2 \to \mathcal{M}_{<\infty}$ belong to $\mathfrak{M}^f_{\kappa_2} \cup \mathfrak{M}_{\kappa_2}$, and assume that* $\sup I_1 = \inf I_2$ *so that $\omega_1 \uplus \omega_2$ is well-defined. Assume that the following condition does not hold:*

(¬link) *ω_1 ends with an indivisible interval of type $\phi \in [0, \pi)$ and ω_2 starts with an indivisible interval of the same type ϕ.*

Then $\omega_1 \uplus \omega_2 \in \mathfrak{M}^f_{\kappa_1+\kappa_2}$ or $\omega_1 \uplus \omega_2 \in \mathfrak{M}_{\kappa_1+\kappa_2}$, depending whether $\omega_2 \in \mathfrak{M}^f_{\kappa_2}$ or $\omega_2 \in \mathfrak{M}_{\kappa_2}$. ❑

3.18 Remark.

(i) The fact that ω_2 starts with an indivisible interval of type ϕ is equivalent to the fact that for some (and hence for all) $t \in I_2 \setminus \{\inf I_2\}$ there exists a nonzero element $\xi_\phi \in \mathfrak{K}(\omega_2(t))$, see (II.5.10), Remark III.3.2.

Applying this fact to $t = -\inf I_1$ and $\operatorname{rev} \omega_1$ we obtain from Lemma 2.7 that the fact that ω_1 ends with an indivisible interval of type ϕ is equivalent to $\omega_1(\sup I_1)\xi_\phi \in \mathfrak{K}(\omega_1(\sup I_1))$.

Therefore, the condition ($\neg$link) is equivalent to the following condition:

($\neg$link$'$) For some $t \in I_2$ there exists a nonzero element $u \in \mathfrak{K}(\omega_2(t))$ such that $\omega_1(\sup I_1)u \in \mathfrak{K}(\omega_1(\sup I_1))$.

(ii) If, in the situation of Proposition 3.17, $\omega_2 \in \mathfrak{M}_{<\infty}$ then

$$q_\infty(\omega_1 \uplus \omega_2) = \omega_1(\sup I_1) \star q_\infty(\omega_2)\,.$$

//

The operations of splitting and pasting are converses of each other. The following statements are easily seen from Proposition 3.17 and Proposition 3.8, (iii). We will thus not elaborate their proofs.

3.19 Lemma. *Assume that $\omega : I \to \mathcal{M}_{<\infty}$ is a finite maximal (or maximal) chain, and let F be a finite subset of I_{reg}. Write $F = \{r_1, \ldots, r_m\}$ with $\sigma_0 < r_1 < \ldots < r_m < \sigma_{n+1}$, and set $r_0 := \sigma_0$, $r_{m+1} := \sigma_{n+1}$.*

Then $\omega_{r_j \leftrightarrow r_{j+1}} \in \mathfrak{M}^f_{<\infty}$, $j = 0, \ldots, m-1$, and $\omega_{r_m \leftrightarrow r_{m+1}}$ belongs to $\mathfrak{M}^f_{<\infty}$ or $\mathfrak{M}_{<\infty}$ depending whether ω has the corresponding property. For each two consecutive chains $\omega_{r_{j-1} \leftrightarrow r_j}, \omega_{r_j \leftrightarrow r_{j+1}}$, the condition ($\neg$link) *fails, and we have*

$$\omega = \biguplus_{j=0}^{m} \omega_{r_j \leftrightarrow r_{j+1}}\,.$$

❑

3.20 Lemma. *Let $\sigma_0, \ldots, \sigma_{n+1} \in \mathbb{R} \cup \{\pm\infty\}$, $\sigma_0 < \ldots < \sigma_{n+1}$, set $I := \bigcup_{i=0}^n (\sigma_i, \sigma_{i+1})$, and let F be a finite subset of I. Write $F = \{r_1, \ldots, r_m\}$ with $\sigma_0 < r_1 < \ldots < r_m < \sigma_{n+1}$, and set $r_0 := \sigma_0$, $r_{m+1} := \sigma_{n+1}$.*

Assume that there are given finite maximal chains $\omega_j : [r_j, r_{j+1}] \cap I \to \mathcal{M}_{<\infty}$, $j = 0, \ldots, m-1$, and a finite maximal (or maximal) chain $\omega_m : [r_m, r_{m+1}] \to \mathcal{M}_{<\infty}$ (or $\omega_m : [r_m, r_{m+1}) \to \mathcal{M}_{<\infty}$, respectively). Assume that for each two consequtive chains ω_j and ω_{j+1} the condition ($\neg$link) *fails, and set $\omega := \biguplus_{j=0}^m \omega_j$, so that ω is a finite maximal or maximal chain depending whether ω_m has the corresponding property.*

Then $F := \{r_1, \ldots, r_m\} \subseteq I_{\mathrm{reg}}$ and

$$\omega_{r_j \leftrightarrow r_{j+1}} = \omega_j,\ j = 0, \ldots, m\,.$$

❑

Let $\omega \in \mathfrak{M}^f_{<\infty} \cup \mathfrak{M}_{<\infty}$. Then, by virtue of Proposition 2.3, we obtain a family of boundary triplets, namely $\mathfrak{B}(\omega(t))$, $t \in I$. It follows by induction from Proposition 2.11 and Remark 3.18 that a splitting of the chain ω corresponds to a splitting of the associated boundary triplets.

3.21 Corollary. *Assume that* $\omega : I \to \mathcal{M}_{<\infty}$ *is a finite maximal (or maximal) chain, and let* F *be a finite subset of* I_{reg}. *Write* $F = \{r_1, \ldots, r_m\}$ *with* $\sigma_0 < r_1 < \ldots < r_m < \sigma_{n+1}$, *and set* $r_0 := \sigma_0$, $r_{m+1} := \sigma_{n+1}$. *Moreover, for* $t \in I$, *let* $i(t) \in \{0, \ldots, m\}$ *be such that* $t \in (r_{i(t)}, r_{i(t)+1}]$. *Then we have*

$$\mathfrak{B}(\omega(t)) = \biguplus_{i=1}^{i(t)} \mathfrak{B}_{\omega(r_{i-1})}(\omega_{r_{i-1} \leftrightarrow r_i}(r_i)) \uplus \mathfrak{B}_{\omega(r_{i(t)})}(\omega_{r_{i(t)} \leftrightarrow r_{i(t)+1}}(t)) .$$

❑

d. Positive definite Hamiltonians

Let $I = (s_-, s_+)$ be an interval on the real axis where $s_- < s_+$, $s_-, s_+ \in \mathbb{R} \cup \{\pm\infty\}$. A Hamiltonian on I is a measurable function H defined on I which takes real and nonnegative 2×2-matrices as values, is locally integrable on I, and does not vanish on any set of positive measure.

An important role is played by the primitive $\mathfrak{t}(H)$ of $\operatorname{tr} H$. It is determined up to an additive constant. Since $\operatorname{tr} H$ is nonnegative, locally integrable, and does not vanish on any set of positive measure, $\mathfrak{t}(H)$ is locally absolutely continuous and strictly increasing. Thus $\mathfrak{t}(H)$ maps I bijectively onto some interval (L_-, L_+).

We say that a Hamiltonian H is regular at the endpoint s_- if, for some $\epsilon > 0$,

$$\int_{s_-}^{s_-+\epsilon} \operatorname{tr} H(t)\, dt < \infty . \tag{3.4}$$

If $\int_{s_-}^{s_-+\epsilon} \operatorname{tr} H(t)\, dt = \infty$, it is called singular at s_-. The terminology of regular/singular at the endpoint s_+ is defined analogously. Sometimes one also speakes of Weyl's limit circle and limit point case, instead of regular and singular, respectively.

3.22 Remark. Let $F : [a, b] \to [c, d]$ be increasing, bijective and absolutely continuous, and assume that F' does not vanish identically on any Borel-subset of $[a, b]$ with positive measure. Then $F^{-1} : [c, d] \to [a, b]$ is absolutely continuous, and $(F^{-1})' = \frac{1}{F' \circ F^{-1}}$ a.e. //

Therefore, since $\operatorname{tr} H$ does not vanish on any set of positive measure, also the inverse function $\mathfrak{t}(H)^{-1}$ is locally absolutely continuous.

We also see that H is regular or singular at the endpoint $s_\pm$ in the sense of (3.4) if and only if $L_\pm$ is finite or infinite.

Intervals where H is of a particularly simple form play a special role. An interval $(\alpha_-, \alpha_+) \subseteq I$, $\alpha_- < \alpha_+$, is called H-indivisible of type $\phi \in [0, \pi)$ if

$$\operatorname{ran} H(t) = \operatorname{span}\{\xi_\phi\},\ t \in (\alpha_-, \alpha_+) \text{ a.e.}$$

In this case we have, with an appropriate measurable, scalar and a.e. positive function $h(t)$,

$$H(t) = h(t) \xi_\phi \xi_\phi^T,\ t \in (\alpha_-, \alpha_+) \text{ a.e.}$$

If (α_-, α_+) is H-indivisible, the difference $\mathfrak{t}(H)(\alpha_+) - \mathfrak{t}(H)(\alpha_-) \in (0, \infty]$ is called the length of this H-indivisible interval.

It is clear that, if (α_-, α_+) and (α'_-, α'_+) are H-indivisible intervals with nonempty intersection, then their types must coincide and their union is again

H-indivisible. Hence, every H-indivisible interval is contained in a maximal H-indivisible interval. Similar as in the setting of chains of matrices, we will also here denote by I_{sing} the union of all indivisible intervals, and set $I_{\text{reg}} := I \setminus I_{\text{sing}}$.

Two Hamiltonians H_1 and H_2 which are defined on intervals (s_-^1, s_+^1) and (s_-^2, s_+^2), respectively, are called reparameterizations of each other, if there exists an increasing bijection ϑ of (s_-^2, s_+^2) onto (s_-^1, s_+^1) such that ϑ and ϑ^{-1} are locally absolutely continuous and $H_2(t) = H_1(\vartheta(t))\vartheta'(t)$. In this case we write $H_1 \leftrightsquigarrow H_2$. Clearly, this relation is an equivalence relation on the set of all Hamiltonians.

It is a classical result that positive definite Hamiltonians are related to positive definite maximal chains of matrices. Let us state this fact in a comprehensive formulation suitable for our purposes.

3.23 Proposition ([GK], [HSW], [dB])**.**

(*i*) *Let H be a positive definite Hamiltonian defined on an interval (s_-, s_+) which is regular at s_-. Then there exists a unique solution $W(t,z)$ of the initial value problem*

$$\frac{\partial}{\partial t} W(t,z)J = zW(t,z)H(t), \quad \text{for a.e. } t \in (s_-, s_+), \qquad W(s_-, z) = I\,, \tag{3.5}$$

where z is a complex parameter. Set

$$\omega_H(t) := W(t,.),\ t \in \begin{cases} [s_-, s_+] & , H \text{ regular at } s_+ \\ (s_-, s_+) & , H \text{ singular at } s_+ \end{cases}$$

If H is regular at s_+, then $\omega_H(t)$ belongs to $\mathfrak{M}_0^f$. If H is singular at s_+, then $\omega_H(t) \in \mathfrak{M}_0$.

The function $\mathfrak{t} \circ \omega_H$ and its inverse are both locally absolutely continuous. In fact, $\mathfrak{t} \circ \omega_H = \mathfrak{t}(H)$ when $\mathfrak{t}(H)$ is chosen such that it takes the value 0 *at s_-.*

If $H_1 \leftrightsquigarrow H_2$, and $\omega_{H_1}, \omega_{H_2}$ are defined correspondingly, then $\omega_{H_1} \leftrightsquigarrow \omega_{H_2}$.

(*ii*) *Let $\omega \in \mathfrak{M}_0^f \cup \mathfrak{M}_0$, and assume that $\mathfrak{t} \circ \omega$ and its inverse are locally absolutely continuous. Then there exists a unique Hamiltonian H which is regular at s_-, and regular or singular at s_+ depending whether $\omega \in \mathfrak{M}_0^f$ or $\omega \in \mathfrak{M}_0$, such that $\omega = \omega_H$, i.e. such that $\omega(t)$ is the solution of* (3.5)*.*

If ω_1 and ω_2 are maximal chains or finite maximal chains which give rise to Hamiltonians H_1 and H_2, respectively, and if $\omega_1 \leftrightsquigarrow \omega_2$, then $H_1 \leftrightsquigarrow H_2$.

❑

Each equivalence class of chains modulo reparameterization contains elements which do have the property that $\mathfrak{t} \circ \omega$ and $(\mathfrak{t} \circ \omega)^{-1}$ are locally absolutely continuous, and hence give rise to a Hamiltonian. Thus Proposition 3.23 can be stated, in a somewhat less detailed form, as follows.

3.24 Remark. Denote by $\mathfrak{H}_0$ the set of all Hamiltonians which are regular at their left endpoint. Then the assignment $H \mapsto \omega_H$ induces a bijective correspondence between $\mathfrak{H}_0/_{\leftrightsquigarrow}$ and $(\mathfrak{M}_0^f \cup \mathfrak{M}_0)/_{\leftrightsquigarrow}$, where Hamiltonians which are regular at s_+ correspond to finite maximal chains, and Hamiltonians which are singular at s_+ correspond to maximal chains. //

A classical result, which lies at the basis of the operator theory of canonical systems, says that a Hamiltonian H generates a boundary triplet $\mathfrak{B}(H) = (L^2(H), T_{\max}(H), \Gamma(H))$. Thereby $\operatorname{mul}\Gamma(H) \neq \{0\}$ if and only if (s_-, s_+) is H-indivisible. These facts were formulated in a suitable way for our present purposes in [KW/IV, §2.1]. If H is regular at s_- and at s_+, the boundary triplet $\mathfrak{B}(H)$ has defect 2, if H is regular at s_- and singular at s_+ it has defect 1. In any case, it satisfies (E).

The boundary triplet generated by H can be related to q_H or $\omega_H(s_+)$, respectively. We start with the case of a singular Hamiltonian.

3.25 Proposition ([HSW]). *Let H be a Hamiltonian which is regular at s_- and singular at s_+. Then the Weyl-coefficient q_H is a Q-function of the symmetry $S(H) := T(H)^*$.* ❑

Assume that H is regular at s_- and at s_+. It is well-known that then there exists an isomorphism between the Hilbert spaces $L^2(H)$ and $\mathfrak{K}(\omega_H(s_+))$. Let us complete the picture and show that this isomorphism actually is an isomorphism of boundary triplets.

3.26 Proposition. *Let H be a Hamiltonian which is regular at both endpoints s_- and s_+. Denote*

$$(\Theta f)(z) := \int_{s_-}^{s_+} \omega_H(t)(z)H(t)f(t)\,dt, \quad f \in L^2(H)\,.$$

Then the pair $(\Theta, \mathrm{id}_{\mathbb{C}^4})$ is an isomorphism between the boundary triplets $\mathfrak{B}(H)$ and $\mathfrak{B}(\omega_H(s_+))$.

Proof. Assume first that $\operatorname{mul}\Gamma(H) = \{0\}$, i.e. that not the whole interval (s_-, s_+) is indivisible. Note that this is equivalent to assuming that $\omega_H(s_+) \neq W_{(l,\phi)}$ with some l, ϕ, and hence equivalent to $\operatorname{mul}\Gamma(\omega_H(s_+)) = \{0\}$.

Since $T_{\max}(H)^*$ is completely nonselfadjoint, and since for each $w \in \mathbb{C}$ the space $\ker(T_{\max}(H) - w)$ is spanned by the functions $t \mapsto \omega_H(t)(w)^T u$, $u \in \mathbb{C}^2$, we have

$$\begin{aligned}
&L^2(H) = \operatorname{cls}\big\{\omega_H(.)(w)^T u :\, u \in \mathbb{C}^2, w \in \mathbb{C}\big\}\,,\\
&T_{\max}(H) = \operatorname{cls}\big\{\big(\omega_H(.)(w)^T u; w\omega_H(.)(w)^T u\big) :\, u \in \mathbb{C}^2, w \in \mathbb{C}\big\}\,,\\
&\Gamma(H) = \operatorname{cls}\big\{\big((\omega_H(.)(w)^T u; w\omega_H(.)(w)^T u); (u; \omega_H(s_+)(w)^T u)\big) :\\
&\hspace{9cm} u \in \mathbb{C}^2, w \in \mathbb{C}\big\}\,.
\end{aligned}$$

The function ω_H satisfies the differential equation (3.5), and hence, as a computation shows, we have

$$H_{\omega_H(s_+)}(w,z) = \int_{s_-}^{s_+} \omega_H(t)(z)H(t)\omega_H(t)(w)^*\,dt\,,$$

cf. [dB]. Therefore,

$$\big(\Theta\,\omega_H(.)(w)^T u\big)(z) = \int_{s_-}^{s_+} \omega_H(t)(z)H(t)\omega_H(t)(\overline{w})^* u\,dt = H_{\omega_H(s_+)}(\overline{w}, z)u\,, \tag{3.6}$$

and we obtain

$$\Theta\big(\operatorname{span}\{\omega_H(.)(w)^T u\colon u\in\mathbb{C}^2, w\in\mathbb{C}\}\big) = \operatorname{span}\{H_{\omega_H(s_+)}(\overline{w},.)\colon u\in\mathbb{C}^2, w\in\mathbb{C}\}\,.$$

Moreover, by the abstract Green's identity in $\mathfrak{B}(H)$,

$$\begin{aligned}
(w_1-\overline{w_2})\big[\omega_H(.)(w_1)^T u_1, \omega_H(.)(w_2)^T u_2\big]_{L^2(H)} &= \\
= u_2^* J u_1 - \big(\omega_H(s_+)(w_2)^T u_2\big)^* J\big(\omega_H(s_+)(w_1)^T u_1\big) &= \\
= -u_2^*\big(\omega_H(s_+)(\overline{w_2})u_2 J \omega_H(s_+)(\overline{w_1})^* u_1 - J\big)u_1 &= \\
= -u_2^*(\overline{w_2}-w_1)H_{\omega_H(s_+)}(\overline{w_1},\overline{w_2})u_1 &= \\
= (w_1-\overline{w_2})\big[H_{\omega_H(s_+)}(\overline{w_1},.)u_1, H_{\omega_H(s_+)}(\overline{w_2},.)u_2\big]_{\mathfrak{K}(\omega_H(s_+))} &= \\
= (w_1-\overline{w_2})\big[\Theta\,\omega_H(.)(w_1)^T u_1, \Theta\,\omega_H(.)(w_2)^T u_2\big]_{\mathfrak{K}(\omega_H(s_+))} &.
\end{aligned}$$

Thus, Θ is isometric. With a standard continuity argument, we obtain that

$$\Theta(L^2(H)) = \mathfrak{K}(\omega_H(s_+)), \quad (\Theta\boxtimes\Theta)T_{\max}(H) = T(\omega_H(s_+))\,.$$

Moreover, it is clear that Θ is compatible with the respective involutions. From the definition of the boundary relation $\Gamma(\omega_H(s_+))$ we see that also

$$\big((\Theta\boxtimes\Theta)\boxtimes\operatorname{id}_{\mathbb{C}^4}\big)\Gamma(H) = \Gamma(\omega_H(s_+))\,.$$

The case that (s_-,s_+) is indivisible can be checked explicitly from the form of $L^2(H)$ and $\mathfrak{K}(W_{(l,\phi)})$, cf. [KW/IV, §2.1.e], Proposition 2.8. We will not carry out the details. ❑

Later on we will need a more general formulation of this result.

3.27 Corollary. *Let H be a Hamiltonian defined on (s_-,s_+), and let $\omega : (s_-,s_+)\to\mathcal{M}_{<\infty}$ be a solution of the differential equation in* (3.5) *(without imposing any conditions on boundary values, they even need not necessarily exist). Moreover, let $r_-,r_+\in[s_-,s_+]$, $r_-<r_+$, be such that $H|_{(r_-,r_+)}$ is regular at both of its endpoints $r_\pm$, and denote*

$$(\Theta_{r_-,r_+}f)(z) := \int_{r_-}^{r_+}\omega(r_-,z)^{-1}\omega(t,z)H(t)f(t)\,dt, \quad f\in L^2(H|_{(r_-,r_+)})\,.$$

Then the pair $(\Theta_{r_-,r_+},\operatorname{id}_{\mathbb{C}^4})$ is an isomorphism between the boundary triplets $\mathfrak{B}(H|_{(r_-,r_+)})$ and $\mathfrak{B}(\omega(r_-)^{-1}\omega(r_+))$.

Proof. This follows immediately from Proposition 3.26, since $\omega(r_-)^{-1}\omega(t)$, $t\in(r_-,r_+)$, is the solution of the initial value problem (3.5) for the Hamiltonian $H|_{(r_-,r_+)}$. ❑

The operations $\circlearrowleft_\alpha$ and rev, defined above on the level of chains of matrices, have their analogues for Hamiltonians.

3.28 Definition. Let H be a Hamiltonian defined on (s_-,s_+).

(*i*) For $\alpha\in\mathbb{R}$ define $\circlearrowleft_\alpha H : (s_-,s_+)\to\mathbb{R}^{2\times 2}$ as

$$(\circlearrowleft_\alpha H)(t) := \circlearrowleft_\alpha (H(t)),\ t\in(s_-,s_+)\,.$$

(*ii*) Define $\operatorname{rev} H : (-s_+, -s_-) \to \mathbb{R}^{2\times 2}$ as

$$(\operatorname{rev} H)(t) := VH(-t)V,\ t \in (-s_+, -s_-)\,.$$

//

3.29 Lemma. *Let H be a Hamiltonian defined on (s_-, s_+). Then also $\circlearrowleft_\alpha H$ and $\operatorname{rev} H$ are Hamiltonians. If H is regular/singular at $s_\pm$, then $\circlearrowleft_\alpha H$ is regular/singular at $s_\pm$, and $\operatorname{rev} H$ is regular/singular at $s_\mp$.*

If H is regular at s_-, then $\omega_{\circlearrowleft_\alpha H} = \circlearrowleft_\alpha \omega_H$. If H is regular at s_- and at s_+, then $\omega_{\operatorname{rev} H} = \operatorname{rev} \omega_H$.

Proof. The proof of these assertions is done by elementary computation, namely by checking that the functions $\circlearrowleft_\alpha \omega_H$ and $\operatorname{rev} \omega_H$ satisfy the respective differential equations. We will not carry out the details. ❑

3.30 Remark. The construction of the boundary triplet $\mathfrak{B}(H)$ in [KW/IV, §2.1] also shows that $(\varpi, \nu_\alpha \boxtimes \nu_\alpha)$ is an isomorphism of the boundary triplets $\mathfrak{B}(H)$ and $\mathfrak{B}(\circlearrowleft_\alpha H)$. Hereby, $\nu_\alpha : \mathbb{C}^2 \to \mathbb{C}^2$ is the map $\nu_\alpha x := N_\alpha x$ and $(\varpi f)(t) := N_\alpha f(t)$ for $f \in L^2(H)$.

Similarly, (ϖ, ϕ) is an isomorphism of the boundary triplets $\mathfrak{B}(H)$ and $\mathfrak{B}(\operatorname{rev} H)$, where $(\varpi f)(t) = Vf(-t)$, and where ϕ and V are as in Definition 2.6. //

Also the splitting-and-pasting method has an analogue for Hamiltonians. Let (s_-, s_+) be given, and let F be a finite subset of $[s_-, s_+]$ with $s_-, s_+ \in F$. Write $F = \{r_0, \dots, r_{m+1}\}$ with $r_0 < \dots < r_{m+1}$. If H is a Hamiltonian defined on (s_-, s_+), set

$$H_{r_i \leftrightarrow r_{i+1}} := H|_{(r_i, r_{i+1})}\,.$$

Conversely, if H_i are Hamiltonians on (r_i, r_{i+1}), $i = 0, \dots, m$, define a function $\biguplus_{i=0}^m H_i : (s_-, s_+) \to \mathbb{R}^{2\times 2}$ a.e. by

$$\Big(\biguplus_{i=0}^m H_i\Big)(t) := H_i(t), \quad t \in (r_i, r_{i+1}),\ i = 0, \dots, m\,.$$

The following is immediate from the definitions:

3.31 Remark. If H is a Hamiltonian on (s_-, s_+), then $H_{r_i \leftrightarrow r_{i+1}}$ are Hamiltonians on (r_i, r_{i+1}), which are regular at the endpoints r_j, $j = 1, \dots, m$, and regular/singular at r_0 or r_{m+1} depending whether H is regular/singular at s_- or s_+. If H is regular at s_-, then we have (see Definition 3.16)

$$\omega_H = \biguplus_{i=0}^m \omega_{H_{r_i \leftrightarrow r_{i+1}}}\,. \tag{3.7}$$

Conversely, if H_i are Hamiltonians on (r_i, r_{i+1}), which are regular at the endpoints r_j, $j = 0, \dots, m$, then $H := \biguplus_{i=0}^m H_i$ is a Hamiltonian on (s_-, s_+) which is regular at s_-, and (3.7) holds. //

Similar as in Corollary 3.21, we can pass to boundary triplets. It is straightforward to verify the following remark.

3.32 Remark. Let H be a Hamiltonian defined on (s_-, s_+), and let $F = \{r_0, \dots, r_{m+1}\}$, $r_0 < \dots < r_{m+1}$, be a finite subset of $[s_-, s_+]$ with $s_-, s_+ \in F$. Assume that $r_i \notin I_{\text{sing}}$, $i = 1, \dots, m$. Then

$$\mathfrak{B}(H) = \biguplus_{i=0}^{m} \mathfrak{B}(H_{r_i \leftrightarrow r_{i+1}}) .$$

//

3.33 Remark. Let H be a Hamiltonian defined on (s_-, s_+). From (3.5) it is easily seen that H starts (ends) with an indivisible interval of type ϕ if and only ω_H does.

Therefore, the condition that $r \in (s_-, s_+)$ is contained in I_{reg} is equivalent to the fact that ($\neg$link) from Proposition 3.17 for $\omega_1 = \omega_{H_{s_- \leftrightarrow r}}$ and $\omega_2 = \omega_{H_{r \leftrightarrow s_+}}$ fails. //

It is interesting to note the following compatibility with the isomorphisms Θ.

3.34 Lemma. *Let H be a Hamiltonian defined on (s_-, s_+), and let $\omega : (s_-, s_+) \to \mathcal{M}_{<\infty}$ be a solution of the differential equation in* (3.5) *(without imposing any conditions on boundary values). Moreover, let $r_-, r_+, r'_-, r'_+ \in [s_-, s_+]$, $r_- \leq r'_- < r'_+ \leq r_+$, be such that $H|_{(r_-, r_+)}$ is regular at both of its endpoints $r_\pm$, and assume that $r'_\pm \notin I_{\text{sing}}$. Let us denote*

(i) by $\iota_{r'_-, r'_+} : L^2(H_{r'_- \leftrightarrow r'_+}) \to L^2(H_{r_- \leftrightarrow r_+})$ the natural embedding operator,

(ii) by $\rho_{r'_-, r'_+} : L^2(H_{r_- \leftrightarrow r_+}) \to L^2(H_{r'_- \leftrightarrow r'_+})$ the restriction operator (we then have $\rho_{r'_-, r'_+} = \iota^*_{r'_-, r'_+}$),

(iii) by $P_{r'_-, r'_+}$ the orthogonal projection of $\mathfrak{K}(\omega(r_-)^{-1}\omega(r_+))$ onto its subspace $\omega(r_-)^{-1}\omega(r'_-)\mathfrak{K}(\omega(r'_-)^{-1}\omega(r'_+))$,

(iv) and by

$$\Theta_{r_-, r_+} : L^2(H_{r_- \leftrightarrow r_+}) \to \mathfrak{K}(\omega(r_-)^{-1}\omega(r_+)) ,$$
$$\Theta_{r'_-, r'_+} : L^2(H_{r'_- \leftrightarrow r'_+}) \to \mathfrak{K}(\omega(r'_-)^{-1}\omega(r'_+))$$

the respective isomorphisms as in Corollary 3.27.

Then we have

$$\begin{array}{ccc} L^2\big(H_{r_- \leftrightarrow r_+}\big) & \underset{\iota_{r'_-, r'_+}}{\overset{\rho_{r'_-, r'_+}}{\rightleftarrows}} & L^2\big(H_{r'_- \leftrightarrow r'_+}\big) \\ \Big\downarrow {\scriptstyle \Theta_{r_-, r_+}} & & \Big\downarrow {\scriptstyle \Theta_{r'_-, r'_+}} \\ \mathfrak{K}\big(\omega(r_-)^{-1}\omega(r_+)\big) & \underset{f \mapsto \omega(r_-)^{-1}\omega(r'_-) f}{\overset{\omega(r'_-)^{-1}\omega(r_-) P_{r'_-, r'_+}}{\rightleftarrows}} & \mathfrak{K}\big(\omega(r'_-)^{-1}\omega(r'_+)\big) \end{array}$$

Proof. The fact that the restriction operator $\rho_{r'_-,r'_+} : f \mapsto f|_{(r'_-,r'_+)}$ is the adjoint of the natural embedding operator $\iota_{r'_-,r'_+}$ is obvious.

By Corollary 3.27 we have for $f \in L^2\big(H_{r'_-\leftrightarrow r'_+}\big)$

$$\Theta_{r_-,r_+} \circ \iota_{r'_-,r'_+}(f) = \int_{r_-}^{r_+} \omega(r_-,z)^{-1}\omega(t,z)H(t)\iota_{r'_-,r'_+}(f)(t)\,dt =$$

$$= \omega(r_-,z)^{-1}\omega(r'_-,z)\int_{r'_-}^{r'_+} \omega(r'_-,z)^{-1}\omega(t,z)H(t)f(t)\,dt =$$

$$= \omega(r_-,z)^{-1}\omega(r'_-,z)\Theta_{r'_-,r'_+}(f)\,.$$

Since $\iota_{r'_-,r'_+}$ is isometric and both operators Θ_{r_-,r_+} and $\Theta_{r'_-,r'_+}$ are unitary, also the assignment $f \mapsto \omega(r_-)^{-1}\omega(r'_-)f$ maps $\mathfrak{K}\big(\omega(r'_-)^{-1}\omega(r'_+)\big)$ isometrically into $\mathfrak{K}\big(\omega(r_-)^{-1}\omega(r_+)\big)$. Its range, namely

$$\omega(r_-)^{-1}\omega(r'_-)\mathfrak{K}\big(\omega(r'_-)^{-1}\omega(r'_+)\big) = \big\{\omega(r_-)^{-1}\omega(r'_-)f : f \in \mathfrak{K}\big(\omega(r'_-)^{-1}\omega(r'_+)\big)\big\}\,,$$

is thus a closed subspace of $\mathfrak{K}\big(\omega(r_-)^{-1}\omega(r_+)\big)$. Hence, the adjoint of $f \mapsto \omega(r_-)^{-1}\omega(r'_-)f$ is the projection $P_{r'_-,r'_+}$ onto $\omega(r_-)^{-1}\omega(r'_-)\mathfrak{K}\big(\omega(r'_-)^{-1}\omega(r'_+)\big)$ followed by multiplication with $[\omega(r_-)^{-1}\omega(r'_-)]^{-1}$. ❑

e. General Hamiltonians

The notion of positive definite Hamiltonians admits a generalization to an indefinite setting. The definition of this generalization requires some preliminary notation. Let H be a Hamiltonian defined on the interval (s_-,s_+).

⇝ If H is regular at s_- a number $\Delta(H) \in \mathbb{N}\cup\{0,\infty\}$ is associated with H which measures in a certain sense the growth of H towards s_+, cf. Definition IV.3.1.

⇝ If H is regular at s_- and singular at s_+, we say that H satisfies the condition (HS) if resolvents of selfadjoint extensions of $T_{\max}(H)^*$ are Hilbert-Schmidt operators, cf. [KW/IV, §2.3.a]. In this case there exists a unique number $\phi(H)$ such that $\int_{s_-}^{s_+} \xi_{\phi(H)}^T H(t)\xi_{\phi(H)}\,dt < \infty$, cf. [KW] or Theorem IV.2.27.

⇝ Let H be singular at both endpoints s_- and s_+, and fix $s_0 \in (s_-,s_+)$. We say that H satisfies the condition (HS$_+$) or (HS$_-$), if $H|_{(s_0,s_+)}$ or $H|_{(s_-,s_0)}(-x)$, respectively, satisfies (HS). Moreover, we define

$$\Delta_+(H) := \Delta(H|_{(s_0,s_+)}),\ \Delta_-(H) := \Delta(H|_{(s_-,s_0)}(-x))\,,$$

$$\phi_+(H) := \phi(H|_{(s_0,s_+)}),\ \phi_-(H) := \phi(H|_{(s_-,s_0)}(-x))\,.$$

It was shown in [KW/IV, §2.3.c, (i)] and Lemma IV.3.12 that these numbers do not depend on the choice of $s_0 \in (s_-,s_+)$.

Now we can state the definition of a general Hamiltonian. It consists of a Hamiltonian function H, which has in a sense certain singularities, and some additional data associated with each singularity.

3.35 Definition. A general Hamiltonian $\mathfrak{h}$ is a collection of data of the following kind:

(i) $n \in \mathbb{N} \cup \{0\}$, $\sigma_0, \dots, \sigma_{n+1} \in \mathbb{R} \cup \{\pm\infty\}$ with $\sigma_0 < \sigma_1 < \dots < \sigma_{n+1}$.

(ii) Hamiltonians $H_i : (\sigma_i, \sigma_{i+1}) \to \mathbb{R}^{2\times 2}$ for $i = 0, \dots, n$ (as defined in the beginning of §3.d),

(iii) numbers $\ddot{o}_1, \dots, \ddot{o}_n \in \mathbb{N} \cup \{0\}$ and $b_{i,1}, \dots, b_{i,\ddot{o}_i+1} \in \mathbb{R}$, $i = 1, \dots, n$, with $b_{i,1} \neq 0$ in case $\ddot{o}_i \geq 1$,

(iv) numbers $d_{i,0}, \dots, d_{i,2\Delta_i-1} \in \mathbb{R}$, where $\Delta_i := \max\{\Delta_+(H_{i-1}), \Delta_-(H_i)\}$ for $i = 1, \dots, n$,

(v) a finite subset E of $\{\sigma_0, \sigma_{n+1}\} \cup \bigcup_{i=0}^n (\sigma_i, \sigma_{i+1})$,

which is assumed to be subject to the following conditions

(H1) The Hamiltonian H_0 is regular at σ_0 and, if $n \geq 1$, singular at σ_1. Each Hamiltonian H_i, $i = 1, \dots, n-1$, is singular at both endpoints σ_i and σ_{i+1}. If $n \geq 1$, then H_n is singular at σ_n.

(H2) None of the intervals (σ_i, σ_{i+1}), $i = 1, \dots, n-1$, is indivisible†. If H_n is singular at σ_{n+1}, then also (σ_n, σ_{n+1}) is not indivisible.

(H3) We have $\Delta_i < \infty$, $i = 1, \dots, n$. Moreover, H_0 satisfies (HS$_+$), H_i satisfies (HS$_-$) and (HS$_+$), $i = 1, \dots, n-1$, and H_n satisfies (HS$_-$).

(H4) We have $\phi_+(H_{i-1}) = \phi_-(H_i)$, $i = 1, \dots, n$.

(H5) Let $i \in \{1, \dots, n\}$. If both of H_{i-1} and H_i end with an indivisible interval towards σ_i, then $d_1 = 0$. If additionally $b_{i,1} = 0$, then also $d_0 < 0$.

(E1) $\sigma_0, \sigma_{n+1} \in E$, and $E \cap (\sigma_i, \sigma_{i+1}) \neq \emptyset$ for $i = 1, \dots, n-1$. If H_n is singular at σ_{n+1}, then also $E \cap (\sigma_n, \sigma_{n+1}) \neq \emptyset$. Moreover, E contains all endpoints of indivisible intervals of infinite length which lie in $\bigcup_{i=0}^n (\sigma_i, \sigma_{i+1})$.

(E2) No point of E is an inner point of an indivisible interval.

The common value of $\phi_+(H_{i-1})$ and $\phi_-(H_i)$ will be denoted by ϕ_i.

The general Hamiltonian $\mathfrak{h}$ is called definite if $n = 0$, and indefinite otherwise. It is called regular or singular, if H_n is regular or singular, respectively, at σ_{n+1}. Moreover, with the numbers $c_{i,j}$ for each i linked with the numbers $b_{i,j}$ $\ddot{o}_i$ as in (IV.4.2) we set

$$\operatorname{ind}_- \mathfrak{h} := \sum_{i=1}^n \big(\Delta_i + [\frac{\ddot{o}_i}{2}]\big) + \big|\big\{1 \leq i \leq n : \ddot{o}_i \text{ odd}, c_{i,1} < 0\big\}\big| . \tag{3.8}$$

We will denote the set of all general Hamiltonians by $\mathfrak{H}_{<\infty}$, and set $\mathfrak{H}_\kappa := \{\mathfrak{h} \in \mathfrak{H}_{<\infty} : \operatorname{ind}_- \mathfrak{h} = \kappa\}$. //

†No typo: The interval (σ_0, σ_1) may be indivisible.

Let us introduce some more generic notation. Let $\mathfrak{h}$ be a general Hamiltonian. The subset E is called an admissible partition and will be written as $E = \{s_0, \dots, s_{N+1}\}$ with $s_0 < \dots < s_{N+1}$. The function $H : I \to \mathbb{R}^{2\times 2}$, where $I := \bigcup_{i=0}^{n} (\sigma_i, \sigma_{i+1})$, which is defined by

$$H(t) := H_i(t),\ t \in (\sigma_i, \sigma_{i+1}), i = 0, \dots, n\,,$$

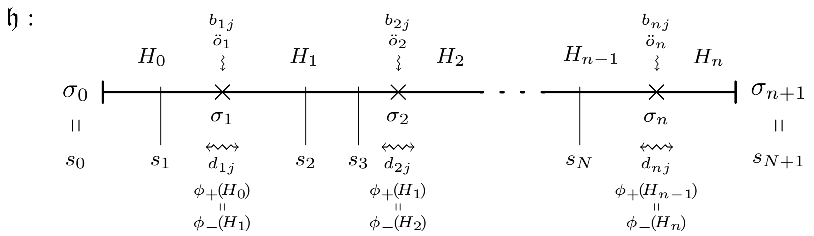
is referred to as the Hamiltonian function of $\mathfrak{h}$. For technical reasons we add to I the points σ_0 and σ_{n+1}. If t is one of these points we choose $H(t)$ to be the identity matrix. Since we consider H only almost everywhere this choice is not relevant.

We will denote by I_{sing} the union of all H-indivisible intervals, and set $I_{\text{reg}} = I \setminus I_{\text{sing}}$. Moreover, we will often write a Hamiltonian $\mathfrak{h}$ which is given by the data n, $\sigma_0, \dots, \sigma_{n+1}$, $H_1, \dots, H_n$, $\ddot{o}_1, \dots, \ddot{o}_n$, $b_{i,j}$, $d_{i,j}$ and E as

$$\mathfrak{h} = (H, \mathfrak{c}, \mathfrak{d})$$

where the Hamiltonian function H includes the number n and the points σ_i, where $\mathfrak{c}$ represents the numbers $\ddot{o}_i$ and $b_{i,j}$, and where $\mathfrak{d}$ represents the numbers $d_{i,j}$ and the subset E.

Intuitively speaking, a general Hamiltonian models a canonical system on $[\sigma_0, \sigma_{n+1})$ whose Hamiltonian is allowed to have singularities, namely $\sigma_1, \dots, \sigma_n$, and which is in the limit circle or limit point case at σ_{n+1} depending whether $\mathfrak{h}$ is regular or singular. The behaviour of H at a singularity is not too bad in the sense of (H3). A singularity itself contributes to the equation in two ways: Firstly, a contribution concentrated inside the singularity; passing the singularity influences the solution. This is modelled by the parameters $\mathfrak{c}$. Secondly, interface conditions which connect before and after each singularity. This is modelled by the parameters $\mathfrak{d}$, and by the condition (H4). We can picture the situation as follows ($E = \{s_0, \dots, s_{N+1}\}$):



Let $\mathfrak{h}$ be a general Hamiltonian. In [KW/IV] a boundary triplet $\mathfrak{B}(\mathfrak{h}) = (\mathcal{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h}))$ has been associated to $\mathfrak{h}$, cf. Definition IV.8.5.

3.36 Remark. Let us briefly recall the construction of $\mathfrak{B}(\mathfrak{h}) = (\mathcal{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h}))$, cf. Lemma IV.8.4.

For $l \in \{0, \dots, N\}$ let $i(l) \in \{0, \dots, n\}$ be such that either $(s_l, s_{l+1}) \subseteq (\sigma_{i(l)}, \sigma_{i(l)+1})$ or $\sigma_{i(l)} \in (s_l, s_{l+1})$ and $\sigma_i \notin (s_l, s_{l+1})$, $i \neq i(l)$.

In the first – definite – case $\mathfrak{h}^l := H_{i(l)}|_{(s_l, s_{l+1})}$ is a positive definite Hamiltonian which is regular at s_l. It is also regular at s_{l+1} if and only if it is not true that $\mathfrak{h}$ is singular and $l = N$.

If the second – indefinite – case occurs, then the data

$$\circlearrowleft_{\phi_{i(l)}} H_{i(l)-1}|_{(s_l, \sigma_{i(l)})}, \circlearrowleft_{\phi_{i(l)}} H_{i(l)}|_{(\sigma_{i(l)}, s_{l+1})}\,,$$

$$\ddot{o}_{i(l)}, c_{i(l),1}, \dots, c_{i(l),\ddot{o}_{i(l)}},\ d_{i(l),0}, \dots, d_{i(l),2\Delta_{i(l)}-1},$$

constitutes an elementary indefinite Hamiltonian $\mathfrak{h}^l$ of kind (A), (B) or (C), see Definition IV.4.1.

For each $l \in \{0, \dots, N\}$ boundary triplets $\mathfrak{B}(\mathfrak{h}^l) := (\mathcal{P}(\mathfrak{h}^l), T(\mathfrak{h}^l), \Gamma(\mathfrak{h}^l))$ are well defined by [KW/IV, §2.1, §4.1]. If $\mathfrak{h}$ is regular, then all these boundary triplets are of defect 2. If $\mathfrak{h}$ is singular, then $\mathfrak{B}(\mathfrak{h}^0), \dots, \mathfrak{B}(\mathfrak{h}^{N-1})$ are of defect 2, and $\mathfrak{B}(\mathfrak{h}^N)$ is of defect 1.

The boundary triplet associated to $\mathfrak{h}$ is defined as

$$\mathfrak{B}(\mathfrak{h}) = \big(\mathcal{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h})\big) := \biguplus_{l=0}^{N} \circlearrowleft_{\gamma_l} \mathfrak{B}(\mathfrak{h}^l), \tag{3.9}$$

where

$$\gamma_l := \begin{cases} 0 & , \ \mathfrak{h}^l \text{ is positive definite} \\ -\phi_{i(l)} & , \ \mathfrak{h}^l \text{ is elementary indefinite} \end{cases}$$

remember Definition 2.4. //

Also a mapping $\psi(\mathfrak{h}) : \mathcal{P}(\mathfrak{h}) \to \mathcal{M}(I)/_{=_H}$ has been defined. Here $\mathcal{M}(I)$ is the set of all measurable functions $f : I \to \mathbb{C}^2$ such that on any indivisible interval of H of type ϕ the complex valued function $\xi_\phi^T f$ is constant a.e., and $\mathcal{M}(I)/_{=_H}$ denotes the set of equivalence classes of $\mathcal{M}(I)$ induced by the equivalence relation $f =_H g \Leftrightarrow H(f-g) = 0$ a.e.

The mapping $\psi(\mathfrak{h})$ has a finite dimensional kernel. Hence, $\psi(\mathfrak{h})(f)$ reflects the major part of the information about a given element $f \in \mathcal{P}(\mathfrak{h})$. Nevertheless $\psi(\mathfrak{h})(f)$ does not describe f entirely. Some information is hidden in the singularities.

The following facts have been established in Theorem IV.8.6 and Theorem IV.8.7:

⇝ We have

$$\operatorname{ind}_- \mathcal{P}(\mathfrak{h}) = \sum_{i=1}^{n} \big(\Delta_i + [\frac{\ddot{o}}{2}]\big) + \big|\big\{1 \leq i \leq n : \ddot{o}_i \text{ odd}, c_{i,1} < 0\big\}\big|. \tag{3.10}$$

⇝ The triple $\mathfrak{B}(\mathfrak{h})$ is a boundary triplet which has defect 2 or 1, depending whether $\mathfrak{h}$ is regular or singular. Moreover, it satisfies the condition (E).

⇝ If $\mathfrak{h} = (H, \mathfrak{c}, \mathfrak{d})$ is regular, the adjoint $S(\mathfrak{h}) := T(\mathfrak{h})^*$ is a completely nonselfadjoint symmetric operator which satisfies (CR), cf. Definition IV.2.15, and has the property that $r(S(\mathfrak{h})) = \mathbb{C}$. Moreover, $S(\mathfrak{h})$ has defect index $(2,2)$ and $\operatorname{mul}\Gamma(\mathfrak{h}) = \{0\}$ unless the Hamiltonian function H is almost everywhere of the form $h(t)\xi_\gamma\xi_\gamma^T$ for a constant γ. In the later case we have $n \in \{0,1\}$, $S(\mathfrak{h})$ has defect index $(1,1)$ and $\operatorname{mul}\Gamma(\mathfrak{h}) \neq \{0\}$, where

$$\operatorname{mul}\Gamma(\mathfrak{h}) = \operatorname{span}\Big\{\big(J\xi_\gamma; J\xi_\gamma\big)\Big\}. \tag{3.11}$$

Note that if $h(t)\xi_\gamma\xi_\gamma^T$ almost everywhere, $n = 0$ means that $\mathfrak{h}$ is definite and indivisible, and $n = 1$ means that $(\circlearrowleft_{\phi_1} H, \mathfrak{c}, \mathfrak{d})$ is elementary indefinite of kind (B) or (C). Hereby $\phi_1 = \gamma - \frac{\pi}{2}$.

⇝ If $\mathfrak{h}$ is singular, the adjoint $S(\mathfrak{h}) := T(\mathfrak{h})^*$ is a symmetric operator[†]. Moreover, $S(\mathfrak{h})$ has defect index $(1,1)$ and $\operatorname{mul}\Gamma(\mathfrak{h}) = \{0\}$ unless the Hamiltonian function H is almost everywhere of the form $h(t)\xi_\phi\xi_\phi^T$ for a constant ϕ. In the later case $n=0$, $S(\mathfrak{h})$ is selfadjoint and $\operatorname{mul}\Gamma(\mathfrak{h}) \neq \{0\}$.

In the case that the Hamiltonian function H is not almost everywhere of the form $h(t)\xi_\phi\xi_\phi^T$ for some constant ϕ, i.e. $\Gamma(\mathfrak{h})$ is a function, in Remark IV.8.9 a mapping $\Psi^{ac}(\mathfrak{h}) : T(\mathfrak{h}) \to AC(I \cup \{\sigma_0\}) \times \mathcal{M}(I)/{=_H}$ was defined such that

$$\Psi^{ac}(\mathfrak{h})((f;g))_1' = JH\Psi^{ac}(\mathfrak{h})((f;g))_2, \ \textit{a.e.} \ \text{ on } I\,.$$

We have $\Psi^{ac}(\mathfrak{h})((f;g))_1(\sigma_0) = \Gamma(\mathfrak{h})(f;g)_1$ and, in case of a regular Hamiltonian also $\Psi^{ac}(\mathfrak{h})((f;g))_1(\sigma_{n+1}) = \Gamma(\mathfrak{h})(f;g)_2$. Moreover, $\Psi^{ac}(\mathfrak{h})(f;g)$ is such that its entries are equivalent (with respect to $=_H$) to $\psi(\mathfrak{h})(f)$ and $\psi(\mathfrak{h})(g)$, respectively.

For later use we bring the following assertion. We will say that $\mathfrak{h} = (H, \mathfrak{c}, \mathfrak{d})$ starts with an indivisible interval of type α if H starts with an indivisible interval of type α.

3.37 Lemma. *A general Hamiltonian $\mathfrak{h} = (H, \mathfrak{c}, \mathfrak{d})$, such that it is not the case, that it is positive definite and just one indivisible interval of infinite length, starts with an indivisible interval of type α if and only if there exists $g \in \mathcal{P}(\mathfrak{h})$, such that $((0;g);(J\xi_\alpha;0)) \in \Gamma(\mathfrak{h})$.*

Proof. The proof is similar to the arguments in the proof of Theorem IV.8.6. We first construct elementary general Hamiltonians $\mathfrak{h}^0, \dots, \mathfrak{h}^N$ from $\mathfrak{h}$ as in Remark 3.36.

Assume that for some $g \in \mathcal{P}(\mathfrak{h})$ we have $(0;g) \in T(\mathfrak{h})$. We write $g = g_0 + \dots + g_N$ according to (3.9), i.e. $\mathcal{P}(\mathfrak{h}) = \mathcal{P}(\mathfrak{h}^0) \oplus \dots \oplus \mathcal{P}(\mathfrak{h}^N)$. Then $(0;g_l) \in T(\mathfrak{h}^l)$ and there exist $a_0, a_1, \dots, a_N \in \mathbb{C}^2$ with

$$(a_0;a_1) \in (\nu_{\gamma_0} \boxtimes \nu_{\gamma_0}) \circ \Gamma(\mathfrak{h}^0)(0;g_0), (a_1;a_2) \in (\nu_{\gamma_1} \boxtimes \nu_{\gamma_1}) \circ \Gamma(\mathfrak{h}^1)(0;g_1),$$
$$\dots, (a_N;a_{N+1}) \in (\nu_{\gamma_N} \boxtimes \nu_{\gamma_N})\Gamma(\mathfrak{h}^N)(0;g_N)\,,$$

or equivalently,

$$(N_{-\gamma_0}a_0; N_{-\gamma_0}a_1) \in \Gamma(\mathfrak{h}^0)(0;g_0), (N_{-\gamma_1}a_1; N_{-\gamma_1}a_2) \in \Gamma(\mathfrak{h}^1)(0;g_1),$$
$$\dots, (N_{-\gamma_N}a_N; N_{-\gamma_N}a_{N+1}) \in \Gamma(\mathfrak{h}^N)(0;g_N)\,.$$

In the case that $\mathfrak{h}^l$ is positive definite we have $\gamma_l = 0$ and we know from Corollary IV.2.25, that

$$a_l \in \begin{cases} \operatorname{span}\{J\xi_{\phi_l^-}\} & , \ {\scriptstyle \alpha_1^-(H|_{(s_l,s_{l+1})})>s_l,\ \phi_l^- \text{ type of } (s_l,\alpha_1^-(H|_{(s_l,s_{l+1})}))} \\ \{0\} & , \ \alpha_1^-(H|_{(s_l,s_{l+1})}) = s_l \end{cases} \tag{3.12}$$

$$a_{l+1} \in \begin{cases} \operatorname{span}\{J\xi_{\phi_l^+}\} & , \ {\scriptstyle \alpha_1^+(H|_{(s_l,s_{l+1})})<s_{l+1},\ \phi_l^+ \text{ type of } (\alpha_1^+(H|_{(s_l,s_{l+1})}),s_{l+1})} \\ \{0\} & , \ \alpha_1^+(H|_{(s_l,s_{l+1})}) = s_{l+1} \end{cases} \tag{3.13}$$

[†]In Corollary 6.5 we will see that $S(\mathfrak{h})$ is also completely nonselfadjoint.

In the case that $\mathfrak{h}^l$ is indefinite we see from Proposition IV.5.16, that

$$N_{-\gamma_l} a_l \in \begin{cases} \operatorname{span}\{J\xi_{\psi_l^-}\} & , \begin{array}{c} \alpha_1^-(\circlearrowleft_{-\gamma_l} H|_{(s_l,s_{l+1})})>s_l, \\ \psi_l^- \text{ type of } (s_l,\alpha_1^-(\circlearrowleft_{-\gamma_l} H|_{(s_l,s_{l+1})})) \end{array} \\ \{0\} & , \ \alpha_1^-(\circlearrowleft_{-\gamma_l} H|_{(s_l,s_{l+1})}) = s_l \end{cases}$$

$$N_{-\gamma_l} a_{l+1} \in \begin{cases} \operatorname{span}\{J\xi_{\psi_l^+}\} & , \begin{array}{c} \alpha_1^+(\circlearrowleft_{-\gamma_l} H|_{(s_l,s_{l+1})})<s_{l+1}, \\ \psi_l^+ \text{ type of } (\alpha_1^+(\circlearrowleft_{-\gamma_l} H|_{(s_l,s_{l+1})}),s_{l+1}) \end{array} \\ \{0\} & , \ \alpha_1^+(\circlearrowleft_{-\gamma_l} H|_{(s_l,s_{l+1})}) = s_{l+1} \end{cases}$$

Note that if $\mathfrak{h}^l$ is indefinite of kind (B) or (C), then the type of (s_l, s_{l+1}) for $\circlearrowleft_{-\gamma_l} H|_{(s_l,s_{l+1})}$ is always $\frac{\pi}{2}$.

Applying $\circlearrowleft_{\gamma_l}$ we see that in fact (3.12) and (3.13) hold for all $l = 0, \dots, N$.

If we have $\alpha_1^+(H|_{(s_l,s_{l+1})}) < s_{l+1}$ and $\alpha_1^-(H|_{(s_{l+1},s_{l+2})}) > s_{l+1}$ for some index $l \in \{0, \dots, N-1\}$, then the types of the indivisible intervals $(\alpha_1^+(H|_{(s_l,s_{l+1})}), s_{l+1})$ and of $(s_{l+1}, \alpha_1^-(H|_{(s_{l+1},s_{l+2})}))$ must be different because $s_{l+1} \in I_{\text{reg}}$. Hence, $a_1 = \cdots = a_N = 0$.

By (3.12) for $l = 0$ the fact that $((0; g); (J\xi_\alpha; 0)) \in \Gamma(\mathfrak{h})$ implies that $\mathfrak{h}$ starts with an indivisible interval of type α.

If, conversely, $\mathfrak{h}$ starts with an indivisible interval of type α then $\mathfrak{h}^0$ starts with an indivisible interval of type $\alpha + \gamma_0$.

According to Corollary IV.2.25, Proposition IV.2.24 (see also Section 2.1.e of [KW/IV]) in the definite case or Proposition IV.5.16 in the indefinite case we have $((0; g); (J\xi_{\alpha+\gamma_0}; 0)) \in \Gamma(\mathfrak{h}^0)$ for some $g \in \mathcal{P}$. As $N_{\gamma_0} J\xi_{\alpha+\gamma_0} = J\xi_\alpha$ we obtain $((0; g); (J\xi_\alpha; 0)) \in (\nu_{\gamma_0} \boxtimes \nu_{\gamma_0}) \circ \Gamma(\mathfrak{h}^0)$, and further

$$((0; g); (J\xi_\alpha; 0)) \in \Gamma(\mathfrak{h}).$$

❑

3.38 Remark. Positive definite Hamiltonians which are reparameterizations of each other share their important properties. This fact holds true also for indefinite Hamiltonians, only, that the definition of 'reparameterization' is a bit more tricky. It proceeds in several steps:

(i) First we define a relation $\sim_1$ which directly generalizes what we are familiar with from the positive definite case. We say that $\mathfrak{h} \sim_1 \mathfrak{h}'$, if $n = n'$, if there exists an increasing bijection ϑ of $[\sigma_0', \sigma_{n+1}']$ onto $[\sigma_0, \sigma_{n+1}]$ such that ϑ and ϑ^{-1} are locally absolutely continuous,

$$\vartheta(\sigma_i') = \sigma_i, \ i = 1, \dots, n, \quad H' = (H \circ \vartheta) \circ \vartheta' \ \text{ a.e. },$$

and if $\mathfrak{c} = \mathfrak{c}'$ and $\mathfrak{d} = \mathfrak{d}'$.

(ii) Next we write $\mathfrak{h} \sim_2 \mathfrak{h}'$, if all the data of $\mathfrak{h}$ and $\mathfrak{h}'$ with exception of the numbers $d_{i,2\Delta_i-1}, d'_{i,2\Delta_i-1}$ and $b_{i,\ddot{o}_i+1}, b'_{i,\ddot{o}_i+1}$ are the same. These parameters should satisfy

$$d'_{i,2\Delta_i-1} - b'_{i,\ddot{o}_i+1} = d_{i,2\Delta_i-1} - b_{i,\ddot{o}_i+1} \,.$$

(iii) Finally, and this is the most involved step, we write $\mathfrak{h} \sim_3 \mathfrak{h}'$, if $H = H'$ and $\mathfrak{c} = \mathfrak{c}'$, but the sets E and E' may differ, and the numbers d'_{ij} are those used in the proof of Proposition IV.8.11 to perform the change from E to E' as admissible partitions.

It is obvious that each of these relations is reflexive and symmetric. Hence, the transitive closure $\leftrightsquigarrow$ of $(\sim_1 \cup \sim_2 \cup \sim_3)$ is an equivalence relation. If $\mathfrak{h} \leftrightsquigarrow \mathfrak{h}'$, we say that $\mathfrak{h}$ and $\mathfrak{h}'$ are reparameterizations of each other. Inspecting the proof of Proposition IV.8.11, and using the relation between $\tilde{d}_k$ and d_k given in [KW/IV, §7, p.812], shows that

$$\mathfrak{h} \leftrightsquigarrow \mathfrak{h}' \quad \Longleftrightarrow \quad \exists\, \mathfrak{h}_1, \mathfrak{h}_2 \in \mathfrak{H}_{<\infty} : \; \mathfrak{h} \sim_1 \mathfrak{h}_1 \sim_2 \mathfrak{h}_2 \sim_3 \mathfrak{h}'$$

In particular, if $\mathfrak{h} \leftrightsquigarrow \mathfrak{h}'$, there exists an increasing bijection $\vartheta : I' \to I$ such that ϑ and ϑ^{-1} are locally absolutely continuous and $H' = (H \circ \vartheta) \cdot \vartheta'$. //

The following statement is an immediate consequence of Proposition IV.8.11 and Proposition IV.8.13. It indicates that general Hamiltonians which are reparameterizations of each other will behave essentially the same.

3.39 Remark. Let $\mathfrak{h}$ and $\mathfrak{h}'$ be general Hamiltonians, and let $\mathfrak{B}(\mathfrak{h})$ and $\mathfrak{B}(\mathfrak{h}')$ be the boundary triplets associated to $\mathfrak{h}$ and $\mathfrak{h}'$, respectively, by Definition IV.8.5. Assume that $\mathfrak{h} \leftrightsquigarrow \mathfrak{h}'$, and denote by ϑ the increasing bijection of $[\sigma_0', \sigma_{n+1}']$ onto $[\sigma_0, \sigma_{n+1}]$ which satsifies $H' = (H \circ \vartheta) \cdot \vartheta'$.

Then there exists an isomorphism of $\mathfrak{B}(\mathfrak{h})$ to $\mathfrak{B}(\mathfrak{h}')$, which has the form (ϖ, id) and satisfies $\psi(\mathfrak{h}')(\varpi(x)) = \psi(\mathfrak{h})(x) \circ \vartheta$, $x \in \mathcal{P}(\mathfrak{h})$. If H is not almost everywhere of the form $h(t)\xi_\phi \xi_\phi^T$ for a constant ϕ, then $\Psi^{ac}(\mathfrak{h})(f;g) = \Psi^{ac}(\mathfrak{h}')(\varpi(f);\varpi(g))$ for all $(f;g) \in T(\mathfrak{h})$. //

We have already encountered the operations $\circlearrowleft_\alpha$ and rev in various settings. It is no surprise that these operations also have their analogues for general Hamiltonians.

3.40 Definition. Let $\mathfrak{h} = (H, \mathfrak{c}, \mathfrak{d}) \in \mathfrak{H}_{<\infty}$.

(*i*) For $\alpha \in \mathbb{R}$ define a general Hamiltonian $\circlearrowleft_\alpha \mathfrak{h}$ as

$$\circlearrowleft_\alpha \mathfrak{h} := (\circlearrowleft_\alpha H, \mathfrak{c}, \mathfrak{d})\,.$$

(*ii*) If $\mathfrak{h}$ is regular, define $\operatorname{rev} \mathfrak{h}$ as $\operatorname{rev} \mathfrak{h} = (\operatorname{rev} H, \tilde{\mathfrak{c}}, \tilde{\mathfrak{d}})$, where $\operatorname{rev} H$ is defined as in Definition 3.28 together with the singularities $\tilde{\sigma}_i := -\sigma_{n+1-i}$ for $i = 0, \dots, n+1$, where the data $\tilde{\mathfrak{c}}$ consist of the numbers $\tilde{\ddot{o}}_i := \ddot{o}_{n+1-i}$, $\tilde{b}_{ij} := b_{n+1-i,j}$ for $i = 0, \dots, n+1$, and where the data $\tilde{\mathfrak{d}}$ consist of the numbers $\tilde{d}_{ij} := d_{n+1-i,j}$ for $i = 0, \dots, n+1$ and of the subset $\tilde{E} := -E$.

//

3.41 Lemma.

(*i*) *Let* $\mathfrak{h} \in \mathfrak{H}_{<\infty}$*, then* $\circlearrowleft_\alpha \mathfrak{h} \in \mathfrak{H}_{<\infty}$ *and* $\operatorname{ind}_- \circlearrowleft_\alpha \mathfrak{h} = \operatorname{ind}_- \mathfrak{h}$. $\mathfrak{h}$ *and* $\circlearrowleft_\alpha \mathfrak{h}$ *are together regular or singular. Moreover,* $\mathfrak{h} \leftrightsquigarrow \mathfrak{h}'$ *implies* $\circlearrowleft_\alpha \mathfrak{h} \leftrightsquigarrow \circlearrowleft_\alpha \mathfrak{h}'$.

(*ii*) *Let* $\mathfrak{h} \in \mathfrak{H}_{<\infty}$ *be regular, then* $\operatorname{rev} \mathfrak{h}$ *is regular and belongs* $\mathfrak{H}_{<\infty}$*. We have* $\operatorname{ind}_- \operatorname{rev} \mathfrak{h} = \operatorname{ind}_- \mathfrak{h}$*. Moreover,* $\mathfrak{h} \leftrightsquigarrow \mathfrak{h}'$ *implies* $\operatorname{rev} \mathfrak{h} \leftrightsquigarrow \operatorname{rev} \mathfrak{h}'$.

Proof. It is elementary to check that with $\mathfrak{h}$ also $\circlearrowleft_\alpha$ and, in case of a regular Hamiltonian, $\operatorname{rev} \mathfrak{h}$ satisfy all conditions in Definition 3.35. Also the compatibility with $\leftrightsquigarrow$ is verified in a straightforward manner. Moreover, it is immediately seen from (3.10) that $\operatorname{ind}_- \circlearrowleft_\alpha \mathfrak{h} = \operatorname{ind}_- \mathfrak{h}$ and $\operatorname{ind}_- \operatorname{rev} \mathfrak{h} = \operatorname{ind}_- \mathfrak{h}$. ❑

3.42 Remark. The boundary triplets $\mathfrak{B}(\circlearrowleft_\alpha \mathfrak{h})$ and $\mathfrak{B}(\mathfrak{h})$ are isomorphic. To see this, we first construct elementary general Hamiltonians $\mathfrak{h}^0, \dots, \mathfrak{h}^N$ from $\mathfrak{h}$ and $(\circlearrowleft_\alpha \mathfrak{h})^0, \dots, (\circlearrowleft_\alpha \mathfrak{h})^N$ from $\circlearrowleft_\alpha \mathfrak{h}$ as in Remark 3.36; see in particular (3.9).

If $\mathfrak{h}^l$ is definite, then obviously $(\circlearrowleft_\alpha \mathfrak{h})^l = \circlearrowleft_\alpha (\mathfrak{h}^l)$. By Remark 3.30 $(\varpi^l, \nu_\alpha \boxtimes \nu_\alpha)$ with $(\varpi^l f)(t) := N_\alpha f(t)$ for $f \in L^2(\mathfrak{h}^l)$ is an isomorphism from $\mathfrak{B}(\mathfrak{h}^l)$ onto $\mathfrak{B}((\circlearrowleft_\alpha \mathfrak{h})^l)$.

If $\mathfrak{h}^l$ is indefinite, then $(\circlearrowleft_\alpha \mathfrak{h})^l = \mathfrak{h}^l$ because $\phi_{i(l)}(\circlearrowleft_\alpha \mathfrak{h}) = \phi_{i(l)}(\mathfrak{h}) - \alpha$ (see Remark 3.30). Hence $\mathfrak{B}(\mathfrak{h}^l) = \mathfrak{B}((\circlearrowleft_\alpha \mathfrak{h})^l)$, and further

$$\circlearrowleft_\alpha \circlearrowleft_{-\phi_{i(l)}(\mathfrak{h})} \mathfrak{B}(\mathfrak{h}^l) = \circlearrowleft_{-\phi_{i(l)}(\circlearrowleft_\alpha \mathfrak{h})} \mathfrak{B}((\circlearrowleft_\alpha \mathfrak{h})^l).$$

Thus, $(\varpi^l, \nu_\alpha \boxtimes \nu_\alpha)$ with $\varpi^l = \mathrm{id}|_{\mathcal{P}(\mathfrak{h}^l)}$ is an isomorphism from $\circlearrowleft_{-\phi_{i(l)}(\mathfrak{h})} \mathfrak{B}(\mathfrak{h}^l)$ onto $\circlearrowleft_{-\phi_{i(l)}(\circlearrowleft_\alpha \mathfrak{h})} \mathfrak{B}((\circlearrowleft_\alpha \mathfrak{h})^l)$.

Defining $\varpi : \mathcal{P}(\mathfrak{h}) \to \mathcal{P}(\circlearrowleft_\alpha \mathfrak{h})$ by $\varpi = \boxtimes_{l=0}^N \varpi^l$ we see that $(\varpi, \nu_\alpha \boxtimes \nu_\alpha)$ is an isomorphism from $\mathfrak{B}(\mathfrak{h})$ onto $\mathfrak{B}(\circlearrowleft_\alpha \mathfrak{h})$.

Moreover, by the definition of $\psi(\mathfrak{h})$ and $\psi(\circlearrowleft_\alpha \mathfrak{h})$ in Definition IV.8.5 we find $\psi(\circlearrowleft_\alpha \mathfrak{h}) \circ \varpi = N_\alpha \psi(\mathfrak{h})$, and by Remark IV.8.9 we find $\Psi^{ac}(\circlearrowleft_\alpha \mathfrak{h}) \circ (\varpi \boxtimes \varpi)|_{T(\mathfrak{h})} = (N_\alpha \boxtimes N_\alpha) \Psi^{ac}(\mathfrak{h})$. //

3.43 Remark. Assume that $\mathfrak{h}$ is regular. Similarly as in Remark 3.42 the boundary triplets $\mathfrak{B}(\operatorname{rev} \mathfrak{h})$ and $\mathfrak{B}(\mathfrak{h})$ are isomorphic. As before we first construct elementary general Hamiltonians $\mathfrak{h}^0, \dots, \mathfrak{h}^N$ from $\mathfrak{h}$ and $(\operatorname{rev} \mathfrak{h})^0, \dots, (\operatorname{rev} \mathfrak{h})^N$ from $\operatorname{rev} \mathfrak{h}$ as in Remark 3.36.

If $\mathfrak{h}^{N-l}$ is definite, then obviously $(\operatorname{rev} \mathfrak{h})^l = \operatorname{rev}(\mathfrak{h}^{N-l})$. By Remark 3.30 (ϖ^l, ϕ) with $(\varpi^l f)(t) := V f(-t)$ for $f \in L^2(\mathfrak{h}^{N-l})$ and ϕ, V as in Definition 2.6 is an isomorphism from $\mathfrak{B}(\mathfrak{h}^{N-l})$ onto $\mathfrak{B}((\operatorname{rev} \mathfrak{h})^l)$.

If $\mathfrak{h}^{N-l}$ is (elementary) indefinite, then $(\operatorname{rev} \mathfrak{h})^l = \operatorname{rev}(\mathfrak{h}^{N-l})$, since by Remark 3.30 $\phi_{i(l)}(\operatorname{rev} \mathfrak{h}) = -\phi_{i(N-l)}(\mathfrak{h})$ and $\operatorname{rev} \circlearrowleft_{-\phi_{i(l)}(\operatorname{rev} \mathfrak{h})} = \circlearrowleft_{\phi_{i(l)}(\operatorname{rev} \mathfrak{h})} \operatorname{rev}$.

If $\mathfrak{h}^{N-l}$ is elementary indefinite of kind (B) or (C), then by Definition IV.4.1 (see also Remark IV.4.2) $\operatorname{rev}(\mathfrak{h}^{N-l}) = \mathfrak{h}^{N-l}$. From the construction of the corresponding boundary triplet in [KW/IV, §4.2], one can easily derive that (ϖ^l, ϕ) with $\varpi^l = -\mathrm{id}|_{\mathcal{P}(\mathfrak{h}^{N-l})}$ is an isomorphism from $\mathfrak{B}(\mathfrak{h}^{N-l})$ onto $\mathfrak{B}(\mathfrak{h}^{N-l}) = \mathfrak{B}((\operatorname{rev} \mathfrak{h})^l)$, and because of $\phi_{i(l)}(\operatorname{rev} \mathfrak{h}) = -\phi_{i(N-l)}(\mathfrak{h})$ also from $\circlearrowleft_{\phi_{i(N-l)}(\mathfrak{h})} \mathfrak{B}(\mathfrak{h}^{N-l})$ onto $\circlearrowleft_{\phi_{i(l)}(\operatorname{rev} \mathfrak{h})} \mathfrak{B}((\operatorname{rev} \mathfrak{h})^l)$.

Assume now that $\mathfrak{h}^{N-l}$ is elementary indefinite of kind (A). We know from Remark 3.30, that $f(t) \mapsto V f(-t)$ is an isomorphism from $L^2(H|_{(s_{N-l}, s_{N-l+1})})$ onto $L^2(\operatorname{rev} H|_{(s_{N-l}, s_{N-l+1})})$. If $\mathfrak{w}_k(\mathfrak{h}^{N-l})$ and $\mathfrak{w}_k(\operatorname{rev} \mathfrak{h}^{N-l})$ are defined as in the end of [KW/IV, §4.1] for $\mathfrak{h}^{N-l}$ and $\operatorname{rev} \mathfrak{h}^{N-l}$, respectively, then it follows from Lemma IV.3.10, that $V \mathfrak{w}_k(\mathfrak{h}^{N-l})(-t) = -\mathfrak{w}_k(\operatorname{rev} \mathfrak{h}^{N-l})(t)$.

From this it follows that ϖ^l defined by (for the notation see [KW/IV, §4.2])

$$f(t) \mapsto V f(-t), \ f \in X_L(\mathfrak{h}^{N-l}),$$

$$p_j(\mathfrak{h}^{N-l}) \mapsto -p_j(\operatorname{rev} \mathfrak{h}^{N-l}), \ j = 0, \dots, \Delta(\mathfrak{h}^{N-l}) - 1,$$

$$\delta_k(\mathfrak{h}^{N-l}) \mapsto -\delta_k(\operatorname{rev} \mathfrak{h}^{N-l}), \ k = \Delta(\mathfrak{h}^{N-l}), \dots, \Delta(\mathfrak{h}^{N-l}) + \ddot{o}(\mathfrak{h}^{N-l}) - 1,$$

extends to an isometric isomorphism from $\mathcal{P}(\mathfrak{h}^{N-l})$ onto $\mathcal{P}(\operatorname{rev}(\mathfrak{h}^{N-l}))$. Moreover, (ϖ^l, ϕ) is an isomorphism from $\mathfrak{B}(\mathfrak{h}^{N-l})$ onto $\mathfrak{B}((\operatorname{rev} \mathfrak{h})^l)$, and because of $\phi_{i(l)}(\operatorname{rev} \mathfrak{h}) = -\phi_{i(N-l)}(\mathfrak{h})$ also from $\circlearrowleft_{\phi_{i(N-l)}(\mathfrak{h})} \mathfrak{B}(\mathfrak{h}^{N-l})$ onto $\circlearrowleft_{\phi_{i(l)}(\operatorname{rev} \mathfrak{h})} \mathfrak{B}((\operatorname{rev} \mathfrak{h})^l)$.

Defining $\varpi : \mathcal{P}(\mathfrak{h}) \to \mathcal{P}(\operatorname{rev}\mathfrak{h})$ by $\varpi = \boxtimes_{l=0}^{N} \varpi^{N-l}$ we see that (ϖ, ϕ) is an isomorphism from $\mathfrak{B}(\mathfrak{h})$ onto $\mathfrak{B}(\operatorname{rev}\mathfrak{h})$. Thus, $(\varpi, \mathrm{id}_{\mathbb{C}^2\times\mathbb{C}^2})$ is an isomorphism from $\operatorname{rev}\mathfrak{B}(\mathfrak{h})$ onto $\mathfrak{B}(\operatorname{rev}\mathfrak{h})$

Moreover, by the definition of $\psi(\mathfrak{h})$ and $\psi(\operatorname{rev}\mathfrak{h})$ in Definition IV.8.5 we find $(\psi(\operatorname{rev}\mathfrak{h}) \circ \varpi f)(t) = V(\psi(\mathfrak{h})(f)(-t))$ for $f \in \mathcal{P}(\mathfrak{h})$, and by Remark IV.8.9 we find $\Psi^{ac}(\operatorname{rev}\mathfrak{h}) \circ (\varpi \boxtimes \varpi)(f;g)(t) = (V \boxtimes V)\Psi^{ac}(\mathfrak{h})(f;g)(-t)$ for $(f;g) \in T(\mathfrak{h})$. //

Also for general Hamiltonians a splitting-and-pasting method can be introduced. However, in this setting, splitting up is the more involved matter. It is obtained by an inductive application of Corollary IV.8.12.

3.44 Lemma. *Assume that $\mathfrak{h}$ is a general Hamiltonian, and let $F = \{r_0, \ldots, r_{m+1}\}$, $r_0 < r_1 < \ldots < r_{m+1}$, be a finite subset of $\overline{I}$ such that*

$$r_0 = \sigma_0,\ r_{m+1} = \sigma_{n+1}, \quad r_i \in I_{\mathrm{reg}},\ i = 1, \ldots, m\,.$$

Then there exist general Hamiltonians $\mathfrak{h}^i$, $i = 0, \ldots, m$, defined on (r_i, r_{i+1}), respectively, such that there exists an isomorphism (ϖ, id) between the boundary triplets

$$\mathfrak{B}(\mathfrak{h}) \quad and \quad \biguplus_{i=0}^{m} \mathfrak{B}(\mathfrak{h}^i)\,, \tag{3.14}$$

which has the property that

$$\psi(\mathfrak{h}) = \Big(\boxtimes_{i=0}^{m} \psi(\mathfrak{h}^i)\Big) \circ \varpi, \quad \Psi^{ac}(\mathfrak{h}) = \Big(\boxtimes_{i=0}^{m} \Psi^{ac}(\mathfrak{h}^i)\Big) \circ (\varpi \boxtimes \varpi)|_{T(\mathfrak{h})}\,. \tag{3.15}$$

Moreover, $\mathfrak{h}^i$, is regular for $i = 0, \ldots, m-1$ and $\mathfrak{h}^m$ is regular (singular) if and only if $\mathfrak{h}$ is regular (singular).

Proof. We use induction on $|F|$. If $|F| = 2$, i.e. $F = \{\sigma_0, \sigma_{n+1}\}$, we set $\mathfrak{h}^0 := \mathfrak{h}$. Then the desired properties are trivially satisfied.

Assume that $|F| = m+2 > 2$, and consider the set $F' := F \setminus \{r_m\}$. By the inductive hypothesis there exist $\mathfrak{h}^i$, $i = 0, \ldots, m-1$, which are defined on (r_i, r_{i+1}), $i = 0, \ldots, m-2$, and on (r_{m-1}, r_{m+1}) for $i = m-1$, and possess the stated properties. By Corollary IV.8.12 we can further split $\mathfrak{h}^{m-1}$ in two general Hamiltonians $\mathfrak{h}_0^{m-1}$, $\mathfrak{h}_1^{m-1}$, defined on (r_{m-1}, r_m) and (r_m, r_{m+1}), respectively, such that (3.14) and (3.15) hold for $\mathfrak{h}^{m-1}$, and $\mathfrak{h}_0^{m-1}$, $\mathfrak{h}_1^{m-1}$. Note here that, by (3.11), the condition (LI) is satisfied. ❑

More interesting than the proof just given is that we actually know the general Hamiltonians $\mathfrak{h}^i$ quite explicitly. The following notice is obtained from the formulas given in Corollary IV.8.12 by carrying out the above inductive process step by step.

3.45 Remark. For the Hamiltonians $\mathfrak{h}^i$, $i = 0, \ldots, m$, in Lemma 3.44 we can choose the ones described as follows:

For $i \in \{0, \ldots, m\}$ let $k(i)$ denote the smallest number such that $\sigma_{k(i)} > r_i$. Then

$$(r_i, r_{i+1}) \setminus I = \{\sigma_{k(i)}, \ldots, \sigma_{k(i+1)-1}\}\,.$$

Note that this set might be empty, actually this is the case if and only if $\sigma_{k(i)} > r_{i+1}$ or, equivalently, $k(i) = k(i+1)$. In this case the Hamiltonian $\mathfrak{h}^i$ shall be positive definite and given by the Hamiltonian function

$$H(\mathfrak{h}^i) := H_{k(i)-1}|_{(r_i,r_{i+1})} \,.$$

If $k(i) < k(i+1)$, then $\mathfrak{h}^i$ shall be given by the data

$$k(i+1)-k(i)\in\mathbb{N},\quad r_i,\sigma_{k(i)},\dots,\sigma_{k(i+1)-1},r_{i+1}\in\mathbb{R}\cup\{\pm\infty\}\,,$$

$$H(\mathfrak{h}^i)_l := \begin{cases} H_{k(i)-1}|_{(r_i,\sigma_{k(i)})} & ,\ l=0 \\ H_{k(i)+l-1} & ,\ l=1,\dots,k(i+1)-k(i)-1 \\ H_{k(i+1)-1}|_{(\sigma_{k(i+1)-1},r_{i+1})} & ,\ l=k(i+1)-k(i)>0 \end{cases}$$

$$\ddot{o}(\mathfrak{h}^i)_l := \ddot{o}_{k(i)+l-1},\quad l=1,\dots,k(i+1)-k(i)\,,$$

$$b(\mathfrak{h}^i)_{l,j} := b_{k(i)+l-1,j},\quad l=1,\dots,k(i+1)-k(i),\ j=1,\dots,\ddot{o}_{k(i)+l-1}+1\,,$$

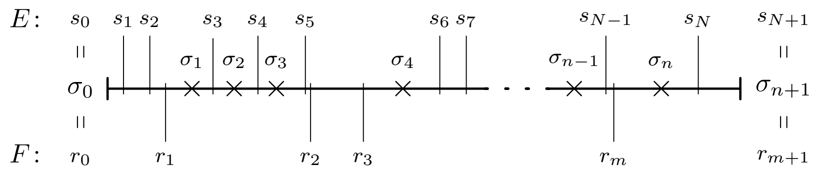
$$E^i := \{r_i,r_{i+1}\}\cup\big(E\cap(r_i,r_{i+1})\big)\,,$$

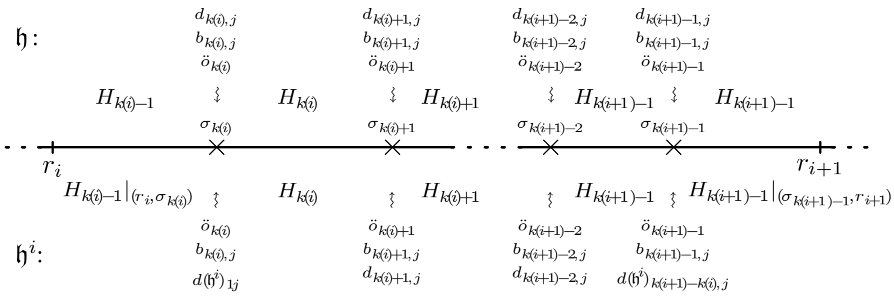
$$d(\mathfrak{h}^i)_{l,j} := d_{k(i)+l-1,j},\quad l=2,\dots,k(i+1)-k(i)-1,\ j=0,\dots,2\Delta(\mathfrak{h}^i)_l-1\,,$$

and some appropriate numbers

$$d(\mathfrak{h}^i)_{l,j},\quad l=1 \text{ and } l=k(i+1)-k(i),\ j=0,\dots,2\Delta(\mathfrak{h}^i)_l-1$$

according to Corollary IV.8.12. Note here that $\Delta(\mathfrak{h}^i)_l = \Delta(\mathfrak{h})_{k(i)+l-1}$, $l = 1,\dots,k(i+1)-k(i)$. We can picture the situation as follows:





//

The following observation, which is obtained by a closer inspection of the construction in [KW/IV, §7], turns out to be quite important.

3.46 Remark. The numbers $d(\mathfrak{h}^i)_{l,j}$ for $l=1$ and $l=k(i+1)-k(i)$, depend only on the points r_i and r_{i+1}, respectively. More exactly: Let $F=\{r_0,\dots,r_m\}$ and $\hat{F}=\{\hat{r}_0,\dots,\hat{r}_{\hat{m}}\}$ be two finite subsets of $\overline{I}$ which qualify for the above construction, and denote the corresponding general Hamiltonians resulting from Remark 3.45 by $\mathfrak{h}^i$, $i=0,\dots,m$, and $\hat{\mathfrak{h}}^k$, $k=0,\dots,\hat{m}$, respectively. If, for some i and k, we have $r_i=\hat{r}_k$, $r_{i+1}=\hat{r}_{k+1}$, then $\mathfrak{h}^i=\hat{\mathfrak{h}}^k$. //

Let us now introduce notation similar as we did for maximal chains.

3.47 Definition. Let $r \in I \cup \{\sigma_0\}$ and $s \in \overline{I}$, $r < s$, and assume that $r, s \notin I_{\text{sing}}$. We distinguish five cases:

(*i*) $r = \sigma_0, s \in I$: Let $\mathfrak{h}^0$ and $\mathfrak{h}^1$ be the general Hamiltonians constructed in Remark 3.45 for the set $F := \{\sigma_0, s, \sigma_{n+1}\}$, and set

$$\mathfrak{h}_{r \leftrightarrow s} := \mathfrak{h}^0 .$$

(*ii*) $r > \sigma_0, s = \sigma_{n+1}$: Let $\mathfrak{h}^0$ and $\mathfrak{h}^1$ be the general Hamiltonians constructed in Remark 3.45 for the set $F := \{\sigma_0, r, \sigma_{n+1}\}$, and set

$$\mathfrak{h}_{r \leftrightarrow s} := \mathfrak{h}^1 .$$

(*iii*) $r > \sigma_0$ and $s \in I$: Let $\mathfrak{h}^0, \mathfrak{h}^1, \mathfrak{h}^2$ be the general Hamiltonians constructed in Remark 3.45 for the set $F := \{\sigma_0, r, s, \sigma_{n+1}\}$, and set

$$\mathfrak{h}_{r \leftrightarrow s} := \mathfrak{h}^1 .$$

(*iv*) $r = \sigma_0$ and $s \notin I$: Let $n' \in \{1, \ldots, n+1\}$ be such that $s = \sigma_{n'}$, and let $\mathfrak{h}_{\sigma_0 \leftrightarrow \sigma_{n'}}$ be the general Hamiltonian comprised of the data

$$\begin{array}{lll} n' - 1 \in \mathbb{N}, & \sigma_0, \ldots, \sigma_{n'} \in \mathbb{R} \cup \{\pm\infty\}, & H_i,\ i = 0, \ldots, n'-1, \\ \ddot{o}_i,\ i = 1, \ldots, n'-1, & b_{ij},\ i = 1, \ldots, n'-1, j = 1, \ldots, \ddot{o}_i - 1 & \\ (E \cap [\sigma_0, \sigma_{n'}]) \cup \{\sigma_{n'}\}, & d_{ij},\ i = 1, \ldots, n'-1, j = 0, \ldots, 2\Delta_i - 1 & \end{array}$$

(*v*) $r > \sigma_0$ and $s \notin I$: Set $\mathfrak{h}_{r \leftrightarrow s} := (\mathfrak{h}_{r \leftrightarrow \sigma_{n+1}})_{r \leftrightarrow s}$.

We will also write $\mathfrak{h}_{\sigma_0 \leftrightarrow s} =: \mathfrak{h}_{\Lsh s}$ and $\mathfrak{h}_{r \leftrightarrow \sigma_{n+1}} =: \mathfrak{h}_{r \Rsh}$. //

According to the definition of the negative index of a general Hamiltonian, we have

$$\operatorname{ind}_- \mathfrak{h}_{r \leftrightarrow s} := \sum_{\substack{i=1,\ldots,n \\ \sigma_i \in (r,s)}} \big(\Delta_i + [\frac{\ddot{o}_i}{2}]\big) + \big|\big\{1 \leq i \leq n : \ddot{o}_i \text{ odd}, c_{i,1} < 0\big\}\big| .$$

3.48 Remark. It immediately follows from our construction that

(*i*) If $s, t \in I_{\text{reg}}$, $s < t$, then

$$(\mathfrak{h}_{\Lsh t})_{\Lsh s} = \mathfrak{h}_{\Lsh s}, \quad (\mathfrak{h}_{s \Rsh})_{\Lsh t} = (\mathfrak{h}_{\Lsh t})_{s \Rsh} = \mathfrak{h}_{s \leftrightarrow t} .$$

(*ii*) If $s, t, u \in I_{\text{reg}}$, $s < t < u$, then

$$(\mathfrak{h}_{\Lsh u})_{s \leftrightarrow t} = (\mathfrak{h}_{s \leftrightarrow u})_{\Lsh t} = \mathfrak{h}_{s \leftrightarrow t}, \quad (\mathfrak{h}_{s \Rsh})_{t \leftrightarrow u} = (\mathfrak{h}_{s \leftrightarrow u})_{t \Rsh} = \mathfrak{h}_{t \leftrightarrow u} .$$

(*iii*) From the construction which led to Definition 3.47 and from Definition 3.40 it is immediate that for $s, t \in I_{\text{reg}}, s < t$ and $\alpha \in \mathbb{R}$, we have

$$(\circlearrowleft_\alpha \mathfrak{h})_{\Lsh s} = \circlearrowleft_\alpha (\mathfrak{h}_{\Lsh s}),\ (\circlearrowleft_\alpha \mathfrak{h})_{\Rsh s} = \circlearrowleft_\alpha (\mathfrak{h}_{\Rsh s}),\ (\circlearrowleft_\alpha \mathfrak{h})_{s \leftrightarrow t} = \circlearrowleft_\alpha (\mathfrak{h}_{s \leftrightarrow t}),$$

$$(\operatorname{rev} \mathfrak{h})_{\Lsh -s} = \operatorname{rev}(\mathfrak{h}_{\Rsh s}),\ (\operatorname{rev} \mathfrak{h})_{\Rsh -s} = \operatorname{rev}(\mathfrak{h}_{\Lsh s}),\ (\operatorname{rev} \mathfrak{h})_{-t \leftrightarrow -s} = \operatorname{rev}(\mathfrak{h}_{s \leftrightarrow t}).$$

//

Also an operation of pasting of general Hamiltonians can be defined in a natural way.

3.49 Definition. Let $\mathfrak{h}_1 = (H_1, \mathfrak{c}_1, \mathfrak{d}_1)$ and $\mathfrak{h}_2 = (H_2, \mathfrak{c}_2, \mathfrak{d}_2)$ be general Hamiltonians, and assume that $\mathfrak{h}_1$ is regular. Let their respective domains be such that $\sigma_{n+1}^{(1)} = \sigma_0^{(2)}$, and assume that the following does not hold true:

(¬paste) $\mathfrak{h}_1$ ends with an indivisible interval of type $\phi \in [0, \pi)$ and $\mathfrak{h}_2$ starts with an indivisible interval of the same type ϕ

Then let $\mathfrak{h}_1 \uplus \mathfrak{h}_2$ be the general Hamiltonian constituted by the data

$$n := n_1 + n_2 \in \mathbb{N}_0, \quad \sigma_0^{(1)}, \dots, \sigma_{n_1}^{(1)}, \sigma_1^{(2)}, \dots, \sigma_{n_2+1}^{(2)} \in \mathbb{R} \cup \{\pm\infty\},$$
$$H(t) := \begin{cases} H_1(t) & , \ t \in I_1 \\ H_2(t) & , \ t \in I_2 \end{cases}$$
$$\ddot{o}_l := \ddot{o}(\mathfrak{h}_1)_l, \ l = 1, \dots, n_1, \quad \ddot{o}_l := \ddot{o}(\mathfrak{h}_2)_{l-n_1}, \ l = n_1 + 1, \dots, n,$$
$$b_{l,j} := b(\mathfrak{h}_1)_{l,j}, \ l = 1, \dots, n_1, \quad b_{l,j} := b(\mathfrak{h}_2)_{l-n_1,j}, \ l = n_1 + 1, \dots, n,$$
$$E := E(\mathfrak{h}_1) \cup E(\mathfrak{h}_2),$$
$$d_{l,j} := d(\mathfrak{h}_1)_{l,j}, \ l = 1, \dots, n_1, \quad d_{l,j} := d(\mathfrak{h}_2)_{l-n_1,j}, \ l = n_1 + 1, \dots, n,$$

//

The fact that actually $\mathfrak{h}_1 \uplus \mathfrak{h}_2$ is a general Hamiltonian, is obvious. Note here that the assumption that (¬paste) fails is necessary to ensure (H2). Moreover, $\mathfrak{h}_1 \uplus \mathfrak{h}_2$ is regular or singular depending whether $\mathfrak{h}_2$ is regular or singular.

The operations of splitting and pasting are converses of each other. The following statement is easily seen from the definitions.

3.50 Remark. If $\mathfrak{h} \in \mathfrak{H}_{<\infty}$ and $F = \{r_0, \dots, r_{m+1}\}$, $r_j < r_{j+1}$, be a finite subset of $I_{\text{reg}} \cup \{\sigma_0, \sigma_{n+1}\}$ with $\sigma_0, \sigma_{n+1} \in F$, then $\mathfrak{h}_{r_i \leftrightarrow r_{i+1}} \in \mathfrak{H}_{<\infty}$ for $i = 0, \dots, m$. For each two consecutive general Hamiltonians the condition (¬paste) fails, and we have

$$\mathfrak{h} = \biguplus_{i=0}^{m} \mathfrak{h}_{r_i \leftrightarrow r_{i+1}}.$$

//

3.51 Remark. Let $\mathfrak{h}_i$ be general Hamiltonians defined on $(r_i, r_{i+1}) \setminus \{\sigma_{i,j} : j = 1, \dots, n_i\}$. If (¬paste) fails for each two consequtive Hamiltonians, then $\biguplus_{i=0}^m \mathfrak{h}_i \in \mathfrak{H}_{<\infty}$, and

$$\Big(\biguplus_{i=0}^{m} \mathfrak{h}_i\Big)_{r_i \leftrightarrow r_{i+1}} = \mathfrak{h}_i, \ i = 0, \dots, m.$$

In particular, we obtain from (3.14) that

$$\mathfrak{B}\Big(\biguplus_{i=0}^{m} \mathfrak{h}_i\Big) \quad \text{and} \quad \biguplus_{i=0}^{m} \mathfrak{B}(\mathfrak{h}_i),$$

are isomorphic. Moreover, it is straight forward to check that ($\alpha \in \mathbb{R}$)

$$\circlearrowleft_\alpha \biguplus_{i=0}^{m} \mathfrak{h}_i = \biguplus_{i=0}^{m} \circlearrowleft_\alpha \mathfrak{h}_i, \quad \operatorname{rev} \biguplus_{i=0}^{m} \mathfrak{h}_i = \biguplus_{i=0}^{m} \operatorname{rev} \mathfrak{h}_{m-i}.$$

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4 Boundary triplets and matrix functions of the class $\mathcal{M}_{<\infty}$

In this section we will construct for each boundary triplet $\mathfrak{B} = (\mathcal{P}, T, \Gamma)$ of defect 2 in the sense of Definition IV.2.7, Definition IV.2.8, which satisfies the condition (E) of Definition IV.2.16, that is

$$z \in \mathbb{C}, (f; zf) \in T, f \neq 0, ((f; zf); (a; b)) \in \Gamma \Rightarrow a \neq 0 \ \text{ and } \ b \neq 0,$$

a 2×2-matrix function $\omega(\mathfrak{B})$ which is analytic on $r(T^*)$ and is such that the kernel $H_{\omega(\mathfrak{B})}$ has a finite number of negative squares. As for a converse, we are content to show that each matrix function $W \in \mathcal{M}_{<\infty}$ is realized as $\omega(\mathfrak{B}(W))$. For boundary triplets $\mathfrak{B} = (\mathcal{P}, T, \Gamma)$ of defect 1 an object $v(\mathfrak{B})$ playing a similar role will be constructed.

The content of this section is arranged in five subsections:

a. Here the definition of $\omega(\mathfrak{B})$ is given, and some compatibilities of the assignment $\mathfrak{B} \mapsto \omega(\mathfrak{B})$ are provided.
b. Here the definition of $v(\mathfrak{B})$ is given, and some of its properties are discussed.
c. In this subsection we show that $\omega(\mathfrak{B})$ can be considered as a u-resolvent matrix in the sense of [KW/0]; this is a central result.
d. We show that $\omega(\mathfrak{B}(W)) = W$ for $W \in \mathcal{M}_{<\infty}$.
e. The construction of $\omega(.)$ will later be applied to the boundary triplet $\mathfrak{B}(\mathfrak{h})$ associated with a general Hamiltonian $\mathfrak{h}$. Here we collect some properties specific for this situation.

Throughout this section we will keep the following notation: If $\mathfrak{B} = (\mathcal{P}, T, \Gamma)$ is a boundary triplet, the adjoint of T will be denoted by $S := T^*$. Moreover, $\pi_l, \pi_r, \pi_j, \pi_{l,j}, \pi_{r,j}$ denote the following projections of $\mathbb{C}^2 \times \mathbb{C}^2$ (or $\mathbb{C}^2$) onto $\mathbb{C}^2$ (or $\mathbb{C}$):

$$\pi_l\Big(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}; \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\Big) := \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \ \pi_r\Big(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}; \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\Big) := \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

$$\pi_j \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} := a_j, \ \pi_{l,j}\Big(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}; \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\Big) := a_j, \ \pi_{r,j}\Big(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}; \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\Big) := b_j, \ j = 1, 2\,.$$

a. Construction of $\omega(\mathfrak{B})$.

The definition of $\omega(\mathfrak{B})$ is based on the following observation.

4.1 Lemma. *Let $\mathfrak{B} = (\mathcal{P}, T, \Gamma)$ be a boundary triplet which has defect 2 and satisfies (E). Moreover, let $z \in r(S)$. Then there exist unique elements*

$\phi(z), \psi(z) \in \mathcal{P}$ and unique vectors $\alpha(z), \beta(z) \in \mathbb{C}^2$, such that

$$\begin{aligned} &\Big((\phi(z); z\phi(z)); (\begin{pmatrix}1\\0\end{pmatrix}; \alpha(z))\Big) \in \Gamma\,, \\ &\Big((\psi(z); z\psi(z)); (\begin{pmatrix}0\\1\end{pmatrix}; \beta(z))\Big) \in \Gamma\,. \end{aligned} \tag{4.1}$$

We have $\ker(T-z) = \operatorname{span}\{\phi(z), \psi(z)\}$.

Proof. Put $N_z := \{(f; zf) \in \mathcal{P}^2 : f \in \ker(T-z)\}$, and consider the linear relation $\pi_l \circ \Gamma|_{N_z}$. By the condition (E) we have

$$\ker\big(\pi_l \circ \Gamma|_{N_z}\big) = \{0\}\,. \tag{4.2}$$

This already shows the uniqueness of $\phi(z)$ and $\psi(z)$.

Let us first consider the case that $\operatorname{mul}\Gamma = \{0\}$. Then S has defect index $(2,2)$, i.e. $\dim\ker(T-z) = 2$ for all $z \in r(S)$. In this case $\pi_l \circ \Gamma|_{N_z}$ is an injective linear map between spaces of dimension 2. Therefore, it is bijective, and hence elements $\phi(z), \psi(z), \alpha(z), \beta(z)$ with the desired property (4.1) exist. Moreover, $\phi(z)$ and $\psi(z)$ span $\ker(T-z)$, since their images span $\mathbb{C}^2$.

In the case that $\operatorname{mul}\Gamma \neq \{0\}$, $\operatorname{mul}\Gamma =: \operatorname{span}\{(m; m)\}$, the symmetry S has defect index $(1,1)$, i.e. $\dim N_z = 1$, $z \in r(S)$. Choose $f_0 \in N_z \setminus \{0\}$ and $a_0, b_0 \in \mathbb{C}^2$ such that $((f_0; zf_0); (a_0; b_0)) \in \Gamma$. Clearly, then

$$\big((\mu f_0; z\mu f_0); (\mu a_0 + \lambda m; \mu b_0 + \lambda m)\big) \in \Gamma, \quad \lambda, \mu \in \mathbb{C}\,.$$

By (4.2), the elements a_0 and m are linearly independent and thus span $\mathbb{C}^2$. Again we see that elements $\phi(z), \psi(z), \alpha(z), \beta(z)$ with the desired property (4.1) exist. If both $\phi(z)$ and $\psi(z)$ were equal to 0, we would obtain the contradiction

$$\big(\begin{pmatrix}1\\0\end{pmatrix}; \alpha(z)\big), \big(\begin{pmatrix}0\\1\end{pmatrix}; \beta(z)\big) \in \operatorname{mul}\Gamma = \operatorname{span}\{(m; m)\}\,.$$

Thus, also in the present case, $\ker(T-z) = \operatorname{span}\{\phi(z), \psi(z)\}$.

Finally, note that $\alpha(z)$ and $\beta(z)$ are uniquely determined by (4.1) since $\operatorname{mul}\Gamma$ has the form $\operatorname{span}\{(m; m)\}$. ❑

4.2 Corollary. *Let notation be as in Lemma 4.1, and let $\rho_1, \rho_2 \in \mathbb{C}$ be given. Then there exist unique vectors $\chi(z) \in \ker(T-z)$ and $c(z) \in \mathbb{C}^2$ such that*

$$\Big((\chi(z); z\chi(z)); (\begin{pmatrix}\rho_1\\\rho_2\end{pmatrix}; c(z))\Big) \in \Gamma\,. \tag{4.3}$$

In fact, $\chi(z) = \rho_1\phi(z) + \rho_2\psi(z)$ and $c(z) = \rho_1\alpha(z) + \rho_2\beta(z)$.

Proof. Set $\chi(z) := \rho_1\phi(z) + \rho_2\psi(z)$ and $c(z) := \rho_1\alpha(z) + \rho_2\beta(z)$. Then the relation (4.3) follows immediately from Lemma 4.1.

In order to see uniqueness, assume that (4.3) also holds for elements $\tilde{\chi}(z)$ and $\tilde{c}(z)$. Then

$$\Big(((\chi(z) - \tilde{\chi}(z)); z(\chi(z) - \tilde{\chi}(z))); \big(0; c(z) - \tilde{c}(z)\big)\Big) \in \Gamma\,.$$

By property (E) we obtain $\chi(z) - \tilde{\chi}(z) = 0$, and the fact that $\operatorname{mul}\Gamma$ is spanned by a vector of the form $(m; m)$ gives $c(z) - \tilde{c}(z) = 0$. ❑

4.3 Definition. Let $\mathfrak{B}$ be a boundary triplet which has defect 2 and satisfies (E). Let

$$\alpha(z) = \begin{pmatrix} \alpha(z)_1 \\ \alpha(z)_2 \end{pmatrix} \in \mathbb{C}^2, \quad \beta(z) = \begin{pmatrix} \beta(z)_1 \\ \beta(z)_2 \end{pmatrix} \in \mathbb{C}^2$$

be the elements constructed in Lemma 4.1. Then we define

$$\omega(\mathfrak{B})(z) := \big(\alpha(z) \,|\, \beta(z)\big)^T = \begin{pmatrix} \alpha(z)_1 & \alpha(z)_2 \\ \beta(z)_1 & \beta(z)_2 \end{pmatrix}, \; z \in r(S)\,.$$

//

To start with let us collect some properties of $\omega(\mathfrak{B})$ which follow by elementary computation.

4.4 Proposition. *Let $\mathfrak{B}$ be a boundary triplet which has defect 2 and satisfies (E). Then $\phi(\overline{z}) = \overline{\phi(z)}$, $\psi(\overline{z}) = \overline{\psi(z)}$, $\alpha(\overline{z}) = \overline{\alpha(z)}$ and $\beta(\overline{z}) = \overline{\beta(z)}$. Moreover,* $\det \omega(\mathfrak{B})(z) = 1$ *and*

$$\omega(\mathfrak{B})(z) J \omega(\mathfrak{B})(w)^* - J = (z - \overline{w}) \begin{pmatrix} [\phi(z), \phi(w)] & [\phi(z), \psi(w)] \\ [\psi(z), \phi(w)] & [\psi(z), \psi(w)] \end{pmatrix}. \tag{4.4}$$

Proof. The relation Γ is real with respect to the involution $\bar{.}$ in the sense of (IV.2.5). Hence we obtain from (4.1) that

$$\Big((\overline{\phi(z)}; \overline{z}\overline{\phi(z)}); (\begin{pmatrix} 1 \\ 0 \end{pmatrix}; \overline{\alpha(z)}) \Big) \in \Gamma.$$

The uniqueness statement of Lemma 4.1 yields $\phi(\overline{z}) = \overline{\phi(z)}$ and $\alpha(\overline{z}) = \overline{\alpha(z)}$. The relations $\psi(\overline{z}) = \overline{\psi(z)}$ and $\beta(\overline{z}) = \overline{\beta(z)}$ follow in the same way.

In order to show (4.4), we use the abstract Green's identity (IV.2.6). It gives:

$$(z - \overline{w})[\phi(z), \phi(w)] = [z\phi(z), \phi(w)] - [\phi(z), w\phi(w)] =$$
$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* J \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \alpha(w)^* J \alpha(z) = \alpha(z)_2 \overline{\alpha(w)_1} - \alpha(z)_1 \overline{\alpha(w)_2}\,,$$

$$(z - \overline{w})[\phi(z), \psi(w)] = [z\phi(z), \psi(w)] - [\phi(z), w\psi(w)] =$$
$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* J \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \beta(w)^* J \alpha(z) = \alpha(z)_2 \overline{\beta(w)_1} - \alpha(z)_1 \overline{\beta(w)_2} + 1\,,$$

$$(z - \overline{w})[\psi(z), \phi(w)] = [z\psi(z), \phi(w)] - [\psi(z), w\phi(w)] =$$
$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* J \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \alpha(w)^* J \beta(z) = \beta(z)_2 \overline{\alpha(w)_1} - \beta(z)_1 \overline{\alpha(w)_2} - 1\,,$$

$$(z - \overline{w})[\psi(z), \psi(w)] = [z\psi(z), \psi(w)] - [\psi(z), w\psi(w)] =$$
$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* J \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \beta(w)^* J \beta(z) = \beta(z)_2 \overline{\beta(w)_1} - \beta(z)_1 \overline{\beta(w)_2}\,.$$

Computing $\omega(\mathfrak{B})(z) J \omega(\mathfrak{B})(w)^* - J$ from the definition of $\omega(\mathfrak{B})$, cf. (I.8.2) with '$S = 1$', and comparing with the above relations yields (4.4).

If we put $w = \overline{z}$ in (4.4) and use that α and β are symmetric with respect to the real line, it follows in particular that $\det \omega(\mathfrak{B})(z) = 1$. ❑

4.5 Corollary. *We have* $\operatorname{ind}_- H_{\omega(\mathfrak{B})} \leq \operatorname{ind}_- \mathcal{P}$.

Proof. By (4.4) the map Θ defined by linearity and

$$\Theta : H_{\omega(\mathfrak{B})}(w,.)\begin{pmatrix}\lambda\\ \mu\end{pmatrix} \mapsto \lambda\phi(\overline{w}) + \mu\psi(\overline{w}) \tag{4.5}$$

is an isometry of the linear space $\operatorname{span}\{H_{\omega(\mathfrak{B})}(w,.)\binom{\lambda}{\mu} : \lambda,\mu \in \mathbb{C}, w \in r(S)\}$ into $\mathcal{P}$. ❑

4.6 Remark. Assume that S is completely non-selfadjoint, i.e. that we have $\operatorname{cls}\bigcup_{z\in r(S)} \ker(T-z) = \mathcal{P}$ or, equivalently,

$$\operatorname{cls}\big\{\phi(z),\psi(z) : z \in \mathbb{C}\big\} = \mathcal{P}\,, \tag{4.6}$$

compare Lemma 4.1. Then the map Θ defined above has dense range and therefore admits a continuation to a unitary mapping $\Theta : \mathfrak{K}(\omega(\mathfrak{B})) \to \mathcal{P}$. //

Our next task is to establish two compatibilities of the assignment $\mathfrak{B} \mapsto \omega(\mathfrak{B})$. The first one deals with isomorphisms (ϖ,ϕ) of boundary triplets, where ϕ is of the form $\hat\phi \boxtimes \hat\phi$, cf. Definition IV.2.12 and Remark IV.2.13, (*iii*). The second one with pasting of boundary triplets.

4.7 Proposition. *Let* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *be boundary triplets, and let* $(\varpi,\hat\phi\boxtimes\hat\phi)$ *be an isomorphism between* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$. *Denote by* $N_{\hat\phi} \in \mathbb{C}^{2\times 2}$ *the matrix such that* $\hat\phi(x) = N_{\hat\phi}x$, $x \in \mathbb{C}^2$. *If* $\mathfrak{B}_1$ *has defect* 2 *and satisfies (E), so does* $\mathfrak{B}_2$, *and we have*

$$\omega(\mathfrak{B}_2) = N_{\hat\phi}^{-T}\omega(\mathfrak{B}_1)N_{\hat\phi}^{T}\,.$$

Proof. As it was noted in Remark IV.2.13, (*iii*), and Remark IV.2.17, the presence of the isomorphism $(\varpi,\hat\phi\boxtimes\hat\phi)$ implies that also $\mathfrak{B}_2$ has defect 2 and satisfies (E).

Let $\phi_j(z),\psi_j(z),\alpha_j(z),\beta_j(z)$ be defined for $\mathfrak{B}_j$, $j=1,2$, as in Lemma 4.1, and let $(m_{ij})_{i,j=1,2}$ be the inverse of $N_{\hat\phi}$. Then

$$\big((m_{1j}\phi_1(z)+m_{2j}\psi_1(z); z[m_{1j}\phi_1(z)+m_{2j}\psi_1(z)]); \\ (\begin{pmatrix}m_{1j}\\ m_{2j}\end{pmatrix}; m_{1j}\alpha_1(z)+m_{2j}\beta_1(z))\big) \in \Gamma_1, \quad j=1,2\,.$$

Hence

$$\big((\varpi[m_{1j}\phi_1(z)+m_{2j}\psi_1(z)]; z\varpi[m_{1j}\phi_1(z)+m_{2j}\psi_1(z)]); \\ (\begin{pmatrix}\delta_{1j}\\ \delta_{2j}\end{pmatrix}; N_{\hat\phi}(m_{1j}\alpha_1(z)+m_{2j}\beta_1(z)))\big) \in \Gamma_2, \quad j=1,2\,, \tag{4.7}$$

and we obtain from the uniqueness assertion in Lemma 4.1

$$\phi_2(z) = \varpi[m_{11}\phi_1(z)+m_{21}\psi_1(z)],\ \psi_2(z) = \varpi[m_{12}\phi_1(z)+m_{22}\psi_1(z)]\,,$$

$$\alpha_2(z) = N_{\hat\phi}\big(\alpha_1(z)\,|\,\beta_1(z)\big)\begin{pmatrix}m_{11}\\ m_{21}\end{pmatrix},\ \beta_2(z) = N_{\hat\phi}\big(\alpha_1(z)\,|\,\beta_1(z)\big)\begin{pmatrix}m_{12}\\ m_{22}\end{pmatrix}.$$

By uniqueness in Lemma 4.1, it follows that

$$\big(\alpha_2(z)\,|\,\beta_2(z)\big) = N_{\hat\phi}\big(\alpha_1(z)\,|\,\beta_1(z)\big)N_{\hat\phi}^{-1}\,.$$

Taking transposes we obtain the desired result. ❑

The above Proposition 4.7 applies to the isomorphism $(\mathrm{id}_{\mathcal P}, \nu_\gamma \boxtimes \nu_\gamma)$ from $\mathfrak B$ to the rotated boundary triplet $\circlearrowleft_\gamma \mathfrak B$, see (2.9). Recall also the notation $\circlearrowleft_\gamma W$ from (2.8).

4.8 Corollary. *We have* $\omega(\circlearrowleft_\gamma \mathfrak B) = \circlearrowleft_\gamma \omega(\mathfrak B)$.

Proof. As already noted, $(\mathrm{id}, \nu_\gamma \boxtimes \nu_\gamma)$ is an isomorphism from $\mathfrak B$ to $\circlearrowleft_\gamma \mathfrak B$. The desired equality follows from Proposition 4.7 and the fact that $N_\gamma^{-1} = N_\gamma^T$. ❑

For the following recall Definition 2.6.

4.9 Lemma. *Assume that the boundary triplet* $\mathfrak B$ *has defect* 2 *and satisfies (E). Then also* rev $\mathfrak B$ *has these properties. Moreover, in this case, we have*

$$\omega(\mathrm{rev}\,\mathfrak B) = \mathrm{rev}(\omega(\mathfrak B))\,. \tag{4.8}$$

Proof. Assume that $\mathfrak B = (\mathcal P, T, \Gamma)$ has defect 2 and satisfies (E). The base space and the relation T in rev $\mathfrak B$ is just the same as in $\mathfrak B$, and

$$\mathrm{mul}\,\Gamma = \mathrm{span}\{(m;m)\} \Rightarrow \mathrm{mul}(\phi\circ\Gamma) = \mathrm{span}\{(Vm;Vm)\}\,.$$

Thus also the boundary triplet rev $\mathfrak B$ has defect 2. Since $((f;zf);(a;b)) \in \phi\circ\Gamma$ implies that $((f;zf);(Vb;Va)) \in \Gamma$, also the condition (E) transfers to rev $\mathfrak B$.

Denote by $\phi'(z), \psi'(z), \alpha'(z), \beta'(z)$ the elements constructed in Lemma 4.1 for rev $\mathfrak B$, and let

$$\begin{pmatrix} c_{11}(z) & c_{12}(z)\\ c_{21}(z) & c_{22}(z)\end{pmatrix} := \big(\alpha(z)|\beta(z)\big)^{-1}\,,$$

so that $c_{11}\alpha + c_{21}\beta = \binom{1}{0}$ and $c_{12}\alpha + c_{22}\beta = \binom{0}{1}$. We have

$$\big((\phi; z\phi); (\begin{pmatrix}1\\0\end{pmatrix}; \alpha)\big),\ \big((\psi; z\psi); (\begin{pmatrix}0\\1\end{pmatrix}; \beta)\big) \in \Gamma$$

and hence

$$\Big(((c_{11}\phi + c_{21}\psi); z(c_{11}\phi + c_{21}\psi)); (\begin{pmatrix}c_{11}\\c_{21}\end{pmatrix}; \begin{pmatrix}1\\0\end{pmatrix})\Big) \in \Gamma\,,$$

$$\Big(((c_{12}\phi + c_{22}\psi); z(c_{12}\phi + c_{22}\psi)); (\begin{pmatrix}c_{12}\\c_{22}\end{pmatrix}; \begin{pmatrix}0\\1\end{pmatrix})\Big) \in \Gamma\,.$$

Thus

$$\Big(((c_{11}\phi + c_{21}\psi); z(c_{11}\phi + c_{21}\psi)); (\begin{pmatrix}1\\0\end{pmatrix}; \begin{pmatrix}c_{11}\\-c_{21}\end{pmatrix})\Big) \in \phi\circ\Gamma\,,$$

$$\Big(((c_{12}\phi + c_{22}\psi); z(c_{12}\phi + c_{22}\psi)); (\begin{pmatrix}0\\-1\end{pmatrix}; \begin{pmatrix}c_{12}\\-c_{22}\end{pmatrix})\Big) \in \phi\circ\Gamma\,,$$

and we conclude that

$$\phi' = c_{11}\phi + c_{21}\psi,\ \alpha' = \begin{pmatrix} c_{11} \\ -c_{21} \end{pmatrix}, \quad \psi' = -(c_{12}\phi + c_{22}\psi),\ \beta' = \begin{pmatrix} -c_{12} \\ c_{22} \end{pmatrix}.$$

Hence,

$$\big(\alpha|\beta\big)^{-1} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = V\big(\alpha'|\beta'\big)V\,.$$

Taking transposes yields (4.8). ❑

Though elementary, it is important that $\mathfrak{B} \mapsto \omega(\mathfrak{B})$ is compatible with pasting of boundary triplets.

4.10 Proposition. *Let $\mathfrak{B}_1 = (\mathcal{P}_1, T_1, \Gamma_1)$ and $\mathfrak{B}_2 = (\mathcal{P}_2, T_2, \Gamma_2)$ be boundary triplets which have defect 2 and satisfy (E). Assume that the condition (LI) of Proposition IV.6.2 holds true, so that the pasting $\mathfrak{B} := \mathfrak{B}_1 \uplus \mathfrak{B}_2$ is well-defined, has defect 2, and satisfies (E), cf. Definition IV.6.1, Proposition IV.6.2 and Lemma IV.6.7, (iii). Denote by $\phi_i(z), \psi_i(z) \in \mathcal{P}_i$ and $\alpha_i(z)_j, \beta_i(z)_j \in \mathbb{C}$ those elements and numbers, such that*

$$\big((\phi_i(z); z\phi_i(z)); (\begin{pmatrix} 1 \\ 0 \end{pmatrix}; \begin{pmatrix} \alpha_i(z)_1 \\ \alpha_i(z)_2 \end{pmatrix})\big) \in \Gamma_i,\ i = 1, 2\,,$$

$$\big((\psi_i(z); z\psi_i(z)); (\begin{pmatrix} 0 \\ 1 \end{pmatrix}; \begin{pmatrix} \beta_i(z)_1 \\ \beta_i(z)_2 \end{pmatrix})\big) \in \Gamma_i,\ i = 1, 2\,.$$

Then the elements $\phi(z), \psi(z) \in \mathcal{P}_1 \uplus \mathcal{P}_2$ defined by Lemma 4.1 for the boundary triplet $\mathfrak{B}$, are given as

$$\begin{aligned} \phi(z) &= \phi_1(z) + \big(\alpha_1(z)_1\phi_2(z) + \alpha_1(z)_2\psi_2(z)\big) \\ \psi(z) &= \psi_1(z) + \big(\beta_1(z)_1\phi_2(z) + \beta_1(z)_2\psi_2(z)\big) \end{aligned} \tag{4.9}$$

Moreover, we have

$$\omega(\mathfrak{B})(z) = \omega(\mathfrak{B}_1)(z)\omega(\mathfrak{B}_2)(z),\ z \in r(S) \cap r(S_1) \cap r(S_2)\,.$$

Proof. By the definition of $\Gamma_1 \uplus \Gamma_2$, we have

$$\big((\phi_1(z) + [\alpha_1(z)_1\phi_2(z) + \alpha_1(z)_2\psi_2(z)]; z\phi_1(z) + [\alpha_1(z)_1 z\phi_2(z) + \alpha_1(z)_2 z\psi_2(z)]);$$
$$(\begin{pmatrix} 1 \\ 0 \end{pmatrix}; \begin{pmatrix} \alpha_1(z)_1\alpha_2(z)_1 + \alpha_1(z)_2\beta_2(z)_1 \\ \alpha_1(z)_1\alpha_2(z)_2 + \alpha_1(z)_2\beta_2(z)_2 \end{pmatrix})\big) \in \Gamma_1 \uplus \Gamma_2$$

$$\big((\psi_1(z) + [\beta_1(z)_1\phi_2(z) + \beta_1(z)_2\psi_2(z)]; z\psi_1(z) + [\beta_1(z)_1 z\phi_2(z) + \beta_1(z)_2 z\psi_2(z)]);$$
$$(\begin{pmatrix} 0 \\ 1 \end{pmatrix}; \begin{pmatrix} \beta_1(z)_1\alpha_2(z)_1 + \beta_1(z)_2\beta_2(z)_1 \\ \beta_1(z)_1\alpha_2(z)_2 + \beta_1(z)_2\beta_2(z)_2 \end{pmatrix})\big) \in \Gamma_1 \uplus \Gamma_2$$

By Lemma 4.1 this shows that the relation (4.9) holds, and that

$$\omega(\mathfrak{B})(z) = \begin{pmatrix} \alpha_1(z)_1\alpha_2(z)_1 + \alpha_1(z)_2\beta_2(z)_1 & \alpha_1(z)_1\alpha_2(z)_2 + \alpha_1(z)_2\beta_2(z)_2 \\ \beta_1(z)_1\alpha_2(z)_1 + \beta_1(z)_2\beta_2(z)_1 & \beta_1(z)_1\alpha_2(z)_2 + \beta_1(z)_2\beta_2(z)_2 \end{pmatrix} =$$

$$= \begin{pmatrix} \alpha_1(z)_1 & \alpha_1(z)_2 \\ \beta_1(z)_1 & \beta_1(z)_2 \end{pmatrix} \begin{pmatrix} \alpha_2(z)_1 & \alpha_2(z)_2 \\ \beta_2(z)_1 & \beta_2(z)_2 \end{pmatrix} = \omega(\mathfrak{B}_1)(z)\omega(\mathfrak{B}_2)(z)\,.$$

❑

b. Construction of $v(\mathfrak{B})$.

In Definition 4.3 we have associated to each boundary triplets $\mathfrak{B}$ with defect 2 satisfying (E) a 2×2-matrix function $\omega(\mathfrak{B})$. In this subsection we will carry out a similar construction for boundary triplets with defect 1.

4.11 Definition. Let $\mathfrak{B} = (\mathcal{P}, T, \Gamma)$ be a boundary triplet with defect 1 satisfying (E). For $z \in r(S)$, let $v(\mathfrak{B})(z)$ be the subspace of all vectors u in $\mathbb{C}^2$ such that there exists $(f; zf) \in T$ such that $((f; zf); (-Ju; 0)) \in \Gamma$, i.e.

$$v(\mathfrak{B})(z) = J\pi_l \Gamma\big(\{(f; zf) : f \in \ker(T - z)\}\big) .$$

We call $v(\mathfrak{B})(z)$ the Titchmarsh-Weyl subspace of $\mathfrak{B}$. //

Note that, by property (E), $r(S)$ is not empty.

4.12 Remark. We immediately see, that $v(\mathfrak{B})(\bar{z}) = \overline{v(\mathfrak{B})(z)}$. //

4.13 Remark. In case that $\operatorname{mul}\Gamma = \{0\}$, S is a symmetric relation with defect $(1, 1)$. If in this situation $(f; zf) \in T$ and $z \in r(S)$, then $(f; zf) \notin S = \ker \Gamma$. Hence $v(\mathfrak{B})(z)$ is one-dimensional.

In case that $\operatorname{mul}\Gamma \neq \{0\}$ we have $T = T^* = S$ and $\operatorname{mul}\Gamma = \operatorname{span}\{(m; 0)\}$. Therefore, $\ker(T - z) = \{0\}$ for $z \in r(S)$. We obtain $v(\mathfrak{B})(z) = \operatorname{span}\{Jm\}$. Hence also in this case $v(\mathfrak{B})(z)$ is one-dimensional. //

4.14 Lemma. *Let $\mathfrak{B} = (\mathcal{P}, T, \Gamma)$ be a boundary triplet with defect 1 satisfying (E). Set $\mathring{A} := \ker(\pi_{l,1} \circ \Gamma)$. Then $\mathring{A}$ is selfadjoint. If $\operatorname{mul}\Gamma = \{0\}$ or, more generally, $\operatorname{mul}\Gamma \neq \operatorname{span}\{(\binom{0}{1}; 0)\}$, then*

$$\rho(\mathring{A}) = \big\{z \in r(S) : \pi_2 v(\mathfrak{B})(z) \neq \{0\}\big\}.$$

Moreover, $\rho(\mathring{A}) = \emptyset$ implies $\operatorname{mul} S \neq \{0\}$ and $\operatorname{mul}\Gamma = \{0\}$.

Under the condition that $\operatorname{mul}\Gamma \neq \operatorname{span}\{(\binom{0}{1}; 0)\}$ for every $z \in \rho(\mathring{A})$ there exist $f_z \in \ker(T - z)$ and $q(z) \in \mathbb{C}$ such that $\binom{q(z)}{1}$ spans $v(\mathfrak{B})(z)$. Moreover, $((f_z - f_w); (zf_z - wf_w)) \in \mathring{A}$.

If, in addition, $\operatorname{mul}\Gamma = \{0\}$, then f_z is a defect family for $(S, \mathring{A})$ and $q(z)$ is the corresponding Q-function.

Proof. Selfadjointness of $\mathring{A}$ is easily checked using Green's identity. Note also that $\rho(\mathring{A}) \subseteq r(S)$ and that a point $z \in r(S)$ belongs to $\sigma(\mathring{A})$ if and only if $\ker(\mathring{A} - z) \neq \{0\}$.

Assume first that $\pi_2 v(\mathfrak{B})(z) = \{0\}$. Let $w \in \mathbb{C}^2$ be such that $\operatorname{span}\{Jw\} = v(\mathfrak{B})(z)$. By definition there exists $(f; zf) \in T$ such that $((f; zf); (w; 0)) \in \Gamma$, and as $\pi_1 w = 0$ we have $(f; zf) \in \mathring{A}$, i.e. $f \in \ker(\mathring{A} - z)$. If we had $f = 0$, then $(\binom{0}{1}; 0) \in \operatorname{mul}\Gamma = \operatorname{span}\{(m; 0)\}$, a contradiction to our assumption. Hence $\ker(\mathring{A} - z) \neq \{0\}$.

Conversely, assume that $f \in \ker(\mathring{A} - z) \setminus \{0\}$. Then there exists $w \in \mathbb{C}^2 \setminus \{0\}$ such that $((f; zf); (w; 0)) \in \Gamma$ with $\pi_1 w = 0$, hence $\pi_2 v(\mathfrak{B})(z) = \{0\}$.

If $\operatorname{mul}\Gamma \neq \{0\}$, then because of $\mathring{A} = S$ we have $\rho(\mathring{A}) \neq \emptyset$. Now assume that $\operatorname{mul}\Gamma = \{0\}$ and that $\rho(\mathring{A}) = \emptyset$. It is well-known, e.g. from [DS], that then $(\lambda f; \mu f) \in \mathring{A}$ for some $f \neq 0$ and all $\lambda, \mu \in \mathbb{C}$. As S is a subspace of $\mathring{A}$ with codimension one, we get $(\lambda f; \mu f) \in S$ for some $(\lambda; \mu) \in \mathbb{C}^2 \setminus \{(0; 0)\}$. By property (E) we must have $\lambda = 0$ and, hence, $\operatorname{mul} S \neq \{0\}$.

As $v(\mathfrak{B})(z)$ is one-dimensional the condition $\pi_2 v(\mathfrak{B})(z) \neq \{0\}$ implies that $v(\mathfrak{B})(z)$ is spanned by a vector of the form $\binom{q(z)}{1}$, and hence

$$((f_z; zf_z); (\begin{pmatrix} 1 \\ -q(z) \end{pmatrix}; 0)) \in \Gamma.$$

for some $f_z \in \ker(T - z)$. As $(((f_z - f_w); (zf_z - wf_w)); (\binom{0}{-q(z)+q(w)}; 0)) \in \Gamma$, we find $((f_z - f_w); (zf_z - wf_w)) \in \mathring{A}$.

If $\operatorname{mul}\Gamma = \{0\}$, then S is symmetric with defect $(1,1)$ with $S^* = T$ and $f_z \neq 0$. Hence $f_z \in \ker(S^* - z)$ is a defect family for $(S, \mathring{A})$. Moreover, by Green's identity

$$(z - w)[f_z, f_w] = [zf_z, f_w] - [f_z, wf_w] = \begin{pmatrix} 1 \\ -q(w) \end{pmatrix}^* J \begin{pmatrix} 1 \\ -q(z) \end{pmatrix} = q(z) - \overline{q(w)}.$$

This shows that $q(z)$ is a Q-function of $(S, \mathring{A})$ corresponding to the defect family f_z. ❑

The following proposition is the analogue of Proposition 4.7.

4.15 Proposition. *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be boundary triplets, and let $(\varpi, \hat{\phi} \boxtimes \hat{\phi})$ be an isomorphism between $\mathfrak{B}_1$ and $\mathfrak{B}_2$. Denote by $N_{\hat{\phi}} \in \mathbb{C}^{2\times 2}$ the matrix such that $\hat{\phi}(x) = N_{\hat{\phi}} x$, $x \in \mathbb{C}^2$. If $\mathfrak{B}_1$ has defect 1 and satisfies (E), so does $\mathfrak{B}_2$, and we have*

$$v(\mathfrak{B}_2) = N_{\hat{\phi}} v(\mathfrak{B}_1)\,.$$

Proof. As it was noted in Remark IV.2.13, (*iii*), and Remark IV.2.17, the presence of the isomorphism $(\varpi, \hat{\phi}\boxtimes\hat{\phi})$ implies that also $\mathfrak{B}_2$ has defect 1 and satisfies (E).

The relation $v(\mathfrak{B}_2) = N_{\hat{\phi}} v(\mathfrak{B}_1)$ immediately follows from $((f; zf); (u; 0)) \in \Gamma_1 \Leftrightarrow ((\varpi f; z\varpi f); (N_{\hat{\phi}} u; 0)) \in \Gamma_2$. ❑

The above Proposition 4.15 applies in particular to the isomorphism $(\mathrm{id}_{\mathcal{P}}, \nu_\gamma \boxtimes \nu_\gamma)$ from $\mathfrak{B}$ to $\circlearrowleft_\gamma \mathfrak{B}$, cf. (2.9).

4.16 Corollary. *We have $v(\circlearrowleft_\gamma \mathfrak{B}) = N_\gamma v(\mathfrak{B})$.* ❑

For Titchmarsh-Weyl subspaces a similar multiplicativity property as in Proposition 4.10 holds true.

4.17 Lemma. *Let $\mathfrak{B}_1$ be a boundary triplet with defect 2 and $\mathfrak{B}_2$ be a boundary triplet with defect 1 both satisfying property (E).*

Assume that the condition (LI) of Proposition IV.6.2 holds true, so that the pasting $\mathfrak{B} = (\mathcal{P}, T, \Gamma) := \mathfrak{B}_1 \uplus \mathfrak{B}_2$ is well-defined, has defect 1, and satisfies (E). Then

$$v(\mathfrak{B}_1 \uplus \mathfrak{B}_2)(z) = \omega(\mathfrak{B}_1)(z) v(\mathfrak{B}_2)(z),\ z \in r(S) \cap r(S_1) \cap r(S_2)\,.$$

Proof. For $v \in v(\mathfrak{B}_2)(z)$ we have $((f; zf); (-Jv; 0)) \in \Gamma_2$ for some $f \in \ker(T_2 - z)$. Corollary 4.2 applied to $\mathfrak{B}_1$ gives we have

$$((w_1\phi(z) + w_2\psi(z); w_1 z\phi(z) + w_2 z\psi(z)); (\omega(\mathfrak{B}_1)(z)^{-T}(-Jv); -Jv)) \in \Gamma_1$$

with $(w_1\ w_2)^T = \omega(\mathfrak{B}_1)(z)^{-T}(-Jv)$. As $\omega(\mathfrak{B}_1)(z)J\omega(\mathfrak{B}_1)(z)^T = J$ we have $\omega(\mathfrak{B}_1)(z)^{-T} = -J\omega(\mathfrak{B}_1)(z)J$ and hence $(w_1\ w_2)^T = -J\omega(\mathfrak{B}_1)(z)v$. As $\Gamma = \Gamma_1 \uplus \Gamma_2$ we have

$$((w_1\phi(z) + w_2\psi(z) + f; w_1 z\phi(z) + w_2 z\psi(z) + zf); (-J\omega(\mathfrak{B}_1)(z)v; 0)) \in \Gamma.$$

Therefore, $\omega(\mathfrak{B}_1)(z)v \in v(\mathfrak{B}_1 \uplus \mathfrak{B}_2)(z)$. ❑

Note that a boundary triplet of the form $\mathfrak{B} = \mathfrak{B}_1 \uplus \mathfrak{B}_2$ the assumptions of Lemma 4.14 are satisfied, since $\operatorname{mul}\Gamma = \{0\}$, cf. Proposition IV.6.2.

4.18 Remark. With the same notation and assumptions as in Lemma 4.17 consider $\mathring{A} := \ker(\pi_{l,1} \circ \Gamma)$. By Lemma 4.14, for $z \in r(S)$, we have $z \in \rho(\mathring{A})$ if and only if $\pi_2 v(\mathfrak{B})(z) \neq \{0\}$. In our situation this means that for the entries $\beta_1(z)$ and $\beta_2(z)$ in the lower row of $\omega(\mathfrak{B}_1)(z)$ we have $\beta_1(z)\nu_1(z) + \beta_2(z)\nu_2(z) = 0$, where $(\nu_1(z)\ \nu_2(z))^T$ is any non-zero element of $v(\mathfrak{B}_2)(z)$. This, in turn, is the same as $\frac{\nu_2(z)}{\nu_1(z)} = -\frac{\beta_1(z)}{\beta_2(z)}$. In particular,

$$\rho(\mathring{A}) = \emptyset \quad \Longleftrightarrow \quad \frac{\nu_2(z)}{\nu_1(z)} = -\frac{\beta_1(z)}{\beta_2(z)},\ z \in r(S)\,.$$

//

c. Realization as a u-resolvent matrix.

We will now prove that the matrix $\omega(\mathfrak{B})$ can be viewed as a u-resolvent matrix in the sense of [KW/0]. This is a crucial result; it provides us with several conclusions of great value. Also, it establishes a connection between the viewpoint of the classical theory of differential equations and the viewpoint of the operator theory of the associated symmetry, namely the following: In the particular situation that the boundary triplet $\mathfrak{B}$ under consideration is the maximal operator of a positive definite canonical system, cf. Theorem IV.2.18, the matrix $\omega(\mathfrak{B})$ above is defined as the boundary values at the right endpoint of the pair of fundamental solutions. The below theorem then says nothing else but the well-known fact that this matrix is a u-resolvent matrix of the symmetry associated to the problem.

4.19 Theorem. *Let $\mathfrak{B} = (\mathcal{P}, T, \Gamma)$ be a boundary triplet which has defect 2 and satisfies (E), and assume that the symmetric relation $S := T^*$ is completely nonselfadjoint. Moreover, assume that*

$$\begin{gathered} \operatorname{mul}\Gamma \neq \operatorname{span}\Big\{\Big(\begin{pmatrix}0\\1\end{pmatrix}; \begin{pmatrix}0\\1\end{pmatrix}\Big)\Big\}\,, \\ \exists\, z_0 \in r(S):\ \beta(z_0)_2 \neq 0\,. \end{gathered} \tag{4.10}$$

Then the restriction S_1 of T defined as $S_1 := \ker\big((\pi_{l,1} \times \pi_r) \circ \Gamma\big)$ is a symmetric and real extension of S with defect index $(1,1)$. Its adjoint $T_1 := S_1^*$ is given as $T_1 = \ker(\pi_{l,1} \circ \Gamma)$.

Let $\mathcal{P}_-$ be the space constructed from S_1 as in [KW/0, §3], and let u be the element of $\mathcal{P}_-$ which is defined by

$$\big[(f; g), u\big]_{\pm} = (\pi_{l,2} \circ P \circ \Gamma)(f; g),\ \ (f; g) \in T_1\,, \tag{4.11}$$

where

$$P\big((a;b)\big) := \begin{cases} (a;b) & ,\ \operatorname{mul}\Gamma = \{0\} \\ (a;b) - \frac{\pi_1 a}{\pi_1 m}(m;m) & ,\ \operatorname{mul}\Gamma = \operatorname{span}\{(m;m)\} \neq \{0\} \end{cases} .$$

Then $r_u(S_1) = r(S)$[†] *and* $\omega(\mathfrak{B})$ *is a u-resolvent matrix of* S_1. *Moreover,* $\operatorname{ind}_- H_{\omega(\mathfrak{B})} = \operatorname{ind}_- \mathcal{P}$, *and the map* Ξ *defined as*

$$(\Xi f)(z) := \begin{pmatrix} [f, \phi(\overline{z})] \\ [f, \psi(\overline{z})] \end{pmatrix},\ f \in \mathcal{P}\,, \tag{4.12}$$

is an isometric isomorphism of $\mathcal{P}$ *onto the reproducing kernel space* $\mathfrak{K}(\omega(\mathfrak{B}))$. *The relation* $(\Xi \boxtimes \Xi)(S)$ *is the multiplication operator* $\mathbf{f}(z) \mapsto z\mathbf{f}(z)$ *in this space with domain* $\{\mathbf{f} \in \mathfrak{K}(\omega(\mathfrak{B})) :\ z\mathbf{f}(z) \in \mathfrak{K}(\omega(\mathfrak{B}))\}$.

Proof. The proof of this theorem is quite elaborate and will be carried out in several steps.
Step 1: Due to the abstract Green's identity (IV.2.6), the relation S_1 is symmetric. Moreover, S_1 is real with respect to the involution $\bar{.}$. In particular, the defect indices of S_1 are equal.

Consider first the case that $\operatorname{mul}\Gamma = \{0\}$. Then Γ is an isomorphism of T/S onto $\mathbb{C}^2 \times \mathbb{C}^2$, and the abstract Green's identity yields $T_1 = \ker(\pi_{l,1} \circ \Gamma)$. We also see that $\dim T_1/S_1 = 2$, i.e. S_1 has defect index $(1,1)$. The relation Γ is a closed, and thus bounded, operator defined on T. Hence, the right hand side of (4.11) is bounded linear functional on $T_1 = S_1^*$, which is, by definition, nothing else but $\mathcal{P}_+$, c.f. [KW/0, §3]. By its definition, [KW/0, p.290], $[.,.]_\pm$ is a duality between $\mathcal{P}_+$ and $\mathcal{P}_-$. Therefore, an element $u \in \mathcal{P}_-$ is well-defined by (4.11).

Assume now that $\operatorname{mul}\Gamma = \operatorname{span}\{(m;m)\} \neq \{0\}$. By our assumption, $\pi_1 m \neq 0$. This implies $T = \ker(\pi_{l,1} \circ \Gamma)$. Together with the abstract Green's identity we get

$$S \subseteq S_1 \subseteq \ker(\pi_{l,1} \circ \Gamma)^* = T^* = S\,,$$

and conclude that $S_1 = S$ and $T_1 = T = \ker(\pi_{l,1} \circ \Gamma)$. Thus, also in this case S_1 has defect index $(1,1)$. Since P is nothing else but the projection of $\mathbb{C}^2 \times \mathbb{C}^2$ onto $(\{0\} \times \mathbb{C}) \times \mathbb{C}^2$ with kernel $\operatorname{span}\{(m;m)\}$, we have $\operatorname{mul}(P \circ \Gamma) = \{0\}$. The same reasoning as above yields that the right hand side of (4.11) is bounded linear functional on T_1, and hence that $u \in \mathcal{P}_-$ is well-defined by the relation (4.11).

Clearly, $r(S_1) \subseteq r(S)$. If $z \in r(S)$, then $\operatorname{ran}(S - z)$ is closed, and hence $\operatorname{ran}(S_1 - z)$ is closed. By (E), we have $\ker(S_1 - z) = \{0\}$ for all $z \in \mathbb{C}$, hence $z \in r(S_1)$. Let $f \in \ker(T_1 - z)$, and assume that $[(f; zf), u]_\pm = 0$. Then $((f; zf); (0; b)) \in \Gamma$ and hence, again by (E), $f = 0$. Thus, $r_u(S_1) = r(S_1)$.

Step 2: Consider the relation

$$A := \ker\big((\pi_{l,1} \times \pi_{r,2}) \circ \Gamma\big)\,.$$

By the abstract Green's identity we have $A \subseteq A^*$, and thus certainly $A \subsetneq T_1$. However, $\dim T_1/A \leq 1$. Hence, A is selfadjoint. Let us show that

$$\rho(A) = \big\{z \in r(S) :\ \beta(z)_2 \neq 0\big\}\,.$$

[†] $r_u(S_1)$ is the set of all $z \in r(S_1)$ (points of regular type of S_1) such that $[u, (f; \bar{z}f)]_\pm \neq 0$ for all $f \in \ker(S_1^* - \bar{z}) \setminus \{0\}$.

In particular, by (4.10), it will follow that $\rho(A) \neq \emptyset$. To see this formula for $\rho(A)$, note that a point $z \in r(S)$ belongs to $\sigma(A)$ if and only if $\ker(A-z) \neq \{0\}$, and that $\rho(A) \subseteq r(S)$. Assume first that $\beta(z)_2 = 0$. Then $(\psi(z); z\psi(z)) \in A$, i.e. $\psi(z) \in \ker(A-z)$. If we had $\psi(z) = 0$, then $(\binom{0}{1}; \beta(z)) \in \operatorname{mul}\Gamma = \operatorname{span}\{(m;m)\}$, a contradiction to (4.10). Hence $\ker(A-z) \neq \{0\}$.

Conversely, assume that $f \in \ker(A-z) \setminus \{0\}$. Then there exist $a, b \in \mathbb{C}^2$ such that $((f; zf); (a; b)) \in \Gamma$ and $\pi_1 a = \pi_2 b = 0$. By (E) we have $\pi_2 a \neq 0$. Again by (E) the fact that

$$(f; zf) - \pi_2 a \cdot (\psi(z); z\psi(z)) \in \ker(\pi_l \circ \Gamma),$$

implies $f = \pi_2 a \cdot \psi(z)$. Thus,

$$(0; b - \pi_2 a \cdot \beta(z)) = (a; b) - \pi_2 a \cdot \Big(\begin{pmatrix} 0 \\ 1 \end{pmatrix}; \beta(z)\Big) \in \operatorname{mul}\Gamma = \operatorname{span}\{(m;m)\},$$

and we conclude that $b - \pi_2 a \cdot \beta(z) = 0$. In particular, $0 = \pi_2 b = \pi_2 a \cdot \beta(z)_2$, and thus $\beta(z)_2 = 0$.

Step 3: For $z \in \rho(A)$ define elements $\varphi(z)$ and $\gamma(z)$ as

$$\varphi(z) := -\frac{1}{\beta(z)_2}\psi(z), \quad \gamma(z) := \frac{\alpha(z)_2}{\beta(z)_2}\psi(z) - \phi(z).$$

We show that

$$\big(I - (w-z)(A-z)^{-1}\big)\varphi(w) = \varphi(z), \ z, w \in \rho(A), \tag{4.13}$$

where $0 \neq \varphi(z) \in \ker(T_1 - z)$ and

$$\big(I - (w-z)(A-z)^{-1}\big)\gamma(w) = \gamma(z), \ z, w \in \rho(A) \tag{4.14}$$

and $\gamma(z) \in \ker(\ker(\pi_{r,2} \circ \Gamma) - z)$.

For the first relation note that, for $z \in \rho(A)$,

$$\Big(\frac{-1}{\beta(z)_2}(\psi(z); z\psi(z)) \ ; \Big(\begin{pmatrix} 0 \\ -\frac{-1}{\beta(z)_2} \end{pmatrix}; \begin{pmatrix} -\frac{\beta(z)_1}{\beta(z)_2} \\ -1 \end{pmatrix}\Big)\Big) \in \Gamma, \tag{4.15}$$

which yields $\varphi(z) \in \ker(T_1 - z)$. As $\varphi(z) = 0$ would give $\psi(z) = 0$ and further $\operatorname{mul}\Gamma = \operatorname{span}\{(\binom{0}{1}; \binom{0}{1})\}$, we conclude that $\varphi(z) \neq 0$. Moreover, for $z, w \in \rho(A)$

$$\Big(\frac{-1}{\beta(w)_2}(\psi(w); w\psi(w)) - \frac{-1}{\beta(z)_2}\big(\psi(z); z\psi(z)\big) \ ; \\ \Big(\begin{pmatrix} 0 \\ \frac{-1}{\beta(w)_2} - \frac{-1}{\beta(z)_2} \end{pmatrix}; \begin{pmatrix} -\frac{\beta(w)_1}{\beta(w)_2} + \frac{\beta(z)_1}{\beta(z)_2} \\ 0 \end{pmatrix}\Big)\Big) \in \Gamma.$$

Hence

$$\big(\varphi(w) - \varphi(z); w\varphi(w) - z\varphi(z)\big) \in A,$$

which implies (4.13).

As $\det \omega(\mathcal{B})(z) = 1$ we have

$$\Big(\frac{\alpha(z)_2}{\beta(z)_2}\big(\psi(z); z\psi(z)\big) - \big(\phi(z); z\phi(z)\big); \Big(\begin{pmatrix} -1 \\ \frac{\alpha(z)_2}{\beta(z)_2} \end{pmatrix}; \begin{pmatrix} \frac{-1}{\beta(z)_2} \\ 0 \end{pmatrix}\Big)\Big) \in \Gamma, \tag{4.16}$$

which implies $\gamma(z) \in \ker(\ker(\pi_{r,2} \circ \Gamma) - z)$.

$$\Big(\frac{\alpha(w)_2}{\beta(w)_2}\big(\psi(w); w\psi(w)\big) - \big(\phi(w); w\phi(w)\big) - \frac{\alpha(z)_2}{\beta(z)_2}\big(\psi(z); z\psi(z)\big) + \big(\phi(z); z\phi(z)\big)\ ; \\ \Big(\begin{pmatrix} 0 \\ \frac{\alpha(w)_2}{\beta(w)_2} - \frac{\alpha(z)_2}{\beta(z)_2} \end{pmatrix}; \begin{pmatrix} \frac{-1}{\beta(w)_2} + \frac{1}{\beta(z)_2} \\ 0 \end{pmatrix}\Big)\Big) \in \Gamma\,,$$

and hence

$$\big(\gamma(w) - \gamma(z); w\gamma(w) - z\gamma(z)\big) \in A\,,$$

which yields (4.14).

Step 4: By the previous step $\varphi(z), \gamma(z)$ span the defect space $\ker(T - z)$. The corresponding Q-function is

$$Q(z) = \begin{pmatrix} \frac{\alpha(z)_2}{\beta(z)_2} & -\frac{1}{\beta(z)_2} \\ -\frac{1}{\beta(z)_2} & -\frac{\beta(z)_1}{\beta(z)_2} \end{pmatrix}\,. \tag{4.17}$$

In fact, if we consider ($z \in r(S)$)

$$W(z) := -\omega(\mathfrak{B})(z)J = \begin{pmatrix} -\alpha(z)_2 & \alpha(z)_1 \\ -\beta(z)_2 & \beta(z)_1 \end{pmatrix}\,, \tag{4.18}$$

then for $z \in \rho(A)$ its Potapov-Ginzburg transform is $Q(z)$, cf. [KW/0, §6]. Note here that $\det \omega(\mathfrak{B}) = 1$. By Lemma 0.6.2, for all $z, w \in \rho(A)$,

$$\frac{W(z)JW(w)^* - J}{z - \overline{w}} = \begin{pmatrix} -1 & -\alpha(z)_2 \\ 0 & -\beta(z)_2 \end{pmatrix} \frac{Q(z) - Q(w)^*}{z - \overline{w}} \begin{pmatrix} -1 & -\alpha(w)_2 \\ 0 & -\beta(w)_2 \end{pmatrix}^*\,.$$

However, by (4.4),

$$\frac{W(z)JW(w)^* - J}{z - \overline{w}} = \frac{\omega(\mathfrak{B})(z)J\omega(\mathfrak{B})(w)^* - J}{z - \overline{w}} = \begin{pmatrix} [\phi(z), \phi(w)] & [\phi(z), \psi(w)] \\ [\psi(z), \phi(w)] & [\psi(z), \psi(w)] \end{pmatrix}\,,$$

and, hence,

$$\frac{Q(z) - Q(w)^*}{z - \overline{w}} =$$

$$= \begin{pmatrix} 1 & -\frac{\alpha(z)_2}{\beta(z)_2} \\ 0 & \frac{1}{\beta(z)_2} \end{pmatrix} \begin{pmatrix} [\phi(z), \phi(w)] & [\phi(z), \psi(w)] \\ [\psi(z), \phi(w)] & [\psi(z), \psi(w)] \end{pmatrix} \begin{pmatrix} 1 & -\frac{\alpha(w)_2}{\beta(w)_2} \\ 0 & \frac{1}{\beta(w)_2} \end{pmatrix}^* =$$

$$= \begin{pmatrix} \begin{matrix} [\phi(z), \phi(w)] + \frac{\alpha(z)_2}{\beta(z)_2}\overline{(\frac{\alpha(w)_2}{\beta(w)_2})}[\psi(z), \psi(w)] - \\ \overline{(\frac{\alpha(w)_2}{\beta(w)_2})}[\phi(z), \psi(w)] - \frac{\alpha(z)_2}{\beta(z)_2}[\psi(z), \phi(w)] \end{matrix} & \begin{matrix} \overline{(\frac{1}{\beta(w)_2})}[\phi(z), \psi(w)] - \\ \overline{(\frac{1}{\beta(w)_2})}\frac{\alpha(z)_2}{\beta(z)_2}[\psi(z), \psi(w)] \end{matrix} \\ \begin{matrix} \frac{1}{\beta(z)_2}[\psi(z), \phi(w)] - \\ \frac{1}{\beta(z)_2}\overline{(\frac{\alpha(w)_2}{\beta(w)_2})}[\psi(z), \psi(w)] \end{matrix} & \frac{1}{\beta(z)_2}\overline{(\frac{1}{\beta(w)_2})}[\psi(z), \psi(w)] \end{pmatrix} =$$

$$= \begin{pmatrix} [\gamma(z),\gamma(w)] & [\gamma(z),\varphi(w)] \\ [\varphi(z),\gamma(w)] & [\varphi(z),\varphi(w)] \end{pmatrix} = \begin{pmatrix} [\gamma(z),\gamma(w)] & [\varphi(z),\gamma(w)] \\ [\gamma(z),\varphi(w)] & [\varphi(z),\varphi(w)] \end{pmatrix}.$$

The last equality sign holds since the off-diagonal entries of $\Psi(W(z))$ are equal. If we compare this formula with the expression for $\frac{Q(z)-Q(w)^*}{z-\overline{w}}$ obtained from the explicit form (4.17) of Q, and keep in mind that α and β are symmetric with respect to the real line, we obtain

$$[\gamma(z),\gamma(w)] = \frac{1}{z-\overline{w}}\Big(\frac{\alpha(z)_2}{\beta(z)_2} - \frac{\alpha(\overline{w})_2}{\beta(\overline{w})_2}\Big), \tag{4.19}$$

$$[\gamma(z),\varphi(w)] = [\varphi(z),\gamma(w)] = \frac{1}{z-\overline{w}}\Big(\frac{-1}{\beta(z)_2} - \frac{-1}{\beta(\overline{w})_2}\Big), \tag{4.20}$$

$$[\varphi(z),\varphi(w)] = \frac{1}{z-\overline{w}}\Big(\frac{-\beta(z)_1}{\beta(z)_2} - \frac{-\beta(\overline{w})_1}{\beta(\overline{w})_2}\Big). \tag{4.21}$$

In particular, we obtain from (4.21), that $\frac{-\beta(z)_1}{\beta(z)_2}$ is a Q-function of (S_1, A) corresponding to the defect family $\varphi(z)$, and (4.19) shows that $\frac{\alpha(z)_2}{\beta(z)_2}$ is a Q-function of $(\ker(\pi_{r,2}\circ\Gamma)^*, A)$ corresponding to the defect family $\gamma(z)$.

Step 5: We show that $\gamma(w) = R_w^- u$, $w \in \rho(A)$, and that $\frac{\alpha(z)_2}{\beta(z)_2}$ is a generalized u-resolvent of S_1 induced by A where R_w^- is defined as in [KW/0, §3] and generalized u-resolvents as in Definition 0.4.2: By the definition of R_z^+ and relations (4.13) and (4.14),

$$R_{\overline{w}}^+\varphi(z) = \frac{1}{z-\overline{w}}\big(\varphi(z)-\varphi(\overline{w}); z\varphi(z)-\overline{w}\varphi(\overline{w})\big),$$

$$R_{\overline{w}}^+\gamma(z) = \frac{1}{z-\overline{w}}\big(\gamma(z)-\gamma(\overline{w}); z\gamma(z)-\overline{w}\gamma(\overline{w})\big).$$

We compute for $z, w \in \rho(A)$

$$[\gamma(z), R_w^- u] = [R_{\overline{w}}^+\gamma(z), u]_\pm = \Big[\frac{1}{z-\overline{w}}\big(\gamma(z)-\gamma(\overline{w}); z\gamma(z)-\overline{w}\gamma(\overline{w})\big), u\Big]_\pm =$$
$$= \frac{1}{z-\overline{w}}\Big(\frac{\alpha(z)_2}{\beta(z)_2} - \frac{\alpha(\overline{w})_2}{\beta(\overline{w})_2}\Big) = [\gamma(z),\gamma(w)],$$

$$[\varphi(z), R_w^- u] = [R_{\overline{w}}^+\varphi(z), u]_\pm = \Big[\frac{1}{z-\overline{w}}\big(\varphi(z)-\varphi(\overline{w}); z\varphi(z)-\overline{w}\varphi(\overline{w})\big), u\Big]_\pm =$$
$$= \frac{1}{z-\overline{w}}\Big(\frac{-1}{\beta(z)_2} - \overline{\Big(\frac{-1}{\beta(w)_2}\Big)}\Big) = [\varphi(z),\gamma(w)].$$

Since S is completely nonselfadjoint, by Lemma 4.1, the linear span of all elements $\varphi(z), \gamma(z)$, $z \in \rho(A)$, is dense in $\mathcal{P}$. Hence, $\gamma(z) = R_z^- u$. The fact, that $\frac{\alpha(z)_2}{\beta(z)_2}$ is a generalized u-resolvent of S_1 induced by A now follows from (4.19), Proposition 0.4.5 and its proof, where we saw that any such generalized u-resolvent is a Q-function corresponding to the defect family $R_z^- u$.

Step 6: Because of

$$[(\varphi(z); z\varphi(z)), u]_\pm = \frac{-1}{\beta(z)_2},\ z \in \rho(A),$$

the u-resolvent matrix of S_1 constructed by means of Definition 0.4.8 with the selfadjoint extension A, the regularized u-resolvent $\frac{\alpha(z)_2}{\beta(z)_2}$, and the Q-function $\frac{-\beta(z)_1}{\beta(z)_2}$ corresponding to the defect family $(\varphi(z))_{z\in\rho(A)}$ is now nothing else, but $W(z)$ in (4.18) and, hence, $\omega(\mathfrak{B})(z) = W(z)J$, $z \in \rho(A)$.

Step 7: We know from [KW/0, §4], that the matrix $W(z)$ has an analytic continuation $\tilde{W}(z)$ to $r_u(S_1) = r(S)$. Hence, also the functions

$$\alpha(z)_1|_{\rho(A)},\ \alpha(z)_2|_{\rho(A)},\ \beta(z)_1|_{\rho(A)},\ \beta(z)_2|_{\rho(A)}\,,$$

are analytic and have analytic continuations $\tilde{\alpha}(z)_1$, $\tilde{\alpha}(z)_2$, $\tilde{\beta}(z)_1$, $\tilde{\beta}(z)_2$ to $r(S)$.

Consider the functionals $\mathcal{P}(z), \mathcal{Q}(z)$, $z \in \rho(A)$, as defined in (0.5.1) and (0.5.2). We compute for $f \in \mathcal{P}$ and $z \in \rho(A)$

$$\mathcal{P}(z)f = \frac{[f,\varphi(\overline{z})]}{[u,(\varphi(\overline{z});\overline{z}\varphi(\overline{z}))]_\pm} = [f,(-\beta(\overline{z})_2)\varphi(\overline{z})] = [f,\psi(\overline{z})]\,. \tag{4.22}$$

$$\mathcal{Q}(z)f = [R_z^+ f, u]_\pm - (\mathcal{P}(z)f)r(z) = [f, R_{\overline{z}}^- u] - (\mathcal{P}(z)f)r(z) =$$

$$= [f,\gamma(\overline{z})] - (\mathcal{P}(z)f)r(z) = \frac{\alpha(z)_2}{\beta(z)_2}[f,\psi(\overline{z})] - [f,\phi(\overline{z})] - (\mathcal{P}(z)f)r(z)\,.$$

By Lemma 0.6.4, (4.17), and $Q(z) = \hat{Q}(z)$ we have

$$\frac{\alpha(z)_2}{\beta(z)_2} = r(z)\,.$$

Together with (4.22) this yields

$$\mathcal{Q}(z)f = -[f,\phi(\overline{z})],\ f \in \mathcal{P}, z \in \rho(A)\,.$$

By Lemma 0.5.1, the functions $\mathcal{P}(z)f$ and $\mathcal{Q}(z)f$ have analytic continuations to $r_u(S_1) = r(S)$. Hence, for every $f \in \mathcal{P}$, the functions

$$z \mapsto [f,\psi(\overline{z})], \quad z \mapsto [f,\phi(\overline{z})], \qquad z \in \rho(A)\,,$$

have analytic continuations to $r(S)$. Therefore, also the functions

$$z \mapsto \psi(z), \quad z \mapsto \phi(z), \qquad z \in \rho(A)\,,$$

have analytic continuations to $r(S)$. We shall denote these continuations by $\tilde{\psi}(z)$ and $\tilde{\phi}(z)$, respectively.

Since $\rho(A)$ is dense in $r(S)$ and Γ is closed, we obtain

$$\big((\tilde{\phi}(z); z\tilde{\phi}(z)); (\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}\tilde{\alpha}(z)_1\\ \tilde{\alpha}(z)_2\end{pmatrix})\big) \in \Gamma, \quad z \in r(S)\,,$$

$$\big((\tilde{\psi}(z); z\tilde{\psi}(z)); (\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}\tilde{\beta}(z)_1\\ \tilde{\beta}(z)_2\end{pmatrix})\big) \in \Gamma, \quad z \in r(S)\,.$$

By the uniqueness assertion in Lemma 4.1 we get

$$\phi(z) = \tilde{\phi}(z),\ \psi(z) = \tilde{\psi}(z), \quad z \in r(S)\,,$$

$$\alpha(z)_1=\tilde\alpha(z)_1,\ \alpha(z)_2=\tilde\alpha(z)_2,\ \beta(z)_1=\tilde\beta(z)_1,\ \beta(z)_2=\tilde\beta(z)_2,\quad z\in r(S)\,.$$

We conclude that $\omega(\mathfrak{B})$ is analytic on $r(S)$ and is a u-resolvent matrix of S_1, namely

$$\omega(\mathfrak{B})(z)=\tilde W(z)J,\ z\in r(S)\,.$$

Since S is completely non-selfadjoint, the function $Q(z)=\Psi(W(z))$ is a generalized Nevanlinna function with $\operatorname{ind}_- Q=\operatorname{ind}_-\mathcal{P}$. Therefore, $\operatorname{ind}_- H_{\tilde W}=\operatorname{ind}_- H_{\omega(\mathfrak{B})}=\operatorname{ind}_-\mathcal{P}$.

Finally, by Theorem 0.5.4,

$$f\mapsto\begin{pmatrix}-\mathcal{Q}(z)f\\ \mathcal{P}(z)f\end{pmatrix}=\begin{pmatrix}[f,\phi(\overline{z})]\\ [f,\psi(\overline{z})]\end{pmatrix}.$$

is an isometric isomorphism from $\mathcal{P}$ onto $\mathfrak{K}(\tilde W)$. Since the kernels $H_{\tilde W}(w,z)$ and $H_{\omega(\mathfrak{B})}(w,z)$ coincide, we however have $\mathfrak{K}(\tilde W)=\mathfrak{K}(\omega(\mathfrak{B}))$.

The fact that, S is mapped to the multiplication operator by $\Xi\boxtimes\Xi$, follows from Theorem 0.5.3 and its proof. ❏

Using rotation isomorphisms it is easy to deduce a variant of Theorem 4.19 which is not bound to $\binom{0}{1}$ and $\beta(z)_2$. The original formulation corresponds to the case $\gamma=0$ in the following statement. Recall the notation ξ_γ from (2.16).

4.20 Theorem. *Let $\mathfrak{B}=(\mathcal{P},T,\Gamma)$ be a boundary triplet which has defect 2 and satisfies (E), and assume that the symmetric relation $S:=T^*$ is completely nonselfadjoint. Moreover, let $\gamma\in\mathbb{R}$, and assume that*

$$\begin{aligned}\operatorname{mul}\Gamma\neq\operatorname{span}\big\{(\xi_{\gamma+\frac{\pi}{2}};\xi_{\gamma+\frac{\pi}{2}})\big\}\,,\\ \exists\, z\in r(S):\ \xi^T_{\gamma+\frac{\pi}{2}}\omega(\mathfrak{B})(z)\xi_{\gamma+\frac{\pi}{2}}\neq 0\,.\end{aligned}\tag{4.23}$$

Then the restriction S_1^γ of T defined as $S_1^\gamma:=\ker(([\xi_\gamma^T\pi_l]\times\pi_r)\circ\Gamma)$, is a symmetric and real extension of S with defect index $(1,1)$. Its adjoint $T_1^\gamma:=(S_1^\gamma)^*$ is given as $\ker(\xi_\gamma^T\pi_l\circ\Gamma)$.

Let $\mathcal{P}_-$ be the space constructed from S_1^γ as in [KW/0, §3], and let u^γ be the element of $\mathcal{P}_-$ which is defined by

$$\big[(f;g),u^\gamma\big]_\pm=\big(\xi^T_{\gamma+\frac{\pi}{2}}\pi_l\circ P^\gamma\circ\Gamma\big)(f;g),\ (f;g)\in T_1^\gamma\,,\tag{4.24}$$

where P^γ is the identity, if $\operatorname{mul}\Gamma=\{0\}$, and the projection of $\mathbb{C}^2\times\mathbb{C}^2$ onto $\operatorname{span}\{\xi_{\gamma+\frac{\pi}{2}}\}\times\mathbb{C}^2$ with kernel $\operatorname{mul}\Gamma=\operatorname{span}\{(m;m)\}$, otherwise.

Then $r_{u^\gamma}(S_1^\gamma)=r(S)$ and $\circlearrowleft_\gamma\omega(\mathfrak{B})$ is a u^γ-resolvent matrix of S_1^γ. Moreover, $\operatorname{ind}_- H_{\circlearrowleft_\gamma\omega(\mathfrak{B})}=\operatorname{ind}_-\mathcal{P}$, and map Ξ^γ defined as

$$(\Xi^\gamma f)(z):=N_\gamma\begin{pmatrix}[f,\phi(\overline{z})]\\ [f,\psi(\overline{z})]\end{pmatrix},\ f\in\mathcal{P}\,,$$

is an isometric isomorphism of $\mathcal{P}$ onto the reproducing kernel space $\mathfrak{K}(\circlearrowleft_\gamma\omega(\mathfrak{B}))$. The relation $(\Xi^\gamma\boxtimes\Xi^\gamma)(S)$ is the multiplication operator $\mathbf{f}(z)\mapsto z\mathbf{f}(z)$ in this space with domain $\{\mathbf{f}\in\mathfrak{K}(\circlearrowleft_\gamma\omega(\mathfrak{B})):\ z\mathbf{f}(z)\in\mathfrak{K}(\circlearrowleft_\gamma\omega(\mathfrak{B}))\}$.

Proof. We wish to apply Theorem 4.19 to the boundary triplet $\circlearrowleft_\gamma\mathfrak{B}$, cf. (2.9). In order to do so, we must make sure that the corresponding hypothesis (4.10) is satisfied: Since $\operatorname{mul}[(\nu_\gamma\boxtimes\nu_\gamma)\circ\Gamma]=(N_\gamma\boxtimes N_\gamma)\operatorname{mul}\Gamma$, the first condition in

(4.10) for $\circlearrowleft_\gamma \mathfrak{B}$ is equivalent to $\operatorname{mul}\Gamma \neq (N_\gamma^T \boxtimes N_\gamma^T)\operatorname{span}\{(\binom{0}{1};\binom{0}{1})\}$, and hence equivalent to the first condition in (4.23). We have

$$\begin{pmatrix}0\\1\end{pmatrix}^T \circlearrowleft_\gamma \omega(\mathfrak{B})\begin{pmatrix}0\\1\end{pmatrix} = \xi^T_{\gamma+\frac{\pi}{2}}\omega(\mathfrak{B})\xi_{\gamma+\frac{\pi}{2}}\,,$$

and hence, the second condition in (4.10) for $\circlearrowleft_\gamma \mathfrak{B}$ is equivalent to the second condition in (4.23).

Next we identify all the quantities appearing in Theorem 4.19 for $\circlearrowleft_\gamma \mathfrak{B}$. First

$$S_1(\circlearrowleft_\gamma \mathfrak{B}) = \ker\big((\pi_{l,1}\times\pi_r)\circ(\nu_\gamma\boxtimes\nu_\gamma)\circ\Gamma\big) = \ker\big(([(1\;0)N_\gamma]\times N_\gamma)\circ(\pi_l\boxtimes\pi_r)\circ\Gamma\big) =$$

$$= \ker\big(([(\cos\gamma\;\;\sin\gamma)\pi_l]\times\pi_r)\circ\Gamma\big) = S_1^\gamma\,,$$

and

$$T_1(\circlearrowleft_\gamma \mathfrak{B}) = \ker(\pi_{l,1}\circ(\nu_\gamma\boxtimes\nu_\gamma)\circ\Gamma) = \ker((1\;0)\pi_l\circ(\nu_\gamma\boxtimes\nu_\gamma)\circ\Gamma) =$$

$$= \ker((1\;0)N_\gamma\pi_l\circ\Gamma) = \ker((\cos\gamma\;\;\sin\gamma)\pi_l\circ\Gamma) = (S_1^\gamma)^* = T_1^\gamma\,.$$

The equation (4.11) reads as

$$\big[(f;g),u(\circlearrowleft_\gamma \mathfrak{B})\big]_\pm = (\pi_{l,2}\circ P\circ(\nu_\gamma\boxtimes\nu_\gamma)\circ\Gamma)(f;g),\;\;(f;g)\in T_1^\gamma\,,$$

where P is the identity map, if $\operatorname{mul}\Gamma = \{0\}$, and the projection of $\mathbb{C}^2\times\mathbb{C}^2$ onto $(\{0\}\times\mathbb{C})\times\mathbb{C}^2$ with kernel $\operatorname{mul}[(\nu_\gamma\boxtimes\nu_\gamma)\circ\Gamma] = \operatorname{span}\{(N_\gamma m; N_\gamma m)\}$, otherwise. Moreover, we have

$$(\pi_{l,2}\circ P\circ(\nu_\gamma\boxtimes\nu_\gamma)\circ\Gamma) = ((0\;1)N_\gamma\circ\pi_l\circ(N_\gamma^T\boxtimes N_\gamma^T)P(N_\gamma\boxtimes N_\gamma)\circ\Gamma).\quad(4.25)$$

The linear mapping $P^\gamma := (N_\gamma^T\times N_\gamma^T)P(N_\gamma\times N_\gamma)$ obviously is the projection with $\ker P^\gamma = \operatorname{span}\{(m;m)\}$ and $\operatorname{ran} P^\gamma = \operatorname{span}\{\xi_{\gamma+\frac{\pi}{2}}\}\times\mathbb{C}^2$. The relation (4.25) thus shows that the right side of (4.24) for $\mathfrak{B}$ equals the right side of (4.11) for $\circlearrowleft_\gamma \mathfrak{B}$ and, hence,

$$u(\circlearrowleft_\gamma \mathfrak{B}) = u^\gamma\,.$$

Finally, by (4.7), we have

$$\phi(\circlearrowleft_\gamma \mathfrak{B}) = \cos\gamma\cdot\phi+\sin\gamma\cdot\psi =: \phi^\gamma,\quad \psi(\circlearrowleft_\gamma \mathfrak{B}) = -\sin\gamma\cdot\phi+\cos\gamma\cdot\psi =: \psi^\gamma\,,$$

and hence

$$(\Xi(\circlearrowleft_\gamma \mathfrak{B}))f(z) = \begin{pmatrix}[f,\phi(\circlearrowleft_\gamma \mathfrak{B})(\overline{z})]\\ [f,\psi(\circlearrowleft_\gamma \mathfrak{B})(\overline{z})]\end{pmatrix} = N_\gamma\begin{pmatrix}[f,\phi(\overline{z})]\\ [f,\psi(\overline{z})]\end{pmatrix}.$$

Theorem 4.19 now yields that $r_{u^\gamma}(S_1^\gamma) = r(S)$, that $\omega(\circlearrowleft_\gamma \mathfrak{B})$ is a u^γ-resolvent matrix of S_1^γ, that $\operatorname{ind}_- H_{\omega(\circlearrowleft_\gamma\mathfrak{B})} = \operatorname{ind}_-\mathcal{P}$, and that Ξ^γ is an isomorphism of $\mathcal{P}$ onto $\mathfrak{K}(\omega(\circlearrowleft_\gamma \mathfrak{B}))$. However, by Corollary 4.8, we have $\omega(\circlearrowleft_\gamma \mathfrak{B}) = \circlearrowleft_\gamma \omega(\mathfrak{B})$. ❑

Looking at Theorem 4.20 from a little different angle, we obtain the following corollary.

4.21 Corollary. *Let $\mathfrak{B} = (\mathcal{P}, T, \Gamma)$ be a boundary triplet which has defect 2 and satisfies (E), and assume that the symmetric relation $S := T^*$ is completely nonselfadjoint. Moreover, assume that for some $\gamma \in \mathbb{R}$ the conditions* (4.23) *are satisfied.*

Then $\operatorname{ind}_- H_{\omega(\mathfrak{B})} = \operatorname{ind}_- \mathcal{P}$ *and the map Ξ defined as in* (4.12) *is an isometric isomorphism of $\mathcal{P}$ onto the reproducing kernel space $\mathfrak{K}(\omega(\mathfrak{B}))$. The relation $(\Xi \boxtimes \Xi)(S)$ is the multiplication operator $\mathbf{f}(z) \mapsto z\mathbf{f}(z)$ in this space with domain $\{\mathbf{f} \in \mathfrak{K}(\omega(\mathfrak{B})) : z\mathbf{f}(z) \in \mathfrak{K}(\omega(\mathfrak{B}))\}$. In fact, $\Xi = \Theta^{-1}$, where Θ is defined in Remark 4.6 by* (4.5).

Proof. We can apply Theorem 4.20 with the number γ given by the present hypothesis. The fact that Ξ is then an isometric isomorphism follows from (2.11) and the remark made after it. Finally, by the definition of Ξ and by the fact that $\mathfrak{K}(\omega(\mathfrak{B}))$ is a reproducing kernel space Ξ maps $\lambda\phi(\overline{w}) + \mu\psi(\overline{w})$ onto $H_{\omega(\mathfrak{B})}(w, .)\binom{\lambda}{\mu}$. Therefore, $\Xi = \Theta^{-1}$. ❑

4.22 Remark. Let us have a closer look at the set $\mathcal{E}$ of all values $\gamma \in \mathbb{R}$ for which (4.23) fails. If $\omega(\mathfrak{B}) = \pm J$, trivially, $\xi^T_{\gamma+\frac{\pi}{2}}(\pm J)\xi_{\gamma+\frac{\pi}{2}} \equiv 0$, and thus $\mathcal{E} = \mathbb{R}$.

Let us show that otherwise $|[0, \pi) \cap \mathcal{E}| \leq 3$. The condition in the first line of (4.23) fails for at most one value of $\gamma \in [0, \pi)$. Assume that the condition in the second line fails for three different values $\gamma_1, \gamma_2, \gamma_3 \in [0, \pi)$, and let $z \in r(S)$. Since $\operatorname{span}\{\xi_{\gamma+\frac{\pi}{2}}\}^\perp = \operatorname{span}\{J\xi_{\gamma+\frac{\pi}{2}}\}$, this implies that $\xi_{\gamma_j+\frac{\pi}{2}}$, $j = 1, 2, 3$, are eigenvalues of the matrix $\omega(\mathfrak{B})(z)$. Since each two of these vectors are linearly independent, this yields that $J\omega(\mathfrak{B})(z) = \pm I$, i.e. $\omega(\mathfrak{B})(z) = \pm J$.

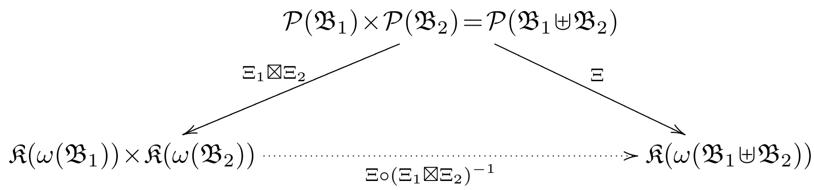
Finally note that if $\omega(\mathfrak{B}) = \pm J$ and if $S = T^*$ is completely non-selfadjoint, then due to Remark 4.6 $\mathcal{P} = \{0\}$. //

As a consequence of Theorem 4.20, we can compute the reproducing kernel space generated by the matrix constructed from a pasting of two boundary triplets.

4.23 Corollary. *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be boundary triplets which have defect 2 and satisfy (E) and (LI). Assume that $S(\mathfrak{B}_1)$, $S(\mathfrak{B}_2)$, and $S(\mathfrak{B}_1 \uplus \mathfrak{B}_2)$ are completely nonselfadjoint, and that none of the matrices $\omega(\mathfrak{B}_1)$, $\omega(\mathfrak{B}_2)$, and $\omega(\mathfrak{B}_1 \uplus \mathfrak{B}_2)$ is equal to $\pm J$. Then*

$$\mathfrak{K}(\omega(\mathfrak{B}_1 \uplus \mathfrak{B}_2)) = \mathfrak{K}(\omega(\mathfrak{B}_1)) \oplus \left[\omega(\mathfrak{B}_1) \cdot \mathfrak{K}(\omega(\mathfrak{B}_2))\right].$$

Proof. Due to Remark 4.22, we can choose a value $\gamma \in \mathbb{R}$ such that (4.23) holds for each of $\mathfrak{B}_1$, $\mathfrak{B}_2$ and $\mathfrak{B} := \mathfrak{B}_1 \uplus \mathfrak{B}_2$. Denoting the corresponding isomorphisms given in Corollary 4.21 by Ξ_1, Ξ_2 and Ξ, we have



However, if $f_1 \in \mathcal{P}_1$, $f_2 \in \mathcal{P}_2$, we obtain from (4.9) that

$$\Xi(f_1; f_2)(z) = \begin{pmatrix} [(f_1; f_2), \phi(\overline{z})] \\ [(f_1; f_2), \psi(\overline{z})] \end{pmatrix} = \begin{pmatrix} [f_1, \phi_1(\overline{z})] \\ [f_1, \psi_1(\overline{z})] \end{pmatrix} + \omega(\mathfrak{B}_1)(z) \begin{pmatrix} [f_2, \phi_2(\overline{z})] \\ [f_2, \psi_2(\overline{z})] \end{pmatrix} =$$

$$= \Xi_1 f_1(z) + \omega(\mathfrak{B}_1)(z)\Xi_2 f_2(z)\,,$$

and, hence,

$$\big(\Xi \circ (\Xi_1 \boxtimes \Xi_2)^{-1}\big)(F;G)(z) = F(z) + \omega(\mathfrak{B}_1)(z)G(z),\ \ F \in \mathfrak{K}(\mathfrak{B}_1)), g \in \mathfrak{K}(\mathfrak{B}_2))\,.$$

The assertion follows. ❑

4.24 Theorem. *Let $\mathfrak{B}_1 = (\mathcal{P}_1, T_1, \Gamma_1)$ be a boundary triplet which satisfies the hypothesis of Theorem 4.19. Moreover, let $\mathfrak{B}_2$ be a boundary triplet with defect* 1 *satisfying property (E) such that the condition (LI) of Proposition IV.6.2 holds true. Set*

$$\mathfrak{B} = (\mathcal{P}, T, \Gamma) := \mathfrak{B}_1 \uplus \mathfrak{B}_2\,,$$

and $\mathring{A} := \ker(\pi_{l,1} \circ \Gamma)$, and assume that $\rho(\mathring{A}) \neq \emptyset$. For $z \in \rho(\mathring{A})$ let

$$\begin{pmatrix} q(z) \\ 1 \end{pmatrix} \in v(\mathfrak{B})(z) \quad \textit{and} \quad f_z \in \ker(T - z)$$

be as in Lemma 4.14. Let $z \in r(S_2)$, and let $(\nu_1(z)\ \nu_2(z))^T$ be any nonzero element of $v(\mathfrak{B}_2)(z)$. Then †

$$q(z) = \omega(\mathfrak{B})(z) \star \frac{\nu_1(z)}{\nu_2(z)}\,, \tag{4.26}$$

and this function is a generalized u-resolvent in the sense of Definition 0.4.2 of $S_1 := \ker\big((\pi_{l,1} \times \pi_r) \circ \Gamma_1\big)$ induced by $\mathring{A}$. Here u is the same element as in Theorem 4.19. Moreover, $f_z = \xi R_z^- u$ for some $\xi \in \mathbb{C} \setminus \{0\}$.

If it is not the case that $q(z) = \frac{\nu_1(z)}{\nu_2(z)}$, $z \in r(S)$, and that this function is a real constant θ such that $\operatorname{mul}\Gamma_1 = \operatorname{span}\big\{\big(\binom{-1}{-\theta}; \binom{-1}{-\theta}\big)\big\}$, then $\xi = -1$.

Proof.
Step 1: First of all (4.26) is an immediate consequence of Lemma 4.17.

Step 2: Note that $\mathring{A} := \ker(\pi_{l,1} \circ (\Gamma_1 \uplus \Gamma_2))$ is in fact a selfadjoint extension of $S_1 = \ker((\pi_{l,1} \times \pi_r) \circ \Gamma_1)$. Let $A = \ker((\pi_{l,1} \times \pi_{r,2}) \circ \Gamma_1)$ be as in Step 2 of the proof of Theorem 4.19.

In the following R_z^- denotes the extension of the resolvent as defined in [KW/0] on page 290, where now $\mathfrak{P}$ is $\mathcal{P}_1$ and $\tilde{\mathfrak{P}}$ is $\mathcal{P}$ and where $A = \mathring{A}$.

Because of

$$[g - \bar{z}f, R_z^- u] = [R_z^+(g - \bar{z}f), u]_\pm = [(f;g), u]_\pm\,,$$

cf. Proposition 0.4.5, $R_z^- u$ is an appropriate parametrization of the defect spaces of the symmetric restriction

$$S_u = \{(f;g) \in \mathring{A} : [(f;g), u]_\pm = 0\}$$

of $\mathring{A}$. As $\Gamma = \Gamma_1 \uplus \Gamma_2$ and consequently $\tilde{\mathfrak{P}}_- = \mathfrak{P}_- \oplus (\tilde{\mathfrak{P}} \ominus \mathfrak{P})^2$ and as $u \in \mathfrak{P}_-$ we see that

$$S_u = \{(f_1 + f_2; g_1 + g_2) : \exists\ ((f_1;g_1);(a_1;b_1)) \in \Gamma_1, ((f_2;g_2);(a_2;0)) \in \Gamma_2,$$
$$b_1 = a_2,\ \pi_1 a_1 = 0,\ [(f_1;g_1), u]_\pm = 0\}.$$

† Recall the notion of $W \star \tau$ from short after Definition 2.17.

Since u is defined such that

$$[(f_1;g_1),u]_\pm = \pi_{l,2}\circ P\circ \Gamma_1(f_1;g_1)\,,$$

cf. (4.11), where $P=I$ in case $\operatorname{mul}\Gamma_1=\{0\}$ and P is the projection of $\mathbb{C}^2\times\mathbb{C}^2$ onto $(\{0\}\times\mathbb{C})\times\mathbb{C}^2$ with kernel $\operatorname{mul}\Gamma_1$. As $\pi_1 a_1 = 0$ in the above equation it follows that

$$S_u = \{(f;g)\in T : \pi_l\circ\Gamma(f;g)=0\} = T^* =: S.$$

Thus $R_z^- u\in\ker(T-z)$. By Lemma 4.14 and Lemma 0.4.3 $R_z^- u$ is a constant multiple of f_z.

Note that, since $\operatorname{mul}\Gamma=\{0\}$, S has defect $(1,1)$, and hence, both $R_z^- u$ and f_z do not vanish.

Step 3: For $z\in\rho(A)\cap\rho(\mathring{A})$ let $g\in\mathcal{P}_1\backslash\operatorname{ran}(S_1-z)$ and set $f=(A-z)^{-1}g$ $(\in\mathcal{P}_1)$ and $\mathring{f}=(\mathring{A}-z)^{-1}g$. According to the decomposition $\mathcal{P}=\mathcal{P}_1\oplus\mathcal{P}_2$ we write $\mathring{f}=f_1+f_2$. It follows that

$$((f;g+zf);(\begin{pmatrix}0\\a(z)\end{pmatrix};\begin{pmatrix}b(z)\\0\end{pmatrix}))\in\Gamma_1, \tag{4.27}$$

for some $a(z),b(z)\in\mathbb{C}$. Moreover, $((\mathring{f};g+z\mathring{f});(\binom{0}{\mathring{a}(z)};0)\in\Gamma=\Gamma_1\uplus\Gamma_2$, and hence,

$$((f_1;g+zf_1);(\begin{pmatrix}0\\\mathring{a}(z)\end{pmatrix};\begin{pmatrix}c_1(z)\\c_2(z)\end{pmatrix}))\in\Gamma_1,\quad ((f_2;zf_2);(\begin{pmatrix}c_1(z)\\c_2(z)\end{pmatrix};0))\in\Gamma_2 \tag{4.28}$$

for some $\mathring{a}(z),c_1(z),c_2(z)\in\mathbb{C}$. By Definition 4.11 $J\binom{c_1(z)}{c_2(z)}\in v(\mathfrak{B}_2)(z)$. Moreover, from (4.27) and (4.28) we obtain

$$((f_1-f;z(f_1-f));(\begin{pmatrix}0\\\mathring{a}(z)-a(z)\end{pmatrix};\begin{pmatrix}c_1(z)-b(z)\\c_2(z)\end{pmatrix}))\in\Gamma_1. \tag{4.29}$$

By the definition of $\omega(\mathfrak{B}_1)(z)=(\binom{\alpha_1(z)}{\alpha_2(z)}|\binom{\beta_1(z)}{\beta_2(z)})^T$ (see Lemma 4.1 and Corollary 4.2)

$$\begin{pmatrix}c_1(z)-b(z)\\c_2(z)\end{pmatrix}=(\mathring{a}(z)-a(z))\begin{pmatrix}\beta_1(z)\\\beta_2(z)\end{pmatrix}. \tag{4.30}$$

On the other hand we have by Green's identity ($\psi(z)$ is defined in Lemma 4.1 applied to $\mathfrak{B}_1$)

$$[g,\psi(\bar z)] = [g+zf,\psi(\bar z)]-[f,\bar z\psi(\bar z)] =$$

$$(0\ 1)J\begin{pmatrix}0\\a(z)\end{pmatrix}-\begin{pmatrix}\beta_1(\bar z)\\\beta_2(\bar z)\end{pmatrix}^*J\begin{pmatrix}b(z)\\0\end{pmatrix}=-\overline{\beta_2(\bar z)}b(z)=-\beta_2(z)b(z).$$

Hence, with the notation from the proof of Theorem 4.19, Step 3, we have $[g,\varphi(\bar z)]=b(z)$. For $\mathring{a}(z)-a(z)\neq 0$ we obtain from (4.30)

$$c_1(z)=[g,\varphi(\bar z)]+(\mathring{a}(z)-a(z))\beta_1(z),\quad c_2(z)=(\mathring{a}(z)-a(z))\beta_2(z).$$

As $z\in\rho(A)$ we have $\beta_2(z)\neq 0$, compare the proof of Theorem 4.19, Step 2, and hence $c_2(z)\neq 0$. Therefore,

$$\frac{c_1(z)}{c_2(z)}=\frac{[g,\varphi(\bar z)]}{(\mathring{a}(z)-a(z))\ \beta_2(z)}+\frac{\beta_1(z)}{\beta_2(z)},$$

und further

$$(\mathring{a}(z) - a(z)) = \frac{[g, \varphi(\bar{z})]}{\frac{c_1(z)}{c_2(z)} - \frac{\beta_1(z)}{\beta_2(z)}} \cdot \frac{1}{\beta_2(z)}.$$

By (4.29) and Lemma 4.1

$$f_1 = f + \frac{[g, \varphi(\bar{z})]}{\frac{c_1(z)}{c_2(z)} - \frac{\beta_1(z)}{\beta_2(z)}} \cdot \frac{1}{\beta_2(z)} \psi(z) = f - \frac{[g, \varphi(\bar{z})]}{\frac{c_1(z)}{c_2(z)} - \frac{\beta_1(z)}{\beta_2(z)}} \cdot \varphi(z). \tag{4.31}$$

If $\mathring{a}(z) - a(z) = 0$ for some $z \in \rho(A) \cap \rho(\mathring{A})$, then by property (E) of $\mathfrak{B}_1$ we have $f_1 = f$ and $c_1(z) = b(z),\ c_2(z) = 0$. By property (E) of $\mathfrak{B}_2$ we have $c_1(z) \neq 0$. Thus, also in this case (4.31) holds true.

However, since $-\frac{\beta_1(z)}{\beta_2(z)}$ is the Q-function of (S_1, A) corresponding to the defect family $\varphi(z)$, cf. Step 3 of the proof of Theorem 4.19, the formula (4.31) is nothing else, but Krein's formula for generalized resolvents of S_1, see (0.2.4).

Step 4: Let $W(z)$ be the u-resolvent matrix of S_1 constructed by means of Definition 0.4.8 with the selfadjoint extension A, the regularized u-resolvent $r(z) = \frac{\alpha_2(z)}{\beta_2(z)}$, and the Q-function $-\frac{\beta_1(z)}{\beta_2(z)}$, see Step 5 of the proof of Theorem 4.19.

By Theorem 0.4.9 and (4.31) we get that

$$\mathring{r}(z) = W(z) \star \frac{c_1(z)}{c_2(z)}$$

is a generalized u-resolvent of S_1 induced by $\mathring{A}$. Moreover, by Step 6 of the proof of Theorem 4.19 $W(z)J = \omega(\mathfrak{B}_1)(z)$, and hence,

$$\mathring{r}(z) = \omega(\mathfrak{B}_1)(z) \star -\frac{c_2(z)}{c_1(z)} = \omega(\mathfrak{B}_1)(z) \star \frac{\nu_1(z)}{\nu_2(z)} = q(z). \tag{4.32}$$

Step 5: In Step 2 of the present proof we saw that $f_z = \xi R_z^- u$ for some $\xi \in \mathbb{C} \setminus \{0\}$.

To compute the actual value of ξ we employ (0.3.8) and the fact that $((A - z)^{-1})^- u = \gamma(z)$, cf. Step 5 of the proof of Theorem 4.19. This gives

$$P_1 R_z^- u = \gamma(z) - \frac{[u, (\varphi(\bar{z}); \bar{z}\varphi(\bar{z}))]_\pm}{\frac{c_1(z)}{c_2(z)} - \frac{\beta_1(z)}{\beta_2(z)}} \cdot \varphi(z) =$$

$$\gamma(z) + \frac{1}{\beta_2(z)} \cdot \frac{[u, (\psi(\bar{z}); \bar{z}\psi(\bar{z}))]_\pm}{\frac{c_1(z)}{c_2(z)} - \frac{\beta_1(z)}{\beta_2(z)}} \cdot \varphi(z) = \gamma(z) + \frac{1}{\beta_2(z)\frac{c_1(z)}{c_2(z)} - \beta_1(z)} \cdot \varphi(z).$$

Since $\gamma(z), \varphi(z) \in \mathcal{P}_1$ we obtain

$$[\gamma(z), P_1 R_w^- u] = [\gamma(z), \gamma(w)] + \frac{1}{\overline{\left(\frac{c_1(w)}{c_2(w)}\right)}\beta_2(\bar{w}) - \beta_1(\bar{w})} \cdot [\gamma(z), \varphi(w)]$$

and

$$[\varphi(z), P_1 R_w^- u] = [\varphi(z), \gamma(w)] + \frac{1}{\overline{\left(\frac{c_1(w)}{c_2(w)}\right)}\beta_2(\bar{w}) - \beta_1(\bar{w})} \cdot [\varphi(z), \varphi(w)].$$

On the other hand, by the definition of f_z and by Corollary 4.2, we have

$$((P_1 f_z; zP_1 f_z); (\begin{pmatrix} 1 \\ -q(z) \end{pmatrix}; \begin{pmatrix} \alpha_1(z) - q(z)\beta_1(z) \\ \alpha_2(z) - q(z)\beta_2(z) \end{pmatrix})) \in \Gamma_1 .$$

Hence, by Green's identity, and by (4.15) and (4.16),

$$(z-\bar{w})[\gamma(z), P_1 f_w] = [z\gamma(z), P_1 f_w] - [\gamma(z), wP_1 f_w] =$$

$$\begin{pmatrix} 1 \\ -q(w) \end{pmatrix}^* J \begin{pmatrix} -1 \\ \frac{\alpha_2(z)}{\beta_2(z)} \end{pmatrix} - \begin{pmatrix} \alpha_1(w) - q(w)\beta_1(w) \\ \alpha_2(w) - q(w)\beta_2(w) \end{pmatrix}^* J \begin{pmatrix} -\frac{1}{\beta_2(z)} \\ 0 \end{pmatrix} =$$

$$-\frac{\alpha_2(z)}{\beta_2(z)} + q(\bar{w}) - (\alpha_2(\bar{w}) - q(\bar{w})\beta_2(\bar{w})) \cdot \frac{1}{\beta_2(z)},$$

and $(z-\bar{w})[\varphi(z), P_1 f_w]$ coincides with

$$\begin{pmatrix} 1 \\ -q(w) \end{pmatrix}^* J \begin{pmatrix} 0 \\ -\frac{1}{\beta_2(z)} \end{pmatrix} - \begin{pmatrix} \alpha_1(w) - q(w)\beta_1(w) \\ \alpha_2(w) - q(w)\beta_2(w) \end{pmatrix}^* J \begin{pmatrix} -\frac{\beta_1(z)}{\beta_2(z)} \\ -1 \end{pmatrix} =$$

$$\frac{1}{\beta_2(z)} - \alpha_1(\bar{w}) + q(\bar{w})\beta_1(\bar{w}) + (\alpha_2(\bar{w}) - q(\bar{w})\beta_2(\bar{w}))\frac{\beta_1(z)}{\beta_2(z)}.$$

From (4.32) and the fact that $\det \omega(\mathfrak{B}_1)(\bar{w}) = 1$ we get

$$q(\bar{w}) = \frac{\frac{\alpha_2(\bar{w})}{\beta_2(\bar{w})}(\beta_2(\bar{w})\frac{c_1(\bar{w})}{c_2(\bar{w})} - \beta_1(\bar{w})) + \frac{\alpha_2(\bar{w})}{\beta_2(\bar{w})}\beta_1(\bar{w}) - \alpha_1(\bar{w})}{\beta_2(\bar{w})\frac{c_1(\bar{w})}{c_2(\bar{w})} - \beta_1(\bar{w})} =$$

$$= \frac{\alpha_2(\bar{w})}{\beta_2(\bar{w})} - \frac{1}{\beta_2(\bar{w})} \frac{1}{\beta_2(\bar{w})\frac{c_1(\bar{w})}{c_2(\bar{w})} - \beta_1(\bar{w})}.$$

Hence,

$$(z-\bar{w})[\gamma(z), P_1 f_w] = -\frac{\alpha_2(z)}{\beta_2(z)} + \frac{\alpha_2(\bar{w})}{\beta_2(\bar{w})} -$$

$$-\frac{1}{\beta_2(\bar{w})} \cdot \frac{1}{\beta_2(\bar{w})\frac{c_1(\bar{w})}{c_2(\bar{w})} - \beta_1(\bar{w})} - \frac{1}{\beta_2(\bar{w})\frac{c_1(\bar{w})}{c_2(\bar{w})} - \beta_1(\bar{w})} \cdot \frac{1}{\beta_2(z)} =$$

$$= -\left(\frac{\alpha_2(z)}{\beta_2(z)} - \frac{\alpha_2(\bar{w})}{\beta_2(\bar{w})}\right) - \frac{1}{\beta_2(\bar{w})\frac{c_1(\bar{w})}{c_2(\bar{w})} - \beta_1(\bar{w})} \left(\frac{-1}{\beta_2(z)} - \frac{-1}{\beta_2(\bar{w})}\right).$$

$$(z-\bar{w})[\varphi(z), P_1 f_w] = \frac{1}{\beta_2(z)} - \alpha_1(\bar{w}) + \frac{\alpha_2(\bar{w})\beta_1(\bar{w})}{\beta_2(\bar{w})} -$$

$$-\frac{\beta_1(\bar{w})}{\beta_2(\bar{w})} \cdot \frac{1}{\beta_2(\bar{w})\frac{c_1(\bar{w})}{c_2(\bar{w})} - \beta_1(\bar{w})} + \frac{1}{\beta_2(\bar{w})\frac{c_1(\bar{w})}{c_2(\bar{w})} - \beta_1(\bar{w})} \cdot \frac{\beta_1(z)}{\beta_2(z)}$$

$$= -\left(\frac{-1}{\beta_2(z)} - \frac{-1}{\beta_2(w)}\right) - \left(\frac{-\beta_1(z)}{\beta_2(z)} - \frac{-\beta_1(\bar{w})}{\beta_2(\bar{w})}\right) \frac{1}{\beta_2(\bar{w})\frac{c_1(\bar{w})}{c_2(\bar{w})} - \beta_1(\bar{w})}.$$

Therefore, according to relations (4.19) and (4.20), $[\gamma(z), P_1 f_w]$ coincides with $[\gamma(z), -P_1 R_w^- u]$ and $[\varphi(z), P_1 f_w]$ coincides with $[\varphi(z), -P_1 R_w^- u]$. By the complete non-selfadjointness the vectors $\gamma(z), \varphi(z)$ span $\mathcal{P}_1$ and, hence, $-P_1 R_w^- u = P_1 f_w$. If this vector does not vanish for at least one $w \in r(S)$, we get $\xi = -1$.

If this vector always vanishes, then $((0;0);(\binom{-1}{q(w)};\binom{b_1}{b_2})) \in \Gamma_1$ and $(((I-P_1)f_w; w(I-P_1)f_w);(\binom{b_1}{b_2};0)) \in \Gamma_2$ for some $b_1, b_2 \in \mathbb{C}$. Therefore $\binom{-1}{q(w)} = \binom{b_1}{b_2}$ and $\operatorname{mul}\Gamma_1 = \operatorname{span}\{(\binom{-1}{-q(w)};\binom{-1}{-q(w)})\}$ for all $w \in r(S)$. From this we see that $q(w)$ is constant. Moreover, $J\binom{-1}{q(w)}$ spans $v(\mathfrak{B}_2)(z)$ and, hence, $q(w) = \frac{\nu_1(w)}{\nu_2(w)}$. ❑

4.25 Remark. Assume that in the previous theorem it is the case that $q(z) = \frac{\nu_1(z)}{\nu_2(z)}$, $z \in r(S)$, and that this function is a real constant such that $\operatorname{mul}\Gamma_1 = \operatorname{span}\{(\binom{-1}{-q(z)};\binom{-1}{-q(z)})\}$.

If in addition $\operatorname{mul}\Gamma_2 = \{(0;0)\}$, then, since according to Lemma 4.14 $q(z) = \frac{\nu_1(z)}{\nu_2(z)}$ is a Q-function corresponding to the defect family $(I-P_1)f_z \in \ker(T_2 - z)$, the span of these defect vectors is a neutral subspace. In particular, $\mathcal{P}_2$ must have at least one negative square.

If $\operatorname{mul}\Gamma_2 \neq \{(0;0)\}$, then according to Remark 4.13 $\operatorname{mul}\Gamma_2$ is spanned by $\binom{-1}{q(w)}$. But this is impossible since we impose condition (LI) of Proposition IV.6.2. //

4.26 Remark. By Remark 4.18 the fact that $\rho(\mathring{A}) \neq \emptyset$ is equivalent to the fact that the denominator in (4.26) does not vanish identically. //

d. The matrix $\omega(\mathfrak{B}(W))$ induced by $W \in \mathcal{M}_{<\infty}$.

We will now show that the constructions of $\omega(.)$ and $\mathfrak{B}(.)$ are in a sense converse to each other. First, let $W \in \mathcal{M}_{<\infty}$ be given. Then the construction of $\omega(.)$ can be applied to the boundary triplet $\mathfrak{B}(W)$. It is easy to compute $\omega(\mathfrak{B}(W))$.

4.27 Proposition. *Let $W \in \mathcal{M}_{<\infty}$, $W \neq I$. Then $\omega(\mathfrak{B}(W)) = W$.*

Proof. We have

$$((H_W(\overline{z},.)\begin{pmatrix}1\\0\end{pmatrix}; zH_W(\overline{z},.)\begin{pmatrix}1\\0\end{pmatrix});(\begin{pmatrix}1\\0\end{pmatrix}; W(\overline{z})^*\begin{pmatrix}1\\0\end{pmatrix})) \in \Gamma(W),$$

$$((H_W(\overline{z},.)\begin{pmatrix}0\\1\end{pmatrix}; zH_W(\overline{z},.)\begin{pmatrix}0\\1\end{pmatrix});(\begin{pmatrix}0\\1\end{pmatrix}; W(\overline{z})^*\begin{pmatrix}0\\1\end{pmatrix})) \in \Gamma(W).$$

Thus

$$\phi(z) = H_W(\overline{z},.)\begin{pmatrix}1\\0\end{pmatrix}, \quad \psi(z) = H_W(\overline{z},.)\begin{pmatrix}0\\1\end{pmatrix}$$

$$\alpha(z) = W(\overline{z})^*\begin{pmatrix}1\\0\end{pmatrix}, \quad \beta(z) = W(\overline{z})^*\begin{pmatrix}0\\1\end{pmatrix}$$

and it follows that

$$\omega(\mathfrak{B}(W))(z) = \big(\alpha(z)|\beta(z)\big)^T = \big(W(\overline{z})^*\big)^T = W(z).$$

❑

Secondly, let a boundary triplet $\mathfrak{B}$ which has defect 2 and satisfies (E) be given. Then we can consider the boundary triplet $\mathfrak{B}(\omega(\mathfrak{B}))$. It is equally easy to relate it with $\mathfrak{B}$.

4.28 Proposition. *Let $\mathfrak{B}$ be a boundary triplet which has defect 2 and satisfies (E). Let $\phi(z)$ and $\psi(z)$ be the elements constructed in Lemma 4.1, assume that S is completely non-selfadjoint. Moreover, let $\Theta : \mathfrak{K}(\omega(\mathfrak{B})) \to \mathcal{P}$ be the unitary operator defined in Remark 4.6.*

Then the pair $(\Theta; \mathrm{id})$ is an isomorphism between the boundary triplets $\mathfrak{B}(\omega(\mathfrak{B}))$ and $\mathfrak{B}$.

Proof. By Lemma 2.12 and the definition of $\mathfrak{B}(\omega(\mathfrak{B}))$ it is enough to show that

$$\Big((\Theta\times\Theta)\big(H_{\omega(\mathfrak{B})}(w,.)v;\overline{w}H_{\omega(\mathfrak{B})}(w,.)v\big);\big(v;\omega(\mathfrak{B})(w)^*v\big)\Big)\in\Gamma,\ w\in\mathbb{C},v\in\mathbb{C}^2\,. \tag{4.33}$$

By linearity, it is enough to consider the cases $v=\binom{1}{0}$ and $v=\binom{0}{1}$. However, we have

$$\Theta\big(H_{\omega(\mathfrak{B})}(w,.)\begin{pmatrix}1\\0\end{pmatrix}\big)=\phi(\overline{w}),\ \omega(\mathfrak{B})(w)^*\begin{pmatrix}1\\0\end{pmatrix}=\alpha(\overline{w})\,,$$

$$\Theta\big(H_{\omega(\mathfrak{B})}(w,.)\begin{pmatrix}0\\1\end{pmatrix}\big)=\psi(\overline{w}),\ \omega(\mathfrak{B})(w)^*\begin{pmatrix}0\\1\end{pmatrix}=\beta(\overline{w})\,,$$

and thus (4.33) follows from the defining relations for α and β. ❑

e. The matrix $\omega(\mathfrak{B}(\mathfrak{h}))$ induced by a general Hamiltonian.

Let $\mathfrak{h}\in\mathfrak{H}_{<\infty}$. The construction of $\omega(.)$ can be applied to the boundary triplet $\mathfrak{B}(\mathfrak{h})$. Properties of the boundary triplet $\mathfrak{B}(\mathfrak{h})$ reflect in properties of $\omega(\mathfrak{B}(\mathfrak{h}))$.

4.29 Proposition. *Let $\mathfrak{h}$ be a regular general Hamiltonian. Then $\omega(\mathfrak{B}(\mathfrak{h}))\in\mathcal{M}_{<\infty}$ and*

$$\mathrm{ind}_-\,\omega(\mathfrak{B}(\mathfrak{h}))=\sum_{i=1}^n\big(\Delta_i+[\frac{\ddot{o}}{2}]\big)+\big|\big\{1\le i\le n:\ \ddot{o}_i\ odd,c_{i,1}<0\big\}\big|\,.$$

The map Ξ defined by (4.12) *for $f\in\mathcal{P}(\mathfrak{h})$ is an isometric isomorphism of $\mathcal{P}(\mathfrak{h})$ onto $\mathfrak{K}(\omega(\mathfrak{B}(\mathfrak{h})))$.*

Proof. We have $r(S(\mathfrak{h}))=\mathbb{C}$, hence $\omega(\mathfrak{B}(\mathfrak{h}))$ is entire. Moreover, $S(\mathfrak{h})$ is completely nonselfadjoint. We need to show that

$$\omega(\mathfrak{B}(\mathfrak{h}))(0)=I\,. \tag{4.34}$$

In order to establish this relation we first consider positive definite and elementary indefinite general Hamiltonians, and then use the standard pasting argument.

If $\mathfrak{h}$ is positive definite, i.e. $\mathfrak{h}$ is just a Hamiltonian $H(t)$, $t\in(s_-,s_+)$, then any constant $\mathbb{C}^2$-valued function $f(t):=\binom{a}{b}$ satisfies $f'=JH\binom{0}{0}$. Hence, $((f;0);(\binom{a}{b};\binom{a}{b}))\in\Gamma(H)$, cf. Subsections IV.2.1.c and IV.2.1.d. If $\mathfrak{h}$ is an elementary indefinite Hamiltonian of type (A), then we see from Definition IV.4.11 and Definition IV.4.12 that

$$((\chi_-\begin{pmatrix}1\\0\end{pmatrix}+\chi_+\begin{pmatrix}1\\0\end{pmatrix};0);(\begin{pmatrix}1\\0\end{pmatrix};\begin{pmatrix}1\\0\end{pmatrix})),((p_0;0);(\begin{pmatrix}0\\1\end{pmatrix};\begin{pmatrix}0\\1\end{pmatrix}))\in\Gamma(\mathfrak{h})\,.$$

For the elements such as p_0 appearing here see Subsection IV.4.2.

Finally, if $\mathfrak{h}$ is an elementary indefinite Hamiltonain of type (B) or (C), then we see from Definition IV.4.3 and Definition IV.4.5 that

$$\big((0;0);(\begin{pmatrix}1\\0\end{pmatrix};\begin{pmatrix}1\\0\end{pmatrix})\big),\big((p_0;0);(\begin{pmatrix}0\\1\end{pmatrix};\begin{pmatrix}0\\1\end{pmatrix})\big)\in\Gamma(\mathfrak{h})\,.$$

In all these cases we have $\omega(\mathfrak{B}(\mathfrak{h}))(0)=I$.

Next let $\mathfrak{h}$ be an arbitrary regular general Hamiltonian, and let Hamiltonians $\mathfrak{h}^l$, $l=0,\dots,N$, be defined as in Remark 3.36. By Corollary 4.8 and Proposition 4.10 we conclude from (3.9) that

$$\omega(\mathfrak{B}(\mathfrak{h}))(z)=\circlearrowleft_{\gamma_0}\omega(\mathfrak{B}(\mathfrak{h}^0))(z)\cdot\ldots\cdot\circlearrowleft_{\gamma_N}\omega(\mathfrak{B}(\mathfrak{h}^N))(z)\,.$$

Evaluating at $z=0$ yields (4.34).

The relation (4.34) gives

$$\xi_{\gamma+\frac{\pi}{2}}^T\omega(\mathfrak{B}(\mathfrak{h}))(0)\xi_{\gamma+\frac{\pi}{2}}=1\,,$$

and hence the hypothesis (4.23) is satisfied for all but at most one value of $\gamma\in[0,\pi)$. An application of Corollary 4.21 yields the desired assertion. ❑

From Remark 3.42, Proposition 4.7 for $\circlearrowleft_\alpha$ and from Remark 3.43 together with Lemma 4.9 und Proposition 4.7 for rev we immediately obtain the following compatibility result.

4.30 Lemma. *For $\mathfrak{h}$ be a regular general Hamiltonian and $\alpha\in\mathbb{R}$ we have $\omega(\mathfrak{B}(\circlearrowleft_\alpha\mathfrak{h}))=\circlearrowleft_\alpha\omega(\mathfrak{B}(\mathfrak{h}))$ and $\omega(\mathfrak{B}(\operatorname{rev}\mathfrak{h}))=\operatorname{rev}\omega(\mathfrak{B}(\mathfrak{h}))$.* ❑

For elementary indefinite Hamiltonians $\mathfrak{h}$ of kind (B) or (C), the matrix $\omega(\mathfrak{B}(\mathfrak{h}))$ can be computed explicitly.

4.31 Proposition. *Let $\mathfrak{h}$ be an elementary indefinite Hamiltonian of kind (B) or (C) which consists of the data*

$$H(t),\ t\in[s_-,\sigma)\cup(\sigma,s_+]\,,$$

$$\ddot{o}\in\mathbb{N}\cup\{0\},\ b_1,\dots,b_{\ddot{o}+1}\in\mathbb{R},\ d_0\in\mathbb{R},\ d_1=0\,,$$

subject to the conditions of Definition IV.4.1. Then

$$\omega(\mathfrak{B}(\mathfrak{h}))(z)=\begin{pmatrix}1 & 0\\ -zd_0+z^2b_{\ddot{o}+1}+\ldots+z^{\ddot{o}+2}b_1 & 1\end{pmatrix}.$$

Proof. In both cases, kind (B) or (C), we have $\operatorname{mul}\Gamma(\mathfrak{h})=\operatorname{span}\{(\binom{1}{0};\binom{1}{0})\}$, cf. Lemma IV.4.19. Hence,

$$\phi(z)=0,\ \alpha(z)=\begin{pmatrix}1\\0\end{pmatrix}.$$

If $\mathfrak{h}$ is of kind (C), trivially $\ker(T-z)=\operatorname{span}\{p_0\}$. In fact, $(p_0;zp_0)=(p_0;0)+z(0;p_0)$, which shows that $\Lambda(p_0;zp_0)=\binom{zd_0}{1}$, and thus

$$\big((p_0;zp_0);(\begin{pmatrix}0\\1\end{pmatrix};\begin{pmatrix}-zd_0\\1\end{pmatrix})\big)\in\Gamma(\mathfrak{h})\,. \tag{4.35}$$

Hence

$$\psi(z) = p_0,\ \beta(z) = \begin{pmatrix} -zd_0 \\ 1 \end{pmatrix}.$$

Assume now that $\mathfrak{h}$ is of kind (B). In this case (for the notation see again Subsection IV.4.2)

$$\ker(T - z) = \operatorname{span}\big\{p_0 + \sum_{j=0}^{\ddot{o}} \mu_j \delta_j\big\},$$

with

$$\mu_j := \sum_{k=0}^{\ddot{o}-j} z^{k+1} b_{1+\ddot{o}-j-k},\ j = 0, \dots, \ddot{o}.$$

To see this, note that by the definition of μ_j we have $\mu_{j-1} - z\mu_j = zb_{\ddot{o}-j+2}$, $j = 1, \dots, \ddot{o}$, and remember $\mathfrak{b} = \sum_{l=1}^{\ddot{o}+1} b_l \delta_{1+\ddot{o}-l}$. Hence,

$$\big(p_0 + \sum_{j=0}^{\ddot{o}} \mu_j \delta_j; zp_0 + \sum_{j=0}^{\ddot{o}} z\mu_j \delta_j\big) =$$

$$= (p_0; 0) + \sum_{j=1}^{\ddot{o}} z\mu_j (\delta_{j-1}; \delta_j) + z(\mathfrak{b}; p_0 + d_0\delta_0) + z(\mu_0 - d_0)(0; \delta_0).$$

Hence

$$\Lambda\big(p_0 + \sum_{j=0}^{\ddot{o}} \mu_j \delta_j; zp_0 + \sum_{j=0}^{\ddot{o}} z\mu_j \delta_j\big) = \begin{pmatrix} z(d_0 - \mu_0) \\ 1 \end{pmatrix},$$

and therefore

$$\Big(\big(p_0 + \sum_{j=0}^{\ddot{o}} \mu_j \delta_j; zp_0 + \sum_{j=0}^{\ddot{o}} z\mu_j \delta_j\big); \big(\begin{pmatrix} 0 \\ 1 \end{pmatrix}; \begin{pmatrix} -zd_0 + z\mu_0 \\ 1 \end{pmatrix}\big)\Big) \in \Gamma(\mathfrak{h}).$$

It follows that

$$\psi(z) = p_0 + \sum_{j=0}^{\ddot{o}} \mu_j \delta_j,\ \beta(z) = \begin{pmatrix} -zd_0 + \sum_{k=0}^{\ddot{o}} z^{k+2} b_{1+\ddot{o}-k} \\ 1 \end{pmatrix}.$$

❑

4.32 Corollary. *Let $\mathfrak{h}$ be a regular general Hamiltonian given by the data*

$$h(t)\xi_\phi \xi_\phi^T,\ t \in [s_-, \sigma) \cup (\sigma, s_+],$$

$$\ddot{o} \in \mathbb{N} \cup \{0\},\ b_1, \dots, b_{\ddot{o}+1} \in \mathbb{R},\ d_0 \in \mathbb{R},\ d_1 = 0,$$

where h is locally integrable on $[s_-, \sigma) \cup (\sigma, s_+]$, $\int_{s_-}^{\sigma} h = \int_{\sigma}^{s_+} = \infty$, and where either $b_1 \neq 0$ or $d_0 < 0$, $\ddot{o} = 0$, $b_1 = 0$. Then

$$\omega(\mathfrak{B}(\mathfrak{h}))(z) = \circlearrowleft_{\frac{\pi}{2}-\phi} \begin{pmatrix} 1 & 0 \\ -zd_0 + z^2 b_{\ddot{o}+1} + \dots + z^{\ddot{o}+2} b_1 & 1 \end{pmatrix}$$

Proof. Under the present hypothesis the general Hamiltonian $\circlearrowleft_{\phi-\frac{\pi}{2}} \mathfrak{h}$ is elementary indefinite of kind (B) or (C). The assertion follows from Proposition 4.31 and compatibility with $\circlearrowleft_\gamma$, Lemma 4.30. ❑

4.33 Remark. Let $\mathfrak{h}$ be a positive definite regular Hamiltonian which consists of just one indivisible interval, i.e. let $\mathfrak{h} = H$ where $H(t) = h(t)\xi_\phi\xi_\phi^T$, $t \in (s_-, s_+)$, where h is some non-negative integrable scalar function. Let us show that

$$\omega(\mathfrak{B}(\mathfrak{h}))(z) = \circlearrowleft_{\frac{\pi}{2}-\phi} \begin{pmatrix} 1 & 0 \\ -z\int_{s_-}^{s_+} h & 1 \end{pmatrix} = W_{(\int_{s_-}^{s_+} h,\phi)}(z)\,.$$

The same argument as in Corollary 4.32 shows that it is enough to consider the case $\phi = \frac{\pi}{2}$. Hence, assume that $\phi = \frac{\pi}{2}$. Then, by the considerations in [KW/IV, §2.1.e], we have $\operatorname{mul}\Gamma(H) = \operatorname{span}\{(\binom{1}{0};\binom{1}{0})\}$, and therefore

$$\phi(z) = 0,\ \alpha(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Moreover,

$$(\Big(\begin{pmatrix} -z\int_{s_-}^{x} h \\ 1 \end{pmatrix}; z\begin{pmatrix} -z\int_{s_-}^{x} h \\ 1 \end{pmatrix}\Big); \Big(\begin{pmatrix} 0 \\ 1 \end{pmatrix}; \begin{pmatrix} -z\int_{s_-}^{s_+} h \\ 1 \end{pmatrix}\Big)) \in \Gamma(H)\,,$$

and hence

$$\psi(z) = \begin{pmatrix} -z\int_{s_-}^{x} h \\ 1 \end{pmatrix},\ \beta(z) = \begin{pmatrix} -z\int_{s_-}^{s_+} h \\ 1 \end{pmatrix}.$$

We conclude that

$$\omega(\mathfrak{B}(\mathfrak{h})) = \begin{pmatrix} 1 & 0 \\ -z\int_{s_-}^{s_+} h & 1 \end{pmatrix}$$

//

4.34 Remark. Let $\mathfrak{h}$ be a positive definite singular Hamiltonian which consists of just one indivisible interval, i.e. let $\mathfrak{h} = H$ where $H(t) = h(t)\xi_\phi\xi_\phi^T$, $t \in (s_-, s_+)$, where h is non-negative scalar function, which is integrable on all subintervals $(s_-, t) \subseteq (s_-, s_+)$. Let us show that

$$v(\mathfrak{B}(\mathfrak{h}))(z) = \circlearrowleft_{\frac{\pi}{2}-\phi} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \operatorname{span}\{\xi_\phi\}.$$

Note that the boundary triplet $\mathfrak{B}(\mathfrak{h})$ induced by the singular positive definite Hamiltonian has defect 1 and satisfies property (E), see [KW/IV, §2.1.e]. Therefore, $v(\mathfrak{B}(\mathfrak{h}))(z)$ is well-defined, cf. Definition 4.11.

The same argument as in Corollary 4.32 shows that it is enough to consider the case $\phi = \frac{\pi}{2}$. Hence, assume that $\phi = \frac{\pi}{2}$. By the considerations in [KW/IV, §2.1.e], we have $\operatorname{mul}\Gamma(H) = \operatorname{span}\{(\binom{1}{0}; 0)\}$, and therefore (see Remark 4.13)

$$v(\mathfrak{B}(\mathfrak{h}))(z) = \operatorname{span}\{J\begin{pmatrix} 1 \\ 0 \end{pmatrix}\} = \operatorname{span}\{\xi_{\frac{\pi}{2}}\}.$$

//

Let us make explicit the following compatibility of $\omega(.)$ with splitting of $\mathfrak{h}$.

4.35 Lemma. *Assume that $\mathfrak{h} \in \mathfrak{H}_{<\infty}$ is regular, and let $\{r_0, \ldots, r_{m+1}\}$, $r_0 < r_1 < \ldots < r_{m+1}$, be a finite subset of $\overline{I}$ such that*

$$r_0 = \sigma_0,\ r_{m+1} = \sigma_{n+1}, \quad r_i \in I_{\text{reg}},\ i = 1, \ldots, m\,.$$

Then we have

$$\omega(\mathfrak{B}(\mathfrak{h})) = \prod_{i=0}^{m} \omega(\mathfrak{B}(\mathfrak{h}_{r_i \leftrightarrow r_{i+1}}))\,, \tag{4.36}$$

and

$$\mathfrak{K}\big(\omega(\mathfrak{B}(\mathfrak{h}))\big) = \mathfrak{K}\big(\omega(\mathfrak{B}(\mathfrak{h}_{r_0 \leftrightarrow r_1}))\big) \oplus \omega(\mathfrak{B}(\mathfrak{h}_{r_0 \leftrightarrow r_1})) \cdot \mathfrak{K}\big(\omega(\mathfrak{B}(\mathfrak{h}_{r_1 \leftrightarrow r_2}))\big) \oplus \ldots$$
$$\ldots \oplus \prod_{i=0}^{m-1} \omega(\mathfrak{B}(\mathfrak{h}_{r_i \leftrightarrow r_{i+1}})) \cdot \mathfrak{K}\big(\omega(\mathfrak{B}(\mathfrak{h}_{r_m \leftrightarrow r_{m+1}}))\big)\,.$$

Proof. With the notation from Definition 3.47 by Lemma 3.44 we may apply Proposition 4.7 with $\hat{\phi} = \text{id}$ and then Proposition 4.10 to conclude that (4.36) holds. The relation between reproducing kernel spaces follows from Corollary 4.23. ❑

4.36 Remark. According to Remark 3.51 we obtain from Lemma 4.35 that pasting of general Hamiltonians is also compatible with building $\omega(.)$:

Let $\mathfrak{h}_i,\ i = 1, \ldots, m$, be general Hamiltonians. If (¬paste) fails for each two consequtive Hamiltonians, then

$$\omega(\mathfrak{B}(\biguplus_{i=0}^{m} \mathfrak{h}_i)) = \prod_{i=0}^{m} \omega(\mathfrak{B}(\mathfrak{h}_i)).$$

Moreover, the reproducing kernel spaces are connected as in Lemma 4.35. //

Let $\mathfrak{h}$ be a regular general Hamiltonian. Observe that, by Proposition 4.31, Corollary 4.32, and Remark 4.33, we have computed $\omega(\mathfrak{B}(\mathfrak{h}))$ explicitly in those cases when $\operatorname{mul}\Gamma(\mathfrak{h}) \neq \{0\}$. If $\operatorname{mul}\Gamma(\mathfrak{h}) = \{0\}$, this will in general not be possible. However, $\omega(\mathfrak{B}(\mathfrak{h}))$ can be represented with help of the function $\Psi^{ac}(\mathfrak{h}) : T(\mathfrak{h}) \to \text{AC}(I) \times \mathcal{M}(I)/_{=_H}$, which was defined in Remark IV.8.9.

4.37 Remark. Let $\mathfrak{h}$ be a regular general Hamiltonian defined on the set $I = \bigcup_{i=0}^{n}(\sigma_i, \sigma_{i+1})$, and assume that $\operatorname{mul}\Gamma(\mathfrak{h}) = \{0\}$. Let $\phi(z)$ and $\psi(z)$ be as in Lemma 4.1. Then

$$\omega(\mathfrak{B}(\mathfrak{h}))(z) = \big(\pi_l \Psi^{ac}(\mathfrak{h})(\phi(z); z\phi(z))(\sigma_0) \,|\, \pi_l \Psi^{ac}(\mathfrak{h})(\psi(z); z\psi(z))(\sigma_{n+1})\big)^T\,.$$

This is obvious from the last lines of Remark IV.8.9, cf. the bottom line on [KW/IV, p.827]. //

5 Construction of the maximal chain

In this section we turn to the actual construction of the chain $\omega_{\mathfrak{h}} : I \cup \{\sigma_0\} \to \mathcal{M}_{<\infty}$ associated with the general Hamiltonian $\mathfrak{h}$. After the definition of $\omega_{\mathfrak{h}}$, we will prove the following theorem:

5.1 Theorem. *Let $\mathfrak{h}$ be a general Hamiltonian defined on $I = \bigcup_{i=0}^{n}(\sigma_i, \sigma_{i+1})$. If $\mathfrak{h}$ is regular, then $\omega_{\mathfrak{h}}$ is the finite maximal chain going downwards from $\omega(\mathfrak{B}(\mathfrak{h}))$. If $\mathfrak{h}$ is singular, then $\omega_{\mathfrak{h}}|_I$ is a maximal chain. In either case, we have* $\operatorname{ind}_- \mathfrak{h} = \operatorname{ind}_- \omega_{\mathfrak{h}}$.

Moreover, by means of this result, the following definition is meaningful, cf. Lemma II.8.2:

5.2 Definition. Let $\mathfrak{h}$ be a singular general Hamiltonian. The function

$$q_{\mathfrak{h}} := q_\infty(\omega_{\mathfrak{h}})$$

is called the Weyl coefficient of $\mathfrak{h}$. //

The content of this section is arranged in five subsections:

a. We give the definition of $\omega_{\mathfrak{h}}$, and prove some of its properties.
b. In this subsection we give the proof of Theorem 5.1 for the case that $\mathfrak{h}$ is regular. This is done by invoking Proposition 3.10.
c. The case of a singular general Hamiltonian can easily be reduced to the regular case.
d. We show compatibility of the assignment $\mathfrak{h} \mapsto \omega_{\mathfrak{h}}$ with the previously defined operations $\circlearrowleft_\gamma$, rev, and with pasting.
e. We give a representation of $q_{\mathfrak{h}}$ as a Q-function.

a. Construction of $\omega_{\mathfrak{h}}$.

In the following let $\mathfrak{h}$ always denote a (regular or singular) general Hamiltonian.

5.3 Definition. In the singular case let $t \in I \cup \{\sigma_0\}$ be given and in the regular case let $t \in I \cup \{\sigma_0, \sigma_{n+1}\}$ be given. For the definition of $\omega_{\mathfrak{h}}(t)$ we distinguish the following cases. First we deal with the cases that t is not contained in an indivisible interval.

(*i*) If $t = \sigma_0$, put $\omega_{\mathfrak{h}}(t) := I$.

(*ii*) If $\mathfrak{h}$ is regular and $t = \sigma_{n+1}$, put $\omega_{\mathfrak{h}}(t) := \omega(\mathfrak{B}(\mathfrak{h}))$.

(*iii*) If $t \in I \setminus I_{\text{sing}}$, define $\omega_{\mathfrak{h}}(t) := \omega(\mathfrak{B}(\mathfrak{h}\urcorner_t))$.

It remains to define $\omega_{\mathfrak{h}}(t)$ if t is inner point of some indivisible interval. Assume that $(t_-, t_+) \subseteq I$ is the maximal indivisible interval which contains t, and write $H(t) = h(t)\xi_\phi\xi_\phi^T$, $t \in (t_-, t_+)$, where $\phi \in [0, \pi)$ is the type of the indivisible interval and h is some locally integrable scalar function.

(*iv*) If $t_- \in I \cup \{\sigma_0\}$, and hence belongs to $I_{\text{reg}} \cup \{\sigma_0\}$, define

$$\omega_{\mathfrak{h}}(t) := \omega_{\mathfrak{h}}(t_-) \cdot W_{(l_-(t), \phi)}\,, \tag{5.1}$$

where $l_-(t) := \int_{t_-}^{t} h(t)\,dt$.

(*v*) Assume that $t_- \notin I \cup \{\sigma_0\}$. Then we must have $t_+ \in I \cup \{\sigma_{n+1}\}$, cf. axiom (H2) in Definition 3.35, and hence $t_+ \in I_{\text{reg}} \cup \{\sigma_{n+1}\}$, where the case $t = \sigma_{n+1}$ can occur only if $\mathfrak{h}$ is regular. In this case set $l_+(t) := \int_{t_+}^{t} h(t)\,dt$ (< 0), and define

$$\omega_{\mathfrak{h}}(t) := \omega_{\mathfrak{h}}(t_+) \cdot W_{(l_+(t), \phi)}\,. \tag{5.2}$$

//

First of all note that $\omega_{\mathfrak{h}}(t)$ actually always does belong to $\mathcal{M}_{<\infty}$. If $t \in (I \cup \{\sigma_0\}) \setminus I_{\text{sing}}$, this was said in Proposition 4.29, otherwise it follows because $\mathcal{M}_{<\infty}$ is closed with respect to products and $W_{(l,\phi)} \in \mathcal{M}_0 \cup \mathcal{M}_1$.

We will frequently employ our standard splitting-and-pasting method. In order to do so, it is practical to note explicitly:

5.4 Lemma. *Let $s, s' \in (I \cup \{\sigma_0\}) \setminus I_{\text{sing}}$, $s < s'$, be given. Then*

$$\omega_{\mathfrak{h}}(t) = \omega_{\mathfrak{h}}(s)\omega_{\mathfrak{h}_{s\leftrightarrow s'}}(t),\ t \in (I \cup \{\sigma_0\}) \cap [s, s'] . \tag{5.3}$$

Proof. Let $t \in (I \cup \{\sigma_0\}) \cap [s, s']$ be fixed. If $t = s$, we have $\omega_{\mathfrak{h}_{s\leftrightarrow s'}}(t) = I$ and hence (5.3) holds.

Assume next that $t \in I_{\text{reg}}$, $t > s$. We employ Remark 3.48 to obtain

$$\omega_{\mathfrak{h}_{s\leftrightarrow s'}}(t) = \omega\big(\mathfrak{B}\big((\mathfrak{h}_{s\leftrightarrow s'})_{\urcorner t}\big)\big) = \omega(\mathfrak{B}(\mathfrak{h}_{s\leftrightarrow t})) ,$$

and hence

$$\omega_{\mathfrak{h}}(t) = \omega(\mathfrak{B}(\mathfrak{h}_{\urcorner t})) = \omega(\mathfrak{B}((\mathfrak{h}_{\urcorner t})_{\urcorner s})) \cdot \omega(\mathfrak{B}((\mathfrak{h}_{\urcorner t})_{s\ulcorner})) =$$

$$= \omega(\mathfrak{B}(\mathfrak{h}_{\urcorner s})) \cdot \omega(\mathfrak{B}(\mathfrak{h}_{s\leftrightarrow t})) = \omega_{\mathfrak{h}}(s) \cdot \omega_{\mathfrak{h}_{s\leftrightarrow s'}}(t) .$$

Note here that

$$I_{\text{reg}}(\mathfrak{h}_{s\leftrightarrow s'}) = I_{\text{reg}}(\mathfrak{h}) \cap (s, s') . \tag{5.4}$$

since the Hamiltonian functions of $\mathfrak{h}_{s\leftrightarrow s'}$ are just restrictions of the Hamiltonian functions of $\mathfrak{h}$.

Assume now that we are in the case (iv) if Definition 5.3. Note that, by (5.4) and $s \in I_{\text{reg}}$, case (iv) prevails for $\mathfrak{h}$ if and only if it does for $\mathfrak{h}_{s\leftrightarrow s'}$. Moreover, we must have $t_- \geq s$. Clearly, the magnitudes ϕ and $h(t)$ are the same for $\mathfrak{h}$ and $\mathfrak{h}_{s\leftrightarrow s'}$. Hence, by what we have already proved,

$$\omega_{\mathfrak{h}}(t) = \omega_{\mathfrak{h}}(t_-)W_{(l_-(t),\phi)} = \omega_{\mathfrak{h}}(s) \cdot \omega_{\mathfrak{h}_{s\leftrightarrow s'}}(t_-)W_{(l_-(t),\phi)} = \omega_{\mathfrak{h}}(s) \cdot \omega_{\mathfrak{h}_{s\leftrightarrow s'}}(t) .$$

If we are in case (v) of Definition 5.3 for $\mathfrak{h}$ and thus also for $\mathfrak{h}_{s\leftrightarrow s'}$, we can proceed in the same manner. We must have $t_+ \leq s'$, and the magnitudes ϕ and $h(t)$ are the same for $\mathfrak{h}$ and $\mathfrak{h}_{s\leftrightarrow s'}$. Hence

$$\omega_{\mathfrak{h}}(t) = \omega_{\mathfrak{h}}(t_+)W_{(l_+(t),\phi)} = \omega_{\mathfrak{h}}(s) \cdot \omega_{\mathfrak{h}_{s\leftrightarrow s'}}(t_+)W_{(l_+(t),\phi)} = \omega_{\mathfrak{h}}(s) \cdot \omega_{\mathfrak{h}_{s\leftrightarrow s'}}(t) .$$

❑

With the help of Proposition 4.31 – Remark 4.37, we can determine $\omega_{\mathfrak{h}}$ explicitly if $\operatorname{mul}\Gamma(\mathfrak{h}) \neq \{0\}$, and at least represent it in terms of $\Psi^{ac}(\mathfrak{h})$ otherwise.

5.5 Proposition. *Let $\mathfrak{h}$ be a regular general Hamiltonian.*

(*i*) *Assume that $\mathfrak{h}$ is positive definite and consists of just one indivisible interval, and write $\mathfrak{h} = H$ with $H(t) = h(t)\xi_\phi\xi_\phi^T$, $t \in [s_-, s_+]$. Then*

$$\omega_{\mathfrak{h}}(t) = W_{(l_-(t),\phi)},\ t \in [s_-, s_+] ,$$

where $l_-(t) := \int_{s_-}^t h$.

(ii) *Assume that* $\mathfrak{h}$ *is given by the data*

$$h(t)\xi_\phi\xi_\phi^T,\ t\in[s_-,\sigma)\cup(\sigma,s_+]\,,$$

$$\ddot{o}\in\mathbb{N}\cup\{0\},\ b_1,\dots,b_{\ddot{o}+1}\in\mathbb{R},\ d_0\in\mathbb{R}, d_1=0\,,$$

where h *is locally integrable on* $[s_-,\sigma)\cup(\sigma,s_+]$, $\int_{s_-}^\sigma h=\int_\sigma^{s_+}h=\infty$, *and where either* $b_1\neq 0$ *or* $d_0<0$, $\ddot{o}=0$, $b_1=0$. *Then*

$$\omega_{\mathfrak{h}}(t)(z)=W_{(l_-(t),\phi)}(z)=\circlearrowleft_{\frac{\pi}{2}-\phi}\begin{pmatrix}1 & 0\\ -zl_-(t) & 1\end{pmatrix},\ t\in[s_-,\sigma)\,.$$

$$\omega_{\mathfrak{h}}(t)(z)=\circlearrowleft_{\frac{\pi}{2}-\phi}\begin{pmatrix}1 & 0\\ -z(l_+(t)+d_0)+z^2b_{\ddot{o}+1}+\ldots+z^{\ddot{o}+2}b_1 & 1\end{pmatrix},\ t\in(\sigma,s_+]\,,$$

where $l_+(t):=\int_{s_+}^t h$.

(iii) *Assume that* $\operatorname{mul}\Gamma(\mathfrak{h})=\{0\}$. *Let* $\phi(z)$ *and* $\psi(z)$ *be as in Lemma 4.1 for* $\mathfrak{B}(\mathfrak{h})$. *Then*

$$\omega_{\mathfrak{h}}(t)(z)=\big(\pi_l\Psi^{ac}(\mathfrak{h})(\phi(z);z\phi(z))(t)\,|\,\pi_l\Psi^{ac}(\mathfrak{h})(\psi(z);z\psi(z))(t)\big)^T,\quad t\in I\cup\{\sigma_0\}\,.$$

Proof. The assertions in (i) and (ii) are immediate from the definition of $\omega_{\mathfrak{h}}$ and Remark 4.33, Corollary 4.32.

In order to show (iii), consider first the case that $t\in(I\cup\{\sigma_0\})\setminus I_{\text{sing}}$. For $t=\sigma_0$, we have by Remark 4.37,

$$\omega_{\mathfrak{h}}(\sigma_0)(z)=I=\big(\pi_l\Psi^{ac}(\mathfrak{h})(\phi(z);z\phi(z))(\sigma_0)\,|\,\pi_l\Psi^{ac}(\mathfrak{h})(\psi(z);z\psi(z))(\sigma_0)\big)^T\,.$$

If $t>\sigma_0$, we obtain from (3.15) and Remark 4.37 that

$$\omega_{\mathfrak{h}}(t)(z)=\omega(\mathfrak{B}(\mathfrak{h}_{\urcorner t}))(z)=$$

$$=\big(\pi_l\Psi^{ac}(\mathfrak{h}_{\urcorner t})(\phi(\mathfrak{h}_{\urcorner t})(z);z\phi(\mathfrak{h}_{\urcorner t})(z))(t)\,|\,\pi_l\Psi^{ac}(\mathfrak{h}_{\urcorner t})(\psi(\mathfrak{h}_{\urcorner t})(z);z\psi(\mathfrak{h}_{\urcorner t})(z))(t)\big)^T=$$

$$=\big(\pi_l\Psi^{ac}(\mathfrak{h})(\phi(\mathfrak{h})(z);z\phi(\mathfrak{h})(z))(t)\,|\,\pi_l\Psi^{ac}(\mathfrak{h})(\psi(\mathfrak{h})(z);z\psi(\mathfrak{h})(z))(t)\big)^T\,.$$

Next observe that, by Remark IV.8.9, Ψ^{ac} maps $T(\mathfrak{h})$ into $T(H)$, and that the definition of Ψ^{ac} ensures that

$$\pi_r\Psi^{ac}(\mathfrak{h})(f;zf)=z\pi_l\Psi^{ac}(\mathfrak{h})(f;zf),\ f\in\ker(T(\mathfrak{h})-z)\,.$$

Thus the matrix function

$$M(t,z):=\big(\pi_l\Psi^{ac}(\mathfrak{h})(\phi(z);z\phi(z))(t)\,|\,\pi_l\Psi^{ac}(\mathfrak{h})(\psi(z);z\psi(z))(t)\big)^T,\quad t\in I\cup\{\sigma_0\}, z\in\mathbb{C}\,,$$

satisfies the differential equation

$$\frac{\partial}{\partial t}M(t,z)J=zM(t,z)H(t),\ t\in I\,.$$

Assume now that $t \in I_{\text{sing}}$ is given, so that either case (iv) or (v) of Definition 5.3 prevails. If we have $t_- \in I \cup \{\sigma_0\}$, then $M(t,z)$ is the (unique) solution of the initial value problem

$$\frac{\partial}{\partial t} W(t,z)J = zW(t,z)H(t),\ t \in [t_-, t_+), \qquad W(t_-, z) = M(t_-, z)\,.$$

Since (t_-, t_+) is indivisible, however, this equation is easily solved on this interval and we obtain

$$M(t,z) = M(t_-, z)W_{(l_-(t),\phi)},\ t \in [t_-, t_+)\,.$$

By what we have already shown, it now follows that

$$\omega_{\mathfrak{h}}(t) = \omega_{\mathfrak{h}}(t_-) \cdot W_{(l_-(t),\phi)} = M(t_-, .) \cdot W_{(l_-(t),\phi)} = M(t, .),\ t \in (t_-, t_+)\,.$$

The case that $t_- \notin I \cup \{\sigma_0\}$ can be treated in a completely similar manner, since then the function $M(t,z)$ is the (unique) solution of

$$\frac{\partial}{\partial t} W(t,z)J = zW(t,z)H(t),\ t \in (t_-, t_+], \qquad W(t_+, z) = M(t_+, z)\,.$$

❑

5.6 Corollary. *The function $\omega_{\mathfrak{h}} : I \cup \{\sigma_0\} \to \mathcal{M}_{<\infty}$ is locally absolutely continuous and satisfies the differential equation*

$$\frac{\partial}{\partial t} \omega_{\mathfrak{h}}(t)(z)J = z\omega_{\mathfrak{h}}(t)(z)H(t),\ t \in I, \qquad \omega_{\mathfrak{h}}(\sigma_0) = I\,, \tag{5.5}$$

and $\omega_{\mathfrak{h}}(\sigma_{n+1}) = \omega(\mathfrak{B}(\mathfrak{h}))$ in case $\mathfrak{h}$ is regular.

Proof. If $\mathfrak{h}$ is regular, this is obvious from the above proposition and its proof.

Consider the case that $\mathfrak{h}$ is singular. If $s \in I_{\text{reg}}$, then

$$\omega_{\mathfrak{h}}(t) = \omega_{\mathfrak{h}\urcorner_s}(t),\ t \leq s\,,$$

and hence $\omega_{\mathfrak{h}}$ satisfies (5.5) on $I \cap (\sigma_0, s)$. If $\sup I_{\text{reg}} = \sigma_{n+1}$, we are done. Otherwise, put $s := \sup I_{\text{reg}}$. Then $s \in I_{\text{reg}}$ and the interval (s, σ_{n+1}) is a maximal indivisible interval, cf. axiom (H2) in Definition 3.35. By what we know for the regular case, $\omega_{\mathfrak{h}}|_{(I \cup \{\sigma_0\}) \cap [\sigma_0, s]}$ is locally absolutely continuous and satisfies (5.5). However, it is apparent from the definition of $\omega_{\mathfrak{h}}$ that $\omega_{\mathfrak{h}}|_{[s, \sigma_{n+1})}$ is locally absolutely continuous and satisfies (5.5). Thus also in this case the desired assertion follows. ❑

5.7 Remark.

- (i) Although the equation (5.5) looks like an initial value problem, actually it is not: If $\mathfrak{h}$ is indefinite, the set I is not connected. In particular, $\omega_{\mathfrak{h}}$ is not uniquely determined by (5.5). This is of course no surprise, since a general Hamiltonian also contains the data $\ddot{o}_i, b_{ij}, d_{ij}$, and this data does not appear in (5.5).
- (ii) We already see that the function $\omega_{\mathfrak{h}}$ is closely related to the general Hamiltonian $\mathfrak{h}$, namely via the equation (5.5).

(iii) If $\mathfrak{h} = H$ is positive definite, then (5.5) is a proper initial value problem. Hence, in this case, $\omega_\mathfrak{h}$ coincides with the chain ω_H previously defined, cf. Proposition 3.23, (i).

//

b. Proof of Theorem 5.1, $\mathfrak{h}$ regular.

We will employ Proposition 3.10 to show that, for a regular general Hamiltonian $\mathfrak{h}$, the function $\omega_\mathfrak{h}$ is a finite maximal chain.

The function $\omega_\mathfrak{h}$ maps the set $I \cup \{\sigma_0\}$, which is of the form described in Proposition 3.10, into $\mathcal{M}_{<\infty}$, the matrix $\omega(\mathfrak{B}(\mathfrak{h}))$ belongs to $\mathcal{M}_{<\infty}$, and as we have already noted the equalities in hypothesis (i) of Proposition 3.10 hold true. The validity of hypothesis (iii) is an immediate consequence of the differential equation (5.5), axiom (H1), and the classical theory of canonical systems:

5.8 Corollary. *If s, s' belong to the same connected component of I, $s \le s'$, then*

$$\mathfrak{t}(\omega_\mathfrak{h}(s')) = \mathfrak{t}(\omega_\mathfrak{h}(s)) + \int_s^{s'} \operatorname{tr} H \,.$$

In particular, the function $\mathfrak{t}(\omega_\mathfrak{h}(t))$ is locally absolutely continuous and strictly increasing on I and

$$\lim_{t \nearrow \sigma_i} \mathfrak{t}(\omega_\mathfrak{h}(t)) = +\infty, \ \lim_{t \searrow \sigma_i} \mathfrak{t}(\omega_\mathfrak{h}(t)) = -\infty, \quad i = 1, \ldots, n \,.$$

❑

It is more exhausting to show that hypothesis (ii) of Proposition 3.10 holds true.

5.9 Proposition. *Let $\mathfrak{h}$ be a regular general Hamiltonian, and let $t \in I \cup \{\sigma_0\}$. Then*

$$\operatorname{ind}_- \omega_\mathfrak{h}(t) =$$

$$= \sum_{\substack{i=1,\ldots,n \\ \sigma_i < t}} \big(\Delta_i + [\tfrac{\ddot{o}_i}{2}]\big) + \big|\big\{1 \le i \le n : \sigma_i < t, \ddot{o}_i \textit{ odd}, c_{i,1} < 0\big\}\big| \,, \tag{5.6}$$

$$\operatorname{ind}_- \big(\omega_\mathfrak{h}(t)^{-1} \omega(\mathfrak{B}(\mathfrak{h}))\big) =$$

$$= \sum_{\substack{i=1,\ldots,n \\ \sigma_i > t}} \big(\Delta_i + [\tfrac{\ddot{o}_i}{2}]\big) + \big|\big\{1 \le i \le n : \sigma_i > t, \ddot{o}_i \textit{ odd}, c_{i,1} < 0\big\}\big| \,. \tag{5.7}$$

In particular, $\operatorname{ind}_- \omega_\mathfrak{h}(t)$ *is constant on each component of $I \cup \{\sigma_0\}$, and*

$$\operatorname{ind}_- \omega(\mathfrak{B}(\mathfrak{h})) = \operatorname{ind}_- \omega_\mathfrak{h}(t) + \operatorname{ind}_- \big(\omega_\mathfrak{h}(t)^{-1} \omega(\mathfrak{B}(\mathfrak{h}))\big) \,.$$

Proof (of Proposition 5.9, Case $t \in (I \cup \{\sigma_0\}) \setminus I_{\text{sing}}$). We know from Proposition 4.29 that

$$\operatorname{ind}_- \omega(\mathfrak{B}(\mathfrak{h})) = \sum_{i=1,\ldots,n} \big(\Delta_i + [\tfrac{\ddot{o}_i}{2}]\big) + \big|\big\{1 \le i \le n : \ddot{o}_i \text{ odd}, c_{i,1} < 0\big\}\big| \,.$$

In the cases that $t=\sigma_0$ or $t=\sigma_{n+1}$, the desired formulas (5.6) and (5.7) readily follow.

Assume that $t\in(\sigma_0,\sigma_{n+1})$. By the definition of $\omega_{\mathfrak h}$ we have $\omega_{\mathfrak h}(t)=\omega_{\mathfrak h_{\urcorner t}}$, and by Lemma 5.4 we have $\omega_{\mathfrak h}(t)^{-1}\omega(\mathfrak B(\mathfrak h))=\omega_{\mathfrak h_{t\ulcorner}}$. Hence (5.6) and (5.7) follow from Remark 3.45 and Proposition 4.29 applied to $\mathfrak h_{\urcorner t}$ and $\mathfrak h_{t\ulcorner}$, respectively. ❑

If $t\in I_{\text{sing}}$, three cases may occur: If (t_-,t_+) is the maximal indivisible interval which contains t, then

$$t_-,t_+\in I\cup\{\sigma_0\}\quad\text{or}\quad t_-\in I\cup\{\sigma_0\},t_+\not\in I\cup\{\sigma_0\}\quad\text{or}$$

$$t_-\not\in I\cup\{\sigma_0\},t_+\in I\cup\{\sigma_0\}$$

Proof (of Proposition 5.9, Case $t_-,t_+\in I\cup\{\sigma_0\}$). Write $H(t)=h(t)\xi_\phi\xi_\phi^T$, $t\in(t_-,t_+)$, and put $l_1:=\int_{t_-}^t h$, $l_2:=\int_t^{t_+}h$. Then, by the definition of $\omega_{\mathfrak h}$ and by (5.5), respectively,

$$\omega_{\mathfrak h}(t)=\omega_{\mathfrak h}(t_-)W_{(l_1,\phi)},\quad \omega_{\mathfrak h}(t_+)=\omega_{\mathfrak h}(t)W_{(l_2,\phi)}\,.$$

Since $l_1,l_2\geq 0$ and hence $W_{(l_i,\phi)}\in\mathcal M_0$, it follows from the subadditivity of ind_- that

$$\operatorname{ind}_-\omega_{\mathfrak h}(t_+)\leq\operatorname{ind}_-\omega_{\mathfrak h}(t)\leq\omega_{\mathfrak h}(t_-)\,.$$

However, by what we have already proved, both of $\omega_{\mathfrak h}(t_-)$ and $\omega_{\mathfrak h}(t_+)$ are equal to the number written on the right side of (5.6).

Similarly, we have

$$\omega_{\mathfrak h}(t)^{-1}\omega(\mathfrak B(\mathfrak h))=W_{(l_2,\phi)}\omega_{\mathfrak h}(t_+)^{-1}\omega(\mathfrak B(\mathfrak h))\,,$$

$$\omega_{\mathfrak h}(t_-)^{-1}\omega(\mathfrak B(\mathfrak h))=W_{(l_1,\phi)}\omega_{\mathfrak h}(t)^{-1}\omega(\mathfrak B(\mathfrak h))\,,$$

and hence

$$\operatorname{ind}_-\big(\omega_{\mathfrak h}(t_-)^{-1}\omega(\mathfrak B(\mathfrak h))\big)\leq\operatorname{ind}_-\big(\omega_{\mathfrak h}(t)^{-1}\omega(\mathfrak B(\mathfrak h))\big)\leq$$

$$\leq\operatorname{ind}_-\big(\omega_{\mathfrak h}(t_+)^{-1}\omega(\mathfrak B(\mathfrak h))\big)\,.$$

Both numbers $\operatorname{ind}_-(\omega_{\mathfrak h}(t_-)^{-1}\omega(\mathfrak B(\mathfrak h)))$ and $\operatorname{ind}_-(\omega_{\mathfrak h}(t_+)^{-1}\omega(\mathfrak B(\mathfrak h)))$ are equal to the number written on the right side of (5.7). ❑

The treatment of the case $t_-\in I\cup\{\sigma_0\},t_+\not\in I\cup\{\sigma_0\}$ is based on the following result.

5.10 Lemma. *Let (t_-,t_+) be a maximal indivisible interval with $t_-\in I\cup\{\sigma_0\},t_+\not\in I\cup\{\sigma_0\}$, and let $\phi\in[0,\pi)$ and $h(t)$ be such that $H(t)=h(t)\xi_\phi\xi_\phi^T$, $t\in(t_-,t_+)$. Then*

$$\xi_\phi\in\mathfrak K\big(\omega_{\mathfrak h}(t_-)^{-1}\omega(\mathfrak B(\mathfrak h))\big),\quad [\xi_\phi,\xi_\phi]\leq 0\,.$$

Proof. Let $s:=\min(E\cap(t_+,\sigma_{n+1}])$, then $\hat{\mathfrak h}:=\circlearrowleft_{\phi-\frac{\pi}{2}}\mathfrak h_{t_-\leftrightarrow s}$ is elementary indefinite, cf. Lemma IV.8.4.

If $\hat{\mathfrak h}$ is of kind (B) or (C), then by Proposition 4.31 and Proposition 2.8 we have

$$\begin{pmatrix}0\\1\end{pmatrix}\in\mathfrak K\big(\omega(\mathfrak B(\hat{\mathfrak h}))\big)\,,$$

$$\Big[\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\Big]=\begin{cases}0 & ,\ \hat{\mathfrak{h}}\text{ kind (B)}\\ \frac{1}{d_0} & ,\ \hat{\mathfrak{h}}\text{ kind (C)}\end{cases}$$

Assume that $\hat{\mathfrak{h}}$ is of kind (A). Since (t_-,t_+) is indivisible, we have $\chi_-\binom{0}{1}=0$, and hence

$$\Big((0;-\delta_0);(\begin{pmatrix}1\\0\end{pmatrix};0)\Big)\in\Gamma(\hat{\mathfrak{h}})\,.$$

By the abstract Green's identity, cf. (IV.2.6),

$$\begin{aligned}&[-\delta_0,\phi(\overline{z})]-[0,\overline{z}\phi(\overline{z})]=\begin{pmatrix}1\\0\end{pmatrix}^*J\begin{pmatrix}1\\0\end{pmatrix}-\alpha(\overline{z})^*J0=0\,,\\&[-\delta_0,\psi(\overline{z})]-[0,\overline{z}\psi(\overline{z})]=\begin{pmatrix}0\\1\end{pmatrix}^*J\begin{pmatrix}1\\0\end{pmatrix}-\beta(\overline{z})^*J0=1\,.\end{aligned}\tag{5.8}$$

Let Ξ be the isomorphism of $\mathcal{P}(\hat{\mathfrak{h}})$ onto $\mathfrak{K}(\omega(\mathfrak{B}(\hat{\mathfrak{h}})))$ defined by (4.12). Then the formulas (5.8) give

$$\begin{pmatrix}0\\1\end{pmatrix}=\Xi(-\delta_0)\,.$$

In particular, $\binom{0}{1}\in\mathfrak{K}(\omega(\mathfrak{B}(\hat{\mathfrak{h}})))$ and

$$\Big[\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\Big]=[-\delta_0,-\delta_0]=0\,.$$

Referring to Lemma 2.5, we find that

$$\xi_\phi=N_{-\phi+\frac{\pi}{2}}\begin{pmatrix}0\\1\end{pmatrix}\in\mathfrak{K}(\omega(\mathfrak{B}(\mathfrak{h}_{t_-\leftrightarrow s}))),\ [\xi_\phi,\xi_\phi]\leq 0\,.$$

Finally, since $t_-,s\in I_{\text{reg}}$, we may apply Lemma 3.44 with the general Hamiltonian $\mathfrak{h}_{t_-\leftrightarrow s}$ and the set $\{t_-,s,\sigma_{n+1}\}$. It follows that $\mathfrak{K}(\omega(\mathfrak{B}(\mathfrak{h}_{t_-\leftrightarrow s})))$ is contained isometrically in $\mathfrak{K}(\omega(\mathfrak{B}(\mathfrak{h}_{t_-\Rsh})))$. However, $\omega_{\mathfrak{h}}(t_-)^{-1}\omega(\mathfrak{B}(\mathfrak{h}))=\omega(\mathfrak{B}(\mathfrak{h}_{t_-\Rsh})))$. ❑

Proof (of Proposition 5.9, Case $t_-\in I\cup\{\sigma_0\},t_+\notin I\cup\{\sigma_0\}$). Let $t\in(t_-,t_+)$, then $\omega_{\mathfrak{h}}(t_-)^{-1}\omega_{\mathfrak{h}}(t)=W_{(l,\phi)}$ where $l:=\int_{t_-}^t h>0$. We see from Lemma 5.10 and Proposition 2.8 that the space $\mathfrak{K}(\omega_{\mathfrak{h}}(t_-)^{-1}\omega_{\mathfrak{h}}(t))$ is contained in $\mathfrak{K}(\omega_{\mathfrak{h}}(t_-)^{-1}\omega(\mathfrak{B}(\mathfrak{h})))$ and that the inclusion map is contractive. By [ADSR, Theorem 1.5.6], it follows that

$$\begin{aligned}&\text{ind}_-\big(H_{\omega_{\mathfrak{h}}(t_-)^{-1}\omega(\mathfrak{B}(\mathfrak{h}))}-H_{\omega_{\mathfrak{h}}(t_-)^{-1}\omega_{\mathfrak{h}}(t)}\big)=\\&=\text{ind}_-\,H_{\omega_{\mathfrak{h}}(t_-)^{-1}\omega(\mathfrak{B}(\mathfrak{h}))}-\text{ind}_-\,H_{\omega_{\mathfrak{h}}(t_-)^{-1}\omega_{\mathfrak{h}}(t)}=\text{ind}_-\,H_{\omega_{\mathfrak{h}}(t_-)^{-1}\omega(\mathfrak{B}(\mathfrak{h}))}\,.\end{aligned}$$

However, we have

$$\begin{aligned}&H_{\omega_{\mathfrak{h}}(t_-)^{-1}\omega(\mathfrak{B}(\mathfrak{h}))}(w,z)-H_{\omega_{\mathfrak{h}}(t_-)^{-1}\omega_{\mathfrak{h}}(t)}(w,z)=\\&=\omega_{\mathfrak{h}}(t_-)^{-1}(z)\omega_{\mathfrak{h}}(t)(z)\cdot H_{\omega_{\mathfrak{h}}(t)^{-1}\omega(\mathfrak{B}(\mathfrak{h}))}(w,z)\cdot\omega_{\mathfrak{h}}(t)(w)^*\omega_{\mathfrak{h}}(t_-)^{-*}(w)\,.\end{aligned}$$

Since it is already known that the formula (5.7) holds for t_-, it now follows that (5.7) also holds for t.

From $\omega_{\mathfrak{h}}(t)=\omega_{\mathfrak{h}}(t_-)W_{(l,\phi)}$, we have $\operatorname{ind}_-\omega_{\mathfrak{h}}(t)\leq\operatorname{ind}_-\omega_{\mathfrak{h}}(t_-)$. On the other hand

$$\omega(\mathfrak{B}(\mathfrak{h}))=\omega_{\mathfrak{h}}(t)\cdot\big[\omega_{\mathfrak{h}}(t)^{-1}\omega(\mathfrak{B}(\mathfrak{h}))\big]=\omega_{\mathfrak{h}}(t_-)\cdot\big[\omega_{\mathfrak{h}}(t_-)^{-1}\omega(\mathfrak{B}(\mathfrak{h}))\big]\,.$$

Using (5.7), which is already known for t and t_-, and (5.6) for t_-, it follows that

$$\operatorname{ind}_-\omega(\mathfrak{B}(\mathfrak{h}))\leq\operatorname{ind}_-\omega_{\mathfrak{h}}(t)+\operatorname{ind}_-\big[\omega_{\mathfrak{h}}(t)^{-1}\omega(\mathfrak{B}(\mathfrak{h}))\big]\leq$$
$$\leq\operatorname{ind}_-\omega_{\mathfrak{h}}(t_-)+\operatorname{ind}_-\big[\omega_{\mathfrak{h}}(t_-)^{-1}\omega(\mathfrak{B}(\mathfrak{h}))\big]=\operatorname{ind}_-\omega(\mathfrak{B}(\mathfrak{h}))\,.$$

Thus equality must hold throughout, and we conclude that (5.6) holds also for t. ❑

In the case $t_-\notin I\cup\{\sigma_0\},t_+\in I\cup\{\sigma_0\}$ we proceed along similar lines.

5.11 Lemma. *Let (t_-,t_+) be a maximal indivisible interval with $t_-\notin I\cup\{\sigma_0\},t_+\in I\cup\{\sigma_0\}$, and let $\phi\in[0,\pi)$ and $h(t)$ be such that $H(t)=h(t)\xi_\phi\xi_\phi^T$, $t\in(t_-,t_+)$. Then*

$$\omega_{\mathfrak{h}}(t_+)\xi_\phi\in\mathfrak{K}(\omega_{\mathfrak{h}}(t_+)),\quad\big[\omega_{\mathfrak{h}}(t_+)\xi_\phi,\omega_{\mathfrak{h}}(t_+)\xi_\phi\big]\leq 0\,.$$

Proof. Let $s:=\max(E\cap[\sigma_0,t_-))$, then $\hat{\mathfrak{h}}:=\circlearrowleft_{\phi-\frac{\pi}{2}}\mathfrak{h}_{s\leftrightarrow t_+}$ is elementary indefinite.

Assume first that $\hat{\mathfrak{h}}$ is of kind (B) or (C), then by Proposition 4.31

$$\omega(\mathfrak{B}(\hat{\mathfrak{h}}))\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}.$$

Combining this with Proposition 2.8, we obtain the desired assertion.

Next, assume that $\hat{\mathfrak{h}}$ is of kind (A). Since (t_-,t_+) is indivisible, we have $\chi_+\binom{0}{1}=0$, and hence

$$\big((0;\delta_0);(0;\begin{pmatrix}1\\0\end{pmatrix})\big)\in\Gamma(\hat{\mathfrak{h}})\,.$$

By the abstract Green's identity,

$$\begin{aligned}[]
[\delta_0,\phi(\overline{z})]-[0,\overline{z}\phi(\overline{z})]&=\begin{pmatrix}1\\0\end{pmatrix}^*J0-\alpha(\overline{z})^*J\begin{pmatrix}1\\0\end{pmatrix}=-\alpha(z)_2\,,\\
[\delta_0,\psi(\overline{z})]-[0,\overline{z}\psi(\overline{z})]&=\begin{pmatrix}0\\1\end{pmatrix}^*J0-\beta(\overline{z})^*J\begin{pmatrix}1\\0\end{pmatrix}=-\beta(z)_2\,.
\end{aligned}\tag{5.9}$$

Let Ξ be the isomorphism (4.12), then (5.9) yields

$$\omega(\mathfrak{B}(\hat{\mathfrak{h}}))\begin{pmatrix}0\\1\end{pmatrix}=\Xi(-\delta_0)\,.$$

We conclude that $\omega(\mathfrak{B}(\hat{\mathfrak{h}}))\binom{0}{1}\in\mathfrak{K}(\omega(\mathfrak{B}(\hat{\mathfrak{h}})))$ and

$$\Big[\omega(\mathfrak{B}(\hat{\mathfrak{h}}))\begin{pmatrix}0\\1\end{pmatrix},\omega(\mathfrak{B}(\hat{\mathfrak{h}}))\begin{pmatrix}0\\1\end{pmatrix}\Big]=[-\delta_0,-\delta_0]=0\,.$$

It follows that

$$\omega(\mathfrak{B}(\mathfrak{h}_{s\leftrightarrow t_+}))\xi_\phi=N_{-\phi+\frac{\pi}{2}}\omega(\mathfrak{B}(\hat{\mathfrak{h}}))\begin{pmatrix}0\\1\end{pmatrix}\in\mathfrak{K}(\omega(\mathfrak{B}(\mathfrak{h}_{s\leftrightarrow t_+})))\,,$$

and $[\xi_\phi, \xi_\phi] \leq 0$.

Since $t_+, s \in I_{\text{reg}}$, we may apply Lemma 3.44 with the general Hamiltonian $\mathfrak{h}\urcorner_{t_+}$ and the set $\{\sigma_0, s, t_+\}$. It follows that $\omega(\mathfrak{B}(\mathfrak{h}\urcorner_s))\mathfrak{K}(\omega(\mathfrak{B}(\mathfrak{h}_{s\leftrightarrow t_+})))$ is contained isometrically in $\mathfrak{K}(\omega(\mathfrak{B}(\mathfrak{h}\urcorner_{t_+})))$. Since $\omega_\mathfrak{h}(t_+) = \omega(\mathfrak{B}(\mathfrak{h}\urcorner_s))\omega(\mathfrak{B}(\mathfrak{h}_{s\leftrightarrow t_+}))$, the assertion follows. ❑

Proof (of Proposition 5.9, Case $t_- \notin I \cup \{\sigma_0\}, t_+ \in I \cup \{\sigma_0\}$). Let $t \in (t_-, t_+)$, then $\omega_\mathfrak{h}(t)^{-1}\omega_\mathfrak{h}(t_+) = W_{(l,\phi)}$ with $l := \int_t^{t_+} h > 0$. Thus, by Lemma 5.11, the space $\omega_\mathfrak{h}(t_+)\mathfrak{K}(\omega_\mathfrak{h}(t)^{-1}\omega_\mathfrak{h}(t_+))$ is contained contractively in $\omega_\mathfrak{h}(t_+)$. However,

$$\omega_\mathfrak{h}(t)\xi_\phi = \omega_\mathfrak{h}(t_+) \cdot \underbrace{\omega_\mathfrak{h}(t_+)^{-1}\omega_\mathfrak{h}(t)}_{=W_{(-l,\phi)}}\xi_\phi = \omega_\mathfrak{h}(t_+)\xi_\phi\,,$$

and hence also $\omega_\mathfrak{h}(t)\mathfrak{K}(\omega_\mathfrak{h}(t)^{-1}\omega_\mathfrak{h}(t_+))$ is contained contractively in $\mathfrak{K}(\omega_\mathfrak{h}(t_+))$.

Appealing to [ADSR, Theorem 1.5.6], we find

$$\begin{aligned}\operatorname{ind}_-\big(H_{\omega_\mathfrak{h}(t_+)}(w,z) - \omega_\mathfrak{h}(t)(z)H_{\omega_\mathfrak{h}(t)^{-1}\omega_\mathfrak{h}(t_+)}\omega_\mathfrak{h}(t)(w)^*\big) = \\ = \operatorname{ind}_- H_{\omega_\mathfrak{h}(t_+)}(w,z) - \operatorname{ind}_- \omega_\mathfrak{h}(t)(z)H_{\omega_\mathfrak{h}(t)^{-1}\omega_\mathfrak{h}(t_+)}\omega_\mathfrak{h}(t)(w)^* = \\ = \operatorname{ind}_- H_{\omega_\mathfrak{h}(t_+)}(w,z)\,.\end{aligned}$$

However,

$$H_{\omega_\mathfrak{h}(t_+)}(w,z) - \omega_\mathfrak{h}(t)(z)H_{\omega_\mathfrak{h}(t)^{-1}\omega_\mathfrak{h}(t_+)}\omega_\mathfrak{h}(t)(w)^* = H_{\omega_\mathfrak{h}(t)}(w,z)\,,$$

and (5.6) follows. We have $\omega_\mathfrak{h}(t)^{-1}\omega(\mathfrak{B}(\mathfrak{h})) = W_{(l,\phi)} \cdot \omega_\mathfrak{h}(t_+)^{-1}\omega(\mathfrak{B}(\mathfrak{h}))$, and therefore

$$\begin{aligned}\operatorname{ind}_- \omega(\mathfrak{B}(\mathfrak{h})) \leq \operatorname{ind}_- \omega_\mathfrak{h}(t) + \operatorname{ind}_- \omega_\mathfrak{h}(t)^{-1}\omega(\mathfrak{B}(\mathfrak{h})) \leq \\ \leq \operatorname{ind}_- \omega_\mathfrak{h}(t_+) + \operatorname{ind}_- \omega_\mathfrak{h}(t_+)^{-1}\omega(\mathfrak{B}(\mathfrak{h})) = \operatorname{ind}_- \omega(\mathfrak{B}(\mathfrak{h}))\,.\end{aligned}$$

We see that also (5.7) holds. ❑

Finally, we will establish condition (*iv*) of Proposition 3.10. Thereby we will use the following simple computation:

5.12 Lemma. *Let $\phi, \psi \in \mathbb{C}^2$, $M = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathbb{C}^2$, and $q \in \mathbb{C}$. Assume that* $\det M = 1$, *and*

$$\lambda := d - cq \neq 0\,. \tag{5.10}$$

Then[†]

$$(\phi \,|\, \psi)M^{-T}\begin{pmatrix}1\\q\end{pmatrix} = \lambda\big[\phi - (M \star (-q))\psi\big]\,.$$

Proof. We have

$$M^{-T} = \begin{pmatrix} d & -c \\ -b & a\end{pmatrix}\,,$$

and hence

$$\begin{aligned}(\phi;\psi)M^{-T}\begin{pmatrix}1\\q\end{pmatrix} = (\phi;\psi)\begin{pmatrix} d-cq \\ -b+aq\end{pmatrix} = (d-cq)\phi + (-b+aq)\psi = \\ = (d-cq)\big[\phi - \psi\frac{b-aq}{d-cq}\big] = \lambda\big[\phi - \big(M \star (-q)\big)\psi\big]\,.\end{aligned} \tag{5.11}$$

❑

[†]Recall the notion of $W \star \tau$ from short after Definition 2.17.

5.13 Proposition. *Let $\sigma = \sigma_i$ be a singularity of $\mathfrak{h} = (H, \mathfrak{c}, \mathfrak{d})$. Then there exists a number $\tau \in \mathbb{R} \cup \{\infty\}$ such that*

$$\lim_{t \nearrow \sigma} \omega_{\mathfrak{h}}(t)(z) \star \tau = \lim_{t \searrow \sigma} \omega_{\mathfrak{h}}(t)(z) \star \tau \,.$$

Proof. Put $r := \max(E \cap [\sigma_0, \sigma))$ and $s := \min(E \cap (\sigma, \sigma_{n+1}])$. Then $\hat{\mathfrak{h}} :=\circlearrowleft_{\phi_i} \mathfrak{h}_{r \leftrightarrow s}$ is elementary indefinite. For $t \in [r, s] \setminus \{\sigma\}$ we have

$$\omega_{\mathfrak{h}}(t) = \omega_{\mathfrak{h}}(r) \cdot \circlearrowleft_{-\phi_i} \omega_{\hat{\mathfrak{h}}}(t) = \omega_{\mathfrak{h}}(r) N_{-\phi_i} \cdot \omega_{\hat{\mathfrak{h}}}(t) N_{-\phi_i}^{-1} \,,$$

and hence

$$\omega_{\mathfrak{h}}(t) \star (N_{-\phi_i} \star \tau) = \omega_{\mathfrak{h}}(r) N_{-\phi_i} \star \big(\omega_{\hat{\mathfrak{h}}}(t) \star \tau\big) \,.$$

Hence, it is enough to prove the stated assertion for elementary indefinite Hamiltonians.

Let $\mathfrak{h}$ be an elementary indefinite Hamiltonian defined on $[r, s] \setminus \{\sigma\}$. If $\mathfrak{h}$ is of kind (B) or (C), then it is apparent from Proposition 5.5 that

$$\omega_{\mathfrak{h}}(t) \star 0 = 0, \; t \in [r, s] \setminus \{\sigma\} \,.$$

The situation is more delicate if $\mathfrak{h}$ is of type (A). By Corollary 5.6 the matrix function $\omega_{\mathfrak{h}}(t)(z)$, $t \in [r, \sigma)$, is the solution of the initial value problem

$$\frac{\partial}{\partial t} W(t, z) J = z W(t, z) H_-(t), \; t \in [r, \sigma), \quad W(r, z) = I \,, \tag{5.12}$$

and $\omega_{\mathfrak{h}}(s)(z)^{-1} \omega_{\mathfrak{h}}(t)(z)$, $t \in (\sigma, s]$, is the solution of

$$\frac{\partial}{\partial t} W(t, z) J = z W(t, z) H_+(t), \; t \in (\sigma, s], \quad W(s, z) = I \,.$$

For each $\tau \in \mathbb{R} \cup \{\infty\}$ the limits

$$q_\sigma(z) := \lim_{t \nearrow \sigma} \omega_{\mathfrak{h}}(t) \star \tau, \; q(z) := - \lim_{t \searrow \sigma} \big[\omega_{\mathfrak{h}}(s)(z)^{-1} \omega_{\mathfrak{h}}(t)(z) \star \tau\big]$$

exist and do not depend on τ. By [HSW, Theorem 2.1] and [HSW, (6.5),(6.6)], respectively, we have (as functions of $t \in [r, \sigma)$ or $t \in (\sigma, s]$, respectively)

$$\begin{gathered} \omega_{\mathfrak{h}}(t)(\overline{z})^* \begin{pmatrix} 1 \\ -q_\sigma(z) \end{pmatrix} \in L^2(H_-) \,, \\ [\omega_{\mathfrak{h}}(s)(\overline{z})^{-1} \omega_{\mathfrak{h}}(t)(\overline{z})]^* \begin{pmatrix} 1 \\ q(z) \end{pmatrix} \in L^2(H_+) \,, \end{gathered} \tag{5.13}$$

where $H_- = H|_{[r,\sigma)}$ and $H_+ := H|_{(\sigma,s]}$. Moreover, again by [HSW, (6.5)] the function $q(z)$ is identically equal to ∞ only if H_+ is of the form $h(t)\binom{1}{0}(1,0)$ for $t \in (\sigma, s]$. However, since the generalized Hamiltonian $\mathfrak{h}$ is elementary of kind (A) this is impossible, cf. (IV.4.1). Similarly, $q_\sigma(z)$ is not identically equal to ∞.

Put $\tilde{\phi}(z)(t) := \Psi^{ac}((\phi(z); z\phi(z)))_1(t)$ and $\tilde{\psi}(z)(t) := \Psi^{ac}((\psi(z); z\psi(z)))_1(t)$. By Proposition 5.5 we have $(\tilde{\phi}(z)(t) | \tilde{\psi}(z)(t)) = \omega_{\mathfrak{h}}(t)(z)^T$.

Consider the element $\phi(z) - q_\sigma(z)\psi(z) \in \ker(T(\mathfrak{h}) - z)$. The relations (5.12) and (5.13) imply $\tilde{\phi}(z)|_{[r,\sigma)} - q_\sigma(z)\tilde{\psi}(z)|_{[r,\sigma)} \in L^2(H_-)$. By Lemma IV.5.13 this implies that also $\tilde{\phi}(z)|_{(\sigma,s]} - q_\sigma(z)\tilde{\psi}(z)|_{(\sigma,s]} \in L^2(H_+)$.

On the other hand, writing

$$\omega_{\mathfrak{h}}(s)(z) =: \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} ,$$

we obtain from (5.11) that ($t \in (\sigma, s]$)

$$0 \neq [\omega_{\mathfrak{h}}(s)(\overline{z})^{-1}\omega_{\mathfrak{h}}(t)(\overline{z})]^* \begin{pmatrix} 1 \\ q(z) \end{pmatrix} = \omega_{\mathfrak{h}}(t)(\overline{z})^* \omega_{\mathfrak{h}}(s)(\overline{z})^{-*} \begin{pmatrix} 1 \\ q(z) \end{pmatrix} =$$

$$= \big(\tilde{\phi}(z)(t); \tilde{\psi}(z)(t)\big)\omega_{\mathfrak{h}}(s)(z)^{-T} \begin{pmatrix} 1 \\ q(z) \end{pmatrix} =$$

$$= (d(z) - c(z)q(z))\tilde{\phi}(z)(t) + (-b(z) + a(z)q(z))\tilde{\psi}(z)(t).$$

By (5.13) we see that this function belongs to $L^2(H_+)$. Hence, $\tilde{\phi}(z)|_{(\sigma,s]} - q_\sigma(z)\tilde{\psi}(z)|_{(\sigma,s]}$ and $(d(z) - c(z)q(z))\tilde{\phi}(z)|_{(\sigma,s]} + (-b(z) + a(z)q(z))\tilde{\psi}(z)|_{(\sigma,s]}$ both belong to $\ker(T(H_+) - z)$. We know from [HSW] that this space is one-dimensional. Thus, these functions must be collinear. In particular, $\lambda(z) := d(z) - c(z)q(z) \neq 0$, and by Lemma 5.12

$$\tilde{\phi}(z)|_{(\sigma,s]} - q_\sigma(z)\tilde{\psi}(z)|_{(\sigma,s]} = \tilde{\phi}(z)|_{(\sigma,s]} - \big(\omega_{\mathfrak{h}}(s)(z) \star (-q(z))\big)\tilde{\psi}(z)|_{(\sigma,s]} ,$$

i.e. $q_\sigma(z) = \omega_{\mathfrak{h}}(s)(z) \star (-q(z))$. By the definition of q, however,

$$\omega_{\mathfrak{h}}(s)(z) \star (-q(z)) = \lim_{t \searrow \sigma} \omega_{\mathfrak{h}}(t)(z) \star \tau = q(z) .$$

❑

We have by now shown that all the assumptions of Proposition 3.10 are satisfied, and conclude that $\omega_{\mathfrak{h}}$ is a finite maximal chain. ☺

c. Reduction to the regular case.

It is easy to deduce the desired assertion of Theorem 5.1 for singular general Hamiltonians from the already established regular case. To this end, let us note the following: If $\mathfrak{h}$ is a general Hamiltonian and $s \in I_{\text{reg}}$, then

$$\omega_{\mathfrak{h}_{\urcorner s}}(x) = \omega_{\mathfrak{h}}(x),\ x \in (I \cup \{\sigma_0\}) \cap [\sigma_0, s] .$$

In case $x \in (I \cup \{\sigma_0\}) \setminus I_{\text{sing}}$, this is immediate from Remark 3.48, (i). For $x \in I_{\text{sing}}$, it follows from the fact that $\omega_{\mathfrak{h}_{\urcorner s}}(t)$ and $\omega_{\mathfrak{h}}(t)$ are both solutions of the differential equation (5.5) together with the fact that at least one endpoint of the corresponding indivisible interval belongs to I_{reg} and that these two functions agree at this endpoint by the previous case.

Let a singular general Hamiltonian $\mathfrak{h}$ be given. By what we already know, for each $s \in (I \cup \{\sigma_0\}) \setminus I_{\text{sing}}$, the function $\omega_{\mathfrak{h}}|_{(I\cup\{\sigma_0\})\cap[\sigma_0,s]}$ is a finite maximal chain. If $\sup I_{\text{reg}} = \sigma_{n+1}$, it follows from Remark 3.15 that $\omega_{\mathfrak{h}}$ is a maximal chain. Otherwise, put $s := \sup I_{\text{reg}}$, then $\omega_{\mathfrak{h}}|_{(I\cup\{\sigma_0\})\cap[\sigma_0,s]}$ is a finite maximal chain, and $\omega_{\mathfrak{h}}|_{[s,\sigma_{n+1})}$ is a maximal chain which consists of just one indivisible interval. Since $s \in I_{\text{reg}}$, the assumption of Proposition 3.17 is fullfilled, and we conclude that $\omega_{\mathfrak{h}} = \omega_{\mathfrak{h}}|_{(I\cup\{\sigma_0\})\cap[\sigma_0,s]} \uplus \omega_{\mathfrak{h}}|_{[s,\sigma_{n+1})}$ is a maximal chain.

Finally, by Theorem 4.20 and (3.8), we have for any $s \in I_{\text{reg}} \cap (\sigma_n, \sigma_{n+1})$,

$$\text{ind}_-\,\mathfrak{h} = \text{ind}_-\,\mathfrak{h}_{\urcorner s} = \text{ind}_-\,\omega_{\mathfrak{h}_{\urcorner s}}(s) = \text{ind}_-\,\omega_{\mathfrak{h}}(s) = \text{ind}_-\,\omega_{\mathfrak{h}} .$$

d. Compatibilities.

It is important to know that the construction of $\omega_{\mathfrak{h}}$ is compatible with rotation, reversing, and pasting.

5.14 Lemma. *If* $\mathfrak{h}$ *is a general Hamiltonian and* $\alpha \in \mathbb{R}$*, then* $\omega_{\circlearrowleft_\alpha \mathfrak{h}} = \circlearrowleft_\alpha \omega_{\mathfrak{h}}$*.*

Proof. If $t \in (I \cup \{\sigma_0\}) \setminus I_{\text{sing}}$, $t > \sigma_0$ or if $t = \sigma_{n+1}$ in the regular case, then by Definition 5.3, $\omega_{\circlearrowleft_\alpha \mathfrak{h}}(t) = \omega(\mathfrak{B}((\circlearrowleft_\alpha \mathfrak{h})_{\urcorner t}))$. By Remark 3.48 we have $(\circlearrowleft_\alpha \mathfrak{h})_{\urcorner t} = (\circlearrowleft_\alpha \mathfrak{h}_{\urcorner t})$. Hence by Lemma 4.30

$$\omega(\mathfrak{B}((\circlearrowleft_\alpha \mathfrak{h})_{\urcorner t})) = \circlearrowleft_\alpha \omega(\mathfrak{B}(\mathfrak{h}_{\urcorner t})) = \circlearrowleft_\alpha \omega_{\mathfrak{h}}(t).$$

If t is an inner point of an indivisible interval of type ϕ for $\mathfrak{h}$, then t is an inner point of an indivisible interval of type $\phi - \alpha$ for $\circlearrowleft_\alpha \mathfrak{h}$. Therefore, using (2.10) and (3.1) we see from (5.1) and (5.2), that also in this case we have $\omega_{\circlearrowleft_\alpha \mathfrak{h}}(t) = \circlearrowleft_\alpha \omega_{\mathfrak{h}}(t)$. ❑

5.15 Lemma. *If* $\mathfrak{h}$ *is a regular general Hamiltonian, then* $\omega_{\operatorname{rev} \mathfrak{h}} = \operatorname{rev} \omega_{\mathfrak{h}}$*.*

Proof. If $t \in (I \cup \{\sigma_0, \sigma_{n+1}\}) \setminus I_{\text{sing}}$, $-t > \sigma_0$, then by Definition 5.3, $\omega_{\operatorname{rev} \mathfrak{h}}(-t) = \omega(\mathfrak{B}((\operatorname{rev} \mathfrak{h})_{\urcorner -t}))$. By Remark 3.48 we have $(\operatorname{rev} \mathfrak{h})_{\urcorner -t} = \operatorname{rev}(\mathfrak{h}_{\ulcorner t})$ and according to Remark 3.50 and Remark 3.51 we obtain

$$\operatorname{rev}(\mathfrak{h}) = \operatorname{rev}(\mathfrak{h}_{\urcorner t} \uplus \mathfrak{h}_{\ulcorner t}) = \operatorname{rev}(\mathfrak{h}_{\ulcorner t}) \uplus \operatorname{rev}(\mathfrak{h}_{\urcorner t}).$$

From Lemma 4.35 and Lemma 4.30 we obtain

$$\operatorname{rev} \omega(\mathfrak{B}(\mathfrak{h})) = \omega(\mathfrak{B}(\operatorname{rev}(\mathfrak{h}))) = \omega(\mathfrak{B}(\operatorname{rev}(\mathfrak{h}_{\ulcorner t}))) \cdot \omega(\mathfrak{B}(\operatorname{rev}(\mathfrak{h}_{\urcorner t}))) =$$

$$= \omega(\mathfrak{B}(\operatorname{rev}(\mathfrak{h}_{\ulcorner t}))) \cdot \operatorname{rev} \omega(\mathfrak{B}(\mathfrak{h}_{\urcorner t})),$$

and hence by Definition 3.12

$$\omega(\mathfrak{B}(\operatorname{rev}(\mathfrak{h}_{\ulcorner t})))(z) = \operatorname{rev} \omega(\mathfrak{B}(\mathfrak{h}))(z) \cdot (\operatorname{rev} \omega(\mathfrak{B}(\mathfrak{h}_{\urcorner t}))(z))^{-1} =$$

$$= \operatorname{rev} \Big((\omega(\mathfrak{B}(\mathfrak{h}_{\urcorner t}))(z))^{-1} \cdot \omega(\mathfrak{B}(\mathfrak{h}))(z) \Big) = (\operatorname{rev} \omega_{\mathfrak{h}})(-t)(z).$$

If t is an inner point of an indivisible interval of type ϕ for $\mathfrak{h}$, then $-t$ is an inner point of an indivisible interval of type $-\phi$ for $\operatorname{rev} \mathfrak{h}$. Therefore, using the fact that $\operatorname{rev} W_{(l,\phi)} = W_{(l,-\phi)}$ we see from (5.1) and (5.2), that also in the case we have $\omega_{\operatorname{rev} \mathfrak{h}}(t) = \operatorname{rev} \omega_{\mathfrak{h}}(t)$. ❑

In order to consider pasting of general Hamiltonians and linking of chains we need the following statement.

5.16 Proposition. *The following assertions are equivalent:*

(*i*) $\mathfrak{h}$ *starts with an indivisible interval of type* α*.*

(*ii*) *For some* $r \in I$ *with* $(\sigma_0, r) \subseteq I$ *we have* $\omega_{\mathfrak{h}}(t) = W_{(\int_{\sigma_0}^t h, \alpha)}$*,* $t \in [\sigma_0, r)$*.*

(*iii*) $\xi_\alpha \in \mathfrak{K}(\omega_{\mathfrak{h}}(t))$ *for some* $t \in I$*.*

(*iv*) $\xi_\alpha \in \mathfrak{K}(\omega_{\mathfrak{h}}(t))$ *for all* $t \in I$*.*

Proof. If $\mathfrak{h}$ starts with an indivisible interval of type α, then on a certain interval $(\sigma_0, r) \subseteq I$ we have $H(t) = h(t)\xi_\alpha \xi_\alpha^T$. As $t \mapsto W_{(\int_{\sigma_0}^t h,\alpha)}$, $t \in [\sigma_0, r)$, satisfies

$$\frac{\partial}{\partial t} W_{(\int_{\sigma_0}^t h,\alpha)}(z)J = zW_{(\int_{\sigma_0}^t h,\alpha)}(z)H(t), \text{ a.e. on } [\sigma_0, r), \quad W_{(\int_{\sigma_0}^{\sigma_0} h,\alpha)}(z) = I,$$

it follows from (5.5), that $\omega_\mathfrak{h}(t) = W_{(\int_{\sigma_0}^t h,\alpha)}$ on $[\sigma_0, r)$.

If $\omega_\mathfrak{h}(t) = W_{(\int_{\sigma_0}^t h,\alpha)} = \circlearrowleft_{\frac{\pi}{2}-\alpha} W_{(\int_{\sigma_0}^t h,\frac{\pi}{2})}$ for at least one $t \in I$, then due to Corollary 2.9 we have $\xi_\alpha \in \mathfrak{K}(\omega_\mathfrak{h}(t))$.

Since $\omega_\mathfrak{h}$ is a maximal chain, we obtain from Corollary II.5.15 that if $\xi_\alpha \in \mathfrak{K}(\omega_\mathfrak{h}(t))$ for one $t \in I$, then $\xi_\alpha \in \mathfrak{K}(\omega_\mathfrak{h}(t))$ for all $t \in I$.

Assume now that $\xi_\alpha \in \mathfrak{K}(\omega_\mathfrak{h}(t))$ for all $t \in I_{\text{reg}}$. Fix $t \in I_{\text{reg}}$.

If $\operatorname{mul}\Gamma(\circlearrowleft_\alpha \ \mathfrak{h}_{\urcorner t}) = \operatorname{span}\{(\binom{0}{1}; \binom{0}{1})\}$, then due to Theorem IV.8.6, and Lemma IV.4.19 and Section 2.e of [KW/IV], $\circlearrowleft_{\alpha-\frac{\pi}{2}} \ \mathfrak{h}_{\urcorner t}$ must be elemenatry indefinite of kind (B) or (C) or positive and indivisible of type $\frac{\pi}{2}$. In any case of these cases $\mathfrak{h}_{\urcorner t}$ starts with an indivisible interval of type α.

If $\operatorname{mul}\Gamma(\circlearrowleft_\alpha \ \mathfrak{h}_{\urcorner t}) \neq \operatorname{span}\{(\binom{0}{1}; \binom{0}{1})\}$, then $\xi_0 \in \mathfrak{K}(\omega_{\circlearrowleft_\alpha \mathfrak{h}}(t))$ and we can apply Theorem 4.19.

Due to Theorem 0.5.3 $(S_1 - a)^{-1}\Xi^{-1}(\xi_0) = 0$ for all $a \in \mathbb{C}$, where $S_1 = \ker((\pi_{l,1} \times \pi_r) \circ \Gamma(\omega_{\circlearrowleft_\alpha \mathfrak{h}_{\urcorner t}}))$. Hence, $\Xi^{-1}(\xi_0) \in \operatorname{mul} S_1$ or equivalently $((0; \Xi^{-1}(\xi_0)); (\xi_{\frac{\pi}{2}}; 0)) \in \Gamma(\omega_{\circlearrowleft_\alpha \mathfrak{h}_{\urcorner t}})$. As $\xi_{\frac{\pi}{2}} = J\xi_0$ we obtain from Lemma 3.37 that $\circlearrowleft_\alpha \ \mathfrak{h}_{\urcorner t}$ starts with an indivisible interval of type 0, and, therefore, $\mathfrak{h}_{\urcorner t}$ starts with an indivisible interval of type α. ❑

5.17 Remark. Let us state explicitly one consequence of the previous proof. We saw that, if $\operatorname{mul}\Gamma(\circlearrowleft_\alpha \ \mathfrak{h}_{\urcorner t}) = \operatorname{span}\{(\binom{0}{1}; \binom{0}{1})\}$, then $\mathfrak{h}_{\urcorner t}$ starts with an indivisible interval of type α. //

5.18 Lemma. *Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be general Hamiltonians such that $\mathfrak{h}_1$ is regular. Then condition (¬paste) from Definition 3.49 fails for $\mathfrak{h}_1$ and $\mathfrak{h}_2$ if and only condition (¬paste) from Proposition 3.17 fails for the correspooding chains $\omega_{\mathfrak{h}_1}$ and $\omega_{\mathfrak{h}_2}$. In this case we have*

$$\omega_{\mathfrak{h}_1 \uplus \mathfrak{h}_2} = \omega_{\mathfrak{h}_1} \uplus \omega_{\mathfrak{h}_2} .$$

Proof. By Proposition 5.16 $\mathfrak{h}_2$ starts with an indivisible interval of type α if and only if $\omega_{\mathfrak{h}_2}$ does, and $\operatorname{rev}\mathfrak{h}_1$ starts with an indivisible interval of type $-\alpha$ if and only if $\operatorname{rev}\omega_{\mathfrak{h}_1} = \omega_{\operatorname{rev}\mathfrak{h}_1}$ does, see Lemma 5.15.Therefore, $\mathfrak{h}_1$ ends with an indivisible interval of type α if and only if $\omega_{\mathfrak{h}_1}$ does. Hence, the conditions (¬paste) from Definition 3.49 and from Proposition 3.17 correspond to each other. Finally, $\omega_{\mathfrak{h}_1 \uplus \mathfrak{h}_2} = \omega_{\mathfrak{h}_1} \uplus \omega_{\mathfrak{h}_2}$ follows easily from Definition 5.3 using Remark 4.36. ❑

e. The Weyl coefficient as Q-function.

If $\mathfrak{h}$ is a singular positive definite Hamiltonian, $\mathfrak{h} = H : (\sigma_0, \sigma_1) \to \mathbb{R}^{2\times 2}$ which does not consist of just one indivisible interval, it was proved in [HSW, Theorem 4.3] that the Weyl coefficient q of H is a Q-function of $S(\mathfrak{h})$. Actually, the selfadjoint extension of $S(\mathfrak{h})$ which gives rise to q as a Q-function is the one determined by the boundary condition $\pi_1 f(\sigma_0) = 0$. Using the concept of Titchmarsh-Weyl subspaces introduced in §4.b we are able to establish the exact analogue of this result for singular general Hamiltonians.

First of all note that for a singular general Hamiltonian $\mathfrak{h}$ the boundary triplet $\mathfrak{B}(\mathfrak{h})$ has defect 1 and satisfies (E), see Theorem IV.8.7. Hence, $v(\mathfrak{B}(\mathfrak{h}))(z)$ is well-defined.

Moreover, by the same theorem, we have $\operatorname{mul} S(\mathfrak{h}) = \{0\}$. Therefore, except in the case $\operatorname{mul} \Gamma(\mathfrak{h}) = \operatorname{span}\{(\binom{0}{1}; 0)\}$ we may apply Lemma 4.14 to $\mathfrak{B}(\mathfrak{h})$ and see that $\mathring{A} := \ker(\pi_{l,1} \circ \Gamma(\mathfrak{h}))$ has non-empty resolvent set.

5.19 Proposition. *Let $\mathfrak{h}$ be a singular general Hamiltonian, and assume that $S(\mathfrak{h})$ has defect index $(1,1)$, i.e. that $\mathfrak{h}$ is not a positive definite Hamiltonian which consists of just one indivisible interval.*

With the notation from Lemma 4.14 applied to $\mathfrak{B}(\mathfrak{h})$ we have

$$q_{\mathfrak{h}}(z) = q(z)\,.$$

Thus the Titchmarsh-Weyl coefficient of $\mathfrak{h}$ is a Q-function of $(S(\mathfrak{h}), \mathring{A})$.

Proof. Let $s \in I_{\text{reg}}$, then there exists at most one number $\alpha_1 \in [0, \pi)$ such that $\operatorname{mul} \Gamma \circlearrowleft_{\alpha_1} \mathfrak{h}_{\upharpoonleft s} = \operatorname{span}\big\{(\binom{0}{1}; \binom{0}{1})\big\}$. Since $q_{\circlearrowleft_\alpha \mathfrak{h}} = \circlearrowleft_\alpha q_{\mathfrak{h}}$, cf. Lemma 3.13, and since $v(\mathfrak{B}(\circlearrowleft_\alpha \mathfrak{h})) = N_\alpha v(\mathfrak{B}(\mathfrak{h}))$, cf. Corollary 4.16, we may thus assume without loss of generality that Theorem 4.24 is applicable.

First we consider the case that there exists a largest number $s \in I_{\text{reg}} \cap (\sigma_n, \sigma_{n+1})$. Since $\mathfrak{h}_{\upharpoonright s}$ is positive and consists of just one indivisible interval of some type ϕ we see from Remark 4.34 that

$$v(\mathfrak{B}(\mathfrak{h}_{\upharpoonright s}))(z) = \operatorname{span}\{\xi_\phi\}.$$

Hence, by (4.26)

$$q(z) = \omega_{\mathfrak{h}}(s) \star \cot \phi.$$

By [HSW, Example 2.2] we know that $\cot \phi$ is the Titchmarsh-Weyl coefficient of $\mathfrak{h}_{\upharpoonright s}$. Hence, $q(z)$ is the Titchmarsh-Weyl coefficient of $\mathfrak{h}$.

If there is no largest number $s \in I_{\text{reg}} \cap (\sigma_n, \sigma_{n+1})$, there is a strictly increasing sequence s_k, $k \in \mathbb{N} \cup \{0\}$, in $I_{\text{reg}} \cap (\sigma_n, \sigma_{n+1})$. As $\mathfrak{B}(\mathfrak{h}_{\upharpoonleft s_k}) = \mathfrak{B}(\mathfrak{h}_{\upharpoonleft s}) \uplus \mathfrak{B}(\mathfrak{h}_{s_0 \leftrightarrow s_k})$ we have $\operatorname{mul} \Gamma_{\upharpoonleft s_k} = \{0\}$, $k \in \mathbb{N}$, see Proposition IV.6.2.

Thus we can apply Theorem 4.24 with $\mathfrak{B}_1 = \mathfrak{B}(\mathfrak{h}_{\upharpoonleft s_k})$ and $\mathfrak{B}_2 = \mathfrak{B}(\mathfrak{h}_{\upharpoonright s_k})$, $k \in \mathbb{N}$. By (4.26)

$$q(z) = \omega_{\mathfrak{h}}(s_k) \star \frac{\nu_1^k(z)}{\nu_2^k(z)}, \tag{5.14}$$

where $(\nu_1^k(z)\ \nu_2^k(z))^T$ is any non-zero element from $v(\mathfrak{B}(\mathfrak{h}_{\upharpoonright s_k}))(z)$. By Lemma 4.14 either $\frac{\nu_1^k(z)}{\nu_2^k(z)} \equiv \infty$ or $\frac{\nu_1^k(z)}{\nu_2^k(z)}$ is a Q-function in the Hilbert space $\mathcal{P}(\mathfrak{h}_{\upharpoonright s_k})$.

In any case $\frac{\nu_1^k(z)}{\nu_2^k(z)}$ is a Nevanlinna function. Hence, letting k tend to ∞ in (5.14) we obtain $q(z) = q_{\mathfrak{h}}(z)$. ❑

6 The Fourier transform

Let H be a singular positive definite Hamiltonian which does not start with an indivisible interval of type zero and is not just one indivisible interval with infinite length. Denote by μ the measure in the Herglotz–integral representation of its Weyl–coefficient q_H. Then there exists an isomorphism, the Fourier–transform associated with H, of $L^2(H)$ onto $L^2(\mu)$. Thereby, the preimage of the multiplication operator in $L^2(\mu)$ is a certain selfadjoint extension of $S(H)$.

In this section we will establish the analogous result for singular general Hamiltonians $\mathfrak{h}$. The space $\mathfrak{h}$ is thereby naturally substituted by $\mathfrak{P}(\mathfrak{h})$. On the other side, the measure μ is substituted by a certain distribution ρ associated to $q_\mathfrak{h}$, and the space $L^2(\mu)$ by the Pontryagin space $\Pi(\rho)$, cf. [KW/II, §2, §3].

Before we turn to the definition and investigation of the Fourier transform, we need a preparatory result.

6.1 Proposition. *Let $\mathfrak{h}$ be a general singular Hamiltonian which is not positive definite and consist of just one indivisible interval. Then $S(\mathfrak{h})$ is densely defined if and only if $\mathfrak{h}$ does not start with an indivisible interval.*

Moreover, the following assertions are equivalent

- *(i) $\mathfrak{h}$ does not start with an indivisible interval of type 0.*
- *(ii) The selfadjoint extension $\mathring{A} := \ker(\pi_{l,1} \circ \Gamma(\mathfrak{h}))$ of $S(\mathfrak{h})$ is an operator.*
- *(iii) For all $t \in I$ the projection π_2, which assignes to any function in $\mathfrak{K}(\omega_\mathfrak{h}(t))$ the second entry of this function, is injective.*
- *(iv) ∞ is regular for the corresponding Weyl-coefficient $q_\mathfrak{h}$, i.e. we have $\lim_{y\to+\infty} \frac{q_\mathfrak{h}(iy)}{y} = 0$.*

Proof. First note that, under our assumption on $\mathfrak{h}$, we have $\operatorname{mul}\Gamma(\mathfrak{h}) = \{0\}$.

The relation $S(\mathfrak{h})$ is densely defined if and only if $S^*(\mathfrak{h}) = T(\mathfrak{h})$ is an operator. Since $\operatorname{mul} S(\mathfrak{h}) = \{0\}$, cf. Theorem IV.8.7, by Lemma 3.37 this happens if and only if $\mathfrak{h}$ does not start with an indivisible interval.

The multivalued part of $\mathring{A}$ consists of all elements $g \in \operatorname{mul} T(\mathfrak{h})$ such that $\pi_{l,1}(\Gamma(\mathfrak{h})(0;g)) = 0$. Thus, again by $\operatorname{mul} S(\mathfrak{h}) = \{0\}$ and Lemma 3.37, the assertion that $\operatorname{mul}\mathring{A} = \{0\}$ is equivalent to the fact that $\mathfrak{h}$ does not start with an indivisible interval of type 0.

Finally, by Proposition 5.16 this is equivalent to $\xi_0 \notin \mathfrak{K}(\omega_\mathfrak{h}(t))$ for all $t \in I$. However, by Corollary I.9.7 together with Proposition I.8.3, this is equivalent to $\ker\pi_2 = \{0\}$. By Lemma I.8.6 this means that $\mathfrak{K}(\omega_\mathfrak{h}(t)) = \mathfrak{K}_-(\omega_\mathfrak{h}(t))$ and by Theorem II.5.7 nothing else, but the fact that ∞ is regular for $q_\mathfrak{h}$. ❑

6.2 Definition. Let $\mathfrak{h}$ be singular generalized Hamiltonian. We say that an element $f \in \mathcal{P}(\mathfrak{h})$ is right finite, if for some $r \in I_{\text{reg}}$ we have $f \in \mathcal{P}(\mathfrak{h}_{\urcorner r})$, when we consider $\mathcal{P}(\mathfrak{h}_{\urcorner r})$ as subspace of $\mathcal{P}(\mathfrak{h})$, cf. Lemma 3.44. The set of all right finite elements of $\mathfrak{P}(\mathfrak{h})$ will be denoted by $\mathcal{P}^{\text{fin}}(\mathfrak{h})$, i.e.

$$\mathcal{P}^{\text{fin}}(\mathfrak{h}) := \bigcup_{r\in I_{\text{reg}}} \mathcal{P}(\mathfrak{h}_{\urcorner r}) .$$

//

6.3 Lemma. *The space $\mathcal{P}^{\text{fin}}(\mathfrak{h})$ of right finite elements is a dense subspace of $\mathcal{P}(\mathfrak{h})$.*

Proof. As the last interval (σ_n, σ_{n+1}) of I contains regular points we have

$$\mathcal{P}^{\text{fin}}(\mathfrak{h}) = \bigcup_{r \in I_{\text{reg}} \cap (\sigma_n, \sigma_{n+1})} \mathcal{P}(\mathfrak{h}_{\urcorner r}).$$

If $I_{\text{reg}} \cap (\sigma_n, \sigma_{n+1})$ contains a maximal element, say r, then $\mathcal{P}(\mathfrak{h}_{\ulcorner r}) = L^2(H)$ with some purely indivisible Hamiltonian H. Hence $\mathcal{P}(\mathfrak{h}_{\ulcorner r}) = \{0\}$ and $\mathcal{P}(\mathfrak{h}_{\urcorner r}) = \mathcal{P}^{\text{fin}}(\mathfrak{h}) = \mathcal{P}(\mathfrak{h})$.

If $I_{\text{reg}} \cap (\sigma_n, \sigma_{n+1})$ does not possess a maximal element, then for each fixed $r \in I_{\text{reg}} \cap (\sigma_n, \sigma_{n+1})$ the Hamiltonian $\mathfrak{h}_{\ulcorner r}$ is not purely indivisible. Still,

$$\mathcal{P}(\mathfrak{h}_{\ulcorner r}) = L^2(H|_{[r,+\infty]}) = \overline{\bigcup_{r<s \in I_{\text{reg}}} L^2(H|_{[r,s]})}\,.$$

Therefore, for our fixed r

$$\overline{\mathcal{P}^{\text{fin}}(\mathfrak{h})} = \overline{\bigcup_{r<s \in I_{\text{reg}}} \mathcal{P}(\mathfrak{h}_{\urcorner s})} = \mathcal{P}(\mathfrak{h}_{\urcorner r}) \oplus \overline{\bigcup_{r<s \in I_{\text{reg}}} \mathcal{P}(\mathfrak{h}_{r \leftrightarrow s})} = \mathcal{P}(\mathfrak{h})\,.$$

❑

Assume that $\mathfrak{h}$ is a singular generalized Hamiltonian which does not start with an indivisible interval of type 0. Due to Remark 5.17 we have $\operatorname{mul}\Gamma(\mathfrak{h}_{\urcorner t}) \neq \operatorname{span}\{(\binom{0}{1}; \binom{0}{1})\}$, $t \in I_{\text{reg}}$. Therefore, we Theorem 4.19 is applicable, and we have the isometric isomorphism $\Xi_t : \mathcal{P}(\mathfrak{h}_{\urcorner t}) \to \mathfrak{K}(\omega(\mathfrak{B}(\mathfrak{h}_{\urcorner t})))$ defined by (4.12).

Moreover, by Proposition 6.1, the projection π_2 onto the second entry is injective. This implies that condition (2.23) is satisfied, cf. Lemma I.8.6. Hence, $\pi_2 : \mathfrak{K}(\omega(\mathfrak{B}(\mathfrak{h}_{\urcorner t}))) \to \mathfrak{P}(E_t)$ is an isometric isomorphism, where $E_t = w_{21} - iw_{22}$ and $\omega(\mathfrak{B}(\mathfrak{h}_{\urcorner t})) = (w_{i,j})_{i,j=1,2}$. Thus also the map

$$\Theta_t : \begin{cases} \mathcal{P}(\mathfrak{h}_{\urcorner t}) & \to & \mathfrak{P}(E_t) \\ f & \mapsto & \pi_2(\Xi_t f) \end{cases}$$

is an isometric isomorphism. For $s, t \in I_{\text{reg}}, s < t$, we obtain from Corollary 4.23 that $\Theta_t|_{\mathcal{P}(\mathfrak{h}_{\urcorner s})} = \Theta_s$. Therefore a map $\Theta : \mathcal{P}^{\text{fin}}(\mathfrak{h}) \to \bigcup_{t \in I_{\text{reg}}} \mathfrak{P}(E_t)$ is well-defined by $\Theta f := \Theta_t f$, $f \in \mathfrak{P}(\mathfrak{h}_{\urcorner t})$. Clearly, Θ is surjective.

Let ϱ be the distribution in the representation Corollary II.3.5 of the Weyl-coefficient $q_{\mathfrak{h}}$, so that we have

$$q_{\mathfrak{h}}(z) = c + \varrho\Big(\big(\frac{1}{t-z} - \frac{t - \operatorname{Re} z_0}{|t-z_0|^2}\big)|t-z_0|^2\Big)\,. \tag{6.1}$$

with some $z_0 \in \rho(\mathring{A}) \setminus \mathbb{R}$. Note that, by Proposition 5.19, z_0 belongs to the domain of holomorphy of $q_{\mathfrak{h}}$. Finally, let $\Pi(\rho)$ denote the Pontryagin space generated by ρ, cf. Proposition II.3.1.

6.4 Theorem. *Let $\mathfrak{h}$ be a general singular Hamiltonian which is not positive definite and consist of just one indivisible interval. Assume that $\mathfrak{h}$ does not start with an indivisible interval of type 0. Then $S(\mathfrak{h})$ is completely nonselfadjoint symmetric operator with defect index $(1,1)$. The map*

$$\Lambda : f \mapsto \Theta(f) \cdot (. - z_0)$$

is an isometry of the dense subspace $\mathcal{P}^{\text{fin}}(\mathfrak{h})$ of $\mathcal{P}(\mathfrak{h})$ onto a certain dense subset of $\Pi(\varrho)$. We have

$$\Lambda(\mathring{A}f)(z) = z\Theta(f)(z) \cdot (z - z_0),\ \ f \in \mathcal{P}^{\text{fin}}(\mathfrak{h}) \cap \operatorname{dom} \mathring{A}\,.$$

Proof. From Theorem IV.8.7 we know that $S(\mathfrak{h})$ is a symmetric operator with defect index $(1, 1)$.

Let $t \in I_{\text{reg}} \cap (\sigma_n, \sigma_{n+1})$. By Remark 5.17 we can apply Theorem 4.19 to the boundary triplet $\mathfrak{B}_1 = \mathfrak{B}(\mathfrak{h}_{\urcorner t})$ and see that $\omega(\mathfrak{B}_1)$ is a generalized u-resolvent matrix, where $u \in (\mathcal{P}_1)_-$ is given by (4.11).

Now Ξ_t maps $\psi(\bar{w})$ to $H_{\omega(\mathfrak{B}_1)}(w, .)\binom{0}{1}$ and, hence, Θ_t maps $\psi(\bar{w})$ to $K_{E_t}(w, .)$ (see (2.22)).

The continuation $(\Theta_t)_-$ of Θ_t to $(\mathcal{P}_1)_-$ maps u to a function in $\mathfrak{P}(E_t)_-$. From Proposition I.10.2 we know that $\mathfrak{P}(E_t)_-$ can be identified with $\operatorname{Assoc}\mathfrak{P}(E_t)$ by the relation (I.10.1). Hereby,

$$(\Theta_t)_-(u)(w) = [(\Theta_t)_-(u), \begin{pmatrix} K_{E_t}(w, .) \\ \bar{w}K_{E_t}(w, .) \end{pmatrix}]_\pm =$$
$$= [u, \begin{pmatrix} \psi(\bar{w}) \\ \bar{w}\psi(\bar{w}) \end{pmatrix}]_\pm = (\pi_{l,2} \circ P \circ \Gamma_1)(\psi(\bar{w})) = 1.$$

Therefore, $(\Theta_t)_-(u)$ is the constant one-function on $\mathbb{C}$.

The selfadjoint extension

$$\mathring{A} \supseteq S_1(\mathfrak{h}_{\urcorner t}) = \ker\big((\pi_{l,1} \times \pi_r) \circ \Gamma(\mathfrak{h}_{\urcorner t})\big)$$

acting in $\mathcal{P}(\mathfrak{h})$ satisfies $\rho(\mathring{A}) \neq \emptyset$, because of Lemma 4.14 which is applicable since $\operatorname{mul}\Gamma(\mathfrak{h}) = \{0\}$. Let us choose $z_0 \in \rho(\mathring{A})$ such that $q_\mathfrak{h}$ is holomorphic there.

By Proposition 6.1 $\mathring{A}$ is in fact an operator. Therefore, we can apply Proposition II.4.4 with $\tilde{A} = \mathring{A}$ and see that $\mathring{A}$ is $R^-_{z_0}u$-minimal, or equivalently

$$\operatorname{cls}\{R^-_z u : z \in \rho(\mathring{A})\} = \mathcal{P}(\mathfrak{h}),$$

where R^-_z denotes the extension of the resolvent as defined in [KW/0, p.290], where now $\mathfrak{P}$ is $\mathcal{P}_1$ and $\tilde{\mathfrak{P}}$ is $\mathcal{P}(\mathfrak{h})$ and where $A = \mathring{A}$

By Theorem 4.24 we have $R^-_z u = -f_z$, where f_z is the defect family from Lemma 4.14. Here we can be sure that actually $\xi = -1$ because of Remark 4.25 since for $\Gamma_2 = \Gamma(\mathfrak{h}_{\ulcorner t})$ either $\operatorname{mul}\Gamma_2 \neq \{(0; 0)\}$ or $\mathcal{P}(\mathfrak{h}_{\ulcorner t}) \neq \{0\}$ is a Hilbert space.

Thus we showed that $S(\mathfrak{h})$ is completely non-selfadjoint. Since $q_\mathfrak{h}$ is the Q-function of $(S(\mathfrak{h}), \mathring{A})$ corresponding to the defect family f_z, we see from Proposition II.3.4 that there is an isometric isomorphism $\Lambda : \mathcal{P}(\mathfrak{h}) \to \Pi(\varrho)$, where $\Pi(\varrho)$ is the model space constructed in [KW/II, §3] from the distribution in (6.1).

In particular, $\Lambda \boxtimes \Lambda(\mathring{A})$ is just the multiplication operator by z. Moreover, by Corollary II.6.1 it satisfies

$$\Lambda(f) = \Theta_t(f) \cdot (. - z_0)$$

for any $f \in \mathcal{P}(\mathfrak{h}_{\urcorner t})$, where $t \in I_{\text{reg}} \cap (\sigma_n, \sigma_{n+1})$. As $\mathcal{P}^{\text{fin}}(\mathfrak{h})$ coincides with the union of all $\mathcal{P}(\mathfrak{h}_{\urcorner t})$, $I_{\text{reg}} \cap (\sigma_n, \sigma_{n+1})$, we see that

$$f \mapsto \Theta(f) \cdot (. - z_0)$$

is an isometric mapping from the dense subseteq $\mathcal{P}^{\mathrm{fin}}(\mathfrak{h})$ of $\mathcal{P}(\mathfrak{h})$ onto a certain dense subseteq of $\Pi(\varrho)$. ❑

As a consequence, we obtain one property of $S(\mathfrak{h})$ which was missing in Theorem IV.8.7, cf. Remark IV.8.8.

6.5 Corollary. *Let $\mathfrak{h}$ be a singular generalized Hamiltonian, which is not just one positive definite indivisible interval. Then $S(\mathfrak{h})$ is completely non-selfadjoint.*

Proof. Apply Theorem 6.4 to the general Hamiltonian $\circlearrowleft_\alpha \mathfrak{h}$ with an appropriate choice of α. ❑

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