Finite-dimensional de Branges subspaces generated by majorants

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Abstract. If $\mathcal{H}(E)$ is a de Branges space and ω is a nonnegative function on $\mathbb{R}$, define a de Branges subspace of $\mathcal{H}(E)$ by

$$\mathcal{R}_\omega(E) = \operatorname{Clos}_{\mathcal{H}(E)} \big\{F \in \mathcal{H}(E) :\ \exists C > 0 : |E^{-1}F| \leq C\omega \text{ on } \mathbb{R}\big\}\,.$$

It is known that one-dimensional de Branges subspaces generated in this way are related to minimal majorants. We investigate finite-dimensional de Branges subspaces, their representability in terms of majorants, and their relation to minimal majorants.

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1. Introduction

The theory of Hilbert spaces $\mathcal{H}(E)$ of entire functions founded by L.de Branges is an important branch of analysis. After its foundation in [dB1]–[dB6], it was further developed by many authors. It is an example for a fruitful interplay of function theory and operator theory, and has applications in mathematical physics, see e.g. [R].

Recently, in the context of model subspaces $H^2(\mathbb{C}^+)\ominus\Theta H^2(\mathbb{C}^+)$ of the Hardy space $H^2(\mathbb{C}^+)$, V. Havin and J. Mashregi introduced the notion of *admissible majorants*, i.e. functions ω on the real line which majorize a nonzero element of the space, cf. [HM1], [HM2]. The interest to this problem was motivated by the famous Beurling–Malliavin Multiplier Theorem, cf. [HJ], [K]. The approach suggested in [HM1], [HM2] has led to a new, essentially simpler, proof of the Beurling–Malliavin Theorem cf. [HMN]. These ideas were further developped in [BH], [BBH], where interesting connections between majorization and other problems of function theory (polynomial approximation, quasianalitycity) were discovered.

A de Branges space $\mathcal{H}(E)$ is isomorphic to the model subspace $H^2(\mathbb{C}^+) \ominus \frac{E^\#}{E} H^2(\mathbb{C}^+)$. Hence the theory of admissible majorants can be applied to de Branges spaces. Due to the rich (analytic) structure of de Branges spaces, however, much more specific results than in the general case can be expected.

In de Branges' theory the notion of *de Branges subspaces*, i.e. subspaces of a space $\mathcal{H}(E)$ which are themselves de Branges spaces, plays an outstanding role. At this point a link with the theory of admissible majorants occurs: Given an admissible majorant ω for a de Branges space $\mathcal{H}(E)$, the space

$$\mathcal{R}_\omega(E) := \mathrm{Clos}_{\mathcal{H}(E)} \big\{ F \in \mathcal{H}(E) : \ \exists C > 0 : |E^{-1}F| \leq C\omega \text{ on } \mathbb{R} \big\} \tag{1.1}$$

is a de Branges subspace of $\mathcal{H}(E)$. This relationship was investigated in [BW]. There the set of all those de Branges subspaces of a given space $\mathcal{H}(E)$ which can be represented in this way was determined, and it was shown that minimal majorants correspond to one-dimensional de Branges subspaces.

In the present paper we investigate the representability of finite-dimensional de Branges subspaces by means of admissible majorants. These considerations are based on our previous work [BW]. Moreover, the relation of finite-dimensional de Branges subspaces with minimal majorants is made explicit.

Let us briefly describe the content of this paper. In the preliminary Section 2, we set up some notation and recall some basic facts on de Branges spaces which are essential for furher use. In Section 3 we prove our main result, Theorem 3.8. Besides the results of [BW], it is based on a thorough understanding of the family of majorants $\omega^{[k]}(x) := (1+|x|)^k \omega(x)$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Finally, in Section 4, we turn to the case of infinite-dimensional de Branges subspaces. In this general setting, the situation is more complicated. However, some positive result can be established, cf. Proposition 4.2.

2. Preliminaries on de Branges spaces

An entire function E is said to belong to the *Hermite-Biehler class* $\mathcal{H}B$, if it satisfies

$$|E(\bar{z})| < |E(z)|, \quad z \in \mathbb{C}^+ .$$

In what follows, for any function F, we denote by $F^\#$ the function $F^\#(z) := \overline{F(\bar{z})}$.

2.1. Definition. If $E \in \mathcal{H}B$, the *de Branges space* $\mathcal{H}(E)$ is defined as the set of all entire functions F which have the property that

$$\frac{F}{E}, \frac{F^\#}{E} \in H^2(\mathbb{C}^+) .$$

Moreover, $\mathcal{H}(E)$ will be endowed with the norm

$$\|F\|_E := \Big(\int_{\mathbb{R}} \Big| \frac{F(t)}{E(t)} \Big|^2 \, dt \Big)^{1/2}, \quad F \in \mathcal{H}(E) .$$

It is shown in [dB7, Theorem 21] that $\mathcal{H}(E)$ is a Hilbert space with respect to the norm $\|.\|_E$.

2.2. Definition. A subset $\mathcal{L}$ of a de Branges space $\mathcal{H}(E)$ is called a *de Branges subspace*, if it is itself, with the norm inherited from $\mathcal{H}(E)$, a de Branges space. The set of all de Branges subspaces of a given space $\mathcal{H}(E)$ will be denoted as $\operatorname{Sub}(E)$.

The fact that $\mathcal{L}$ is a de Branges subspace of $\mathcal{H}(E)$ thus means that there exists $E_1 \in \mathcal{H}B$ such that $\mathcal{L} = \mathcal{H}(E_1)$ and $\|F\|_{E_1} = \|F\|_E$, $F \in \mathcal{L}$.

If F is an entire function, denote by $\mathfrak{d}(F)$ its zero-divisor, i.e. the map $\mathfrak{d}(F) : \mathbb{C} \to \mathbb{N}_0$ which assigns to each point w its multiplicity as a zero of F. If $\mathcal{H}(E)$ is a de Branges space, set

$$\mathfrak{d}(\mathcal{H}(E))(w) := \min\left\{\mathfrak{d}(F)(w) : F \in \mathcal{H}(E)\right\}.$$

It is shown in [dB7] that $\mathfrak{d}(\mathcal{H}(E))(w) = 0$, $w \in \mathbb{C}\setminus\mathbb{R}$, and $\mathfrak{d}(\mathcal{H}(E))(w) = \mathfrak{d}(E)(w)$, $w \in \mathbb{R}$.

If $E \in \mathcal{H}B$ and $\mathfrak{d} : \mathbb{R} \to \mathbb{N}_0$ are given, we denote

$$\operatorname{Sub}^{\mathfrak{d}}(E) := \left\{\mathcal{L} \in \operatorname{Sub}(E) : \mathfrak{d}(\mathcal{L}) = \mathfrak{d}\right\}.$$

Those subspaces $\mathcal{L} \in \operatorname{Sub}(E)$ with $\mathfrak{d}(\mathcal{L}) = \mathfrak{d}(\mathcal{H}(E))$ are the most interesting ones. To shorten notation we put $\operatorname{Sub}^s(E) := \operatorname{Sub}^{\mathfrak{d}(\mathcal{H}(E))}(E)$.

A milestone in de Branges' theory is the Ordering Theorem for subspaces of a space $\mathcal{H}(E)$, cf. [dB7, Theorem 35] (we state only a somewhat weaker version which suffices for our needs):

2.3. de Branges' Ordering Theorem: *Let $\mathcal{H}(E)$ be a de Branges space and let $\mathfrak{d} : \mathbb{R} \to \mathbb{N}_0$. Then $\operatorname{Sub}^{\mathfrak{d}}(E)$ is totally ordered with respect to set-theoretic inclusion.*

Even more about the structure of the chain $\operatorname{Sub}^{\mathfrak{d}}(E)$ is known. For every $\mathcal{H} \in \operatorname{Sub}^{\mathfrak{d}}(E)$, put

$$\mathcal{H}_- := \operatorname{Clos} \bigcup_{\substack{\mathcal{L}\in\operatorname{Sub}^{\mathfrak{d}}(E)\\ \mathcal{L}\subsetneq\mathcal{H}}} \mathcal{L}\,, \quad \mathcal{H}_+ := \bigcap_{\substack{\mathcal{L}\in\operatorname{Sub}^{\mathfrak{d}}(E)\\ \mathcal{L}\supsetneq\mathcal{H}}} \mathcal{L}.$$

Then $\mathcal{H}_-, \mathcal{H}_+ \in \operatorname{Sub}^{\mathfrak{d}}(E)$ and

$$\dim(\mathcal{H}/\mathcal{H}_-),\ \dim(\mathcal{H}/\mathcal{H}_+) \in \{0,1\}.$$

2.4. *Example.* Fundamental examples of de Branges spaces come from canonical systems of differential equations, cf. e.g. [dB7, Theorems 37,38], [GK], [HSW]. Let H be a 2×2-matrix valued function defined for $t \in [0,l]$, such that $H(t)$ is real and nonnegative, the entries of $H(t)$ belong to $L^1([0,l])$ and $H(t)$ does not vanish on any nonempty interval. We call an interval $(\alpha,\beta) \subseteq [0,l]$ H-indivisible, if for some $\varphi \in \mathbb{R}$ and some scalar function $h(t)$ we have

$$H(t) = h(t)\begin{pmatrix}\cos\varphi & \sin\varphi\end{pmatrix}^T \begin{pmatrix}\cos\varphi & \sin\varphi\end{pmatrix}, \quad \text{a.e. } t \in (\alpha,\beta).$$

Let $W(t,z)$ be the (unique) solution of the initial value problem

$$\frac{\partial}{\partial t}W(t,z)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = zW(t,z)H(t),\ t\in[0,l], \qquad W(0,z)=I\,,$$

and put $E_t(z) := A_t(z) - iB_t(z)$, $t\in[0,l]$, where $(A_t(z),B_t(z)) := (1,0)W(t,z)$. Then

(*i*) $E_t\in\mathcal{H}B$, $t\in(0,l]$, and $E_0=1$.

(*ii*) If $0<s\le t\le l$, then $\mathcal{H}(E_s)\subseteq\mathcal{H}(E_t)$ and the set-theoretic inclusion map is contractive. If s is not an inner point of an H-indivisible interval, it is actually isometric.

(*iii*) We have

$$\operatorname{Sub}^s(\mathcal{H}(E_l)) = \big\{\mathcal{H}(E_t) : t \text{ not inner point of an } H\text{-indivisible interval}\big\}\,.$$

2.5. *Example.* The *Paley-Wiener space* $\mathcal{P}W_a$, $a>0$, is defined as the space of all entire functions of exponential type at most a, whose restrictions to the real axis belong to $L^2(\mathbb{R})$. The norm in $\mathcal{P}W_a$ is given by the usual L^2-norm,

$$\|F\| := \Big(\int_{\mathbb{R}}|F(t)|^2\,dt\Big)^{1/2}, \quad F\in\mathcal{P}W_a\,.$$

By a theorem of Paley and Wiener, the space $\mathcal{P}W_a$ is the image under the Fourier transform of $L^2(-a,a)$. If in Example 2.4 we take $H(t)=I$, $t\in[0,l]$, we obtain $E_t(z)=e^{-itz}$. It is a consequence of a theorem of M.G. Krein, cf. [RR, Examples/Addenda 2, p. 134], that the space $\mathcal{H}(e^{-itz})$ coincides with $\mathcal{P}W_t$.

We see from Example 2.4, (*iii*), that $\operatorname{Sub}^s(\mathcal{P}W_a)=\{\mathcal{P}W_b : 0<b\le a\}$.

In the present paper we will mainly deal with finite-dimensional de Branges subspace of a given space $\mathcal{H}(E)$.

2.6. Definition. Let $E\in\mathcal{H}B$ and $\mathfrak{d}:\mathbb{R}\to\mathbb{N}_0$. Define

$$\operatorname{FSub}(E) := \big\{\mathcal{L}\in\operatorname{Sub}(E) : \dim\mathcal{L}<\infty\big\}\,,$$

$$\operatorname{FSub}^{\mathfrak{d}}(E) := \operatorname{FSub}(E)\cap\operatorname{Sub}^{\mathfrak{d}}(E), \quad \operatorname{FSub}^s(E) := \operatorname{FSub}(E)\cap\operatorname{Sub}^s(E)\,.$$

Moreover, put

$$\delta(\mathfrak{d},E) := \sup\big\{\dim\mathcal{L} : \mathcal{L}\in\operatorname{FSub}^{\mathfrak{d}}(E)\big\}\,.$$

The structure of the chain $\operatorname{FSub}^{\mathfrak{d}}(E)$ is very simple. This can be deduced from the following statement which, in particular, applies to a finite-dimensional de Branges subspace $\mathcal{H}(E_1)$ of a given space $\mathcal{H}(E)$.

2.7. *Example.* If $\mathcal{H}(E_1)$ is any finite-dimensional de Branges space, $n:=\dim\mathcal{H}(E_1)$, then there exists a function $S\in\mathcal{H}(E_1)$, $S=S^\#$, such that

$$\mathcal{H}(E_1) = S\cdot\operatorname{span}\{1,z,\dots,z^{n-1}\}\,.$$

The chain $\operatorname{Sub}^s(E_1)$ is given as

$$\operatorname{Sub}^s(E_1) = \Big\{S\cdot\operatorname{span}\{1\}, S\cdot\operatorname{span}\{1,z\},\dots,S\cdot\operatorname{span}\{1,z,\dots,z^{n-1}\}\Big\}\,.$$

3. Representation of finite-dimensional subspaces by majorants

3.1. Definition. Let $E \in \mathcal{H}B$. A nonnegative function ω on the real axis $\mathbb{R}$ is called an *admissible majorant* for the space $\mathcal{H}(E)$, if there exists a function $F \in \mathcal{H}(E) \setminus \{0\}$ such that $|E(x)^{-1}F(x)| \leq \omega(x)$, $x \in \mathbb{R}$. The set of all admissible majorants for $\mathcal{H}(E)$ is denoted by $\operatorname{Adm}(E)$.

If $\omega \in \operatorname{Adm}(E)$, the space $\mathcal{R}_\omega(E)$ defined by (1.1) is a de Branges subspace of $\mathcal{H}(E)$, cf. [BW, Proposition 3.2]. Moreover, by [BW, Theorem 3.4], a de Branges subspace $\mathcal{H}(E_1)$ of $\mathcal{H}(E)$ is of the form $\mathcal{R}_\omega(E)$ for some majorant ω if and only if $\operatorname{mt} \frac{E_1}{E} = 0$. Here $\operatorname{mt} f$ is the mean type of a function f in the class $N(\mathbb{C}^+)$ of functions of bounded type in the upper half plane:

$$\operatorname{mt} f := \limsup_{y \to +\infty} \frac{1}{y} \log |f(iy)| ,$$

For a function $\omega : \mathbb{R} \to [0, \infty)$, we define $\mathfrak{d}(\omega) : \mathbb{R} \to \mathbb{N}_0 \cup \{\infty\}$ as the function which assigns to a point $v \in \mathbb{R}$ the minimum of all numbers $n \in \mathbb{N}_0$ such that there exists a neighbourhood $U \subseteq \mathbb{R}$ of v with the property

$$\inf_{\substack{z \in U \\ |z-v|^n \neq 0}} \frac{|\omega(z)|}{|z - v|^n} > 0 .$$

For functions $\omega_1, \omega_2 : \mathbb{R} \to [0, \infty)$ we will write

$$\omega_1 \lesssim \omega_2 \ :\Leftrightarrow \ \exists C > 0 : \omega_1(x) \leq C\omega_2(x), x \in \mathbb{R} ,$$

$$\omega_1 \asymp \omega_2 \ :\Leftrightarrow \ \omega_1 \lesssim \omega_2 \text{ and } \omega_2 \lesssim \omega_1 .$$

3.2. Lemma. *Let $E \in \mathcal{H}B$ and $\omega \in \operatorname{Adm}(E)$. Then $\mathfrak{d}(\mathcal{R}_\omega(E)) = \mathfrak{d}(\mathcal{H}(E)) + \mathfrak{d}(\omega)$.*

Proof. Let $v \in \mathbb{R}$ be fixed. If $F \in \mathcal{H}(E)$, then $E^{-1}F$ is analytic in a neighbourhood of v, and $\mathfrak{d}_{E^{-1}F}(v) = \mathfrak{d}_F(v) - \mathfrak{d}_E(v)$. Hence, for some sufficiently small neighbourhood $U \subseteq \mathbb{R}$ of v, we have

$$\Big|\frac{F(x)}{E(x)}\Big| \asymp |x - v|^{\mathfrak{d}_F(v) - \mathfrak{d}_E(v)}, \quad x \in U .$$

If $F \in R_\omega(E)$, we obtain $|x - v|^{\mathfrak{d}_F(v) - \mathfrak{d}_E(v)} \lesssim \omega(x)$, $x \in U$, and thus $\mathfrak{d}(\omega)(v) \leq \mathfrak{d}_F(v) - \mathfrak{d}_E(v)$. Since $\mathfrak{d}(\mathcal{R}_\omega(E))(v) = \min\{\mathfrak{d}_F(v) : F \in R_\omega(E)\}$, this yields

$$\mathfrak{d}(\mathcal{R}_\omega(E))(v) \geq \mathfrak{d}(\omega)(v) + \mathfrak{d}(\mathcal{H}(E))(v) .$$

In order to prove the converse inequality, let $F \in R_\omega(E)$ with $\mathfrak{d}_F(v) > \mathfrak{d}(\omega)(v) + \mathfrak{d}(\mathcal{H}(E))(v)$ be given. Since $\mathfrak{d}_F(v) > \mathfrak{d}(\mathcal{H}(E))(v)$ the function $G(z) := (z - v)^{-1}F(z)$ belongs to the space $\mathcal{H}(E)$. Let $U \subseteq \mathbb{R}$ be a neighbourhood of v such that

$$\inf_{\substack{z \in U \cap D \\ |z-w|^{\mathfrak{d}(\omega)(v)} \neq 0}} \frac{|\omega(z)|}{|z - w|^{\mathfrak{d}(\omega)(v)}} > 0 .$$

Moreover, let U be chosen so small that $E^{-1}F$ is analytic at every point of U. Since $\mathfrak{d}(E^{-1}F)(v) > \mathfrak{d}(\omega)(v)$, the function $\frac{F(z)}{(z-v)^{\mathfrak{d}(\omega)(v)+1}E(z)}$ is analytic, and hence bounded, on U. It follows that

$$\Big|\frac{F(x)}{(x-v)^{\mathfrak{d}(\omega)(v)+1}E(x)}\Big| \lesssim \frac{\omega(x)}{|x-v|^{\mathfrak{d}(\omega)(v)}}, \quad x \in U,$$

and hence

$$\Big|\frac{G(x)}{E(x)}\Big| = \Big|\frac{F(x)}{(x-v)E(x)}\Big| \lesssim \omega(x), \quad x \in U.$$

For $x \notin U$, we have $\frac{1}{|x-v|} \lesssim 1$, and hence

$$\Big|\frac{G(x)}{E(x)}\Big| = \Big|\frac{F(x)}{(x-v)E(x)}\Big| \lesssim |F(x)| \lesssim \omega(x), \quad x \in \mathbb{R}\setminus U.$$

Altogether we see that $G \in R_\omega(E)$.

Since $\mathfrak{d}(\mathcal{R}_\omega(E))(v) = \min\{\mathfrak{d}_F(v) : F \in R_\omega(E)\}$, we conclude that

$$\mathfrak{d}(\mathcal{R}_\omega(E))(v) \leq \mathfrak{d}(\omega)(v) + \mathfrak{d}(\mathcal{H}(E))(v).$$

❑

If $\mathfrak{d} : \mathbb{R} \to \mathbb{N}_0$ is given, we shall denote $\mathrm{Adm}^{\mathfrak{d}}(E) := \{\omega \in \mathrm{Adm}(E) : \mathfrak{d}(\omega) = \mathfrak{d}\}$. The majorants with $\mathfrak{d}(\omega) = 0$ are those of biggest interest. By Lemma 3.2 they generate de Branges subspaces in $\mathrm{Sub}^s(E)$.

We now proceed to the study of a family $\omega^{[k]}$ of majorants, which is vital for our consideration of finite-dimensional de Branges subspaces.

3.3. Definition. Let $\omega : \mathbb{R} \to [0,\infty)$. For $k \in \mathbb{Z}$ define

$$\omega^{[k]}(x) := (1+|x|)^k \omega(x), \ x \in \mathbb{R}.$$

If, additionally, $E \in \mathcal{H}B$ is given, put

$$\alpha(\omega, E) := \inf\big\{k \in \mathbb{Z} : \omega^{[k]} \in \mathrm{Adm}(E)\big\} \in \mathbb{Z}\cup\{\pm\infty\}.$$

Note that, clearly, $(\omega^{[k]})^{[l]} = \omega^{[k+l]}$, $\mathfrak{d}(\omega^{[k]}) = \mathfrak{d}(\omega)$, and $\omega^{[k]} \leq \omega^{[l]}$ for $k \leq l$.

As the following (trivial) example shows, $\alpha(\omega, E)$ may assume any prescribed value in $\mathbb{Z}\cup\{\pm\infty\}$.

3.4. *Example.* Let $E(z) := z + i$, then $\mathcal{H}(E) = \mathrm{span}\{1\}$. Hence a function ω is an admissible majorant for $\mathcal{H}(E)$ if and only if it is bounded away from zero. It is obvious that for any given $n \in \mathbb{Z}\cup\{\pm\infty\}$ we can find $\omega : \mathbb{R} \to (0,\infty)$ such that $\alpha(\omega, E) = n$.

3.5. Lemma. *Let $E \in \mathcal{H}B$ and $\omega \in \mathrm{Adm}(E)$.*

(*i*) *We have* $\dim\big(\mathcal{R}_{\omega^{[1]}}(E)/\mathcal{R}_\omega(E)\big) \leq 1$.

(*ii*) *Assume that* $\dim \mathcal{R}_\omega(E) < \infty$ *and that*

$$\exists\, \mathcal{L} \in \operatorname{Sub}^{\mathfrak{d}(\omega)}(E) : \dim\big(\mathcal{L}/\mathcal{R}_\omega(E)\big) = 1\,. \tag{3.1}$$

Then $\dim(\mathcal{R}_{\omega^{[1]}}(E)/\mathcal{R}_\omega(E)) = 1$*, i.e.* $\mathcal{L} = \mathcal{R}_{\omega^{[1]}}(E)$*.*

Proof. Let G be entire and not identically zero, let $v \in \mathbb{C} \setminus \mathbb{R}$ such that $G(v) \neq 0$, and consider the difference quotient operator

$$\rho : F(z) \mapsto \frac{F(z) - \frac{F(v)}{G(v)} G(z)}{z - v}\,.$$

If $\mathcal{H}$ is a de Branges space which contains the function G, then $\rho|_{\mathcal{H}}$ is a bounded linear operator of $\mathcal{H}$ into itself, and

$$\ker\big(\rho|_{\mathcal{H}}\big) = \operatorname{span}\{G\}, \quad \operatorname{ran}\big(\rho|_{\mathcal{H}}\big) = \operatorname{dom} \mathcal{S}(\mathcal{H})\,.$$

Here $\mathcal{S}(\mathcal{H})$ denotes the operator of multiplication by z in $\mathcal{H}$. In particular, by [dB7, Theorem 29], we have $\dim(\mathcal{H}/\overline{\operatorname{ran}(\rho|_{\mathcal{H}})}) \in \{0, 1\}$.

Proof of (*i*): Choose $G \in R_\omega(E) \setminus \{0\}$. We have the estimate

$$|(\rho F)(x)| \leq \frac{1}{|x - v|} \cdot |F(x)| + \frac{1}{|x - v|} \Big|\frac{F(v)}{G(v)}\Big| \cdot |G(x)|,\ x \in \mathbb{R}\,.$$

Hence $\rho(R_{\omega^{[1]}}(E)) \subseteq R_\omega(E)$ which, by the continuity of $\rho|_{\mathcal{R}_{\omega^{[1]}}(E)}$, implies that $\rho(\mathcal{R}_{\omega^{[1]}}(E)) \subseteq \mathcal{R}_\omega(E)$. Thus also $\overline{\rho(\mathcal{R}_{\omega^{[1]}}(E))} \subseteq \mathcal{R}_\omega(E)$. We conclude that

$$\dim(\mathcal{R}_{\omega^{[1]}}(E)/\mathcal{R}_\omega(E)) \in \{0, 1\}\,.$$

Proof of (*ii*): Choose G in $R_{\omega^{[1]}}(E) \setminus \{0\}$. By the already proved part (*i*) of the present lemma, and the fact that $\operatorname{Sub}^{\mathfrak{d}(\omega)}(E)$ is a chain, we have $\mathcal{R}_{\omega^{[1]}}(E) \subseteq \mathcal{L}$. The estimate

$$|F(x)| \leq \Big|\frac{F(v)}{G(v)}\Big| \cdot |G(x)| + |x - v| \cdot |(\rho F)(x)|,\ x \in \mathbb{R}\,,$$

shows that

$$(\rho|_{\mathcal{L}})^{-1}(R_\omega(E)) \subseteq R_{\omega^{[1]}}(E)\,.$$

However, by finite-dimensionality,

$$\operatorname{ran}(\rho|_{\mathcal{L}}) = \operatorname{dom} \mathcal{S}(\mathcal{L}) = \overline{\operatorname{dom} \mathcal{S}(\mathcal{L})} = \mathcal{R}_\omega(E) = R_\omega(E)\,.$$

It follows that

$$\mathcal{L} = (\rho|_{\mathcal{L}})^{-1}\big(\operatorname{ran}(\rho|_{\mathcal{L}})\big) \subseteq R_{\omega^{[1]}}(E) = \mathcal{R}_{\omega^{[1]}}(E)\,.$$

❑

An inductive application of this lemma yields the following result.

3.6. Proposition. *Let $E \in \mathcal{H}B$, $\omega \in \operatorname{Adm}(E)$, and assume that* $\dim \mathcal{R}_\omega(E) < \infty$. *Then*

$$\dim\left(\mathcal{R}_{\omega^{[k+1]}}(E)/\mathcal{R}_{\omega^{[k]}}(E)\right) = \begin{cases} 1 & , \ 0 \leq k < \delta(\mathfrak{d}(\omega)) - \dim \mathcal{R}_\omega(E) \\ 0 & , \ k \geq \delta(\mathfrak{d}(\omega)) - \dim \mathcal{R}_\omega(E) \end{cases}$$

Proof. We will show by induction on k that

$$\dim \mathcal{R}_{\omega^{[k]}}(E) = \begin{cases} \dim \mathcal{R}_\omega(E) + k, & 0 \leq k \leq \delta(\mathfrak{d}(\omega)) - \dim \mathcal{R}_\omega(E), \\ \delta(\mathfrak{d}(\omega)), & k > \delta(\mathfrak{d}(\omega)) - \dim \mathcal{R}_\omega(E). \end{cases}$$

For $k = 0$ this is trivial. Assume that $0 < k \leq \delta(\mathfrak{d}(\omega)) - \dim \mathcal{R}_\omega(E)$ and that $\dim \mathcal{R}_{\omega^{[k-1]}}(E) = \dim \mathcal{R}_\omega(E) + (k-1)$. Then, by the definition of $\delta(\mathfrak{d}(\omega))$, there exists $\mathcal{L} \in \operatorname{FSub}^{\mathfrak{d}(\omega)}(E)$ with $\mathcal{L} \supsetneq \mathcal{R}_{\omega^{[k-1]}}(E)$. By the structure of the chain $\operatorname{FSub}^{\mathfrak{d}(\omega)}(E)$ we can choose $\mathcal{L}$ such that $\dim(\mathcal{L}/\mathcal{R}_{\omega^{[k-1]}}(E)) = 1$. Lemma 3.5, (ii), implies that $\dim \mathcal{R}_{\omega^{[k]}}(E) = \dim \mathcal{R}_{\omega^{[k-1]}}(E) + 1$.

If $\delta(\mathfrak{d}(\omega)) = \infty$, we are done. Otherwise, by the already proved, we have $\dim \mathcal{R}_{\omega^{[k_0]}}(E) = \delta(\mathfrak{d}(\omega))$ for $k_0 := \delta(\mathfrak{d}(\omega)) - \dim \mathcal{R}_\omega(E)$. It follows from Lemma 3.5, (i), and the definition of $\delta(\mathfrak{d}(\omega))$ that $\mathcal{R}_{\omega^{[k]}}(E) = \mathcal{R}_{\omega^{[k_0]}}(E)$ for all $k \geq k_0$.

❑

The relation $\lesssim$ is reflexive and transitive, and hence induces an order on the factor set $\operatorname{Adm}(E)/_{\asymp}$. If we speak of minimal elements we always refer to this order.

3.7. Lemma. *Let $E \in \mathcal{H}B$ and $\mathfrak{d} : \mathbb{R} \to \mathbb{N}_0$.*

(i) *Let $\omega \in \operatorname{Adm}^{\mathfrak{d}}(E)$. Then $\omega/_{\asymp}$ is minimal in $\operatorname{Adm}^{\mathfrak{d}}(E)/_{\asymp}$ if and only if $\omega/_{\asymp}$ is minimal in $\operatorname{Adm}(E)/_{\asymp}$.*

(ii) *The set $\operatorname{Adm}^{\mathfrak{d}}(E)/_{\asymp}$ contains at most one minimal element.*

Proof. The assertion (i) is seen in exactly the same way as [BW, Lemma 4.7]. The second item is then a consequence of [BW, Corollary 4.4].

❑

We can now settle the question when, and in which way, finite-dimensional de Branges subspaces can be represented by majorants. Put

$$\mathfrak{R}(E) := \left\{ \mathcal{R}_\omega(E) : \omega \in \operatorname{Adm}(E) \right\}.$$

3.8. Theorem. *Let $E \in \mathcal{H}B$ and $\mathfrak{d} : \mathbb{R} \to \mathbb{N}_0$. Then the following conditions are equivalent:*

(i) $\operatorname{Adm}^{\mathfrak{d}}(E)/_{\asymp}$ *contains a minimal element;*

(ii) $\operatorname{FSub}^{\mathfrak{d}}(E) \cap \mathfrak{R}(E) \neq \emptyset$;

(iii) $\operatorname{FSub}^{\mathfrak{d}}(E) \neq \emptyset$ *and* $\operatorname{Sub}^{\mathfrak{d}}(E) \subseteq \mathfrak{R}(E)$.

In this case, if $\mathcal{L} \in \operatorname{FSub}^{\mathfrak{d}}(E)$, we have

$$\mathcal{L} = \mathcal{R}_{\omega_0^{[k]}}(E),$$

where ω_0 *is any representant of the minimal element of* $\operatorname{Adm}^{\mathfrak{d}}(E)/_{\asymp}$ *and* $k = \dim \mathcal{L} - 1$.

Proof. $(i) \Rightarrow (ii)$: Let $\omega/_{\asymp}$ be the minimal element of $\operatorname{Adm}^{\mathfrak{d}}(E)$ (unique by Lemma 3.7). Then, by Lemma 3.7, $\omega/_{\asymp}$ is minimal in $\operatorname{Adm}(E)$. By [BW, Theorem 4.2], $\dim \mathcal{R}_\omega(E) = 1$.

$(ii) \Rightarrow (iii)$: Pick $\mathcal{H}(E_1) \in \operatorname{FSub}^{\mathfrak{d}}(E) \cap \mathfrak{R}(E)$. Then, by [BW, Theorem 3.4], we have $\operatorname{mt} \frac{E_1}{E} = 0$. Let $\mathcal{H}(E_2) \in \operatorname{Sub}^{\mathfrak{d}}(E)$ be given. If $\mathcal{H}(E_1) \subseteq \mathcal{H}(E_2)$, we conclude from

$$\operatorname{mt} \frac{E_2}{E} = \max_{F \in \mathcal{H}(E_2)} \operatorname{mt} \frac{F}{E} \leq 0, \quad \operatorname{mt} \frac{E_1}{E_2} = \max_{F \in \mathcal{H}(E_1)} \operatorname{mt} \frac{F}{E_2} \leq 0\,,$$

that $\operatorname{mt} \frac{E_2}{E} = 0$. Otherwise, if $\mathcal{H}(E_2) \subseteq \mathcal{H}(E_1)$, we see from Example 2.7 that also $\operatorname{mt} \frac{E_2}{E} = 0$. It follows from [BW, Theorem 3.4] that in either case $\mathcal{H}(E_2) \in \mathfrak{R}(E)$.

$(iii) \Rightarrow (i)$: Since $\operatorname{FSub}^{\mathfrak{d}}(E) \neq \emptyset$, there exists a one-dimensional subspace $\mathcal{L} \in \operatorname{Sub}^{\mathfrak{d}}(E)$. Since $\mathcal{L} \in \mathfrak{R}(E)$, we obtain from [BW, Theorem 4.2] a minimal element $\omega/_{\asymp}$ of $\operatorname{Adm}(E)/_{\asymp}$ such that $\mathcal{L} = \mathcal{R}_\omega(E)$. Since $\mathfrak{d}(\omega) = \mathfrak{d}$, $\omega/_{\asymp}$ minimal in $\operatorname{Adm}^{\mathfrak{d}}(E)$.

Representation of $\mathcal{L}$: Assume that one (and hence all) of (i)–(iii) hold and that $\mathcal{L} \in \operatorname{FSub}^{\mathfrak{d}}(E)$. Let $\omega_0/_{\asymp}$ be the minimal element of $\operatorname{Adm}^{\mathfrak{d}}(E)$. Then $\dim \mathcal{R}_{\omega_0}(E) = 1$. Since, clearly, $\delta(\mathfrak{d}) \geq \dim \mathcal{L}$, we obtain from Proposition 3.6 that

$$\mathcal{L} = \mathcal{R}_{\omega_0^{[\dim \mathcal{L} - 1]}}(E)\,.$$

❑

3.9. *Remark.* Let us note that the (equivalent) conditions of Theorem 3.8 are not always satisfied. This is seen for example by taking $E(z) := (z+i)e^{-iz}$. For this function E, the space $\mathcal{H}(E)$ contains the function 1. Thus $\operatorname{span}\{1\} \in \operatorname{FSub}^s(E)$. However, by [BW, Theorem 3.4], we have $\mathfrak{R}(E) = \{\mathcal{H}(E)\}$.

In Proposition 3.6 we have clarified the behaviour of $\mathcal{R}_{\omega^{[k]}}(E)$ for $k \geq 0$. It seems that the situation for $k < 0$ is more complicated. In this place let us only note the following corollary of Proposition 3.6.

3.10. Corollary. *Let* $E \in \mathcal{H}B$ *and* $\omega \in \operatorname{Adm}(E)$. *Assume that* $\dim \mathcal{R}_\omega(E) < \infty$ *and that* (3.1) *holds. Then* $\alpha(\omega, E) \leq 1 - \dim \mathcal{R}_\omega(E)$ *and*

$$\dim \big(\mathcal{R}_{\omega^{[k+1]}}(E)/\mathcal{R}_{\omega^{[k]}}(E)\big) = 1,\ 1 - \dim \mathcal{R}_\omega(E) \leq k < 0\,.$$

Proof. The space $\mathcal{R}_{\omega^{[-1]}}(E)$ is finite dimensional and the assumption (3.1) holds for it since it holds for the bigger space $\mathcal{R}_\omega(E)$. Hence we may apply Proposition 3.6, and obtain that

$$\dim \mathcal{R}_\omega(E) = \dim \mathcal{R}_{\omega^{[-1]}}(E) + 1\,.$$

The assertion follows by induction.

❑

4. The family $\omega^{[k]}$ for $\dim \mathcal{R}_\omega(E) = \infty$

Our treatment of finite-dimensional de Branges subspaces in $\mathfrak{R}(E)$ was based on Lemma 3.5. Let us show by an example that the assumption '$\dim \mathcal{R}_\omega(E) < \infty$' in part *(ii)* of this lemma cannot be dropped.

4.1. *Example.* We shall construct $E \in \mathcal{H}B$ and $\omega \in \operatorname{Adm}^0(E)$ such that (3.1) holds, but $\mathcal{R}_{\omega^{[k]}}(E) = \mathcal{R}_\omega(E)$ for all $k \in \mathbb{N} \cup \{0\}$.

Consider the canonical system on $[0,2]$ with Hamiltonian

$$H(t) := \begin{cases} \quad I \quad , & t \in [0,1), \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & t \in [1,2], \end{cases}$$

and put $E := E_2$, cf. Example 2.4. Then $E \in \mathcal{H}B$, $\mathfrak{d}(E) \equiv 0$, and E is explicitly given as

$$E(z) = \cos z + i(\sin z + z\cos z)\,.$$

We have

$$\operatorname{Sub}^s(\mathcal{H}(E)) = \big\{\mathcal{P}W_a :\, 0 < a \le 1\big\} \cup \{\mathcal{H}(E)\}\,,$$

and $\operatorname{codim}_{\mathcal{H}(E)} \mathcal{P}W_1 = 1$. In fact,

$$\mathcal{H}(E) = \mathcal{P}W_1 \dotplus \operatorname{span}\{\cos z\}\,.$$

It follows from [BW, Theorem 3.4] that $\mathcal{R}_\omega(E) \supseteq \mathcal{P}W_1$ for all $\omega \in \operatorname{Adm}^0(E)$, i.e. $\mathcal{R}_\omega(E) \in \{\mathcal{P}W_1, \mathcal{H}(E)\}$.

We give some estimates on E. Let $x \in \mathbb{R}$, $|x| \ge 1$, and assume that $|\cos x| \le \frac{1}{2|x|}$. Then $|\sin x| \ge \sqrt{3}/2$ and

$$|\sin x + x\cos x| \ge \big||\sin x| - |x\cos x|\big| \ge \frac{\sqrt{3}-1}{2}\,.$$

It follows that

$$|E(x)| \ge \min\Big\{\min_{t\in[-1,1]} |E(t)|, \frac{\sqrt{3}-1}{2}, \frac{1}{2|x|}\Big\} \gtrsim \frac{1}{1+|x|},\ x \in \mathbb{R}\,. \tag{4.1}$$

Trivially, we have the following estimate from above:

$$|E(x)| \le 1 + |x|,\ x \in \mathbb{R}\,. \tag{4.2}$$

We show that for all $\beta \in (0,1)$ the function $\omega_\beta(x) := e^{-|x|^\beta}$ belongs to $\operatorname{Adm}^0(E)$.

Choose $\beta' \in (\beta,1)$. It is well known (see [HJ, p. 276] or [K, p. 159]) that there exists $F \in \mathcal{P}W_1 \setminus \{0\}$ such that

$$|F(x)| \lesssim e^{-|x|^{\beta'}},\ x \in \mathbb{R}\,.$$

It follows that $F \in \mathcal{H}(E)$ and, by (4.1),

$$\Big|\frac{F(x)}{E(x)}\Big| \lesssim |F(x)|(1+|x|) \lesssim e^{-|x|^{\beta'}}(1+|x|) \lesssim e^{-|x|^{\beta}},\ x \in \mathbb{R}\,.$$

We show that $\mathcal{R}_{\omega_\beta}(E) = \mathcal{P}W_1$. We already know that $\mathcal{R}_{\omega_\beta}(E) \supseteq \mathcal{P}W_1$. Assume that $\mathcal{R}_{\omega_\beta}(E) \supsetneq \mathcal{P}W_1$, then also $R_{\omega_\beta}(E) \supsetneq \mathcal{P}W_1$. Hence there exist $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and $F_0 \in \mathcal{P}W_1$, such that $F_0 + \lambda \cos z \in R_{\omega_\beta}(E)$, i.e.

$$\Big|\frac{F_0(x)+\lambda\cos x}{E(x)}\Big| \lesssim \omega_\beta(x)\,.$$

By (4.2),

$$|\cos x| \lesssim (1+|x|)\,\omega_\beta(x) + |F_0(x)|,\ x \in \mathbb{R}\,.$$

We have reached a contradiction, since both F_0 and $(1+|x|)\omega_\beta(x)$ belong to $L^2(\mathbb{R})$.

For all $\beta \in (0,1)$ *and* $k \in \mathbb{N}\cup\{0\}$ *we have* $\mathcal{R}_{\omega^{[k]}}(E) = \mathcal{P}W_1$. Choose $\beta' \in (0,\beta)$, then $(1+|x|)^k e^{-|x|^\beta} \lesssim e^{-|x|^{\beta'}}$, i.e. $\omega_\beta^{[k]} \lesssim \omega_{\beta'}$. Hence $\mathcal{R}_{\omega_\beta^{[k]}}(E) \subseteq \mathcal{R}_{\omega_{\beta'}}(E) = \mathcal{P}W_1$.

Although Example 4.1 shows that the statement of Proposition 3.6 is not true without the assumption that $\dim \mathcal{R}_\omega(E) < \infty$, still there always exist some representing majorants which behave nicely in this respect.

For $\mathcal{L} \in \operatorname{Sub}(E)$ define

$$\delta_+(\mathcal{L}) := \sup\big\{ \dim \mathcal{H}/\mathcal{L} :\ \mathcal{H} \in \operatorname{Sub}^{\mathfrak{d}(\mathcal{L})}(E), \dim \mathcal{H}/\mathcal{L} < \infty \big\}\,.$$

4.2. Proposition. *Let* $E \in \mathcal{H}B$ *and* $\mathcal{L} \in \mathfrak{R}(E) \cap \operatorname{Sub}(E)$. *Then there exists* $\omega \in \operatorname{Adm}(E)$ *such that* $\mathcal{R}_\omega(E) = \mathcal{L}$ *and*

$$\dim\big(\mathcal{R}_{\omega^{[k+1]}}(E)/\mathcal{R}_{\omega^{[k]}}(E)\big) = \begin{cases} 1, & 0 \le k < \delta_+(\mathcal{L}),\\ 0, & k \ge \delta_+(\mathcal{L}). \end{cases}$$

Proof. Write $\mathcal{L} = \mathcal{H}(E_0)$. Assume that $\mathcal{H}(E_1) \in \operatorname{Sub}^{\mathfrak{d}(\mathcal{L})}(E)$ is such that $n := \dim(\mathcal{H}(E_1)/\mathcal{H}(E_0)) < \infty$. Then there exists a 2×2-matrix polynomial $M(z)$ of degree n, such that ($A_j := \frac{1}{2}(E_j + E_j^\#)$, $B_j := \frac{i}{2}(B_j - B_j^\#)$ for $j = 0,1$)

$$\big(A_1(z), B_1(z)\big) = \big(A_0(z), B_0(z)\big)M(z)\,. \tag{4.3}$$

Put

$$\omega(x) := \frac{|E_0(x)|}{(1+|x|)|E(x)|},\ \omega_1(x) := \frac{|E_1(x)|}{(1+|x|)|E(x)|}\,.$$

Then, by the proof of sufficiency in [BW, Theorem 3.4], we have

$$\mathcal{L} = \mathcal{H}(E_0) = \mathcal{R}_\omega(E),\ \mathcal{H}(E_1) = \mathcal{R}_{\omega_1}(E)\,.$$

However, we see from (4.3) that $\omega_1 \lesssim \omega^{[n]}$. This implies

$$\mathcal{H}(E_1) = \mathcal{R}_{\omega_1}(E) \subseteq \mathcal{R}_{\omega^{[n]}}(E)\,.$$

By Lemma 3.5, (i), we have $\dim(\mathcal{R}_{\omega^{[n]}}(E)/\mathcal{R}_{\omega}(E)) \leq n$, and hence it follows that $\mathcal{H}(E_1) = \mathcal{R}_{\omega^{[n]}}(E)$.

❑

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