# On semibounded canonical systems

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**Abstract.** We present two inverse spectral relations for canonical differential equations $Jy'(x) = -zH(x)y(x)$, $x \in [0, L)$: Denote by $Q_H$ the Titchmarsh-Weyl coefficient associated with this equation. We show: If the Hamiltonian $H$ is on some interval $[0, \epsilon)$ of the form

$$H(x) = \begin{pmatrix} v(x)^2 & v(x) \\ v(x) & 1 \end{pmatrix} ,$$

with a nondecreasing function $v$, then $\lim_{x \searrow 0} v(x) = \lim_{y \to +\infty} Q_H(iy)$. If $H$ is of the above form on some interval $[l, L)$, then $\lim_{x \nearrow L} v(x) = \lim_{z \nearrow 0} Q_H(z)$. In particular, these results are applicable to semibounded canonical systems, or canonical systems with a finite number of negative eigenvalues, respectively.

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## 1. Introduction

A canonical (or Hamiltonian) system is an boundary value problem of the form

$$Jy'(x) = -zH(x)y(x), \; x \in [0, L), \quad y_1(0) = 0, \tag{1.1}$$

where $L \in (0, \infty]$, and where $H$ is a function which takes real, symmetric and nonnegative $2 \times 2$-matrices as values, does not vanish on any set of positive measure, and belongs to $L^1_{loc}([0, L))$. Moreover, $z$ is a complex parameter and

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

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The function $H$ is called the Hamiltonian of the system (1.1). Canonical systems occur in mathematical physics and were intensively investigated, see e.g. [1], [3], [4], [7], [8].

The condition

$$\int_0^L \operatorname{trace} H(x)\,dx = +\infty \tag{1.2}$$

plays a crucial role in the spectral theory of canonical systems. In fact, (1.2) says that the so-called Weyl's limit point case prevails. To a system (1.1) which satisifes (1.2) there is associated a function $Q_H(z)$, its Titchmarsh-Weyl coefficient, which belongs to the Nevanlinna class $\mathcal{N}$. This is the set of all functions $Q$ analytic on $\mathbb{C}\setminus\mathbb{R}$, $Q(\bar{z}) = \overline{Q(z)}$, with Im $Q(z) \geq 0$ for Im $z > 0$. The Inverse Spectral Theorem of L.de Branges states that the assignment $H \mapsto Q_H$ yields, up to changes of scale, a bijection of the set of all Hamiltonians which satsify (1.2) onto $\mathcal{N} \cup \{\infty\}$.

Inverse spectral relations are statements which relate properties of $Q_H$ to preoperties of $H$. In this paper we establish two statements of this kind. We show that, if the Hamiltonian is on some interval $[0, \epsilon)$ of the form

$$H(x) = \begin{pmatrix} v(x)^2 & v(x) \\ v(x) & 1 \end{pmatrix} ,$$

where $v$ is nondecreasing, then $\lim_{x\searrow 0} v(x) = \lim_{y\to+\infty} Q_H(iy)$, cf. Theorem 3.3, and that, if $H$ is of the above form on some interval $[l, L)$, then $\lim_{x\nearrow L} v(x) = \lim_{z\nearrow 0} Q_H(z)$, cf. Theorem 3.9.

Our investigations are motivated by the study of semibounded canonical systems, that are systems with the property that their Titchmarsh-Weyl coefficient has an analytic continuation to some set of the form $\mathbb{C} \setminus [M, \infty)$, cf. Theorem 2.3, Corollary 3.5. Proofs are based on the theory of strings, cf. [15]. The statement in Corollary 3.5 also finds some application in the extension theory of symmetric relations, for, it shows a straightforward way to determine the Friedrichs extension in terms of the Hamiltonian, see [9], [11] and [23] for details.

In the preliminary Section 2 we set up our notation and recall some results which will be used later on. In Section 3 we prove and discuss our main results Theorem 3.3 and Theorem 3.9.

## 2. Preliminaries

### A. Nevanlinna functions

By the Herglotz representation theorem, a Nevanlinna function $Q$ has an integral representation of the form

$$Q(z) = bz + a + \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) d\sigma(\lambda)\,, \tag{2.1}$$

with $b \geq 0$, $a \in \mathbb{R}$, and a measure $\sigma$ satisfying $\int_{\mathbb{R}} (1+\lambda^2)^{-1}\,d\sigma(\lambda) < \infty$. Thereby $a, b$ and $\sigma$ are uniquely determined by $Q$. Many interesting subclasses of $\mathcal{N}$ can be defined, or characterized, in terms of $a, b$ and $\sigma$. In our context two subclasses will play an important role: the Kac class $\mathcal{N}_1$ and the Stieltjes class $\mathcal{S}$.

The Kac class $\mathcal{N}_1$ is defined as the set of all $Q \in \mathcal{N}$ with

$$b = 0, \ \int_{\mathbb{R}} \frac{d\sigma(\lambda)}{1+|\lambda|} < \infty \,.$$

This means that $Q \in \mathcal{N}_1$ if and only if it can be represented as

$$Q(z) = \alpha + \int_{\mathbb{R}} \frac{d\sigma(\lambda)}{\lambda - z} \tag{2.2}$$

with some $\alpha \in \mathbb{R}$ and $\int_{\mathbb{R}} (1+|\lambda|)^{-1}\, d\sigma(\lambda) < \infty$. An analytic characterization of $\mathcal{N}_1$ was given in [13], see also [14, Theorem S1.3.1]: A Nevanlinna function $Q$ belongs to $\mathcal{N}_1$ if and only if

$$\int_1^\infty \frac{\operatorname{Im}\, Q(iy)}{y} dy < \infty \,. \tag{2.3}$$

For a closer investigation of Kac classes and related subjects see also [2], [10] or [23].

The Stieltjes class $\mathcal{S}$ is defined as the set of all functions $Q$ which are analytic in $\mathbb{C} \setminus [0,\infty)$, satisfy $\operatorname{Im}\, Q(z) \geq 0$, $z \in \mathbb{C}^+$, and $Q(z) \geq 0$, $z \in (-\infty, 0)$. Clearly, $\mathcal{S} \subseteq \mathcal{N}$. The history of the class $\mathcal{S}$ goes back to some investigations of T.J.Stieltjes on the moment problem and continued fractions, cf. [19]. Also the class $\mathcal{S}$ can be characterized in various ways, cf. [14, Theorem S1.5.1, Lemma S1.5.1]. In fact, for a function $Q$ which is analytic in $\mathbb{C}\setminus[0,\infty)$ and satisfies $Q(\bar{z}) = \overline{Q(z)}$, the following conditions are equivalent:

1. $Q \in \mathcal{S}$.
2. $Q \in \mathcal{N}_1$, $\operatorname{supp}\sigma \subseteq [0,\infty)$, and the constant $\alpha$ in (2.2) is nonnegative.
3. $Q(z) \in \mathcal{N}$ and $zQ(z) \in \mathcal{N}$.
4. $zQ(z^2) \in \mathcal{N}$.

Further investigations and generalizations of the Stieltjes class can be found e.g. in [2], [5], or [16].

**B. Canonical systems**

Let us recall the construction of the Titchmarsh-Weyl coefficient associated to a Hamiltonian $H$: Denote by

$$W(x,z) = \begin{pmatrix} w_{11}(x,z) & w_{12}(x,z) \\ w_{21}(x,z) & w_{22}(x,z) \end{pmatrix}, \ W(0,z) = I \,,$$

the transposed of the fundamental matrix solution of the system (1.1). That is, $W(x,z)$ is the unique solution of $\frac{\partial}{\partial x} W(x,z)J = zW(x,z)H(x)$, $W(0,z) = I$. Then, since we assume that (1.2) holds, for each $\omega \in \mathcal{N} \cup \{\infty\}$ and $z \in \mathbb{C}^+$ the limit

$$Q_H(z) := \lim_{x \to L} \frac{w_{11}(x,z)\omega(z) + w_{12}(x,z)}{w_{21}(x,z)\omega(z) + w_{22}(x,z)} \tag{2.4}$$

exists, is independent of $\omega$, and, as a function of $z$, belongs to $\mathcal{N} \cup \{\infty\}$, see e.g. [4]. This is the Titchmarsh-Weyl coefficient associated with $H$. The measure $\sigma_H$ in the integral representation (2.1) of $Q_H$ is called the spectral measure of $H$.

Two Hamiltonians $H_1$ on $[0, L_1)$ and $H_2$ on $[0, L_2)$ are said to be reparameterizations of each other, $H_1 \sim H_2$, if there exists a strictly increasing bijection $\lambda$ of $[0, L_1)$ onto $[0, L_2)$ such that $H_1(x) = H_2(\lambda(x))\lambda'(x)$, $x \in [0, L_1)$. It is easy to see that, if $H_1 \sim H_2$, then $Q_{H_1} = Q_{H_2}$.

The basic inverse result of L. de Branges is, cf. [4], [20]:

**Theorem 2.1 (Inverse Spectral Theorem).** *The assignment $H \mapsto Q_H$ sets up a bijection between the set of all Hamiltonians modulo $\sim$ and $\mathcal{N} \cup \{\infty\}$.*

To illustrate the nature of inverse spectral relations, let us mention two results of this kind, which will also be of good use later on:

*Remark* 2.2.

1. If we assume that trace $H(t) \equiv 1$, which can always be achieved by a suitable reparameterization, then the constant $b$ in the integral representation of $Q_H$ is the maximal number such that $H|_{[0,b)} = \operatorname{diag}(1, 0)$, cf. [15].
2. Let $\sigma$ be the measure in the integral representation of $Q_H$. Then
$$\int_0^L (0,1) H(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} dx = \frac{1}{\sigma(\{0\})}, \tag{2.5}$$
where the right hand side is understood as $+\infty$ if $\sigma(\{0\}) = 0$. This fact was proved in [22, Theorem 2.2].

**C. Transformation of canonical systems**

We will employ two transformations of Hamiltonians. These, and others, were investigated in [21].

Let $H$ be a Hamiltonian defined on $[0, L)$. Then also
$$\widehat{H} := J H J^T \tag{2.6}$$
is a Hamiltonian on $[0, L)$. Clearly $H$ and $\widehat{H}$ together do or do not satisfy (1.2). The fundamental matrix $\widehat{W}$ corresponding to $\widehat{H}$ satisfies the relation $\widehat{W} = J W J^T$. Hence, by (2.4), we have
$$Q_{\widehat{H}}(z) = -Q_H(z)^{-1}. \tag{2.7}$$
Let again $H$ be a Hamiltonian defined on $[0, L)$ and let $c \in \mathbb{R}$. Then also
$$\widehat{H} := C H C^T, \tag{2.8}$$
where $C := \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$, is a Hamiltonian on $[0, L)$ and satisfies together with $H$ the condition (1.2). In this situation we have $Q_{\widehat{H}}(z) = Q_H(z) + c$.

**D. Semibounded canonical systems**

One of the main objects of our studies are canonical systems whose spectral measure is semibounded from below. Recall the following result which was proved, in a slightly different formulation, in [22].

**Theorem 2.3.** *Let* $Q \in \mathcal{N}$ *be a Nevanlinna function with* $\inf\operatorname{supp}\sigma > -\infty$. *Then there exists a number* $L \in (0,\infty]$ *and a nondecreasing and right-continuous function* $\nu\colon [0,L) \to [0,+\infty)$ *such that, with* $v(x) := -\cot\nu(x)$, $\nu(x) \notin \pi\mathbb{Z}$, *the Hamiltonian*

$$H(x) = \begin{cases} \begin{pmatrix} v(x)^2 & v(x) \\ v(x) & 1 \end{pmatrix} & \text{if } \nu(x) \notin \pi\mathbb{Z} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \nu(x) \in \pi\mathbb{Z} \end{cases} \tag{2.9}$$

*satsifies* (1.2) *and* $Q_H = Q$. *If* $\nu$ *is normalized in such a way that* $\nu(0) \in [0,\pi)$ *and* $\nu(x) - \nu(x-) < \pi$, *then* $L$ *and* $\nu$ *are unique.*

*The function* $\nu$ *(if normalized as above) is bounded if and only if* $(-\infty,0) \cap \operatorname{supp}\sigma$ *is finite. If* $\nu(L-)/\pi \in \mathbb{N}$, *then* $Q$ *has* $n-1$ *poles on* $(-\infty,0)$, *otherwise the number of poles of* $Q$ *on* $(-\infty,0)$ *is equal to the integer part of* $\nu(L-)/\pi$.

It is not known to the authors whether or not the converse of this result holds. However, in a particular case a converse can be proved, cf. [22].

**Theorem 2.4.** *Let* $H$ *be a Hamiltonian of the form* (2.9), *and assume that* $\nu$ *is bounded. Then, for the spectral measure* $\sigma$ *of* $H$, *we have* $\inf\operatorname{supp}\sigma > -\infty$.

Let $Q \in \mathcal{N}$, $\inf\operatorname{supp}\sigma > -\infty$, and let $\nu$ be as in Theorem 2.3. Then the constant $b$ in the integral representation (2.1) of $Q$ is determined by

$$b = \sup\left(\{x \geq 0:\ \nu(x) = 0\} \cup \{0\}\right).$$

Hence, if $b = 0$, there exists a nonempty interval $(0,\epsilon)$, such that $\nu(x) \notin \pi\mathbb{Z}$, $x \in (0,\epsilon)$.

A case of particular importance occurs if $b = 0$ and $\inf\operatorname{supp}\sigma \geq 0$. Then $\nu(x) \subseteq (0,\pi)$ and $L \geq \sigma(\{0\})^{-1}$. Thereby $L > \sigma(\{0\})^{-1}$ if and only if $\sigma(\{0\}) > 0$ and $\int_0^{\sigma(\{0\})^{-1}} v(x)^2\,dx < \infty$, and in this case $H(x) \sim \operatorname{diag}(1,0)$, $x \in (\sigma(\{0\})^{-1}, L)$.

**E. Strings**

A string is a pair consisting of a number $L \in [0,\infty]$, and a Borel measure $\mathfrak{m}$ on $\mathbb{R}$ with $\operatorname{supp}\mathfrak{m} \subseteq [0,L]$ such that $\mathfrak{m}([0,x]) < \infty$ for $x \in [0,L)$ and, in case $L < \infty$, $\mathfrak{m}(\{L\}) = 0$. We shall denote the string given by $L$ and $\mathfrak{m}$ by $S[L,\mathfrak{m}]$. The number $L$ in $S[L,\mathfrak{m}]$ is referred to as the length of the string.

Define a function $m$ as

$$m(x) := \mathfrak{m}((-\infty,x)),\ x \in (-\infty,L). \tag{2.10}$$

Then $m$ is non-decreasing and left-continuous, and we have $m(x) = 0$ if $x \leq 0$. Consider the following boundary value problem:

$$y'(x) + \int_{[0,x]} zy(u)d\mathfrak{m}(u) = 0,\ x \in [0, L)\,, \tag{2.11}$$

with boundary condition $y'(0-) = 0$ and, in case $L + m(L) < \infty$, $y(L) = 0$. Thereby $z$ is a complex parameter. Also in this context a notion of Titchmarsh-Weyl coefficient is of significance: It was shown in [15] that there exist unique solutions $\varphi(x, z)$ and $\psi(x, z)$ of (2.11) which satisfy the initial conditions

$$\varphi(0, z) = 1,\ \varphi'(0-, z) = 0, \quad \psi(0, z) = 0,\ \psi'(0-, z) = 1\,, \tag{2.12}$$

and that, for all $z \in \mathbb{C} \setminus [0, \infty)$, the limit

$$q_S(z) := \lim_{x \to L} \frac{\psi(x, z)}{\varphi(x, z)} \tag{2.13}$$

exists. This function is called the Principal Titchmarsh-Weyl coefficient of the string $S[L, \mathfrak{m}]$.

Let $S[L, \mathfrak{m}]$ be a string. Then $q_S$ admits a representation

$$q_S(z) := b + \int_0^\infty \frac{d\sigma_S(t)}{t - z}\,, \tag{2.14}$$

where $\sigma_S$ is some non-negative measure with $\int_0^\infty \frac{d\sigma_S(t)}{1+t} < \infty$, and $b \geq 0$. In fact, $b = \min \operatorname{supp} \mathfrak{m}$. Hence, the Principal Titchmarsh-Weyl coefficient of any string belongs to the Stieltjes class $\mathcal{S}$.

A basic inverse result going back to M.G.Krein is the following, cf. [17], [6], [18]:

**Theorem 2.5 (Inverse Spectral Theorem; Strings).** *The mapping $S[L, \mathfrak{m}] \mapsto q_S$ is a bijection of the set of all strings onto the Stieltjes class $\mathcal{S}$.*

## 3. Inverse spectral relations

We start with an investigation of the limit $\lim_{z \to -\infty} Q_H(z)$.

**Lemma 3.1.** *Let $Q \in \mathcal{N}$ and let $a, b, \sigma$ be as in* (2.1). *Assume that* $\inf \operatorname{supp} \sigma \geq 0$ *and* $b = 0$. *Let $v(x)$ be the (unique) function which corresponds to $Q$ by means of Theorem* 2.3, (2.9). *Then*

$$\lim_{z \to -\infty} Q(z) = \lim_{x \searrow 0} v(x)\,. \tag{3.1}$$

*Proof.* Note that both limits in (3.1) exist in $\mathbb{R} \cup \{-\infty\}$. We show that, for any $a \in \mathbb{R}$, $\lim_{z \to -\infty} Q(z) = a$ if and only if $\lim_{x \to 0} v(x) = a$. Once this is proved, it will also follow that $\lim_{z \to -\infty} Q(z) = -\infty$ if and only if $\lim_{x \to 0} v(x) = -\infty$.

Assume that $\lim_{z\to-\infty} Q(z) = a \in \mathbb{R}$ and choose $c > -a$. Then the function $Q(z) + c$ is positive on the negative real axis, and hence belongs to the Stieltjes class. It follows that also $z(Q(z) + c) \in \mathcal{N}$ and hence that

$$Q_1(z) := \frac{-1}{z(Q(z)+c)} \in \mathcal{N}\,. \tag{3.2}$$

Clearly, $zQ_1(z) \in \mathcal{N}$, and thus $Q_1 \in \mathcal{S}$. Moreover, $\lim_{z\to-\infty} Q_1(z) = 0$. Hence $Q_1$ can be represented as $Q_1(z) = \int_{[0,+\infty)} \frac{d\tau(\lambda)}{\lambda - z}$, and

$$\int_{[0,+\infty)} d\tau(\lambda) = -\lim_{z\to-\infty} zQ_1(z) = \frac{1}{a+c}\,. \tag{3.3}$$

Let $S[L, \mathfrak{m}]$ be the (unique) string whose Principal Titchmarsh-Weyl coefficient $q_S$ is equal to $Q_1$. Then, by [15] and [18],

$$\lim_{x\searrow 0} m(x) = \left(\int_{[0,+\infty)} d\tau(\lambda)\right)^{-1} = a + c\,.$$

Let $H_1$ be a Hamiltonian with $Q_{H_1}(z) = zQ_1(z)$. It was shown in [18] that, if $H_1$ is parameterized appropriately, there exists $l > 0$ such that

$$H_1(x) = \begin{pmatrix} 1 & -m(x) \\ -m(x) & m(x)^2 \end{pmatrix},\ 0 \le x \le l\,. \tag{3.4}$$

By (2.6) and (2.8), the Hamiltonian

$$H_2(x) := \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} JH_1(x)J^T \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}$$

has Titchmarsh-Weyl coefficient $Q$. Hence there exists a reparameterization $\lambda$ with $H(x) = H_2(\lambda(x))\lambda'(x)$. For $x \in [0, l]$ we have

$$H_2(x) = \begin{pmatrix} (m(x)-c)^2 & m(x)-c \\ m(x)-c & 1 \end{pmatrix}\,.$$

Comparing the right lower corners of $H$ and $H_2$ yields that $\lambda|_{[0,l]} = \mathrm{id}$, and hence that $v(x) = m(x) - c$, $x \in [0, l]$. It follows that $\lim_{x\searrow 0} v(x) = a$.

Conversely, if $\lim_{x\searrow 0} v(x) = a$ and $c + a > 0$, the function $v(x) + c$ is the mass function of the string with Principal Titchmarsh-Weyl coefficient $Q_1$ given by (3.2). According to [15], the fact that $\lim_{x\searrow 0} v(x) + c > 0$ implies that $\lim_{z\to-\infty} Q_1(z) = 0$, and that the relation (3.3) holds. By the definition of $Q_1$, we find $\lim_{z\to-\infty} Q(z) = a$. □

This lemma already has a noteworthy corollary.

**Corollary 3.2.** *Let $Q$ and $v$ be as in Lemma* 3.1. *Then $Q \in \mathcal{S}$ if and only if $\lim_{x\to 0} v(x) \ge 0$ In this case $v$ is the mass function of the string whose Principal Titchmarsh-Weyl coefficient is equal to $-(zQ(z))^{-1}$. That is, $v$ is the mass function of the dual string of the string whose Principal Titchmarsh Weyl coefficient is $Q$.*

Now we are in position to prove our first main result.

**Theorem 3.3.** *Let $H$ be a Hamiltonian defined on $[0,L)$ and assume that for some $\epsilon\in(0,L)$ we have*

$$H(x)=\begin{pmatrix} v(x)^2 & v(x)\\ v(x) & 1\end{pmatrix},\ x\in(0,\epsilon)\,,$$

*with a nondecreasing function $v:(0,\epsilon)\to\mathbb{R}$. Then the limit $\lim_{y\to+\infty}Q_H(iy)$ exists in $\mathbb{R}\cup\{-\infty\}$ and in fact*

$$\lim_{y\to+\infty}Q_H(iy)=\lim_{x\searrow 0}v(x)\,. \tag{3.5}$$

*Proof.* Define Hamiltonians $H_1$ and $H_2$ as

$$H_1(x):=\begin{cases} H(x) & ,\ x\in(0,\epsilon)\\ \operatorname{diag}(1,0) & ,\ x\in[\epsilon,+\infty)\end{cases}$$

$$H_2(x):=H(x+\epsilon),\ x\in[0,L-\epsilon)\,.$$

Denote by $W(x,z)$ the transposed of the fundamental matrix solution of the canonical system with Hamiltonian $H$. Then $Q_{H_1}(z)=\frac{w_{11}(\epsilon,z)}{w_{21}(\epsilon,z)}$, and $Q$ is given by

$$Q(z)=\frac{w_{11}(\epsilon,z)Q_{H_2}(z)+w_{12}(\epsilon,z)}{w_{21}(\epsilon,z)Q_{H_2}(z)+w_{22}(\epsilon,z)}\,.$$

A straightforward calculation, using the fact that $\det W(x,z)=1$, will show that

$$Q(z)-Q_{H_1}(z)=-w_{21}(\epsilon,z)^{-2}\Big(\frac{w_{22}(\epsilon,z)}{w_{21}(\epsilon,z)}+Q_{H_2}(z)\Big)^{-1}=$$

$$=\frac{-1}{z}\cdot\Big(\frac{z}{w_{21}(\epsilon,z)}\Big)^2\cdot\frac{1}{z\big(\frac{w_{22}(\epsilon,z)}{w_{21}(\epsilon,z)}+Q_{H_2}(z)\big)} \tag{3.6}$$

Since $w_{22}(\epsilon,z)w_{21}(\epsilon,z)^{-1}+Q_{H_2}(z)\in\mathcal{N}$, the function

$$f(y):=y\operatorname{Im}\Big(\frac{w_{22}(\epsilon,iy)}{w_{21}(\epsilon,iy)}+Q_{H_2}(iy)\Big)$$

is nondecreasing for $y>0$. In particular, the last factor in (3.6) is bounded for $z\in i[1,\infty)$. The function $g(z):=z^{-1}w_{12}(\epsilon,z)$ is a real entire function of exponential type, and all its zeros lie in $\mathbb{R}\setminus\{0\}$. Thus its Weierstrass product representation is of the form

$$g(z)=Ce^{Az}\prod\left(1-\frac{z}{z_n}\right)e^{z/z_n}$$

where $C$ and $A$ are real constants and $z_n\in\mathbb{R}$. Hence $|g(iy)|$ is a nondecreasing function of $y>0$. In particular, the second factor in (3.6) is bounded for $z\in i[1,\infty)$. We see that

$$\big|Q(iy)-Q_{H_1}(iy)\big|=O\big(\frac{1}{y}\big),\ y\geq 1\,. \tag{3.7}$$

This relation, and the fact that $Q_{H_1}$ is by Theorem 2.3 analytic on $\mathbb{C}\setminus[0,\infty)$, implies that

$$\lim_{y\to+\infty}Q_H(iy)=\lim_{y\to+\infty}Q_{H_1}(iy)=\lim_{x\to-\infty}Q_{H_1}(x)\,.$$

By our definition of $H_1$ the function $Q_{H_1}$ satisfies the hypothesis of Lemma 3.1, and we conclude that $\lim_{y\to+\infty} Q_H(iy) = \lim_{x\searrow 0} v(x)$. □

*Remark* 3.4. Assume that, for some $\epsilon > 0$, we have $H(x) = \operatorname{diag}(1,0)$, $x \in (0,\epsilon)$. Then $\lim_{z\to-\infty} Q_H(z) = -\infty$. This tells us that Theorem 3.3 remains true if we, formally, have $v(x) = -\infty$.

As a particular case of Theorem 3.3 we obtain that the assumption $\inf\operatorname{supp}\sigma \geq 0$ in Lemma 3.1 can be relaxed.

**Corollary 3.5.** *Let $Q \in \mathcal{N}$ and let $a, b, \sigma$ be as in* (2.1)*. Assume that* $\inf\operatorname{supp}\sigma > -\infty$ *and $b = 0$. Let $v(x)$ be the (unique) function which corresponds to $Q$ by means of Theorem* 2.3*,* (2.9)*. Then $\lim_{z\to-\infty} Q(z) = \lim_{x\searrow 0} v(x)$.*

*Proof.* According to Theorem 2.3, the assumptions of Theorem 3.3 are satisfied. To establish the present assertion it suffices to note that, since $\inf\operatorname{supp}\sigma > -\infty$, the relation $\lim_{y\to+\infty} Q(iy) = \lim_{z\to-\infty} Q(z)$ holds. □

**Corollary 3.6.** *Let $Q \in \mathcal{N}$ be such that some Hamiltonian $H$ with $Q_H = Q$ satisfies the hypothesis of Theorem* 3.3*. Then $Q \in \mathcal{N}_1$ if and only if* $\lim_{y\to+\infty} Q(iy) \in \mathbb{R}$.

*Proof.* Assume that $\lim_{y\to+\infty} Q(iy) =: a \in \mathbb{R}$. Consider the Hamiltonian $H_1$ as in the proof of Theorem 3.3. Then $Q_{H_1} - a \in \mathcal{S} \subseteq \mathcal{N}_1$, and hence also $Q_{H_1} \in \mathcal{N}_1$. The relations (3.7) and (2.3) now imply that also $Q \in \mathcal{N}_1$. □

Note that in general only the implication "$Q \in \mathcal{N}_1 \Rightarrow \lim_{y\to+\infty} Q(iy) \in \mathbb{R}$" holds.

*Remark* 3.7. The canonical system (1.1) with the boundary condition $y_1(0) = 0$ corresponds to a selfadjoint extension of a symmetric operator with Dirichlet boundary conditions. In [9] the concept of a generalized Friedrichs extension is introduced and characterized by the condition that its $Q$-function does not belong to $\mathcal{N}_1$, but $-Q^{-1} \in \mathcal{N}_1$. If the assumptions of Theorem 3.3 are satisfied, the condition $\lim_{x\searrow 0} v(x) = -\infty$ characterizes the generalized Friedrichs extension, which is equal to the common Friedrichs extension of semibounded symmetric operators under the assumptions of Corollary 3.5, see [11], [23] for more details.

Next we turn to an investigation of the limit $\lim_{z\nearrow 0} Q_H(z)$.

**Lemma 3.8.** *Let $Q \in \mathcal{N}$ and let $a, b, \sigma$ be as in* (2.1)*. Assume that* $\inf\operatorname{supp}\sigma \geq 0$*, $b = 0$, and that $\sigma(\{0\}) = 0$. Let $v(x)$ be the (unique) function which corresponds to $Q$ by means of Theorem* 2.3*,* (2.9)*. Then*

$$\lim_{z\nearrow 0} Q(z) = \lim_{x\nearrow L} v(x)\,. \tag{3.8}$$

*Proof.* Note that both limits in (3.8) exist in $\mathbb{R}\cup\{+\infty\}$. Again we shall show that for any $a \in \mathbb{R}$ we have $\lim_{z\nearrow 0} Q(z) = a$ if and only if $\lim_{x\nearrow L} v(x) = a$.

Assume that $\lim_{z\nearrow 0} Q(z) = a$. Choose $c < -a$, then $Q(x) + c < 0$ for $x \in (-\infty, 0)$, and hence $-\frac{1}{Q(z)+c} \in \mathcal{S}$. Thus $Q_1(z) := z^{-1}(Q(z) + c) \in \mathcal{N}$. Since, clearly, $zQ_1(z) \in \mathcal{N}$, it follows that $Q_1 \in \mathcal{S}$.

Let $S[L,\mathfrak{m}]$ be the string with $q_S = Q_1$. According to [18], the first part of the Hamiltonian corresponding to $Q(z)+z$ is of the form (3.4). Denoting the independent variable in (3.4) by $u$, a scale transformation of the form $x(u) = \int_{[0,u)} m(t)^2 dt$ brings the first part of the Hamiltonian corresponding to $Q(z)+c$ into the form

$$\widetilde{H}(x) = \begin{pmatrix} \widetilde{m}(x)^{-2} & -\widetilde{m}(x)^{-1} \\ -\widetilde{m}(x)^{-1} & 1 \end{pmatrix},$$

with $\widetilde{m}(x) = m(u)$, and it follows that $-\widetilde{m}(x)^{-1} = v(x)+c$. The assumption $\sigma(\{0\}) = 0$ implies that $v$ is defined on $(0,\infty)$, hence $m(L) = \widetilde{m}(\infty)$, and $L + \int_{[0,L)} m(t)^2 dt = \infty$. Let $Q_2(z) = zQ_1(z^2)$. Then, by [18], the trace-normed Hamiltonian $H$ corresponding to $Q_2$ is of diagonal form, and the relation $m(L) = \int_{[0,+\infty)} (0,1)H(t)(0,1)^T\, dt$ holds. By (2.5), we have $\int_{[0,+\infty)} (0,1)H(t)(0,1)^T\, dt = -(\lim_{y\searrow 0} iyQ_2(iy))^{-1}$. Note that $\lim_{y\searrow 0} iyQ_2(iy) = \lim_{z\nearrow 0} Q(z)+c$. Summing up, the last relations imply that $\lim_{z\nearrow 0} Q(z) = \lim_{x\nearrow L} v(x)$.

Conversely, assume that $\lim_{x\nearrow L} v(x) = a$. Again choose $c < -a$, and denote $\widetilde{v}(x) = v(x)+c$. The Hamiltonian corresponding to $Q(z)+c$ is then of the form (2.9) with $\widetilde{v}$ instead of $v$, and a scale transformation of the form $x(u) = \int_{[0,u)} v(t)^2 dt$ brings it into the form (3.4) with $m(x) = -\widetilde{v}(x)^{-1}$, which implies that $m$ is a mass distribution function of a string. It follows that $Q_1(z) = \frac{Q(z)+c}{z}$ is a Stieltjes function, and we find that $\lim_{z\nearrow 0} Q(z) = a$ by the first part of the proof. □

**Theorem 3.9.** *Let $H$ be a Hamiltonian defined on $[0,L)$ and assume that for some $l \in (0,L)$ we have*

$$H(x) = \begin{pmatrix} v(x)^2 & v(x) \\ v(x) & 1 \end{pmatrix}, \ x \in (l,L),$$

*with a nondecreasing function $v : (l,L) \to \mathbb{R}$. Then $Q_H$ is meromorphic in $\mathbb{C} \setminus [0,+\infty)$, the negative real poles of $Q_H$ cannot accumulate at $0$, and the limit $\lim_{z\nearrow 0} Q_H(z)$ exists in $\mathbb{R} \cup \{+\infty\}$. In fact we have*

$$\lim_{z\nearrow 0} Q_H(z) = \lim_{x\nearrow L} v(x). \tag{3.9}$$

*Proof.* Consider the Hamiltonian $H_1(x) := H(x+l)$, $x \in (0, L-l)$. Let $a_1, b_1, \sigma_1$ be the data in the integral representation of $Q_{H_1}$. By Theorems 2.4, 2.3, and Remark 2.2, $(i)$, we have $b_1 = 0$ and $\operatorname{supp}\sigma_1 \in [0,\infty)$. Thus $Q_{H_1}$ is analytic in $\mathbb{C} \setminus [0,\infty)$ and the limit $\lim_{z\nearrow 0} Q_{H_1}(z)$ exists in $\mathbb{R} \cup \{+\infty\}$.

If $W$ denotes the transposed of the fundamental matrix solution of the canonical system with Hamiltonian $H$, we have

$$Q_H(z) = \frac{w_{11}(l,z)Q_{H_1}(z) + w_{12}(l,z)}{w_{21}(l,z)Q_{H_1}(z) + w_{22}(l,z)}.$$

Hence $Q_H$ is meromorphic in $\mathbb{C} \setminus [0,\infty)$ and the limit $\lim_{z\nearrow 0} Q(z)$ exists, in fact $\lim_{z\nearrow 0} Q(z) = \lim_{z\nearrow 0} Q_{H_1}(z)$.

Consider the case that $\sigma_1(\{0\}) = 0$. Then $Q_{H_1}$ satisfies the assumptions of Lemma 3.8. The relation (3.8) implies together with the last formula that (3.9) holds. Assume now that $\sigma_1(\{0\}) > 0$. Then, certainly, $\lim_{z\nearrow 0} Q_{H_1} = +\infty$. The relation (2.5) yields that $L < \infty$, and hence, since Weyl's limit point prevails, $\int_l^L v(x)^2\,dx = \infty$. In particular, $\lim_{x\nearrow L} v(x) = +\infty$. This shows that also in this case (3.9) holds. □

*Remark* 3.10. Assume that, for some $l < L$, we have $H(x) = \operatorname{diag}(1,0)$, $x \in (l,L)$. Then $\lim_{z\nearrow 0} Q_H(z) = +\infty$. This follows, since in the described situation, we have $Q_H(z) = w_{21}(l,z)^{-1}w_{11}(l,z)$, where $W$ is as in the above proof. Hence $Q_H$ is meromorphic in $\mathbb{C}$ and has a pole at 0. This statement just says that the assertion of Theorem 3.9 remains true when we, formally, have $v(x) = +\infty$.

**Corollary 3.11.** *Let $Q \in \mathcal{N}$ and let $a, b, \sigma$ be as in* (2.1). *Assume that* $\operatorname{supp}\sigma \cap (-\infty,0)$ *is a finite set. Let $v(x)$ be the (unique) function which corresponds to $Q$ by means of Theorem* 2.3, (2.9). *Then* $\lim_{z\nearrow 0} Q(z) = -\lim_{x\nearrow L} \cot\nu(x)$, *where we understand* $\cot\phi = -\infty$ *for* $\phi \in \pi\mathbb{Z}$.

*Proof.* By Theorem 2.3 $\nu$ is bounded. That is, there are at most finitely many intervals where the Hamiltonian $H$ is of the form $\operatorname{diag}(1,0)$, and there are at most finitely many points where $v$ has a negative jump or becomes singular. By (2.5), $\int_{(0,L)}(0,1)H(t)(0,1)^T\,dt = \sigma(\{0\})^{-1}$. If $\sigma(\{0\}) = 0$, then $L = +\infty$, and $v$ is nondecreasing on some interval $(l,+\infty)$. Hence, the assumptions of Theorem 3.9 are satisfied. If $\sigma(\{0\}) > 0$, then either $L < +\infty$ and there is some $l < L$ such that $v$ is nondecreasing on $(l,L)$ and $\int_{(l,L)} v(x)^2 dx = +\infty$, that is, $v(L-) = -\cot\nu(L-) = +\infty$, or $H = \operatorname{diag}(1,0)$ on some interval $(l_0,+\infty)$, that is $-\cot\nu(L-) = +\infty$ on $(l_0,+\infty)$. Clearly, if $\sigma(\{0\}) > 0$ then $Q(0-) = +\infty$. □

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