A function space model for canonical systems with an inner singularity

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Abstract

Recently, a generalization to the Pontryagin space setting of the notion of canonical (Hamiltonian) systems which involves a finite number of inner singularities has been given. The spectral theory of indefinite canonical systems was investigated with help of an operator model. This model consists of a Pontryagin space boundary triple and was constructed in an abstract way. Moreover, the construction of this operator model involves a procedure of splitting-and-pasting which is technical but at the present stage of development in general inevitable.

In this paper we provide an isomorphic form of this operator model which acts in a finite dimensional extension of a function space naturally associated with the given indefinite canonical system. We give explicit formulae for the model operator and the boundary relation. Moreover, we show that under certain asymptotic hypotheses the procedure of splitting-and-pasting can be avoided by employing a limiting process.

We restrict attention to the case of one singularity. This is the core of the theory, and by making this restriction we can significantly reduce the technical effort without losing sight of the essential ideas.

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1 Introduction

A canonical system is a 2×2-system of differential equations of the form

$$y'(t) = zJH(t)y(t), \quad t \in (s_-, s_+)\,, \tag{1.1}$$

with a locally integrable, real-valued and non-negative 2×2-matrix valued function $H(t)$, a complex parameter z, and the signature matrix

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The function $H(t)$ is also called the Hamiltonian of the system (1.1). Equations of this form frequently occur in analysis and natural sciences; for example in Hamiltonian mechanics, cf. [Ar], [Fl], as generalizations of Sturm–Liouville equations, cf. [R], or in the study of strings, cf. [At], [KK], [Ka3].

Canonical systems can be viewed from an operator theoretic perspective as a boundary triple $\mathfrak{B}(H) = (L^2(H), T_{\max}(H), \Gamma(H))$. Here the Hilbert space $L^2(H)$ is a weighted L^2-space of 2-vector valued functions, the operator $T_{\max}(H)$ is the natural maximal differential operator in $L^2(H)$ associated with (1.1) (actually, it can be a linear relation, i.e. a 'multi-valued operator'), and $\Gamma(H)$ is the

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natural boundary map. This construction goes back to [Ka1], [Ka2]; see also, e.g. [HSW], [Sa], [O], [GK].

In various contexts generalizations of canonical systems appear which include a finite number of singularities (point-interaction type singularities or non-integrability of H). Such examples are found, e.g. in Sturm–Liouville equations with singular potentials, cf. [FuL], [GZ], [DS], [EGZ], indefinite versions of the Hamburger and the Stieltjes power moment problems, cf. [KL1], [RS1], the extension problem of positive definite functions, cf. [KL2], [LLS], [KW1], or the theory of generalized strings, cf. [LW], [KWW]. In many examples a large part of the spectral theory particular to canonical systems (Titchmarsh–Weyl coefficient, Fourier transform, spectral multiplicity, etc.) can be carried over. The reason lying behind this fact is that for many problems an operator model which acts in a Pontryagin space (instead of a Hilbert space) can be constructed; see, e.g. [RS2], [RS3], [RS4], [vDT], [P], [Sh], [DL], [DLSZ], [KuLu], [AK].

In [KW2] an indefinite analogue of the equation (1.1) which includes finitely many singularities was considered, a notion of a generalized Hamiltonian $\mathfrak{h}$ was defined, and a corresponding Pontryagin space model consisting of a boundary triple $\mathfrak{B}(\mathfrak{h}) = (\mathfrak{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h}))$ was constructed. This notion of indefinite canonical systems covers the known examples and, actually, goes as far as Pontryagin space theory can possibly lead.

One drawback of the construction in [KW2] is that it is rather abstract and not easy to work with in particular instances of the theory. Our aim in the present paper is to give a more concrete form of this Pontryagin space model. We identify the model space with a finite-dimensional extension of a function space and the model operator as a finite-dimensional perturbation of the natural maximal differential operator in this function space.

Throughout this paper we restrict considerations to the case of one singularity only. This restriction is made for two reasons. First, as a general rule, results obtained for the case of one singularity will transfer to the general case by sufficient technical labour. Secondly, many previously studied instances of indefinite canonical systems actually do involve only one singularity. For example, Sturm–Liouville equations with singular potentials like the Bessel equation, cf. [DS], or the canonical system arising from the positive definite function studied in [LLS]. Altogether, we may say that the restriction to the case of one singularity significantly reduces the technical effort without losing sight of the essential ideas and still covers a range of examples.

Let us outline the contents of this paper. In the second part of the present introductory section we recall the notion of generalized Hamiltonians $\mathfrak{h}$ as introduced in [KW2] and make precise our overall assumptions on $\mathfrak{h}$. In Section 2 we associate with $\mathfrak{h}$ a boundary triple $\mathring{\mathring{\mathfrak{B}}}_{x_0}(\mathfrak{h}) = (\mathring{\mathring{\mathfrak{P}}}_{x_0}(\mathfrak{h}), \mathring{\mathring{T}}_{x_0}(\mathfrak{h}), \mathring{\mathring{\Gamma}}_{x_0}(\mathfrak{h}))$ which is isomorphic to the boundary triple $\mathfrak{B}(\mathfrak{h})$ originally constructed in [KW2]. The space $\mathring{\mathring{\mathfrak{P}}}_{x_0}(\mathfrak{h})$ is a finite-dimensional extension of a certain function space. Our first main result is an explicit description of the model operator $\mathring{\mathring{T}}_{x_0}(\mathfrak{h})$ as a finite-dimensional perturbation of the natural maximal differential operator in this function space, cf. Theorem 2.15, Corollary 2.20, Remark 2.21.

The model $\mathring{\mathring{\mathfrak{B}}}_{x_0}(\mathfrak{h})$ (in particular, the operator $\mathring{\mathring{T}}_{x_0}(\mathfrak{h})$) depends on a splitting-point x_0 to the right of the singularity. The question whether the x_0-dependence can be removed is natural. It is our aim in Section 3 to show that under certain asymptotic conditions on $\mathfrak{h}$, this is possible. We apply a limiting procedure to obtain an x_0-independent model boundary triple $\overset{\infty}{\mathfrak{B}}(\mathfrak{h})$, cf. Proposition 3.9 and the paragraph preceding it. The underlying Pontryagin space $\overset{\infty}{\mathfrak{P}}(\mathfrak{h})$ is a finite-dimensional extension of the same function space which appears in Section 2, and the model operator $\overset{\infty}{T}(\mathfrak{h})$ is a finite-dimensional perturbation of the maximal differential operator in this space. Our second main result, besides the fact that $\overset{\infty}{\mathfrak{B}}(\mathfrak{h})$ is well defined, is the explicit description of $\overset{\infty}{T}(\mathfrak{h})$ given in Theorem 3.12.

The paper closes with an appendix, where we prove some technical formulae related to [KW2, §7]. These formulae are needed in the asymptotic considerations of Section 3. However, in order to not disturb the line of thoughts in Section 3, we shifted their proof to the appendix.

Throughout the paper the notation '(IV.2.3)' refers to equation (2.3) in [KW2] and 'Proposition IV.4.14' to Proposition 4.14 in [KW2].

The notion of general Hamiltonians.

First we have to introduce (or recall) some preliminary notation.

We call a function H a *Hamiltonian* if it is defined on some interval (L_-, L_+), takes real and non-negative 2×2-matrices as values, is locally integrable on (L_-, L_+) and does not vanish on any set of positive measure.

We say that H is in *limit circle* or *limit point case* at L_+ if for one (and hence for all) $\alpha \in (L_-, L_+)$ we have

$$\int_\alpha^{L_+} \operatorname{tr} H(t)\, dt < \infty \quad \text{or} \quad \int_\alpha^{L_+} \operatorname{tr} H(t)\, dt = \infty, \quad \text{respectively.}$$

Similarly, we distinguish limit circle/point case at the endpoint L_-, depending whether $\int_{L_-}^{\alpha} \operatorname{tr} H(t)\, dt$ is finite or infinite.

An interval (α, β) is called H-*indivisible* of type ϕ if

$$H(t) = h(t) \xi_\phi \xi_\phi^T, \quad t \in (\alpha, \beta)\,,$$

where $\xi_\phi := (\cos\phi, \sin\phi)^T$ and $h(t)$ is some scalar function that is positive almost everywhere.

We recall the definition of the space $L^2(H)$ and the maximal relation $T_{\max}(H)$; for details see, e.g. [KW2, §2]. Let H be a Hamiltonian defined on (L_-, L_+). Then $L^2(H)$ is the space of measurable functions f defined on (L_-, L_+) with values in $\mathbb{C}^2$ which satisfy

(*i*) $\displaystyle\int_{L_-}^{L_+} f^* H f < \infty$,

(*ii*) $\xi_\phi^T f$ is constant a.e. on every indivisible interval of type ϕ,

factorized with respect to the equivalence relation $=_H$ where

$$f =_H g \quad \Longleftrightarrow \quad H(f-g) = 0 \quad \text{a.e.}$$

and endowed with the inner product

$$(f, g) := \int_{L_-}^{L_+} g^* H f.$$

In the space $L^2(H)$ the *maximal relation* $T_{\max}(H)$ is defined as

$$T_{\max}(H) := \Big\{ (f;g) \in \big(L^2(H)\big)^2 : \exists \text{ representatives } \hat{f},\ \hat{g} \text{ of } f,\ g \text{ such that } \\ \hat{f} \text{ is locally absolutely continuous and } \\ \hat{f}' = JH\hat{g} \quad \text{a.e. on } (L_-, L_+) \Big\}.$$

The *minimal relation* is the adjoint of the maximal: $T_{\min}(H) := (T_{\max}(H))^*$.

Next we recall the notion of boundary triples, which is quite useful and has recently been studied by many people; see, e.g. [B], [D], [DHMS1], [DHMS2]. The definition we use is taken from [KW2]. A *boundary triple* is a triple $(\mathcal{P}, T, \Gamma)$ where $(\mathcal{P}, [\cdot,\cdot])$ is a Pontryagin space and $T \subseteq \mathcal{P} \times \mathcal{P}$ and $\Gamma \subseteq T \times (\mathbb{C}^2 \times \mathbb{C}^2)$ are closed linear relations with $\operatorname{dom}\Gamma = T$ that satisfy

(*i*)
$$[g, h] - [f, k] = y_1^* J x_1 - y_2^* J x_2 \tag{1.2}$$
for all $\big((f;g);(x_1;x_2)\big)$, $\big((h;k);(y_1,y_2)\big) \in \Gamma$;

(*ii*) $\ker\Gamma = T^*$.

Moreover it is assumed that $\mathcal{P}$ carries a conjugate linear, anti-isometric involution $\bar{\cdot} : \mathcal{P} \to \mathcal{P}$ such that T and Γ are compatible with this conjugation. For details see Definition IV.2.7.

An *isomorphism* from a boundary triple $(\mathcal{P}, T, \Gamma)$ to a boundary triple $(\tilde{\mathcal{P}}, \tilde{T}, \tilde{\Gamma})$ is a pair (ϖ, ϕ) where ϖ and ϕ are isometric isomorphisms from $\mathcal{P}$ onto $\tilde{\mathcal{P}}$, and in $\big(\mathbb{C}^2 \times \mathbb{C}^2, \big(\left(\begin{smallmatrix} J & 0 \\ 0 & -J \end{smallmatrix}\right)\cdot, \cdot\big)\big)$, respectively, where ϖ is compatible with conjugation and the relations

$$(\varpi \times \varpi)(T) = \tilde{T}, \qquad \tilde{\Gamma} \circ (\varpi \times \varpi)|_T = \phi \circ \Gamma$$

are valid; see Definition IV.2.12.

The *pasting*
$$(\mathcal{P}, T, \Gamma) = (\mathcal{P}_1, T_1, \Gamma_1) \uplus (\mathcal{P}_2, T_2, \Gamma_2)$$
of two boundary triples $(\mathcal{P}_1, T_1, \Gamma_1)$ and $(\mathcal{P}_2, T_2, \Gamma_2)$ is a boundary triple where, roughly speaking, elements in T are combinations of elements of T_1 and T_2 where the right boundary value of the first must coincide with the left boundary value of the second element. For details, see Definition IV.6.1.

With a Hamiltonian H one can associate a boundary triple $\mathfrak{B}(H) = (L^2(H), T_{\max}(H), \Gamma(H))$ where $\Gamma(H)$ is the boundary relation defined as follows: $\Gamma(H)$ consists of all pairs $\big((f;g);(a;b)\big) \in T \times (\mathbb{C}^2 \times \mathbb{C}^2)$ for which there exists a locally absolutely continuous representative $\hat{f}$ of f such that

$$a = \begin{cases} \lim\limits_{x \to L_-} \hat{f}(x), & H \text{ in limit circle case at } L_-, \\ 0, & H \text{ in limit point case at } L_-, \end{cases}$$

$$b = \begin{cases} \lim\limits_{x \to L_+} \hat{f}(x), & H \text{ in limit circle case at } L_+, \\ 0, & H \text{ in limit point case at } L_+. \end{cases}$$

With a Hamiltonian H, which is in limit circle case at L_-, in [KW2, Definition 3.1] a number $\Delta(H)\in\mathbb{N}\cup\{0,\infty\}$ was associated. This number measures in some sense the growth of H towards L_+. For example, $\Delta(H)=0$ means that $\int_{L_-}^{L_+}\operatorname{tr}H(t)\,dt<\infty$; or if $\int_{L_-}^{L_+}\operatorname{tr}H(t)\,dt=\infty$ and for some $L_1<L_+$ the interval (L_1,L_+) is H-indivisible, then $\Delta(H)=1$; see [KW2, §3] for details.

Assume that H is in limit circle case at L_- and in limit point case at L_+. Then we say that H satisfies the condition (HS) if the resolvents of one (and hence of all) self-adjoint extensions of the minimal relation $T_{\min}(H)$ associated with H are Hilbert–Schmidt operators. In this case, the growth of H towards L_+ is bounded in one (and extremal in another) direction in the sense that for a unique angle $\phi(H)\in[0,\pi)$ we have

$$\int_{L_-}^{L_+}\xi_{\phi(H)}^T H(t)\xi_{\phi(H)}\,dt<\infty\,,$$

cf. [KW3, Theorem 2.4]. The direction of 'extremal growth' is then $\xi_{\phi(H)+\frac{\pi}{2}}$.

If H is a Hamiltonian on (L_-,L_+) and $\alpha\in(L_-,L_+)$, then $H_+(t):=H|_{(\alpha,L_+)}(t)$ and $H_-(t):=H|_{(L_-,\alpha)}(-t)$ are Hamiltonians defined on (α,L_+) or $(-\alpha,-L_-)$, respectively. Both, H_+ and H_-, are in limit circle case at their left endpoint. At their right endpoint limit circle or limit point case prevails depending on the behaviour of H at L_+ or L_-, respectively.

Numbers $\Delta_\pm(H)$ are defined as $\Delta_\pm(H):=\Delta(H_\pm)$. Moreover, we say that H satisfies (HS$_+$) or (HS$_-$) if H_+ or H_-, respectively, satisfies (HS). Numbers $\phi_\pm(H)$ are defined correspondingly. Let us note that each of these notions is independent of the choice of α in the definition of $H_\pm$, cf. Lemma IV.3.12.

1.1 Definition. A *general Hamiltonian* $\mathfrak{h}$ is a collection of data of the following kind:

(*i*) $n\in\mathbb{N}\cup\{0\}$, $\sigma_0,\dots,\sigma_{n+1}\in\mathbb{R}\cup\{\pm\infty\}$ with $\sigma_0<\sigma_1<\dots<\sigma_{n+1}$,

(*ii*) Hamiltonians H_i, $i=0,\dots,n$, defined on the intervals (σ_i,σ_{i+1}), respectively,

(*iii*) numbers $\ddot{o}_1,\dots,\ddot{o}_n\in\mathbb{N}\cup\{0\}$ and $b_{i,1},\dots,b_{i,\ddot{o}_i+1}\in\mathbb{R}$, $i=1,\dots,n$, with $b_{i,1}\neq 0$ in the case $\ddot{o}_i\geq 1$,

(*iv*) numbers $d_{i,0},\dots,d_{i,2\Delta_i-1}\in\mathbb{R}$, $i=1,\dots,n$, where $\Delta_i:=\max\{\Delta_+(H_{i-1}),\Delta_-(H_i)\}$,

(*v*) a finite subset E of $\{\sigma_0,\sigma_{n+1}\}\cup\bigcup_{i=0}^n(\sigma_i,\sigma_{i+1})$,

which is assumed to be subject to the following conditions:

(H1) H_0 is in limit circle case at σ_0 and, if $n\geq 1$, in limit point case at σ_1. H_i is in limit point case at both endpoints σ_i and σ_{i+1}, $i=1,\dots,n-1$. If $n\geq 1$, then H_n is in limit point case at σ_n.

(H2) For $i=1,\dots,n-1$ the interval (σ_i,σ_{i+1}) is not H_i-indivisible. If H_n is in limit point case at σ_{n+1}, then also (σ_n,σ_{n+1}) is not H_n-indivisible.

(H3) We have $\Delta_i < \infty$, $i = 1, \dots, n$. Moreover, H_0 satisfies (HS$_+$), H_i satisfies (HS$_-$) and (HS$_+$) for $i = 1, \dots, n-1$, and H_n satisfies (HS$_-$).

(H4) We have $\phi_+(H_{i-1}) = \phi_-(H_i)$, $i = 1, \dots, n$.

(H5) Let $i \in \{1, \dots, n\}$. If for some $\varepsilon > 0$, the interval $(\sigma_i - \varepsilon, \sigma_i)$ is H_{i-1}-indivisible and the interval $(\sigma_i, \sigma_i + \varepsilon)$ is H_i-indivisible, then $d_{i,1} = 0$. If additionally $b_{i,1} = 0$, then also $d_{i,0} < 0$.

(E1) $\sigma_0, \sigma_{n+1} \in E$, and $E \cap (\sigma_i, \sigma_{i+1}) \neq \varnothing$ for $i = 1, \dots, n-1$. If H_n is in limit point case at σ_{n+1}, then also $E \cap (\sigma_n, \sigma_{n+1}) \neq \varnothing$. Let $i \in \{0, \dots, n\}$; if (α, σ_{i+1}) or (σ_i, α) is a maximal H_i-indivisible interval, then $\alpha \in E$.

(E2) No point of E is an inner point of an indivisible interval.

The number

$$\operatorname{ind}_- \mathfrak{h} := \sum_{i=1}^{n} \left(\Delta_i + \left[\frac{\ddot{o}_i}{2} \right] \right) + \left| \left\{ 1 \leq i \leq n : \ddot{o}_i \text{ odd}, b_{i,1} > 0 \right\} \right|$$

is called the *negative index* of the general Hamiltonian $\mathfrak{h}$. Moreover, $\mathfrak{h}$ is called *definite* if $\operatorname{ind}_- \mathfrak{h} = 0$, and *indefinite* otherwise. We say that $\mathfrak{h}$ is in limit point case or limit circle case if H_n has the respective property at σ_{n+1}. //

In order to shorten notation we shall write a general Hamiltonian $\mathfrak{h}$ which is given by the data n, $\sigma_0, \dots, \sigma_{n+1}$, $H_0, \dots, H_n$, $\ddot{o}_1, \dots, \ddot{o}_n$, $b_{i,j}$, $d_{i,j}$, E, as a triple

$$\mathfrak{h} = (H, \mathfrak{b}, \mathfrak{d}),$$

where H represents the Hamiltonians H_i, including their number n and their domains of definition (σ_i, σ_{i+1}), $\mathfrak{b}$ represents the numbers $\ddot{o}_i$ and $b_{i,j}$, and $\mathfrak{d}$ represents the numbers $d_{i,j}$ and the subset E. Apparently, we may also identify H with the function defined on $\bigcup_{i=0}^{n} (\sigma_i, \sigma_{i+1})$ by

$$H|_{(\sigma_i, \sigma_{i+1})} = H_i, \quad i = 0, \dots, n\,. \tag{1.3}$$

We will speak of H as the *Hamiltonian function* of $\mathfrak{h}$. The boundary triple associated with $\mathfrak{h}$ by means of Definition IV.8.5 will be denoted as $\mathfrak{B}(\mathfrak{h}) = (\mathfrak{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h}))$.

1.2 Remark. Intuitively, the notion of a general Hamiltonian can be understood as follows: we deal with the differential equation $f' = zJHf$ given on an interval (σ_0, σ_{n+1}) which involves some kind of singularities located at the points σ_i, $i = 1, \dots, n$. Condition (H1) says that the differential equation is regular at σ_0, so that the initial value problem at σ_0 is well posed, but that $\sigma_1, \dots, \sigma_n$ actually are singularities. Moreover, and this is the condition (H2), two adjacent singularities σ_i and σ_{i+1} must be separated by more than just a single indivisible interval. The meaning of (H3) is that the growth of H_i towards a singularity is not too fast. Moreover, (H4) is an interface condition at σ_i.

The numbers $\ddot{o}_i \in \mathbb{N} \cup \{0\}$ and $b_{i,1}, \dots, b_{i,\ddot{o}_i+1}$ model the part of the singularity σ_i which is concentrated at σ_i, whereas the numbers $d_{i,0}, \dots, d_{i,2\Delta_i-1}$ model the part of this singularity which is in interaction with the local behaviour around σ_i. The elements of E in the vicinity of σ_i determine quantitatively what 'local' here means. The freedom of this interaction is, by the first part of (H5), restricted if to both sides of σ_i indivisible intervals adjoin. The possibility that on both sides of σ_i indivisible intervals adjoin and at the same time $b_{i,1} = 0$, can occur by the second part of (H5) only in the case of 'indivisible intervals of negative length', the simplest possible kind of a singularity. //

For the reasons already mentioned we consider singular general Hamiltonians having only one singularity. More precisely, whenever the notation $\mathfrak{h}$ appears or we speak of a general Hamiltonian, we will understand that $\mathfrak{h}$ is subject to the following conditions.

1.3. Form of $\mathfrak{h}$.
Let $\mathfrak{h}$ be a singular general Hamiltonian such that

(*i*) $\mathfrak{h}$ has only one singularity $\sigma_1 =: \sigma$.

Due to this assumption, $\mathfrak{h}$ is given by data H, $\ddot{o}, b_j, d_j$, and E. Besides (*i*), we assume that the following conditions are satisfied.

(*ii*) The singularity σ may be the endpoint of an indivisible interval adjoining from the right or adjoining from the left, but not both.

(*iii*) The Hamiltonian function H of $\mathfrak{h}$ is defined on a set $I = (s_-, \sigma) \cup (\sigma, s_+)$ where $-\infty < s_- < \sigma < s_+ \leq +\infty$.

(*iv*) For one (and hence for all) $x_0 \in (\sigma, s_+)$, the function $\binom{1}{0}$ belongs to the space $L^2(H|_{(s_-,\sigma)\cup(\sigma,x_0)})$.

(*v*) We have $b_{\ddot{o}+1} = 0$.

//

Some remarks concerning these conditions are in order. Let us explain that actually (*ii*) is only an insignificant restriction and (*iii*)–(*v*) are no loss of generality at all.

1.4 Remark. Let $\mathfrak{h}$ be a singular general Hamiltonian with only one singularity.

Condition (*ii*)*:* If indivisible intervals adjoin to both sides of σ, then the model $\mathfrak{B}(\mathfrak{h})$ is very simple. Actually, we only need to use the cases (B) or (C) of Definition IV.4.1. The contribution of the singularity to the model is finite dimensional and explicitly described by Definition IV.4.3 and Definition IV.4.5. Hence, requiring (*ii*) just rules out some more or less trivial cases. We require (*ii*) in order to avoid repeated distinction of cases.

Condition (*iii*)*:* The Hamiltonian function of $\mathfrak{h}$ is defined on some set of the form $(s_-, \sigma) \cup (\sigma, s_+)$ where $-\infty \leq s_- < \sigma < s_+ \leq +\infty$. By an obvious reparameterization we can achieve that $s_- \neq -\infty$. Hence, assuming (*iii*) is no loss of generality.

Condition (*iv*)*:* We know that the space $L^2(H|_{(s_-,\sigma)\cup(\sigma,x_0)})$, $x_0 \in (\sigma, s_+)$, contains the constant function $\xi_{\phi(H)}$ where $\phi(H) := \phi_+(H|_{s_-,\sigma})$. Using rotation isomorphisms, cf. Remark IV.2.28, it is no loss of generality to assume that $\phi(H) = 0$, i.e. that (*iv*) holds. The procedure of rotation is actually already implemented in the very definition of $\mathfrak{B}(\mathfrak{h})$, cf. Definition IV.8.5.

Moreover, if an indivisible interval adjoins at σ, then its type equals $\frac{\pi}{2}$ by condition (iv).

Condition (v)*:* By Proposition IV.8.13 it is no loss of generality to assume that $b_{\ddot{o}+1} = 0$. //

2 Function space realization

Elements of various model spaces under consideration will be tuples whose entries are either (equivalence classes of) functions, or elements of $\mathbb{C}^\Delta$ or $\mathbb{C}^{\ddot{o}}$. In order to shorten notation, we agree on the following.

2.1. Notational conventions.

(1) Elements of $\mathbb{C}^\Delta$ or $\mathbb{C}^{\ddot{o}}$ will be denoted by upright Greek letters, like, e.g. $\upalpha, \upbeta, \upxi$, etc. There coordinates will be denoted by the corresponding normal font Greek letter. Indices range between 0 and $\Delta - 1$ for elements of $\mathbb{C}^\Delta$ and between 1 and $\ddot{o}$ for elements of $\mathbb{C}^{\ddot{o}}$. Whether a vector belongs to $\mathbb{C}^\Delta$ or $\mathbb{C}^{\ddot{o}}$ will always be clear from the context and thus not be indicated explicitly (often $\upalpha, \upbeta \in \mathbb{C}^{\ddot{o}}$ and vectors denoted by other Greek letters are in $\mathbb{C}^\Delta$). So, for example, we would have

$$\upxi = (\xi_j)_{j=0}^{\Delta-1},\ \uplambda = (\lambda_j)_{j=0}^{\Delta-1} \ \text{ or } \ \upalpha = (\alpha_j)_{j=1}^{\ddot{o}}\,.$$

Complex conjugation will be denoted accordingly, e.g. we will use $\overline{\upxi} := (\overline{\xi_j})_{j=0}^{\Delta-1}$.

(2) The k-th canonical basis vector of either $\mathbb{C}^\Delta$ or $\mathbb{C}^{\ddot{o}}$ will be denoted by $\upvarepsilon_k$. That is, we write

$$\upvarepsilon_k := (\delta_{kj})_{j=0}^{\Delta-1} \qquad \Big(\text{or } \upvarepsilon_k := (\delta_{kj})_{j=1}^{\ddot{o}}, \text{ respectively}\Big),$$

where δ_{kj} denotes the Kronecker delta symbol

$$\delta_{kj} := \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases}$$

(3) We will deal with elements $(\upxi, \upalpha)$ of $\mathbb{C}^\Delta \times \mathbb{C}^{\ddot{o}}$. The number Δ is always at least 1, and hence the component $\upxi$ is always present. The number $\ddot{o}$, however, may be equal to zero in which case the second component $\upalpha$ is not present at all. Still, in order to unify notation, we will always write $(\upxi, \upalpha) \in \mathbb{C}^\Delta \times \mathbb{C}^{\ddot{o}}$ and understand that $\upalpha$ is empty if $\ddot{o} = 0$.

(4) If F is a function defined on some subset D of the real line $\mathbb{R}$, and $x_0 \in \mathbb{R}$, then we set

$$F_{\Lsh x_0} := F|_{D\cap(-\infty,x_0]}, \quad F_{x_0 \Rsh} := F|_{D\cap[x_0,+\infty)}\,.$$

In the same spirit, we let $\chi_{\Lsh x_0}$ and $\chi_{x_0\Rsh}$ denote the indicator functions

$$\chi_{\Lsh x_0} := \chi_{(-\infty,x_0]}, \quad \chi_{x_0\Rsh} := \chi_{[x_0,+\infty)}\,.$$

If we are given two functions f_1, f_2, then we understand by $f := f_1\chi_{\Lsh x_0} + f_2\chi_{x_0\Rsh}$ the function

$$f(x) := \begin{cases} f_1(x), & x < x_0,\ x \in \operatorname{dom} f_1, \\ f_2(x), & x > x_0,\ x \in \operatorname{dom} f_2. \end{cases}$$

no matter what the original domains of definition of f_1 and f_2 are.

(5) Let H be a Hamiltonian function (more precisely, a collection of two Hamiltonian functions in the sense of (1.3)) defined on a set of the form $I = (s_-, \sigma) \cup (\sigma, s_+)$. Then we denote by I_+ the set of all points $x_0 \in (\sigma, s_+)$ which are not inner points of an H-indivisible interval and, correspondingly, by I_- the set of all points $x_0 \in (s_-, \sigma)$ which are not inner points of an H-indivisible interval. //

a. Identification of $\mathfrak{B}(\mathfrak{h})$ as pasting of two components.

Building blocks for a general Hamiltonian $\mathfrak{h}$ are positive definite and elementary indefinite Hamiltonians, cf. Definition IV.4.1, and building blocks for the model $\mathfrak{B}(\mathfrak{h})$ are the boundary triples associated with such Hamiltonians, cf. [KW2, §2.1], Definition IV.4.10, IV.4.11, IV.4.12. In [KW2, §7] the following fact was shown.

2.2. Splitting of elementary indefinite Hamiltonians.
Let $\mathfrak{h}_{s_-}^{s_+} = (H; \ddot{o}, b_j; d_{s_-,j}^{s_+})$ be an elementary indefinite Hamiltonian of kind (A) defined on $(s_-, \sigma) \cup (\sigma, s_+)$, and let $s_0 \in I_-$ be given. Then there exist numbers $d_{s_0,j}^{s_+}$ such that the boundary triples $\mathfrak{B}(\mathfrak{h}_{s_-}^{s_+})$ and $\mathfrak{B}(H_{\upharpoonleft s_0}) \uplus \mathfrak{B}(\mathfrak{h}_{s_0}^{s_+})$ are isomorphic, when $\mathfrak{h}_{s_0}^{s_+}$ is the elementary indefinite Hamiltonian of kind (A) defined on $(s_0, \sigma) \cup (\sigma, s_+)$ given by the data

$$\mathfrak{h}_{s_0}^{s_+} := (H_{s_0 \upharpoonright}; \ddot{o}, b_j; d_{s_0,j}^{s_+}) .$$

Here $\mathfrak{B}(H_{\upharpoonleft s_0})$ denotes the boundary triple $(L^2(H_{\upharpoonleft s_0}), T_{\max}(H_{\upharpoonleft s_0}), \Gamma(H_{\upharpoonleft s_0}))$ associated naturally with $H_{\upharpoonleft s_0}$, and $\mathfrak{B}(\mathfrak{h}_{s_-}^{s_+})$ and $\mathfrak{B}(\mathfrak{h}_{s_0}^{s_+})$ denote the boundary triples associated with the elementary indefinite Hamiltonians $\mathfrak{h}_{s_-}^{s_+}$ and $\mathfrak{h}_{s_0}^{s_+}$, respectively.

The isomorphism between these boundary triples is of the form $(\gamma_{s_0,s_-}; \mathrm{id}_{\mathbb{C}^4})$ where

$$\gamma_{s_0,s_-} : \mathfrak{P}(\mathfrak{h}_{s_-}^{s_+}) \to L^2(H|_{(s_-,s_0)}) \, [\dotplus] \, \mathfrak{P}(\mathfrak{h}_{s_0}^{s_+}) .$$

The map γ_{s_0,s_-} is compatible with conjugation and satisfies

$$\begin{array}{ccc}
L^2(H|_{(s_-,s_0)}) \, [\dotplus] \, \mathfrak{P}(\mathfrak{h}_{s_0}^{s_+}) & \xleftarrow[\gamma_{s_0,s_-}]{} & \mathfrak{P}(\mathfrak{h}_{s_-}^{s_+}) \\
\Big\downarrow {\scriptstyle \mathrm{id}\,[\dotplus]\,\psi(\mathfrak{h}_{s_0}^{s_+})} & & \Big\downarrow {\scriptstyle \psi(\mathfrak{h}_{s_-}^{s_+})} \\
\mathcal{M}((s_-,s_0))/_{=_H} \times \mathcal{M}((s_0,s_+)\backslash\{\sigma\})/_{=_H} & = & \mathcal{M}((s_-,s_+)\backslash\{\sigma\})/_{=_H}
\end{array}$$

where $\psi(\mathfrak{h}_{s_0}^{s_+})$ and $\psi(\mathfrak{h}_{s_-}^{s_+})$ are the respective maps defined on [KW2, p.760] and $\mathcal{M}(\hat{I})$ is the set of measurable $\mathbb{C}^2$-valued functions on a set $\hat{I} \subseteq \mathbb{R}$; see [KW2, §2].

The analogous statement is true when $s_0 \in I_+$. Then we find an elementary indefinite Hamiltonian $\mathfrak{h}_{s_-}^{s_0} = (H_{\upharpoonleft s_0}; \ddot{o}, b_j; d_{s_-,j}^{s_0})$ defined on $(s_-, \sigma) \cup (\sigma, s_0)$ together with an isomorphism $\gamma_{s_0,s_+} : \mathfrak{P}(\mathfrak{h}_{s_-}^{s_+}) \to \mathfrak{P}(\mathfrak{h}_{s_-}^{s_0}) \, [\dotplus] \, L^2(H|_{(s_0,s_+)})$. //

The statement made in §2.2 can also be read the other way.

2.3. Pasting.
Let $\mathfrak{h}_{s_0}^{s_+} = (H_1; \ddot{o}, b_j; d_{s_0,j}^{s_+})$ be an elementary indefinite Hamiltonian of kind (A) defined on $(s_0, \sigma) \cup (\sigma, s_+)$, and let H_2 be a Hamiltonian function defined on

(s_-, s_0). Assume that, if H_2 ends with an indivisible interval and H_1 starts with an indivisible interval, these indivisible intervals are not of the same type. Set

$$H(t) := \begin{cases} H_2(t), & t \in (s_-, s_0), \\ H_1(t), & t \in (s_0, \sigma) \cup (\sigma, s_+). \end{cases}$$

Then there exist numbers $d_{s_-,j}^{s_+}$ such that the boundary triples $\mathfrak{B}(\mathfrak{h}_{s_-}^{s_+})$ and $\mathfrak{B}(H_{\urcorner s_0}) \uplus \mathfrak{B}(\mathfrak{h}_{s_0}^{s_+})$ are isomorphic, where $\mathfrak{h}_{s_-}^{s_+}$ is the elementary indefinite Hamiltonian of kind (A) defined on $(s_-, \sigma) \cup (\sigma, s_+)$ by the data

$$\mathfrak{h}_{s_-}^{s_+} := (H; \ddot{o}, b_j; d_{s_-,j}^{s_+}) .$$

The isomorphism

$$\kappa_{s_0,s_-} : L^2(H|_{(s_-,s_0)}) \, [\dot{+}] \, \mathfrak{P}(\mathfrak{h}_{s_0}^{s_+}) \to \mathfrak{P}(\mathfrak{h}_{s_-}^{s_+})$$

between these boundary triples has the same properties as the one in §2.2.

The analogous statement is true when $s_0 \in I_+$. In this case we obtain $\mathfrak{h}_{s_-}^{s_+}$ and an isomorphism

$$\kappa_{s_0,s_+} : \mathfrak{P}(\mathfrak{h}_{s_-}^{s_0}) \, [\dot{+}] \, L^2(H|_{(s_0,s_+)}) \to \mathfrak{P}(\mathfrak{h}_{s_-}^{s_+})$$

out of $\mathfrak{h}_{s_-}^{s_0}$ plus a Hamiltonian function H_2 defined on (s_0, s_+). //

It follows from Proposition IV.5.18 and our overall assumption that $b_{\ddot{o}+1} = 0$ that the numbers $d_{s_0,j}^{s_+}, d_{s_-,j}^{s_0}$ obtained in §2.2 and $d_{s_-,j}^{s_+}$ obtained in §2.3, respectively, are uniquely determined.

2.4 Remark. The explained splitting and pasting procedures are converse to each other in the following sense.

Pasting after splitting: Let $\mathfrak{h}_{s_-}^{s_+}$ be an elementary indefinite Hamiltonian defined on $(s_-, \sigma) \cup (\sigma, s_+)$, let $s_0 \in I_-$, and let $\mathfrak{h}_{s_0}^{s_+}$ be the elementary indefinite Hamiltonian defined on $(s_0, \sigma) \cup (\sigma, s_+)$ which is obtained by splitting $\mathfrak{h}_{s_-}^{s_+}$ at s_0. Since $s_0 \in I_-$, the hypothesis required in §2.3 in order to paste $\mathfrak{h}_{s_0}^{s_+}$ with $H|_{(s_-,s_0)}$ is satisfied. Let $\tilde{\mathfrak{h}}_{s_-}^{s_+}$ be the elementary indefinite Hamiltonian defined on $(s_-, \sigma) \cup (\sigma, s_+)$ obtained by means of §2.3. Then we have $\tilde{\mathfrak{h}}_{s_-}^{s_+} = \mathfrak{h}_{s_-}^{s_+}$.

Splitting after pasting: Let $\mathfrak{h}_{s_0}^{s_+}$ be an elementary indefinite Hamiltonian defined on $(s_0, \sigma) \cup (\sigma, s_+)$, let H_2 be a Hamiltonian function defined on (s_-, s_0), and assume that the hypothesis of §2.3 is satisfied. Let $\mathfrak{h}_{s_-}^{s_+}$ be the elementary indefinite Hamiltonian defined on $(s_-, \sigma) \cup (\sigma, s_+)$ obtained by pasting $\mathfrak{h}_{s_0}^{s_+}$ with H_2. Then the number s_0 belongs to I_- (for the Hamiltonian $\mathfrak{h}_{s_-}^{s_+}$). Let $\tilde{\mathfrak{h}}_{s_0}^{s_+}$ be the elementary indefinite Hamiltonian defined on $(s_0, \sigma) \cup (\sigma, s_+)$ which is obtained by splitting $\mathfrak{h}_{s_-}^{s_+}$ at s_0. Then we have $\tilde{\mathfrak{h}}_{s_0}^{s_+} = \mathfrak{h}_{s_0}^{s_+}$.

In both situations we have $\kappa_{s_0,s_-} = \gamma_{s_0,s_-}^{-1}$. //

The similar statements of course hold true when the point s_0 is located to the right of σ. We revisit the splitting/pasting procedure in more detail in the appendix, where we give explicit formulae for the numbers $d_{s_0,j}^{s_+}, d_{s_-,j}^{s_0}, d_{s_-,j}^{s_+}$ and for the action of the isomorphisms κ_{s_0,s_-} and κ_{s_0,s_+}.

By repeated application of the splitting/pasting procedure, we can reduce the set of splitting points of a given general Hamiltonian.

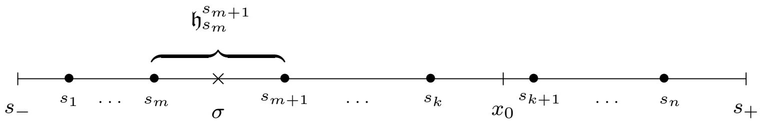
2.5 Lemma. *Let $\mathfrak{h} = (H; \ddot{o}, b_j; d_j; E)$, $E = \{s_-, s_1, \dots, s_n, s_+\}$, be a general Hamiltonian (of the form 1.3). Moreover, let $x_0 \in I_+$ be given. Then there exist unique numbers $d_j^{x_0}$ such that the boundary triple $\mathfrak{B}(\mathfrak{h})$ is isomorphic to*

$$\mathfrak{B}_{x_0}(\mathfrak{h}) = (\mathfrak{P}_{x_0}(\mathfrak{h}), T_{x_0}(\mathfrak{h}), \Gamma_{x_0}(\mathfrak{h})) := \mathfrak{B}(\mathfrak{h}_{x_0}) \uplus \mathfrak{B}(H_{x_0\ulcorner}),$$

where $\mathfrak{h}_{x_0}$ is the elementary indefinite Hamiltonian of kind (A) given by the data

$$\mathfrak{h}_{x_0} := (H_{\urcorner x_0}; \ddot{o}, b_j; d_j^{x_0}). \tag{2.1}$$

Proof. We are in the situation



where $\mathfrak{h}_{s_m}^{s_{m+1}}$ is the elementary indefinite Hamiltonian

$$\mathfrak{h}_{s_m}^{s_{m+1}} = (H|_{(s_m, s_{m+1})}; \ddot{o}, b_j; d_j).$$

First we apply successively the isomorphisms $\kappa_{s_m, s_{m-1}}, \dots, \kappa_{s_1, s_-}$ starting from $\mathfrak{h}_{s_m}^{s_{m+1}}$, and paste in the corresponding pieces with the Hamiltonian functions $H|_{(s_m, s_{m-1})}, \dots, H|_{(s_1, s_-)}$. In this way, we obtain an elementary indefinite Hamiltonian $\mathfrak{h}_{s_-}^{s_{m+1}}$ of the form $(H|_{\urcorner s_{m+1}}; \ddot{o}, b_j; d_{s_-,j}^{s_{m+1}})$ such that

$$\mathfrak{B}(\mathfrak{h}) \cong \mathfrak{B}(\mathfrak{h}_{s_-}^{s_{m+1}}) \uplus \mathfrak{B}(H_{s_{m+1}\ulcorner}).$$

If $x_0 > s_{m+1}$, we apply successively the isomorphism $\kappa_{s_{m+1}, s_{m+2}}, \dots, \kappa_{s_k, x_0}$ where we paste in the corresponding pieces with the Hamiltonian functions $H|_{(s_{m+1}, s_{m+2})}, \dots, H|_{(s_k, x_0)}$. By this procedure, we obtain the desired elementary indefinite Hamiltonian $\mathfrak{h}_{x_0} = (H|_{\urcorner x_0}; \ddot{o}, b_j; d_j^{x_0})$. If $x_0 \in (\sigma, s_{m+1})$, we use the splitting isomorphism $\gamma_{x_0, s_{m+1}}$ and again obtain $\mathfrak{h}_{x_0}$ as desired. ❑

Note the difference in the notation between $\mathfrak{B}_{x_0}(\mathfrak{h})$ and $\mathfrak{B}(\mathfrak{h}_{x_0})$. The latter is a boundary triple connected with a Hamiltonian on (s_-, x_0), the former is a boundary triple connected with a Hamiltonian on (s_-, s_+) with splitting point x_0.

b. The function space $L^2_\Delta(H)$.

If $\mathfrak{h}$ is a general Hamiltonian, the relation $T(\mathfrak{h})$ can be mapped to a relation acting in a certain space of functions, which is actually fully determined by the Hamiltonian function H of $\mathfrak{h}$, cf. (IV.4.18), Proposition IV.4.17. In this subsection we treat this space more systematically.

2.6. Form of H.
We deal with Hamiltonian functions H which are subject to the following conditions.

(i) The Hamiltonian function H is defined on $I = (s_-, \sigma) \cup (\sigma, s_+)$ (in the sense of (1.3). Moreover, $H_{\urcorner\sigma}$ is in limit circle case at s_- and in limit point case at σ, and $H_{\sigma\ulcorner}$ is in limit point case at both endpoints.

(i') The Hamiltonian $H_{\urcorner\sigma}$ satisfies the condition (HS$_+$) and $\Delta_+(H_{\urcorner\sigma}) < \infty$.

(i'') The Hamiltonian $H_{\sigma\upharpoonright}$ satisfies the condition (HS$_-$) and $\Delta_-(H_{\sigma\upharpoonright}) < \infty$.

(ii) The point σ is not both left and right endpoint of an indivisible interval.

(iii) We have $s_- > -\infty$.

(iv) We have $\binom{1}{0} \in L^2(H)$.

If H is of this form, set $\Delta := \max\{\Delta_+(H_{\urcorner\sigma}), \Delta_-(H|_{\sigma\upharpoonright})\}$. //

The significance of Hamiltonian functions of this form in the present context is that H is the Hamiltonian function of a general Hamiltonian $\mathfrak{h}$ of the form 1.3 if and only if H is of the form 2.6.

If $x_0 \in (\sigma, s_+)$, we denote by $\mathfrak{w}_j^{x_0}$, $j \geq 0$, the unique (see Lemma IV.3.10) absolutely continuous 2-vector functions on $[s_-, \sigma) \cup (\sigma, s_+)$ with

$$\begin{aligned}
&\mathfrak{w}_0^{x_0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
&(\mathfrak{w}_k^{x_0})' = JH\mathfrak{w}_{k-1}^{x_0}, \qquad k \geq 1, \\
&\mathfrak{w}_k^{x_0}(s_-), \mathfrak{w}_k^{x_0}(x_0) \in \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}, \qquad k \in \mathbb{N}_0, \\
&\mathfrak{w}_k^{x_0} \in L^2(H_{\urcorner x_0}), \qquad k \geq \Delta.
\end{aligned}$$

For notational convenience, we set $\mathfrak{w}_{-1}^{x_0} := 0$, and let $\omega_k^{x_0}$ denote the second component of the vector $\mathfrak{w}_k^{x_0}(x_0)$, i.e.

$$\mathfrak{w}_k^{x_0}(x_0) = \begin{pmatrix} 0 \\ \omega_k^{x_0} \end{pmatrix}.$$

By Lemma IV.3.6, the functions $\mathfrak{w}_0^{x_0}, \ldots, \mathfrak{w}_{\Delta-1}^{x_0}$ are linearly independent modulo $L^2(H)$. Let us remark that $(\mathfrak{w}_j^{x_0})_{\urcorner\sigma}$ does not depend on x_0, whereas $(\mathfrak{w}_j^{x_0})_{\sigma\upharpoonright}$ does.

2.7 Definition. Let H be of the form 2.6, and choose $x_0 \in (\sigma, s_+)$. Then we set

$$L^2_\Delta(H) := L^2(H) \dot{+} \operatorname{span}\left\{\mathfrak{w}_k^{x_0}\chi_{\urcorner x_0} \colon k = 0, \ldots, \Delta - 1\right\},$$

$$T_{\Delta,\max}(H) := \big\{(f;g) \in L^2_\Delta(H) \times L^2_\Delta(H) : \exists \hat{f} \text{ absolutely continuous representative of } f \text{ s.t. } \hat{f}' = JHg\big\}.$$

//

Note that, by Lemma IV.3.12, the space $L^2_\Delta(H)$ does not depend on the choice of x_0.

2.8 Lemma. *Let H be of the form 2.6, choose $x_0 \in I_+$ and a pair $(f_0; g_0) \in T_{\max}(H_{x_0\upharpoonright})$ with $\Gamma(H_{x_0\upharpoonright})(f_0; g_0) = \binom{0}{1}$. Moreover, let*

$$(u_k; v_k) := (\mathfrak{w}_k^{x_0}\chi_{\urcorner x_0}; \mathfrak{w}_{k-1}^{x_0}\chi_{\urcorner x_0}) + \omega_k^{x_0}(f_0; g_0), \quad k \geq 0\,. \tag{2.2}$$

Then

$$T_{\Delta,\max}(H) = T_{\max}(H) \dot{+} \operatorname{span}\left\{(u_k; v_k) : k = 0, \ldots, \Delta\right\}. \tag{2.3}$$

Proof. Obviously, $(u_k;v_k)\in T_{\Delta,\max}$ and $\{(u_k;v_k):\ k=0,\dots,\Delta\}$ is linearly independent modulo $L^2(H)\times L^2(H)$. It already follows that the inclusion '$\supseteq$' in (2.3) holds and that the sum on the right-hand side is direct.

To show the converse inclusion, let $(f;g)\in T_{\Delta,\max}(H)$. There exist constants $\lambda_0,\dots,\lambda_{\Delta-1}$ and $\mu_0,\dots,\mu_{\Delta-1}$ such that

$$f-\sum_{k=0}^{\Delta-1}\lambda_k\mathfrak{w}_k^{x_0}\chi_{\Lsh x_0},\ g-\sum_{k=0}^{\Delta-1}\mu_k\mathfrak{w}_k^{x_0}\chi_{\Lsh x_0}\in L^2(H).$$

Set

$$(\hat f;\hat g):=(f;g)-\sum_{k=0}^{\Delta-1}\lambda_k(u_k;v_k)-\mu_{\Delta-1}(u_\Delta;v_\Delta). \tag{2.4}$$

Then $(\hat f;\hat g)\in T_{\Delta,\max}(H)$ and $\hat f\in L^2(H)$, $\hat g\chi_{x_0\Rsh}\in L^2(H)$. It is sufficient to show that $\hat g\chi_{\Lsh x_0}\in L^2(H)$. It follows from (2.4) that

$$\hat g\in L^2(H)\dotplus\operatorname{span}\{\mathfrak{w}_0^{x_0}\chi_{\Lsh x_0},\dots,\mathfrak{w}_{\Delta-2}^{x_0}\chi_{\Lsh x_0}\}.$$

Hence there exist scalars $\gamma_0,\dots,\gamma_{\Delta-2}$ such that

$$\tilde g:=\hat g-\sum_{l=0}^{\Delta-2}\gamma_l\mathfrak{w}_l^{x_0}\chi_{\Lsh x_0}\in L^2(H)\,.$$

By the definition of the number Δ, at least one of the sets

$$\big\{\mathfrak{w}_0^{x_0}\chi_{(s_-,\sigma)},\dots,\mathfrak{w}_{\Delta-1}^{x_0}\chi_{(s_-,\sigma)}\big\},\quad \big\{\mathfrak{w}_0^{x_0}\chi_{(\sigma,x_0)},\dots,\mathfrak{w}_{\Delta-1}^{x_0}\chi_{(\sigma,x_0)}\big\}$$

is linearly independent modulo $L^2(H)$. Consider the case when the first one has this property; the other case is treated completely analogously.

Denote by $\mathcal{I}$ the operator

$$(\mathcal{I}h)(x):=\int_{s_-}^x JHh,\quad x\in(s_-,\sigma)\,.$$

Since $(\hat f;\hat g)\in T_{\Delta,\max}$, there exist scalars ϵ_+,ϵ_- such that

$$(\mathcal{I}\hat g)(x)+\binom{\epsilon_+}{\epsilon_-}=\hat f(x),\quad x\in(s_-,\sigma)\,.$$

Moreover, we know from [KW2, §2.b] and the construction preceding Definition IV.3.7, that there exist scalars $\epsilon,\epsilon_0,\dots,\epsilon_{\Delta-2}$ such that

$$\Big[\mathcal{I}\tilde g+\binom{0}{\epsilon}\Big]\chi_{(s_-,\sigma)}\in L^2(H),$$

$$\Big[\mathcal{I}\mathfrak{w}_l^{x_0}+\binom{0}{\epsilon_l}\Big]\chi_{(s_-,\sigma)}=\mathfrak{w}_{l+1}^{x_0}\chi_{(s_-,\sigma)},\quad l=0,\dots,\Delta-2.$$

It follows that (note that $\mathfrak{w}_0^{x_0}(x) = \binom{0}{1}$)

$$\sum_{l=0}^{\Delta-2} \gamma_l \mathfrak{w}_{l+1}^{x_0} \chi_{(s_-,\sigma)} + \big(\epsilon_- - \epsilon - \sum_{l=0}^{\Delta-2} \gamma_l \epsilon_l\big) \mathfrak{w}_0^{x_0} \chi_{(s_-,\sigma)}$$

$$= \left[\sum_{l=0}^{\Delta-2} \gamma_l \left(\mathfrak{w}_{l+1}^{x_0} - \begin{pmatrix} 0 \\ \epsilon_l \end{pmatrix}\right) + \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix} - \begin{pmatrix} \epsilon_+ \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \epsilon \end{pmatrix}\right] \chi_{(s_-,\sigma)}$$

$$= \left[\sum_{l=0}^{\Delta-2} \gamma_l \mathcal{I} \mathfrak{w}_l^{x_0} + \hat{f} - \mathcal{I}\hat{g} - \begin{pmatrix} \epsilon_+ \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \epsilon \end{pmatrix}\right] \chi_{(s_-,\sigma)}$$

$$= \left[-\left(\mathcal{I}\tilde{g} + \begin{pmatrix} 0 \\ \epsilon \end{pmatrix}\right) + \hat{f} - \begin{pmatrix} \epsilon_+ \\ 0 \end{pmatrix}\right] \chi_{(s_-,\sigma)} \in L^2(H),$$

and hence that $\gamma_l = 0$, $l = 0, \dots, \Delta - 2$. We conclude that $\hat{g}\chi_{\urcorner x_0} = \tilde{g}\chi_{\urcorner x_0}$ and hence belongs to $L^2(H)$. ❑

2.9 Corollary. *With the notation as in Lemma 2.8, we have*

$$\operatorname{dom} T_{\Delta,\max}(H) = \operatorname{dom} T_{\max}(H) \dot{+} \operatorname{span}\big\{\mathfrak{w}_k^{x_0}\chi_{\urcorner x_0} + \omega_k^{x_0} f_0 : k = 0, \dots, \Delta\big\}.$$

Proof. In view of (2.3), it is enough to note that $\{\mathfrak{w}_k^{x_0}\chi_{\urcorner x_0} : k = 0, \dots, \Delta\}$ is linearly independent modulo $\operatorname{dom} T_{\max}(H)$, cf. Lemma IV.3.11. ❑

2.10 Corollary. *Let $(f; g) \in T_{\Delta,\max}(H)$, and let λ and μ be the unique constants such that*

$$f - \sum_{l=0}^{\Delta-1} \lambda_l \mathfrak{w}_l^{x_0} \chi_{\urcorner x_0} \in L^2(H), \quad g - \sum_{l=0}^{\Delta-1} \mu_l \mathfrak{w}_l^{x_0} \chi_{\urcorner x_0} \in L^2(H).$$

Moreover, let u_l, v_l be as in (2.2). Then we have

$$\lambda_{l+1} = \mu_l, \quad l = 0, \dots, \Delta - 2,$$

and

$$(f; g) - \sum_{l=0}^{\Delta-1} \lambda_l (u_l; v_l) - \mu_{\Delta-1}(u_\Delta; v_\Delta) \in T_{\max}(H).$$

Proof. Let $\gamma_0, \dots, \gamma_\Delta$ be the unique constants, such that

$$(f; g) - \sum_{l=0}^{\Delta} \gamma_l (u_l; v_l) \in T_{\max}(H).$$

Then, in particular,

$$\big(f - \sum_{l=0}^{\Delta} \gamma_l \mathfrak{w}_l^{x_0}\big)\chi_{\urcorner x_0} \in L^2(H), \quad \big(g - \sum_{l=0}^{\Delta} \gamma_l \mathfrak{w}_{l-1}^{x_0}\big)\chi_{\urcorner x_0} \in L^2(H),$$

and we conclude that

$$\lambda_l = \gamma_l, \ \mu_l = \gamma_{l+1}, \quad l = 0, \dots, \Delta - 1. \tag{2.5}$$

❑

2.11 Remark. Let $\mathfrak{h}$ be a general Hamiltonian, let H be its Hamiltonian function, and let $\psi(\mathfrak{h})$ be the map defined on [KW2, p.760]. Then

$$\big(\psi(\mathfrak{h}) \times \psi(\mathfrak{h})\big)\big(T(\mathfrak{h})\big) = T_{\Delta,\max}(H)\,.$$

In view of Proposition IV.4.17 (*iii*), and Definition IV.4.11, this is an immediate consequence of (2.3). //

c. Definition of the isomorphic copies $\mathring{\mathfrak{B}}_{x_0}(\mathfrak{h})$ and $\mathring{\mathring{\mathfrak{B}}}_{x_0}(\mathfrak{h})$.

From the parameters b_j, $j = 1, \ldots, \ddot{o}$, of $\mathfrak{h}$ we define numbers c_j, $j \in \mathbb{Z}$, by

$$(c_1, \ldots, c_{\ddot{o}}) \begin{pmatrix} b_1 & \cdots & b_{\ddot{o}} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_1 \end{pmatrix} = (-1, 0, \ldots, 0)\,, \quad c_j := 0 \text{ otherwise,}$$

cf. (IV.4.2).

In a first step we construct the isomorphic copy $\mathring{\mathfrak{B}}_{x_0}(\mathfrak{h})$ of a given boundary triple $\mathfrak{B}_{x_0}(\mathfrak{h})$.

2.12 Definition (of $\mathring{\mathfrak{B}}_{x_0}(\mathfrak{h})$, Part 1)**.** Denote by $\mathring{\mathfrak{P}}_{x_0}(\mathfrak{h})$ the linear space

$$\mathring{\mathfrak{P}}_{x_0}(\mathfrak{h}) := L^2(H) \times (\mathbb{C}^\Delta \times \mathbb{C}^\Delta) \times \mathbb{C}^{\ddot{o}}$$

equipped with the inner product $[\,\cdot\,,\,\cdot\,]$ defined by means of the Gram matrix

$$G_{\mathring{\mathfrak{P}}_{x_0}(\mathfrak{h})} := \left(\begin{array}{c|cc|c} I & 0 & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & (c_{k+l-\ddot{o}})_{k,l=1}^{\ddot{o}} \end{array}\right), \tag{2.6}$$

i.e.

$$[F, G] = \big(G_{\mathring{\mathfrak{P}}_{x_0}(\mathfrak{h})} F, G\big)_{L^2(H) \times (\mathbb{C}^\Delta \times \mathbb{C}^\Delta) \times \mathbb{C}^{\ddot{o}}}$$

for $F, G \in \mathring{\mathfrak{P}}_{x_0}(\mathfrak{h})$. Moreover, define $\bar{\cdot} : \mathring{\mathfrak{P}}_{x_0}(\mathfrak{h}) \to \mathring{\mathfrak{P}}_{x_0}(\mathfrak{h})$ by

$$\overline{(f; \xi, \lambda, \alpha)} := \big(\overline{f}; \overline{\xi}, \overline{\lambda}, \overline{\alpha}\big)\,.$$

//

Choose $x_0 \in I_+$, and let $\mathfrak{h}_{x_0}$ be an elementary indefinite Hamiltonian of kind (A) as in (2.1). The isometric isomorphism ι from $\mathfrak{P}(\mathfrak{h}_{x_0})$ onto $L^2(H_{\urcorner x_0}) \times (\mathbb{C}^\Delta \times \mathbb{C}^\Delta) \times \mathbb{C}^{\ddot{o}}$ constructed in (IV.4.10) naturally extends to an isometric isomorphism

$$\iota_{x_0} : \mathfrak{P}_{x_0}(\mathfrak{h}) = \mathfrak{P}(\mathfrak{h}_{x_0})[\dot{+}]L^2(H_{x_0 \ulcorner}) \to \mathring{\mathfrak{P}}_{x_0}(\mathfrak{h})\,,$$

namely by

$$\iota_{x_0}\big(x \dot{+} g\big) := \iota x + (g\chi_{x_0 \ulcorner}; 0, 0, 0), \quad x \in \mathfrak{P}(\mathfrak{h}_{x_0}),\ g \in L^2(H_{x_0 \ulcorner})\,.$$

2.13 Definition (of $\mathring{\mathfrak{B}}_{x_0}(\mathfrak{h})$, Part 2)**.** Let $x_0 \in I_+$. Denote by $\mathring{T}_{x_0}(\mathfrak{h}) \subseteq \mathring{\mathfrak{P}}_{x_0}(\mathfrak{h})^2$ and $\mathring{\Gamma}_{x_0}(\mathfrak{h}) \subseteq \mathring{\mathfrak{P}}_{x_0}(\mathfrak{h})^2 \times (\mathbb{C}^2 \times \mathbb{C}^2)$ the linear relations

$$\mathring{T}_{x_0}(\mathfrak{h}) := (\iota_{x_0} \times \iota_{x_0}) T_{x_0}(\mathfrak{h}) ,$$
$$\mathring{\Gamma}_{x_0}(\mathfrak{h}) := \big((\iota_{x_0} \times \iota_{x_0}) \times \mathrm{id}_{\mathbb{C}^4}\big) \Gamma_{x_0}(\mathfrak{h}) .$$

//

In a second step we construct another isomorphic copy of $\mathfrak{B}_{x_0}(\mathfrak{h})$ where the space component $L^2(H) \times \mathbb{C}^\Delta$ is replaced by $L^2_\Delta(H)$. To this end consider the map

$$\mathring{\iota}_{x_0} : \begin{cases} \mathring{\mathfrak{P}}_{x_0}(\mathfrak{h}) & \to \quad L^2_\Delta(H) \times \mathbb{C}^\Delta \times \mathbb{C}^{\ddot{o}} \\ (f; \xi, \lambda, \alpha) & \mapsto \quad \big(f + \sum_{k=0}^{\Delta-1} \lambda_k \mathfrak{w}_k^{x_0} \chi_{\urcorner x_0}; \xi, \alpha\big) \end{cases}$$

Then, clearly, $\mathring{\iota}_{x_0}$ is bijective.

2.14 Definition (of $\mathring{\mathring{\mathfrak{B}}}_{x_0}(\mathfrak{h})$)**.** Let $x_0 \in I_+$. Denote by $\mathring{\mathring{\mathfrak{P}}}_{x_0}(\mathfrak{h})$ the linear space

$$\mathring{\mathring{\mathfrak{P}}}_{x_0}(\mathfrak{h}) := L^2_\Delta(H) \times \mathbb{C}^\Delta \times \mathbb{C}^{\ddot{o}} ,$$

endowed with the inner product

$$[F, G]_{x_0} := \big[\mathring{\iota}_{x_0}^{-1} F, \mathring{\iota}_{x_0}^{-1} G\big]_{\mathring{\mathfrak{P}}_{x_0}(\mathfrak{h})}, \quad F, G \in \mathring{\mathring{\mathfrak{P}}}_{x_0}(\mathfrak{h}) ,$$

and the conjugate linear involution

$$\overline{(f; \xi, \alpha)} := (\overline{f}; \overline{\xi}, \overline{\alpha}) .$$

Denote by $\mathring{\mathring{T}}_{x_0}(\mathfrak{h}) \subseteq \mathring{\mathring{\mathfrak{P}}}_{x_0}(\mathfrak{h})^2$ and $\mathring{\mathring{\Gamma}}_{x_0}(\mathfrak{h}) \subseteq \mathring{\mathring{\mathfrak{P}}}_{x_0}(\mathfrak{h})^2 \times (\mathbb{C}^2 \times \mathbb{C}^2)$ the linear relations

$$\mathring{\mathring{T}}_{x_0}(\mathfrak{h}) := (\mathring{\iota}_{x_0} \times \mathring{\iota}_{x_0}) \mathring{T}_{x_0}(\mathfrak{h}) ,$$
$$\mathring{\mathring{\Gamma}}_{x_0}(\mathfrak{h}) := \big((\mathring{\iota}_{x_0} \times \mathring{\iota}_{x_0}) \times \mathrm{id}_{\mathbb{C}^4}\big) \mathring{\Gamma}_{x_0}(\mathfrak{h}) .$$

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With these definitions, the triples

$$\mathring{\mathfrak{B}}_{x_0}(\mathfrak{h}) := \big(\mathring{\mathfrak{P}}_{x_0}(\mathfrak{h}), \mathring{T}_{x_0}(\mathfrak{h}), \mathring{\Gamma}_{x_0}(\mathfrak{h})\big),$$
$$\mathring{\mathring{\mathfrak{B}}}_{x_0}(\mathfrak{h}) := \big(\mathring{\mathring{\mathfrak{P}}}_{x_0}(\mathfrak{h}), \mathring{\mathring{T}}_{x_0}(\mathfrak{h}), \mathring{\mathring{\Gamma}}_{x_0}(\mathfrak{h})\big)$$

are boundary triples isomorphic to $\mathfrak{B}(\mathfrak{h})$. Actually, $(\iota_{x_0}; \mathrm{id}_{\mathbb{C}^4})$ is an isomorphism from $\mathfrak{B}_{x_0}(\mathfrak{h})$ to $\mathring{\mathfrak{B}}_{x_0}(\mathfrak{h})$, and $(\mathring{\iota}_{x_0}; \mathrm{id}_{\mathbb{C}^4})$ is an isomorphism from $\mathring{\mathfrak{B}}_{x_0}(\mathfrak{h})$ to $\mathring{\mathring{\mathfrak{B}}}_{x_0}(\mathfrak{h})$.

d. Description of $\mathring{\mathring{\mathfrak{B}}}_{x_0}(\mathfrak{h})$.

In this subsection we establish the following intrinsic description of the boundary triple $\mathring{\mathring{\mathfrak{B}}}_{x_0}(\mathfrak{h})$. The following theorem is the main result of this section.

2.15 Theorem. *Let $\mathfrak{h}$ be a general Hamiltonian (of the form 1.3) and let $x_0 \in I_+$, and $d_j^{x_0}$ as in Lemma 2.5. Moreover, let $F = (f;\xi,\alpha), G = (g;\eta,\beta) \in \mathring{\mathfrak{P}}_{x_0}(\mathfrak{h})$, and denote by λ and μ the unique coefficients such that*

$$\tilde{f} := f - \sum_{l=0}^{\Delta-1} \lambda_l \mathfrak{w}_l^{x_0} \chi_{\urcorner x_0} \in L^2(H), \quad \tilde{g} := g - \sum_{l=0}^{\Delta-1} \mu_l \mathfrak{w}_l^{x_0} \chi_{\urcorner x_0} \in L^2(H).$$

Then

$$[F,G] = (\tilde{f},\tilde{g})_{L^2(H)} + \sum_{k=0}^{\Delta-1} \lambda_k \overline{\eta_k} + \sum_{k=0}^{\Delta-1} \xi_k \overline{\mu_k} + \sum_{k,l=1}^{\ddot{o}} c_{k+l-\ddot{o}} \alpha_k \overline{\beta_l}.$$

Moreover, $(F;G) \in \mathring{T}_{x_0}(\mathfrak{h})$ if and only if

(i) $(f;g) \in T_{\Delta,\max}(H)$;

(ii) *for each $k \in \{0,\dots,\Delta-2\}$,*

$$\xi_k = \eta_{k+1} + \frac{1}{2}\mu_{\Delta-1} d_{\Delta+k}^{x_0} + \frac{1}{2}\lambda_0 d_k^{x_0} + \omega_{k+1}^{x_0} f(x_0)_1 - \begin{cases} \mathfrak{w}_{k+1}^{x_0}(s_-)_2 f(s_-)_1, & (s_-,\sigma) \text{ not indivisible}, \\ 0, & (s_-,\sigma) \text{ indivisible}; \end{cases}$$

(iii)

$$\xi_{\Delta-1} = \int_{s_-}^{x_0} (\mathfrak{w}_\Delta^{x_0})^* H \tilde{g} + \frac{1}{2}\sum_{l=0}^{\Delta-1} \lambda_l d_{l+\Delta-1}^{x_0} + \mu_{\Delta-1} d_{2\Delta-1}^{x_0} - \begin{cases} \beta_1, & \ddot{o} > 0, \\ 0, & \ddot{o} = 0, \end{cases} + \omega_\Delta^{x_0} f(x_0)_1 - \begin{cases} \mathfrak{w}_\Delta^{x_0}(s_-)_2 f(s_-)_1, & (s_-,\sigma) \text{ not indivisible}, \\ 0, & (s_-,\sigma) \text{ indivisible}; \end{cases}$$

(iv) *if (s_-,σ) is not indivisible, then*

$$\eta_0 = f(s_-)_1 - f(x_0)_1 - \frac{1}{2}\sum_{l=0}^{\Delta-1} \mu_l d_l^{x_0};$$

(v) *if $\ddot{o} > 0$, then*

$$\alpha_j = -\mu_{\Delta-1} b_{\ddot{o}-j+1} + \begin{cases} \beta_{j+1}, & j = 1,\dots,\ddot{o}-1, \\ 0, & j = \ddot{o}. \end{cases}$$

Here $f(s_-)$ and $f(x_0)$ denote the values of the unique absolutely continuous representative with $f' = JHg$, which always exists on (σ,s_+), and exists on (s_-,σ) if this interval is not indivisible.

We have $\operatorname{mul}\mathring{T}_{x_0}(\mathfrak{h}) \neq \{0\}$ if and only if H starts with an indivisible interval at s_-. In this case, when $s_0 \in (s_-,\sigma]$ denotes the right endpoint of *the maximal indivisible interval with left endpoint s_-, ϕ denotes its type, and $l := \int_{s_-}^{s_0} \operatorname{tr} H(t)\,dt$ its length, the multi-valued part of $\overset{\circ\circ}{T}_{x_0}(\mathfrak{h})$ is given by*

$$\operatorname{mul} \overset{\circ\circ}{T}_{x_0}(\mathfrak{h}) = \begin{cases} \operatorname{span}\big\{\big(0; (\xi_\phi \chi_{\restriction s_0}; (-\mathfrak{w}_k^{x_0}(s_-)_2 l \sin\phi)_{j=0}^{\Delta-1}, 0)\big)\big\}, & s_0 < \sigma, \\ \operatorname{span}\big\{\big(0; (0; \varepsilon_0, 0)\big)\big\}, & s_0 = \sigma. \end{cases}$$

Whenever $(F;G) \in \overset{\circ\circ}{T}_{x_0}(\mathfrak{h})$, we have

$$\overset{\circ\circ}{\Gamma}_{x_0}(F;G) = \begin{cases} f(s_-), & (s_-,\sigma) \text{ not indivisible}, \\ \begin{pmatrix} \eta_0 + f(x_0)_1 + \frac{1}{2}\sum_{l=0}^{\Delta-1} \mu_l d_l^{x_0} \\ \lambda_0 \end{pmatrix}, & (s_-,\sigma) \text{ indivisible}. \end{cases}$$

2.16 Remark. Note that by Corollary 2.10, condition (i) implies that

$$\mu_k = \lambda_{k+1}, \quad k = 0, \dots, \Delta - 2.$$

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The proof of this result is done in two steps; first we deal with $\mathring{\mathfrak{B}}_{x_0}(\mathfrak{h})$, and then transfer the obtained knowledge to $\overset{\circ\circ}{\mathfrak{B}}_{x_0}(\mathfrak{h})$. The description of $\mathring{\mathfrak{B}}_{x_0}(\mathfrak{h})$ reads as follows.

2.17 Proposition. *Let $\mathfrak{h}$ be a general Hamiltonian (of the form 1.3), and let $x_0 \in I_+$. Moreover, let $d_j^{x_0}$ be as in Lemma 2.5, and fix an element $(f_0; g_0) \in T_{\max}(H_{x_0 \Rsh})$ with $\Gamma(H_{x_0 \Rsh})(f_0; g_0) = \binom{0}{1}$.*

Let $F, G \in \mathring{\mathfrak{P}}_{x_0}(\mathfrak{h})$, and write $F := (f; \xi, \lambda, \alpha)$ and $G := (g; \eta, \mu, \beta)$. Then $(F;G) \in \mathring{T}_{x_0}(\mathfrak{h})$ if and only if

(i')
$$\Big(f - \sum_{l=0}^{\Delta-1} \lambda_l \omega_l^{x_0} f_0 - \mu_{\Delta-1}\big(\mathfrak{w}_\Delta^{x_0} \chi_{\restriction x_0} + \omega_\Delta^{x_0} f_0\big); \qquad g - \Big(\sum_{l=0}^{\Delta-1} \lambda_l \omega_l^{x_0} + \mu_{\Delta-1}\omega_\Delta^{x_0}\Big) g_0\Big) \in T_{\max}(H)$$

$$\mu_k = \lambda_{k+1}, \quad k = 0, \dots, \Delta - 2\,,$$

and $(F;G)$ satisfies (ii)–(v) of Theorem 2.15 where in (iii) the function $\tilde{g}$ is replaced by g. If $(F;G) \in \mathring{T}_{x_0}(\mathfrak{h})$, then

$$\mathring{\Gamma}_{x_0}(F;G) = \begin{cases} f(s_-) + \sum_{l=0}^{\Delta-1} \lambda_l \mathfrak{w}_l^{x_0}(s_-), & (s_-,\sigma) \text{ not indivisible}, \\ \begin{pmatrix} \eta_0 + f(x_0)_1 + \frac{1}{2}\sum_{l=0}^{\Delta-1} \mu_l d_l^{x_0} \\ \lambda_0 \end{pmatrix}, & (s_-,\sigma) \text{ indivisible}. \end{cases} \tag{2.7}$$

For the proof we start with identifying some particular elements of $\mathring{T}_{x_0}(\mathfrak{h})$.

2.18 Lemma.

(i) Let $a_k, b_k \in \mathring{\mathfrak{P}}_{x_0}(\mathfrak{h})$, $k = 0, \ldots, \Delta$, *be defined as*

$$a_0 := \big(f_0; (\frac{1}{2} d_j^{x_0})_{j=0}^{\Delta-1}, \varepsilon_0, 0\big),$$
$$b_0 := \big(g_0; 0, 0, 0\big),$$

$$a_k := \big(\omega_k^{x_0} f_0; (\frac{1}{2} d_{k+j}^{x_0})_{j=0}^{\Delta-1}, \varepsilon_k, 0\big),$$
$$b_k := \big(\omega_k^{x_0} g_0; (\frac{1}{2} d_{k-1+j}^{x_0})_{j=0}^{\Delta-1} - d_{k-1}^{x_0} \varepsilon_0, \varepsilon_{k-1}, 0\big),$$
$$k = 1, \ldots, \Delta - 1,$$

$$a_\Delta := \big(\mathfrak{w}_\Delta^{x_0} \chi_{\urcorner x_0} + \omega_\Delta^{x_0} f_0; (d_{\Delta+j}^{x_0})_{j=0}^{\Delta-1}, 0, -(b_{\ddot{o}+1-j})_{j=1}^{\ddot{o}}\big),$$
$$b_\Delta := \big(\omega_\Delta^{x_0} g_0; (\frac{1}{2} d_{\Delta-1+j}^{x_0})_{j=0}^{\Delta-1} - d_{\Delta-1}^{x_0} \varepsilon_0, \varepsilon_{\Delta-1}, 0\big).$$

Then $(a_k; b_k) \in \mathring{T}_{x_0}(\mathfrak{h})$, $k = 0, \ldots, \Delta$, *and* $\mathring{\Gamma}_{x_0}(\mathfrak{h})(a_k; b_k) = \mathfrak{w}_k^{x_0}(s_-)$.

(ii) Let a'_k, $k = 0, \ldots, \Delta + \ddot{o} - 1$, *be defined as*

$$a'_k := \begin{cases} (0; -\varepsilon_k, 0, 0), & k = 0, \ldots, \Delta - 1, \\ (0; 0, 0, \varepsilon_{k-\Delta+1}), & k = \Delta, \ldots, \Delta + \ddot{o} - 1. \end{cases}$$

Then $(a'_k; a'_{k+1}) \in \mathring{T}_{x_0}(\mathfrak{h})$, $k = 0, \ldots, \Delta + \ddot{o} - 2$, *and* $\mathring{\Gamma}_{x_0}(\mathfrak{h})(a'_k; a'_{k+1}) = 0$.

Proof. Let p_k, $k = 0, \ldots, \Delta - 1$, and δ_k, $k = 0, \ldots, \Delta + \ddot{o} - 1$, be defined as in the paragraphs before and after Lemma IV.4.9 and in Definition IV.4.10. Then according to Remark IV.7.5, equation (IV.4.8), Proposition IV.4.7, top of page 760 in [KW2] and Definition IV.4.10, the following relations are valid:

$$\iota_{x_0} p_k = \Big(0; \big(\frac{1}{2} d_{k+j}^{x_0}\big)_{j=0}^{\Delta-1}, \varepsilon_k, 0\Big), \qquad k = 0, \ldots, \Delta - 1,$$

$$\iota_{x_0} \mathfrak{w}_\Delta^{x_0} = \Big(\mathfrak{w}_\Delta^{x_0} \chi_{\urcorner x_0}; \big(d_{\Delta+j}^{x_0}\big)_{j=0}^{\Delta-1}, 0, 0\Big),$$

$$\iota_{x_0} \delta_k = \begin{cases} (0; -\varepsilon_k, 0, 0), & k = 0, \ldots, \Delta - 1, \\ (0; 0, 0, \varepsilon_{k-\Delta+1}), & k = \Delta, \ldots, \Delta + \ddot{o} - 1. \end{cases}$$

Hence

$$(a_0; b_0) = (\iota_{x_0} \times \iota_{x_0})\big((p_0; 0) + (f_0; g_0)\big),$$
$$(a_k; b_k) = (\iota_{x_0} \times \iota_{x_0})\big((p_k; p_{k-1} + d_{k-1}^{x_0} \delta_0) + \omega_k^{x_0}(f_0; g_0)\big),$$
$$k = 1, \ldots, \Delta - 1,$$
$$(a_\Delta; b_\Delta) = (\iota_{x_0} \times \iota_{x_0})\big((\mathfrak{w}_\Delta^{x_0} + \mathfrak{b}; p_{\Delta-1} + d_{\Delta-1}^{x_0} \delta_0) + \omega_\Delta^{x_0}(f_0; g_0)\big),$$

$$a'_k = \iota_{x_0}(\delta_k), \qquad k = 0, \ldots, \Delta + \ddot{o} - 1,$$

where (see Definition IV.4.11)

$$\mathfrak{b} := \sum_{l=1}^{\ddot{o}+1} b_l \delta_{\Delta+\ddot{o}-l} = \sum_{j=1}^{\ddot{o}} b_{\ddot{o}+1-j} \delta_{\Delta+j-1}$$

since $b_{\ddot{o}+1} = 0$. Now the assertions follow from the fact that the pairs

$$\begin{aligned}
&(p_0; 0), \\
&(p_k; p_{k-1} + d^{x_0}_{k-1}\delta_0), && k = 1, \dots, \Delta - 1, \\
&\big(\mathfrak{w}^{x_0}_\Delta + \mathfrak{b}; p_{\Delta-1} + d^{x_0}_{\Delta-1}\delta_0\big), \\
&(\delta_k; \delta_{k+1}), && k = 0, \dots, \Delta + \ddot{o} - 2
\end{aligned}$$

all belong to $T(\mathfrak{h}_{x_0})$ (see Definition IV.4.11 and Proposition IV.4.17 (*iv*)). The form of the boundary mappings follows from Definition IV.4.12 and the two preceding paragraphs. ❑

Now we are in position to treat the case when both elements F and G belong to $L^2(H) \times (\mathbb{C}^\Delta \times \{0\}) \times \{0\}$.

2.19 Lemma. *Let $F := (f; \xi, 0, 0)$ and $G := (g; \eta, 0, 0)$ be elements of $\mathring{\mathfrak{P}}_{x_0}(\mathfrak{h})$. Then $(F; G) \in \mathring{T}_{x_0}(\mathfrak{h})$ if and only if*

- (*i*) $(f; g) \in T_{\max}(H)$;
- (*ii*) *for each* $k \in \{0, \dots, \Delta - 2\}$,

$$\xi_k = \eta_{k+1} + \omega^{x_0}_{k+1} f(x_0)_1 - \begin{cases} \mathfrak{w}^{x_0}_{k+1}(s_-)_2 f(s_-)_1, & (s_-, \sigma) \textit{ not indivisible}, \\ 0, & (s_-, \sigma) \textit{ indivisible}; \end{cases}$$

- (*iii*)

$$\xi_{\Delta-1} = \int_{s_-}^{x_0} (\mathfrak{w}^{x_0}_\Delta)^* H g + \omega^{x_0}_\Delta f(x_0)_1 - \begin{cases} \mathfrak{w}^{x_0}_\Delta(s_-)_2 f(s_-)_1, & (s_-, \sigma) \textit{ not indivisible}, \\ 0, & (s_-, \sigma) \textit{ indivisible}; \end{cases}$$

- (*iv*) *if (s_-, σ) is not indivisible, then $\eta_0 = f(s_-)_1 - f(x_0)_1$.*

In this case,

$$\mathring{\Gamma}_{x_0}(\mathfrak{h})(F; G) = \begin{cases} f(s_-), & (s_-, \sigma) \textit{ not indivisible}, \\ \begin{pmatrix} \eta_0 + f(x_0)_1 \\ 0 \end{pmatrix}, & (s_-, \sigma) \textit{ indivisible}. \end{cases} \tag{2.8}$$

Proof. Assume first that $(F; G) \in \mathring{T}_{x_0}(\mathfrak{h})$. Since

$$L^2(H) \times (\mathbb{C}^\Delta \times \{0\}) \times \{0\} = \operatorname{span}\big\{\iota_{x_0}(\delta_0), \dots, \iota_{x_0}(\delta_{\Delta+\ddot{o}-1})\big\}^\perp,$$

it follows that

$$(\iota_{x_0}^{-1} \times \iota_{x_0}^{-1})(F; G) \in T_{x_0}(\mathfrak{h}) \cap \big(\operatorname{span}\{\delta_0, \dots, \delta_{\Delta+\ddot{o}-1}\}^\perp\big)^2.$$

We obtain from Proposition IV.4.17 (*iii*), that

$$(f; g) = \big[(\psi \times \psi) \circ (\iota_{x_0}^{-1} \times \iota_{x_0}^{-1})\big](F; G) \in T_{\max}(H),$$

i.e. (i) holds.

In order to obtain the formulae asserted in (ii)–(iv), we apply the abstract Green's identity (1.2) with various elements in the spaces $\mathring{\mathfrak{P}}_{x_0}(\mathfrak{h})$ and $L^2(H_{x_0\upharpoonright})$. First we compute the boundary values of $(F;G)$. We have

$$\big(\mathring{\Gamma}_{x_0}(\mathfrak{h})(F;G)\big)_1 = \mathfrak{w}_0^{x_0}(s_-)^* J\mathring{\Gamma}_{x_0}(\mathfrak{h})(F;G) = [G,a_0]-[F,b_0] =$$

$$= \eta_0 + \underbrace{\int_{x_0}^{s_+} f_0^* Hg - \int_{x_0}^{s_+} g_0^* Hf}_{=f_0(x_0)^* Jf(x_0)} = \eta_0 + f(x_0)_1\,.$$

If (s_-,σ) is not indivisible, by the definition of $\Gamma_{x_0}(\mathfrak{h})$ (see Definition IV.4.12),

$$\mathring{\Gamma}_{x_0}(\mathfrak{h})(F;G) = \Gamma_{x_0}(\mathfrak{h})\circ(\iota_{x_0}^{-1}\times\iota_{x_0}^{-1})(F;G) = f(s_-)\,.$$

We see that (2.8) and (iv) hold. Next, let $k\in\{0,\dots,\Delta-2\}$; then

$$\begin{aligned}
&\mathfrak{w}_{k+1}^{x_0}(s_-)_2\big(\eta_0+f(x_0)_1\big)\\
&= \mathfrak{w}_{k+1}^{x_0}(s_-)^* J\mathring{\Gamma}_{x_0}(\mathfrak{h})(F;G) = [G,a_{k+1}]-[F,b_{k+1}]\\
&= \int_{x_0}^{s_+}(\omega_{k+1}^{x_0}f_0)^* Hg + \eta_{k+1} - \int_{x_0}^{s_+}(\omega_{k+1}^{x_0}g_0)^* Hf - \xi_k\\
&= \eta_{k+1}-\xi_k+\omega_{k+1}^{x_0}f_0(x_0)^* Jf(x_0) = \eta_{k+1}-\xi_k+\omega_{k+1}^{x_0}f(x_0)_1\,.
\end{aligned}$$

If (s_-,σ) is not indivisible, this relation combined with the already established relation (iv) gives (ii). If (s_-,σ) is indivisible, we know from Remark IV.3.8 that $\mathfrak{w}_{k+1}^{x_0}(s_-)_2=0$. Hence, also in this case (ii) holds. Finally, we compute

$$\begin{aligned}
&\mathfrak{w}_\Delta^{x_0}(s_-)_2\big(\eta_0+f(x_0)_1\big) = \mathfrak{w}_\Delta^{x_0}(s_-)^* J\mathring{\Gamma}_{x_0}(\mathfrak{h})(F;G) = [G,a_\Delta]-[F,b_\Delta] =\\
&= \int_{s_-}^{x_0}(\mathfrak{w}_\Delta^{x_0})^* Hg + \omega_\Delta^{x_0}\bigg[\int_{x_0}^{s_+} f_0^* Hg - \int_{x_0}^{s_+} g_0^* Hf\bigg] - \xi_{\Delta-1}\\
&= \int_{s_-}^{x_0}(\mathfrak{w}_\Delta^{x_0})^* Hg + \omega_\Delta^{x_0} f(x_0)_1 - \xi_{\Delta-1},
\end{aligned}$$

which yields (iii).

For the converse, assume that F and G satisfy (i)–(iv). By Proposition IV.4.17 (iii), there exists an element $(\tilde{F};\tilde{G})\in\mathring{T}_{x_0}(\mathfrak{h})\cap(L^2(H)\times(\mathbb{C}^\Delta\times\{0\})\times\{0\})$ with $(\psi\circ\iota_{x_0}^{-1})(\tilde{F})=f$ and $(\psi\circ\iota_{x_0}^{-1})(\tilde{G})=g$. By the first part of this proof, the elements $\tilde{F}$ and $\tilde{G}$ satisfy the conditions (i)–(iv).

Write $\tilde{F}=(f;\tilde{\xi},0,0)$ and $\tilde{G}=(g;\tilde{\eta},0,0)$. By Lemma 2.18 (ii), we can make the choice of $(\tilde{F};\tilde{G})$ such that $\tilde{\eta}_k=\eta_k$, $k=1,\dots,\Delta-1$. If (s_-,σ) is not indivisible, by condition (iv), we must have $\tilde{\eta}_0=\eta_0$. If (s_-,σ) is indivisible, we have $(0;\delta_0)\in T_{x_0}(\mathfrak{h})$ (see Definition IV.4.11 (4.14)), and again $(\tilde{F};\tilde{G})$ can be chosen in this way. The conditions (ii) and (iii) now imply that $\tilde{F}=F$ and $\tilde{G}=G$. ❑

With the help of Lemma 2.18, we can reduce the general case to the case treated in Lemma 2.19.

Proof (of Proposition 2.17).
Step 1. Define an element $(\tilde{F};\tilde{G})$ by

$$(\tilde{F};\tilde{G}) := (F;G) - \sum_{k=0}^{\Delta-1} \lambda_k (a_k;b_k) - \mu_{\Delta-1}(a_\Delta;b_\Delta) - \sum_{j=0}^{\ddot{o}-1} \beta_{j+1}(a'_{\Delta-1+j};a'_{\Delta+j})$$

By Lemma 2.18 we have $(F;G) \in \mathring{T}_{x_0}(\mathfrak{h})$ if and only if $(\tilde{F};\tilde{G}) \in \mathring{T}_{x_0}(\mathfrak{h})$.
Write $\tilde{F} = (\tilde{f};\tilde{\xi},\tilde{\lambda},\tilde{\alpha})$ and $\tilde{G} = (\tilde{g};\tilde{\eta},\tilde{\mu},\tilde{\beta})$, then

$$\begin{aligned}
\tilde{f} &= f - \sum_{l=0}^{\Delta-1} \lambda_l \omega_l^{x_0} f_0 - \mu_{\Delta-1}\big(\mathfrak{w}_\Delta^{x_0}\chi_{\Lsh x_0} + \omega_\Delta^{x_0} f_0\big),\\
\tilde{\xi}_k &= \xi_k - \frac{1}{2}\sum_{l=0}^{\Delta-1} \lambda_l d_{l+k}^{x_0} - \mu_{\Delta-1} d_{\Delta+k}^{x_0}, \qquad k = 0,\dots,\Delta-2,\\
\tilde{\xi}_{\Delta-1} &= \xi_{\Delta-1} - \frac{1}{2}\sum_{l=0}^{\Delta-1} \lambda_l d_{\Delta-1+l}^{x_0} - \mu_{\Delta-1} d_{2\Delta-1}^{x_0} + \begin{cases} \beta_1, & \ddot{o} > 0,\\ 0, & \ddot{o} = 0,\end{cases}\\
\tilde{\lambda}_k &= \lambda_k - \sum_{l=0}^{\Delta-1} \lambda_l \delta_{lk} = 0, \qquad k = 0,\dots,\Delta-1,\\
\tilde{\alpha}_j &= \alpha_j + \mu_{\Delta-1} b_{\ddot{o}+1-j} - \begin{cases} \beta_{j+1}, & j = 1,\dots,\ddot{o}-1,\\ 0, & j = \ddot{o},\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{g} &= g - \left(\sum_{l=0}^{\Delta-1} \lambda_l \omega_l^{x_0} + \mu_{\Delta-1}\omega_\Delta^{x_0}\right) g_0,\\
\tilde{\eta}_0 &= \eta_0 + \frac{1}{2}\sum_{l=1}^{\Delta-1} \lambda_l d_{l-1}^{x_0} + \frac{1}{2}\mu_{\Delta-1} d_{\Delta-1}^{x_0},\\
\tilde{\eta}_k &= \eta_k - \frac{1}{2}\sum_{l=1}^{\Delta-1} \lambda_l d_{l-1+k}^{x_0} - \frac{1}{2}\mu_{\Delta-1} d_{\Delta-1+k}^{x_0}, \qquad k = 1,\dots,\Delta-1,\\
\tilde{\mu}_k &= \mu_k - \sum_{l=1}^{\Delta-1} \lambda_l \delta_{l-1,k} - \mu_{\Delta-1}\delta_{\Delta-1,k} = \begin{cases} \mu_k - \lambda_{k+1}, & k = 0,\dots,\Delta-2,\\ 0, & k = \Delta-1,\end{cases}\\
\tilde{\beta}_j &= 0, \quad j = 1,\dots,\ddot{o}.
\end{aligned}$$

Step 2: assume that $(F;G) \in \mathring{T}_{x_0}(\mathfrak{h})$.
The abstract Green's identity applied to the pairs $(\tilde{F};\tilde{G})$ and $(a'_l;a'_{l+1})$, $l = 0,\dots,\Delta+\ddot{o}-2$, gives

$$\begin{aligned}
&\tilde{\mu}_k = \tilde{\mu}_k - \tilde{\lambda}_{k+1} = -[\tilde{G},a'_k] + [\tilde{F},a'_{k+1}] = 0, \quad k = 0,\dots,\Delta-2,\\
&[\tilde{F},a'_\Delta] = [\tilde{F},a'_\Delta] + \tilde{\mu}_{\Delta-1} = [\tilde{F},a'_\Delta] - [\tilde{G},a'_{\Delta-1}] = 0,\\
&[\tilde{F},a'_{k+1}] = [\tilde{F},a'_{k+1}] - [\tilde{G},a'_k] = 0, \quad k = \Delta,\dots,\Delta+\ddot{o}-2.
\end{aligned}$$

The last two lines imply that $\tilde{\alpha} = 0$, and we see that

$$\tilde{F}, \tilde{G} \in L^2(H) \times (\mathbb{C}^\Delta \times \{0\}) \times \{0\}\,.$$

Plugging the expressions for $\tilde{f}, \tilde{\xi}, \ldots$ from Step 1 into the formulae of Lemma 2.19, shows that the relations (i') of Proposition 2.17 and (ii)–(v) of Theorem 2.15 are satisfied.

Step 3: assume that (i') of Proposition 2.17 and (ii)–(v) of Theorem 2.15 hold. Then we have $\tilde{F}, \tilde{G} \in L^2(H) \times (\mathbb{C}^\Delta \times \{0\}) \times \{0\}$, and hence Lemma 2.19 is applicable. However, (i'), (ii)–(iv) exactly correspond to (i)–(iv) of Lemma 2.19. We conclude that $(\tilde{F}; \tilde{G}) \in \mathring{T}_{x_0}(\mathfrak{h})$, and hence also $(F; G) \in \mathring{T}_{x_0}(\mathfrak{h})$.

Step 4: computation of boundary values.
Assume that $(F; G) \in \mathring{T}_{x_0}(\mathfrak{h})$, and let $(\tilde{F}; \tilde{G})$ be as above. Then, by (2.8),

$$\mathring{\Gamma}_{x_0}(\mathfrak{h})(\tilde{F}; \tilde{G})$$
$$= \begin{cases} f(s_-) - \mu_{\Delta-1}\mathfrak{w}_\Delta^{x_0}(s_-), & (s_-, \sigma) \text{ not indivisible}, \\ \begin{pmatrix} \eta_0 + \frac{1}{2}\sum_{l=1}^{\Delta-1} \lambda_l d_{l-1}^{x_0} + \frac{1}{2}\mu_{\Delta-1} d_{\Delta-1}^{x_0} + f(x_0)_1 \\ 0 \end{pmatrix}, & (s_-, \sigma) \text{ indivisible}. \end{cases}$$

Using the definition of $\tilde{F}, \tilde{G}$, the knowledge about the boundary values of the pairs appearing in Lemma 2.18, and the fact that $\mu_k = \lambda_{k+1}$, $k = 0, \ldots, \Delta - 2$, we obtain (2.7). Remember here also that $\mathfrak{w}_k^{x_0}(s_-) = 0$, $k \geq 1$, if (s_-, σ) is indivisible. ❑

Proof (of Theorem 2.15). The formula for the inner product is clear. Now consider the elements F, G as given in the statement of the theorem. Then

$$\tilde{F} := \mathring{\iota}_{x_0}^{-1} F = (\tilde{f}; \xi, \lambda, \alpha), \;\; \tilde{G} := \mathring{\iota}_{x_0}^{-1} G = (\tilde{g}; \eta, \mu, \beta)\,.$$

By the definition of $\mathring{\mathring{T}}_{x_0}(\mathfrak{h})$, we have $(F; G) \in \mathring{\mathring{T}}_{x_0}(\mathfrak{h})$ if and only if $(\tilde{F}; \tilde{G}) \in \mathring{T}_{x_0}(\mathfrak{h})$, and in turn if and only if $\tilde{F}$ and $\tilde{G}$ satisfy the conditions (i') of Proposition 2.17 and (ii)–(v) of Theorem 2.15 with the function $\tilde{g}$ in (iii). Since

$$\mathfrak{w}_l^{x_0}(x_0), \mathfrak{w}_l^{x_0}(s_-) \in \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

$\tilde{F}, \tilde{G}$ satisfying (ii)–(v) of Theorem 2.15 is equivalent to F, G satisfying these conditions.

We show that Proposition 2.17, (i'), for $\tilde{F}, \tilde{G}$ is equivalent to (i) of Theorem 2.15 for F, G. Clearly, this will finish the proof of the asserted equivalence.

Let $(f_0; g_0)$ and $(u_l; v_l)$ be as in (2.2). Then the element $(f; g)$ belongs to $T_{\Delta,\max}(H)$ if and only if

$$(f; g) - \sum_{l=0}^{\Delta-1} \lambda_l (u_l; v_l) - \mu_{\Delta-1}(u_\Delta; v_\Delta) \in T_{\Delta,\max}(H)\,. \tag{2.9}$$

Using the relations $F=\mathring{\iota}_{x_0}\tilde{F}$, $G=\mathring{\iota}_{x_0}\tilde{G}$ we compute

$$\begin{aligned}
f-&\sum_{l=0}^{\Delta-1}\lambda_l u_l-\mu_{\Delta-1}u_\Delta\\
&=\tilde{f}+\sum_{k=0}^{\Delta-1}\lambda_k\mathfrak{w}_k^{x_0}\chi_{\urcorner x_0}-\sum_{l=0}^{\Delta-1}\lambda_l\big(\mathfrak{w}_l^{x_0}\chi_{\urcorner x_0}+\omega_l^{x_0}f_0\big)-\mu_{\Delta-1}\big(\mathfrak{w}_\Delta^{x_0}\chi_{\urcorner x_0}+\omega_\Delta^{x_0}f_0\big)\\
&=\tilde{f}-\sum_{l=0}^{\Delta-1}\lambda_l\omega_l^{x_0}f_0-\mu_{\Delta-1}\big(\mathfrak{w}_\Delta^{x_0}\chi_{\urcorner x_0}+\omega_\Delta^{x_0}f_0\big)
\end{aligned}$$

and

$$\begin{aligned}
g-&\sum_{l=0}^{\Delta-1}\lambda_l v_l-\mu_{\Delta-1}v_\Delta\\
&=\tilde{g}+\sum_{k=0}^{\Delta-1}\mu_k\mathfrak{w}_k^{x_0}\chi_{\urcorner x_0}-\sum_{l=0}^{\Delta-1}\lambda_l\big(\mathfrak{w}_{l-1}^{x_0}\chi_{\urcorner x_0}+\omega_l^{x_0}g_0\big)-\mu_{\Delta-1}\big(\mathfrak{w}_{\Delta-1}^{x_0}\chi_{\urcorner x_0}+\omega_\Delta^{x_0}g_0\big)\\
&=\tilde{g}-\sum_{l=0}^{\Delta-1}\lambda_l\omega_l^{x_0}g_0-\mu_{\Delta-1}\omega_\Delta^{x_0}g_0+\sum_{l=0}^{\Delta-2}(\mu_l-\lambda_{l+1})\mathfrak{w}_l^{x_0}\chi_{\urcorner x_0}.
\end{aligned}$$

Hence, if (i') holds, the pair in (2.9) will belong to $T_{\Delta,\max}(H)$. Conversely, if (i) holds, we obtain from Corollary 2.10 that $\mu_l=\lambda_{l+1}$, $l=0,\dots,\Delta-2$. In turn, it follows that also the first condition in (i') holds.

Next we determine $\operatorname{mul}\overset{\infty}{T}_{x_0}(\mathfrak{h})$. The boundary triple $\mathfrak{B}_{x_0}(\mathfrak{h})$ is obtained from pasting the boundary triples associated with an elementary indefinite Hamiltonian of kind (A) and a positive definite one. Hence, it follows from Proposition IV.5.16 that $\operatorname{mul}T_{x_0}(\mathfrak{h})\neq\{0\}$ if and only if H starts with an indivisible interval at s_-, and that in this case $\dim\operatorname{mul}T_{x_0}(\mathfrak{h})=1$. The case when (s_-,σ) is indivisible is easily settled. It suffices to observe that in this case the pair

$$(F;G):=\big(0;(0;\varepsilon_0,0)\big)$$

satisfies the conditions (i)–(v), and hence belongs to $\overset{\infty}{T}_{x_0}(\mathfrak{h})$.

Assume that $s_0<\sigma$, and write

$$H(t)=h(t)\xi_\phi\xi_\phi^T,\quad t\in(s_-,s_0)\,.$$

Then $l=\int_{s_-}^{s_0}\operatorname{tr}H(t)\,dt=\int_{s_-}^{s_0}h(t)\,dt$. Set

$$f(t):=\int_{s_0}^{t}h(x)\,dx\cdot J\xi_\phi\chi_{\urcorner s_0},\quad g(t):=\xi_\phi\chi_{\urcorner s_0}\,,$$

then

$$f'=JHg,\quad f=_H0,\quad f(s_-)=l\begin{pmatrix}\sin\phi\\-\cos\phi\end{pmatrix}.$$

Moreover, since $f,g\in L^2(H)$, the numbers λ_l and μ_l all vanish. Consider the pair

$$(F;G):=\big(0;(g;(\mathfrak{w}_k^{x_0}(s_-)_2 l\sin\phi)_{k=0}^{\Delta-1},0)\big)\,.$$

By what we just said, $(F;G)$ satisfies the condition (i). Moreover, (ii), (iv), and (v) hold by the definition of G. It remains to consider (iii). However, since $\xi_\phi^T \mathfrak{w}_\Delta^{x_0}$ is constant on (s_-, s_0), we have

$$\int_{s_-}^{x_0} (\mathfrak{w}_\Delta^{x_0})^* Hg = \int_{s_-}^{s_0} (\mathfrak{w}_\Delta^{x_0})^* H\xi_\phi = \int_{s_-}^{s_0} \big(\xi_\phi^T \mathfrak{w}_\Delta^{x_0}\big)^* h(t)\xi_\phi^T \xi_\phi dt$$

$$= \big(\xi_\phi^T \mathfrak{w}_\Delta^{x_0}(s_0)\big)^* \int_{s_-}^{s_0} h(t)\xi_\phi^T \xi_\phi dt = \mathfrak{w}_\Delta^{x_0}(s_-)^* \xi_\phi \cdot l$$

$$= \mathfrak{w}_\Delta^{x_0}(s_-)_2 \sin\phi \cdot l = \mathfrak{w}_\Delta^{x_0}(s_-)_2 f(s_-)_1 \,.$$

Thus also (iii) holds, and we conclude that $(F;G) \in \overset{\circ\circ}{T}_{x_0}(\mathfrak{h})$.

In order to compute boundary values, assume that $(F;G) \in \overset{\circ\circ}{T}_{x_0}(\mathfrak{h})$. Then $(\tilde{F};\tilde{G}) \in \mathring{T}_{x_0}(\mathfrak{h})$, and by Proposition 2.17 and the definition of $\overset{\circ\circ}{\Gamma}_{x_0}(\mathfrak{h})$ thus

$$\overset{\circ\circ}{\Gamma}_{x_0}(\mathfrak{h})(F;G) = \mathring{\Gamma}_{x_0}(\mathfrak{h})(\tilde{F};\tilde{G})$$

$$= \begin{cases} \tilde{f}(s_-) + \sum_{l=0}^{\Delta-1} \lambda_l \mathfrak{w}_l^{x_0}(s_-), & (s_-,\sigma) \text{ not indivisible,} \\ \begin{pmatrix} \eta_0 + \tilde{f}(x_0)_1 + \frac{1}{2}\sum_{l=0}^{\Delta-1} \mu_l d_l^{x_0} \\ \lambda_0 \end{pmatrix}, & (s_-,\sigma) \text{ indivisible,} \end{cases}$$

$$= \begin{cases} f(s_-), & (s_-,\sigma) \text{ not indivisible,} \\ \begin{pmatrix} \eta_0 + f(x_0)_1 + \frac{1}{2}\sum_{l=0}^{\Delta-1} \mu_l d_l^{x_0} \\ \lambda_0 \end{pmatrix}, & (s_-,\sigma) \text{ indivisible.} \end{cases}$$

❑

As a consequence of Theorem 2.15 we obtain a description of $\overset{\circ\circ}{T}_{x_0}(\mathfrak{h})$ in terms of its domain and action.

2.20 Corollary. *Let $\mathfrak{h}$ be a general Hamiltonian (of the form 1.3) and $x_0 \in I_+$. Moreover, let $F = (f;\xi,\alpha) \in \overset{\circ\circ}{\mathfrak{P}}_{x_0}(\mathfrak{h})$. Then $F \in \operatorname{dom} \overset{\circ\circ}{T}_{x_0}(\mathfrak{h})$ if and only if in case $\ddot{o} = 0$ the condition '$(i) \wedge (ii_{\ddot{o}=0})$', and in case $\ddot{o} > 0$ the condition '$(i) \wedge (ii_{\ddot{o}>0})$' holds.*

(i) *$f \in \operatorname{dom} T_{\Delta,\max}(H)$.*

$(ii_{\ddot{o}=0})$ *Under the assumption that f satisfies (i), let $\gamma_0, \ldots, \gamma_\Delta \in \mathbb{C}$ be the unique constants such that*

$$\hat{f} := \Big(f - \sum_{l=0}^{\Delta} \gamma_l \mathfrak{w}_l^{x_0}\Big)\chi_{\rceil x_0} \in \operatorname{dom} T_{\max}(H_{\rceil x_0})$$

and set

$$L_+ := \lim_{x\searrow\sigma} \Big[\mathfrak{w}_\Delta^{x_0}(x)^* J\hat{f}(x) + \int_x^{x_0} (\mathfrak{w}_{\Delta-1}^{x_0})^* H\hat{f}\Big],$$

$$L_- := \lim_{x\nearrow\sigma} \Big[-\mathfrak{w}_\Delta^{x_0}(x)^* J\hat{f}(x) + \int_{s_-}^{x} (\mathfrak{w}_{\Delta-1}^{x_0})^* H\hat{f}\Big].$$

Then

$$\xi_{\Delta-1} = L_+ + L_- + \frac{1}{2}\sum_{l=0}^{\Delta-1} \gamma_l d^{x_0}_{l+\Delta-1}\gamma_\Delta d^{x_0}_{2\Delta-1}\,.$$

$(ii_{\ddot{o}>0})$ *We have* $\alpha_{\ddot{o}} = 0$.

2.21 Remark. If $F \in \operatorname{dom}\overset{\circ\circ}{T}_{x_0}(\mathfrak{h})$ and $(F;G) \in \overset{\circ\circ}{T}_{x_0}(\mathfrak{h})$ with $G = (g;\eta,\beta)$, then the numbers η,β can be computed immediately by solving the equations in Theorem 2.15, (ii)–(v).

Hence, the action of $\overset{\circ\circ}{T}_{x_0}(\mathfrak{h})$ is easily understood, provided the action of the differential operator $T_{\Delta,\max}(H)$ in the function space $L^2_\Delta(H)$ is. This operator, in turn, is explicitly related to $T_{\max}(H)$ via Lemma 2.8. Altogether, we see that $\overset{\circ\circ}{T}_{x_0}(\mathfrak{h})$ is a finite dimensional perturbation of $T_{\max}(H)$ which is given in a very explicit way. //

Proof (of Corollary 2.20). The case that $\ddot{o} > 0$ is easily settled. If $F \in \operatorname{dom}\overset{\circ\circ}{T}_{x_0}(\mathfrak{h})$, then by Theorem 2.15 $(i),(v)$, we have $f \in \operatorname{dom} T_{\Delta,\max}(H)$ and $\alpha_{\ddot{o}} = 0$. Conversely, if F satisfies the present conditions (i) and $(ii_{\ddot{o}=0})$, then we can first choose $g \in L^2_\Delta(H)$ with $(f;g) \in T_{\Delta,\max}(H)$ and then choose η,β such that Theorem 2.15 (ii)–(v) hold.

Assume for the rest of the proof that $\ddot{o} = 0$. Let $f \in \operatorname{dom} T_{\Delta,\max}(H)$ be given. Choose $g \in L^2_\Delta(H)$ such that $(f;g) \in T_{\Delta,\max}(H)$ and let $\mu_0,\dots,\mu_{\Delta-1}$ be the unique constants such that

$$\hat{g} := \Big(g - \sum_{l=0}^{\Delta-1} \mu_l \mathfrak{w}^{x_0}_l\Big)\chi_{\urcorner x_0} \in L^2(H).$$

First of all, let us verify that the limits $L_\pm$ in $(ii_{\ddot{o}=0})$ do exist. If $x \in (\sigma, x_0)$, Green's identity in $L^2(H_{\urcorner x_0})$ yields

$$\int_x^{x_0} (\mathfrak{w}^{x_0}_\Delta)^* H\hat{g} - \int_x^{x_0} (\mathfrak{w}^{x_0}_{\Delta-1})^* H\hat{f} = \mathfrak{w}^{x_0}_\Delta(x)^* J\hat{f}(x) - \mathfrak{w}^{x_0}_\Delta(x_0)^* J\hat{f}(x_0)\,.$$

Since $\mathfrak{w}_\Delta$ and $\hat{g}$ both belong to $L^2(H|_{(\sigma,x_0)})$, we may pass to the limit $x \searrow \sigma$ to obtain

$$\begin{aligned} L_+ &= \lim_{x\searrow\sigma}\Big[\mathfrak{w}^{x_0}_\Delta(x)^* J\hat{f}(x) + \int_x^{x_0} (\mathfrak{w}^{x_0}_{\Delta-1})^* H\hat{f}\Big] \\ &= \int_\sigma^{x_0} (\mathfrak{w}^{x_0}_\Delta)^* H\hat{g} + \mathfrak{w}^{x_0}_\Delta(x_0)^* J\hat{f}(x_0). \end{aligned}$$

In the same way, we obtain that

$$L_- = \int_{s_-}^{\sigma} (\mathfrak{w}^{x_0}_\Delta)^* H\hat{g} - \mathfrak{w}^{x_0}_\Delta(s_-)^* J\hat{f}(s_-).$$

Since $\mathfrak{w}^{x_0}_l(x_0), \mathfrak{w}^{x_0}_l(x_0) \in \operatorname{span}\{\binom{0}{1}\}$, we have

$$\mathfrak{w}^{x_0}_\Delta(x_0)^* J\hat{f}(x_0) = \omega^{x_0}_\Delta f(x_0)_1, \quad \mathfrak{w}^{x_0}_\Delta(s_-)^* J\hat{f}(s_-) = \mathfrak{w}^{x_0}_\Delta(s_-)_2 f(s_-)_1\,.$$

Moreover, if (s_-, σ) is indivisible, we have $\mathfrak{w}_\Delta^{x_0}(s_-) = 0$. Remembering (2.5), we see that under the assumption of (i), statement $(ii_{\ddot{o}=0})$ is equivalent to Theorem 2.15 (iii).

Assume that F satisfies the present conditions (i) and $(ii_{\ddot{o}=0})$. Then we can choose $g \in L^2_\Delta(H)$ with $(f; g) \in T_{\Delta,\max}(H)$ and Theorem 2.15 (iii). Clearly, η can be chosen such that Theorem 2.15 (ii) and (iv) hold. Thus $(F; (g; \eta)) \in \overset{\circ\circ}{T}_{x_0}(\mathfrak{h})$, and we have $F \in \operatorname{dom} \overset{\circ\circ}{T}_{x_0}(\mathfrak{h})$.

Conversely, if there exists $G \in \overset{\circ\circ}{\mathfrak{P}}_{x_0}$ with $(F; G) \in \overset{\circ\circ}{T}_{x_0}$, then Theorem 2.15 (i) and (iii), immediately give the present conditions (i) and $(ii_{\ddot{o}=0})$. ❑

3 An x_0-independent form of the model

The Pontryagin space $\mathring{\mathfrak{P}}_{x_0}(\mathfrak{h})$ underlying the boundary triple $\mathring{\mathfrak{B}}_{x_0}(\mathfrak{h})$ does not depend on the particular choice of x_0. However, as it is seen from Proposition 2.17, the relations $\mathring{T}_{x_0}(\mathfrak{h})$ and $\mathring{\Gamma}_{x_0}(\mathfrak{h})$ do in general depend on x_0.

A similar remark applies to the boundary triple $\overset{\circ\circ}{\mathfrak{B}}_{x_0}(\mathfrak{h})$. Due to Lemma IV.3.12 (ii), the linear space underlying the Pontryagin space $\overset{\circ\circ}{\mathfrak{P}}_{x_0}(\mathfrak{h})$ does not depend on x_0. Moreover, due to the existence of the isometric isomorphism $\mathring{\iota}_{x_2} \circ \mathring{\iota}_{x_1}^{-1} : \overset{\circ\circ}{\mathfrak{P}}_{x_1}(\mathfrak{h}) \to \overset{\circ\circ}{\mathfrak{P}}_{x_2}(\mathfrak{h})$, and the fact that $\overset{\circ\circ}{\mathfrak{P}}_{x_1}(\mathfrak{h})$ and $\overset{\circ\circ}{\mathfrak{P}}_{x_2}(\mathfrak{h})$ are both Pontryagin spaces, the topology on this linear space induced by the inner product $[\cdot,\cdot]_{x_0}$ does not depend on x_0. However, the inner product $[\cdot,\cdot]_{x_0}$ on $\overset{\circ\circ}{\mathfrak{P}}_{x_0}(\mathfrak{h})$ and the relations $\overset{\circ\circ}{T}_{x_0}(\mathfrak{h})$ and $\overset{\circ\circ}{\Gamma}_{x_0}(\mathfrak{h})$ do in general depend on x_0.

One idea how to remove the dependence on x_0 is the following: the point x_0 is the point where the given general Hamiltonian is cut into two pieces, and the model is then obtained by pasting the two corresponding models. Is it possible to shift this cutting point to s_+ and thus completely avoid cutting and pasting? Our aim in this section is to show that under certain assumptions on the asymptotics of H, limiting its growth towards s_+, the answer is 'yes'. To this end, we have to study the dependence on x_0 in some more detail. A big portion of technically complicated computations has been shifted to the appendix, so that in this section we can concentrate on the essential ideas.

3.1 Remark. If $\mathfrak{h}$ ends with an indivisible interval towards s_+, i.e. $x_{\max} := \sup I_+ < s_+$ (for the definition of I_+ see §2.1 (5)), then it is of course impossible to shift the cutting point to s_+. However, in this case, the boundary triple $\mathfrak{B}(\mathfrak{h})$ can be described without cutting/pasting anyway. Namely, we have $\mathfrak{P}(\mathfrak{h}) = \mathfrak{P}(\mathfrak{h}_{x_{\max}})$, $T(\mathfrak{h})$ is a certain one-dimensional restriction of $T(\mathfrak{h}_{x_{\max}})$ depending on the type of the indivisible interval $(x_{\max}, s_+)$, and $\Gamma(\mathfrak{h})$ is the restriction of $\Gamma(\mathfrak{h}_{x_{\max}})$ to $T(\mathfrak{h})$. //

In view of this remark we will, throughout this section, assume that $\mathfrak{h}$ does not end with an indivisible interval towards s_+. Limits $x \to s_+$ will be understood such that x tends to s_+ inside I_+.

Let $x_1, x_2 \in I_+$, $x_1 < x_2$, and let $\mathfrak{h}_{x_1}$ and $\mathfrak{h}_{x_2}$ be the elementary indefinite Hamiltonians as in Lemma 2.5. We know from [KW2, §7] (cf. also §2.2, §2.3, and Remark 2.4) that there exists an isometric isomorphism

$$\kappa_{x_1,x_2} : \mathfrak{P}(\mathfrak{h}_{x_1}) \,[\dot{+}]\, L^2(H|_{(x_1,x_2)}) \to \mathfrak{P}(\mathfrak{h}_{x_2}),$$

such that $(\kappa_{x_1,x_2}, \mathrm{id}_{\mathbb{C}^4})$ is an isomorphism of the corresponding boundary triples. This map naturally lifts to an isomorphism of $\mathfrak{P}_{x_1}(\mathfrak{h}) = \mathfrak{P}(\mathfrak{h}_{x_1})\,[\dot{+}]\,L^2(H_{x_1\ulcorner})$ onto $\mathfrak{P}_{x_2}(\mathfrak{h}) = \mathfrak{P}(\mathfrak{h}_{x_2})\,[\dot{+}]\,L^2(H_{x_2\ulcorner})$ which will again be denoted by κ_{x_1,x_2}, namely by

$$F \dot{+} f \;\mapsto\; \kappa_{x_1,x_2}(F \dot{+} f\chi_{(x_1,x_2)}) \dot{+} f\chi_{x_2\ulcorner}, \quad F \in \mathfrak{P}(\mathfrak{h}_{x_1}),\ f \in L^2(H_{x_1\ulcorner}).$$

Clearly, $(\kappa_{x_1,x_2}, \mathrm{id}_{\mathbb{C}^4})$ will again be an isomorphism of the corresponding boundary triples $\mathfrak{B}_{x_1}(\mathfrak{h})$ and $\mathfrak{B}_{x_2}(\mathfrak{h})$.

The isomorphisms ι_{x_1}, ι_{x_2} and $\mathring{\iota}_{x_1}, \mathring{\iota}_{x_2}$ can be used to transport the map κ_{x_1,x_2}: let $\mathring{\kappa}_{x_1,x_2} : \mathring{\mathfrak{P}}_{x_1}(\mathfrak{h}) \to \mathring{\mathfrak{P}}_{x_2}(\mathfrak{h})$ and $\overset{\circ\circ}{\kappa}_{x_1,x_2} : \overset{\circ\circ}{\mathfrak{P}}_{x_1}(\mathfrak{h}) \to \overset{\circ\circ}{\mathfrak{P}}_{x_2}(\mathfrak{h})$ be the isometric isomorphisms defined by

$$\begin{array}{ccc}
\mathfrak{P}_{x_1}(\mathfrak{h}) & \xrightarrow{\kappa_{x_1,x_2}} & \mathfrak{P}_{x_2}(\mathfrak{h}) \\
\downarrow{\scriptstyle \iota_{x_1}} & & \downarrow{\scriptstyle \iota_{x_2}} \\
\mathring{\mathfrak{P}}_{x_1}(\mathfrak{h}) & \xrightarrow[\mathring{\kappa}_{x_1,x_2}]{} & \mathring{\mathfrak{P}}_{x_2}(\mathfrak{h}) \\
\downarrow{\scriptstyle \mathring{\iota}_{x_1}} & & \downarrow{\scriptstyle \mathring{\iota}_{x_2}} \\
\overset{\circ\circ}{\mathfrak{P}}_{x_1}(\mathfrak{h}) & \xrightarrow[\overset{\circ\circ}{\kappa}_{x_1,x_2}]{} & \overset{\circ\circ}{\mathfrak{P}}_{x_2}(\mathfrak{h})
\end{array} \tag{3.1}$$

Note that $\mathring{\kappa}_{x_1,x_2}$ is just the natural extension of the map that is also named $\mathring{\kappa}_{x_1,x_2}$ and considered in the appendix, cf. (A.1), to a map

$$\begin{aligned}
\mathring{\mathfrak{P}}_{x_1}(\mathfrak{h}) = \big(L^2(H_{\urcorner x_1})\,[\dot{+}]\,(\mathbb{C}^\Delta \dot{+} \mathbb{C}^\Delta)\,[\dot{+}]\,\mathbb{C}^{\ddot{o}}\big)\,[\dot{+}]\,\big(L^2(H_{(x_1,x_2)})\,[\dot{+}]\,L^2(H_{x_2\ulcorner})\big) \\
\to\ \mathring{\mathfrak{P}}_{x_2}(\mathfrak{h}) = \big(L^2(H_{\urcorner x_2})\,[\dot{+}]\,(\mathbb{C}^\Delta \dot{+} \mathbb{C}^\Delta)\,[\dot{+}]\,\mathbb{C}^{\ddot{o}}\big)\,[\dot{+}]\,L^2(H_{x_2\ulcorner}).
\end{aligned}$$

Using Proposition A.6, it follows that the action of the map $\mathring{\kappa}_{x_1,x_2}$ defined by (3.1) is given by linearity and the formulae

$$\begin{aligned}
&\mathring{\kappa}_{x_1,x_2}(f;0,0,0) = \Big(f;\big(\int_{x_1}^{x_2}(\mathfrak{w}_j^{x_2})^* Hf + \int_\sigma^{x_1}(\mathfrak{w}_j^{x_2} - \mathfrak{w}_j^{x_1})^* Hf\big)_{j=0}^{\Delta-1}, 0, 0\Big), \\
&\mathring{\kappa}_{x_1,x_2}(0;\xi,0,\alpha) = (0;\xi,0,\alpha), \\
&\mathring{\kappa}_{x_1,x_2}(0;0,\varepsilon_k,0) = \Big(-\mathfrak{w}_k^{x_2}\chi_{(x_1,x_2)} - (\mathfrak{w}_k^{x_2} - \mathfrak{w}_k^{x_1})\chi_{\urcorner x_1};(h_{kj}^{x_1,x_2})_{j=0}^{\Delta-1}, \varepsilon_k, 0\Big),
\end{aligned}$$

where we set

$$h_{kj}^{x_1,x_2} := -\frac{1}{2}\sum_{l=j+1}^{k+j+1} \omega_{k+j+1-l}^{x_1}\mathfrak{w}_l^{x_2}(x_1)_1 + \frac{1}{2}\sum_{l=1}^{j} \omega_{k+j+1-l}^{x_1}\mathfrak{w}_l^{x_2}(x_1)_1\,.$$

Using these formulae, we can easily deduce how $\overset{\circ\circ}{\kappa}_{x_1,x_2}$ acts.

3.2 Lemma. *Let $x_1, x_2 \in I_+$, $x_1 < x_2$, and let $F = (f;\xi,\alpha) \in \overset{\circ\circ}{\mathfrak{P}}_{x_1}(\mathfrak{h})$, let λ be the unique coefficients such that*

$$f - \sum_{l=0}^{\Delta-1} \lambda_l \mathfrak{w}_l^{x_1}\chi_{\urcorner x_1} \in L^2(H)\,.$$

Then

$$\mathring{\kappa}_{x_1,x_2}F = \Big(f;\big(\xi_j + \int_{x_1}^{x_2}(\mathfrak{w}_j^{x_2})^* Hf + \int_{\sigma}^{x_1}(\mathfrak{w}_j^{x_2}-\mathfrak{w}_j^{x_1})^* H\big(f-\sum_{l=0}^{\Delta-1}\lambda_l\mathfrak{w}_l^{x_1}\big) + \sum_{l=0}^{\Delta-1}\lambda_l h_{lj}^{x_1,x_2}\big)_{j=0}^{\Delta-1},\alpha\Big)$$

Proof. Let an element $(f;\xi,\alpha)\in\mathring{\mathfrak{P}}_{x_1}(\mathfrak{h})$ be given. Then

$$\mathring{\iota}_{x_1}^{-1}(f;\xi,\alpha) = \big(f-\sum_{l=0}^{\Delta-1}\lambda_l\mathfrak{w}_l^{x_1}\chi_{\urcorner x_1};\xi,\lambda,\alpha\big)\,.$$

Hence, we obtain

$$\begin{aligned}&(\mathring{\kappa}_{x_1,x_2}\circ\mathring{\iota}_{x_1}^{-1})(f;\xi,\alpha)\\ &=\Big(\underbrace{\big(f-\sum_{l=0}^{\Delta-1}\lambda_l\mathfrak{w}_l^{x_1}\chi_{\urcorner x_1}\big)-\sum_{l=0}^{\Delta-1}\lambda_l\big(\mathfrak{w}_l^{x_2}\chi_{(x_1,x_2)}+(\mathfrak{w}_l^{x_2}-\mathfrak{w}_l^{x_1})\chi_{\urcorner x_1}\big)}_{=f-\sum_{l=0}^{\Delta-1}\lambda_l\mathfrak{w}_l^{x_2}\chi_{\urcorner x_2}};\\ &\Big(\int_{x_1}^{x_2}(\mathfrak{w}_j^{x_2})^*Hf+\int_{\sigma}^{x_1}(\mathfrak{w}_j^{x_2}-\mathfrak{w}_j^{x_1})^*H\big(f-\sum_{l=0}^{\Delta-1}\lambda_l\mathfrak{w}_l^{x_1}\big)+\xi_j+\sum_{l=0}^{\Delta-1}\lambda_l h_{lj}^{x_1,x_2}\Big)_{j=0}^{\Delta-1},\\ &\lambda,\alpha\Big).\end{aligned}$$

Applying $\mathring{\iota}_{x_2}$ we obtain the desired formula. ❑

a. Asymptotic conditions on H; the elements $\mathfrak{v}_k$.

3.3. Asymptotic conditions on H. Let H be a Hamiltonian of the form 2.6 with $\sup I_+ = s_+$, and choose a base point $x_0\in I_+$. For $N,M\in\mathbb{N}_0$ we consider the following conditions:

($\mathbf{A}_N$) The limits

$$\lim_{x\to s_+}\mathfrak{w}_k^x\chi_{(x_0,x)},\quad \lim_{x\to s_+}(\mathfrak{w}_k^x-\mathfrak{w}_k^{x_0})\chi_{\urcorner x_0},\qquad k=0,\dots,N,$$

exist in the norm of $L^2(H)$.

($\mathbf{B}_M$) The limits

$$v_k^{x_0}:=\lim_{x\to s_+}\mathfrak{w}_k^x(x_0)_1,\quad k=1,\dots,M,$$

exist in $\mathbb{R}$.

//

For $k\geq\Delta$ the existence of the two limits in (A_N) is equivalent to the existence of the limit $\lim_{x\to s_+}\mathfrak{w}_k^x\chi_{(\sigma,x)}$. However, for $k<\Delta$, it is necessary to introduce the splitting point x_0, since the function $\mathfrak{w}_k^x\chi_{(\sigma,x_0)}$ does not belong to $L^2(H)$ in general.

Note that since $\mathfrak{w}_0^x = \binom{0}{1}$, the limit

$$v_0^{x_0} := \lim_{x \to s_+} \mathfrak{w}_0^x(x_0)_1$$

always trivially exists and is equal to 0. Hence (B_0) is always satisfied.

If H satisfies the condition (A_N), we set

$$\mathfrak{v}_k := \Big(\mathfrak{w}_k^{x_0}\chi_{\upharpoonleft x_0} + \lim_{x \to s_+} (\mathfrak{w}_k^x - \mathfrak{w}_k^{x_0})\chi_{\upharpoonleft x_0}\Big) + \lim_{x \to s_+} \mathfrak{w}_k^x \chi_{(x_0,x)}, \quad k = 0, \dots, N.$$

Since on the interval (s_-, σ) the function $\mathfrak{w}_k^x$ does not depend on x at all, we have $\mathfrak{v}_k(t) := \mathfrak{w}_k^x(t)$, $t \in (s_-, \sigma)$, $x \in (\sigma, s_+)$.

First of all we shall justify these notions by showing that they do not depend on the choice of the base point x_0.

3.4 Lemma. *The validity of the conditions* (A_N) *or* (B_M) *does not depend on the choice of the base point* x_0. *Also the actual value of* $\mathfrak{v}_k$ *is independent of* x_0, *in fact*

$$\lim_{x \to s_+} (\mathfrak{v}_k - \mathfrak{w}_k^x \chi_{\upharpoonleft x}) = 0, \quad k = 0, \dots, N, \tag{3.2}$$

in the norm of $L^2(H)$ *if* (A_N) *satisfied.*

Conversely, if there exist functions $\mathfrak{v}_k$, $k = 0, \dots, N$, *such that* $\mathfrak{v}_k - \mathfrak{w}_k^x \chi_{\upharpoonleft x} \in L^2(H)$ *for every* $x \in (\sigma, s_+)$ *and* (3.2) *is valid, then condition* (A_N) *is satisfied.*

Proof. Let $x_0, x_1 \in I_+$, and assume that (A_N) holds with the choice of x_0 as a base point. Consider the case when $x_1 > x_0$. The existence of the limit $\lim_{x \to s_+} \mathfrak{w}_k^x \chi_{(x_0,x)}$ implies the existence of the limits

$$\lim_{x \to s_+} \mathfrak{w}_k^x \chi_{(x_0,x_1)}, \ \lim_{x \to s_+} \mathfrak{w}_k^x \chi_{(x_1,x)} \,.$$

This implies that also the limit

$$\lim_{x \to s_+} (\mathfrak{w}_k^x - \mathfrak{w}_k^{x_1})\chi_{\upharpoonleft x_1} = \lim_{x \to s_+} (\mathfrak{w}_k^x - \mathfrak{w}_k^{x_0})\chi_{\upharpoonleft x_0} + \mathfrak{w}_k^{x_0}\chi_{\upharpoonleft x_0} - \mathfrak{w}_k^{x_1}\chi_{\upharpoonleft x_1} + \lim_{x \to s_+} \mathfrak{w}_k^x \chi_{(x_0,x_1)}$$

exists. The case $x_1 < x_0$ is treated in a completely similar way; in either case (A_N) also holds with the choice of x_1 as a base point.

In order to show (3.2), let us denote by $\mathfrak{v}_k$ the functions constructed with base point x_0. Then, for arbitrary $x \in I_+$ with $x > x_0$, we have

$$\mathfrak{v}_k - \mathfrak{w}_k^x \chi_{\upharpoonleft x} = (\mathfrak{w}_k^{x_0} - \mathfrak{w}_k^x)\chi_{\upharpoonleft x_0} + \lim_{t \to s_+} (\mathfrak{w}_k^t - \mathfrak{w}_k^{x_0})\chi_{\upharpoonleft x_0} + \lim_{t \to s_+} \mathfrak{w}_k^t \chi_{(x_0,t)} - \mathfrak{w}_k^x \chi_{(x_0,x)} .$$

It follows that $\mathfrak{v}_k - \mathfrak{w}_k^x \chi_{\upharpoonleft x} \in L^2(H)$. Passing to the limit $x \to s_+$ gives (3.2).

Let $x_0, x_1 \in I_+$, and assume that (B_M) holds with the choice of x_0 as a base point. Choose arbitrary real numbers $d_k^{x_0}$, $k = 0, \dots, 2\Delta - 1$, and set

$$d_k^{x_0} := \int_{s_-}^{x_0} (\mathfrak{w}_\Delta^{x_0})^* H \mathfrak{w}_{k-\Delta}^{x_0}, \quad k \geq 2\Delta \,.$$

Moreover, set $\ddot{o} := 0$, $b_1 := 0$. Then the data $\mathfrak{h}_{x_0} := (H_{\upharpoonleft x_0}; \ddot{o}, b_j; d_k^{x_0})$ is an elementary indefinite Hamiltonian of kind (A) defined on $[s_-, \sigma) \cup (\sigma, x_0]$.

Let $x \in I_+$ with $x > \max\{x_0, x_1\}$ be given. Let d_k^x, $k = 0, \dots, 2\Delta - 1$, be the parameters of the elementary indefinite Hamiltonian $\mathfrak{h}_x$ obtained in §2.3 when pasting $\mathfrak{h}_{x_0}$ with $H|_{(x_0,x)}$, and let $d_k^{x_1}$, $k=0,\dots,2\Delta-1$, be the parameters of the elementary indefinite Hamiltonian $\mathfrak{h}_{x_1}$ obtained in §2.2 when splitting $\mathfrak{h}_x$ at the point x_1. Then, by Proposition A.6 and Proposition IV.5.17, we have

$$d_k^{x_0}+\sum_{j=0}^{k+1}\omega_{k+1-j}^{x_0}\mathfrak{w}_j^x(x_0)_1=d_k^x=d_k^{x_1}+\sum_{j=0}^{k+1}\omega_{k+1-j}^{x_1}\mathfrak{w}_j^x(x_1)_1 \tag{3.3}$$

for all $k\in\mathbb{N}_0$. Moreover, again by Proposition A.6 and Proposition IV.5.17, the numbers $d_k^{x_0}$ and $d_k^{x_1}$ are in case $x_0<x_1$ related as

$$d_k^{x_1}=d_k^{x_0}+\sum_{j=0}^{k+1}\omega_{k+1-j}^{x_0}\mathfrak{w}_j^{x_1}(x_0)_1\,.$$

If $x_0>x_1$, we apply (A.2) with the elements

$$0,\mathfrak{w}_0^{x_0},\dots,\mathfrak{w}_{k+1}^{x_0};\ 0,\mathfrak{w}_0^{x_1},\dots,\mathfrak{w}_{k+1}^{x_1}\in L^2(H|_{(x_1,x_0)})$$

to conclude that the same formula holds true. We see that the numbers $d_k^{x_1}$, which were originally constructed via the point x, actually do not depend on x. Hence, we may pass to the limit $x\to s_+$ in (3.3). Since $\omega_0^{x_1}=1$, it follows inductively that the limits

$$v_j^{x_1}=\lim_{x\to s_+}\mathfrak{w}_j^x(x_1)_1,\quad j=1,\dots,M\,,$$

exist, i.e. (B_M) holds with the choice of x_1 as a base point.

For the last assertion of the lemma, observe that the first condition in (A_N) follows immediately from the relation

$$\lim_{x\to s_+}(\mathfrak{v}_k-\mathfrak{w}_k^x\chi_{\urcorner x})\chi_{x_0\ulcorner}=0.$$

For the second condition in (A_N), note that

$$(\mathfrak{w}_k^x-\mathfrak{w}_k^{x_0})\chi_{\urcorner x_0}=(\mathfrak{w}_k^x-\mathfrak{v}_k)\chi_{\urcorner x_0}-(\mathfrak{w}_k^{x_0}-\mathfrak{v}_k)\chi_{\urcorner x_0}.$$

Both terms are in $L^2(H)$ for $x\geq x_0$; the second one is independent of x. The convergence of the first term follows from (3.2). ❑

3.5 Remark. Assume that, in addition to a Hamiltonian H of the form 2.6, also a point $x_0\in I_+$ and real numbers $d_k^{x_0}$, $k=0,\dots,2\Delta-1$, are given. Again we set

$$d_k^{x_0}:=\int_{s_-}^{x_0}(\mathfrak{w}_\Delta^{x_0})^*H\mathfrak{w}_{k-\Delta}^{x_0},\quad k\geq 2\Delta.$$

For $x\in I_+$ let numbers d_k^x be defined as

$$d_k^x:=d_k^{x_0}+\sum_{j=0}^{k+1}\omega_{k+1-j}^{x_0}\mathfrak{w}_j^x(x_0)_1,\quad k\geq 0.$$

Then H satisfies (B_M) if and only if the limits

$$D_k:=\lim_{x\to s_+}d_k^x,\quad k=0,\dots,M-1, \tag{3.4}$$

exist in $\mathbb{R}$.

This remark becomes interesting if we remember that the numbers d_k^x, $k = 0, \dots, 2\Delta - 1$, are exactly the parameters of the elementary indefinite Hamiltonian obtained in Lemma 2.5 when using x as cutting point. //

We trivially have '(A_N)⇒(A_{N-1})' and '(B_M)⇒(B_{M-1})'. It is more interesting to note that a condition 'type A' implies a condition 'type B'.

3.6 Lemma. *Let H be a Hamiltonian of the form 2.6 with $\sup I_+ = s_+$. If H satisfies the condition (A_N), then also (B_{N+1}).*

Proof. Let $1 \leq k \leq N+1$ be given. We apply the abstract Green's identity with the elements

$$(\mathfrak{w}_k^x; \mathfrak{w}_{k-1}^x), (\begin{pmatrix} 0 \\ 1 \end{pmatrix}; 0) \in T_{\max}(H|_{(x_0,x)}) .$$

This gives

$$\int_{x_0}^{s_+} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* H \mathfrak{w}_{k-1}^x \chi_{\Lsh x} = \int_{x_0}^{x} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* H \mathfrak{w}_{k-1}^x$$
$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* J \mathfrak{w}_k^x(x_0) - \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* J \mathfrak{w}_k^x(x) = \mathfrak{w}_k^x(x_0)_1 ,$$

and hence

$$v_k^{x_0} = \lim_{x \to s_+} \mathfrak{w}_k^x(x_0)_1 = \int_{x_0}^{s_+} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* H \mathfrak{v}_{k-1} .$$

❑

Let us collect some properties of the functions $\mathfrak{v}_k$.

3.7 Lemma. *Let H be a Hamiltonian of the form 2.6 with $\sup I_+ = s_+$, and assume that H satisfies (A_N). For notational convenience, set $\mathfrak{v}_{-1} := 0$.*

(*i*) *For each $x_0 \in (\sigma, s_+)$ we have*

$$\big(\mathfrak{v}_k \chi_{x_0 \Rsh}; \mathfrak{v}_{k-1} \chi_{x_0 \Rsh}\big) \in T_{\max}(H_{x_0 \Rsh}), \quad k = 0, \dots, N.$$

(*ii*) *For each $x_0, x_1 \in (\sigma, s_+)$ we have*

$$\big((\mathfrak{v}_k - \mathfrak{w}_k^{x_1})\chi_{\Lsh x_0}; (\mathfrak{v}_{k-1} - \mathfrak{w}_{k-1}^{x_1})\chi_{\Lsh x_0}\big) \in T_{\max}(H_{\Lsh x_0}), \quad k = 0, \dots, N.$$

Due to the above items there exist unique absolutely continuous representatives of $\mathfrak{v}_k$, $0 \leq k \leq N$, which will again be denoted by $\mathfrak{v}_k$, such that $\mathfrak{v}_{k+1}'(t) = JH\mathfrak{v}_k(t)$, $t \in (s_-, \sigma) \cup (\sigma, s_+)$.

(*iii*) *For $k > N$, there exist absolutely continuous functions $\mathfrak{v}_k$ on $(s_-, \sigma) \cup (\sigma, s_+)$ such that the assertion made in (ii) holds for all $k \geq 0$.*

The following limit relations hold:

(*iv*) *For each $x_0 \in (\sigma, s_+)$ we have*

$$\lim_{x \to s_+} \mathfrak{w}_k^x(x_0) = \mathfrak{v}_k(x_0), \quad k = 0, \dots, N.$$

(*v*) *We have*
$$\lim_{x\to s_+} \mathfrak{w}_l^x(x)^* J\mathfrak{v}_k(x) = 0, \quad 0 \le l,k \le N.$$

Proof. By their definition the functions $\mathfrak{v}_k\chi_{x_0\upharpoonright}$ and $(\mathfrak{v}_k - \mathfrak{w}_k^{x_1})\chi_{\urcorner x_0}$, $k = 0,\dots,N$, belong to $L^2(H_{x_0\upharpoonright})$ or $L^2(H_{\urcorner x_0})$, respectively. For the proof of (*i*) and (*ii*), it is therefore enough to show that for each interval (a,b) with $[a,b] \subseteq (\sigma, s_+)$,

$$\big(\mathfrak{v}_k\chi_{(a,b)}; \mathfrak{v}_{k-1}\chi_{(a,b)}\big) \in T_{\max}(H_{(a,b)})\,. \tag{3.5}$$

However, for each $x \in I_+$, we have $\big(\mathfrak{w}_k^x\chi_{(a,b)}; \mathfrak{w}_{k-1}^x\chi_{(a,b)}\big) \in T_{\max}(H_{(a,b)})$, and hence (3.5) follows from (3.2).

Next it is easy to construct functions $\mathfrak{v}_k$, $k > N$, with the desired properties. To this end, fix $x \in I_+$, and remember that for each function $f \in L^2(H|_{(\sigma,x)})$ there exists a unique constant $\alpha(f)$ such that

$$(Bf)(t) := \int_x^t JHf + \begin{pmatrix} 0 \\ \alpha(f) \end{pmatrix} \in L^2(H|_{(\sigma,x)}),$$

cf. [KW2, §2.b]. On (σ, s_+) we define functions $\mathfrak{v}_k$, $k > N$, inductively by

$$\mathfrak{v}_k := B(\mathfrak{v}_{k-1} - \mathfrak{w}_{k-1}^x) + \mathfrak{w}_k^x, \quad k > N.$$

On (s_-, σ) we set $\mathfrak{v}_k := \mathfrak{w}_k^x$, $k > N$; note that on (s_-, σ), $\mathfrak{w}_k^x$ does not depend on x. Then, clearly, $(\mathfrak{v}_k - \mathfrak{w}_k^x)' = JH(\mathfrak{v}_{k-1} - \mathfrak{w}_{k-1}^x)$ and $(\mathfrak{v}_k - \mathfrak{w}_k^x) \in L^2(H|_{(\sigma,x)})$. The asserted properties are now immediate from the known properties of $\mathfrak{w}_k^x$.

We come to the proof of (*iv*). Set $a := x_0$ and choose $b \in (x_0, s_+)$ such that (a,b) is not indivisible. Then the boundary relation $\Gamma(H|_{(a,b)})$ is a continuous operator from $T_{\max}(H|_{(a,b)})$ onto $\mathbb{C}^2 \times \mathbb{C}^2$. Hence

$$\begin{aligned}\big(\mathfrak{v}_k(a); \mathfrak{v}_k(b)\big) &= \Gamma(H|_{(a,b)})\big(\mathfrak{v}_k\chi_{(a,b)}; \mathfrak{v}_{k-1}\chi_{(a,b)}\big)\\ &= \Gamma(H|_{(a,b)})\big(\lim_{x\to s_+}(\mathfrak{w}_k^x\chi_{(a,b)}; \mathfrak{w}_{k-1}^x\chi_{(a,b)})\big) = \lim_{x\to s_+}(\mathfrak{w}_k^x(a); \mathfrak{w}_k^x(b))\,.\end{aligned}$$

For the proof of (*v*), we apply the abstract Green's identity with the elements $(x_0, x \in (\sigma, s_+)$ with $x_0 < x)$

$$\begin{gathered}(\mathfrak{v}_k; \mathfrak{v}_{k-1}), (\mathfrak{v}_l; \mathfrak{v}_{l-1}) \in T_{\max}(H_{x_0\upharpoonright}), \quad 0 \le l,k \le N,\\ (\mathfrak{v}_k; \mathfrak{v}_{k-1}), (\mathfrak{w}_l^x; \mathfrak{w}_{l-1}^x) \in T_{\max}(H|_{(x_0,x)}), \quad 0 \le l,k \le N.\end{gathered}$$

This gives

$$\int_{x_0}^{s_+} \mathfrak{v}_l^* H\mathfrak{v}_{k-1} - \int_{x_0}^{s_+} \mathfrak{v}_{l-1}^* H\mathfrak{v}_k = \mathfrak{v}_l(x_0)^* J\mathfrak{v}_k(x_0), \tag{3.6}$$

$$\int_{x_0}^{x} (\mathfrak{w}_l^x)^* H\mathfrak{v}_{k-1} - \int_{x_0}^{x} (\mathfrak{w}_{l-1}^x)^* H\mathfrak{v}_k = \mathfrak{w}_l^x(x_0)^* J\mathfrak{v}_k(x_0) - \mathfrak{w}_l^x(x)^* J\mathfrak{v}_k(x). \tag{3.7}$$

When $x \to s_+$, the left-hand of side of (3.7) tends to the left-hand side of (3.6). By the already proved item (*iv*), the first summand on the right-hand side of (3.7) approaches the right-hand side of (3.6). Thus, we must have $\lim_{x\to s_+} \mathfrak{w}_l^x(x)^* J\mathfrak{v}_k(x) = 0$. ❑

Let us note that the functions $\mathfrak{v}_k$, $k > N$, in the above item (*iii*) are not unique.

b. Construction of limit boundary triples.

By Proposition A.7 we have

$$\mathring{\kappa}_{x_2,x_3} \circ \mathring{\kappa}_{x_1,x_2} = \mathring{\kappa}_{x_1,x_3}, \quad x_1, x_2, x_3 \in I_+,\ x_1 < x_2 < x_3 \,.$$

Hence, thinking of the totality of all maps $\mathring{\kappa}_{x_1,x_2}$, we have a chain of isomorphisms (which of course also transports via the isomorphisms $\mathring{\iota}_x$)

$$\begin{array}{ccccccccc} & & & & \overset{\mathring{\kappa}_{x_1,x_3}}{\frown} & & & & \\ \cdots \longrightarrow & \mathring{\mathfrak{P}}_{x_1}(\mathfrak{h}) & \xrightarrow{\mathring{\kappa}_{x_1,x_2}} & \mathring{\mathfrak{P}}_{x_2}(\mathfrak{h}) & \xrightarrow{\mathring{\kappa}_{x_2,x_3}} & \mathring{\mathfrak{P}}_{x_3}(\mathfrak{h}) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle \mathring{\iota}_{x_1}} & & \downarrow{\scriptstyle \mathring{\iota}_{x_2}} & & \downarrow{\scriptstyle \mathring{\iota}_{x_3}} & \\ \cdots \longrightarrow & \overset{\circ\circ}{\mathfrak{P}}_{x_1}(\mathfrak{h}) & \xrightarrow[\overset{\circ\circ}{\kappa}_{x_1,x_2}]{} & \overset{\circ\circ}{\mathfrak{P}}_{x_2}(\mathfrak{h}) & \xrightarrow[\overset{\circ\circ}{\kappa}_{x_2,x_3}]{} & \overset{\circ\circ}{\mathfrak{P}}_{x_3}(\mathfrak{h}) & \longrightarrow \cdots \\ & & & & \underset{\overset{\circ\circ}{\kappa}_{x_1,x_3}}{\smile} & & & & \end{array} \tag{3.8}$$

It is a central result for our present purposes that, assuming the asymptotics $(\mathrm{A}_{\Delta-1})$ and $(\mathrm{B}_{2\Delta})$, we may pass to the limit $x_3 \to s_+$. Actually, these conditions are chosen exactly to allow this limiting procedure.

As a first, trivial, step towards passing to the limit, let us emphasize also in notation that some parts of the boundary triples $\mathring{\mathfrak{B}}_{x_0}(\mathfrak{h})$ and $\overset{\circ\circ}{\mathfrak{B}}_{x_0}(\mathfrak{h})$ do not depend on x_0 at all.

3.8 Definition.

(*i*) Denote by $\mathring{\mathfrak{P}}(\mathfrak{h})$ the linear space $L^2(H) \times (\mathbb{C}^\Delta \times \mathbb{C}^\Delta) \times \mathbb{C}^{\ddot{o}}$, endowed with the inner product induced by the Gram matrix (2.6).

(*ii*) Denote by $\overset{\circ\circ}{\mathfrak{P}}(\mathfrak{h})$ the linear space $L^2_\Delta(H) \times \mathbb{C}^\Delta \times \mathbb{C}^{\ddot{o}}$, endowed with the Banach space topology common to all the spaces $\overset{\circ\circ}{\mathfrak{P}}_{x_0}(\mathfrak{h})$, $x_0 \in I_+$.

//

All maps in (3.8) are bijective and bicontinuous operators if considered between these Banach spaces:

$$\mathring{\kappa}_{x_1,x_2} : \mathring{\mathfrak{P}}(\mathfrak{h}) \to \mathring{\mathfrak{P}}(\mathfrak{h}),\ \overset{\circ\circ}{\kappa}_{x_1,x_2} : \overset{\circ\circ}{\mathfrak{P}}(\mathfrak{h}) \to \overset{\circ\circ}{\mathfrak{P}}(\mathfrak{h}),\ \mathring{\iota}_{x_1} : \mathring{\mathfrak{P}}(\mathfrak{h}) \to \overset{\circ\circ}{\mathfrak{P}}(\mathfrak{h}) \,.$$

Since $\mathring{\iota}_{x_1}$ maps the subspace $L^2(H) \subseteq \mathring{\mathfrak{P}}(\mathfrak{h})$ onto $L^2(H) \subseteq \overset{\circ\circ}{\mathfrak{P}}(\mathfrak{h})$, the space $L^2(H)$ is a closed subspace of $\overset{\circ\circ}{\mathfrak{P}}(\mathfrak{h})$. In particular, the topology induced on $L^2(H)$ by $\overset{\circ\circ}{\mathfrak{P}}(\mathfrak{h})$ is equal to the topology induced by the $L^2(H)$-norm.

We can now establish the existence of limits.

3.9 Proposition. *Assume that the general Hamiltonian* $\mathfrak{h}$ *satisfies the asymptotic conditions* $(\mathrm{A}_{\Delta-1})$ *and* $(\mathrm{B}_{2\Delta})$, *and let* $x_0 \in I_+$. *Then the limits*

$$\mathring{\kappa}_{x_0,s_+} := \lim_{x \to s_+} \mathring{\kappa}_{x_0,x}, \quad \overset{\circ\circ}{\kappa}_{x_0,s_+} := \lim_{x \to s_+} \overset{\circ\circ}{\kappa}_{x_0,x}, \quad \mathring{\iota}_{s_+} := \lim_{x \to s_+} \mathring{\iota}_x \,, \tag{3.9}$$

exist in the operator norm. We have

$$
\begin{array}{ccc}
\mathring{\mathfrak{P}}(\mathfrak{h}) & \xrightarrow{\mathring{\kappa}_{x_0,s_+}} & \mathring{\mathfrak{P}}(\mathfrak{h}) \\
\Big\downarrow{\scriptstyle \mathring{\imath}_{x_0}} & & \Big\downarrow{\scriptstyle \mathring{\imath}_{s_+}} \\
\overset{\circ\circ}{\mathfrak{P}}(\mathfrak{h}) & \xrightarrow{\overset{\circ\circ}{\kappa}_{x_0,s_+}} & \overset{\circ\circ}{\mathfrak{P}}(\mathfrak{h})
\end{array}
\tag{3.10}
$$

The map $\mathring{\kappa}_{x_0,s_+}$ *is an isometric isomorphism of* $\mathring{\mathfrak{P}}(\mathfrak{h})$ *onto itself, and its action is given by linearity and*

$$
\begin{aligned}
&\mathring{\kappa}_{x_0,s_+}(f;0,0,0) = \Big(f;\big(\int_{x_0}^{s_+} \mathfrak{v}_j^* Hf + \int_{\sigma}^{x_0} (\mathfrak{v}_j - \mathfrak{w}_j^{x_0})^* Hf\big)_{j=0}^{\Delta-1}, 0, 0\Big), \\
&\mathring{\kappa}_{x_0,s_+}(0;\xi,0,\alpha) = (0;\xi,0,\alpha), \\
&\mathring{\kappa}_{x_0,s_+}(0;0,\varepsilon_k,0) = \Big(-\chi_{x_0\urcorner}\mathfrak{v}_k - \chi_{\llcorner x_0}(\mathfrak{v}_k - \mathfrak{w}_k^{x_0}); (h_{kj}^{x_0,s_+})_{j=0}^{\Delta-1}, \varepsilon_k, 0\Big)
\end{aligned}
$$

where we set

$$
h_{kj}^{x_0,s_+} := -\frac{1}{2}\sum_{l=j+1}^{k+j+1} \omega_{k+j+1-l}^{x_0} v_l^{x_0} + \frac{1}{2}\sum_{l=1}^{j} \omega_{k+j+1-l}^{x_0} v_l^{x_0}.
$$

The maps $\mathring{\imath}_{s_+} : \mathring{\mathfrak{P}}(\mathfrak{h}) \to \overset{\circ\circ}{\mathfrak{P}}(\mathfrak{h})$ *and* $\overset{\circ\circ}{\kappa}_{x_0,s_+} : \overset{\circ\circ}{\mathfrak{P}}_{x_0}(\mathfrak{h}) \to \overset{\circ\circ}{\mathfrak{P}}(\mathfrak{h})$ *act as follows*

$$
\begin{aligned}
&\mathring{\imath}_{s_+}(f;\xi,\lambda,\alpha) = \big(f + \sum_{l=0}^{\Delta-1} \lambda_l \mathfrak{v}_l; \xi, \alpha\big), \\
&\overset{\circ\circ}{\kappa}_{x_0,s_+}(f;\xi,\alpha) = \Big(f; \big(\xi_j + \int_{x_0}^{s_+} \mathfrak{v}_j^* Hf \\
&\qquad + \int_{\sigma}^{x_0} (\mathfrak{v}_j - \mathfrak{w}_j^{x_0})^* H\big(f - \sum_{l=0}^{\Delta-1} \lambda_l \mathfrak{w}_l^{x_0}\big) + \sum_{l=0}^{\Delta-1} \lambda_l h_{lj}^{x_0,s_+}\big)_{j=0}^{\Delta-1}, \alpha\Big)
\end{aligned}
$$

where λ *are the unique coefficients so that* $f - \sum_{l=0}^{\Delta-1} \lambda_l \mathfrak{w}_l^{x_0} \chi_{\llcorner x_0} \in L^2(H)$.

Proof. Letting x tend to s_+ in the formulae describing the action of $\mathring{\kappa}_{x_0,x}$, it follows that the limit (3.9) exists in the strong operator topology and is given by the asserted formulae. Strong convergence also implies that $\mathring{\kappa}_{x_0,s_+}$ is isometric, and hence injective.

In order to show convergence in the operator norm, we compute

$$
\begin{aligned}
&(\mathring{\kappa}_{x_0,s_+} - \mathring{\kappa}_{x_0,x})(f;0,0,0) \\
&\qquad = \Big(0; \big(\int_{x_0}^{x} (\mathfrak{v}_j - \mathfrak{w}_j^x)^* Hf + \int_{\sigma}^{x_0} (\mathfrak{v}_j - \mathfrak{w}_j^x)^* Hf + \int_{x}^{s_+} \mathfrak{v}_j^* Hf \big)_{j=0}^{\Delta-1}, 0, 0\Big), \\
&(\mathring{\kappa}_{x_0,s_+} - \mathring{\kappa}_{x_0,x})(0;\xi,0,\alpha) = 0, \\
&(\mathring{\kappa}_{x_0,s_+} - \mathring{\kappa}_{x_0,x})(0;0,\varepsilon_k,0) = \Big(- (\chi_{x_0\Rsh}\mathfrak{v}_k - \chi_{(x_0,x)}\mathfrak{w}_k^x) - \chi_{\Lsh x_0}(\mathfrak{v}_k - \mathfrak{w}_k^x); \\
&\qquad\qquad \big(-\frac{1}{2}\sum_{l=j+1}^{k+j+1} \omega_{k+j+1-l}^{x_0}(v_l^{x_0} - \mathfrak{w}_l^x(x_0)_1) \\
&\qquad\qquad + \frac{1}{2}\sum_{l=1}^{j} \omega_{k+j+1-l}^{x_0}(v_l^{x_0} - \mathfrak{w}_l^x(x_0)_1)\big)_{j=0}^{\Delta-1}, 0, 0\Big).
\end{aligned}
$$

It follows, with the help of (3.2), that $\lim_{x\to s_+} \|\mathring{\kappa}_{x_0,s_+} - \mathring{\kappa}_{x_0,x}\| = 0$, where $\|\cdot\|$ is a norm that is compatible with the indefinite inner product on $\mathring{\mathfrak{P}}(\mathfrak{h})$. The surjectivity of $\mathring{\kappa}_{x_1,s_+}$ is obvious.

From the definition of $\mathring{\iota}_x$ and (3.2) we immediately obtain that $\mathring{\iota}_{s_+} := \lim_{x\to s_+} \mathring{\iota}_x$ exists in the operator norm and acts as

$$
\mathring{\iota}_{s_+}(f;\xi,\lambda,\alpha) = \big(f + \sum_{k=0}^{\Delta-1} \lambda_k \mathfrak{v}_k; \xi, \alpha\big).
$$

Clearly, $\mathring{\iota}_{s_+}$ is bijective. Finally, existence of the limit $\lim_{x\to s_+} \overset{\circ\circ}{\kappa}_{x_0,x}$ and commutativity of (3.10) follow since for each $x \in I_+$

$$
\overset{\circ\circ}{\kappa}_{x_0,x} = \mathring{\iota}_x \circ \mathring{\kappa}_{x_0,x} \circ \mathring{\iota}_{x_0}^{-1}\,.
$$

The form of $\overset{\circ\circ}{\kappa}_{x_0,s_+}$ follows from taking the strong limit in Lemma 3.2. ❑

3.10 Corollary. *We have* $\lim_{x\to s_+} \mathring{\kappa}_{x,s_+} = I$ *with respect to the operator norm. In particular,*

$$
\lim_{x\to s_+} h_{kj}^{x,s_+} = 0, \quad 0 \le k,j \le \Delta - 1\,.
$$

Proof. By (3.8), we have

$$
\mathring{\kappa}_{x_1,x} = \mathring{\kappa}_{x_2,x} \circ \mathring{\kappa}_{x_1,x_2}, \quad x_1,x_2,x \in I_+,\ x_1 < x_2 < x\,.
$$

Letting $x \to s_+$, gives $\mathring{\kappa}_{x_1,s_+} = \mathring{\kappa}_{x_2,s_+} \circ \mathring{\kappa}_{x_1,x_2}$, i.e. $\mathring{\kappa}_{x_1,s_+} \circ \mathring{\kappa}_{x_1,x_2}^{-1} = \mathring{\kappa}_{x_2,s_+}$. Since $\mathring{\kappa}_{x_1,s_+}$ is boundedly invertible and $\|\mathring{\kappa}_{x_1,x_2}\| = \|\mathring{\kappa}_{x_1,x_2}^{-1}\| = 1$, we may pass to the limit $x_2 \to s_+$ and obtain

$$
\lim_{x_2\to s_+} \mathring{\kappa}_{x_2,s_+} = \mathring{\kappa}_{x_1,s_+} \circ \lim_{x_2\to s_+} (\mathring{\kappa}_{x_1,x_2}^{-1}) = \mathring{\kappa}_{x_1,s_+} \circ (\lim_{x_2\to s_+} \mathring{\kappa}_{x_1,x_2})^{-1} = I\,.
$$

The asserted limit relation for h_{kj}^{x,s_+} follows immediately taking into account the formula for $\mathring{\kappa}_{x,s_+}(0;0,\varepsilon_k;0)$. ❑

We can use the isomorphisms $\mathring{\kappa}_{x_0,s_+}$ and $\overset{\circ\circ}{\kappa}_{x_0,s_+}$ to transport boundary triples.

3.11 Definition. Let $\mathfrak{h}$ be a singular indefinite Hamiltonian (of the form 1.3 with $\sup I_+ = s_+$) which satisfies the asymptotic conditions $(\mathrm{A}_{\Delta-1})$ and $(\mathrm{B}_{2\Delta})$, and choose a base point $x_0 \in I_+$. Set

$$\mathring{T}(\mathfrak{h}) := \big(\mathring{\kappa}_{x_0,s_+} \times \mathring{\kappa}_{x_0,s_+}\big)\mathring{T}_{x_0}(\mathfrak{h}), \quad \mathring{\Gamma}(\mathfrak{h}) := \big((\mathring{\kappa}_{x_0,s_+} \times \mathring{\kappa}_{x_0,s_+}) \times \mathrm{id}_{\mathbb{C}^4}\big)\mathring{\Gamma}_{x_0}(\mathfrak{h}).$$

On the linear space $\mathring{\mathring{\mathfrak{P}}}(\mathfrak{h})$ define an inner product by

$$[F,G] := [\mathring{\mathring{\kappa}}_{x_0,s_+}^{-1}F, \mathring{\mathring{\kappa}}_{x_0,s_+}^{-1}G], \quad F,G \in \mathring{\mathring{\mathfrak{P}}}(\mathfrak{h}),$$

where the inner product on the right-hand side is the one in $\mathring{\mathring{\mathfrak{P}}}_{x_0}(\mathfrak{h})$, and set

$$\mathring{\mathring{T}}(\mathfrak{h}) := \big(\mathring{\mathring{\kappa}}_{x_0,s_+} \times \mathring{\mathring{\kappa}}_{x_0,s_+}\big)\mathring{\mathring{T}}_{x_0}(\mathfrak{h}), \quad \mathring{\mathring{\Gamma}}(\mathfrak{h}) := \big((\mathring{\mathring{\kappa}}_{x_0,s_+} \times \mathring{\mathring{\kappa}}_{x_0,s_+}) \times \mathrm{id}_{\mathbb{C}^4}\big)\mathring{\mathring{\Gamma}}_{x_0}(\mathfrak{h}).$$

//

The triples

$$\mathring{\mathfrak{B}}(\mathfrak{h}) := \big(\mathring{\mathfrak{P}}(\mathfrak{h}), \mathring{T}(\mathfrak{h}), \mathring{\Gamma}(\mathfrak{h})\big) \text{ and } \mathring{\mathring{\mathfrak{B}}}(\mathfrak{h}) := \big(\mathring{\mathring{\mathfrak{P}}}(\mathfrak{h}), \mathring{\mathring{T}}(\mathfrak{h}), \mathring{\mathring{\Gamma}}(\mathfrak{h})\big)$$

are, by definition, boundary triples isomorphic to $\mathfrak{B}(\mathfrak{h})$. It follows immediately from (3.8) and (3.10) that the boundary triples $\mathring{\mathfrak{B}}(\mathfrak{h})$ and $\mathring{\mathring{\mathfrak{B}}}(\mathfrak{h})$ do not depend on the choice of the base point x_0, and that

$$\mathring{\mathring{T}}(\mathfrak{h}) = \big(\mathring{\iota}_{s_+} \times \mathring{\iota}_{s_+}\big)\mathring{T}(\mathfrak{h}), \quad \mathring{\mathring{\Gamma}}(\mathfrak{h}) = \big((\mathring{\iota}_{s_+} \times \mathring{\iota}_{s_+}) \times \mathrm{id}_{\mathbb{C}^4}\big)\mathring{\Gamma}(\mathfrak{h}).$$

c. The x_0-independent description of $\mathring{\mathring{\mathfrak{B}}}(\mathfrak{h})$.

The following description of the relations $\mathring{\mathring{T}}(\mathfrak{h})$ and $\mathring{\mathring{\Gamma}}(\mathfrak{h})$ is the main result of this section.

3.12 Theorem. *Let $\mathfrak{h}$ be a general Hamiltonian (of the form 1.3 with $\sup I_+ = s_+$), and assume that $\mathfrak{h}$ satisfies the asymptotic conditions $(\mathrm{A}_{\Delta-1})$ and $(\mathrm{B}_{2\Delta})$.*

Then, for each $k \in \{0,\dots,\Delta-2\}$, the limit

$$\mathfrak{l}_k := \lim_{x\to s_+}\Big(-\mathfrak{v}_{k+1}(x)^*J\mathfrak{w}_\Delta^x(x) + \int_\sigma^x (\mathfrak{v}_k - \mathfrak{w}_k^x)^* H\mathfrak{w}_\Delta^x\Big)$$

exists in $\mathbb{R}$.

Let $F = (f;\xi,\alpha)$ and $G = (g;\eta,\beta)$ be elements of $\mathring{\mathring{\mathfrak{P}}}(\mathfrak{h})$, and let λ and μ be the unique coefficients with

$$f - \sum_{l=0}^{\Delta-1}\lambda_l\mathfrak{v}_l \in L^2(H), \quad g - \sum_{l=0}^{\Delta-1}\mu_l\mathfrak{v}_l \in L^2(H).$$

Then $(F;G) \in \mathring{\mathring{T}}(\mathfrak{h})$ if and only if

- *(i)* $(f;g) \in T_{\Delta,\max}(H)$;
- *(ii)* *for each $k \in \{0,\dots,\Delta-2\}$ we have*

$$\xi_k = \eta_{k+1} + \frac{1}{2}\lambda_0 D_k + \mu_{\Delta-1}\Big(\frac{1}{2}D_{\Delta+k+1} - \mathfrak{l}_k\Big) - \begin{cases}\mathfrak{v}_{k+1}(s_-)_2 f(s_-)_1, & (s_-,\sigma) \text{ not indivisible},\\ 0, & (s_-,\sigma) \text{ indivisible};\end{cases}$$

(*iii*) *the limit*

$$\mathfrak{l}_{F,G} := \lim_{x\to s_+}\Big(\mathfrak{v}_\Delta(x)^* J\Big(f(x) - \sum_{l=0}^{\Delta-1}\lambda_l \mathfrak{w}_l^x(x) - \mu_{\Delta-1}\mathfrak{w}_\Delta^x(x)\Big) + \mu_{\Delta-1}\int_\sigma^x (\mathfrak{v}_{\Delta-1} - \mathfrak{w}_{\Delta-1}^x)^* H \mathfrak{w}_\Delta^x + \int_{s_-}^x \mathfrak{v}_\Delta^* H\Big(g - \sum_{l=0}^{\Delta-1}\mu_l \mathfrak{w}_l^x\Big)\Big)$$

exists in $\mathbb{R}$, *and we have*

$$\xi_{\Delta-1} = \mathfrak{l}_{F,G} + \frac{1}{2}\sum_{l=0}^{\Delta-1}\lambda_l D_{l+\Delta-1} + \mu_{\Delta-1}D_{2\Delta-1} - \begin{cases}\beta_1, & \ddot{o} > 0,\\ 0, & \ddot{o} = 0,\end{cases} - \begin{cases}\mathfrak{v}_\Delta(s_-)_2 f(s_-)_1, & (s_-,\sigma) \text{ not indivisible},\\ 0, & (s_-,\sigma) \text{ indivisible};\end{cases}$$

(*iv*) *if* (s_-,σ) *is not indivisible, then*

$$\eta_0 = f(s_-)_1 - \frac{1}{2}\sum_{l=0}^{\Delta-1}\mu_l D_l\,;$$

(*v*) *if* $\ddot{o} > 0$, *then*

$$\alpha_j = -\mu_{\Delta-1}b_{\ddot{o}-j+1} + \begin{cases}\beta_{j+1}, & j = 1,\ldots,\ddot{o}-1,\\ 0, & j = \ddot{o}.\end{cases}$$

In this case,

$$\mathring{\Gamma}(F;G) = \begin{cases} f(s_-), & (s_-,\sigma) \text{ not indivisible},\\ \begin{pmatrix}\eta_0 + \frac{1}{2}\sum_{l=0}^{\Delta-1}\mu_l D_l\\ \lambda_0\end{pmatrix}, & (s_-,\sigma) \text{ indivisible}.\end{cases}$$

We have $\operatorname{mul}\mathring{T}(\mathfrak{h}) \neq \{0\}$ *if and only if* H *starts with an indivisible interval at* s_-. *In this case, when* $s_0 \in (s_-,\sigma]$ *denotes the right endpoint of the maximal indivisible interval with left endpoint* s_-, ϕ *denotes its type, and* $l := \int_{s_-}^{s_0}\operatorname{tr} H(t)\,dt$ *its length, we have*

$$\operatorname{mul}\mathring{T}(\mathfrak{h}) = \begin{cases}\operatorname{span}\big\{\big(0;(\xi_\phi\chi_{\urcorner s_0};(-\mathfrak{v}_k(s_-)_2 l\sin\phi)_{j=0}^{\Delta-1},0)\big)\big\}, & s_0 < \sigma,\\ \operatorname{span}\big\{\big(0;(0;\varepsilon_0,0)\big)\big\}, & s_0 = \sigma.\end{cases}$$

Proof. In the proof we use Theorem 2.15. To this end note that, for any $x \in I_+$ and arbitrary elements $(f;\xi,\alpha) \in \mathring{\mathfrak{P}}(\mathfrak{h})$, we have

$$\mathring{\kappa}_{x,s_+}^{-1}(f;\xi,\alpha) = \Big(f;\big(\xi_j - \int_x^{s_+}\mathfrak{v}_j^* H f - \int_\sigma^x (\mathfrak{v}_j - \mathfrak{w}_j^x)^* H\big(f - \sum_{l=0}^{\Delta-1}\lambda_l\mathfrak{w}_l^x\big) - \sum_{l=0}^{\Delta-1}\lambda_l h_{lj}^{x,s_+}\big)_{j=0}^{\Delta-1},\alpha\Big) \tag{3.11}$$

where λ are the unique coefficients with $f - \sum_{l=0}^{\Delta-1} \lambda_l \mathfrak{v}_l \in L^2(H)$.

Step 1: w.l.o.g. $(F;G)$ satisfies (i) and (v).
Let $F = (f;\xi,\alpha), G = (g;\eta,\beta) \in \mathring{\mathfrak{P}}(\mathfrak{h})$ be given. By the definition of $\mathring{T}(\mathfrak{h})$, we have $(F;G) \in \mathring{T}(\mathfrak{h})$ if and only if $(\mathring{\kappa}_{x,s_+}^{-1} F; \mathring{\kappa}_{x,s_+}^{-1} G) \in \mathring{T}_x(\mathfrak{h})$. The relation (3.11) and Theorem 2.15 (i), (v) show that, under the assumption that $(F;G) \in \mathring{T}(\mathfrak{h})$, the present conditions (i) and (v) hold. This implies that, for the proof of the present theorem, we may assume from the start that $(F;G)$ satisfies (i) and (v).

Note that the condition (i) implies that $\mu_k = \lambda_{k+1}$, $k = 0,\ldots,\Delta-2$, cf. Corollary 2.10.

Step 2: computation of the conditions (ii)–(iv) in Theorem 2.15.
Let $F,G \in \mathring{\mathfrak{P}}(\mathfrak{h})$ and μ,λ be as in the formulation of the theorem. Moreover, let $x \in I_+$. We will show that the conditions (ii)–(iv) in Theorem 2.15 for the element $(\mathring{\kappa}_{x,s_+}^{-1} F; \mathring{\kappa}_{x,s_+}^{-1} G)$ read as follows:

(ii_x) for each $k = 0,\ldots,\Delta-2$ we have

$$\begin{aligned}
\xi_k = \eta_{k+1} &+ \frac{1}{2}\lambda_0 d_k^x + \frac{1}{2}\mu_{\Delta-1} d_{\Delta+k}^x \\
&- \begin{cases} \mathfrak{w}_{k+1}^x(s_-)_2 f(s_-)_1, & (s_-,\sigma) \text{ not indivisible,} \\ 0, & (s_-,\sigma) \text{ indivisible,} \end{cases} \\
&- \mathfrak{v}_{k+1}(x)^* J\Big(\sum_{l=0}^{\Delta-1} \lambda_l \mathfrak{w}_l^x(x)\Big) + \sum_{l=0}^{\Delta-1} \lambda_l h_{lk}^{x,s_+} - \sum_{l=0}^{\Delta-1} \mu_l h_{l,k+1}^{x,s_+} \\
&+ \mu_{\Delta-1}\Big(- \mathfrak{v}_{k+1}(x)^* J \mathfrak{w}_\Delta^x(x) + \int_\sigma^x (\mathfrak{v}_k - \mathfrak{w}_k^x)^* H \mathfrak{w}_\Delta^x \Big)\,;
\end{aligned}$$

(iii_x)

$$\begin{aligned}
\xi_{\Delta-1} = \frac{1}{2}\sum_{l=0}^{\Delta-1} \lambda_l d_{l+\Delta-1}^x &+ \mu_{\Delta-1} d_{2\Delta-1}^x - \begin{cases} \beta_1, & \ddot{o} > 0, \\ 0, & \ddot{o} = 0. \end{cases} \\
&+ \mathfrak{v}_\Delta(x)^* J\Big(f(x) - \sum_{l=0}^{\Delta-1} \lambda_l \mathfrak{w}_l^x(x) - \mu_{\Delta-1}\mathfrak{w}_\Delta^x(x)\Big) \\
&+ \mu_{\Delta-1}\int_\sigma^x (\mathfrak{v}_{\Delta-1} - \mathfrak{w}_{\Delta-1}^x)^* H \mathfrak{w}_\Delta^x + \int_{s_-}^x \mathfrak{v}_\Delta^* H\Big(g - \sum_{l=0}^{\Delta-1} \mu_l \mathfrak{w}_l^x\Big) \\
&+ \int_x^{s_+} \mathfrak{v}_{\Delta-1}^* H f + \sum_{l=0}^{\Delta-1} \lambda_l h_{l,\Delta-1}^{x,s_+} \\
&- \begin{cases} \mathfrak{w}_\Delta^x(s_-)_2 f(s_-)_1, & (s_-,\sigma) \text{ not indivisible,} \\ 0, & (s_-,\sigma) \text{ indivisible;} \end{cases}
\end{aligned}$$

(iv_x) if (s_-,σ) is not indivisible, then

$$\eta_0 = f(s_-)_1 - \frac{1}{2}\sum_{l=0}^{\Delta-1} \mu_l d_l^x + \int_x^{s_+} \mathfrak{v}_0^* H g - \mathfrak{v}_0(x)^* J f(x) + \sum_{l=0}^{\Delta-1} \mu_l h_{l0}^{x,s_+}.$$

Let us start with condition (ii_x). Plugging the respective expressions (3.11) for $\mathring{\kappa}_{x,s_+}^{-1} F$ and $\mathring{\kappa}_{x,s_+}^{-1} G$ into Theorem 2.15 (ii) we obtain

$$\begin{aligned}
\xi_k =& \int_x^{s_+} \mathfrak{v}_k^* H f + \int_\sigma^x (\mathfrak{v}_k - \mathfrak{w}_k^x)^* H \big(f - \sum_{l=0}^{\Delta-1} \lambda_l \mathfrak{w}_l^x\big) + \sum_{l=0}^{\Delta-1} \lambda_l h_{lk}^{x,s_+} \\
& + \eta_{k+1} - \int_x^{s_+} \mathfrak{v}_{k+1}^* H g - \int_\sigma^x (\mathfrak{v}_{k+1} - \mathfrak{w}_{k+1}^x)^* H \big(g - \sum_{l=0}^{\Delta-1} \mu_l \mathfrak{w}_l^x\big) - \sum_{l=0}^{\Delta-1} \mu_l h_{l,k+1}^{x,s_+} \\
& + \frac{1}{2}\mu_{\Delta-1} d_{\Delta+k}^k + \frac{1}{2}\lambda_0 d_k^x + \omega_{k+1}^x f(x)_1 \\
& - \begin{cases} \mathfrak{w}_{k+1}^x (s_-)_2 f(s_-)_1, & (s_-,\sigma) \text{ not indivisible}, \\ 0, & (s_-,\sigma) \text{ indivisible}, \end{cases} \\
=\; & \eta_{k+1} - \Big(\int_x^{s_+} \mathfrak{v}_{k+1}^* H g - \int_x^{s_+} \mathfrak{v}_k^* H f \Big) - \bigg(\int_\sigma^x (\mathfrak{v}_{k+1} - \mathfrak{w}_{k+1}^x)^* H \big(g - \sum_{l=0}^{\Delta-1} \mu_l \mathfrak{w}_l^x\big) \\
& - \int_\sigma^x (\mathfrak{v}_k - \mathfrak{w}_k^x)^* H \Big(f - \sum_{l=0}^{\Delta-1} \lambda_l \mathfrak{w}_l^x - \mu_{\Delta-1} \mathfrak{w}_\Delta^x\Big)\bigg) + \mu_{\Delta-1} \int_\sigma^x (\mathfrak{v}_k - \mathfrak{w}_k^x)^* H \mathfrak{w}_\Delta^x \\
& + \sum_{l=0}^{\Delta-1} \lambda_l h_{lk}^{x,s_+} - \sum_{l=0}^{\Delta-1} \mu_l h_{l,k+1}^{x,s_+} + \frac{1}{2}\lambda_0 d_k^x + \frac{1}{2}\mu_{\Delta-1} d_{\Delta+k}^k + \omega_{k+1}^x f(x)_1 \\
& - \begin{cases} \mathfrak{w}_{k+1}^x (s_-)_2 f(s_-)_1, & (s_-,\sigma) \text{ not indivisible}, \\ 0, & (s_-,\sigma) \text{ indivisible}. \end{cases}
\end{aligned} \tag{3.12}$$

The abstract Green's identity applied to the pairs

$$(f; g),\ (\mathfrak{v}_{k+1}; \mathfrak{v}_k) \in T_{\max}(H_{x\urcorner})$$

yields

$$\int_x^{s_+} \mathfrak{v}_{k+1}^* H g - \int_x^{s_+} \mathfrak{v}_k^* H f = \mathfrak{v}_{k+1}(x)^* J f(x)$$

and applied to the pairs (remember that $\mu_k = \lambda_{k+1},\ k = 0, \dots, \Delta - 2$, and $w_{-1}^x := 0$)

$$\Big(f - \sum_{l=0}^{\Delta-1} \lambda_l \mathfrak{w}_l^x - \mu_{\Delta-1} \mathfrak{w}_\Delta^x;\ g - \sum_{l=0}^{\Delta-1} \mu_l \mathfrak{w}_l^x\Big), (\mathfrak{v}_{k+1} - \mathfrak{w}_{k+1}^x; \mathfrak{v}_k - \mathfrak{w}_k^x) \in T_{\max}(H|_{(\sigma,x)}),$$
$$k = 0, \dots, \Delta - 2,$$

it gives

$$\int_\sigma^x (\mathfrak{v}_{k+1}-\mathfrak{w}^x_{k+1})^* H\big(g-\sum_{l=0}^{\Delta-1}\mu_l\mathfrak{w}^x_l\big) - \int_\sigma^x (\mathfrak{v}_k-\mathfrak{w}^x_k)^* H\Big(f-\sum_{l=0}^{\Delta-1}\lambda_l\mathfrak{w}^x_l-\mu_{\Delta-1}\mathfrak{w}^x_\Delta\Big)$$
$$= -\big(\mathfrak{v}_{k+1}(x)-\mathfrak{w}^x_{k+1}(x)\big)^* J\Big(f(x)-\sum_{l=0}^{\Delta-1}\lambda_l\mathfrak{w}^x_l(x)-\mu_{\Delta-1}\mathfrak{w}^x_\Delta(x)\Big)$$
$$= -\big(\mathfrak{v}_{k+1}(x)\big)^* J\Big(f(x)-\sum_{l=0}^{\Delta-1}\lambda_l\mathfrak{w}^x_l(x)-\mu_{\Delta-1}\mathfrak{w}^x_\Delta(x)\Big)+\underbrace{\mathfrak{w}^x_{k+1}(x)^*Jf(x)}_{=\omega^x_{k+1}f(x)_1}. \quad (3.13)$$

Putting this all together we obtain (ii_x).

Next we show the correspondence of (iii_x) and Theorem 2.15 (iii). Plugging (3.11) into Theorem 2.15 (iii) gives

$$\xi_{\Delta-1} = \int_x^{s_+}\mathfrak{v}^*_{\Delta-1}Hf + \int_\sigma^x(\mathfrak{v}_{\Delta-1}-\mathfrak{w}^x_{\Delta-1})^*H\big(f-\sum_{l=0}^{\Delta-1}\lambda_l\mathfrak{w}^x_l\big)+\sum_{l=0}^{\Delta-1}\lambda_l h^{x,s_+}_{l,\Delta-1}$$
$$+\int_{s_-}^x(\mathfrak{w}^x_\Delta)^*H\big(g-\sum_{l=0}^{\Delta-1}\mu_l\mathfrak{w}^x_l\big)+\frac{1}{2}\sum_{l=0}^{\Delta-1}\lambda_l d^x_{l+\Delta-1}+\mu_{\Delta-1}d^x_{2\Delta-1}-\begin{cases}\beta_1, & \ddot{o}>0,\\ 0, & \ddot{o}=0,\end{cases}$$
$$+\omega^x_\Delta f(x)_1-\begin{cases}\mathfrak{w}^x_\Delta(s_-)_2 f(s_-)_1, & (s_-,\sigma)\text{ not indivisible},\\ 0, & (s_-,\sigma)\text{ indivisible}.\end{cases}$$

Let $\mathfrak{v}_\Delta$ be a function as in Lemma 3.7 (iii); then we can apply (3.13) also with $k=\Delta-1$. Using this we obtain (iii_x).

Finally, we plug (3.11) into Theorem 2.15 (iv), and obtain

$$\eta_0-\int_x^{s_+}\mathfrak{v}_0^*Hg-\int_\sigma^x(\underbrace{\mathfrak{v}_0-\mathfrak{w}^x_0}_{=0})^*H\big(g-\sum_{l=0}^{\Delta-1}\mu_l\mathfrak{w}^x_l\big)-\sum_{l=0}^{\Delta-1}\mu_l h^{x,s_+}_{l0}$$
$$= f(s_-)_1-f(x)_1-\frac{1}{2}\sum_{l=0}^{\Delta-1}\mu_l d^x_l,$$

which is exactly (iv_x).

Step 3: $(F;G)\in\mathring{\mathring{T}}(\mathfrak{h})\ \Rightarrow\ (ii)-(iv)$.

For each $x\in I_+$ the pair $(\kappa^{-1}_{x,s_+}F;\kappa^{-1}_{x,s_+}G)$ belongs to $\mathring{\mathring{T}}_x(\mathfrak{h})$. Hence the conditions (ii_x)–(iv_x) are satisfied for all $x\in I_+$ and we may pass to the limit $x\to s_+$.

Lemma 3.7 (v) and Corollary 3.10 imply that the terms in the third line of (ii_x) tend to 0. The expressions in the first line tend to the corresponding expressions in (ii), cf. (3.4). We conclude that the limit

$$\lim_{x\to s_+}\mu_{\Delta-1}\Big(-\mathfrak{v}_{k+1}(x)^*J\mathfrak{w}^x_\Delta(x)+\int_\sigma^x(\mathfrak{v}_k-\mathfrak{w}^x_k)^*H\mathfrak{w}^x_\Delta\Big) \quad (3.14)$$

exists and that (ii) will hold once the existence of the limit $\mathfrak{l}_k$ is established. This, however, follows at once from (3.14) since $\mathring{\mathring{T}}(\mathfrak{h})$ certainly does contain elements with $\mu_{\Delta-1}\neq 0$.

Let us next consider (iii_x). Since, for $x_0 \in I_+$, the elements $v_{\Delta-1}\chi_{x_0\Rsh}$ and $f\chi_{x_0\Rsh}$ both belong to $L^2(H_{x_0\Rsh})$, we have

$$\lim_{x\to s_+}\int_x^{s_+}\mathfrak{v}_{\Delta-1}^*Hf=0\,.$$

Together with Corollary 3.10 this shows that the terms in the fourth line of (iii_x) tend to 0. By (3.4), the terms in the first line approach the corresponding terms in (iii). The expression in the last line does not depend on x. It follows that the limit $\mathfrak{l}_{F,G}$ exists and that the formula (iii) holds.

Finally, let us consider (iv_x). Since for $x_0 \in I_+$, $f\chi_{x_0\Rsh}$ and $\mathfrak{v}_0\chi_{x_0\Rsh}$ belong to $\operatorname{dom} T_{\max}(H_{x_0\Rsh})$, we have

$$\lim_{x\to s_+}\mathfrak{v}_0(x)^*Jf(x)=0,$$

cf. [HSW, Theorem 3.6]. This together with considerations as above show that the right-hand side of (iv_x) converges to the right-hand side of (iv).

Step 4: $(ii)-(iv) \Rightarrow (F;G)\in\overset{\circ\circ}{T}(\mathfrak{h})$.
For each $x \in I_+$, we define elements $F_x, G_x \in \overset{\circ\circ}{\mathfrak{P}}(\mathfrak{h})$ as $F_x := (f;\xi^x,\alpha)$ and $G_x := (g;\eta^x,\beta)$, where

$$\eta_0^x := \begin{cases}\text{right-hand side of } (iv_x), & (s_-,\sigma) \text{ not indivisible,}\\ \eta_0, & (s_-,\sigma) \text{ indivisible,}\end{cases}$$

$$\eta_k^x := \eta_k, \quad k=1,\dots,\Delta-1\,,$$

$$\xi_k := \begin{cases}\text{right-hand side of } (ii_x), & k=0,\dots,\Delta-2,\\ \text{right-hand side of } (iii_x), & k=\Delta-1.\end{cases}$$

Then, by Theorem 2.15 and Step 2, the pair $(\kappa_{x,s_+}^{-1}F_x;\kappa_{x,s_+}^{-1}G_x)$ belongs to $\overset{\circ\circ}{T}_x(\mathfrak{h})$, and hence $(F_x;G_x)\in\overset{\circ\circ}{T}(\mathfrak{h})$. However, as we saw in Step 3, the right-hand sides of (ii_x)–(iv_x) tend to the right-hand sides of (ii)–(iv) for $x\to s_+$. Thus

$$\lim_{x\to s_+}F_x=F,\quad \lim_{x\to s_+}G_x=G\,,$$

and we conclude that $(F;G)\in\overset{\circ\circ}{T}(\mathfrak{h})$ since $\overset{\circ\circ}{T}(\mathfrak{h})$ is a closed linear relation.

Step 5: boundary values.
Let $(F;G)\in\overset{\circ\circ}{T}(\mathfrak{h})$ be given, and let $x\in I_+$. Then

$$\overset{\circ\circ}{\Gamma}(F;G)=\overset{\circ\circ}{\Gamma}_x\big(\kappa_{x,s_+}^{-1}F;\kappa_{x,s_+}^{-1}G\big) = \begin{cases} f(s_-), & (s_-,\sigma) \text{ not indivisible,}\\ \begin{pmatrix}\tilde{\eta}_0^x+f(x)_1+\frac12\sum_{l=0}^{\Delta-1}\mu_l d_l^x\\ \lambda_0\end{pmatrix}, & (s_-,\sigma) \text{ indivisible,}\end{cases} \tag{3.15}$$

where

$$\tilde{\eta}_0^x := \eta_0-\int_x^{s_+}\mathfrak{v}_0^*Hg-\sum_{l=0}^{\Delta-1}\mu_l h_{l0}^{x,s_+}\,.$$

If (s_-,σ) is not indivisible, we are already done. Consider the case when (s_-,σ) is indivisible. Since $f\chi_{x_0\Rsh}$ and $\mathfrak{v}_0\chi_{x_0\Rsh}$ both belong to $\operatorname{dom} T_{\max}(H_{x_0\Rsh})$ for some $x_0 \in I_+$, we have

$$\lim_{x\to s_+} f(x)_1 = \lim_{x\to s_+} \mathfrak{v}_0(x)^* J f(x) = 0.$$

Hence passing to the limit in (3.15) gives the desired formula.

Step 6: multi-valued part.
We have $(0;G) \in \operatorname{mul} \overset{\circ\circ}{T}(\mathfrak{h})$ if and only if $(0;\overset{\circ\circ}{\kappa}{}_{x,s_+}^{-1} G) \in \operatorname{mul} \overset{\circ\circ}{T}_x(\mathfrak{h})$. However, if $(0;\tilde{G}) \in \operatorname{mul} \overset{\circ\circ}{T}_x(\mathfrak{h})$, then by the form of $\tilde{G}$ given in Theorem 2.15 we have $\overset{\circ\circ}{\kappa}_{x,s_+}\tilde{G} = \tilde{G}$. Hence, $\operatorname{mul} \overset{\circ\circ}{T}(\mathfrak{h}) = \operatorname{mul} \overset{\circ\circ}{T}_x(\mathfrak{h})$. ❑

3.13 Remark. Under slightly stronger assumptions on the asymptotics of $\mathfrak{h}$, the limits appearing in Theorem 3.12 can be computed: assume that $\mathfrak{h}$ satisfies (A_Δ) and ($\mathrm{B}_{2\Delta}$). Then, with the notation of Theorem 3.12, we have

$$\mathfrak{l}_k = 0, \quad k = 0,\ldots,\Delta-2\,.$$

Moreover, if $(F;G) \in \overset{\circ\circ}{T}(\mathfrak{h})$, then

$$\mathfrak{l}_{F,G} = \int_{s_-}^{s_+} \mathfrak{v}_\Delta^* H\big(g - \sum_{l=0}^{\Delta-1} \lambda_l \mathfrak{v}_l\big)\,.$$

The first relation follows from Lemma 3.7 (v) and (3.2). For the second relation note in addition that $\mathfrak{v}_\Delta\chi_{x_0} \in \operatorname{dom} T_{\max}(H_{x_0\Rsh})$ and hence $\lim_{x\to s_+} \mathfrak{v}_\Delta(x)^* J f(x) = 0$. //

Appendix A. Splitting of the model for an elementary indefinite Hamiltonian $\mathfrak{h}_{s_-}^{s_+}$

In this appendix we derive some formulae connected with the splitting of an indefinite Hamiltonian. Although we need the formulae in the case that the splitting point is to the right of the singularity, we first derive them for the case that the splitting point is to the left and then apply an order-reversing reparameterization. This is because the formulae we use from [KW2] are for the former case.

Let $\mathfrak{h}_{s_-}^{s_+} = (H;\ddot{o},b_j;d_{s_-,j}^{s_+})$ be an elementary indefinite Hamiltonian of kind (A) defined on $(s_-,\sigma)\cup(\sigma,s_+)$, let $s_0 \in I_-$, and let $\mathfrak{h}_{s_0}^{s_+} = (H;\ddot{o},b_j;d_{s_0,j}^{s_+})$ be the elementary indefinite Hamiltonian obtained by splitting $\mathfrak{h}_{s_-}^{s_+}$ at s_0. Then, by §2.2, §2.3 and Remark 2.4, we have the isomorphism

$$\kappa_{s_0,s_-} : L^2(H_{\Lsh s_0})\,[\dot{+}]\,\mathfrak{P}(\mathfrak{h}_{s_0}^{s_+}) \to \mathfrak{P}(\mathfrak{h}_{s_-}^{s_+})\,.$$

Let

$$\iota_{s_-}^{s_+} : \mathfrak{P}(\mathfrak{h}_{s_-}^{s_+}) \to L^2(H)\,[\dot{+}]\,(\mathbb{C}^\Delta \dot{+} \mathbb{C}^\Delta)\,[\dot{+}]\,\mathbb{C}^{\ddot{o}},$$

$$\iota_{s_0}^{s_+} : \mathfrak{P}(\mathfrak{h}_{s_0}^{s_+}) \to L^2(H_{s_0\Rsh})\,[\dot{+}]\,(\mathbb{C}^\Delta \dot{+} \mathbb{C}^\Delta)\,[\dot{+}]\,\mathbb{C}^{\ddot{o}}$$

be the respective isomorphisms constructed in (IV.4.10). By

$$(f \dotplus x) \mapsto (f; 0, 0, 0) + \iota_{s_0}^{s_+} x, \quad f \in L^2(H_{\Lsh s_0}),\ x \in \mathfrak{P}(\mathfrak{h}_{s_0}^{s_+}),$$

the map $\iota_{s_0}^{s_+}$ extends naturally to an isomorphism of $L^2(H_{\Lsh s_0})\,[\dotplus]\,\mathfrak{P}(\mathfrak{h}_{s_0}^{s_+})$ onto $L^2(H)\,[\dotplus]\,(\mathbb{C}^\Delta \dotplus \mathbb{C}^\Delta)\,[\dotplus]\,\mathbb{C}^{\ddot{o}}$ which again will be denoted by $\iota_{s_0}^{s_+}$.

The isomorphisms $\iota_{s_-}^{s_+}$ and $\iota_{s_0}^{s_+}$ can be used to transport κ_{s_0,s_-}: denote by $\mathring{\kappa}_{s_0,s_-}$ the map defined by

$$\begin{array}{ccc}
L^2(H_{\Lsh s_0})\,[\dotplus]\,\mathfrak{P}(\mathfrak{h}_{s_0}^{s_+}) & \xrightarrow{\ \kappa_{s_0,s_-}\ } & \mathfrak{P}(\mathfrak{h}_{s_-}^{s_+}) \\
\iota_{s_0}^{s_+} \Big\downarrow & & \Big\downarrow \iota_{s_-}^{s_+} \\
L^2(H)\,[\dotplus]\,(\mathbb{C}^\Delta \dotplus \mathbb{C}^\Delta)\,[\dotplus]\,\mathbb{C}^{\ddot{o}} & \xrightarrow[\mathring{\kappa}_{s_0,s_-}]{} & L^2(H)\,[\dotplus]\,(\mathbb{C}^\Delta \dotplus \mathbb{C}^\Delta)\,[\dotplus]\,\mathbb{C}^{\ddot{o}}
\end{array} \tag{A.1}$$

Our aim in the present section is to give explicit formulae for the action of $\mathring{\kappa}_{s_0,s_-}$.

Explicit form of the splitting isomorphism [KW2, §7].

Let $\mathfrak{h}_{s_-}^{s_+}$ and $\mathfrak{h}_{s_0}^{s_+}$, where $s_0 \in (s_-, \sigma)$ is not an inner point of an indivisible interval, be given as above. Let $\mathfrak{w}_{s_-,k}^{s_+}$ and $\mathfrak{w}_{s_0,k}^{s_+}$ denote the unique absolutely continuous functions defined on $[s_-, s_+] \setminus \{\sigma\}$ and $[s_0, s_+] \setminus \{\sigma\}$, respectively, with values in $\mathbb{C}^2$ such that ($k \in \mathbb{N}$)

$$(\mathfrak{w}_{s_-,k}^{s_+})' = JH\mathfrak{w}_{s_-,k-1}^{s_+}, \qquad (\mathfrak{w}_{s_0,k}^{s_+})' = JH\mathfrak{w}_{s_0,k-1}^{s_+},$$

$$\mathfrak{w}_{s_-,0}^{s_+} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \mathfrak{w}_{s_0,0}^{s_+} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\mathfrak{w}_{s_-,k}^{s_+}(s_-), \mathfrak{w}_{s_-,k}^{s_+}(s_+) \in \operatorname{span}\Big\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big\}, \qquad \mathfrak{w}_{s_0,k}^{s_+}(s_0), \mathfrak{w}_{s_0,k}^{s_+}(s_+) \in \operatorname{span}\Big\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big\},$$

$$\mathfrak{w}_{s_-,k}^{s_+} \in L^2(H), \quad k \geq \Delta, \qquad \mathfrak{w}_{s_0,k}^{s_+} \in L^2(H_{s_0 \Rsh}), \quad k \geq \Delta.$$

For notational convenience, we set $\mathfrak{w}_{s_-,-1}^{s_+} = \mathfrak{w}_{s_0,-1}^{s_+} = 0$.

A.1 Remark. The functions $\mathfrak{w}_{s_0,k}^{s_+}$, $k \in \mathbb{N}_0$, are a priori only defined on $[s_0, s_+] \setminus \{\sigma\}$. However, on the interval $[s_0, \sigma)$, they are of the form

$$\mathfrak{w}_{s_0,k}^{s_+}(x) = \sum_{j=0}^{k} \omega_{k-j} \mathcal{I}^j \left(\chi_{[s_0,\sigma)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)(x), \quad x \in [s_0, \sigma)\,,$$

with some sequence $\omega_j \in \mathbb{R}$, $\omega_0 = 1$, and where $\mathcal{I}$ acts as

$$\mathcal{I} f = \int_{s_0}^{x} JHf,\ x \in [s_0, \sigma)\,.$$

Since H is also defined on (s_-, s_0), each function $\mathfrak{w}_{s_0,k}^{s_+}$ admits a natural continuation to $[s_-, s_+] \setminus \{\sigma\}$. Apparently, the relations

$$(\mathfrak{w}_{s_0,k}^{s_+})' = JH\mathfrak{w}_{s_0,k-1}^{s_+},\ k \in \mathbb{N}\,,$$

hold also on the bigger set $(s_-, s_+) \setminus \{\sigma\}$. //

We know from Lemma IV.3.12 that there exist unique real numbers λ_j such that

$$\mathfrak{d}_k^{s_0,s_-}(x) := \mathfrak{w}_{s_-,k}^{s_+}(x) - \mathfrak{w}_{s_0,k}^{s_+}(x) = \sum_{j=0}^{k-1} \lambda_{k-j} B^j \left(\chi_{[s_-,\sigma)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)(x), \quad x \in [s_0, \sigma)\,,$$

where B is the operator constructed in (IV.2.22). The numbers λ_j are in fact given by

$$\lambda_k = \mathfrak{d}_k^{s_0,s_-}(s_-)_1 = -\mathfrak{w}_{s_0,k}^{s_+}(s_-)_1, \quad k \in \mathbb{N}_0.$$

It will be practical to have the following extension of the abstract Green's identity available.

A.2 Remark. Let $(\mathcal{P}, T, \Gamma)$ be a boundary triple in the sense of Definition IV.2.7. Let $n \in \mathbb{N}$ and

$$f_0, \dots, f_n;\ g_0, \dots, g_n \in \mathcal{P}, \quad \alpha_1^\pm, \dots, \alpha_n^\pm;\ \beta_1^\pm, \dots, \beta_n^\pm \in \mathbb{C}^2$$

be such that

$$\big((f_j; f_{j-1}); (\alpha_j^-; \alpha_j^+)\big),\ \big((g_j; g_{j-1}); (\beta_j^-; \beta_j^+)\big) \in \Gamma, \quad j = 1, \dots, n.$$

Then

$$[f_0, g_n] - [f_n, g_0] = \sum_{j=1}^n (\beta_{n+1-j}^-)^* J \alpha_j^- - \sum_{j=1}^n (\beta_{n+1-j}^+)^* J \alpha_j^+\,. \tag{A.2}$$

This relation is obtained from taking the sum of the equalities

$$[f_{j-1}, g_{n+1-j}] - [f_j, g_{n-j}] = (\beta_{n+1-j}^-)^* J \alpha_j^- - (\beta_{n+1-j}^+)^* J \alpha_j^+,$$

for $j = 1, \dots, n$. //

The definition of the parameters $d_{s_0,k}^{s_+}$ of $\mathfrak{h}_{s_0}^{s_-}$ in [KW2, p. 8121 and Proposition 7.8] reads as

$$d_{s_-,k}^{s_+} = d_{s_0,k}^{s_+} + [\mathfrak{d}_k^{s_0,s_-}, p_0] - \lambda_{k+1}, \quad k \in \mathbb{N}_0, \tag{A.3}$$

where p_0 is defined in [KW2, p. 759]. It is essential for our present purposes to give a more explicit relation between $d_{s_-,j}^{s_+}$ and $d_{s_0,j}^{s_+}$.

A.3 Proposition. *We have*

$$d_{s_-,k}^{s_+} = d_{s_0,k}^{s_+} - \sum_{j=1}^{k+1} \mathfrak{w}_{s_0,k+1-j}^{s_+}(s_0)_2 \mathfrak{w}_{s_-,j}^{s_+}(s_0)_1, \quad k \in \mathbb{N}_0.$$

Proof. According to Proposition IV.4.7, we have

$$B^j \left(\chi_{[s_-,\sigma)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = -\mathfrak{w}_{s_-,j+1}^{s_+}(s_-)_2$$

and hence

$$[\mathfrak{d}_k^{s_0,s_-}, p_0] = -\sum_{j=0}^{k-1} \lambda_{k-j} \mathfrak{w}_{s_-,j+1}^{s_+}(s_-)_2 = \sum_{j=0}^{k-1} \mathfrak{w}_{s_0,k-j}^{s_+}(s_-)_1 \mathfrak{w}_{s_-,j+1}^{s_+}(s_-)_2$$
$$= \sum_{j=1}^{k} \mathfrak{w}_{s_0,k+1-j}^{s_+}(s_-)_1 \mathfrak{w}_{s_-,j}^{s_+}(s_-)_2\,.$$

Hence

$$[\mathfrak{d}_k^{s_0,s_-}, p_0] - \lambda_{k+1} = \sum_{j=0}^{k} \mathfrak{w}^{s_+}_{s_0,k+1-j}(s_-)_1 \mathfrak{w}^{s_+}_{s_-,j}(s_-)_2$$

$$= -\sum_{j=0}^{k} \mathfrak{w}^{s_+}_{s_0,k+1-j}(s_-)^* J \mathfrak{w}^{s_+}_{s_-,j}(s_-)$$

$$= -\sum_{j=0}^{k+1} \mathfrak{w}^{s_+}_{s_0,k+1-j}(s_-)^* J \mathfrak{w}^{s_+}_{s_-,j}(s_-)$$

$$\overset{(*)}{=} -\sum_{j=0}^{k+1} \mathfrak{w}^{s_+}_{s_0,k+1-j}(s_0)^* J \mathfrak{w}^{s_+}_{s_-,j}(s_0)$$

$$= -\sum_{j=1}^{k+1} \mathfrak{w}^{s_+}_{s_0,k+1-j}(s_0)^* J \mathfrak{w}^{s_+}_{s_-,j}(s_0) = -\sum_{j=1}^{k+1} \mathfrak{w}^{s_+}_{s_0,k+1-j}(s_0)_2 \mathfrak{w}^{s_+}_{s_-,j}(s_0)_1 \, .$$

The equality sign marked with $(*)$ is obtained from Remark A.2 applied to the elements

$$0, \mathfrak{w}^{s_+}_{s_-,0}, \ldots, \mathfrak{w}^{s_+}_{s_-,k+1};\ 0, \mathfrak{w}^{s_+}_{s_0,0}, \ldots, \mathfrak{w}^{s_+}_{s_0,k+1} \in L^2(H_{\urcorner s_0}).$$

❑

Now we are in position to prove formulae for $\mathring{\kappa}_{s_0,s_-}$.

A.4 Proposition. *The action of $\mathring{\kappa}_{s_0,s_-}$ is determined by linearity and*

$$\mathring{\kappa}_{s_0,s_-}(f;0,0,0) = \Big(f; \Big(\int_{s_-}^{s_0} (\mathfrak{w}^{s_+}_{s_-,j})^* H f + \int_{s_0}^{\sigma} (\mathfrak{d}_j^{s_0,s_-})^* H f\Big)_{j=0}^{\Delta-1}, 0, 0\Big),$$

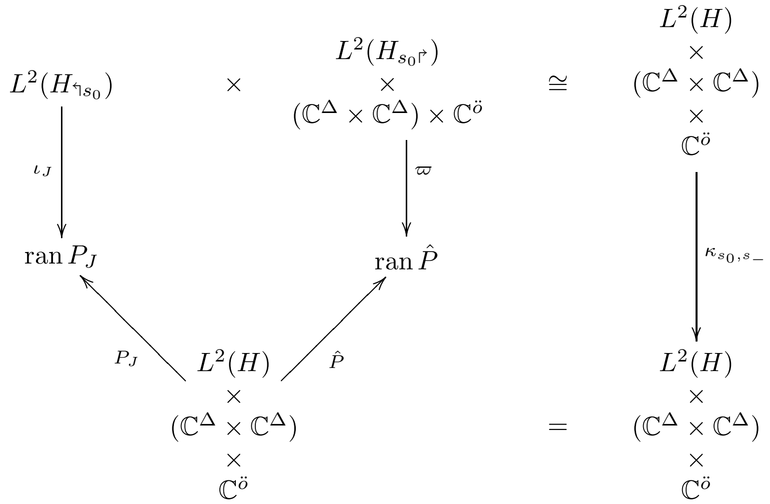
$$\mathring{\kappa}_{s_0,s_-}(0;\xi,0,\alpha) = (0;\xi,0,\alpha),$$

$$\mathring{\kappa}_{s_0,s_-}(0;0,\varepsilon_k,0) = \Big(-\chi_{\urcorner s_0}\mathfrak{w}^{s_+}_{s_-,k} - \chi_{s_0\ulcorner}\mathfrak{d}_k^{s_0,s_-};$$

$$\Big(\frac{1}{2}\sum_{l=j+1}^{k+j+1} \mathfrak{w}^{s_+}_{s_0,k+j+1-l}(s_0)_2 \mathfrak{w}^{s_+}_{s_-,l}(s_0)_1 - \frac{1}{2}\sum_{l=1}^{j} \mathfrak{w}^{s_+}_{s_0,k+j+1-l}(s_0)_2 \mathfrak{w}^{s_+}_{s_-,l}(s_0)_1\Big)_{j=0}^{\Delta-1},$$

$$\varepsilon_k, 0\Big).$$

Proof. The construction of $\mathring{\kappa}_{s_0,s_-}$ in [KW2, §7] was carried out in a two-step procedure according to the following diagram:



where the notation of [KW2, §7] is used. The second formula in the present assertion is apparent, since we have

$$\begin{aligned}\varpi(0;\xi,0,\alpha) &= (0;\xi,0,\alpha),\\ \hat{P}(0;\xi,0,\alpha) &= (0;\xi,0,\alpha).\end{aligned}$$

The first formula is also easy to see: from Lemma IV.7.1 and the proof of Proposition IV.4.14 we obtain

$$\begin{aligned}&P_J\Big(f;\big(\int_{s_-}^{s_0}(\mathfrak{w}_{s_-,j}^{s_+})^*Hf+\int_{s_0}^{\sigma}(\mathfrak{d}_j^{s_0,s_-})^*Hf\big)_{j=0}^{\Delta-1},0,0\Big)\\ &\quad=\Big(\chi_{\urcorner s_0}f;\big(\int_{s_-}^{s_0}(\mathfrak{w}_{s_-,j}^{s_+})^*Hf\big)_{j=0}^{\Delta-1},0,0\Big)=\iota_J(\chi_{\urcorner s_0}f).\end{aligned}$$

Using $\hat{P}=I-P_J$ and the definition (IV.7.2) of ϖ, it follows that

$$\begin{aligned}&\hat{P}\Big(f;\big(\int_{s_-}^{s_0}(\mathfrak{w}_{s_-,j}^{s_+})^*Hf+\int_{s_0}^{\sigma}(\mathfrak{d}_j^{s_0,s_-})^*Hf\big)_{j=0}^{\Delta-1},0,0\Big)\\ &\quad=\Big(\chi_{s_0\ulcorner}f;\big(\int_{s_0}^{\sigma}(\mathfrak{d}_j^{s_0,s_-})^*Hf\big)_{j=0}^{\Delta-1},0,0\Big)=\varpi(\chi_{s_0\ulcorner}f;0,0,0).\end{aligned}$$

Together we obtain the first formula of the present assertion.

We come to the proof of the last asserted formula. By the definition (IV.7.2) of ϖ and the formulae in Remark IV.7.5, we have

$$\varpi(0;0,\varepsilon_k,0) = \varpi(\tilde p_k) - \varpi\Big(0;(\frac{1}{2}d^{s_+}_{s_0,k+j})_{j=0}^{\Delta-1},0,0\Big)$$

$$= \hat P(p_k - \mathfrak{d}^{s_0,s_-}_k) - \Big(0;(\frac{1}{2}d^{s_+}_{s_0,k+j})_{j=0}^{\Delta-1},0,0\Big)$$

$$= \hat P\Big(0;(\frac{1}{2}d^{s_+}_{s_-,k+j})_{j=0}^{\Delta-1},\varepsilon_k,0\Big) - \hat P\Big(\mathfrak{d}^{s_0,s_-}_k;([\mathfrak{d}^{s_0,s_-}_k,p_j])_{j=0}^{\Delta-1},0,0\Big)$$

$$- \Big(0;(\frac{1}{2}d^{s_+}_{s_0,k+j})_{j=0}^{\Delta-1},0,0\Big)$$

$$= \big(-\chi_{\urcorner s_0}\mathfrak{w}^{s_+}_{s_-,k} - \chi_{s_0\ulcorner}\mathfrak{d}^{s_0,s_-}_k;(\xi_{kj})_{j=0}^{\Delta-1},\varepsilon_k,0\big)$$

where

$$\xi_{kj} := \frac{1}{2}(d^{s_+}_{s_-,k+j} - d^{s_+}_{s_0,k+j}) - \int_{s_-}^{s_0}(\mathfrak{w}^{s_+}_{s_-,j})^* H \mathfrak{w}^{s_+}_{s_-,k} - [\mathfrak{d}^{s_0,s_-}_k,p_j]$$

$$+ \int_{s_-}^{s_0}(\mathfrak{w}^{s_+}_{s_-,j})^* H \mathfrak{d}^{s_0,s_-}_k .$$

By Proposition A.3 we have

$$-\frac{1}{2}(d^{s_+}_{s_-,k+j} - d^{s_+}_{s_0,k+j}) = \frac{1}{2}\sum_{l=1}^{k+j+1}\mathfrak{w}^{s_+}_{s_0,k+j+1-l}(s_0)_2\mathfrak{w}^{s_+}_{s_-,l}(s_0)_1 .$$

Remark A.2 applied to the functions

$$\mathfrak{w}^{s_+}_{s_0,k},\dots,\mathfrak{w}^{s_+}_{s_0,k+j+1};\ 0,\mathfrak{w}^{s_+}_{s_-,0},\dots,\mathfrak{w}^{s_+}_{s_-,j} \in L^2(H|_{(s_-,s_0)})$$

gives

$$\int_{s_-}^{s_0}(\mathfrak{w}^{s_+}_{s_-,j})^* H\mathfrak{w}^{s_+}_{s_-,k} - \int_{s_-}^{s_0}(\mathfrak{w}^{s_+}_{s_-,j})^* H\mathfrak{d}^{s_0,s_-}_k = \int_{s_-}^{s_0}(\mathfrak{w}^{s_+}_{s_-,j})^* H\mathfrak{w}^{s_+}_{s_0,k}$$

$$= \sum_{l=1}^{j+1}\mathfrak{w}^{s_+}_{s_-,j+1-l}(s_-)^* J\mathfrak{w}^{s_+}_{s_0,k+l}(s_-) - \sum_{l=1}^{j+1}\mathfrak{w}^{s_+}_{s_-,j+1-l}(s_0)^* J\mathfrak{w}^{s_+}_{s_0,k+l}(s_0) .$$

Moreover, as we saw in the proof of Lemma IV.7.7, and by (A.3)

$$[\mathfrak{d}^{s_0,s_-}_k,p_j] = [\mathfrak{d}^{s_0,s_-}_{k+j},p_0] + \sum_{l=1}^{j}\mathfrak{w}^{s_+}_{s_-,l}(s_-)^* J\mathfrak{d}^{s_0,s_-}_{k+j+1-l}(s_-)$$

$$= d^{s_+}_{s_-,k+j} - d^{s_+}_{s_0,k+j} + \lambda_{k+j+1} + \sum_{l=1}^{j}\mathfrak{w}^{s_+}_{s_-,l}(s_-)_2\mathfrak{d}^{s_0,s_-}_{k+j+1-l}(s_-)_1$$

$$= d^{s_+}_{s_-,k+j} - d^{s_+}_{s_0,k+j} - \mathfrak{w}^{s_+}_{s_0,k+j+1}(s_-)_1 - \sum_{l=1}^{j}\mathfrak{w}^{s_+}_{s_-,l}(s_-)_2\mathfrak{w}^{s_0,s_-}_{k+j+1-l}(s_-)_1$$

$$= d^{s_+}_{s_-,k+j} - d^{s_+}_{s_0,k+j} - \sum_{l=0}^{j}\mathfrak{w}^{s_+}_{s_-,l}(s_-)_2\mathfrak{w}^{s_0,s_-}_{k+j+1-l}(s_-)_1 .$$

Hence we obtain

$$\begin{aligned}
\xi_{kj} &= \frac{1}{2}(d^{s_+}_{s_-,k+j} - d^{s_+}_{s_0,k+j}) - \Big(\sum_{l=1}^{j+1} \mathfrak{w}^{s_+}_{s_-,j+1-l}(s_-)^* J \mathfrak{w}^{s_+}_{s_0,k+l}(s_-) \\
&\quad - \sum_{l=1}^{j+1} \mathfrak{w}^{s_+}_{s_-,j+1-l}(s_0)^* J \mathfrak{w}^{s_+}_{s_0,k+l}(s_0) \Big) \\
&\quad - \Big(d^{s_+}_{s_-,k+j} - d^{s_+}_{s_0,k+j} - \sum_{l=0}^{j} \mathfrak{w}^{s_+}_{s_-,l}(s_-)_2 \mathfrak{w}^{s_0,s_-}_{k+j+1-l}(s_-)_1 \Big) \\
&= -\frac{1}{2}(d^{s_+}_{s_-,k+j} - d^{s_+}_{s_0,k+j}) - \sum_{l=1}^{j+1} \mathfrak{w}^{s_+}_{s_-,j+1-l}(s_-)^* J \mathfrak{w}^{s_+}_{s_0,k+l}(s_-) \\
&\quad - \sum_{l=1}^{j+1} \mathfrak{w}^{s_+}_{s_-,j+1-l}(s_0)_1 \mathfrak{w}^{s_+}_{s_0,k+l}(s_0)_2 + \sum_{l=0}^{j} \mathfrak{w}^{s_+}_{s_-,l}(s_-)_2 \mathfrak{w}^{s_+}_{s_0,k+j+1-l}(s_-)_1 \\
&= \frac{1}{2} \sum_{l=1}^{k+j+1} \mathfrak{w}^{s_+}_{s_0,k+j+1-l}(s_0)_2 \mathfrak{w}^{s_+}_{s_-,l}(s_0)_1 - \sum_{l=1}^{j} \mathfrak{w}^{s_+}_{s_-,j+1-l}(s_0)_1 \mathfrak{w}^{s_+}_{s_0,k+l}(s_0)_2 \\
&= \frac{1}{2} \sum_{l=j+1}^{k+j+1} \mathfrak{w}^{s_+}_{s_0,k+j+1-l}(s_0)_2 \mathfrak{w}^{s_+}_{s_-,l}(s_0)_1 - \frac{1}{2} \sum_{l=1}^{j} \mathfrak{w}^{s_+}_{s_0,k+j+1-l}(s_0)_2 \mathfrak{w}^{s_+}_{s_-,l}(s_0)_1 \,.
\end{aligned}$$

Now the relations

$$\begin{aligned}
& P_J\big(-\chi_{\Lsh s_0} \mathfrak{w}^{s_+}_{s_-,k} - \chi_{s_0 \Rsh} \mathfrak{d}^{s_0,s_-}_k ; (\xi_{kj})_{j=0}^{\Delta-1}, \varepsilon_k, 0 \big) = 0, \\
& \hat{P}\big(-\chi_{\Lsh s_0} \mathfrak{w}^{s_+}_{s_-,k} - \chi_{s_0 \Rsh} \mathfrak{d}^{s_0,s_-}_k ; (\xi_{kj})_{j=0}^{\Delta-1}, \varepsilon_k, 0 \big) \\
& \qquad\qquad\qquad = \big(-\chi_{\Lsh s_0} \mathfrak{w}^{s_+}_{s_-,k} - \chi_{s_0 \Rsh} \mathfrak{d}^{s_0,s_-}_k ; (\xi_{kj})_{j=0}^{\Delta-1}, \varepsilon_k, 0 \big)
\end{aligned}$$

show the third formula of the proposition. ✌

Splitting to the right of $\boldsymbol{\sigma}$.

Of course, similar considerations can be made when the splitting point s_0 belongs to I_+ instead of I_-. As noted in Remark IV.7.9, the isomorphism κ_{s_0,s_+} can be constructed using the previous case $s_0 \in I_-$ and an order-reversing reparameterization. In order to obtain explicit formulae, we have to carry out this argument in some more detail.

A.5 Definition. Let $\mathfrak{h}$ be an elementary indefinite Hamiltonian of kind (A) which is given by the data $H, \ddot{o}, b_j, d_j$ and where H is defined on $[s_-, \sigma) \cup (\sigma, s_+]$. Define an elementary indefinite Hamiltonian $\overleftarrow{\mathfrak{h}}$ of kind (A) on the interval $[-s_+, -\sigma) \cup (-\sigma, -s_-]$ as the collection of data

$$\overleftarrow{H}(t) := H(-t), \quad \overleftarrow{\ddot{o}} := \ddot{o}, \ \overleftarrow{b_j} := (-1)^{\ddot{o}-j} b_j, \quad \overleftarrow{d_j} := (-1)^j d_j \,.$$

//

By Lemma IV.5.19, there exists an isomorphism between $\mathfrak{P}(\mathfrak{h})$ and $\mathfrak{P}(\overleftarrow{\mathfrak{h}})$ which transfers $T(\mathfrak{h})$ to $-T(\overleftarrow{\mathfrak{h}})$ and reverses boundary values. More precisely, working in terms of the isomorphic copies $\mathring{\mathfrak{B}}(\mathfrak{h})$ and $\mathring{\mathfrak{B}}(\overleftarrow{\mathfrak{h}})$, it is easy to see that the map $\overleftarrow{\kappa}_{\mathfrak{h}}\colon \mathring{\mathfrak{P}}(\mathfrak{h}) \to \mathring{\mathfrak{P}}(\overleftarrow{\mathfrak{h}})$ defined as

$$\overleftarrow{\kappa}_{\mathfrak{h}}\colon (f;\xi,\lambda,\alpha) \mapsto \big(f(-t);((-1)^j\xi_j)_{j=0}^{\Delta-1},((-1)^j\lambda_j)_{j=0}^{\Delta-1},((-1)^{j+\Delta-1}\alpha_j)_{j=1}^{\ddot{o}}\big)$$

is an isomorphism and has the property

$$\big((F;G);(a;b)\big)\in\mathring{\Gamma}(\mathfrak{h}) \iff \big((\overleftarrow{\kappa}_{\mathfrak{h}}\,F;-\overleftarrow{\kappa}_{\mathfrak{h}}\,G);(b;a)\big)\in\mathring{\Gamma}(\overleftarrow{\mathfrak{h}}).$$

If H is a positive definite regular Hamiltonian defined on an interval (s_-,s_+), then

$$\overleftarrow{H}(t) := H(-t)$$

is a positive definite regular Hamiltonian defined on the interval $(-s_+,-s_-)$. The map

$$\overleftarrow{\kappa}_H\colon f\mapsto f(-t)$$

is an isomorphism of $L^2(H)$ onto $L^2(\overleftarrow{\mathfrak{h}})$ and satisfies

$$\big((F;G);(a;b)\big)\in\Gamma(H) \iff \big((\overleftarrow{\kappa}_H\,F;-\overleftarrow{\kappa}_H\,G);(b;a)\big)\in\Gamma(\overleftarrow{H}),$$

cf. Lemma IV.2.6.

These order-reversing isomorphisms are compatible with the pasting of boundary triples: let $\mathfrak{h}$ be an elementary indefinite Hamiltonian of kind (A) defined on $[s_-,\sigma)\cup(\sigma,s_0]$, and let H be a positive definite regular Hamiltonian defined on (s_0,s_+). Then the map $\overleftarrow{\kappa}_{\mathfrak{h},H}$ which is defined as

$$\overleftarrow{\kappa}_{\mathfrak{h},H}\colon F\oplus f\mapsto \overleftarrow{\kappa}_H\,f\oplus\overleftarrow{\kappa}_{\mathfrak{h}}\,F$$

is an isomorphism of $\mathring{\mathfrak{P}}(\mathfrak{h})\oplus L^2(H)$ onto $L^2(\overleftarrow{H})\oplus\mathring{\mathfrak{P}}(\overleftarrow{\mathfrak{h}})$, and we have

$$\big((F\oplus f;G\oplus g);(a;b)\big)\in\mathring{\Gamma}(\mathfrak{h})\uplus\Gamma(H) \iff$$
$$\big((\overleftarrow{\kappa}_{\mathfrak{h},H}\,(F\oplus f);-\overleftarrow{\kappa}_{\mathfrak{h},H}\,(G\oplus g));(b;a)\big)\in\Gamma(\overleftarrow{H})\uplus\mathring{\Gamma}(\overleftarrow{\mathfrak{h}})$$

Now we are ready to describe the splitting isomorphism for splitting to the right of the singularity. Functions $\mathfrak{w}^{s_0}_{s_-,k}$ are defined correspondingly, and we set $\mathfrak{d}_k^{s_0,s_+} := \mathfrak{w}^{s_+}_{s_-,k}-\mathfrak{w}^{s_0}_{s_-,k}$. Moreover, let $\mathfrak{h}^{s_0}_{s_-} = (H\restriction_{s_0};\ddot{o},b_j;d^{s_0}_{s_-,j})$ be the elementary indefinite Hamiltonian obtained by splitting $\mathfrak{h}^{s_+}_{s_-}$ at s_0, and let $\mathring{\kappa}_{s,s_+}$ be the isomorphism of $L^2(H)\,[\dot{+}]\,(\mathbb{C}^\Delta\dot{+}\mathbb{C}^\Delta)\,[\dot{+}]\,\mathbb{C}^{\ddot{o}}$ onto itself defined by the diagram corresponding to (A.1).

A.6 Proposition. *Let $\mathfrak{h}^{s_+}_{s_-} = (H;\ddot{o},b_j;d^{s_+}_{s_-,j})$ be an elementary indefinite Hamiltonian defined on $[s_-,\sigma)\cup(\sigma,s_+]$, and let $s_0\in I_+$. With the notation described above, the numbers $d^{s_+}_{s_-,k}$ and $d^{s_0}_{s_-,k}$ are related as follows:*

$$d^{s_0}_{s_-,k} = d^{s_+}_{s_-,k} - \sum_{j=1}^{k+1}\mathfrak{w}^{s_0}_{s_-,k+1-j}(s_0)_2\,\mathfrak{w}^{s_+}_{s_-,j}(s_0)_1,\quad k\in\mathbb{N}_0.$$

The action of the map $\mathring{\kappa}_{s_0,s_+}$ *is given by linearity and*

$$\mathring{\kappa}_{s_0,s_+}(f;0,0,0) = \Big(f;\big(\int_{s_0}^{s_+}(\mathfrak{w}_{s_-,j}^{s_+})^* Hf + \int_{\sigma}^{s_0}(\mathfrak{d}_j^{s_0,s_+})^* Hf\big)_{j=0}^{\Delta-1},0,0\Big)$$

$$\mathring{\kappa}_{s_0,s_+}(0;\xi,0,\alpha) = (0;\xi,0,\alpha)$$

$$\mathring{\kappa}_{s_0,s_+}(0;0,\varepsilon_k,0) = \Big(-\chi_{s_0\upharpoonright}\mathfrak{w}_{s_-,k}^{s_+} - \chi_{\upharpoonleft s_0}\mathfrak{d}_k^{s_0,s_+};$$

$$\big(-\frac{1}{2}\sum_{l=j+1}^{k+j+1}\mathfrak{w}_{s_-,k+j+1-l}^{s_0}(s_0)_2\mathfrak{w}_{s_-,l}^{s_+}(s_0)_1 + \frac{1}{2}\sum_{l=1}^{j}\mathfrak{w}_{s_-,k+j+1-l}^{s_0}(s_0)_2\mathfrak{w}_{s_-,l}^{s_+}(s_0)_1\big)_{j=0}^{\Delta-1},$$

$$\varepsilon_k,0\Big),$$

Proof. Define elementary indefinite Hamiltonians of kind (A) by

$$\mathfrak{h}_{-s_+}^{-s_-} := (\overleftarrow{H};\ddot{o},(-1)^{\ddot{o}-j}b_j;d_{-s_+,j}^{-s_-}),$$
$$\mathfrak{h}_{-s_0}^{-s_-} := (\overleftarrow{H}_{-s_0\upharpoonright};\ddot{o},(-1)^{\ddot{o}-j}b_j;d_{-s_0,j}^{-s_-})$$

where

$$d_{-s_+,k}^{-s_-} := (-1)^k d_{s_-,k}^{s_+},$$
$$d_{-s_0,k}^{-s_-} := d_{-s_+,k}^{-s_-} + \sum_{j=1}^{k+1}\mathfrak{w}_{-s_0,k+1-j}^{-s_-}(-s_0)_2\mathfrak{w}_{-s_+,j}^{-s_-}(-s_0)_1 .$$

Then, apparently, $\mathfrak{h}_{-s_+}^{-s_-} = \overleftarrow{\mathfrak{h}_{s_-}^{s_+}}$, and $\mathfrak{h}_{-s_0}^{-s_-}$ is just the Hamiltonian which appears in the splitting of $\mathfrak{h}_{-s_+}^{-s_-}$ at the point $-s_0$ according to the previous subsection, cf. Proposition A.3. Moreover, since we have

$$\mathfrak{w}_{-s_+,k}^{-s_-}(t) = (-1)^k\mathfrak{w}_{s_-,k}^{s_+}(-t),\ \mathfrak{w}_{-s_0,k}^{-s_-}(t) = (-1)^k\mathfrak{w}_{s_-,k}^{s_0}(-t), \tag{A.4}$$

and thus also $\mathfrak{d}_k^{-s_0,-s_+}(t) = (-1)^k\mathfrak{d}_k^{s_0,s_+}(-t)$, we see that

$$\mathfrak{h}_{-s_0}^{-s_-} = \overleftarrow{\mathfrak{h}_{s_-}^{s_0}} .$$

It follows that we have isomorphisms

$$\mathring{\mathfrak{P}}(\mathfrak{h}_{s_-}^{s_0}) \oplus L^2(H_{s_0\upharpoonright}) \overset{\overleftarrow{\kappa}_{\mathfrak{h}_{s_-}^{s_0},H_{s_0\upharpoonright}}}{\longrightarrow}$$
$$L^2(\overleftarrow{H}_{\upharpoonleft -s_0}) \oplus \mathring{\mathfrak{P}}(\mathfrak{h}_{-s_0}^{-s_-}) \overset{\kappa_{-s_0,-s_-}}{\longrightarrow} \mathring{\mathfrak{P}}(\mathfrak{h}_{-s_+}^{-s_-}) \overset{\overleftarrow{\kappa}_{\mathfrak{h}_{-s_+}^{-s_-}}}{\longrightarrow} \mathring{\mathfrak{P}}(\mathfrak{h}_{s_-}^{s_+}) .$$

Moreover, their composition

$$\kappa := \overleftarrow{\kappa}_{\mathfrak{h}_{-s_+}^{-s_-}} \circ \kappa_{-s_0,-s_-} \circ \overleftarrow{\kappa}_{\mathfrak{h}_{s_-}^{s_0},H_{s_0\upharpoonright}}$$

satisfies

$$\big((\kappa\times\kappa)\times(\mathrm{id}\times\mathrm{id})\big)\big(\mathring{\Gamma}(\mathfrak{h}_{s_-}^{s_0}) \uplus \Gamma(H_{s_0\upharpoonright})\big) = \mathring{\Gamma}(\mathfrak{h}_{s_-}^{s_+}) .$$

Putting together the formulae for $\overleftarrow{\kappa}_{\mathfrak{h}_{-s_+}^{-s_-}}$, $\kappa_{-s_0,-s_-}$, and $\overleftarrow{\kappa}_{\mathfrak{h}_{s_-}^{s_0},H_{s_0}\urcorner}$, as given in their definition and in Proposition A.4, and keeping in mind the relations (A.4), a straightforward calculation yields that $\kappa = \kappa_{s_0,s_+}$ as defined in the statement of the present proposition. ❑

Transitivity of splitting isomorphisms.

Let an elementary indefinite Hamiltonian $\mathfrak{h}_{s_-}^{s_+} = (H; \ddot{o}, b_j; d_{s_-,j}^{s_+})$ of kind (A) be given, and assume that s_0 and s_1 with $\sigma < s_0 < s_1 < s_+$ are both not inner points of indivisible intervals. From Proposition A.6 we obtain numbers $d_{s_-,j}^{s_0}$ and $d_{s_-,j}^{s_1}$, and isomorphisms κ_{s_0,s_+} and κ_{s_1,s_+}. Applying Proposition A.6 with the Hamiltonian $\mathfrak{h}_{s_-}^{s_1}$ and the splitting point s_0 we get a Hamiltonian $\tilde{\mathfrak{h}}_{s_-}^{s_0}$ with corresponding parameters $\tilde{d}_{s_-,j}^{s_0}$ and an isomorphism κ_{s_0,s_1}. Altogether, we find ourselves in the following situation:

$$\begin{array}{ccc}
\begin{array}{c}\mathfrak{P}(\tilde{\mathfrak{h}}_{s_-}^{s_0}) \times L^2(H|_{(s_0,s_1)}) \\ \times \\ L^2(H|_{(s_1,s_+)})\end{array} & \xrightarrow{\kappa_{s_0,s_1} \times \mathrm{id}} & \mathfrak{P}(\mathfrak{h}_{s_-}^{s_1}) \times L^2(H|_{(s_1,s_+)}) \\
 & & \Big\downarrow \kappa_{s_1,s_+} \\
\mathfrak{P}(\mathfrak{h}_{s_-}^{s_0}) \times L^2(H|_{(s_0,s_+)}) & \xrightarrow[\kappa_{s_0,s_+}]{} & \mathfrak{P}(\mathfrak{h}_{s_-}^{s_+})
\end{array}$$

A.7 Proposition. *In the situation described above, we have*

$$\tilde{\mathfrak{h}}_{s_-}^{s_0} = \mathfrak{h}_{s_-}^{s_0} \ \textit{and}\ \kappa_{s_1,s_+} \circ (\kappa_{s_0,s_1} \times \mathrm{id}) = \kappa_{s_0,s_+}$$

Proof. This statement can be deduced easily from Proposition IV.5.18, when one slightly changes the point of view. Define $\tilde{\mathfrak{h}}_{s_-}^{s_+}$ by specifying parameters

$$\tilde{d}_{s_-,k}^{s_+} := d_{s_-,k}^{s_0} + \sum_{j=0}^{k+1} \mathfrak{w}_{s_-,k+1-j}^{s_0}(s_0)_2 \mathfrak{w}_{s_-,j}^{s_+}(s_0)_1 \,.$$

Then Proposition A.6 furnishes us with an isomorphism

$$\tilde{\kappa}_{s_0,s_+} : \mathfrak{P}(\tilde{\mathfrak{h}}_{s_-}^{s_0}) \times L^2(H|_{(s_0,s_+)}) \to \mathfrak{P}(\tilde{\mathfrak{h}}_{s_-}^{s_+}).$$

From the formulae in Proposition A.6 it is apparent that $\tilde{\kappa}_{s_0,s_+} = \kappa_{s_0,s_+}$. Thus $(\kappa_{s_1,s_+} \circ (\kappa_{s_0,s_1} \times \mathrm{id}) \circ \kappa_{s_0,s_+}^{-1}; \mathrm{id})$ is an isomorphism between the boundary triples $\mathfrak{B}(\tilde{\mathfrak{h}}_{s_-}^{s_+})$ and $\mathfrak{B}(\mathfrak{h}_{s_-}^{s_+})$. By Proposition IV.5.18 and its proof it follows that

$$\tilde{d}_{s_-,k}^{s_+} = d_{s_-,k}^{s_+}, \qquad \kappa_{s_1,s_+} \circ (\kappa_{s_0,s_1} \times \mathrm{id}) \circ \kappa_{s_0,s_+}^{-1} = \mathrm{id}\,.$$

The equality $\tilde{d}_{s_-,k}^{s_+} = d_{s_-,k}^{s_+}$ clearly implies that also $\tilde{d}_{s_-,k}^{s_0} = d_{s_-,k}^{s_0}$. ❑

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