Existence of zerofree functions N-associated to a de Branges Pontryagin space

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Abstract

In the theory of de Branges Hilbert spaces of entire functions, so-called 'functions associated to a space' play an important role. In the present paper we deal with a generalization of this notion in two directions, namely with functions N-associated ($N \in \mathbb{Z}$) to a de Branges Pontryagin space.

Let a de Branges Pontryagin space $\mathcal{P}$ and $N \in \mathbb{Z}$ be given. Our aim is to characterize whether there exists a real and zerofree function N-associated to $\mathcal{P}$ in terms of Kreĭn's Q-function associated with the multiplication operator in $\mathcal{P}$. The conditions which appear in this characterization involve the asymptotic distribution of the poles of the Q-function plus a summability condition.

Although this question may seem rather abstract, its answer has a variety of nontrivial consequences. We use it to answer two questions arising in the theory of general (indefinite) canonical systems. Namely, to characterize whether a given generalized Nevanlinna function is the intermediate Weyl-coefficient of some system in terms of its poles and residues, and to characterize whether a given general Hamiltonian ends with a specified number of indivisible intervals in terms of the Weyl-coefficient associated to the system. In addition, we present some applications, e.g., dealing with admissible majorants in de Branges spaces or the continuation problem for hermitian indefinite functions.

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1 Introduction

An entire function $E(z)$ which has no zeros on the real line is said to belong to the Hermite-Biehler class $\mathcal{HB}_0^*$, if it satisfies $|E(\overline{z})| < |E(z)|$ throughout the open upper half-plane $\mathbb{C}^+$. Equivalently, one could require that the kernel K_E which is defined as (for $z = \overline{w}$ this formula has to be interpreted appropriately as a derivative)

$$K_E(w,z) := i\frac{E(z)\overline{E(w)} - \overline{E(\overline{z})}E(\overline{w})}{2(z-\overline{w})}, \quad w,z \in \mathbb{C}\,,$$

does not vanish identically and is positive semidefinite. By this we mean that for each choice of $n \in \mathbb{N}$ and $w_1, \ldots, w_n \in \mathbb{C}$, the quadratic form

$$Q_E(\xi_1, \ldots, \xi_n) := \sum_{i,j=1}^{n} K_E(w_i, w_j)\xi_i\overline{\xi_j} \tag{1.1}$$

is positive semidefinite.

For each function $E \in \mathcal{HB}_0^*$, the kernel K_E generates a reproducing kernel Hilbert space $\mathcal{H}(E)$ whose elements are entire functions. Spaces of this kind were first introduced by L.de Branges, who also developed their deep and rich structure theory, cf. [8], [9]. Therefore, one usually refers to $\mathcal{H}(E)$ as the de Branges space associated with the function E.

A generalization of this concept to an indefinite (Pontryagin space) setting was introduced in [21]. An entire function E is said to belong to the indefinite Hermite-Biehler class $\mathcal{HB}_{<\infty}^*$, if it has no zeros on the real line, no conjugate pairs of nonreal zeros, and if the kernel K_E has a finite number of negative squares. This means that the numbers of negative squares of the quadratic forms (1.1) are bounded independently of n and $w_1, \ldots, w_n$. If $E \in \mathcal{HB}_{<\infty}^*$, we denote the maximal number of negative squares of forms (1.1) by $\operatorname{ind}_- E$. Via the kernel K_E each function $E \in \mathcal{HB}_{<\infty}^*$ generates a reproducing kernel Pontryagin space $\mathcal{P}(E)$ whose elements are entire functions. We will again refer to $\mathcal{P}(E)$ as the de Branges space generated by E.

In the theory of de Branges spaces $\mathcal{P}(E)$, an object of prime importance is the operator $\mathcal{S}_E$ of multiplication by the independent variable in $\mathcal{P}(E)$. This is the linear operator which acts as $F(z) \mapsto zF(z)$, and whose domain consists of all functions $F \in \mathcal{P}(E)$ such that $zF(z)$ belongs to $\mathcal{P}(E)$. It is a closed and symmetric (not necessarily densely defined) operator with defect index $(1,1)$ whose set of regular points equals all of $\mathbb{C}$.

With a space $\mathcal{P}(E)$ we associate a chain of linear spaces. Namely, for $N \in \mathbb{Z}$, we define the set $\operatorname{Assoc}_N \mathcal{P}(E)$ of functions N-associated to the space $\mathcal{P}(E)$ as

$$\operatorname{Assoc}_N \mathcal{P}(E) := \begin{cases} \mathcal{P}(E) + z\mathcal{P}(E) + \ldots + z^N\mathcal{P}(E), & N \geq 0 \\ \operatorname{dom} \mathcal{S}_E^{|N|} \qquad\qquad\qquad\qquad\quad , & N < 0 \end{cases}$$

Obviously,

$$\begin{array}{c} \cdots \subseteq \operatorname{Assoc}_{-1} \mathcal{P}(E) \subseteq \operatorname{Assoc}_0 \mathcal{P}(E) \subseteq \operatorname{Assoc}_1 \mathcal{P}(E) \subseteq \operatorname{Assoc}_2 \mathcal{P}(E) \subseteq \cdots \\ \qquad\qquad\qquad\qquad\quad\; \| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \\ \qquad\qquad\qquad\qquad\; \mathcal{P}(E) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \end{array}$$

This chain is, in its spirit but not in all details, similar to the chain of rigged spaces associated to a selfadjoint operator, see, e.g., [5]. In the Hilbert space case, the set $\operatorname{Assoc}_1 \mathcal{H}(E)$ already played an important role in [9]; elements of $\operatorname{Assoc}_1 \mathcal{H}(E)$ were called 'functions associated to the space $\mathcal{H}(E)$'. For positive values of N larger than 1, and still in the Hilbert space case, the set $\operatorname{Assoc}_N \mathcal{H}(E)$ was investigated in [36].

An entire function U is called real, if it takes real values along the real axis. It is called zerofree, if $U(z) \neq 0$ for all $z \in \mathbb{C}$. Our aim in this paper is

(1) to characterize whether or not $\operatorname{Assoc}_N \mathcal{P}(E)$ contains a real and zerofree function in terms of the poles and residues of the function ($\varphi \in \mathbb{R}$)

$$q_\varphi(z) := i\frac{e^{i\varphi}E(z) + e^{-i\varphi}\overline{E(\overline{z})}}{e^{i\varphi}E(z) - e^{-i\varphi}\overline{E(\overline{z})}},$$

cf. Theorem 3.2,

and to answer two questions arising in the theory of general (indefinite) canonical systems, namely

(2) to characterize intermediate Weyl-coefficients, cf. Theorem 4.8,

(3) to characterize whether a given general Hamiltonian ends with a specified number of indivisible intervals, cf. Theorem 5.4.

The decisive conditions in these respects are some requirements on the asymptotic distribution of the poles of q_φ and a summability condition, see 3.1. The proof of Theorem 3.2 is an elegant combination of Pontryagin space techniques and methods of classical complex analysis. In the proof of Theorem 4.8 we employ the part '$N \geq 1$' of Theorem 3.2; to establish Theorem 5.4 the part '$N \leq 1$' is used.

The present work can be viewed as a complete extension of [9, Problem 70] for real and zerofree functions after the previous work [40] and [36]. In [40] we have dealt with the case that $E \in \mathcal{H}B_0^*$ and $N = 1$; the main achievement made there was that we replaced the 'bounded type condition' in [9, Problem 70] by a more easily accessible asymptotic condition. In [36] we have dealt with the case that $E \in \mathcal{H}B_0^*$ and $N > 0$; the main achievement there was that we have related the fact that $\operatorname{Assoc}_N \mathcal{H}(E)$ contains a real and zerofree function with the theory of general (indefinite) canonical systems, and that we extended the replacement for bounded type given in [40] to the case '$N > 1$'. In the present work we combine the methods of [9, Problem 70], [40], and [36], with some results on the geometry of Pontryagin spaces and some de Branges space theory.

Needless to say, results like Theorem 3.2 have a history besides the already mentioned work. The closest example for a parent result is probably [19, 11.11°], which deals with strings in the sense of M.G.Kreĭn. Another related problem is the (definite or indefinite) power moment problem, cf. [1]. Also in this context summability conditions appear and play a similar role as in the present considerations, cf. [14, Theorem 3.1.4], [32, Proposition 4.4]. In some recent work, results like Theorem 3.2 are used in the context of M.G.Kreĭn's theory of entire operators, cf. [39].

Although the question of existence of real and zerofree elements in $\operatorname{Assoc}_N \mathcal{P}(E)$ may seem rather abstract, its answer has a variety of consequences for various indefinite (and even positive definite) problems. We will present some more applications of Theorem 3.2, Theorem 4.8, and Theorem 5.4, namely the solution of a particular inverse problem for Hamiltonians and their Weyl-coefficients (Corollary 4.10), a characterization that $1 \in \operatorname{Assoc}_N \mathcal{P}(E)$ in the spirit of [9, Theorem 27] (Proposition 6.1), a characterization of existence of minimal (positive) admissible majorants in a de Branges (Hilbert) space $\mathcal{H}(E)$ (Proposition 6.8), and the computation of a characteristic number appearing in the extension problem for a hermitian indefinite function (Proposition 6.10, Proposition 6.11). A striking application of Theorem 5.4 is found in the, classical and positive definite, inverse spectral problem for Kreĭn-strings. This result will be presented in forthcoming work.

The present paper closes with an appendix, where we investigate the polynomial asymptotics of functions belonging to a de Branges space, cf. Proposition A.1. This result is interesting on its own right. Its full strength is not necessary for the proofs of our main theorems; we can do with the weaker (and simpler) Lemma 3.7. However, it points out some interesting peculiarities of some parts of the proof of Theorem 3.2, cf. Remark A.3.

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2 Some geometric preliminaries

a. The role played by associated functions.

For later use, we need to recall some facts concerning $\operatorname{Assoc}_1 \mathcal{P}(E)$, see [21].

Associated functions describe the extensions with nonempty resolvent set of the operator $\mathcal{S}_E$ via their resolvent families as follows: For an entire function G and a point $w \in \mathbb{C}$ with $G(w) \neq 0$, define the difference quotient operator

$$R_{G;w} : F(z) \mapsto \frac{F(z) - \frac{F(w)}{G(w)} G(z)}{z - w} \tag{2.1}$$

which acts on the set of all entire functions. If $G \in \operatorname{Assoc}_1 \mathcal{P}(E) \setminus \{0\}$, then there exists a closed linear relation $T_G \subseteq \mathcal{P}(E) \times \mathcal{P}(E)$ which extends $\mathcal{S}_E$, such that $\rho(T_G) = \{w \in \mathbb{C} : G(w) \neq 0\}$ and $(T_G - w)^{-1} = R_{G;w}$, $w \in \rho(T_G)$. Conversely, if T is an extension of $\mathcal{S}_E$ with $\rho(T_G) \neq \emptyset$, then there exists a function $G \in \operatorname{Assoc}_1 \mathcal{P}(E)$ such that $T = T_G$. The relation T_G is (the graph of) an operator, if and only if $G \notin \mathcal{P}(E)$.

Among all the extensions of $\mathcal{S}_E$, selfadjoint ones are of particular interest. In the above correspondence, these are induced by the functions

$$S_\varphi(z) := \frac{1}{2i}\big(e^{i\varphi} E(z) - e^{-i\varphi}\overline{E(\overline{z})}\big) \in \operatorname{Assoc}_1 \mathcal{P}(E), \quad \varphi \in \mathbb{R}\,. \tag{2.2}$$

Thereby, $T_{S_\varphi} = T_{S_\psi}$ if and only if $\varphi \equiv \psi \mod \pi$. Moreover, Kreĭn's Q-function q_φ of the symmetry $\mathcal{S}_E$ produced by the selfadjoint extension T_{S_φ} is equal to

$$q_\varphi = \frac{S_{\varphi + \frac{\pi}{2}}}{S_\varphi}\,.$$

We will, throughout this paper, use the notation $\mathcal{A}_\varphi := T_{S_\varphi}$.

Let us note that, since E is assumed to have no real zeros and no conjugate pairs of nonreal zeros, the functions S_φ and $S_{\varphi+\frac{\pi}{2}}$ have no common zeros. Moreover, since a symmetry with defect index $(1,1)$ has at most one selfadjoint extension which is not an operator, the function S_φ can belong to the space $\mathcal{P}(E)$ for at most one value of φ modulo π.

b. Extension of isometries.

In this subsection we give a condition which ensures continuity of an isometric map between Pontryagin spaces. This result extends [7, IX.Theorem 3.1] and [2, Theorem 1.4.2].

If $\mathcal{M}$ is a linear subspace of an inner product space, we denote by $\mathcal{M}^\circ$ its isotropic part, that is

$$\mathcal{M}^\circ := \mathcal{M} \cap \mathcal{M}^\perp = \big\{ x \in \mathcal{M} : x \perp \mathcal{M} \big\} .$$

2.1 Proposition. *Let $\langle \mathcal{P}_1, [.,.]_1 \rangle$ and $\langle \mathcal{P}_2, [.,.]_2 \rangle$ be Pontryagin spaces, let*

$$\phi : \operatorname{dom}\phi \subseteq \mathcal{P}_1 \to \mathcal{P}_2$$

be a linear and isometric map, and assume that $\overline{\operatorname{ran}\phi}$ is nondegenerate. Then ϕ is continuous. Its extension by continuity $\tilde{\phi} : \overline{\operatorname{dom}\phi} \subseteq \mathcal{P}_1 \to \mathcal{P}_2$ is a linear and continuous isometry which maps $\overline{\operatorname{dom}\phi}$ surjectively onto $\overline{\operatorname{ran}\phi}$.

Proof.
Step 1; Reduction to the case that $\operatorname{ran}\phi$ *is dense in* $\mathcal{P}_2$*:* Since $\overline{\operatorname{ran}\phi}$ is nondegenerate, it is itself a Pontryagin space, and the topology $\langle \overline{\operatorname{ran}\phi}, [.,.]_2 \rangle$ carries as a Pontryagin space is equal to the restriction to $\overline{\operatorname{ran}\phi}$ of the topology of the Pontryagin space $\langle \mathcal{P}_2, [.,.]_2 \rangle$. Hence, for the proof of the present assertions, we may as well consider the map ϕ as a map of $\operatorname{dom}\phi \subseteq \mathcal{P}_1$ into $\overline{\operatorname{ran}\phi}$.

Step 2; Reduction to the case that $\mathcal{P}_2$ *is a Hilbert space:* According to Step 1, let us assume that $\operatorname{ran}\phi$ is dense in $\mathcal{P}_2$. Then we may choose a maximal negative subspace D_2 of $\mathcal{P}_2$, which is contained in $\operatorname{ran}\phi$, cf. [7, IX.Theorem 1.4]. Choose a negative subspace $D_1 \subseteq \operatorname{dom}\phi$, such that $\phi(D_1) = D_2$. Then we have

$$\operatorname{dom}\phi = D_1 [\dot{+}]_1 \big(D_1^\perp \cap \operatorname{dom}\phi \big), \qquad \operatorname{ran}\phi = D_2 [\dot{+}]_2 \big(D_2^\perp \cap \operatorname{ran}\phi \big) ,$$

and ϕ is decomposed as

$$\begin{pmatrix} \phi|_{D_1} & 0 \\ 0 & \phi|_{D_1^\perp \cap \operatorname{dom}\phi} \end{pmatrix} : \begin{matrix} D_1 \\ [\dot{+}]_1 \\ D_1^\perp \cap \operatorname{dom}\phi \end{matrix} \subseteq \begin{matrix} D_1 \\ [\dot{+}]_1 \\ D_1^\perp \end{matrix} \to \begin{matrix} D_2 \\ [\dot{+}]_2 \\ D_2^\perp \end{matrix}$$

Note here that negative subspaces are finite-dimensional, and hence certainly orthocomplemented. The space $D_1^\perp$ is itself a Pontryagin space, and the topology of $\langle D_1^\perp, [.,.]_1 \rangle$ is equal to the restriction to $D_1^\perp$ of the topology of $\langle \mathcal{P}_1, [.,.]_1 \rangle$. The same holds for the space $D_2^\perp$. Hence, for the proof of the present assertions, it is enough to show that $\phi|_{D_1^\perp}$ is continuous and can be extended in the desired way. However, $\operatorname{ran}\phi|_{D_1^\perp}$ is a dense subset of the Hilbert space $D_2^\perp$.

Step 3; Finish of proof: According to Steps 1 and 2, we may assume that $\mathcal{P}_2$ is a Hilbert space, and that $\operatorname{ran}\phi$ is dense in $\mathcal{P}_2$. Let $\|.\|_1$ be a norm on $\mathcal{P}_1$ induced by some fundamental decomposition, and let $\|.\|_2$ be the norm $\mathcal{P}_2$ carries as a Hilbert space. Then we have

$$\|\phi x\|_2^2 = [\phi x, \phi x]_2 = [x,x]_1 \leq \|x\|_1^2, \quad x \in \operatorname{dom}\phi ,$$

i.e. ϕ is continuous. Let $\tilde{\phi}$ be the extension by continuity of ϕ to a map of $\overline{\operatorname{dom}\phi}$ into $\mathcal{P}_2$. Clearly, $\tilde{\phi}$ is linear, continuous, isometric, and has dense range. In particular, $\overline{\operatorname{dom}\phi}$ is positive semidefinite and $\ker\phi = (\overline{\operatorname{dom}\phi})^\circ$.

Since $\dim(\overline{\operatorname{dom}\phi})^\circ < \infty$, certainly there exists a closed subspace $\mathcal{X}$ of $\mathcal{P}_1$, such that

$$\mathcal{X}\dot{+}(\overline{\operatorname{dom}\phi})^\circ = \overline{\operatorname{dom}\phi}\,.$$

Then $\mathcal{X}$ is positive definite, and hence even uniformly positive, cf. [7, IX.Lemma 2.1]. Thus the topology induced by the inner product $[.,.]_1$ on $\mathcal{X}$ is equal to the restriction to $\mathcal{X}$ of the topology of $\mathcal{P}_1$. This shows that $\langle \mathcal{X}, [.,.]_1\rangle$ is a Hilbert space. Moreover, $\tilde{\phi}|_{\mathcal{X}}$ is an isometry of $\mathcal{X}$ onto a dense subspace of the Hilbert space $\mathcal{P}_2$. Hence, it is a homeomorphism of $\mathcal{X}$ onto $\mathcal{P}_2$. In particular, $\tilde{\phi}$ is surjective. ❑

c. Sums of reproducing kernels.

Our next aim in this preliminary section is to investigate the geometry of reproducing kernel Pontryagin spaces generated by the sum of two kernels. The below Proposition 2.2 extends [2, Theorem 1.5.5]. For the positive definite case see also [10].

To start with, let us briefly recall: A Pontryagin space $\langle \mathcal{P}, [.,.]\rangle$ is called a reproducing kernel Pontryagin space of function on a set Ω, if its elements are complex valued functions defined on Ω, and if for each $w \in \Omega$ the point evaluation functional $f \mapsto f(w)$ is continuous on $\mathcal{P}$. In this case, for each $w \in \Omega$, there exists a unique element $K(w,.) \in \mathcal{P}$ with

$$[f, K(w,.)] = f(w), \quad f \in \mathcal{P}\,.$$

The function $K(w,z) : \Omega \times \Omega \to \mathbb{C}$ is called the reproducing kernel of $\langle \mathcal{P}, [.,.]\rangle$.

If $\langle \mathcal{P}_1, [.,.]_1\rangle$ and $\langle \mathcal{P}_2, [.,.]_2\rangle$ are reproducing kernel Pontryagin spaces on some set Ω, and $K_1(w,z)$, $K_2(w,z)$, denote their respective kernels, then the function

$$K(w,z) := K_1(w,z) + K_2(w,z)$$

is the kernel of some reproducing kernel Pontryagin space $\langle \mathcal{P}, [.,.]\rangle$, cf. [2, Theorem 1.1.3]. The space $\mathcal{P}$ is closely related to $\mathcal{P}_1$ and $\mathcal{P}_2$, however, this relation is not straightforward.

If Ω is an arbitrary set, we denote by Ψ the map

$$\Psi : \begin{cases} \mathbb{C}^\Omega \times \mathbb{C}^\Omega & \to \quad \mathbb{C}^\Omega \\ \quad (f,g) & \mapsto \quad f+g \end{cases}$$

Moreover, if $\langle \mathcal{P}_j, [.,.]_j\rangle$, $j = 1,2$, are Pontryagin spaces, we consider $\mathcal{P}_1 \times \mathcal{P}_2$ as a Pontryagin space endowed with the sum inner product

$$[(f_1,f_2),(g_1,g_2)]_+ := [f_1,g_1]_1 + [f_2,g_2]_2, \quad (f_1,f_2),(g_1,g_2) \in \mathcal{P}_1 \times \mathcal{P}_2\,.$$

2.2 Proposition. *Let $\langle \mathcal{P}_j, [.,.]_j\rangle$, $j = 1,2$, be reproducing kernel Pontryagin spaces of functions on a set Ω, and denote their respective kernels by $K_j(w,z)$, $j = 1,2$. Set $K(w,z) := K_1(w,z) + K_2(w,z)$, $w,z \in \Omega$, and let $\langle \mathcal{P}, [.,.]\rangle$ be the reproducing kernel Pontryagin space with kernel $K(w,z)$. We thus deal with three subspaces $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}$ of $\mathbb{C}^\Omega$. Set*

$$\mathcal{W} := \mathcal{P}_1 \cap \mathcal{P}_2, \qquad \mathcal{Q}_j := \mathcal{P}_j[-]_j\mathcal{W},\ j = 1,2\,,$$

$$\mathcal{D} := \big\{(g,-g) :\, g \in \mathcal{W}\big\} \subseteq \mathcal{P}_1 \times \mathcal{P}_2, \qquad \mathcal{Q} := (\mathcal{P}_1 \times \mathcal{P}_2)[-]_+\mathcal{D}\,.$$

Then the following hold:

(i) $\Psi|_{\mathcal{Q}}$ *maps* $\mathcal{Q}$ *continuously, isometrically, and surjectively onto* $\mathcal{P}$.

(ii) *We have* $\ker \Psi|_{\mathcal{Q}} = \mathcal{D}^{[\circ]_+}$, *and* $\Psi|_{\mathcal{Q}}$ *maps closed subspaces of* $\mathcal{Q}$ *onto closed subspaces of* $\mathcal{P}$.

(iii) *We have* $\mathcal{Q}_1 + \mathcal{Q}_2 \subseteq \mathcal{P}$, *and each of* $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_1 + \mathcal{Q}_2$ *is closed in* $\mathcal{P}$. *Moreover,*

$$\mathcal{Q}_1[\perp]\mathcal{Q}_2, \qquad \mathcal{Q}_1 \cap \mathcal{Q}_2 = \mathcal{W}^{[\circ]_1} \cap \mathcal{W}^{[\circ]_2},$$

$$[f,g]_j = [f,g], \quad f,g \in \mathcal{Q}_j,\ j = 1,2,$$

and the space $\mathcal{W}^{[\circ]_1} + \mathcal{W}^{[\circ]_2}$ *is* $[.,.]$*-neutral.*

(iv) *Assume additionally that*

$$[f,g]_1 = -[f,g]_2, \quad f,g \in \mathcal{W}. \tag{2.3}$$

Then $\mathcal{D}$ *is* $[.,.]_+$*-neutral, and (note that under the assumption* (2.3) *certainly* $\mathcal{W}^{[\circ]_1} = \mathcal{W}^{[\circ]_2}$*)*

$$\mathcal{Q}_1 + \mathcal{Q}_2 = \mathcal{P}[-]\mathcal{W}^{[\circ]_j}. \tag{2.4}$$

In the proof of this statement, we will use the following elementary fact.

2.3 Lemma. *Let* X, Y *be Banach spaces, and let* $A : X \to Y$ *be a bounded linear operator of* X *into* Y. *Assume that* $\operatorname{ran} A$ *is closed, and that* $\dim \ker A < \infty$. *Then* A *maps closed subspaces of* X *onto closed subspaces of* Y.

Proof. Since A is bounded, the final topology on $\operatorname{ran} A$ induced by the one-element family $\{A\}$ is finer than the restriction to $\operatorname{ran} A$ of the topology of Y. Since $\ker A$ is finite-dimensional and hence closed in X, and since $\operatorname{ran} A$ is closed in Y, both are Banach space topologies. By the Open Mapping Theorem, they must coincide.

Let M be a closed subspace of X. Since $\ker A$ is finite dimensional, also

$$A^{-1}(A(M)) = M + \ker A$$

is closed. This says that $A(M)$ is closed in the final topology on $\operatorname{ran} A$. Once again using that $\operatorname{ran} A$ is closed in Y, it follows that $A(M)$ is closed in Y. ❑

Proof (of Proposition 2.2). Consider the linear subspace

$$\mathcal{L} := \operatorname{span}\big\{\big(K_1(w,.), K_2(w,.)\big) : w \in \Omega\big\}$$

of $\mathcal{P}_1 \times \mathcal{P}_2$. By the definition of K, we have

$$\Psi\big(K_1(w,.), K_2(w,.)\big) = K(w,.), \quad w \in \Omega.$$

Hence,

$$[K(w_1,.), K(w_2,.)] = K(w_1,w_2) = K_1(w_1,w_2) + K_2(w_1,w_2) =$$

$$= [K_1(w_1,.), K_1(w_2,.)]_1 + [K_2(w_1,.), K_2(w_2,.)]_2 =$$

$$= \big[(K_1(w_1,.), K_2(w_1,.)), (K_1(w_2,.), K_2(w_2,.))\big]_+,$$

i.e. $\Psi|_{\mathcal{L}}$ is an isometry of $\mathcal{L}$ into $\mathcal{P}$. Clearly, the range of $\Psi|_{\mathcal{L}}$ is dense in $\mathcal{P}$.

By Proposition 2.1, there exists an extension $\tilde{\Psi}$ of $\Psi|_{\mathcal{L}}$ to a linear, continuous, isometric, and surjective map of $\overline{\mathcal{L}}$ onto $\mathcal{P}$. For each element $(f_1, f_2) \in \overline{\mathcal{L}}$ and point $w \in \Omega$, we compute

$$f_1(w) + f_2(w) = \big[(f_1, f_2), \big(K_1(w,.), K_2(w,.)\big)\big]_+ = \\ = \big[\tilde{\Psi}(f_1, f_2), \underbrace{\tilde{\Psi}\big(K_1(w,.), K_2(w,.)\big)}_{=K(w,.)}\big] = \tilde{\Psi}(f_1, f_2)(w)\,,$$

i.e. $\tilde{\Psi} = \Psi|_{\overline{\mathcal{L}}}$. However, since clearly $\mathcal{L}^{[\perp]_+} = \mathcal{D}$, we have

$$\overline{\mathcal{L}} = \mathcal{D}^{[\perp]_+} = \mathcal{Q}\,.$$

This finishes the proof of (i).

For the proof of (ii), first, $\Psi|_{\mathcal{Q}}$ being isometric and $\operatorname{ran}\Psi|_{\mathcal{Q}} = \mathcal{P}$ being nondegenerate implies

$$\ker\Psi|_{\mathcal{Q}} = \mathcal{Q}^{[\circ]_+} = \mathcal{D}^{[\circ]_+}\,.$$

Next, $\ker\Psi|_{\mathcal{Q}}$ being a neutral subspace of the Pontryagin space $\mathcal{P}_1 \times \mathcal{P}_2$, and hence being finite-dimensional implies that $\Psi|_{\mathcal{Q}}$ maps closed subspaces of $\mathcal{Q}$ onto closed subspaces of $\mathcal{P}$, cf. Lemma 2.3.

We proceed to the proof of (iii). First, $\mathcal{Q}_1 \times \mathcal{Q}_2 \subseteq \mathcal{Q}$ and each of $\mathcal{Q}_1 \times \{0\}$, $\{0\} \times \mathcal{Q}_2$, $\mathcal{Q}_1 \times \mathcal{Q}_2$ is a closed subspace of $\mathcal{Q}$. Thus, by the already proved item (ii), each of

$$\mathcal{Q}_1 = \Psi\big(\mathcal{Q}_1 \times \{0\}\big), \quad \mathcal{Q}_2 = \Psi\big(\{0\} \times \mathcal{Q}_2\big), \quad \mathcal{Q}_1 + \mathcal{Q}_2 = \Psi\big(\mathcal{Q}_1 \times \mathcal{Q}_2\big)\,,$$

is a closed subspace of $\mathcal{P}$. Moreover, since $(\mathcal{Q}_1 \times \{0\})[\perp]_+(\{0\} \times \mathcal{Q}_2)$, also $\mathcal{Q}_1[\perp]\mathcal{Q}_2$.

We have

$$\mathcal{W}^{[\circ]_1} = (\mathcal{P}_1[-]_1\mathcal{W}) \cap \mathcal{W} = \mathcal{Q}_1 \cap (\mathcal{P}_1 \cap \mathcal{P}_2) = \mathcal{Q}_1 \cap \mathcal{P}_2\,,$$

and similarly $\mathcal{W}^{[\circ]_2} = \mathcal{Q}_2 \cap \mathcal{P}_1$. Hence,

$$\mathcal{W}^{[\circ]_1} \cap \mathcal{W}^{[\circ]_2} = \big(\mathcal{Q}_1 \cap \mathcal{P}_2\big) \cap \big(\mathcal{P}_1 \cap \mathcal{Q}_2\big) = \mathcal{Q}_1 \cap \mathcal{Q}_2\,.$$

Next, we compute

$$[f_1, g_1]_1 = \big[(f_1, 0), (g_1, 0)\big]_+ = [f_1, g_1], \quad f_1, g_1 \in \mathcal{Q}_1\,,$$

$$[f_2, g_2]_2 = \big[(0, f_2), (0, g_2)\big]_+ = [f_2, g_2], \quad f_2, g_2 \in \mathcal{Q}_2\,.$$

Finally, note that $\mathcal{W}^{[\circ]_j}$ is a $[.,.]_j$-neutral subspace of $\mathcal{Q}_j$, $j = 1, 2$. Hence, by what we already showed, $\mathcal{W}^{[\circ]_j}$, $j = 1, 2$, is contained in $\mathcal{P}$ and is $[.,.]$-neutral. Moreover, $\mathcal{W}^{[\circ]_1}[\perp]\mathcal{W}^{[\circ]_2}$, and we conclude that $\mathcal{W}^{[\circ]_1} + \mathcal{W}^{[\circ]_2}$ is $[.,.]$-neutral. This finishes the proof of (iii).

For the proof of (iv), we first make some computations which hold even without the additional assumption (2.3). Set $\mathcal{M} := \mathcal{Q}_1 \times \mathcal{Q}_2$, then surjectivity and isometry of $\Psi|_{\mathcal{Q}}$ implies

$$(\mathcal{Q}_1 + \mathcal{Q}_2)^{[\perp]} = \Psi(\mathcal{M})^{[\perp]} = \Psi\big(\mathcal{M}^{[\perp]_+} \cap \mathcal{Q}\big) = \Psi\big(\mathcal{M}^{[\perp]_+} \cap \mathcal{D}^{[\perp]_+}\big)\,.$$

Since $\mathcal{Q}_1+\mathcal{Q}_2$ is closed in $\mathcal{P}$, it follows that

$$\mathcal{Q}_1+\mathcal{Q}_2=\Psi\big(\mathcal{M}^{[\perp]_+}\cap\mathcal{D}^{[\perp]_+}\big)^{[\perp]}. \tag{2.5}$$

Next, $\mathcal{M}^{[\perp]_+}=\mathcal{W}\times\mathcal{W}$, and we obtain

$$\mathcal{M}^{[\perp]_+}\cap\mathcal{D}^{[\perp]_+}=\big\{(f_1,f_2)\in\mathcal{W}\times\mathcal{W}:\ [f_1,g]_1=[f_2,g]_2,\ g\in\mathcal{W}\big\}. \tag{2.6}$$

Moreover, obviously,

$$\mathcal{D}^{[\circ]_+}=\big\{(f,-f):\ f\in\mathcal{W},\ [f,g]_1=-[f,g]_2,\ g\in\mathcal{W}\big\}.$$

From now on assume that (2.3) holds. Then we have $\mathcal{D}=\mathcal{D}^{[\circ]_+}$. Moreover, clearly, $\mathcal{W}^{[\circ]_1}=\mathcal{W}^{[\circ]_2}$. In order to show (2.4), it is by (2.5) enough to show $\Psi\big(\mathcal{M}^{[\perp]_+}\cap\mathcal{D}^{[\perp]_+}\big)=\mathcal{W}^{[\circ]_j}$.

If $f\in\mathcal{W}^{[\circ]_j}$, then clearly $(f,0)\in\mathcal{M}^{[\perp]_+}\cap\mathcal{D}^{[\perp]_+}$. Conversely, assume that $(f_1,f_2)\in\mathcal{M}^{[\perp]_+}\cap\mathcal{D}^{[\perp]_+}$. Then, by (2.6) and our hypothesis (2.3), we have $f_1+f_2\in\mathcal{W}$ and

$$[f_1+f_2,g]_1=[f_1,g]_1+[f_2,g]_1=[f_1,g]_1-[f_2,g]_2=0,\quad g\in\mathcal{W}.$$

This gives $f_1+f_2\in\mathcal{W}^{[\circ]_1}$, and we have finished the proof of (iv). ❑

d. Orthogonal sets in de Branges Pontryagin spaces.
Some orthogonal sets in a de Branges Pontryagin space where described in [21, §7]. For our present purposes a more complete and comprehensive formulation of these results is needed.

Let us first recall some facts concerning the spectral structure of the self-adjoint relation $\mathcal{A}_\varphi$ induced by the function S_φ, cf. (2.2). Denote by Z^φ the set of all zeros of S_φ, and let d_α be the multiplicity of the zero α of S_φ. Since S_φ is real, the set Z^φ is symmetric with respect to the real axis, and $d_{\overline{\alpha}}=d_\alpha$, $\alpha\in Z^\varphi$.

2.4 Remark. Let $E\in\mathcal{HB}^*_{<\infty}$ and $\varphi\in\mathbb{R}$.

- (i) The finite spectrum of $\mathcal{A}_\varphi$ is equal to Z^φ.
- (ii) Denote by $\mathfrak{E}^\varphi_\infty$ the algebraic eigenspace of $\mathcal{A}_\varphi$ at infinity, and set $\delta_\varphi:=\dim\mathfrak{E}^\varphi_\infty$. Then

$$\delta_\varphi=\max\big\{k\in\mathbb{N}_0:\ z^kS_\varphi(z)\in\mathcal{P}(E)\big\}+1,$$

and

$$\mathfrak{E}^\varphi_\infty=\operatorname{span}\big\{z^kS_\varphi(z):\ k=0\leq k<\delta_\varphi\big\}.$$

In particular, the relation $\mathcal{A}_\varphi$ has a nontrivial multivalued part if and only if $S_\varphi\in\mathcal{P}(E)$.

- (iii) The Gram matrix of the inner product $[.,.]$ restricted to $\mathfrak{E}^\varphi_\infty$ with respect to the basis $\{z^kS_\varphi(z):\ 0\leq k<\delta_\varphi\}$ has Hankel form. Hence, setting $\delta^\circ_\varphi:=\dim(\mathfrak{E}^\varphi_\infty)^\circ$, we have

$$(\mathfrak{E}^\varphi_\infty)^\circ=\operatorname{span}\{z^kS_\varphi:\ 0\leq k<\delta^\circ_\varphi\}$$

and

$$\delta^\circ_\varphi=\begin{cases}\min\big\{0\leq k<\delta_\varphi:\ [z^{\delta_\varphi-1}S_\varphi,z^kS_\varphi]\neq0\big\}, & (\mathfrak{E}^\varphi_\infty)^\circ\neq\mathfrak{E}^\varphi_\infty\\ \delta_\varphi\hphantom{\min\big\{0\leq k<\delta_\varphi:\ [z^{\delta_\varphi-1}S_\varphi,z^kS_\varphi]\neq0\big\}}, & (\mathfrak{E}^\varphi_\infty)^\circ=\mathfrak{E}^\varphi_\infty\end{cases}$$

(*iv*) The set of finite criticial point of $\mathcal{A}_\varphi$ is equal to the set of multiple real zeros of S_φ. Each of these points is a regular critical point. The point ∞ is a critical point if and only if

$$\delta_\varphi > 1 \quad \text{or} \quad \delta_\varphi = 1 \wedge \dim \mathcal{P}(E) = \infty \wedge [S_\varphi, S_\varphi] < 0\,.$$

It is a singular critical point if and only if $[z^{\delta_\varphi - 1} S_\varphi, S_\varphi] = 0$.

(*v*) The spectral subspace $\mathfrak{E}^\varphi_{\{\alpha\}}$ corresponding to a point $\alpha \in Z^\varphi \cap \mathbb{R}$ is equal to

$$\mathfrak{E}^\varphi_{\{\alpha\}} = \operatorname{span}\Big\{ \frac{S_\varphi(z)}{(z-\alpha)^k} : k = 1, \ldots, d_\alpha \Big\}\,. \tag{2.7}$$

The spectral subspace $\mathfrak{E}^\varphi_{\{\beta,\overline{\beta}\}}$ corresponding to a nonreal conjugate pair $\{\beta, \overline{\beta}\}$, $\beta \in Z^\varphi \cap \mathbb{C}^+$, is equal to

$$\mathfrak{E}^\varphi_{\{\beta,\overline{\beta}\}} = \operatorname{span}\Big(\Big\{ \frac{S_\varphi(z)}{(z-\beta)^k} : k = 1, \ldots, d_\beta \Big\} \cup \Big\{ \frac{S_\varphi(z)}{(z-\overline{\beta})^k} : k = 1, \ldots, d_\beta \Big\} \Big)\,. \tag{2.8}$$

(*vi*) The set of spectral points of positive type is equal to

$$Z^\varphi_+ := \big\{ \alpha \in Z^\varphi \cap \mathbb{R} : S'_\varphi(\alpha) S_{\varphi+\frac{\pi}{2}}(\alpha) < 0 \big\}\,.$$

//

We will rather use a different basis of $\mathfrak{E}^\varphi_{\{\alpha\}}$, $\alpha \in Z^\varphi \cap \mathbb{R}$, and $\mathfrak{E}^\varphi_{\{\beta,\overline{\beta}\}}$, $\beta \in Z^\varphi \cap \mathbb{C}^+$, than the one indicated in (2.7) and (2.8). The reproducing kernel $K_E(w,z)$ of the space $\mathcal{P}(E)$ can be written as

$$K_E(w,z) = \frac{S_\varphi(\overline{w}) S_{\varphi+\frac{\pi}{2}}(z) - S_\varphi(z) S_{\varphi+\frac{\pi}{2}}(\overline{w})}{z - \overline{w}}\,.$$

Again, for $z = \overline{w}$, this formula has to be interpreted appropriately as a derivative. It follows that

$$\frac{\partial^k}{(\partial \overline{w})^k} K_E(w,z)\Big|_{w=\overline{\alpha}} = -\sum_{j=0}^{k} \binom{k}{j} \frac{S_{\varphi+\frac{\pi}{2}}(\alpha)}{(j+1)!} \frac{S_\varphi(z)}{(z-\alpha)^{j+1}}, \quad k = 0, \ldots, d_\alpha - 1\,.$$

Set

$$\mathfrak{b}_{\alpha,k}(z) := \frac{\partial^k}{(\partial \overline{w})^k} K_E(w,z)\Big|_{w=\overline{\alpha}}, \quad \alpha \in Z^\varphi,\ k = 0, \ldots, d_\alpha - 1\,,$$

then

$$\mathfrak{E}^\varphi_{\{\alpha\}} = \operatorname{span}\big\{ \mathfrak{b}_{\alpha,k} : k = 0, \ldots, d_\alpha - 1 \big\}, \quad \alpha \in Z^\varphi \cap \mathbb{R}\,,$$

$$\mathfrak{E}^\varphi_{\{\beta,\overline{\beta}\}} = \operatorname{span}\Big(\big\{ \mathfrak{b}_{\beta,k} : k = 1, \ldots, d_\beta \big\} \cup \big\{ \mathfrak{b}_{\overline{\beta},k} : k = 1, \ldots, d_\beta \big\} \Big), \quad \beta \in Z^\varphi \cap \mathbb{C}^+\,.$$

2.5 Definition. Denote by ℓ^2_φ the weighted ℓ^2-space of sequences $(a_\alpha)_{\alpha \in Z^\varphi_+}$, whose inner product is defined as

$$\big[(a_\alpha)_{\alpha \in Z^\varphi_+}, (b_\alpha)_{\alpha \in Z^\varphi_+} \big] := \sum_{\alpha \in Z^\varphi_+} a_\alpha \overline{b_\alpha} \frac{-1}{S'_\varphi(\alpha) S_{\varphi+\frac{\pi}{2}}(\alpha)}\,.$$

Moreover, set

$$\check{Z}^\varphi := \big\{(\alpha,k) :\ \alpha\in Z^\varphi\setminus Z_+^\varphi, k=0,\ldots,d_\alpha-1\big\}\,,$$

and define an inner product on $\mathbb{C}^{\check{Z}^\varphi}$ by requiring that the linear isomorphism

$$\lambda : \begin{cases} \operatorname{span}\Big(\bigcup\limits_{\substack{\alpha\in Z^\varphi\setminus Z_+^\varphi\\ \alpha\in\mathbb{R}}} \mathfrak{E}^\varphi_{\{\alpha\}} \cup \bigcup\limits_{\beta\in Z^\varphi\cap\mathbb{C}^+} \mathfrak{E}^\varphi_{\{\beta,\overline{\beta}\}}\Big) & \to\ \mathbb{C}^{\check{Z}^\varphi} \\ \qquad\qquad\qquad\qquad\qquad\qquad F & \mapsto\ \big([F,\mathfrak{b}_{\alpha,k}]\big)_{(\alpha,k)\in\check{Z}^\varphi} \end{cases}$$

becomes isometric. Note here that the linear span on the left is nondegenerate. //

Denote by Φ the map

$$\Phi : \begin{cases} \{F :\ F \text{ entire}\} & \to\ \mathbb{C}^{Z_+^\varphi}\times\mathbb{C}^{\check{Z}^\varphi} \\ \qquad\qquad F & \mapsto\ \big((F(\alpha))_{\alpha\in Z_+^\varphi}, (F^{(k)}(\alpha))_{(\alpha,k)\in\check{Z}^\varphi}\big) \end{cases}$$

Note that Φ is continuous if the set of all entire functions is endowed with the topology of locally uniform convergence, and $\mathbb{C}^{Z_+^\varphi}\times\mathbb{C}^{\check{Z}^\varphi}\cong\mathbb{C}^{Z_+^\varphi\dot\cup\check{Z}^\varphi}$ carries the product topology.

2.6 Proposition. *Set $\mathcal{X} := \mathcal{P}(E)[-]\mathfrak{E}^\varphi_\infty$. Then $\Phi|_\mathcal{X}$ is a continuous, isometric, and surjective map of $\mathcal{X}$ onto $\ell^2_\varphi\times\mathbb{C}^{\check{Z}^\varphi}$.*

Proof. Set

$$\mathcal{L}_+ := \operatorname{span}\bigcup_{\alpha\in Z_+^\varphi}\mathfrak{E}^\varphi_{\{\alpha\}},\quad \check{\mathcal{L}} := \operatorname{span}\Big(\bigcup_{\substack{\alpha\in Z^\varphi\setminus Z_+^\varphi\\ \alpha\in\mathbb{R}}}\mathfrak{E}^\varphi_{\{\alpha\}} \cup \bigcup_{\beta\in Z^\varphi\cap\mathbb{C}^+}\mathfrak{E}^\varphi_{\{\beta,\overline{\beta}\}}\Big)\,.$$

Let us show that $\Phi|_{\mathcal{L}_++\check{\mathcal{L}}}$ is isometric. Since $\mathcal{L}_+$ and $\check{\mathcal{L}}$, as well as their images, are orthogonal, it suffices to consider the restrictions $\Phi|_{\mathcal{L}_+}$ and $\Phi|_{\check{\mathcal{L}}}$ separately. For the second one, isometry holds by definition. In order to see isometry of $\Phi|_{\mathcal{L}_+}$, we compute ($\alpha,\alpha'\in Z_+^\varphi$)

$$[\mathfrak{b}_{\alpha,0},\mathfrak{b}_{\alpha',0}] = K_E(\alpha,\alpha') = \begin{cases} -S_\varphi'(\alpha)S_{\varphi+\frac{\pi}{2}}(\alpha)\,, & \alpha=\alpha' \\ 0 \qquad\qquad , & \alpha\neq\alpha' \end{cases}\,.$$

On the other hand,

$$\Phi(\mathfrak{b}_{\alpha,0}) = \big(\mathfrak{b}_{\alpha,0}(\gamma)\big)_{\gamma\in Z_+^\varphi} = \big(\delta_{\alpha\gamma}\cdot K_E(\alpha,\alpha)\big)_{\gamma\in Z_+^\varphi}\,,$$

where $\delta_{\alpha\gamma}$ denotes the Kronecker delta symbol, and hence also

$$\big[\Phi(\mathfrak{b}_{\alpha,0}),\Phi(\mathfrak{b}_{\alpha',0})\big]_{\ell^2_\varphi} = \begin{cases} |K_E(\alpha,\alpha)|^2\frac{-1}{S_\varphi'(\alpha)S_{\varphi+\frac{\pi}{2}}(\alpha)}\,, & \alpha=\alpha' \\ 0 \qquad\qquad , & \alpha\neq\alpha' \end{cases} =$$

$$= \begin{cases} -S_\varphi'(\alpha)S_{\varphi+\frac{\pi}{2}}(\alpha)\,, & \alpha=\alpha' \\ 0 \qquad\qquad , & \alpha\neq\alpha' \end{cases}\,.$$

Clearly, the range of $\Phi|_{\mathcal{L}_+ + \check{\mathcal{L}}}$ is dense in the Pontryagin space $\ell^2_\varphi \times \mathbb{C}^{\check{Z}^\varphi}$. Hence, by Proposition 2.1, there exists an extension $\tilde{\Phi}$ which maps $\overline{\mathcal{L}_+ + \check{\mathcal{L}}}$ continuously, isometrically, and surjectively onto $\ell^2_\varphi \times \mathbb{C}^{\check{Z}^\varphi}$.

Since the topology of $\mathcal{P}(E)$ is stronger than the topology of locally uniform convergence, and the topology of ℓ^2_φ is stronger than pointwise convergence, it follows that

$$\tilde{\Phi} = \Phi|_{\overline{\mathcal{L}_+ + \check{\mathcal{L}}}} .$$

To finish the proof, recall that $\mathfrak{E}^\varphi_\infty = (\mathcal{L}_+ + \check{\mathcal{L}})^{[\perp]}$, cf. [34, Proposition II.5.2], and hence $\overline{\mathcal{L}_+ + \check{\mathcal{L}}} = (\mathfrak{E}^\varphi_\infty)^{[\perp]}$. ❑

e. dB-normable linear spaces.

2.7 Example. Let a function $E \in \mathcal{HB}^*_{<\infty}$ and an integer $N \in \mathbb{Z}$ be given. The space $\operatorname{Assoc}_N \mathcal{P}(E)$ is, ad hoc, just a linear space. However, it can be endowed with an inner product so to become a de Branges space: Choose, according to [21, Theorem 3.3], a function $E_0 \in \mathcal{HB}_0$ such that $\mathcal{P}(E) = \mathcal{H}(E_0)$ as sets. In case $N \geq 0$, set $E_N(z) := (z+i)^N E_0(z)$. Then $E_N \in \mathcal{HB}_0$, and by [36, Corollary 3.4] we have

$$\operatorname{Assoc}_N \mathcal{P}(E) = \operatorname{Assoc}_N \mathcal{H}(E_0) = \mathcal{H}(E_N) .$$

Hence, the inner product $[.,.]_{\mathcal{H}(E_N)}$ turns $\operatorname{Assoc}_N \mathcal{P}(E)$ into a de Branges space.

If $N < 0$, it is easy to see that the graph inner product

$$[F,G]_N := [F,G]_{\mathcal{H}(E_0)} + \big[z^{|N|}F(z), z^{|N|}G(z)\big]_{\mathcal{H}(E_0)}, \quad F,G \in \operatorname{Assoc}_N \mathcal{P}(E) ,$$

turns $\operatorname{Assoc}_N \mathcal{P}(E)$ into a de Branges space. //

The choice of inner products in Example 2.7 is by no means unique, nor in any way canonical. Thus, we shall be interested in properties of a space which follow from its de Branges space structure, but do not depend on the particular choice of the inner product which realizes this structure. In order to stress this fact also notationally, let us introduce the following terminology.

2.8 Definition. Let $\mathcal{X}$ be a linear space of entire functions. We call $\mathcal{X}$ dB-normable, if there exists a positive definite inner product on $\mathcal{X}$ which turns $\mathcal{X}$ into a de Branges space. This amounts to saying that there exists $E \in \mathcal{HB}^*_0$ such that $\mathcal{X} = \mathcal{H}(E)$ as sets. //

Note that, if $\mathcal{X}$ is dB-normable, then there exists a unique Banach space topology on $\mathcal{X}$ such that for each $w \in \mathbb{C}$ the point evaluation functional $F \mapsto F(w)$ is continuous on $\mathcal{X}$. Topological notions will always be understood with respect to this topology.

We will also use the following notation: Let $\mathcal{X}$ be a linear space of entire functions. Then $\mathcal{X}$ is called division invariant, if

$$\forall\, F \in \mathcal{X}, w \in \mathbb{C}: \quad F(w) = 0 \;\Rightarrow\; \frac{F(z)}{z-w} \in \mathcal{X} .$$

The space $\mathcal{X}$ is called reflection invariant, if

$$\forall\, F(z) \in \mathcal{X}: \quad \overline{F(\overline{z})} \in \mathcal{X} .$$

The inner-product-independent objects which are of interest in the present context are the operator of multiplication by the independent variable, the set of N-associated functions, and the chain of de Branges subspaces. Denote by $\mathcal{S}$ the operator which acts as $F(z) \mapsto zF(z)$ on the set of all entire functions.

2.9 Definition. Let $\mathcal{X}$ be a dB-normable linear space of entire functions.

(i) Denote by $\mathcal{S}_\mathcal{X}$ the restriction of $\mathcal{S}$ to

$$\operatorname{dom} \mathcal{S}_\mathcal{X} := \big\{F \in \mathcal{X} : \, zF(z) \in \mathcal{X}\big\} .$$

(ii) For each integer $N \in \mathbb{Z}$, set

$$\operatorname{Assoc}_N \mathcal{X} := \begin{cases} \mathcal{X} + z\mathcal{X} + \ldots + z^N \mathcal{X} & , \quad N \geq 0 \\ \{F \in \mathcal{X} : \, z^{|N|} F(z) \in \mathcal{X}\}, & N < 0 \end{cases}$$

(iii) Denote

$$\operatorname{Sub} \mathcal{X} := \left\{ \mathcal{Y} \subseteq \mathcal{X} : \begin{array}{l} \mathcal{Y} \text{ is a closed nonzero linear subspace} \\ \text{and division and reflection invariant} \end{array} \right\} .$$

An element of $\operatorname{Sub} \mathcal{X}$ is called a dB-subspace of $\mathcal{X}$.

//

Let us recall some facts from de Branges' theory.

2.10 Remark. Let $\mathcal{X}$ be dB-normable. Then the following hold:

(i) The operator $\mathcal{S}_\mathcal{X}$ is closed and every point $w \in \mathbb{C}$ is a point of regular type for $\mathcal{S}_\mathcal{X}$.

(ii) For each $w \in \mathbb{C}$ and $n \in \mathbb{N}$

$$\operatorname{ran}(\mathcal{S}_\mathcal{X} - w)^n = \big\{F \in \mathcal{X} : \, F(w) = \ldots = F^{(n-1)}(w)\big\} .$$

(iii) The set $\operatorname{Sub} \mathcal{X}$ is totally ordered (the 'Ordering Theorem').

(iv) Whenever $N \in \mathbb{Z}$ and $F, G \in \operatorname{Assoc}_N \mathcal{X}$, $G \neq 0$, then the meromorphic function $G^{-1}F$ is of bounded type in both half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$, meaning that in each of these half-planes it can be represented as a quotient of two bounded analytic functions.

//

We will frequently make use of the following simple algebraic properties. They are proved by elementary manipulations; we will not carry out the details.

2.11 Lemma. *Let $\mathcal{X}$ be a linear space of entire functions which is division invariant. Then the following hold:*

(i) *For each $N \in \mathbb{Z}$ the space* $\operatorname{Assoc}_N \mathcal{X}$ *is division invariant. In fact,*

$$\forall\, F \in \operatorname{Assoc}_N \mathcal{X}, w \in \mathbb{C}, k \in \mathbb{N} :$$
$$F(w) = \ldots = F^{(k-1)}(w) = 0 \;\Rightarrow\; \frac{F(z)}{(z-w)^k} \in \operatorname{Assoc}_{N-k} \mathcal{X} . \qquad (2.9)$$

In particular,

$$\forall F,G\in\operatorname{Assoc}_N\mathcal{X}, w\in\mathbb{C}:\quad \frac{F(z)G(w)-G(z)F(w)}{z-w}\in\operatorname{Assoc}_{N-1}\mathcal{X}\,.$$

Conversely, for each $w\in\mathbb{C}$, $N\in\mathbb{Z}$, and $k\in\mathbb{N}$, we have

$$(\mathcal{S}-w)^k\operatorname{Assoc}_N\mathcal{X}=\operatorname{ran}\big(\mathcal{S}_{\operatorname{Assoc}_{N+k}\mathcal{X}}-w\big)^k= \\ =\big\{F\in\operatorname{Assoc}_{N+k}\mathcal{X}:\,F(w)=\ldots=F^{(k-1)}(w)=0\big\}\,. \quad (2.10)$$

(*ii*) *We have*

$$\dim\Big(\mathcal{X}/\operatorname{ran}(\mathcal{S}_{\mathcal{X}}-w)^k\Big)=\begin{cases}k & ,\quad \operatorname{Assoc}_{-k}\mathcal{X}\neq\{0\}\\ \dim\mathcal{X}\,, & \quad\operatorname{Assoc}_{-k}\mathcal{X}=\{0\}\end{cases}$$

where the minimum on the right hand side is understood as k if $\dim\mathcal{X}=\infty$. We have $\operatorname{Assoc}_N\mathcal{X}=\{0\}$ if and only if $-N\geq\dim\mathcal{X}$.

(*iii*) *Let $n,m\in\mathbb{Z}$. Unless $-n\geq\dim\mathcal{X}$ and $-(n+m)<\dim\mathcal{X}$, we have*

$$\operatorname{Assoc}_m\operatorname{Assoc}_n\mathcal{X}=\operatorname{Assoc}_{m+n}\mathcal{X}\,. \quad (2.11)$$

❑

For later reference let us also state the following fact explicitly.

2.12 Lemma. *Let $\mathcal{X}$ be a linear space of entire functions which is reflection invariant. Then for each $N\in\mathbb{Z}$ also the space $\operatorname{Assoc}_N\mathcal{X}$ has this property.* ❑

By means of Lemma 2.11 we have a chain of maps consisting of appropriate restrictions of $\mathcal{S}$:

$$\cdots\xrightarrow{\mathcal{S}}\operatorname{Assoc}_{-1}\mathcal{X}\xrightarrow{\mathcal{S}}\mathcal{X}\xrightarrow{\mathcal{S}}\operatorname{Assoc}_1\mathcal{X}\xrightarrow{\mathcal{S}}\operatorname{Assoc}_2\mathcal{X}\xrightarrow{\mathcal{S}}\cdots$$

2.13 Lemma. *Let $\mathcal{X}$ be dB-normable, and let $N\in\mathbb{Z}$ and $k\in\mathbb{N}$. Then the restriction $\mathcal{S}^k|_{\operatorname{Assoc}_N\mathcal{X}}$ is a homeomorphism of $\operatorname{Assoc}_N\mathcal{X}$ onto $\operatorname{ran}\mathcal{S}^k_{\operatorname{Assoc}_{N+k}\mathcal{X}}$.*

Proof. The map $\mathcal{S}^k|_{\operatorname{Assoc}_N\mathcal{X}}$ is by (2.10) a bijection of the Banach space $\operatorname{Assoc}_N\mathcal{X}$ onto the closed subspace $\operatorname{ran}\mathcal{S}^k_{\operatorname{Assoc}_{N+k}\mathcal{X}}$ of the Banach space $\operatorname{Assoc}_{N+k}\mathcal{X}$. Since point evaluation is continuous in both spaces, it has closed graph. ❑

We arrive at an explicit relation between the chains of dB-subspaces of $\operatorname{Assoc}_N\mathcal{X}$ and $\operatorname{Assoc}_{N+k}\mathcal{X}$, $-N<\dim\mathcal{X}$, $k\in\mathbb{N}$.

2.14 Proposition. *Let $\mathcal{X}$ be a dB-normable space, let $-N<\dim\mathcal{X}$, and $k\in\mathbb{N}$. Then*

$$\big\{\mathcal{Z}\in\operatorname{Sub}\operatorname{Assoc}_{N+k}\mathcal{X}:\,\dim\mathcal{Z}>k\big\}=\big\{\operatorname{Assoc}_k\mathcal{Y}:\,\mathcal{Y}\in\operatorname{Sub}\operatorname{Assoc}_N\mathcal{X}\big\}\,.$$

Proof. Let $\mathcal{Z} \in \operatorname{Sub}\operatorname{Assoc}_{N+k}\mathcal{X}$ with $\dim\mathcal{Z} > k$ be given. Set $\mathcal{Y} := \operatorname{Assoc}_{-k}\mathcal{Z}$, then $\mathcal{Y}$ is a nonzero division- and reflection invariant subspace of $\operatorname{Assoc}_N\mathcal{X}$, which satisfies $\operatorname{Assoc}_k\mathcal{Y} = \mathcal{Z}$. Since $\mathcal{Y} = (\mathcal{S}^k)^{-1}(\operatorname{ran}\mathcal{S}^k_{\mathcal{Z}})$ and $\operatorname{ran}\mathcal{S}^k_{\mathcal{Z}} = \mathcal{Z} \cap \operatorname{ran}\mathcal{S}^k_{\operatorname{Assoc}_{N+k}\mathcal{X}}$, the subspace $\mathcal{Y}$ is also closed in $\operatorname{Assoc}_N\mathcal{X}$.

Conversely, let $\mathcal{Y} \in \operatorname{Sub}\operatorname{Assoc}_N\mathcal{X}$ be given. Then $\mathcal{S}^k\mathcal{Y}$ is a closed subspace of $\operatorname{Assoc}_{N+k}\mathcal{X}$. Since, by Lemma 2.11, (*iii*), we have $\mathcal{Y} = \operatorname{dom}\mathcal{S}^k_{\operatorname{Assoc}_k\mathcal{Y}}$, we obtain that $\dim(\operatorname{Assoc}_k\mathcal{Y}/\mathcal{S}^k\mathcal{Y}) < \infty$. Hence, $\operatorname{Assoc}_k\mathcal{Y}$ is closed in $\operatorname{Assoc}_{N+k}\mathcal{X}$. Moreover, since $\mathcal{Y} \neq \{0\}$, $\dim\operatorname{Assoc}_k\mathcal{Y} > k$. By Lemma 2.11 and Lemma 2.12, $\operatorname{Assoc}_k\mathcal{Y}$ is division- and reflection invariant. ❑

An extension of [9, Problem 72] follows. The proof given below is, however, different to the one suggested in this book. We invoke the Ordering Theorem rather than [9, Theorem 26].

2.15 Corollary. *Let $\mathcal{X}$ be a dB-normable space, let $\mathcal{Y} \in \operatorname{Sub}\mathcal{X}$, and $k \in \mathbb{N}_0$. Moreover, let U be a real and zerofree entire function. If $U \in \operatorname{Assoc}_k\mathcal{X}$, then also $U \in \operatorname{Assoc}_k\mathcal{Y}$.*

Proof. We have

$$\operatorname{span}\{U\}, \operatorname{Assoc}_k\mathcal{Y} \in \operatorname{Sub}\operatorname{Assoc}_k\mathcal{X}\,.$$

The Ordering Theorem implies that $\operatorname{span}\{U\} \subseteq \operatorname{Assoc}_k\mathcal{Y}$. ❑

3 Existence of real and zerofree elements

Let us recall the notion of generalized Nevanlinna functions. If $q : D \to \mathbb{C}$ is an analytic function defined on some open subset D of the complex plane, we define a kernel N_q as

$$N_q(w,z) := \frac{q(z) - \overline{q(w)}}{z - \overline{w}}, \quad z, w \in D\,.$$

Again, for $z = \overline{w}$, this formula has to be interpreted appropriately.

A function q which is meromorphic on $\mathbb{C}\setminus\mathbb{R}$ and satisfies $q(\overline{z}) = \overline{q(z)}$ is said to belong to the class $\mathcal{N}_{<\infty}$ of generalized Nevanlinna function, if the kernel N_q has a finite number of negative squares on the domain of holomorphy of q. If the exact number of its negative squares is equal to $\kappa \in \mathbb{N}_0$, we write $q \in \mathcal{N}_\kappa$ and $\operatorname{ind}_- q = \kappa$. For more on the class $\mathcal{N}_{<\infty}$ see, e.g., [30].

In the formulation of our present results some conditions on the asymptotic distribution of the sequence of poles of a generalized Nevanlinna function appear.

3.1. Some asymptotic conditions: Let $q \in \mathcal{N}_{<\infty}$ and assume that q is meromorphic in the whole plane but not a rational function. Denote by $(\gamma_k)_{k\in\mathbb{N}}$ the sequence of all nonzero real and simple poles of q with negative residuum, and set $\sigma_k := -\operatorname{Res}(q;\gamma_k)$. Let $\alpha_1,\dots,\alpha_r$ be the remaining poles of q, and denote by $d_1,\dots,d_r \in \mathbb{N}$ their multiplicities. Provided the product converges, set

$$A_q(z) := \Big[\prod_{\substack{j=1\\ \alpha_j\neq 0}}^{r}(-\alpha_j)^{d_j}\Big]^{-1} \cdot \prod_{j=1}^{r}(z-\alpha_j)^{d_j} \cdot \lim_{r\to\infty}\prod_{|\gamma_k|\leq r}\Big(1 - \frac{z}{\gamma_k}\Big)\,. \tag{3.1}$$

Finally, let $(\gamma_k^+)_k$ and $(\gamma_k^-)_k$ denote the (finite or infinite) sequences of positive or negative, respectively, elements of $\{\gamma_k : k \in \mathbb{N}\}$, arranged according to increasing modulus. Then we consider the conditions ($N \in \mathbb{Z}$):

(I) The limit

$$\lim_{r\to\infty}\sum_{|\gamma_k|\leq r}\frac{1}{\gamma_k}$$

exists in $\mathbb{R}$.

(II) The limits

$$\lim_{k\to\infty}\frac{k}{\gamma_k^+}\quad\text{and}\quad\lim_{k\to\infty}\frac{k}{\gamma_k^-}$$

exist in $\mathbb{R}$ and are equal. Thereby, if one of $(\gamma_k^+)_k$ or $(\gamma_k^-)_k$ is a finite sequence, the respective limit is understood as 0.

(III$_N$) We have

$$\sum_{k\in\mathbb{N}}\gamma_k^{-2N}\frac{1}{A_q'(\gamma_k)^2\sigma_k}<\infty\,. \tag{3.2}$$

//

The conditions (I) and (II) frequently appear in complex analysis, for example they are named (S) and (D$^+$) in [6, §8.1]. Condition (III$_N$) of course requires that the product (3.1) converges, which will not be the case in general. However, under the assumption that q satisfies (I), this product does converge. Hence, in conjunction with (I), condition (III$_N$) is meaningful.

The next statement is our first main result.

3.2 Theorem. *Let $E\in\mathcal{HB}^*_{<\infty}$ and $N\in\mathbb{Z}$, and assume that* $\dim\mathcal{P}(E)=\infty$. *Then* $\operatorname{Assoc}_N\mathcal{P}(E)$ *contains a real and zerofree function, if and only if for some $\varphi\in\mathbb{R}$ the function $q_\varphi:=S_\varphi^{-1}S_{\varphi+\frac{\pi}{2}}$ satisfies* (I), (II), *and* (III$_N$). *In this case these conditions hold for all $\varphi\in\mathbb{R}$, and the function $A_{q_\varphi}^{-1}S_\varphi$ is the (up to scalar multiples) unique real and zerofree element of $\bigcup_{k\in\mathbb{Z}}\operatorname{Assoc}_k\mathcal{P}(E)$.*

Before we turn to the proof of this theorem, let us discuss some of its aspects.

3.3 Remark. In Theorem 3.2 we impose the condition that $\dim\mathcal{P}(E)=\infty$. The case that $\mathcal{P}(E)$ is finite-dimensional is, however, trivial. Let us elaborate this situation. Denote by $\mathbb{C}[z]_n$, $n\in\mathbb{N}_0$, the set of all polynomials whose degree does not exceed n. If $d:=\dim\mathcal{P}(E)<\infty$, then there exists a real and zerofree function $U(z)$, such that

$$\mathcal{P}(E)=U(z)\cdot\mathbb{C}[z]_{d-1}\,. \tag{3.3}$$

For the Hilbert space case, this is shown in [9, Problem 88]. The indefinite case immediately follows from this by applying [21, Theorem 3.3].

From (3.3) it is obvious that

$$\operatorname{Assoc}_N\mathcal{P}(E)=\begin{cases}U(z)\cdot\mathbb{C}[z]_{d-1+N}\,, & N\geq-(d-1)\\ \{0\}\,, & N\leq -d\end{cases}$$

//

Let $E\in\mathcal{HB}^*_{<\infty}$ and $N\in\mathbb{Z}$. Since the set $\operatorname{Assoc}_N\mathcal{P}(E)$ does not depend on the inner product given on $\mathcal{P}(E)$, but only on the set $\mathcal{P}(E)$, Theorem 3.2 could also be formulated as follows.

3.4. *Theorem 3.2 reformulated:* *Let $\mathcal{X}$ be an infinite dimensional dB-normable space. Then* $\operatorname{Assoc}_N \mathcal{X}$ *contains a real and zerofree function, if and only if for some $E \in \mathcal{HB}^*_{<\infty}$ with $\mathcal{X} = \mathcal{P}(E)$ and some $\varphi \in \mathbb{R}$ the function $q_\varphi := S_\varphi^{-1} S_{\varphi+\frac{\pi}{2}}$ satisfies* (I), (II), *and* (III$_N$). *In this case these conditions hold for all $E \in \mathcal{HB}^*_{<\infty}$ with $\mathcal{X} = \mathcal{P}(E)$ and $\varphi \in \mathbb{R}$, and the function $A_{q_\varphi}^{-1} S_\varphi$ is the (up to scalar multiples) unique real and zerofree element of* $\bigcup_{k\in\mathbb{Z}} \operatorname{Assoc}_k \mathcal{P}(E)$.

3.5 Remark. Let $E \in \mathcal{HB}^*_{<\infty}$ and $N > 0$. Let $E_0 \in \mathcal{HB}^*_0$ be such that $\mathcal{P}(E) = \mathcal{P}(E_0)$ as sets; existence of E_0 is guaranteed by [21, Theorem 3.3]. Assuming knowledge of E_0, we could also decide with help [36, Theorem 5.1] whether or not $\operatorname{Assoc}_N \mathcal{P}(E)$ contains a real and zerofree element. Hence, Theorem 3.2 with $N > 0$ does not give us a wider class of dB-normable spaces with the property that there exists a real and zerofree element in $\operatorname{Assoc}_N \mathcal{P}(E)$ than the class we had obtained previously; it rather gives a wider class of tests whether a given dB-normable space has this property.

However, let us note that in general it is very hard to obtain more knowledge on E_0 than its pure existence. //

3.6 Remark. Let $\mathcal{X}$ be an infinite dimensional dB-normable space. Choose $E \in \mathcal{HB}^*_{<\infty}$ with $\mathcal{X} = \mathcal{P}(E)$ and $\varphi \in \mathbb{R}$.

(*i*) The fact whether or not the function $q_\varphi := S_\varphi^{-1} S_{\varphi+\frac{\pi}{2}}$ satisfies (I), (II), and (III$_N$), does not depend on the choice of E and φ.

(*ii*) If q_φ satisfies (I), (II), and (III$_N$) for some $N \in \mathbb{Z}$, then the function $A_{q_\varphi}^{-1} S_\varphi$ does (up to scalar multiples) not depend on the choice of E and φ.

(*iii*) We have $1 \in \operatorname{Assoc}_N \mathcal{P}(E)$ if and only if $q_\varphi := S_\varphi^{-1} S_{\varphi+\frac{\pi}{2}}$ satisfies (I), (II), and (III$_N$), and $A_{q_\varphi}^{-1} S_\varphi$ is constant.

//

We proceed to the proof of Theorem 3.2. Throughout the remainder of this section, let $E \in \mathcal{HB}^*_{<\infty}$ with $\dim \mathcal{P}(E) = \infty$ be fixed. We first show necessity of the conditions stated in Theorem 3.2.

Proof (of Theorem 3.2, necessity). Let $N \in \mathbb{Z}$, and assume that there exists a real and zerofree function U in $\operatorname{Assoc}_N \mathcal{P}(E)$. Moreover, let $\varphi \in \mathbb{R}$ be given.

The function $U^{-1} S_\varphi$ is real, entire, and of bounded type in both half-planes. Its zeros λ_k coincide with the zeros of S_φ including multiplicities. By [35, V.Lehrsatz 11, p.249] we have:

(*i*) The limit $\lim_{r\to\infty} \sum_{0<|\lambda_k|<r} \frac{1}{\lambda_k}$ exists in $\mathbb{R}$.

(*ii*) The limits
$$\lim_{\substack{k\to\infty\\ |\arg\gamma_k|<\frac{\pi}{4}}} \frac{k}{\gamma_k} \qquad \text{and} \qquad \lim_{\substack{k\to\infty\\ |\arg\gamma_k-\pi|<\frac{\pi}{4}}} \frac{k}{\gamma_k}$$
exist in $\mathbb{R}$ and have the same value.

We already see that $q_\varphi = S_\varphi^{-1} S_{\varphi+\frac{\pi}{2}}$ satisfies (I) and (II). Thus the product A_q converges. Again by [35, V.Lehrsatz 11, p.249], remember here that S_φ is real, we have:

(*iii*) We can factorize

$$S_\varphi(z) = C \cdot U(z) \cdot A_q(z)\,, \tag{3.4}$$

with some constant $C \in \mathbb{C}$.

For the proof of (III_N) we will consider the cases '$N > 0$' and '$N \leq 0$' separately.

Case $N > 0$: Fix $w \in \mathbb{C}^+ \setminus Z^\varphi$ and consider the function $\tilde{U}(z) := (\mathcal{R}_{S_\varphi;w})^N U(z)$, where $\mathcal{R}_{S_\varphi;w}$ is the difference quotient operator (2.1).

Since U is a real and zerofree element of $\operatorname{Assoc}_N \mathcal{P}(E)$, we have $\operatorname{span}\{U\} \in \operatorname{Sub}\operatorname{Assoc}_N \mathcal{P}(E)$. Since $\mathcal{P}(E)[-]\mathfrak{E}^\varphi_\infty$ is a dB-subspace of $\mathcal{P}(E)$, Proposition 2.14 yields that $\operatorname{Assoc}_N(\mathcal{P}(E)[-]\mathfrak{E}^\varphi_\infty) \in \operatorname{Sub}\operatorname{Assoc}_N \mathcal{P}(E)$. The Ordering Theorem implies $U \in \operatorname{Assoc}_N(\mathcal{P}(E)[-]\mathfrak{E}^\varphi_\infty)$, and hence

$$\tilde{U} \in \mathcal{P}(E)[-]\mathfrak{E}^\varphi_\infty\,.$$

By Proposition 2.6, therefore (note that $\sigma_k = S'_\varphi(\gamma_k)^{-1} S_{\varphi+\frac{\pi}{2}}(\gamma_k)$)

$$\sum_{k\in\mathbb{N}} |\tilde{U}(\gamma_k)|^2 \frac{1}{S'_\varphi(\gamma_k)^2 \sigma_k} < \infty\,. \tag{3.5}$$

A straightforward inductive argument shows that

$$\tilde{U}(z) = \frac{U(z)}{(z-w)^N} + S_\varphi(z) \sum_{l=1}^N \frac{\xi_l}{(z-w)^l}\,,$$

with some constants $\xi_l \in \mathbb{C}$. Moreover, by (3.4),

$$|S'_\varphi(\gamma_k)| = |C| \cdot |U(\gamma_k)| \cdot |A'_q(\gamma_k)|, \quad k \in \mathbb{N}\,. \tag{3.6}$$

Hence, the series (3.5) is nothing else but

$$\sum_{k\in\mathbb{N}} \frac{1}{|\gamma_k - w|^{2N} |C|^2 A'_q(\gamma_k)^2 \sigma_k}\,.$$

Convergence of the series (3.2) follows.

Case $N \leq 0$: We use a similar argument. First, $\operatorname{span}\{U\} \in \operatorname{Sub}\operatorname{Assoc}_N \mathcal{P}(E)$. Since $\dim \mathcal{P}(E) = \infty$, also $\dim(\mathcal{P}(E)[-]\mathfrak{E}^\varphi_\infty) = \infty$, and hence there exists $\mathcal{Y} \in \operatorname{Sub}\operatorname{Assoc}_N \mathcal{P}(E)$ such that

$$\operatorname{Assoc}_{-N} \mathcal{Y} = \mathcal{P}(E)[-]\mathfrak{E}^\varphi_\infty\,.$$

By the Ordering Theorem, we have $U \in \mathcal{Y}$ and hence $z^{-N}U(z) \in \mathcal{P}(E)[-]\mathfrak{E}^\varphi_\infty$. Thus

$$\sum_{k\in\mathbb{N}} \left|\gamma_k^{-N} U(\gamma_k)\right|^2 \frac{1}{S'_\varphi(\gamma_k)^2 \sigma_k} < \infty\,,$$

and, using (3.6),

$$\sum_{k\in\mathbb{N}} \gamma_k^{-2N} \frac{1}{A'_q(\gamma_k)^2 \sigma_k} < \infty\,.$$

❑

In the proof of sufficiency, we will employ the below result on polynomial growth. The symbol '$\lim_{z \widehat{\to} i\infty}$' denotes the nontangential limit of z to $i\infty$, i.e. that z may tend to infinity inside an arbitrary Stolz angle $\Gamma_\alpha := \{z \in \mathbb{C} : \alpha \leq \arg z \leq \pi - \alpha\}$ where $\alpha \in (0, \frac{\pi}{2})$.

3.7 Lemma. *Let $E \in \mathcal{H}B^*_{<\infty}$ and $\varphi \in \mathbb{R}$. For some sufficiently large positive integer n_0 we have*

$$\lim_{z \widehat{\to} i\infty} \frac{1}{z^{n_0}} \frac{F(z)}{S_\varphi(z)} = 0, \quad F \in \mathcal{P}(E) .$$

Proof. By [21, Lemma 6.4], a parameterization of the defect elements of $\mathcal{S}_E$ associated with $\mathcal{A}_\varphi$ is given as

$$\chi(z, \zeta) := \frac{1}{S_\varphi(z)} K_E(\overline{z}, \zeta), \quad z \in \mathbb{C},\ S_\varphi(z) \neq 0 .$$

That means we have

$$\chi(z, .) = \big(I + (z - z_0)(\mathcal{A}_\varphi - z)^{-1}\big)\chi(z_0, .), \quad z, z_0 \in \mathbb{C},\ S_\varphi(z), S_\varphi(z_0) \neq 0 .$$

Let $\lambda_0 \in \mathbb{R} \setminus Z^\varphi$, then $(\mathcal{A}_\varphi - \lambda_0)^{-1}$ is a selfadjoint operator in the Pontryagin space $\mathcal{P}(E)$, and thus also definitizable, cf. [34, §I.3]. Using [34, Proposition II.2.1], we find a number $n_0 \in \mathbb{N}$ such that

$$\lim_{z \widehat{\to} i\infty} \frac{1}{z^{n_0}} \big\|(\mathcal{A}_\varphi - z)^{-1}\big\| = 0 .$$

Since $S_\varphi(z)^{-1} F(z) = [F, \chi(z, .)]$, the desired assertion follows. ❑

Proof (of Theorem 3.2, sufficiency). Let $N \in \mathbb{Z}$ and $\varphi \in \mathbb{R}$, and assume that the conditions (I), (II), and (III$_N$) hold for q_φ.

Step 1; The space $\mathcal{P}(\mathring{E})$: The function $A_{q_\varphi}(z)$ is real and has the same zeros as S_φ including multiplicities. Hence, the function

$$U(z) := \frac{S_\varphi(z)}{A_{q_\varphi}(z)}$$

is entire, real, and zerofree. Define

$$B_{q_\varphi}(z) := \frac{S_{\varphi+\frac{\pi}{2}}(z)}{U(z)}, \quad \mathring{E}(z) := A_{q_\varphi}(z) - iB_{q_\varphi}(z) ,$$

then we have

$$U(z) \cdot \mathring{E}(z) = U(z)A_{q_\varphi}(z) - iU(z)B_{q_\varphi}(z) = S_\varphi(z) - iS_{\varphi+\frac{\pi}{2}}(z) = -ie^{i\varphi}E(z) .$$

It follows that $\mathring{E} \in \mathcal{H}B^*_{<\infty}$, and that the map $\mu : F \mapsto UF$ is an isometric isomorphism of $\mathcal{P}(\mathring{E})$ onto $\mathcal{P}(E)$. It is therefore also a linear bijection of $\operatorname{Assoc}_n \mathcal{P}(\mathring{E})$ onto $\operatorname{Assoc}_n \mathcal{P}(E)$ for each $n > 0$. Moreover, since $\mathcal{S}_E \circ \mu = \mu \circ \mathcal{S}_{\mathring{E}}$, μ also maps $\operatorname{dom} \mathcal{S}_{\mathring{E}}^{|n|}$ bijectively onto $\operatorname{dom} \mathcal{S}_E^{|n|}$, $n < 0$.

Let us record that $A_{q_\varphi} = \mathring{S}_{\frac{\pi}{2}}$ and $B_{q_\varphi} = \mathring{S}_\pi$, when $\mathring{S}_\varphi$ denotes the function defined for $\mathring{E}$ as S_φ was defined for E in (2.2).

Step 2; $\mathring{E}$ is of bounded type: We are going to apply [36, Lemma 5.5] with the sequence $(\gamma_k)_{k\in\mathbb{N}}$. Let $\gamma(z)$ denote the product

$$\gamma(z) := \lim_{r\to\infty} \prod_{|\gamma_k|\le r} \Big(1 - \frac{z}{\gamma_k}\Big)\,,$$

and set $p(z) := \prod_{\substack{j=1\\ \alpha_j\neq 0}}^{r} (-\alpha_j)^{d_j} \prod_{j=1}^{r}(z-\alpha_j)^{d_j}$, so that $A_{q_\varphi} = p\gamma$. Moreover, note that

$$\sigma_k = -\frac{S_{\varphi+\frac{\pi}{2}}(\gamma_k)}{S'_\varphi(\gamma_k)} = -\operatorname{Res}(q_\varphi;\gamma_k) = -\frac{B_{q_\varphi}(\gamma_k)}{A'_{q_\varphi}(\gamma_k)}\,. \tag{3.7}$$

Since $q_\varphi \in \mathcal{N}_{<\infty}$, there exists a positive integer n_1 such that

$$\sum_{k\in\mathbb{N}} \frac{\sigma_k}{\gamma_k^{n_1}} < \infty\,,$$

cf. [30, Satz 3.1]. Let $m \in \mathbb{N}$ be such that $2m \ge \max\{2N, n_1\} + \deg p$. Using our assumption (3.2), the fact that $A'_{q_\varphi}(\gamma_k) = p(\gamma_k)\gamma'(\gamma_k)$, and (3.7), we obtain the estimate

$$\sum_{k\in\mathbb{N}} \frac{1}{\gamma_k^{2m}|\gamma'(\gamma_k)|} \le \sum_{k\in\mathbb{N}} \frac{|p(\gamma_k)|}{\gamma_k^{2m}|A'_{q_\varphi}(\gamma_k)|} \underbrace{\Big(|B_{q_\varphi}(\gamma_k)| + \frac{1}{|B_{q_\varphi}(\gamma_k)|}\Big)}_{\ge 1} =$$

$$= \sum_{k\in\mathbb{N}} \frac{|p(\gamma_k)|}{\gamma_k^{2m}}\sigma_k + \sum_{k\in\mathbb{N}} \frac{|p(\gamma_k)|}{\gamma_k^{2m} A'_{q_\varphi}(\gamma_k)^2\sigma_k} < \infty\,.$$

Hence, the hypothesis of [36, Lemma 5.5] is satisfied, and it follows that $\gamma(z)$, and thus also $A_{q_\varphi}(z)$, is of bounded type in $\mathbb{C}^+$. By [30, Satz 6.4] and fractional linear transformation, each generalized Nevanlinna function is a meromorphic function of bounded type in $\mathbb{C}^+$ (meaning a quotient of two bounded analytic functions). Hence, the entire function $\mathring{E}(z) = A_{q_\varphi}(z)(1 - iq_\varphi(z))$ is of bounded type in $\mathbb{C}^+$.

The fact that $\mathring{E}(z)$ is of bounded type, also implies that each element $f \in \bigcup_{n\in\mathbb{Z}} \operatorname{Assoc}_n \mathcal{P}(\mathring{E})$ is of bounded type in both half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$.

Step 3, The functions h and Λ, Case $N \le 0$: Let Φ and $\ell^2_{\frac{\pi}{2}}$ be the map and weighted ℓ^2-space as constructed in Section 2, using the de Branges space $\mathcal{P}(\mathring{E})$ and the angle $\varphi := \frac{\pi}{2}$. Note that, in the notation of Section 2,

$$Z_+^{\frac{\pi}{2}} \setminus \{0\} = \{\gamma_k :\, k\in\mathbb{N}\}\,,$$

$$\check{Z}^{\frac{\pi}{2}} \setminus (\{0\}\times\mathbb{N}) = \{(\alpha_j,k) :\, 1\le j\le r,\, 0\le k< d_j\} \setminus (\{0\}\times\mathbb{N})\,.$$

Moreover, by (3.7), we have $A'_{q_\varphi}(\gamma_k)^2\sigma_k = -A'_{q_\varphi}(\gamma_k)B_{q_\varphi}(\gamma_k)$.

Consider the function $h(z) := z^{-N}$. Then our assumption (3.2) just says that

$$\big(h(\alpha)\big)_{\alpha\in Z_+^{\frac{\pi}{2}}} \in \ell^2_{\frac{\pi}{2}}\,. \tag{3.8}$$

For each $r_0 > 0$, the function $\gamma_{r_0}(z) := \prod_{|\gamma_k|\geq r_0}(1-\gamma_k^{-1}z)$ is entire and real. Moreover, $|\gamma_{r_0}(iy)|$ is nondecreasing for $y > 0$. Since $Z_+^{\frac{\pi}{2}}$ is infinite, remember here that $\dim \mathcal{P}(E) = \infty$, we conclude that

$$\lim_{y\to\pm\infty} \frac{h(iy)}{A_{q_\varphi}(iy)} = 0\,. \tag{3.9}$$

Since h is a polynomial, trivially

$$h \text{ is of bounded type in } \mathbb{C}^+ \text{ and } \mathbb{C}^-\,. \tag{3.10}$$

Due to (3.8) and Proposition 2.6, there exists $f \in \mathcal{P}(\mathring{E})[-]\mathfrak{E}_\infty^{\frac{\pi}{2}}$ with $\Phi f = \Phi h$. Consider the function

$$\Lambda(z) := \frac{h(z) - f(z)}{A_{q_\varphi}(z)}\,.$$

This function is entire by our choice of f, and of bounded type in both half-planes by (3.10). Using Kreĭn's Theorem, cf. [37, Theorem 6.17, 6.18], we obtain that Λ is of exponential type τ, where

$$\tau := \max\Big\{ \lim_{y\to+\infty} \frac{1}{y}\log^+|\Lambda(iy)|, \lim_{y\to-\infty}\frac{1}{|y|}\log^+|\Lambda(iy)|\Big\}\,.$$

By (3.9) and Lemma 3.7, for some sufficiently large positive integer n_0,

$$\lim_{y\to\pm\infty} \frac{1}{|y|^{n_0}}\Lambda(iy) = 0\,. \tag{3.11}$$

Hence $\tau = 0$, and the Phragmén-Lindelöf Principle, applied on the right and left half-plane seperately, implies that Λ is a polynomial.

Step 4, Finish of proof, Case $N \leq 0$: Let $\tilde{p}$ be a nonconstant polynomial with real coefficients, and put

$$\tilde{A} := A_{q_\varphi},\ \tilde{B} := B_{q_\varphi} + \tilde{p}A_{q_\varphi}, \quad \tilde{E} := \tilde{A} - i\tilde{B}\,.$$

Then a short computation shows

$$K_{\tilde{E}}(w,z) = K_{\mathring{E}}(w,z) + \underbrace{A_{q_\varphi}(z)\frac{\tilde{p}(z)-\tilde{p}(\overline{w})}{z-\overline{w}}A_{q_\varphi}(\overline{w})}_{=:K_2(w,z)}\,.$$

In particular, $\tilde{E} \in \mathcal{H}B^*_{<\infty}$. The reproducing kernel space $\mathcal{P}_2$ generated by the kernel $K_2(w,z)$ is of a simple form. Namely, we have

$$\mathcal{P}_2 = A_{q_\varphi}(z)\cdot\mathbb{C}[z]_{\deg\tilde{p}-1}\,,$$

and the inner product $[.,.]_2$ on $\mathcal{P}_2$ is given as

$$[A_{q_\varphi}(z)z^i, A_{q_\varphi}(z)z^j] = \nu_{i+j}, \quad 0 \leq i,j < \deg\tilde{p}\,,$$

with some real numbers ν_l satisfying

$$\nu_0 = \ldots = \nu_{\deg\tilde{p}-2} = 0, \quad \nu_{\deg\tilde{p}-1} \neq 0\,,$$

see, e.g., [25, Proposition 2.8]. We conclude that

$$\mathcal{P}_{2;0} := \operatorname{span}\big\{A_{q_\varphi}(z)z^k : 0 \leq k < [\frac{\deg \tilde{p}}{2}]\big\}$$

is a neutral subspace of $\mathcal{P}_2$.

Let us make the particular choice $\tilde{p}(z) := z^{2\max\{\deg\Lambda,\delta_{\frac{\pi}{2}}\}+2}$, where $\delta_{\frac{\pi}{2}} := \dim \mathfrak{E}_\infty^{\frac{\pi}{2}}$. Then, certainly, $\tilde{p}$ is nonconstant. Moreover, we have

$$\mathcal{P}(\mathring{E}) \cap \mathcal{P}_2 = \mathfrak{E}_\infty^{\frac{\pi}{2}} \subseteq \mathcal{P}_{2;0} = \operatorname{span}\big\{A_{q_\varphi}(z)z^k : 0 \leq k \leq \max\{\deg\Lambda, \delta_{\frac{\pi}{2}}\}\big\},$$

and hence $\mathcal{P}_{2;0} \subseteq \mathcal{P}_2[-]_2(\mathcal{P}(\mathring{E}) \cap \mathcal{P}_2)$. By Proposition 2.2 we have

$$\big(\mathcal{P}(\mathring{E})[-]\mathfrak{E}_\infty^{\frac{\pi}{2}}\big) + \mathcal{P}_{2;0} \subseteq \mathcal{P}(\tilde{E}).$$

It follows that

$$z^{-N} = h(z) = f(z) + A_{q_\varphi}(z)\Lambda(z) \in \mathcal{P}(\tilde{E}),$$

and hence that $\mathbb{C}[z]_{-N} \subseteq \mathcal{P}(\tilde{E})$.

Since $\mathcal{P}(\mathring{E})[-]\mathfrak{E}_\infty^{\frac{\pi}{2}}$ is a dB-subspace of $\mathcal{P}(\mathring{E})$, it is division- and reflection invariant. Proposition 2.2 implies that it is closed in $\mathcal{P}(\tilde{E})$, and hence is a dB-subspace of $\mathcal{P}(\tilde{E})$. By the Ordering Theorem, and the fact that $\dim(\mathcal{P}(\mathring{E})[-]\mathfrak{E}_\infty^{\frac{\pi}{2}}) = \infty$, we obtain

$$\mathbb{C}[z]_{-N} \subseteq \mathcal{P}(\mathring{E})[-]\mathfrak{E}_\infty^{\frac{\pi}{2}} \subseteq \mathcal{P}(\mathring{E}).$$

This shows that $1 \in \operatorname{dom}\mathcal{S}_{\mathring{E}}^{|N|}$. By what we saw in Step 1, this implies $U \in \operatorname{dom}\mathcal{S}_E^{|N|}$. We have finished the proof of sufficiency for the case $N \leq 0$.

Step 5, The functions h and Λ, Case $N > 0$: Choose $w \in \mathbb{C}\setminus Z^\varphi$, so that we have $A_{q_\varphi}(w) \neq 0$, and consider the iterated difference quotient $h := (R_{A_{q_\varphi};w})^N 1$. By elementary induction we see that there exist constants $\xi_l \in \mathbb{C}$, $l = 1, \ldots, N$, such that

$$h(z) = \frac{1}{(z-w)^N} + A_{q_\varphi}(z)\sum_{l=1}^N \frac{\xi_l}{(z-w)^l}.$$

Our assumption (3.2) says that $((\alpha - w)^{-N})_{\alpha\in Z_+^{\frac{\pi}{2}}} \in \ell^2_{\frac{\pi}{2}}$. However, $h(\alpha) = (\alpha - w)^{-N}$, $\alpha \in Z_+^{\frac{\pi}{2}}$, and hence h satisfies (3.8). Clearly, also (3.9) and (3.10) are satisfied. By exactly the same argument as carried out in Step 3, we obtain that for some $f \in \mathcal{P}(\mathring{E})[-]\mathfrak{E}_\infty^{\frac{\pi}{2}}$ the function $\Lambda := A_{q_\varphi}^{-1}(h - f)$ is a polynomial.

Step 6, Finish of proof, Case $N > 0$: We construct the space $\mathcal{P}(\tilde{E})$ in exactly the same way as we did in Step 4, and obtain that $(R_{A_{q_\varphi};w})^N 1 \in \mathcal{P}(\tilde{E})$. This gives $1 \in \operatorname{Assoc}_N \mathcal{P}(\tilde{E})$. Corollary 2.15 used with the dB-subspace $\mathcal{Y} := \mathcal{P}(\mathring{E})[-]\mathfrak{E}_\infty^{\frac{\pi}{2}}$ of $\mathcal{P}(\tilde{E})$ implies that

$$1 \in \operatorname{Assoc}_N \mathcal{Y} \subseteq \operatorname{Assoc}_N \mathcal{P}(\mathring{E}).$$

We conclude from Step 1 that $U \in \operatorname{Assoc}_N \mathcal{P}(E)$, and hence have finished the proof of sufficiency also for the case $N > 0$. ✌

4 Intermediate Weyl-coefficients

The notion of intermediate Weyl-coefficients appears in connection with an indefinite generalization of canonical systems, and was first observed in [23] in the setting of maximal chains of matrices.

In order to explain this notion, we need to recall the definition of a general (indefinite) Hamiltonian, cf. [24]. This requires some background. We call a function H a Hamiltonian, if it is defined on some interval (L_-, L_+), takes real and non-negative 2×2-matrices as values, is locally integrable on (L_-, L_+), and does not vanish on any set of positive measure.

We say that H is in the limit circle case or limit point case at L_+, if for one (and hence for all) $\alpha \in (L_-, L_+)$ we have

$$\int_\alpha^{L_+} \operatorname{tr} H(t)\, dt < \infty \quad \text{or} \quad \int_\alpha^{L_+} \operatorname{tr} H(t)\, dt = \infty, \text{ respectively}.$$

Similarly, we distinguish limit circle/point case at the endpoint L_-, depending whether $\int_{L_-}^{\alpha} \operatorname{tr} H(t)\, dt$ is finite or infinite.

An interval (α, β) is called H-indivisible of type ϕ if

$$H(t) = h(t)\xi_\phi \xi_\phi^T, \quad t \in (\alpha, \beta),$$

where $\xi_\phi := (\cos\phi, \sin\phi)^T$ and $h(t)$ is some scalar function.

With a Hamiltonian H, which is in the limit circle case at L_-, in [24, Definition 3.1] a number $\Delta(H) \in \mathbb{N} \cup \{0, \infty\}$ was associated. This number measures in some sense the growth of H towards L_+. For example, $\Delta(H) = 0$ means that $\int_{L_-}^{L_+} \operatorname{tr} H(t)\, dt < \infty$; or if $\int_{L_-}^{L_+} \operatorname{tr} H(t)\, dt = \infty$ and for some $L_1 < L_+$ the interval (L_1, L_+) is H-indivisible, then $\Delta(H) = 1$.

Assume that H is in the limit circle case at L_- and in the limit point case at L_+. Then we say that H satisfies the condition (HS), if the resolvents of one (and hence of all) self-adjoint extensions of the minimal operator $T_{\min}(H)$ associated with H are Hilbert–Schmidt operators, cf. [24, §2]. In this case, the growth of H towards L_+ is bounded in one (and extremal in another) direction, in the sense that for a unique angle $\phi(H) \in [0, \pi)$ we have

$$\int_{L_-}^{L_+} \xi_{\phi(H)}^T H(t) \xi_{\phi(H)}\, dt < \infty,$$

cf. [29, Theorem 2.4]. The direction of extremal growth is then $\xi_{\phi(H)+\frac{\pi}{2}}$.

If H is a Hamiltonian on (L_-, L_+) and $\alpha \in (L_-, L_+)$, then $H_+(t) := H|_{(\alpha, L_+)}(t)$ and $H_-(t) := H|_{(L_-, \alpha)}(-t)$ are Hamiltonians defined on (α, L_+) or $(-\alpha, -L_-)$, respectively. Both, H_+ and H_-, are in the limit circle case at their left endpoint. At their right endpoint limit circle or limit point case prevails depending on the behaviour of H at L_+ or L_-, respectively.

Numbers $\Delta_\pm(H)$ are defined as $\Delta_\pm(H) := \Delta(H_\pm)$. Moreover, we say that H satifies (HS$_+$) or (HS$_-$) if H_+ or H_-, respectively, satisfies (HS). Numbers $\phi_\pm(H)$ are defined correspondingly. Let us note that each of these notions is independent of the choice of α in the definition of $H_\pm$, cf. [24, Lemma 3.12].

4.1 Definition ([24])**.** A general Hamiltonian $\mathfrak{h}$ is a collection of data of the following kind:

(*i*) $n \in \mathbb{N} \cup \{0\}$, $\sigma_0, \ldots, \sigma_{n+1} \in \mathbb{R} \cup \{\pm\infty\}$ with $\sigma_0 < \sigma_1 < \ldots < \sigma_{n+1}$,

(*ii*) Hamiltonians H_i, $i = 0, \ldots, n$, defined on the respective intervals (σ_i, σ_{i+1}),

(*iii*) numbers $\ddot{o}_1, \ldots, \ddot{o}_n \in \mathbb{N} \cup \{0\}$ and $b_{i,1}, \ldots, b_{i,\ddot{o}_i+1} \in \mathbb{R}$, $i = 1, \ldots, n$, with $b_{i,1} \neq 0$ in the case $\ddot{o}_i \geq 1$,

(*iv*) numbers $d_{i,0}, \ldots, d_{i,2\Delta_i - 1} \in \mathbb{R}$, $i = 1, \ldots, n$, where $\Delta_i := \max\{\Delta_+(H_{i-1}), \Delta_-(H_i)\}$,

(*v*) a finite subset E of $\{\sigma_0, \sigma_{n+1}\} \cup \bigcup_{i=0}^{n} (\sigma_i, \sigma_{i+1})$,

which is assumed to be subject to the following conditions:

(H1) H_0 is in the limit circle case at σ_0 and, if $n \geq 1$, in the limit point case at σ_1. H_i is in the limit point case at both endpoints σ_i and σ_{i+1}, $i = 1, \ldots, n-1$. If $n \geq 1$, then H_n is in the limit point case at σ_n.

(H2) For $i = 1, \ldots, n-1$ the interval (σ_i, σ_{i+1}) is not H_i-indivisible. If H_n is in the limit point case at σ_{n+1}, then also (σ_n, σ_{n+1}) is not H_n-indivisible.

(H3) We have $\Delta_i < \infty$, $i = 1, \ldots, n$. Moreover, H_0 satisfies (HS$_+$), H_i satisfies (HS$_-$) and (HS$_+$) for $i = 1, \ldots, n-1$, and H_n satisfies (HS$_-$).

(H4) We have $\phi_+(H_{i-1}) = \phi_-(H_i)$, $i = 1, \ldots, n$.

(H5) Let $i \in \{1, \ldots, n\}$. If for some $\epsilon > 0$ the interval $(\sigma_i - \epsilon, \sigma_i)$ is H_{i-1}-indivisible and the interval $(\sigma_i, \sigma_i + \epsilon)$ is H_i-indivisible, then $d_1 = 0$. If additionally $b_{i,1} = 0$, then also $d_0 < 0$.

(E1) $\sigma_0, \sigma_{n+1} \in E$, and $E \cap (\sigma_i, \sigma_{i+1}) \neq \emptyset$ for $i = 1, \ldots, n-1$. If H_n is in the limit point case at σ_{n+1}, then also $E \cap (\sigma_n, \sigma_{n+1}) \neq \emptyset$. Let $i \in \{0, \ldots, n\}$; if (α, σ_{i+1}) or (σ_i, α) is a maximal H_i-indivisible interval, then $\alpha \in E$.

(E2) No point of E is an inner point of an indivisible interval.

The number

$$\operatorname{ind}_- \mathfrak{h} := \sum_{i=1}^{n} \left(\Delta_i + \left[\frac{\ddot{o}_i}{2} \right] \right) + \left| \left\{ 1 \leq i \leq n : \ddot{o}_i \text{ odd}, b_{i,1} > 0 \right\} \right|$$

is called the negative index of the general Hamiltonian $\mathfrak{h}$. Moreover, $\mathfrak{h}$ is called definite if $\operatorname{ind}_- \mathfrak{h} = 0$, and indefinite otherwise. We say that $\mathfrak{h}$ is in the limit point case or limit circle case if H_n has the respective property at σ_{n+1}. //

In order to shorten notation we shall write a general Hamiltonian $\mathfrak{h}$ which is given by the data n, $\sigma_0, \ldots, \sigma_{n+1}$, $H_0, \ldots, H_n$, $\ddot{o}_1, \ldots, \ddot{o}_n$, $b_{i,j}$, $d_{i,j}$, E, as a triple

$$\mathfrak{h} = (H, \mathfrak{b}, \mathfrak{d}),$$

where H represents the Hamiltonians H_i, including their number n and their domains of definition (σ_i, σ_{i+1}), $\mathfrak{b}$ represents the numbers $\ddot{o}_i$ and $b_{i,j}$, and $\mathfrak{d}$ represents the numbers $d_{i,j}$ and the subset E. Obviously, we may also identify H with the function defined on $\bigcup_{i=0}^{n}(\sigma_i, \sigma_{i+1})$ by

$$H|_{(\sigma_i,\sigma_{i+1})} = H_i, \quad i = 0, \dots, n\,. \tag{4.1}$$

We will speak of H as the Hamiltonian function of $\mathfrak{h}$.

4.2 Remark. Intuitively, the notion of a general Hamiltonian can be understood as follows: we deal with the differential equation $f' = zJHf$ given on an interval (σ_0, σ_{n+1}) which involves some kind of singularities located at the points σ_i, $i = 1, \dots, n$. Condition (H1) says that the differential equation is regular at σ_0, so that the initial value problem at σ_0 is well posed, but that $\sigma_1, \dots, \sigma_n$ actually are singularities. Moreover, and this is the condition (H2), two adjacent singularities σ_i and σ_{i+1} must be separated by more than just a single indivisible interval. The meaning of (H3) is that the growth of H_i towards a singularity is not too fast. Moreover, (H4) is an interface condition at σ_i.

The numbers $\ddot{o}_i \in \mathbb{N} \cup \{0\}$ and $b_{i,1}, \dots, b_{i,\ddot{o}_i+1}$ model the part of the singularity σ_i which is concentrated at σ_i, whereas the numbers $d_{i,0}, \dots, d_{i,2\Delta_i-1}$ model the part of this singularity which is in interaction with the local behaviour around σ_i. The elements of E in the vicinity of σ_i determine quantitatively what 'local' here means. The freedom of this interaction is, by the first part of (H5), restricted if to both sides of σ_i indivisible intervals adjoin. The possibility that on both sides of σ_i indivisible intervals adjoin and at the same time $b_{i,1} = 0$, can occur by the second part of (H5) only in the case of 'indivisible intervals of negative length', the simplest possible kind of a singularity. //

In the theory of (indefinite) canonical systems a class of entire 2×2-matrix valued functions plays an important role. Let us recall this notion. Let $W = (w_{ij})_{i,j=1,2}$ be an entire 2×2-matrix valued function with $w_{ij}(\overline{z}) = \overline{w_{ij}(z)}$, $\det W(z) = 1$, $z \in \mathbb{C}$, and $W(0) = I$. Then we write $W \in \mathcal{M}_{<\infty}$, if the kernel (for $z = \overline{w}$ this formula has to be interpreted appropriately as a derivative)

$$H_W(w,z) := \frac{W(z)JW(w)^* - J}{z - \overline{w}}, \; z, w \in \mathbb{C}\,,$$

has a finite number of negative squares. If the exact number of negative squares of this kernel is equal to $\kappa \in \mathbb{N}_0$, we write $W \in \mathcal{M}_\kappa$ and $\operatorname{ind}_- W = \kappa$. For more details on the class $\mathcal{M}_{<\infty}$ see, e.g., [25].

To each general Hamiltonian $\mathfrak{h}$ which is in the limit circle case, an entire 2×2-matrix function $W_\mathfrak{h}$ is associated; its monodromy matrix. This matrix function belongs to the class $\mathcal{M}_{<\infty}$. If $\mathfrak{h}$ is in the limit point case, a function $q_\mathfrak{h}(z)$ is associated to $\mathfrak{h}$; its Weyl-coefficient. This function belongs to the class $\mathcal{N}_{<\infty}$. These constructions were carried out in [25]. The following two fundamental results have been proved in [26]:

4.3. *Inverse Spectral Theorem; limit circle case:* *The assignment $\mathfrak{h} \mapsto W_\mathfrak{h}$ establishes a bijective correspondence between the set of all general Hamiltonians in the limit circle case (modulo reparameterization) and the set $\mathcal{M}_{<\infty}$. Thereby* $\operatorname{ind}_- \mathfrak{h} = \operatorname{ind}_- W_\mathfrak{h}$.

4.4. ***Inverse Spectral Theorem; limit point case:*** *The assignment* $\mathfrak h \mapsto q_{\mathfrak h}$ *establishes a bijective correspondence between the set of all general Hamiltonians in the limit point case (modulo reparameterization) and the set* $\mathcal N_{<\infty}$ *of all generalized Nevanlinna functions. Thereby* $\operatorname{ind}_- \mathfrak h = \operatorname{ind}_- q_{\mathfrak h}$.

We turn to intermediate Weyl-coefficients. Let $\mathfrak h$ be a general Hamiltonian given by data as in Definition 4.1, and let σ be one of its singularities, i.e. $\sigma = \sigma_j$ with some $j \in \{1, \ldots, n\}$. Then we may define another general Hamiltonian $\mathfrak h_{\Lsh\sigma} := (H_{\Lsh\sigma}, \mathfrak b_{\Lsh\sigma}, \mathfrak d_{\Lsh\sigma})$ as the collection of data

$$\begin{array}{lll} H_{\Lsh\sigma}: & j-1, \quad \sigma_0, \ldots, \sigma_j, & H_i,\ i = 0, \ldots, j-1, \\ \mathfrak b_{\Lsh\sigma}: & \ddot o_i, b_{i,1}, \ldots, b_{i,\ddot o_i+1},\ i = 1, \ldots, j-1, & \\ \mathfrak d_{\Lsh\sigma}: & d_{i,0}, \ldots, d_{i,2\Delta_i-1},\ i = 1, \ldots, j-1, & \big(E \cap [\sigma_0, \sigma)\big) \cup \{\sigma\}\,. \end{array}$$

Since σ is a singularity of $\mathfrak h$, the general Hamiltonian $\mathfrak h_{\Lsh\sigma}$ is in the limit point case.

4.5 Definition. Let $\mathfrak h$ be a general Hamiltonian given by data as in Definition 4.1, and let σ be one of its singularities. Then the function $q_{\mathfrak h_{\Lsh\sigma}}$ is called the intermediate Weyl-coefficient of $\mathfrak h$ at σ. //

The question which functions $q \in \mathcal N_{<\infty}$ are intermediate Weyl-coefficients of some general Hamiltonian suggests itself. A simple necessary condition is that q is meromorphic in the whole plane. However, this is by far not sufficient.

4.6 Definition. Let $q \in \mathcal N_{<\infty}$. Then q is called an intermediate Weyl-coefficient, if there exists a general Hamiltonian $\mathfrak h$ such that q is the intermediate Weyl-coefficient of $\mathfrak h$ at one of its singularities σ. In this case, the minimum of the numbers $\operatorname{ind}_- \mathfrak h$, when $\mathfrak h$ varies through all general Hamiltonians with this property, is called the weight of q. //

4.7 Remark. Let $q \in \mathcal N_{<\infty}$ and let $\mathfrak h$ be the (unique) general Hamiltonian $\mathfrak h$ with $q_{\mathfrak h} = q$. Expressed in terms of $\mathfrak h$, the function q is an intermediate Weyl-coefficient if and only if $\mathfrak h$ can be prolongued to a 'longer' general Hamiltonian.

It only needs a short look at Definition 4.1 in order to see when this is possible and what the minimal increase of negative index is: *The general Hamiltonian* $\mathfrak h$ *can be prolongued if and only if on the last interval* (σ_n, σ_{n+1}) *of its domain the condition* (HS$_+$) *holds and* $\Delta_+(H_{(\sigma_n,\sigma_{n+1})}) < \infty$. //

In terms of the function q itself it is not that simple to decide whether q is an intermediate Weyl-coefficient. For $q \in \mathcal N_0$ we gave a characterization in [36, Theorem 5.1]. Especially the actual value of the weight of q is a highly sensitive magnitude, interesting examples were given in [36, Corollary 5.10, Corollary 5.11].

Using Theorem 3.2, we obtain complete answers to these questions. This is the second main result of the present paper.

4.8 Theorem. *Let* $q \in \mathcal N_{<\infty}$ *be given. Then* q *is an intermediate Weyl-coefficient if and only if* q *satisfies for some* $N \in \mathbb N$ *the conditions* (I), (II), *and* (III$_N$). *In this case, we have*

$$\text{weight of } q = \begin{cases} 1 & , \quad (\mathrm{III}_1) \text{ holds} \\ \min\big\{N \in \mathbb N : (\mathrm{III}_N) \text{ holds}\big\} - 1\,, & \quad \text{otherwise} \end{cases} \tag{4.2}$$

In the proof we will use the following observation, which is in essence just the same as [36, Lemma 5.6]; we will thus not elaborate the details.

4.9 Lemma. *Let $q \in \mathcal{N}_{<\infty}$ be meromorphic in $\mathbb{C}$ and assume that q is analytic at the point* 0. *Moreover, set*

$$\hat{q}(z) := \frac{-1}{q(z) - \frac{1}{z}} \in \mathcal{N}_{<\infty}\,,$$

and let $N \in \mathbb{Z}$. Then q satisfies (I), (II), *and* (III$_N$) *if and only if $\hat{q}$ satisfies* (I), (II), *and* (III$_{N-1}$). ❑

Next, recall the following notation, cf. [21, §8]: If $W \in \mathcal{M}_{<\infty}$, we denote by $\mathfrak{K}(W)$ the reproducing kernel space generated by the kernel H_W, and by $\mathfrak{K}_\pm(W)$ the subspaces

$$\mathfrak{K}_+(W) := \operatorname{cls}\big\{H_W(w,.)\begin{pmatrix}1\\0\end{pmatrix} \,:\, w \in \mathbb{C}\big\}\,,$$

$$\mathfrak{K}_-(W) := \operatorname{cls}\big\{H_W(w,.)\begin{pmatrix}0\\1\end{pmatrix} \,:\, w \in \mathbb{C}\big\}\,.$$

Proof (of Theorem 4.8). Let N_α denote the matrix

$$N_\alpha := \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}$$

and denote, for each matrix $W = (w_{ij})_{i,j=1,2}$ and scalar τ,

$$W \star \tau := \frac{w_{11}\tau + w_{12}}{w_{21}\tau + w_{22}}\,.$$

Moreover, if $W = (w_{ij})_{i,j=1,2}$ is a invertible 2×2-matrix, set

$$\operatorname{rev} W := VW^{-1}V = \begin{pmatrix}w_{22} & w_{12}\\ w_{21} & w_{11}\end{pmatrix}\,,$$

where

$$V := \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\,.$$

Step 1; A reduction method: Let $q \in \mathcal{N}_{<\infty}$ and $\alpha \in \mathbb{R}$. By [25, Lemma 3.13, (ii)] the function q is an intermediate Weyl-coefficient if and only if $N_\alpha \star q$ has this property. Moreover, in this case, the weights of q and $N_\alpha \star q$ coincide.

Let $q \in \mathcal{N}_{<\infty}$ be meromorphic in $\mathbb{C}$. Write $q = -\frac{A}{B}$ with some real entire functions A, B which have no common zeros, and set $E := A - iB$. Then $E \in \mathcal{HB}^*_{<\infty}$, and a short computation shows that

$$\frac{S_{\varphi+\frac{\pi}{2}}}{S_\varphi} = N_{-\varphi} \star q, \quad \varphi \in \mathbb{R}\,.$$

Hence,

$$N_\alpha \star \frac{S_{\varphi+\frac{\pi}{2}}}{S_\varphi} = \frac{S_{(\varphi-\alpha)+\frac{\pi}{2}}}{S_{(\varphi-\alpha)}}, \quad \varphi, \alpha \in \mathbb{R}\,,$$

in particular $N_\alpha \star q = S_{-\alpha}^{-1} S_{-\alpha+\frac{\pi}{2}}$. By Remark 3.6, (i), the function q satisfies (I), (II), and (III_N), if and only if $N_\alpha \star q$ does.

In order to prove the equivalence asserted in Theorem 4.8 for the given function q, it is thus enough to prove the corresponding equivalence for some function $N_\alpha \star q$ instead of q.

Step 2; The case (III_1)*:* Assume that q satisfies (I), (II), and (III_1). We take advantage of the reduction method in Step 1, and assume additionally that $q(0) = 0$. Note here that there always exists exactly one value of $\alpha \in [0, \pi)$ for which the function $N_\alpha \star q$ vanishes at 0.

Since q satisfies (I) and (II), the product $A_q(z)$ in (3.1) converges, and we may consider

$$B_q := A_q \cdot q, \qquad E_q := A_q - iB_q\,. \tag{4.3}$$

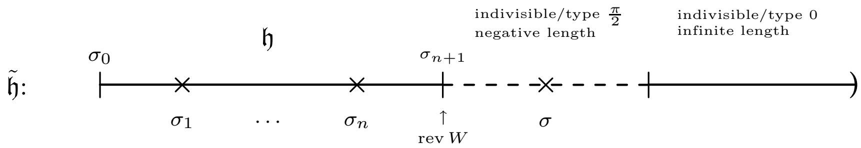
Then $E_q \in \mathcal{HB}^*_{<\infty}$, $\operatorname{ind}_- E_q = \operatorname{ind}_- q$, and by Theorem 3.2 we have $1 \in \operatorname{Assoc}_1 \mathcal{P}(E_q)$. By [21, Corollary 10.4], there exist entire functions $C(z), D(z)$ such that

$$W(z) := \begin{pmatrix} A_q & B_q \\ C & D \end{pmatrix}$$

belongs to $\mathcal{M}_{<\infty}$ and satisfies $\mathfrak{K}_+(W) = \mathfrak{K}(W)$. Consider the matrix $\operatorname{rev} W \in \mathcal{M}_{<\infty}$. Then

$$\operatorname{ind}_-(\operatorname{rev} W) = \operatorname{ind}_- W = \operatorname{ind}_- E_q = \operatorname{ind}_- q, \qquad \operatorname{rev} W \star 0 = q\,.$$

Let $\mathfrak{h}$ be the general Hamiltonian which is in the limit circle case and whose monodromy matrix equal $\operatorname{rev} W$. Then we can prolongue $\mathfrak{h}$ for example as follows:



Note here that the interval (σ_n, σ_{n+1}) cannot be indivisible of type $\frac{\pi}{2}$ for $\mathfrak{h}$, since $\operatorname{ind}_- \operatorname{rev} W = \operatorname{ind}_-(\operatorname{rev} W \star 0)$. The general Hamiltonian $\tilde{\mathfrak{h}}$ is thus well-defined. It has the singularities $\sigma_1, \ldots, \sigma_n, \sigma$, and its intermediate Weyl-coefficient at σ is q. Moreover,

$$\operatorname{ind}_- \tilde{\mathfrak{h}} = \operatorname{ind}_- \mathfrak{h} + 1 = \operatorname{ind}_- q + 1\,.$$

Step 3; The case (III_N)*,* $N \geq 2$*:* Assume that q satisfies (I), (II), and (III_N) with some $N \geq 2$. Again we take advantage of the reduction in Step 1, and assume additionally that

$$\lim_{y\to\infty} \frac{1}{y} q(iy) = 0, \qquad \lim_{y\to 0} yq(iy) = 0\,. \tag{4.4}$$

Note here that there are at most two values of $\alpha \in [0, \pi)$ for which the function $N_\alpha \star q$ does not satisfy one of these limit relations.

We are going to utilize [23, Theorem 7.4]. Consider the sequence of functions q_j which is defined inductively by

$$q_0(z) := q(z), \qquad q_j(z) := \frac{-1}{q_{j-1}(z) - \frac{1}{z}},\ j \geq 1\,. \tag{4.5}$$

By Lemma 4.9 the function q_{N-1} satisfies (I), (II), and (III$_1$). Moreover, $q_{N-1}(0) = 0$ and $q'_{N-1}(0) = 1$. By what we saw in the above Step 2, there exists a matrix $\tilde{W} = (\tilde{w}_{ij})_{i,j=1,2} \in \mathcal{M}_{<\infty}$ with $\operatorname{ind}_- \tilde{W} = \operatorname{ind}_- q_{N-1}$ such that $q_{N-1} = \tilde{W} \star 0$. Clearly, $\tilde{w}'_{12}(0) = q'_{N-1}(0) > 0$. Hence, [23, Theorem 7.4] yields that q is an intermediate Weyl-coefficient. Moreover, as we see from the proof of this theorem, its weight is at most $N-1$.

Step 4; The converse: Again assume that (4.4) holds, and let the sequence q_j, $j \geq 0$, be defined by (4.5). Assume that q is the intermediate Weyl-coefficient at a singularity σ of some general Hamiltonian $\mathfrak{h}$, and let $\Delta \in \mathbb{N}$ be the increase of negative index at the singularity σ. [23, Proposition 6.9] and [23, Lemma 7.1] together imply that the function q_Δ can be represented as

$$q_\Delta = W \star 0$$

with some $W = (w_{ij})_{i,j=1,2} \in \mathcal{M}_{<\infty}$, $\operatorname{ind}_- W = \operatorname{ind}_- q$. Let $A_{q_\Delta}, B_{q_\Delta}, E_{q_\Delta}$ be as in (3.1) and (4.3) using the function q_Δ. Since $\operatorname{ind}_- E_{q_\Delta} = \operatorname{ind}_- q_\Delta = \operatorname{ind}_- W$, we may apply [21, Proposition 10.3] with $\operatorname{rev} W$, and conclude that $1 \in \operatorname{Assoc}_1 \mathcal{P}(E_{q_\Delta})$. By Theorem 3.2, the function q_Δ satisfies (I), (II), and (III$_1$). Lemma 4.9 shows that q satisfies (I), (II), and (III$_{1+\Delta}$).

This proves the asserted equivalence. Since we may choose $\mathfrak{h}$ such that Δ equals the weight of q, also the formula (4.2) follows. ❏

Combining Theorem 4.8 with Remark 4.7, we obtain the following observation which solves an interesting inverse problem already in the positive definite case.

4.10 Corollary. *Let $\mathfrak{h}$ be a general Hamiltonian. Then $\mathfrak{h}$ satisfies on the last interval (σ_n, σ_{n+1}) of its domain the condition* (HS$_+$) *and we have $\Delta_+(H|_{(\sigma_n,\sigma_{n+1})}) < \infty$ if and only if $q_\mathfrak{h}$ satisfies* (I), (II), *and* (III$_N$) *for some $N \in \mathbb{N}$. In this case,*

$$\Delta_+(H|_{(\sigma_n,\sigma_{n+1})}) = \begin{cases} 1 & , \quad (\mathrm{III}_1) \textit{ holds} \\ \min\big\{N \in \mathbb{N} : (\mathrm{III}_N) \textit{ holds}\big\} - 1, & \textit{otherwise} \end{cases}$$

❏

Another immediate consequence is the following, rather astonishing, fact.

4.11 Corollary. *Let $W \in \mathcal{M}_{<\infty}$, $q \in \mathcal{N}_{<\infty}$, and $N \in \mathbb{N}$. Assume that $\operatorname{ind}_-(W \star q) = \operatorname{ind}_- W + \operatorname{ind}_- q$. Then the function q satisfies* (I), (II), *and* (III$_N$), *if and only if $W \star q$ does.*

Proof. The general Hamiltonian whose Weyl-coefficient is $W \star q$ is just the pasting (cf. [25, §3.e]) of the general Hamiltonian whose monodromy matrix is W, with the general Hamiltonian whose Weyl-coefficient is q. ❏

5 Canonical systems ending with indivisible intervals

Although indivisible intervals can be seen as pieces of a Hamiltonian of the most simple form, they often play an important role. For example in such classical topics like the power moment problem or Stieltjes strings, see, e.g., [1].

It is intuitively clear what is meant by saying that a Hamiltonian H defined on an interval (L_-, L_+) 'ends with at least M indivisible intervals'. However, let us reformulate this property in a way suitable for our present purposes.

5.1 Definition. Let $H : (L_-, L_+) \to \mathbb{R}^{2\times 2}$ be a Hamiltonian, and let $M \in \mathbb{N}$. We say that H ends with at least M indivisible intervals, if there exist points $x_1 \ldots, x_\nu \in [L_-, L_+)$ such that ($x_0 := L_+$)

(i) $x_\nu < x_{\nu-1} < \ldots < x_1$, and each of the intervals (x_j, x_{j-1}), $j = 1, \ldots, \nu$, is maximal indivisible.

(ii) $M \leq \nu$.

//

Let H be a Hamiltonian defined on (L_-, L_+) which is in the limit circle case at L_- and in the limit point case at L_+, and let q_H be its Weyl-coefficient. Moreover, let us assume that $\lim_{y\to+\infty} y^{-1} q_H(iy) = 0$, and denote by μ_H the measure in the Herglotz-integral representation of q_H. Then there exists a unique chain of de Branges Hilbert spaces $\mathcal{H}(E_t)$, $t \in J \subseteq \mathbb{R}$, such that

$$\mathcal{H}(E_t) \subseteq \mathcal{H}(E_s) \subseteq L^2(\mu_H), \quad t, s \in J,\ t \leq s\,,$$

where all these inclusions are isometric, and

$$\operatorname{clos} \bigcup_{t\in J} \mathcal{H}(E_t) = L^2(\mu_H)\,.$$

The index set J is actually given as

$$J = (L_-, L_+) \setminus \bigcup \big\{(a_-, a_+) :\ \text{indivisible}\big\}\,.$$

We arrive at the following obvious remark.

5.2 Remark. Let H be a Hamiltonian of the above form and let $M \in \mathbb{N}$. Then H ends with at least M indivisible intervals, if and only if the chain $\{\mathcal{H}(E_t) :\ t \in J\}$ is of the form

$$\{\mathcal{H}(E_t) :\ t \in J\} = \{\mathcal{H}(E_t) :\ t \in J'\} \cup \big\{\mathcal{H}(E_{x_M}), \ldots, \mathcal{H}(E_{x_1})\big\}$$

with some subset $J' \subseteq J$ and

$$\mathcal{H}(E_t) \subseteq \mathcal{H}(E_{x_M}) \subsetneq \ldots \subsetneq \mathcal{H}(E_{x_1}), \quad t \in J'\,.$$

//

Let us now turn our attention to the indefinite situation, i.e. let us consider a general Hamiltonian $\mathfrak{h}$. Then it is not so obvious what should be meant when saying that '$\mathfrak{h}$ ends with at least M indivisible intervals'. The problem is to properly include the contribution of singularities so that the statement analogous to Remark 5.2 holds true.

5.3 Definition. Let $\mathfrak{h}$ be a general Hamiltonian consisting of data as in Definition 4.1 and let $M \in \mathbb{N}$. We say that $\mathfrak{h}$ ends with at least M indivisible intervals, if there exist points $x_1, \ldots, x_\nu \in [\sigma_0, \sigma_{n+1})$ such that ($x_0 := \sigma_{n+1}$)

(*i*) $x_\nu < x_{\nu-1} < \ldots < x_1$, and each of the intervals (x_j, x_{j-1}), $j = 1, \ldots, \nu$, is maximal indivisible (with positive or infinite length).

(*ii*)
$$M \leq \nu + \sum_{i:\sigma_i > x_\nu} \begin{cases} -1\,, & \ddot{o}_i = 0, b_{i,1} = d_{i,1} \\ \ddot{o}_i\ \ , & \text{otherwise} \end{cases} + \\ + \begin{cases} \ddot{o}_j + \Delta_+(H|_{(\sigma_{j-1},\sigma_j)}) - 1\,, & x_\nu = \sigma_j \text{ for some } j \geq 1 \\ 0 \qquad\qquad\qquad\qquad\qquad , & \text{otherwise} \end{cases}$$

//

The following characterization is the third main result of this paper.

5.4 Theorem. *Let $\mathfrak{h}$ be a general Hamiltonian which is in the limit point case, and let $q_\mathfrak{h}$ be its Weyl-coefficient. Moreover, let $M \in \mathbb{N}$. Then $\mathfrak{h}$ ends with at least M indivisible intervals if and only if $q_\mathfrak{h}$ satisfies* (I), (II), *and* (III$_{2-M}$).

The essential observation towards the proof of this result is the below lemma.

If $\mathfrak{h}$ is a general Hamiltonian consisting of data as in Definition 4.1, we define a (finite or infinite) decreasing sequence $(x_j)_j$ of points $x_j \in [\sigma_0, \sigma_{n+1}]$ by the following inductive process which is carried out as long as possible:

$$x_0 := \sigma_{n+1}, \qquad x_j \text{ s.t. } (x_j, x_{j-1}) \begin{array}{l}\text{maximal indivisible} \\ \text{(positive or infinite length)}\end{array}, \ j \geq 1\,. \quad (5.1)$$

5.5 Lemma. *Let $\mathfrak{h}$ be a general Hamiltonian consisting of data as in Definition 4.1 which is in the limit circle case, and let $\omega_\mathfrak{h}(t)$ denote the finite maximal chain associated with $\mathfrak{h}$, cf. [25, §5].*

(*i*) *If x_ν is defined by the inductive process* (5.1) *and does not belong to $\{\sigma_1, \ldots, \sigma_n\}$, then*
$$\dim\Big(\mathfrak{K}\big(\omega_\mathfrak{h}(\sigma_{n+1})\big) \Big/ \mathfrak{K}\big(\omega_\mathfrak{h}(x_\nu)\big)\Big) = \nu + \sum_{i:\sigma_i > x_\nu} \begin{cases} -1\,, & \ddot{o}_i = 0, b_{i,1} = d_{i,1} \\ \ddot{o}_i\ \ , & \textit{otherwise} \end{cases}$$

(*ii*) *If the inductive process* (5.1) *terminates after one step and $x_1 = \sigma_n$ ($n \geq 1$), then*
$$\dim\Big(\mathfrak{K}\big(\omega_\mathfrak{h}(\sigma_{n+1})\big) \Big/ \operatorname{cls} \textstyle\bigcup_{t<\sigma_n} \mathfrak{K}\big(\omega_\mathfrak{h}(t)\big)\Big) = \ddot{o}_n + \Delta_+(H|_{(\sigma_{n-1},\sigma_n)})$$

Proof. Assume that we are in the situation (*i*). Let $j \leq \nu$ and assume that $x_j, x_{j-1} \notin \{\sigma_1, \ldots, \sigma_n\}$. Then the matrix $\omega_\mathfrak{h}(x_j)^{-1}\omega_\mathfrak{h}(x_{j-1})$ is a linear polynomial, cf. [25, Proposition 5.5, (*i*)]. Hence,
$$\dim\Big(\mathfrak{K}\big(\omega_\mathfrak{h}(x_j)\big) \Big/ \mathfrak{K}\big(\omega_\mathfrak{h}(x_{j-1})\big)\Big) = 1\,.$$

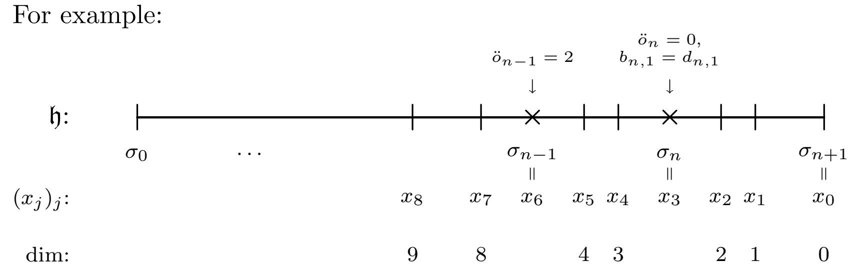
Let $j+1 \leq \nu$ and assume that $x_j = \sigma_i$ for some $i \in \{1, \ldots, n\}$. Then $x_{j-1}, x_{j+1} \notin \{\sigma_1, \ldots, \sigma_n\}$, and the matrix $\omega_\mathfrak{h}(x_{j+1})^{-1}\omega_\mathfrak{h}(x_{j-1})$ is a polynomial of degree
$$\begin{cases} 1 \qquad , & \ddot{o}_i = 0, b_{i,1} = d_{i,1} \\ \ddot{o}_i + 2\,, & \text{otherwise} \end{cases}$$

cf. [25, Proposition 5.5, (ii)]. Thus

$$\dim\Big(\mathfrak{K}\big(\omega_{\mathfrak{h}}(x_{j+1})\big)\Big/\mathfrak{K}\big(\omega_{\mathfrak{h}}(x_{j-1})\big)\Big)=\begin{cases}1\qquad, & \ddot{o}_i=0, b_{i,1}=d_{i,1}\\ \ddot{o}_i+2\,, & \text{otherwise}\end{cases}$$

Together, the desired formula follows.

For example:



We turn to the situation described in (ii). Since $\mathfrak{h}$ is, up to reparameterization, equal to the general Hamiltonian constructed in [26] for the monodromy matrix $\omega_{\mathfrak{h}}(\sigma_{n+1})$, it is sufficient to consider the construction carried out there, especially [26, Definition 2.6]. We just need to match the notation used there with the present one: The dimension we are seeking for is equal to $a+1$, the number $\Delta_+(\omega)$ is equal to 1, and $\Delta_-(\omega)=\Delta_+(H|_{(\sigma_{n-1},\sigma_n)})$, $\ddot{o}(\omega)=\ddot{o}_n$. Thus the desired formula holds. ❑

Proof (of Theorem 5.4).
Step 1; Reduction: For $\alpha\in\mathbb{R}$ denote by $\circlearrowleft_\alpha\mathfrak{h}$ the general Hamiltonian whose Weyl-coefficient is $N_\alpha\star q_{\mathfrak{h}}$; for an explicit definition and a detailed treatment see [25, §3.e, Lemma 5.14]. Clearly, $\mathfrak{h}$ ends with at least M indivisible intervals if and only if $\circlearrowleft_\alpha\mathfrak{h}$ does. Hence, similar as in the proof of Theorem 4.8, we may consider $\circlearrowleft_\alpha\mathfrak{h}$ and $N_\alpha\star q_{\mathfrak{h}}$ instead of $\mathfrak{h}$ and $q_{\mathfrak{h}}$.

We will take advantage of this fact, and assume throughout the following that $q_{\mathfrak{h}}(0)=0$.

Step 2; The case $M=1$: Assume first that $\mathfrak{h}$ ends with at least one indivisible interval, and let x_1 be as in Definition 5.3. Then $\operatorname{ind}_-\omega_{\mathfrak{h}}(x_1)=\operatorname{ind}_-q_{\mathfrak{h}}$ and $q_{\mathfrak{h}}=\omega_{\mathfrak{h}}(x_1)\star 0$. Set $E:=\omega_{\mathfrak{h}}(x_1)_{22}-i\omega_{\mathfrak{h}}(x_1)_{12}$, then $E\in\mathcal{HB}^*_{<\infty}$. Moreover, since

$$(\omega_{\mathfrak{h}}(x_1)_{22},\omega_{\mathfrak{h}}(x_1)_{12})=(1,0)\operatorname{rev}\omega_{\mathfrak{h}}(x_1)\,,$$

and $\operatorname{ind}_-E=\operatorname{ind}_-q_{\mathfrak{h}}=\operatorname{ind}_-\omega_{\mathfrak{h}}(x_1)=\operatorname{ind}_-\operatorname{rev}\omega_{\mathfrak{h}}(x_1)$, we obtain that $1\in\operatorname{Assoc}_1\mathcal{P}(E)$, cf. [21, Proposition 10.3]. However, the function $q_{\frac{\pi}{2}}$ defined for E is equal to $q_{\mathfrak{h}}$, and hence Theorem 3.2 implies that $q_{\mathfrak{h}}$ satisfies (I), (II), and (III_1).

Conversely, assume that $q_{\mathfrak{h}}$ satisfies (I), (II), and (III_1). As we already saw in Step 2 of the proof of Theorem 4.8, $q_{\mathfrak{h}}$ is the Weyl-coefficient of a general Hamiltonian which ends with an indivisible interval (namely $\tilde{\mathfrak{h}}_{\urcorner\sigma}$, in the notation used there).

Step 3; The case $M\geq 2$, necessity: Let $M\geq 2$, and assume that $\mathfrak{h}$ ends with at least M indivisible intervals. Then the inductive process (5.1) for $\mathfrak{h}$ can be carried out at least for two steps. Moreover, by Step 2, the function $q_{\mathfrak{h}}$ satisfies

(I), (II), and (III_1), and we can write $q_{\mathfrak{h}} = \operatorname{rev} W \star 0$ where W is the matrix constructed in Step 2 of the proof of Theorem 4.8.

Let $\hat{\mathfrak{h}}$ be the general Hamiltonian which evolves from $\mathfrak{h}$ by cutting off the interval (x_1, σ_{n+1}). Then the monodromy matrix of $\hat{\mathfrak{h}}$ equals $\operatorname{rev} W$, remember here that $\operatorname{ind}_- \operatorname{rev} W = \operatorname{ind}_- q_{\mathfrak{h}}$. Let $\omega(t) : [\sigma_0, x_1] \setminus \{\sigma_1, \dots, \sigma_n\} \to \mathcal{M}_{<\infty}$ be the finite maximal chain going downwards from $\operatorname{rev} W$, then the chain $\operatorname{rev}\omega$ (given by [25, Definition 3.12]) is the finite maximal chain going downwards from W, cf. [25, Lemma 3.13].

By [25, Lemma 2.7 and Lemma 2.13], there exists an isomorphism $\varpi : \mathfrak{K}(\operatorname{rev} W) \to \mathfrak{K}(W)$, and this isomorphism satisfies $\varpi(\mathfrak{K}(\omega(t))^{\perp}) = \mathfrak{K}([\operatorname{rev}\omega](-t))$. Hence, for each $t_0 \in [\sigma_0, x_1]$, we have

$$\varpi\Big(\Big[\operatorname{cls} \bigcup_{t<t_0} \mathfrak{K}\big(\omega(t)\big)\Big]^{\perp}\Big) = \bigcap_{s>-t_0} \mathfrak{K}\big([\operatorname{rev}\omega](s)\big)\,. \tag{5.2}$$

Let ν be as in Definition 5.3. By possibly increasing ν by 1, we may assume that either x_ν is not a singularity of $\mathfrak{h}$ or the process (5.1) terminates at x_ν. Lemma 5.5 together with (5.2) tells us that

$$\dim \bigcap_{s>-x_\nu} \mathfrak{K}\big([\operatorname{rev}\omega](s)\big) = (\nu-1) + \sum_{i:\sigma_i > x_\nu} \begin{cases} -1\,, & \ddot{o}_i = 0, b_{i,1} = d_{i,1} \\ \ddot{o}_i\ , & \text{otherwise} \end{cases} + \\ + \begin{cases} \ddot{o}_j + \Delta_+(H|_{(\sigma_{j-1},\sigma_j)}) - 1\,, & x_\nu = \sigma_j \text{ for some } j \geq 1 \\ 0 \hfill , & \text{otherwise} \end{cases} \tag{5.3}$$

Denote this number by $M(\nu)$.

By the construction of W, we have $\mathfrak{K}_+(W) = \mathfrak{K}(W)$, and hence the chain of nondegenerate dB-subspaces of $\mathcal{P}(E_{q_{\mathfrak{h}}})$ is given as

$$\big\{\mathcal{P}\big([\operatorname{rev}\omega](t)_{11} - i[\operatorname{rev}\omega](t)_{12}\big) : t \in (-x_1, -\sigma_0] \setminus \{-\sigma_n, \dots, -\sigma_1\}\big\}\,.$$

As we observed above, the chain of all dB-subspaces of $\mathcal{P}(E_{q_{\mathfrak{h}}})$ contains a space whose dimension is equal to $M(\nu) < \infty$. Hence, $\operatorname{Assoc}_{-(M(\nu)-1)} \mathcal{P}(E_{q_{\mathfrak{h}}})$ contains a real and zerofree element, and thus $q_{\mathfrak{h}}$ satisfies the conditions (I), (II), and ($\mathrm{III}_{1-M(\nu)}$). However, $1 - M(\nu) \leq 2 - M$, and hence also (III_{2-M}) holds.

Step 4; The case $M \geq 2$, sufficiency: Let $M \geq 2$ and assume that $q_{\mathfrak{h}}$ satisfies (I), (II), and (III_{2-M}). In particular, (III_1) holds, and hence we may write $q_{\mathfrak{h}} = \operatorname{rev} W \star 0$ as in Step 2. Moreover, the space $\mathcal{P}(E_{q_{\mathfrak{h}}})$ contains a dB-subspace with dimension $M-1$.

Assume that the process (5.1) terminates at a point x_ν (which must be $< \sigma_{n+1}$). Then, since for each $t < x_\nu$ the general Hamiltonian $\mathfrak{h}$ contains an interval right of t which is not indivisible,

$$\dim \Big(\mathfrak{K}\big(\operatorname{rev} W\big) \Big/ \mathfrak{K}\big(\omega(t)\big) \Big) = \infty, \quad t < x_\nu\,,$$

and hence

$$\dim \mathfrak{K}\big([\operatorname{rev}\omega](s)\big) = \infty, \quad s > -x_\nu\,.$$

However, as we proved in Step 3,

$$\dim \bigcap_{s>-x_\nu} \mathfrak{K}\big([\operatorname{rev}\omega](s)\big) = M(\nu)\,,$$

where $M(\nu)$ is the number defined by the right hand side of (5.3). It follows that $M-1\leq M(\nu)$, and this means by definition that $\mathfrak{h}$ ends with at least M indivisible intervals. ❑

6 A selection of applications

In this section we present three selected applications of Theorem 3.2, Theorem 4.8, and Theorem 5.4.

6.1 A characterization that $1\in\operatorname{Assoc}_N\mathcal{P}(E)$

In the theory of de Branges Hilbert spaces the characterization [9, Theorem 27] of associated functions plays a most important role. Similarly, its indefinite analogue [21, Proposition 10.3] is a fundamental result. Applied with the constant function 1 it says: Let $E\in\mathcal{HB}^*_{<\infty}$, $E(0)=1$, then

$$1\in\operatorname{Assoc}_1\mathcal{P}(E)\quad\Longleftrightarrow\quad\exists\, W\in\mathcal{M}_{<\infty},\mathfrak{K}_+(W)^\circ=\{0\}:\,(1,0)W=(S_{\frac{\pi}{2}},S_\pi)$$

This fact is supplemented by [22, Lemm 5.11], where we showed that

$$\exists\, W\in\mathcal{M}_{<\infty},\mathfrak{K}_+(W)^\circ\neq\{0\}:\,(1,0)W=(S_{\frac{\pi}{2}},S_\pi)\quad\Longrightarrow\quad 1\notin\operatorname{Assoc}_1\mathcal{P}(E)$$

We can now prove a beautiful completion of these results.

6.1 Proposition. *Let $E=\in\mathcal{HB}^*_{<\infty}$, $E(0)=1$, and let $N\geq 2$. Then*

$$\begin{aligned}1\in\operatorname{Assoc}_N\mathcal{P}(E)\setminus\operatorname{Assoc}_{N-1}\mathcal{P}(E)\qquad&\Longleftrightarrow\\ &\exists\, W\in\mathcal{M}_{<\infty},\dim\mathfrak{K}_+(W)^\circ=N-1:\,(1,0)W=(S_{\frac{\pi}{2}},S_\pi)\end{aligned}$$

The proof relies on the following two lemmata about the geometry of spaces $\mathfrak{K}(W)$.

6.2 Lemma. *Let $W,W_1\in\mathcal{M}_{<\infty}$, $W,W_1\neq I$, and assume that*

$$\nexists\, u\in\mathbb{C}^2:\quad u\in\mathfrak{K}(W_1)\text{ and }Wu\in\mathfrak{K}(W)\,. \tag{6.1}$$

Moreover, set $\widetilde{W}:=WW_1$, and denote by π_+ and $\tilde\pi_+$ the projections onto the first component in the spaces $\mathfrak{K}(W)$ and $\mathfrak{K}(\widetilde{W})$, respectively. Then we have

$$\ker\pi_+=\ker\tilde\pi_+\,.$$

Proof. Our assumption (6.1) implies, e.g., by [25, Proposition 2.11], that

$$\mathfrak{K}(\widetilde{W})=\mathfrak{K}(W)[\dot+]W\mathfrak{K}(W_1)\,. \tag{6.2}$$

In particular, $\operatorname{ind}_-\widetilde{W}=\operatorname{ind}_-W+\operatorname{ind}_-W_1$ and $\ker\pi_+\subseteq\ker\tilde\pi_+$.

Case 1; W is not a polynomial: We may apply [22, Theorem 5.7], and conclude that $\ker\pi_+=\{0\}$ if and only if $\ker\tilde\pi_+=\{0\}$. In order to prove the desired equality, we thus need to show that $\ker\pi_+\neq\{0\}$ implies $\ker\tilde\pi_+\subseteq\ker\pi_+$.

Let $p\in\ker\tilde\pi_+$ be given, so that $\binom{0}{p}\in\mathfrak{K}(\widetilde{W})$. Recall that, by [21, Corollary I.9.7], p is a polynomial.

Since $\ker\pi_+ \neq \{0\}$ implies that $\mathfrak{K}_-(W) = \mathfrak{K}(W)$, we may apply [22, Corollary 7.4], and obtain that there exists no constant $u \in \mathfrak{K}(W_1)$ such that $\pi_- W u \in \mathcal{P}(E_W)$, where E_W is defined by the second row of W, see, e.g., [25, §2.e], and π_- denotes the projection onto the second component. Thus the conditions (i)–(iii) of [21, Theorem 12.2, Corollary 12.3] are fullfilled. It follows that ($E_{\widetilde{W}}$ is defined similarly by the second row of $\widetilde{W}$)

$$\mathcal{P}(E_{\widetilde{W}}) = \mathcal{P}(E_W)[\dot{+}]\pi_- W\mathfrak{K}(W_1)\,,$$

and that the map $\binom{g_+}{g_-} \mapsto \pi_- W_1\binom{g_+}{g_-}$ is an isometric isomorphism of $\mathfrak{K}(W_1)$ onto $\mathcal{P}(E_{\widetilde{W}})[-]\mathcal{P}(E_W)$.

Since p is a polynomial and belongs to $\mathcal{P}(E_{\widetilde{W}})$, it also belongs to any infinite dimensional dB-subspace of $\mathcal{P}(E_{\widetilde{W}})$. However, $\mathcal{P}(E_W) \cong \mathfrak{K}(W)$ is one such, and it follows that $p \in \mathcal{P}(E_W)$. According to (6.2) we can write

$$\begin{pmatrix} 0 \\ p \end{pmatrix} = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} + W\begin{pmatrix} g_+ \\ g_- \end{pmatrix}$$

with some $\binom{f_+}{f_-} \in \mathfrak{K}(W)$ and $\binom{g_+}{g_-} \in \mathfrak{K}(W_1)$. In particular,

$$\pi_- W\begin{pmatrix} g_+ \\ g_- \end{pmatrix} = p - f_- \in \mathcal{P}(E_W)\,,$$

and we conclude that $\binom{g_+}{g_-} = 0$. Hence $\binom{0}{p} \in \mathfrak{K}(W)$, i.e. $p \in \ker\pi_+$.

Case 2; W is a polynomial: If $\ker\tilde{\pi}_+ = \{0\}$, the inclusion $\ker\tilde{\pi}_+ \subseteq \ker\pi_+$ holds trivially. Hence, assume that $\ker\tilde{\pi}_+ \neq \{0\}$.

We first treat the case that W is of particular form

$$W = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix}$$

with some polynomial q with real coefficients and $q(0) = 0$. Let $\binom{0}{p} \in \ker\widetilde{W}$ and write, according to (6.2),

$$\begin{pmatrix} 0 \\ p \end{pmatrix} = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} + W\begin{pmatrix} g_+ \\ g_- \end{pmatrix}$$

with some $\binom{f_+}{f_-} \in \mathfrak{K}(W)$ and $\binom{g_+}{g_-} \in \mathfrak{K}(W_1)$. However, $\mathfrak{K}(W) = \ker\pi_+$, see, e.g., [25, Proposition 2.8]. Thus this relation reads as

$$\begin{pmatrix} 0 \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ f_- \end{pmatrix} + \begin{pmatrix} g_+ \\ -qg_+ + g_- \end{pmatrix}.$$

It follows that $g_+ = 0$. By our assumption (6.1), we have $\binom{0}{1} \notin \mathfrak{K}(W_1)$, and hence the projection onto the first component in the space $\mathfrak{K}(W_1)$ is injective. Thus also $g_- = 0$, and we see that $\binom{0}{p} \in \mathfrak{K}(W)$.

Next we turn to the case that W is an arbitrary polynomial. Then there exist numbers $n \in \mathbb{N}$, $\alpha_i \in [0, \pi)$, $i = 1, \ldots, n$, and real polynomials q_i, $i = 1, \ldots, n$, with $\alpha_i \neq \alpha_{i+1}$, $i = 1, \ldots, n-1$, and $q_i(0) = 0$, such that

$$W = \prod_{i=1}^{n} N_{\alpha_i}\begin{pmatrix} 1 & 0 \\ -q_i & 1 \end{pmatrix} N_{\alpha_i}^{-1}\,,$$

where

$$N_\alpha := \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

see, e.g., [27, Theorem 3.1]. It follows, e.g., from [25, Corollary 2.9], that

$$\xi_{\frac{\pi}{2}-\alpha_1} \in \mathfrak{K}\Big(N_{\alpha_1}\begin{pmatrix} 1 & 0 \\ -q_1 & 1 \end{pmatrix} N_{\alpha_1}^{-1}\Big) \subseteq \mathfrak{K}(W) \subseteq \mathfrak{K}(\widetilde{W}).$$

Since the space $\mathfrak{K}(\widetilde{W})$ can contain at most one constant (up to scalar multiples), we must have $\alpha_1 = 0$.

Consider the matrix

$$\widehat{W} := \begin{pmatrix} 1 & 0 \\ -q_1 & 1 \end{pmatrix}$$

then we can write $\widetilde{W} = \widehat{W} \cdot [\widehat{W}^{-1} W] \cdot W_1$, and hence $\mathfrak{K}(\widehat{W}) \subseteq \mathfrak{K}(W) \subseteq \mathfrak{K}(\widetilde{W})$. If $\hat{\pi}_+$ denotes the projection onto the first component in the space $\mathfrak{K}(\widehat{W})$, thus

$$\ker\hat{\pi}_+ \subseteq \ker\pi_+ \subseteq \ker\tilde{\pi}_+ .$$

By what we have already shown, $\ker\hat{\pi}_+ = \ker\tilde{\pi}_+$, and thus also $\ker\pi_+ = \ker\tilde{\pi}_+$. ❑

6.3 Lemma. *Let $\mathfrak{h}$ be a general Hamiltonian in the limit circle case (with at least one singularity), and let $W = (w_{ij})_{i,j=1,2} \in \mathcal{M}_{<\infty}$ be its monodromy matrix. Let π_+ be the projection onto the first component defined on the space $\mathfrak{K}(W)$. Then $(\ker\pi_+)^\circ \neq \{0\}$ if and only if the interval (σ_0, σ_1) is indivisible of type $\frac{\pi}{2}$ and σ_1 is not left endpoint of an indivisible interval. In this case, we have*

$$\dim(\ker\pi_+)^\circ = \Delta_-(H|_{(\sigma_1,\sigma_2)}) .$$

Proof. First of all, recall that

$$\ker\pi_+ = \operatorname{span}\Big\{\begin{pmatrix} 0 \\ z^k \end{pmatrix} : k = 0,\dots,d\Big\}, \quad d := \dim\ker\pi_+ - 1 ,$$

$$(\ker\pi_+)^\circ = \operatorname{span}\Big\{\begin{pmatrix} 0 \\ z^k \end{pmatrix} : k = 0,\dots,d^\circ\Big\}, \quad d^\circ := \dim(\ker\pi_+)^\circ - 1 .$$

Clearly, $d = \max\{k \in \mathbb{N}_0 : \binom{0}{z^k} \in \mathfrak{K}(W)\}$.

We are going to work through all possible cases and determine $\ker\pi_+$ and $(\ker\pi_+)^\circ$ in each of them. Again let $\omega_\mathfrak{h}$ denote the finite maximal chain associated with $\mathfrak{h}$.

Case 1; $\binom{0}{1} \notin \mathfrak{K}(W)$: In this case $\ker\pi_+ = \{0\}$, and hence also $(\ker\pi_+)^\circ = \{0\}$.

Case 2; $\binom{0}{1} \in \mathfrak{K}(W)$, $[\binom{0}{1}, \binom{0}{1}] > 0$: By [22, Lemma 7.5], the chain $\omega_\mathfrak{h}$ starts with an indivisible interval of type $\frac{\pi}{2}$ and finite (positive) length. Lemma 6.2 implies that $\ker\pi_+ = \operatorname{span}\{\binom{0}{1}\}$, and hence $(\ker\pi_+)^\circ = \{0\}$.

Case 3; $\binom{0}{1} \in \mathfrak{K}(W)$, $[\binom{0}{1}, \binom{0}{1}] \leq 0$: By [23, Remark 3.2, (*i*)], the interval (σ_0, σ_1) is indivisible. Moreover, by [22, Lemma 7.6, proof of Theorem 7.1], its type is equal to $\frac{\pi}{2}$.

Subcase 3a; σ_1 is left endpoint of indivisible interval: Let $s \in (\sigma_1, \sigma_2)$ be such that (σ_1, s) is maximal indivisible. Then the matrix $\omega_{\mathfrak{h}}(s)$ is the monodromy matrix of an elementary indefinite Hamiltonian of kind (B) or (C), and hence a polynomial.

We can factorize $W = \omega_{\mathfrak{h}}(s) \cdot [\omega_{\mathfrak{h}}(s)^{-1} W]$. The type of the indivisible interval (σ_1, s) is equal to $\frac{\pi}{2}$, and hence the constant $\binom{0}{1}$ cannot belong to $\mathfrak{K}(\omega_{\mathfrak{h}}(s)^{-1} W)$. Thus we may apply Lemma 6.2, and conclude that $\ker \pi_+ = \mathfrak{K}(\omega_{\mathfrak{h}}(s))$. it follows that $(\ker \pi_+)^\circ = \{0\}$.

Subcase 3b; σ_1 is not left endpoint of indivisible interval: We can choose $s_+ \in (\sigma_1, \sigma_2)$ which is not an inner point of an indivisible interval such that the general Hamiltonian $\operatorname{rev}(\mathfrak{h}_{\restriction s_+})$ satisfies [26, 2.1. Overall assumption]. By [25, Lemma 5.15] the monodromy matrix of $\operatorname{rev}(\mathfrak{h}_{\restriction s_+})$ is equal to $\operatorname{rev} \omega_{\mathfrak{h}}(s_+)$.

By [26, Corollary 2.13] we have (using the same notation as there)

$$z^k (\operatorname{rev} \omega_{\mathfrak{h}}(s_+)) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = z^k \pi_2^{-1} D_0 \,.$$

Thus, we obtain the following inequalities:

$$\begin{array}{ccc}
\max\big\{k:\ z^k(\operatorname{rev}\omega_{\mathfrak{h}}(s_+))\binom{0}{1} \in \mathfrak{K}(\operatorname{rev}\omega_{\mathfrak{h}}(s_+))\big\} & \overset{\text{trivial}}{\geq} & \max\big\{k:\ z^k(\operatorname{rev}\omega_{\mathfrak{h}}(s_+))\binom{0}{1} \in \pi_-^{-1}(\operatorname{span}\{\omega_{\mathfrak{h}}(s_+)_{11}\}^\perp)\big\} \\
\text{trivial}\ \rotatebox{90}{$\geq$} & & \text{trivial}\ \rotatebox{90}{$\geq$} \quad \rotatebox{90}{$\leq$}\ \begin{smallmatrix}\text{[26, Corollary 2.13]} \\ \text{[26, Lemma 2.16]}\end{smallmatrix} \\
\max\big\{k:\ z^k\omega_{\mathfrak{h}}(s_+)_{11} \in \mathcal{P}(E_{\omega_{\mathfrak{h}}(s_+)})\big\} & \overset{\text{[22, Lemma 5.19]}}{=} & \max\big\{k:\ z^k\omega_{\mathfrak{h}}(s_+)_{11} \in \operatorname{span}\{\omega_{\mathfrak{h}}(s_+)_{11}\}^\perp\big\} \\
 & & \Vert\ \text{[26, Lemma 2.8]} \\
 & & \Delta + \ddot{o} - 1
\end{array}$$

Since the map $f \mapsto \operatorname{rev} W \cdot Vf$ is an isomorphism of $\mathfrak{K}(\omega_{\mathfrak{h}}(s_+))$ onto $\mathfrak{K}(\operatorname{rev} \omega_{\mathfrak{h}}(s_+))$, we have (keep in mind that by Lemma 6.2 the kernel of the projection onto the first component in the space $\mathfrak{K}(W)$ coincides with the respective kernel in the space $\mathfrak{K}(\omega_{\mathfrak{h}}(s_+))$)

$$d = \max\Big\{k \in \mathbb{N}_0:\ z^k (\operatorname{rev} \omega_{\mathfrak{h}}(s_+)) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathfrak{K}(\operatorname{rev} \omega_{\mathfrak{h}}(s_+))\Big\}\,.$$

Thus we obtain that $d = \Delta_1 + \ddot{o}_1 - 1$ and, remember [26, (2.15)], that $d^\circ = \Delta_1 - 1$. ❑

Proof (of Proposition 6.1). Assume first that $W \in \mathcal{M}_{<\infty}$ with $\mathfrak{K}_+(W)^\circ \neq \{0\}$ is given, and set $(A, B) := (1, 0)W$, $E := A - iB$, $q := A^{-1}B$. Moreover, let $\mathfrak{h}$ be the general Hamiltonian whose monodromy matrix equals W, and denote the data $\mathfrak{h}$ consist of as in Definition 4.1. Since $\mathfrak{K}_+(W)^\circ \neq \{0\}$, by the above lemma, the interval (σ_0, σ_1) of $\mathfrak{h}$ is indivisible of type $\frac{\pi}{2}$ and σ_1 is not left endpoint of an indivisible interval.

Consider the general Hamiltonian $\operatorname{rev} \mathfrak{h}$ whose monodromy matrix is $\operatorname{rev} W$, cf. [25, Definition 3.40, Lemma 4.30]. Then the last interval $(-\sigma_1, -\sigma_0)$ of $\operatorname{rev} \mathfrak{h}$ is indivisible of type $\frac{\pi}{2}$ and $\Delta_+(\operatorname{rev} H|_{(-\sigma_2, -\sigma_1)}) = \Delta_-(H|_{(\sigma_1, \sigma_2)})$. Moreover, by its definition, $\ddot{o}_{-\sigma_1}(\operatorname{rev} \mathfrak{h}) = \ddot{o}_1$. The function q can be written as $q = \operatorname{rev} W \star 0$, and hence is the intermediate Weyl-coefficient of $\operatorname{rev} \mathfrak{h}$ at $-\sigma_1$. Since $\mathfrak{K}_+(W)^\circ \neq \{0\}$, we have $1 \notin \operatorname{Assoc}_1 \mathcal{P}(E)$. By Theorem 3.2, the function q does not satisfy

(III$_1$). Corollary 4.10 implies that q satisfies (I), (II), and (III$_n$) for some $n \in \mathbb{N}$. Moreover, the minimal number N for which (III$_n$) holds is equal to

$$N := \Delta_-(H|_{(\sigma_1,\sigma_2)}) + 1 .$$

Theorem 3.2 gives $1 \in \operatorname{Assoc}_N \mathcal{P}(E) \setminus \operatorname{Assoc}_{N-1} \mathcal{P}(E)$. Again referring to Lemma 6.3, we find $\dim \mathfrak{K}_+(W)^\circ = N - 1$.

Conversely, assume that $1 \in \operatorname{Assoc}_N \mathcal{P}(E) \setminus \operatorname{Assoc}_{N-1} \mathcal{P}(E)$ with some $N \geq 2$. Then the function $q := S_{\frac{\pi}{2}}^{-1} S_\pi$ satisfies (I), (II), and (III$_N$), but not (III$_{N-1}$). Let $\mathfrak{h}$ be the general Hamiltonian with Weyl-coefficient $q_\mathfrak{h} = q$, and denote the data $\mathfrak{h}$ consists of as in Definition 4.1. By Corollary 4.10 the Hamiltonian $H|_{(\sigma_n,\sigma_{n+1})}$ satisfies (HS$_+$) and $\Delta_+(H|_{(\sigma_n,\sigma_{n+1})}) = N - 1$; remember here that $N \geq 2$.

Let $\tilde{\mathfrak{h}}$ be the general Hamiltonian which is defined by the following data ($\phi_0 := \phi_+(H|_{(\sigma_n,\sigma_{n+1})})$):

$$\sigma_0, \dots, \sigma_{n+1}, \quad \sigma_{n+1} + 1 ,$$

$$H|_{(\sigma_j,\sigma_{j+1})},\ j = 0, \dots, n, \quad \frac{1}{t - \sigma_{n+1}} \xi_{\phi_0} \xi_{\phi_0}^T,\ t \in (\sigma_{n+1}, \sigma_{n+1} + 1) ,$$

$$\ddot{o}_1, \dots, \ddot{o}_n, \quad \ddot{o}_{n+1} := 0 ,$$

$$b_{j,1}, \dots, b_{j,\ddot{o}_j+1},\ j = 0, \dots, n, \quad b_{n+1,1} := 0 ,$$

$$d_{j,0}, \dots, d_{j,2\Delta_j-1}, j = 0, \dots, n, \quad d_{n+1,0} = d_{n+1,1} := 0 .$$

Since, by Theorem 5.4, the general Hamiltonian $\mathfrak{h}$ does not end with an indivisible interval, $\tilde{\mathfrak{h}}$ is indeed well-defined by this data.

Let $\tilde{W} = (\tilde{w}_{ij})_{i,j=1,2}$ be the monodromy matrix of $\tilde{\mathfrak{h}}$. Then, by the definition of $\tilde{\mathfrak{h}}$, we have $\tilde{W} \star \cot \phi_0 = q$. Since $q(0) = 0$, it follows that $\phi_0 = \frac{\pi}{2}$. Thus $\tilde{w}_{22}^{-1} \tilde{w}_{12} = S_{\frac{\pi}{2}}^{-1} S_\pi$. The function $\tilde{w}_{22}^{-1} S_{\frac{\pi}{2}}$ is entire, real, zerofree, and of bounded type. Hence, it is a constant. However, $\tilde{w}_{22}(0) = 1 = S_{\frac{\pi}{2}}(0)$, and it follows that $\tilde{w}_{22} = S_{\frac{\pi}{2}}$. Thus also $\tilde{w}_{12} = S_\pi$.

The matrix $\operatorname{rev} \tilde{W}$ belongs to $\mathcal{M}_{<\infty}$ and satisfies $(1, 0) \operatorname{rev} \tilde{W} = (S_{\frac{\pi}{2}}, S_\pi)$. By Lemma 6.3, we have $\dim \mathfrak{K}(\operatorname{rev} \tilde{W})^\circ = \Delta_+(H|_{(\sigma_n,\sigma_{n+1})}) = N - 1$. ❑

Let us explicitly state the following obvious, but still noteworthy, consequence of Proposition 6.1.

6.4 Corollary. *Let $E \in \mathcal{HB}^*_{<\infty}$, $E(0) = 1$. Then the number* $\dim \mathfrak{K}_+(W)^\circ$ *does not depend on W, whenever W is a matrix belonging to $\mathcal{M}_{<\infty}$ with $(1, 0)W = (S_{\frac{\pi}{2}}, S_\pi)$.* ❑

6.2 Polynomials in de Branges Pontryagin spaces

The case $N \leq 0$ in Theorem 3.2 can also be formulated in a slightly different way. Again denote by $\mathbb{C}[z]_m$ the set of all polynomials whose degree does not exceed m.

6.5 Remark. Let $E \in \mathcal{HB}^*_{<\infty}$ and $m \in \mathbb{N}_0$. In order that there exists a real and zerofree function U with

$$U \cdot \mathbb{C}[z]_m \subseteq \mathcal{P}(E) , \tag{6.3}$$

it is necessary and sufficient that for one (and hence for all) $\varphi \in \mathbb{R}$ the function q_φ satisfies (I), (II), and (III$_{-m}$).

This statement indeed is an immediate reformulation of the case '$N \leq 0$' in Theorem 3.2, since the inclusion (6.3) just means that $U \in \operatorname{dom} \mathcal{S}_E^m$. //

We obtain a criterion for the existence of minimal (positive) admissible majorants. Let us recall this notion.

6.6 Definition. Let $E \in \mathcal{H}B_0^*$ and $\mathfrak{m} : \mathbb{R} \to (0, \infty)$. Then $\mathfrak{m}$ is called a (positive) admissible majorant for $\mathcal{H}(E)$, if there exists a function $F \in \mathcal{H}(E) \setminus \{0\}$ such that

$$|F(x)| \leq \mathfrak{m}(x), \quad x \in \mathbb{R}.$$

//

Admissible majorants are a classical object of investigation, going back as far as to the famous Beurling-Malliavin Multiplier Theorem which can be seen as a sufficient condition for a function $\mathfrak{m}$ to be an admissible majorant for each Paley-Wiener space $\mathcal{H}(e^{-iaz})$, $a > 0$. For more on this topic see, e.g., [17], [18], [3], [4], and the literature cited there. Although in general it is very hard to decide whether a given function is an admissible majorant for a space $\mathcal{H}(E)$, minimal majorants are much easier accessible. Recall:

6.7 Definition. Let $E \in \mathcal{H}B_0^*$, and let $\mathfrak{m} : \mathbb{R} \to (0, \infty)$ be a (positive) admissible majorant for $\mathcal{H}(E)$. Then $\mathfrak{m}$ is called minimal, if for each (positive) admissible majorant $\mathfrak{m}'$ for $\mathcal{H}(E)$ the implication

$$\mathfrak{m}'(x) \leq \mathfrak{m}(x),\ x \in \mathbb{R} \quad \Rightarrow \quad \exists C > 0 : \mathfrak{m}(x) \leq C\mathfrak{m}'(x),\ x \in \mathbb{R}$$

holds. //

It is known that minimal majorants precisely correspond to one-dimensional dB-subspaces of the space $\mathcal{H}(E)$ under consideration, cf. [3, Theorem 5.5], [4, Theorem 4.9]. From Theorem 3.2 we thus obtain without further notice the following statement.

6.8 Proposition. *Let $E \in \mathcal{H}B_0^*$. Then there exists a minimal (positive) admissible majorant for $\mathcal{H}(E)$ if and only if for one (and hence for all) $\varphi \in \mathbb{R}$ the function q_φ satisfies* (I), (II), *and* (III$_0$). ❑

6.3 Extension of hermitian indefinite functions

Let $a \in (0, \infty)$ and $\kappa \in \mathbb{N}_0$. The class $\mathfrak{P}_{\kappa,a}$ is defined as the set of all continuous functions $f : [-2a, 2a] \to \mathbb{C}$ with $f(-t) = \overline{f(t)}$, $t \in [-2a, 2a]$, for which the kernel

$$D(s,t) := f(t-s), \quad s,t \in (-a,a), \tag{6.4}$$

has κ negative squares. Moreover, we set

$$\mathfrak{P}_{<\infty,a} := \bigcup_{\kappa \in \mathbb{N}_0} \mathfrak{P}_{\kappa,a}.$$

Similarly, we define classes $\mathfrak{P}_\kappa$ as the sets of continuous functions $f : \mathbb{R} \to \mathbb{C}$ with $f(-t) = \overline{f(t)}$, $t \in \mathbb{R}$, for which the kernel (6.4) has κ negative squares, and set $\mathfrak{P}_{<\infty} := \bigcup_{\kappa\in\mathbb{N}_0} \mathfrak{P}_\kappa$. If $f \in \mathfrak{P}_{<\infty,a}$ or $f \in \mathfrak{P}_{<\infty}$, we write $\operatorname{ind}_- f$ for the actual number of negative squares of the kernel (6.4).

The extension problem for hermitian indefinite functions is the following question: Given $f \in \mathfrak{P}_{<\infty,a}$. Do there exist extensions $\tilde{f}$ of f to the whole real line which belong to $\mathfrak{P}_{<\infty}$? If yes, describe the totality of all of them.

6.9. Solution ([33], [28]): If $f \in \mathfrak{P}_{<\infty,a}$ is given, then there exists a number $\Delta(f) \in \mathbb{N} \cup \{0,\infty\}$ such that

(i) If $\Delta(f) = 0$, then

$$\forall\, \kappa \geq \operatorname{ind}_- f\ \exists\, \infty\text{-many } \tilde{f} \in \mathfrak{P}_\kappa :\ \tilde{f}|_{[-2a,2a]} = f$$

(ii) If $0 < \Delta(f) < \infty$, then

$$\exists!\, \tilde{f} \in \mathfrak{P}_{\operatorname{ind}_- f} :\ \tilde{f}|_{[-2a,2a]} = f$$
$$\forall\, \operatorname{ind}_- f < \kappa < \operatorname{ind}_- f + \Delta(f)\ \nexists\, \tilde{f} \in \mathfrak{P}_\kappa :\ \tilde{f}|_{[-2a,2a]} = f$$
$$\forall\, \kappa \geq \operatorname{ind}_- f + \Delta(f)\ \exists\, \infty\text{-many } \tilde{f} \in \mathfrak{P}_\kappa :\ \tilde{f}|_{[-2a,2a]} = f$$

(iii) If $\Delta(f) = \infty$, then

$$\exists!\, \tilde{f} \in \mathfrak{P}_{\operatorname{ind}_- f} :\ \tilde{f}|_{[-2a,2a]} = f$$
$$\forall\, \kappa > \operatorname{ind}_- f\ \nexists\, \tilde{f} \in \mathfrak{P}_\kappa :\ \tilde{f}|_{[-2a,2a]} = f$$

Moreover, if $\Delta(f) < \infty$, then there exists $W \in \mathcal{M}_{<\infty}$ such that the formula

$$i \int_0^\infty \tilde{f}(t) e^{izt}\, dt = W(z) \star \tau(z), \quad \operatorname{Im} z \geq h_{\tilde{f}} > 0$$

parameterizes all extensions $\tilde{f} \in \mathfrak{P}_{<\infty}$ of f. The parameter $\tau(z)$ thereby runs through a certain class of functions which depends on $\Delta(f)$. //

The proof of existence of the constant $\Delta(f)$ is rather implicit, and in general it is a hard task to determine the number $\Delta(f)$. For example, for in some sense 'smooth' functions $f \in \mathfrak{P}_{<\infty,a}$ in [33, Theorem 2.2, Theorem 2.3] a (quite implicit) criterion to decide whether $\Delta(f) = 0$ or $\Delta(f) > 0$ has been given.

We can use Theorem 4.8 and Theorem 5.4 to determine even the exact value of $\Delta(f)$ in a different (but admittedly also not very explicit) way. This is the content of the following two propositions. If $g \in \mathfrak{P}_{<\infty}$, we set

$$Q_g(z) := i \int_0^\infty g(t) e^{izt}\, dt\,.$$

6.10 Proposition. *Let $f \in \mathfrak{P}_{<\infty,a}$ be given. Then $\Delta(f) = 0$ if and only if for some extension $\tilde{f} \in \mathcal{P}_{\operatorname{ind}_- f}$ of f the function $Q_{\tilde{f}}$ satisfies* (I), (II), (III$_1$).

6.11 Proposition. *Let $f \in \mathfrak{P}_{<\infty,a}$ be given. Assume that $\Delta(f) > 0$, and let $\mathring{f}$ be the unique extension of f in the class $\mathfrak{P}_{\operatorname{ind}_- f}$. Then*

$$\Delta(f) = \inf\big\{ N \in \mathbb{N} :\ Q_{\mathring{f}} \text{ satisfies (I), (II), (III}_N) \big\} - 1\,,$$

where the infimum of the empty set is understood as ∞.

The proofs of these two propositions are based on the same observation, which we shall therefore state separately.

Basic observation: Let $g \in \mathfrak{P}_{<\infty}$ and let $a > 0$. Let $\mathfrak{h}_g$ be the general Hamiltonian whose Weyl-coefficient is equal to Q_g (given by data denoted as in Definition 4.1). Denote by ω_g the maximal chain of matrices corresponding to $\mathfrak{h}_g$, and set

$$t_a(g) := \sup\big\{t \in \{\sigma_0\} \cup \bigcup_{i=0}^{n}(\sigma_i, \sigma_{i+1}) : \operatorname{et}\omega_g(t)_{lk} < a,\ l,k = 1,2\big\}\,,$$

where et denotes the exponential type of the function under consideration. By [20, Proposition 5.4, Lemma 5.8], the chain $\omega_g(t)|_{(\sigma_0, t_a(g))}$, and hence the general Hamiltonian $(\mathfrak{h}_g)_{\urcorner t_a(g)}$, is completely determined by the restriction $g|_{[-2a,2a]}$.

Hence, with a given function $f \in \mathfrak{P}_{<\infty,a}$ we can associate a general Hamiltonian $\mathring{\mathfrak{h}}_f$ by setting $\mathring{\mathfrak{h}}_f := (\mathfrak{h}_g)_{\urcorner t_a(g)}$ where $g \in \mathfrak{P}_{<\infty}$ is any extension of f. Moreover, a function $g \in \mathfrak{P}_{<\infty}$ is an extension of f if and only if $\mathfrak{h}_g$ prolongues $\mathring{\mathfrak{h}}_f$.

Proof (of Proposition 6.10). Let $f \in \mathfrak{P}_{<\infty,a}$ be given. Obviously there exist infinitely many extensions of f in the class $\mathfrak{P}_{\operatorname{ind}_- f}$ if and only if $\mathring{\mathfrak{h}}_f$ is in the limit circle case. This, however, is the case if and only if there exists an extension g of f with $\operatorname{ind}_- g = \operatorname{ind}_- f$ such that the corresponding general Hamiltonian $\mathfrak{h}_g$ ends with an indivisible interval. Note here that, by the definition of $t_a(g)$, the general Hamiltonian $(\mathfrak{h}_g)_{\urcorner t_a(g)}$ can in particular not end with an indivisible interval. The assertion of Proposition 6.10 follows from Theorem 5.4. ❑

Proof (of Proposition 6.11). Let $f \in \mathfrak{P}_{<\infty,a}$ and assume that $\Delta(f) > 0$. Let $\mathring{f}$ be the unique extension of f with $\operatorname{ind}_- \mathring{f} = \operatorname{ind}_- f$. Then $\mathring{\mathfrak{h}}_f = \mathfrak{h}_{\mathring{f}}$ and this general Hamiltonian is in the limit point case. Thus, there exist extensions $g \in \mathfrak{P}_{<\infty}$ of f with $\operatorname{ind}_- g > \operatorname{ind}_- f$ if and only if $Q_{\mathring{f}}$ is an intermediate Weyl-coefficient, and the minimal number of increase of negative squares (i.e. the number $\Delta(f)$) is equal to the weight of $Q_{\mathring{f}}$. The assertion of Proposition 6.11 follows from Theorem 4.8. ❑

Appendix A. Polynomial asymptotics for functions in a de Branges Pontryagin space

Let $E \in \mathcal{HB}^*_{<\infty}$ and $\varphi \in \mathbb{R}$ be fixed. Our aim in this appendix is to give sharp estimates on the polynomial growth of the functions $S_\varphi^{-1}F$, $F \in \mathcal{P}(E)$. Depending on the geometric structure of the algebraic eigenspace at ∞ of the selfadjoint relation $\mathcal{A}_\varphi$, and on how close F is to the orthogonal complement of this space, different situations occur.

The proof of the below Proposition A.1 is purely operator theoretic. However, in order to avoid introduction of even more terms and notation, we will stick to the situation of present interest.

We will use the following notation (remember Remark 2.4, *(ii)*, *(iii)*):

$$\delta_\varphi := \dim \mathfrak{E}_\infty^\varphi, \qquad \delta_\varphi^\circ := \dim(\mathfrak{E}_\infty^\varphi)^\circ\,,$$

$$\mathfrak{j}_F := \min\big\{j \in \{0, \dots, \delta_\varphi - 1\} : F \not\perp z^j S_\varphi\big\}, \quad F \in \mathcal{P}(E), F \notin (\mathfrak{E}_\infty^\varphi)^\perp\,.$$

A.1 Proposition. *Let $E \in \mathcal{H}B^*_{<\infty}$ and $\varphi \in \mathbb{R}$. Then the following hold:*

(*i*) *Assume that $\mathfrak{E}^\varphi_\infty = \{0\}$. Then*

$$\lim_{z \widehat{\to} i\infty} \frac{F(z)}{S_\varphi(z)} = 0\,, \quad F \in \mathcal{P}(E)\,.$$

(*ii*) *Assume that $\delta^\circ_\varphi < \delta_\varphi$. Then*

$$\lim_{z \widehat{\to} i\infty} \frac{1}{z^{\delta^\circ_\varphi}} \frac{F(z)}{S_\varphi(z)} = 0\,, \qquad F \in (\mathfrak{E}^\varphi_\infty)^\perp$$

$$\lim_{z \widehat{\to} i\infty} \frac{1}{z^{\delta^\circ_\varphi+(\delta_\varphi - \mathfrak{j}_F - 1)}} \frac{F(z)}{S_\varphi(z)} = -\frac{[F, z^{\mathfrak{j}_F} S_\varphi]}{[z^{\delta_\varphi - 1} S_\varphi, z^{\delta^\circ_\varphi} S_\varphi]}\,, \quad F \notin (\mathfrak{E}^\varphi_\infty)^\perp$$

(*iii*) *Assume that $\delta^\circ_\varphi = \delta_\varphi > 0$, and that $\lim_{z \widehat{\to} i\infty} |z^{2\delta_\varphi} q_{\varphi+\frac{\pi}{2}}(z)| = \infty$. Then*

$$\lim_{z \widehat{\to} i\infty} \frac{1}{z^{\delta_\varphi}} \frac{F(z)}{S_\varphi(z)} = 0\,, \quad F \in (\mathfrak{E}^\varphi_\infty)^\perp$$

$$\lim_{z \widehat{\to} i\infty} \frac{1}{z^{\delta_\varphi+(\delta_\varphi - \mathfrak{j}_F - 1)}} \frac{F(z)}{S_\varphi(z)} = 0\,, \quad F \notin (\mathfrak{E}^\varphi_\infty)^\perp$$

(*iv*) *Assume that $\delta_\varphi > 0$, and that $s_{2\delta_\varphi - 1} := \lim_{z \widehat{\to} i\infty} z^{2\delta_\varphi} q_{\varphi+\frac{\pi}{2}}(z) \in \mathbb{R}\setminus\{0\}$. In this case, certainly, $\delta^\circ_\varphi = \delta_\varphi$. Then*

$$\lim_{z \widehat{\to} i\infty} \frac{1}{z^{\delta_\varphi}} \frac{F(z)}{S_\varphi(z)} = 0\,, \qquad F \in (\mathfrak{E}^\varphi_\infty)^\perp$$

$$\lim_{z \widehat{\to} i\infty} \frac{1}{z^{\delta_\varphi+(\delta_\varphi - \mathfrak{j}_F - 1)}} \frac{F(z)}{S_\varphi(z)} = -\frac{[F, z^{\mathfrak{j}_F} S_\varphi]}{s_{2\delta_\varphi - 1}}\,, \quad F \notin (\mathfrak{E}^\varphi_\infty)^\perp$$

(*v*) *Assume that $\delta^\circ_\varphi = \delta_\varphi > 0$, and that $s_{2\delta_\varphi - 1} = 0$. Also in this case, certainly, $\delta^\circ_\varphi = \delta_\varphi$. Then*

$$\lim_{z \widehat{\to} i\infty} \frac{1}{z^{\delta_\varphi+1}} \frac{F(z)}{S_\varphi(z)} = 0\,, \quad F \in (\mathfrak{E}^\varphi_\infty)^\perp$$

$$\lim_{z \widehat{\to} i\infty} \frac{1}{z^{\delta_\varphi+(\delta_\varphi - \mathfrak{j}_F - 1)+1}} \frac{F(z)}{S_\varphi(z)} = 0\,, \quad F \notin (\mathfrak{E}^\varphi_\infty)^\perp$$

$$\lim_{z \widehat{\to} i\infty} \left| \frac{1}{z^{\delta_\varphi+(\delta_\varphi - \mathfrak{j}_F - 1)}} \frac{F(z)}{S_\varphi(z)} \right| = \infty\,, \quad F \notin (\mathfrak{E}^\varphi_\infty)^\perp$$

(*vi*) *Assume that $\delta^\circ_\varphi = \delta_\varphi > 0$, and that the limit $\lim_{z \widehat{\to} i\infty} z^{2\delta_\varphi} q_{\varphi+\frac{\pi}{2}}(z)$ does not exist in $\mathbb{R} \cup \{\pm\infty\}$. Put $s := \limsup_{z \widehat{\to} i\infty} |z^{2\delta_\varphi} q_{\varphi+\frac{\pi}{2}}(z)| \in (0, \infty]$. Then*

$$\lim_{z \widehat{\to} i\infty} \frac{1}{z^{\delta_\varphi+1}} \frac{F(z)}{S_\varphi(z)} = 0\,, \qquad F \in (\mathfrak{E}^\varphi_\infty)^\perp$$

$$\lim_{z \widehat{\to} i\infty} \frac{1}{z^{\delta_\varphi+(\delta_\varphi - \mathfrak{j}_F - 1)+1}} \frac{F(z)}{S_\varphi(z)} = 0\,, \qquad F \notin (\mathfrak{E}^\varphi_\infty)^\perp$$

$$\liminf_{z \widehat{\to} i\infty} \left| \frac{1}{z^{\delta_\varphi}} \frac{F(z)}{S_\varphi(z)} \right| = 0\,, \qquad F \in (\mathfrak{E}^\varphi_\infty)^\perp$$

$$\liminf_{z \widehat{\to} i\infty} \left| \frac{1}{z^{\delta_\varphi+(\delta_\varphi - \mathfrak{j}_F - 1)}} \frac{F(z)}{S_\varphi(z)} \right| \leq \frac{|[F, z^{\mathfrak{j}_F} S_\varphi]|}{s}\,, \quad F \notin (\mathfrak{E}^\varphi_\infty)^\perp$$

A.2 Remark. Using the canonical factorization of a generalized Nevanlinna function established in [12], one easily sees the following statement. For each generalized Nevanlinna function q, there exists a (unique) integer n, such that the limit $\lim_{z \hat{\to} i\infty} z^k q(z)$ shows the behaviour

$k \in \mathbb{Z}$	$\dots$	$2n-3$	$2n-2$	$2n-1$	$2n$	$2n+1$	$2n+2$	$2n+3$	$\dots$
$\lim_{z \hat{\to} i\infty} z^k q(z)$		0	0	$\exists$ in $\mathbb{R}$	$?$	$\exists$ in $[-\infty,0)\cup(0,\infty]$	$\pm\infty$	$\pm\infty$	

where '?' means that the limit may or may not exist in $\mathbb{R} \cup \{\pm\infty\}$. //

Proof (of Proposition A.1). Let us consider the selfadjoint extension $\mathcal{A}_\varphi$ of the multiplication operator $\mathcal{S}_E$, and recall some of its properties, cf. [21, Lemma 6.4, Lemma 7.2]. Set

$$\chi(z,\zeta) := \frac{1}{S_\varphi(z)} K_E(\overline{z},\zeta), \quad z \in \mathbb{C}, S_\varphi(z) \neq 0 \,.$$

Then we have $\operatorname{ran}(\mathcal{S}_E - \overline{z})^\perp = \operatorname{span}\{\chi(z,.)\}$, and

$$\chi(z,.) = \big(I + (z-z_0)(\mathcal{A}_\varphi - z)^{-1}\big)\chi(z_0,.), \quad z, z_0 \in \mathbb{C},\ S_\varphi(z), S_\varphi(z_0) \neq 0 \,.$$

In the case that $\mathfrak{E}_\infty^\varphi = \{0\}$, the assertion of Proposition A.1 will immediately follow from this fact.

The case that $\mathfrak{E}_\infty^\varphi \neq \{0\}$ is much more involved, and its proof requires finer arguments. Assume that $S_\varphi \in \mathcal{P}(E)$, and consider another selfadjoint extension of $\mathcal{S}_E$, namely $A := \mathcal{A}_{\varphi+\frac{\pi}{2}}$. Note that, since $\mathcal{A}_\varphi$ is a proper relation, and $\mathcal{S}_E$ has defect index $(1,1)$, the extension A is an operator. Using the definition of A, we compute

$$(A-z)^{-1} S_\varphi(\zeta) = \frac{S_\varphi(\zeta) - \frac{S_{\varphi+\frac{\pi}{2}}(\zeta)}{S_{\varphi+\frac{\pi}{2}}(z)} S_\varphi(z)}{\zeta - z} = -\frac{K_E(\overline{z},\zeta)}{S_{\varphi+\frac{\pi}{2}}(z)} \,.$$

Hence, and this is the crucial formula for the present proof, we have

$$\big[(A-z)^{-1} S_\varphi(.), \overline{F(\overline{.})}\big] = -\frac{F(z)}{S_{\varphi+\frac{\pi}{2}}}(z) = q_{\varphi+\frac{\pi}{2}}(z) \cdot \frac{F(z)}{S_\varphi(z)} \,.$$

In particular,

$$[(A-z)^{-1} S_\varphi, S_\varphi] = q_{\varphi+\frac{\pi}{2}}(z) \,.$$

It now follows, e.g., from [15, Proposition 3.4, (3.10)], that the algebraic eigenspace of A at 0 is the same as the algebraic eigenspace of $\mathcal{A}_\varphi$ at ∞, i.e. equal to $\mathfrak{E}_\infty^\varphi$. More precisely, we have $0 \in \sigma(A)$ if and only if $S_\varphi \in \mathcal{P}(E)$. In this case $S_\varphi \in \operatorname{dom} A^{\delta_\varphi - 1}$, and there exist constants $\lambda_1, \dots, \lambda_{\delta_\varphi - 1} \in \mathbb{C}$ such that

$$A^k S_\varphi = z^k S_\varphi + \sum_{j=0}^{k-1} \lambda_{k-j} z^j S_\varphi, \quad k = 0, \dots, \delta_\varphi - 1 \,. \tag{A.1}$$

We now use the usual trick to obtain asymptotics for $(A-z)^{-1}|_{\operatorname{dom} A^k}$, see, e.g., the proof of [30, Theorem 1.10], and write

$$z^{\delta_\varphi}(A-z)^{-1}S_\varphi = A(A-z)^{-1}A^{\delta_\varphi-1}S_\varphi - \sum_{j=0}^{\delta_\varphi-1} z^{\delta_\varphi-j-1}A^jS_\varphi\,.$$

It follows that

$$\big[A(A-z)^{-1}A^{\delta_\varphi-1}S_\varphi(.),\overline{F(\bar{.})}\big] = z^{\delta_\varphi}\cdot q_{\varphi+\frac{\pi}{2}}(z)\cdot\frac{F(z)}{S_\varphi(z)} + \sum_{j=0}^{\delta_\varphi-1} z^{\delta_\varphi-j-1}[A^jS_\varphi(.),\overline{F(\bar{.})}]\,.$$

Since A is an operator, the left side of this relation tends to 0 if z tends to $i\infty$ nontangentially, see, e.g., again the proof of [30, Theorem 1.10]. Keeping in mind that S_φ is real, we end up with the limit relation

$$\lim_{z\,\widehat{\to}\, i\infty}\Big(z^{\delta_\varphi}\cdot q_{\varphi+\frac{\pi}{2}}(z)\cdot\frac{F(z)}{S_\varphi(z)} + \sum_{j=0}^{\delta_\varphi-1} z^{\delta_\varphi-j-1}[F,A^jS_\varphi]\Big) = 0\,. \tag{A.2}$$

The asymptotic behaviour of the function $q_{\varphi+\frac{\pi}{2}}(z) = [(A-z)^{-1}S_\varphi,S_\varphi]$ is well understood, e.g., from the already cited result [30, Theorem 1.10]. We will in the following rather refer to the more detailed and more up to date exposition [15], see also [11]. By [15, Theorem 3.1, Corollary 4.4], we have

$$q_{\varphi+\frac{\pi}{2}}(z) = \sum_{j=\delta_\varphi+\delta_\varphi^\circ}^{2\delta_\varphi-1}\frac{s_{j-1}}{z^j} + o\Big(\frac{1}{z^{2\delta_\varphi-1}}\Big),\quad z\,\widehat{\to}\, i\infty\,, \tag{A.3}$$

with numbers $s_j\in\mathbb{R}$ where $s_{\delta_\varphi+\delta_\varphi^\circ-1}\neq 0$. In fact, the numbers s_j are given as

$$s_j := [z^{\delta_\varphi-1}S_\varphi, z^jS_\varphi]\,.$$

The asymptotic expansion (A.3) need not be maximal. Still, in the case that $\delta_\varphi^\circ\neq 0$, it is almost maximal in the following sense: The limit

$$s_{2\delta_\varphi-1} := -\lim_{z\,\widehat{\to}\, i\infty} z^{2\delta_\varphi}\Big(q(z) + \sum_{j=\delta_\varphi+\delta_\varphi^\circ}^{2\delta_\varphi-1}\frac{s_{j-1}}{z^j}\Big)$$

may or may not exist in $\mathbb{R}\cup\{\pm\infty\}$. If it either does not exist or is equal to $\pm\infty$, then the asymptotic expansion (A.3) cannot be prolongued. If $s_{2\delta_\varphi-1}$ exists and is a real number, then the expansion (A.3) can be prolongued for one more step, and we have

$$q_{\varphi+\frac{\pi}{2}}(z) = \sum_{j=\delta_\varphi+\delta_\varphi^\circ}^{2\delta_\varphi}\frac{s_{j-1}}{z^j} + o\Big(\frac{1}{z^{2\delta_\varphi}}\Big),\quad z\,\widehat{\to}\, i\infty\,.$$

This expansion cannot be prolongued anymore, in fact

$$\lim_{z\,\widehat{\to}\, i\infty}\Big|z^{2\delta_\varphi+1}\Big(q(z) + \sum_{j=\delta_\varphi+\delta_\varphi^\circ}^{2\delta_\varphi}\frac{s_{j-1}}{z^j}\Big)\Big| = \infty\,. \tag{A.4}$$

Proof of case (i): We have $\mathfrak{E}^{\varphi}_{\infty} = \{0\}$, and hence $\mathcal{A}_{\varphi}$ is an operator. From the relation $I + z(A-z)^{-1} = A(A-z)^{-1}$, and what we have already once noted above, it follows that $\lim_{z \widehat{\to} i\infty}(I + z(A-z)^{-1}) = 0$. Hence $\lim_{z \widehat{\to} i\infty} \chi(z,.) = 0$, and we obtain

$$\lim_{z \widehat{\to} i\infty} \frac{F(z)}{S_{\varphi}(z)} = \lim_{z \widehat{\to} i\infty} [\chi(z,.), \overline{F(\bar{.})}] = 0\,.$$

Proof of case (ii): We have $0 \leq \delta^{\circ}_{\varphi} < \delta_{\varphi}$, in particular, $\mathfrak{E}^{\varphi}_{\infty} \neq \{0\}$. The leading term in the asymptotic expansion (A.3) is $z^{-\delta_{\varphi}-\delta^{\circ}_{\varphi}} s_{\delta_{\varphi}+\delta^{\circ}_{\varphi}-1}$, and we obtain

$$\lim_{z \widehat{\to} i\infty} z^{\delta_{\varphi}+\delta^{\circ}_{\varphi}} q_{\varphi+\frac{\pi}{2}}(z) = s_{\delta_{\varphi}+\delta^{\circ}_{\varphi}-1} \neq 0\,.$$

Assume that $F \in (\mathfrak{E}^{\varphi}_{\infty})^{\perp}$. Then the sum in (A.2) vanishes, and it follows that

$$\lim_{z \widehat{\to} i\infty} \frac{1}{z^{\delta^{\circ}_{\varphi}}} \frac{F(z)}{S_{\varphi}(z)} =$$

$$= \lim_{z \widehat{\to} i\infty} \Big(z^{\delta_{\varphi}+\delta^{\circ}_{\varphi}} q_{\varphi+\frac{\pi}{2}}(z)\Big)^{-1} \cdot \lim_{z \widehat{\to} i\infty} \Big(z^{\delta_{\varphi}+\delta^{\circ}_{\varphi}} q_{\varphi+\frac{\pi}{2}}(z) \cdot \frac{1}{z^{\delta^{\circ}_{\varphi}}} \frac{F(z)}{S_{\varphi}(z)}\Big) = 0\,.$$

This is the desired assertion.

Assume that $F \notin (\mathfrak{E}^{\varphi}_{\infty})^{\perp}$. Then, keeping in mind (A.1), we see that the leading term in the sum in (A.2) is $z^{\delta_{\varphi}-\mathfrak{j}_F-1}[F, z^{\mathfrak{j}_F} S_{\varphi}]$. It follows that

$$0 = \lim_{z \widehat{\to} i\infty} z^{\delta_{\varphi}-\mathfrak{j}_F-1} \cdot \Big(z^{\delta_{\varphi}+\delta^{\circ}_{\varphi}} q_{\varphi+\frac{\pi}{2}}(z) \cdot \frac{1}{z^{\delta^{\circ}_{\varphi}+(\delta_{\varphi}-\mathfrak{j}_F-1)}} \frac{F(z)}{S_{\varphi}(z)} + \sum_{j=\mathfrak{j}_F}^{\delta_{\varphi}-1} \frac{[F, A^j S_{\varphi}]}{z^{j-\mathfrak{j}_F}}\Big)\,,$$

and since $\delta_{\varphi} - \mathfrak{j}_F - 1 \geq 0$, the desired assertion follows.

Proof of case (iii): If $F \in (\mathfrak{E}^{\varphi}_{\infty})^{\perp}$, we write the first summand in (A.2) as

$$z^{2\delta_{\varphi}} q_{\varphi+\frac{\pi}{2}}(z) \cdot \frac{1}{z^{\delta_{\varphi}}} \frac{F(z)}{S_{\varphi}(z)}\,. \tag{A.5}$$

For $F \notin (\mathfrak{E}^{\varphi}_{\infty})^{\perp}$, we write it as

$$z^{\delta_{\varphi}-\mathfrak{j}_F-1} \cdot z^{2\delta_{\varphi}} q_{\varphi+\frac{\pi}{2}}(z) \cdot \frac{1}{z^{\delta_{\varphi}+(\delta_{\varphi}-\mathfrak{j}_F-1)}} \frac{F(z)}{S_{\varphi}(z)}\,. \tag{A.6}$$

Proof of case (iv): Again writing the first summand in (A.2) as in (A.5) or (A.6), respectively, and again keeping in mind (A.1), the assertion follows also in this case.

Proof of case (v): If $F \in (\mathfrak{E}^{\varphi}_{\infty})^{\perp}$, we write the first summand in (A.2) as

$$z^{2\delta_{\varphi}+1} q_{\varphi+\frac{\pi}{2}}(z) \cdot \frac{1}{z^{\delta_{\varphi}+1}} \frac{F(z)}{S_{\varphi}(z)}\,. \tag{A.7}$$

For $F \notin (\mathfrak{E}^{\varphi}_{\infty})^{\perp}$, we write it as

$$z^{\delta_{\varphi}-\mathfrak{j}_F-1} \cdot z^{2\delta_{\varphi}+1} q_{\varphi+\frac{\pi}{2}}(z) \cdot \frac{1}{z^{\delta_{\varphi}+(\delta_{\varphi}-\mathfrak{j}_F-1)+1}} \frac{F(z)}{S_{\varphi}(z)}\,. \tag{A.8}$$

Due to (A.4), the first two formulas follow. In order to see the third formula, we rather use (A.6) and observe that the second summand in (A.2), after being multiplied with $z^{-(\delta_\varphi-\mathfrak{j}_F-1)}$, tends to a nonzero limit.

Proof of case (vi): Due to Remark A.2, we have $\lim_{z\widehat{\to}i\infty}|z^{2\delta_\varphi+1}q_{\varphi+\frac{\pi}{2}}(z)|=\infty$. Hence, the first two formulas follow using (A.7) and (A.8).

For the proof of the second pair of formulas, we use (A.5) and (A.6) instead, and choose a sequence $(z_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty}|z_n^{2\delta_\varphi}q_{\varphi+\frac{\pi}{2}}(z_n)|=\limsup_{z\widehat{\to}i\infty}|z^{2\delta_\varphi}q_{\varphi+\frac{\pi}{2}}(z)|\,.$$

❑

To finish with, let us explain the promised link of the sharp estimates given in Proposition A.1 with the proof of Theorem 3.2.

A.3 Remark. In Steps 3 and 5 of the proof of Theorem 3.2 we have concluded from the estimate (3.11) that Λ is a polynomial. Of course, its degree cannot exceed n_0-1, where n_0 is as in Lemma 3.7. Since the function f belongs to $f\in\mathcal{P}(\mathring{E})[-]\mathfrak{E}_\infty^{\frac{\pi}{2}}$, the stronger estimates provided in Proposition A.1 imply in all cases listed in Proposition A.1 with exception of case (v), that we have

$$\liminf_{z\widehat{\to}i\infty}\frac{1}{z^{\delta_\varphi^\circ}}\frac{f(z)}{A_{q_\varphi}(z)}=0\,.$$

It follows that $\deg\Lambda\leq\delta_\varphi^\circ-1$, and hence that

$$h=f+\Lambda A_{q_\varphi}\in\big(\mathcal{P}(\mathring{E})[-]\mathfrak{E}_\infty^{\frac{\pi}{2}}\big)+\underbrace{\operatorname{span}\{z^kS_\varphi:\,0\leq k<\delta_\varphi^\circ\}}_{=(\mathfrak{E}_\infty^{\frac{\pi}{2}})^\circ}\subseteq\mathcal{P}(\mathring{E})[-]\mathring{\mathfrak{E}}_\infty^{\frac{\pi}{2}}\,.$$

Hence, unless $\lim_{z\widehat{\to}i\infty}z^{2\delta_\varphi}q_{\varphi+\frac{\pi}{2}}(z)=0$, the argument made in Step 4 will not be necessary.

This notice gains interest, if we remember that the conclusion of Steps 4 and 6 relies on the Ordering Theorem, which is very deep and most specific, whereas all other machinery employed before was rather general, of geometric nature, and (comparatively) elementary. //

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