Strings, dual strings, and related canonical systems.

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A symmetric Nevanlinna function Q is of the form $Q(z) = zQ_s(z^2)$ where Q_s and $Q_0(z) = zQ_s(z)$ are also Nevanlinna functions. In such a situation Q_s and $-Q_0^{-1}$ are Stieltjes functions. An inverse result of L. de Branges implies that each Nevanlinna function is the Titchmarsh-Weyl coefficient of a uniquely determined canonical system with some nonnegative Hamiltonian matrix function H, and, according to M.G. Krein, each Stieltjes function is the Titchmarsh-Weyl coefficient of a uniquely determined string. The Hamiltonians corresponding to Q_s, Q_0 and Q are constructed in terms of the string corresponding to Q_s and the dual string corresponding to $-Q_0^{-1}$. The relations between the associated Fourier transformations are described by commuting isometric isomorphisms between the considered spaces.

1 Introduction

Recall [15] that if a function $Q_s(z)$ belongs to the class of Stieltjes functions $\mathcal{S}$ then the functions $Q_0(z) = zQ_s(z)$ and $Q_d(z) = zQ_s(z^2)$ belong to the class $\mathcal{N}$ of Nevanlinna functions. Let $\widehat{Q}(z) = -Q(z)^{-1}$. Then $Q \in \mathcal{N}$ if and only if $\widehat{Q} \in \mathcal{N}$, and the function $-(zQ_s(z))^{-1}$ belongs as well to the Stieltjes class, see [15]. A well known inverse result of L. de Branges (see [7], [8], [23]) states that each Nevanlinna function is the Titchmarsh-Weyl coefficient of a uniquely determined 2-dimensional system of canonical differential equations (2.26) which is characterized by a real, nonnegative and trace normed matrix funtion H, called the Hamiltonian. Moreover, according to M.G. Krein, (see [16], [21]) each Stieltjes function is the principal Titchmarsh-Weyl coefficient of a unique string $S[L,\mathfrak{m}]$ which is characterized by its length L and a measure $\mathfrak{m}$ which may be considered as the mass distribution of a physical string.

Let a string $S[L,\mathfrak{m}]$ with principal Titchmarsh-Weyl coefficient $Q_s(z)$ be given. Then the problem arises how to express the Hamiltonians H_s, H_0 and H_d associated with the Nevanlinna functions Q_s, Q_0 and Q_d, and also the Hamiltonians corresponding to $\widehat{Q}_s$, $\widehat{Q}_0$ and $\widehat{Q}_d$, in terms of the length L and the measure $\mathfrak{m}$ of the string $S[L,\mathfrak{m}]$. Partial results concerning this question are contained in e.g. [11], [8], [20], [19], [21], [18], see also [14], [10], and [1], [2], [3], [22] for n-dimensional canonical systems. The Stieltjes function $Q_{\hat{s}}(z) = -(zQ_s(z))^{-1}$ is the principal Titchmarsh-Weyl coefficient of the dual string of $S[L,\mathfrak{m}]$, that is, roughly speaking, the string which arises if length and mass in $S[L,\mathfrak{m}]$ are interchanged (see, e.g., [16], [9]). It turns out that the Hamiltonian H_s, which has the same Titchmarsh-Weyl coefficient as the string $S[L,\mathfrak{m}]$, can be expressed in terms of the corresponding dual string.

Let σ denote the spectral measure (see (2.1)) of some Nevanlinna function Q. If H is the associated Hamiltonian, there is a Fourier transformation mapping the space L^2_H isometrically onto L^2_σ such that the action of

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the selfadjoint differential operator determined by the canonical system in L^2_H becomes the operator of multiplication with the independent variable in L^2_σ, see e.g. [6], [13], [12]. An aim of this paper is to give a comprehensive description of all the isometric isomorphisms which connect the different Fourier transformations of all the Hamiltonians associated with the Nevanlinna functions Q_s, Q_0 and Q_d, and the Fourier transformations which are determined by the string $S[L, \mathfrak{m}]$ and its dual string. The fact that $Q_d(z) = -Q_d(-z)$ implies that the corresponding spectral measure σ_d is symmetric and that the associated Hamiltonian $H_d = \operatorname{diag}(h_1, h_2)$ is of diagonal form. The function h_1 is related to the length of the string, and the function h_2 is related to the corresponding mass function. It turns out that the decomposition of the vector functions $f = (f_1, f_2)^T \in L^2_{H_d}$ into its components f_1 and f_2 corresponds via the Fourier transformation to the splitting of L^2_σ into the subspaces $L^2_{\sigma,o}$ of the odd, and $L^2_{\sigma,e}$ of the even functions. There are isometric isomorphisms between $L^2_{H_0}$ and $L^2_{\sigma,o}$ and $L^2_{H_s}$ and $L^2_{\sigma,e}$, and also between the space $L^2_\mathfrak{m}$ of the string $S[L, \mathfrak{m}]$ and $L^2_{\sigma,e}$.

The second chapter contains preliminary results from the literature about Nevanlinna and Stieltjes functions, strings and canonical systems. The third chapter is concerned with dual strings, and a couple of partially known results about the interaction between strings and dual strings are formulated for later use. Chapter 4 contains explicit relations for all the Hamiltonians which are connected with the string as mentioned above, and the isometries between the corresponding spaces are described. In chapter 5 the isometric isomorphisms which arise between the different Fourier transformations are presented, see Theorem 5.1 below.

2 Preliminaries

2.1 Subclasses of Nevanlinna functions

A function Q is said to belong to the set of Nevanlinna functions $\mathcal{N}$ if it is analytic on $\mathbb{C}\setminus\mathbb{R}$, satisfies the symmetry condition $Q(\bar{z}) = \overline{Q(z)}$, and maps the upper half-plane $\mathbb{C}^+$ into $\mathbb{C}^+ \cup \mathbb{R}$. In particular, the relation $\operatorname{Im} Q(z) \geq 0$ for $z \in \mathbb{C}^+$ holds. A Nevanlinna function $Q \in \mathcal{N}$ is said to belong to the set $\mathcal{S}$, the class of Stieltjes functions, if additionally the function $\tilde{Q}$ defined by $\tilde{Q}(z) = zQ(z)$ has the property that $\tilde{Q} \in \mathcal{N}$, see [15]. Recall [17] that a Nevanlinna function Q is is called symmetric, $Q \in \mathcal{N}^{sym}$, if Q is odd, that is $Q(z) = -Q(-z)$ for $z \in \mathbb{C} \setminus \mathbb{R}$. Each $Q \in \mathcal{N}$ has a unique representation of the form

$$Q(z) = a + bz + \int_{-\infty}^{+\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\sigma(t), \tag{2.1}$$

with $a \in \mathbb{R},\ b \geq 0$, and a measure σ on $\mathbb{R}$ with the property that

$$\int_{-\infty}^{+\infty} \frac{d\sigma(t)}{1+t^2} < \infty.$$

Theorem 2.1 *Let $Q \in \mathcal{N}^{sym}$ be given. Define functions Q_s and Q_0 by the relations*

$$zQ_s(z^2) = Q(z), \qquad \frac{Q_0(z^2)}{z} = Q(z). \tag{2.2}$$

Then Q_s, $-\dfrac{1}{Q_0} \in \mathcal{S}$. Let σ, σ_s and σ_0 be the spectral measures of Q, Q_s and Q_0, respectively. Then the measures σ_s and σ_0 are given by

$$d\sigma_s(t^2) = 2d\sigma(t), \qquad d\sigma_0(t^2) = 2t^2 d\sigma(t). \tag{2.3}$$

Put

$$L^2_{\sigma,e} := \left\{ h \in L^2_\sigma :\ h(-t) = h(t) \right\}, \tag{2.4}$$

$$L^2_{\sigma,o} := \left\{ k \in L^2_\sigma :\ k(-t) = -k(t) \right\}. \tag{2.5}$$

Then $L^2_\sigma = L^2_{\sigma,e} \oplus L^2_{\sigma,o}$, and we have the following isometric isomorphisms:

$$L^2_{\sigma_s} \mapsto L^2_{\sigma,e}: \qquad f(t) \to h(t) := f(t^2), \tag{2.6}$$

$$L^2_{\sigma_0} \mapsto L^2_{\sigma,o}: \qquad g(t) \to k(t) := tg(t^2). \tag{2.7}$$

P r o o f. As $Q \in \mathcal{N}^{sym}$ is symmetric, the functions Q_0 and Q_s are well-defined. Note that $Q_0(z) = zQ_s(z)$. The proposition 4.6 of [15] implies that Q_s and Q_0 are Nevanlinna functions, and the relations $Q_s, -Q_0^{-1} \in \mathcal{S}$ follow by a result also from the paper [15]. Recall the Stieltjes - Lifschitz inversion formula: Let $Q \in \mathcal{N}$ and $[a, b] \subset \mathbb{R}$, and let ϕ be real and analytic on $[a, b]$. If $\sigma(\{a\}) = \sigma(\{b\}) = 0$, then

$$\int_a^b \phi(t)d\sigma(t) = \lim_{y \searrow 0} \pi^{-1} \int_a^b \operatorname{Im}\,(\phi(t+iy)Q(t+iy))dt. \tag{2.8}$$

In particular, if $\sigma'(t)$ exists, then $\sigma'(t) = \pi^{-1} \lim_{y\searrow 0} \operatorname{Im} Q(t+iy)$. By (2.8), the relations (2.3) follow from (2.2). The measure $d\sigma$ is symmetric, that is $d\sigma(t) = -d\sigma(-t)$. If $f \in L^2_{\sigma_s}$, define the even function h on $\mathbb{R}$ as $h(t) = f(t^2)$. It follows that

$$\|f\|^2_{L^2_{\sigma_s}} = \int_0^\infty |f(t^2)|^2 d\sigma_s(t^2) = 2\int_0^\infty |h(t)|^2 d\sigma(t) = \|h\|^2_{L^2_\sigma}.$$

Analogously, if $g \in L^2_{\sigma_0}$, define k on $\mathbb{R}$ by $k(t) = tg(t^2)$. It follows that

$$\|g\|^2_{L^2_{\sigma_0}} = \int_0^\infty |g(t^2)|^2 d\sigma_0(t^2) = 2\int_0^\infty |k(t)|^2 d\sigma(t) = \|k\|^2_{L^2_\sigma}.$$

□

2.2 Strings

A string $S[L, \mathfrak{m}]$ is given by its length L, $0 \le L \le \infty$, and a non-negative (possibly infinite) Borel measure $\mathfrak{m}$ on $\mathbb{R}$ with $\operatorname{supp}\mathfrak{m} \subseteq [0, L]$ such that $\mathfrak{m}([0, x]) < \infty$ for $x \in [0, L)$ and $\mathfrak{m}(\{L\}) = 0$. Define

$$m(x) := \mathfrak{m}((-\infty, x)),\ x \in (-\infty, L]. \tag{2.9}$$

Then m is a non-decreasing left-continuous function defined on $(-\infty, L]$ if $L < \infty$, or on $(-\infty, \infty)$ if $L = \infty$, such that $m(x) = 0$ if $x \le 0$. Consider the integral equation boundary value problem with complex parameter z:

$$y'(x) + \int_{[0,x]} zy(u)d\mathfrak{m}(u) = 0, \tag{2.10}$$

with the boundary conditions

$$y'(0-) = 0, \qquad \text{and} \qquad y(L) = 0 \text{ if } L + m(L) < \infty. \tag{2.11}$$

This problem arises if Fourier's method is applied to the partial differential equation which describes the vibrations of a string with free left endpoint 0 on the interval $[0, L)$ or $[0, L]$, where $m(x)$ is the mass of the string on the interval $[0, x)$. Let $l := \sup(\operatorname{supp}\mathfrak{m})$. The string $S[L, \mathfrak{m}]$ is called *singular* if $l + m(l) = \infty$, otherwise, if $l + m(l) < \infty$, it is called *regular*.

There exists unique solutions $\varphi(x, z)$ and $\psi(x, z)$ (see [16]) of the equation (2.10) on $[0, L)$ which satisfy the initial conditions

$$\varphi(0, z) = 1,\ \varphi'(0-, z) = 0, \quad \psi(0, z) = 0,\ \psi'(0-, z) = 1. \tag{2.12}$$

Note that if y is a solution of (2.10), then y' is continuous from the right. It follows easily that φ and ψ are the solutions of the following integral equations:

$$\varphi(x,z) = 1 - z\int_{[0,x]}(x-s)\varphi(s,z)d\mathfrak{m}(s), \quad 0 \le x < L, \tag{2.13}$$

$$\psi(x,z) = x - z\int_{[0,x]}(x-s)\psi(s,z)d\mathfrak{m}(s), \quad 0 \le x < L. \tag{2.14}$$

Note that

$$\varphi'(x,z) = -z\int_{[0,x]}\varphi(s,z)d\mathfrak{m}(s), \quad 0 \le x < L, \tag{2.15}$$

$$\varphi'(x-,z) = -z\int_{[0,x)}\varphi(s,z)d\mathfrak{m}(s), \quad 0 \le x < L, \tag{2.16}$$

$$\psi'(x,z) = 1 - z\int_{[0,x]}\psi(s,z)d\mathfrak{m}(s), \quad 0 \le x < L, \tag{2.17}$$

$$\psi'(x-,z) = 1 - z\int_{[0,x)}\psi(s,z)d\mathfrak{m}(s), \quad 0 \le x < L. \tag{2.18}$$

In particular, $\psi'(0,z) = \psi'(0-,z) = 1$ and $\varphi'(0,z) = -zm(0+)$. As $\mathfrak{m}(\{x\}) = 0$ for almost all $x \in [0,L)$ with respect to the Lebesgue measure, the relations

$$\varphi(x,z) - 1 = \int_0^x \varphi'(s,z)ds = \int_0^x \varphi'(s-,z)ds, \tag{2.19}$$

$$\psi(x,z) = \int_0^x \psi'(s,z)ds = \int_0^x \psi'(s-,z)ds, \tag{2.20}$$

follow. Let $b := \min(\operatorname{supp}\mathfrak{m})$, that is, $m(x) = 0$ if $x \le b$ and $m(x) > 0$ if $x > b$. Note that $b \ge 0$. The limit

$$q_S(z) := \lim_{x\to L}\frac{\psi(x,z)}{\varphi(x,z)}, \; z \in \mathbb{C}\setminus[0,\infty), \tag{2.21}$$

exists and admits the representation

$$q_S(z) := b + \int_0^\infty \frac{d\sigma_S(t)}{t-z}, \tag{2.22}$$

where σ_S is a non-negative measure with the property

$$\int_0^\infty \frac{d\sigma_S(t)}{1+t} < \infty, \tag{2.23}$$

that is, $q_S \in \mathcal{S}$ (see [16]). According to M.G. Krein, the measure σ_S is called the *principal spectral measure* of the string $S[L,\mathfrak{m}]$. We shall call the corresponding function q_S the *principal Titchmarsh-Weyl coefficient* of the string $S[L,\mathfrak{m}]$. A basic inverse result of M.G. Krein states (see [9], [21]): *Any function $q \in \mathcal{S}$ is the principal Titchmarsh-Weyl coefficient of a (regular or singular) string $S[L,\mathfrak{m}]$; this string is uniquely determined by q.*

Denote by $L^2_{\mathfrak{m},0}$ the subset of $L^2_{\mathfrak{m}}$ of elements which vanish identically near L if $S[L,\mathfrak{m}]$ is singular and define for $f \in L^2_{\mathfrak{m},0}$ the following kind of Fourier transformation:

$$F_S(f,z) := \int_0^L \varphi(x,z)f(x)d\mathfrak{m}(x). \tag{2.24}$$

It can be shown (see [16]) that the mapping $F_S: \ f \mapsto F(f,\cdot)$ is an isometry from $L^2_{\mathfrak{m},0}$ onto a dense subset of $L^2_{\sigma_S}$. Hence it can be continuously extended to all of $L^2_{\mathfrak{m}}$. The inverse transformation, mapping $L^2_{\sigma_S}$ onto L^2_m, is given by

$$f(x) = \text{l.i.m.}_{N\to\infty} \int_0^N \varphi(x,\lambda) F_S(f,\lambda) d\sigma_S(\lambda), \ x \in [0,L), \tag{2.25}$$

where l.i.m. denotes the limit in the norm of $L^2_{\mathfrak{m}}$. Note that the Parseval identity $[f,g]_{L^2_{\mathfrak{m}}} = [F_S(f,\cdot), F_S(g,\cdot)]_{L^2_{\sigma_S}}$ for $f, g \in L^2_{\mathfrak{m}}$ holds.

2.3 Canonical systems

Let H be a real, symmetric and non-negative 2×2–matrix function on the interval $[0, l_H)$:

$$H(x) = \begin{pmatrix} h_1(x) & h_3(x) \\ h_3(x) & h_2(x) \end{pmatrix}, \quad x \in [0, l_H),$$

with locally integrable functions $h_1, \ h_2$ and h_3. In this section we consider initial value problems of the form

$$Jy'(x) = -zH(x)y(x), \ x \in [0, l_H), \quad y_1(0) = 0, \tag{2.26}$$

with $y(x) = (y_1(x) \ y_2(x))^T$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and a complex parameter z. Here the differential equation in (2.26) is considered to hold almost everywhere on $[0, l_H)$. The *fundamental matrix function*

$$W(x,z) = \begin{pmatrix} w_{11}(x,z) & w_{12}(x,z) \\ w_{21}(x,z) & w_{22}(x,z) \end{pmatrix}$$

of a canonical system (2.26) with Hamiltonian H is the unique solution of the integral equation

$$W(x,z)J - J = z\int_0^x W(s,z)H(s)ds. \tag{2.27}$$

Note that $W(0,z) = I$. At l_H for the canonical system (2.26) Weyl's limit point case prevails if and only if

$$\int_0^{l_H} \operatorname{trace} H(x)dx = \infty, \tag{2.28}$$

and from now on we assume that for each Hamiltonian H the relation (2.28) holds, and that H is not identically equal to diag $(1 \ 0)$ on the interval $[0,\infty)$. Then the limit point case prevails at l_H, and it follows that for each $\omega \in \widetilde{\mathcal{N}} := \mathcal{N} \cup \{\infty\}$ and $z \in \mathbb{C}^+$ the limit

$$Q(z) := \lim_{x\to l_H} \frac{w_{11}(x,z)\omega(z) + w_{12}(x,z)}{w_{21}(x,z)\omega(z) + w_{22}(x,z)} \tag{2.29}$$

exists, is independent of ω, and, as a function of z, belongs to the set of Nevanlinna functions $\mathcal{N}$ (see, e.g., [5]). The function Q is called the *Titchmarsh-Weyl coefficient* of the canonical system (2.26) or of the Hamiltonian H. The measure σ in the representation (2.1) of Q is called the *spectral measure* of the canonical system (2.26) or of the Hamitonian H. In particular, if $H = VV^T$ a.e. on (l,∞) for some constant vector $V = (v_1, v_2)^T \neq 0$ then

$$Q(z) := \frac{v_1 w_{11}(l,z) + v_2 w_{12}(l,z)}{v_1 w_{21}(l,z) + v_2 w_{22}(l,z)}. \tag{2.30}$$

The following intervals play a special role in the sequel (see [13], [8]). Let $\xi_\phi := (\cos\phi, \sin\phi)^T$ for some $\phi \in [0, \pi)$. The open interval $I_\phi \subset [0, l_H)$ is called *H-indivisible of type ϕ* if the relation

$$\xi_\phi^T JH = 0, \text{ a.e. on } I_\phi, \tag{2.31}$$

holds. In particular, $\det H = 0$ a.e. on I_ϕ. An H-indivisible interval is called *maximal* if it is not a proper subset of another H-indivisible interval.

With the Hamiltonian H the following Hilbert space L^2_H is associated (see [13]): It is the set of all (equivalence classes of) 2-vector functions $f(x) = (f_1(x)\ f_2(x))^T$ on $[0, l_H)$ with the properties

(i) $\int\limits_0^{l_H} f(x)^* H(x) f(x) dx < +\infty$,

(ii) for every H-indivisible interval I_ϕ of type ϕ there is a constant $c_{I_\phi,f} \in \mathbb{C}$ such that $\xi_\phi^T f = c_{I_\phi,f}$, a.e. on I_ϕ,

equipped with the inner product

$$(f, g)_{L^2_H} := \int\limits_0^{l_H} g(x)^* H(x) f(x) dx.$$

A Hamiltonian H is called *trace normed* if $\operatorname{trace} H = h_1 + h_2 = 1$ a.e. on $[0, \infty)$. For the class of trace normed Hamiltonians a basic inverse result in [7] can be formulated as follows (see [23], [19]): *Each function $Q \in \mathcal{N}$ is the Titchmarsh-Weyl coefficient of a canonical system with a trace normed Hamiltonian H on $[0, \infty)$ which is not equal to $\operatorname{diag}(1, 0)$ a. e. on $[0, \infty)$; this correspondence is bijective if two Hamiltonians which coincide almost everywhere are identified.*

We mention that in the representation (2.1) of a Titchmarsh-Weyl coefficient Q the number $b \geq 0$ is positive if and only if $(0, b)$ is a maximal indivisible interval of the corresponding trace normed Hamiltonian H such that $H = \operatorname{diag}(1, 0)$ a. e. on $(0, b)$. Let Q_H denote the Titchmarsh-Weyl coefficient corresponding to some Hamiltonian H, and let

$$\widehat{H} = JHJ^T. \tag{2.32}$$

The relation (2.29) implies that $Q_{\widehat{H}}(z) = -(Q_H(z))^{-1}$. For $f \in L^2_H$ the function $\hat{f} := Jf$ belongs to $L^2_{\widehat{H}}$, and it is easy to see that

$$\|f\|_{L^2_H} = \|\hat{f}\|_{L^2_{\widehat{H}}}.$$

Hence, the mapping

$$L^2_H \mapsto L^2_{\widehat{H}} : \quad f \to \hat{f},$$

establishes an isometric isomorphism. We set

$$u(x, z) := \begin{pmatrix} w_{21}(x, z) \\ w_{22}(x, z) \end{pmatrix}.$$

Denote by $L^2_{H,0}$ the subset of L^2_H of elements which vanish identically near l_H, and define for $f \in L^2_{H,0}$ the following sort of Fourier transformation:

$$F_H(f, z) := \int\limits_0^{l_H} u(x, z)^T H(x) f(x) dx. \tag{2.33}$$

It can be shown (see [6]) that the mapping $F_H : f \mapsto F_H(f, \cdot)$ is an isometry from $L^2_{H,0}$ onto a dense subset of L^2_σ. Hence it can be extended by continuity to all of L^2_H. The inverse transformation, mapping L^2_σ onto L^2_H, is given by

$$f(x) = \text{l.i.m.}_{N \to +\infty} \int\limits_{-N}^{+N} u(x, \lambda) F_H(f, \lambda) d\sigma(\lambda), \ x \in [0, l_H), \tag{2.34}$$

where l.i.m. denotes the limit in the norm of L^2_H. Parseval's identity $[f,g]_{L^2_H} = [F_H(f,\cdot), F_H(g,\cdot)]_{L^2_\sigma}$ holds for $f,g \in L^2_H$.

Lemma 2.2 *If H is of diagonal form, that is $H = \operatorname{diag}(h_1,h_2)$, then $Q_H \in \mathcal{N}^{sym}$.*

P r o o f. Let $M = \operatorname{diag}(1,-1)$. Clearly, $M^2 = I$ and $JM = -MJ$, and $MHM = H$ as H is of diagonal form. With the last relations it follows from (2.27) that $W(x,z) = MW(x,-z)M$, and the relation (2.29) implies that $Q_H(z) = -Q_H(-z)$. ☐

3 Dual strings

Let a string $S[L,\mathfrak{m}]$ be given. As m is non-decreasing, its inverse function exists and determines also a string, see, e.g., [16], [9]. Define

$$\hat{L} = \begin{cases} m(L) & \text{if } L + m(L) = \infty, \\ \infty & \text{if } L + m(L) < \infty. \end{cases} \tag{3.1}$$

and

$$\hat{m}(\hat{\xi}) = \inf\{x \geq 0 :\ \hat{\xi} \leq m(x)\}, \qquad \hat{\xi} \in [0, m(L)], \tag{3.2}$$

with the additional conditions that $\hat{m}(\hat{\xi}) = 0$ for $\hat{\xi} \in (-\infty, 0)$ and, if $L + m(L) < \infty$ then

$$\hat{m}(\hat{\xi}) = L \text{ for } \hat{\xi} \in (m(L), \infty).$$

The function $\hat{m}$ is non-decreasing, and if $L + m(L) = \infty$, it follows that $\hat{m}(\hat{L}) = l$, the supremum of growth points of m. The left continuity of m implies that $m(L) = \hat{l}$, the supremum of growth points of $\hat{m}$. To see that $\hat{m}$ is also left-continuous, let $\hat{\xi}_n \nearrow \hat{\xi}$, $a_n = \hat{m}(\hat{\xi}_n)$, $a = \lim a_n$, $b = \hat{m}(\hat{\xi})$. Assume that $a < b$, then $\hat{\xi}_n \leq m(t) < \hat{\xi}$ for $t \in (a,b)$, and it follows that $\hat{\xi} \leq m(t) < \hat{\xi}$, a contradiction. Note that

$$\hat{m}(\hat{\xi}+) = \sup\{x \leq L :\ m(x) \leq \hat{\xi}\}, \qquad \hat{\xi} \in [0, m(L)]. \tag{3.3}$$

To see this, let $\hat{\xi}_n \searrow \hat{\xi}$, $d_n = \hat{m}(\hat{\xi}_n)$, $d = \lim d_n$, $c = \sup\{x :\ m(x) \leq \hat{\xi}\}$. If $c < d$, then $\hat{\xi} < m(t) < \hat{\xi}_n$ for $t \in (c,d)$, and it follows that $\hat{\xi} \leq m(t) < \hat{\xi}$, a contradiction.

Let $\hat{\mathfrak{m}}$ denote the corresponding measure generated by $\hat{m}$. The string $\hat{S}[\hat{L},\hat{\mathfrak{m}}]$ is called the dual string of $S[L,\mathfrak{m}]$, that is, roughly speaking, the string which arises if mass and length change their role in $S[L,\mathfrak{m}]$. Note that $\hat{m}(\hat{L}) + \hat{L} < \infty$ if and only if the string $S[L,\mathfrak{m}]$ is regular with $L = \infty$, in this case the relations $\hat{L} = m(l+)$ and $\hat{m}(\hat{L}) = l$ hold. Moreover, the string $\hat{S}[\hat{L},\hat{\mathfrak{m}}]$ is regular with $\hat{L} = \infty$ if and only if $L + m(L) < \infty$. It follows that the string $S[L,\mathfrak{m}]$ is regular if and only if its dual string $\hat{S}[\hat{L},\hat{\mathfrak{m}}]$ is regular.

Lemma 3.1 *Let $S[L,\mathfrak{m}]$ be some string and $\hat{S}[\hat{L},\hat{\mathfrak{m}}]$ be its dual string. Then the relation $m(\xi+) \in \operatorname{supp}\hat{\mathfrak{m}}$ holds. If $\hat{\xi} < m(\xi)$ then $\hat{m}(\hat{\xi}) < \xi$, and if $m(\xi+) < \hat{\xi}$ then $\xi < \hat{m}(\hat{\xi})$. If $\xi \in \operatorname{supp}\mathfrak{m}$ then $\hat{m}(m(\xi+)) = \xi$. Moreover, if $m(\xi+) > m(\xi)$ then $\hat{m}(\hat{\xi}) = \xi$ for $\hat{\xi} \in (m(\xi), m(\xi+)]$. If $0 \leq \xi \leq L$ and $\xi \notin \operatorname{supp}\mathfrak{m}$, then $\hat{m}$ has a jump at $m(\xi)$ and $\xi \in (\hat{m}(m(\xi)), \hat{m}(m(\xi)+))$.*

P r o o f. The relation $m(\xi+) \in \operatorname{supp}\hat{\mathfrak{m}}$ is a consequence of the other statements of the present Lemma. If $\hat{\xi} < m(\xi)$, the left continuity of m implies that there is some $x < \xi$ such that $\hat{\xi} < m(x) < m(\xi)$, and $\hat{m}(\hat{\xi}) < \xi$ follows from (3.2). Now let $m(\xi+) < \hat{\xi}$. As $m(\xi+)$ is a limit from the right, there is some $x > \xi$ such that $m(x) < \hat{\xi}$, and the relation (3.2) implies that $\hat{m}(\hat{\xi}) \geq x > \xi$. Let $\xi \in \operatorname{supp}\mathfrak{m}$. Then

$$\hat{m}(m(\xi+)) = \inf\{x :\ m(\xi+) \leq m(x)\} = \xi.$$

Let $m(\xi+) > m(\xi)$. If $\hat{\xi} \in (m(\xi), m(\xi+))$, then also

$$\hat{m}(\hat{\xi}) = \inf\{x :\ \hat{\xi} \leq m(x)\} = \xi.$$

Let $0 \leq \xi \neq \operatorname{supp}\mathfrak{m}$, then m is constant in an open neighborhood of ξ, and it follows that

$$\hat{m}(m(\xi)) = \inf\{x: \ m(x) = m(\xi)\} < \xi < \sup\{x: \ m(x) = m(\xi)\} = \hat{m}(m(\xi)+).$$

□

Lemma 3.2 *Each string $S[L,\mathfrak{m}]$ is equal to the dual string of its dual string $\hat{S}[\hat{L},\hat{\mathfrak{m}}]$.*

Proof. Let $S[L,\mathfrak{m}]$ be regular. If $L=\infty$ then $\hat{L}+\hat{m}(\hat{L})<\infty$ and hence $\hat{\hat{L}}=\infty$. Moreover, $\hat{\hat{m}}(\infty)=\hat{L}=m(\infty)$. If $L<\infty$ then $\hat{L}=\infty$ and $\hat{\hat{L}}=\hat{m}(\hat{l}+)=L$ by the relation (3.3). It follows that $\hat{\hat{m}}(L)=\hat{l}=m(L)$. If $S[L,\mathfrak{m}]$ is singular, then

$$\hat{\hat{L}} = \hat{m}(\hat{L}) = \inf\{x \geq 0: \ m(L) \leq m(x)\} = L.$$

The Lemma 3.1 implies that

$$\hat{\hat{m}}(\xi) = \inf\{x \geq 0: \ \xi \leq \hat{m}(x)\} = m(\xi), \qquad \xi \in [0,L]. \tag{3.4}$$

□

The relation $\hat{\hat{S}} = S$ implies that to each result concerning S and $\hat{S}$ a corresponding dual result exists. In particular, the relations

$$m(\xi) = \inf\{\hat{x} \geq 0: \ \xi \leq \hat{m}(\hat{x})\}, \qquad \xi \in [0,l], \tag{3.5}$$

$$m(\xi+) = \sup\{\hat{x} \leq \hat{L}: \ \hat{m}(\hat{x}) \leq \xi\}, \qquad \xi \in [0,l]. \tag{3.6}$$

hold. The dual result of the Lemma 3.1 is

Corollary 3.3 *Let $S[L,\mathfrak{m}]$ be some string and $\hat{S}[\hat{L},\hat{\mathfrak{m}}]$ be its dual string. Then the relation $\hat{m}(\hat{\xi}+) \in \operatorname{supp}\mathfrak{m}$ holds. If $\xi < \hat{m}(\hat{\xi})$ then $m(\xi) < \hat{\xi}$, and if $\hat{m}(\hat{\xi}+) < \xi$ then $\hat{\xi} < m(\xi)$. Let $\hat{\xi} \in \operatorname{supp}\hat{\mathfrak{m}}$. Then $m(\hat{m}(\hat{\xi}+)) = \hat{\xi}$. Moreover, if $\hat{m}(\hat{\xi}+) > \hat{m}(\hat{\xi})$ then $m(\xi) = \hat{\xi}$ for $\xi \in (\hat{m}(\hat{\xi}), \hat{m}(\hat{\xi}+)]$. If $0 \leq \hat{\xi} \leq \hat{L}$ and $\hat{\xi} \notin \operatorname{supp}\hat{\mathfrak{m}}$, then m has a jump at $\hat{m}(\hat{\xi})$ and $\hat{\xi} \in (m(\hat{m}(\hat{\xi})), m(\hat{m}(\hat{\xi})+))$.*

Corollary 3.4 *Let f be a measurable function. Then*

$$\int_{[a,b]} f(\xi) d\mathfrak{m}(\xi) = \int_{[m(a),m(b+)]} f(\hat{m}(\hat{\xi})) d\hat{\xi}, \tag{3.7}$$

$$\int_{[a,b)} f(\xi) d\mathfrak{m}(\xi) = \int_{[m(a),m(b)]} f(\hat{m}(\hat{\xi})) d\hat{\xi}. \tag{3.8}$$

Analogously, if g is a measurable function, then

$$\int_{[a,b]} g(\hat{\xi}) d\hat{\mathfrak{m}}(\hat{\xi}) = \int_{[\hat{m}(a),\hat{m}(b+)]} g(m(\xi)) d\xi, \tag{3.9}$$

$$\int_{[a,b)} g(\hat{\xi}) d\hat{\mathfrak{m}}(\hat{\xi}) = \int_{[\hat{m}(a),\hat{m}(b)]} g(m(\xi)) d\xi. \tag{3.10}$$

Proof. Let $\xi \in \operatorname{supp}\mathfrak{m}$. If $m(\xi+) > m(\xi)$, then $\xi = \hat{m}(\hat{\xi})$ for $\hat{\xi} \in (m(\xi), m(\xi+)]$ by Lemma 3.1. It follows that $f(\xi)\mathfrak{m}(\{\xi\}) = \int_{[m(\xi),m(\xi+)]} f(\hat{m}(\hat{\xi}))d\hat{\xi}$. If $m(\xi+) = m(\xi)$, Lemma 3.1 implies that $\hat{m}(m(\xi)) = \xi$, hence, with $\hat{\xi} = m(\xi)$ the relations $f(\xi) = f(\hat{m}(\hat{\xi}))$ and $d\mathfrak{m}(\xi) = d\hat{\xi}$ follow. This proves the relations (3.7) and (3.8), and a similar argument using the Corrollary 3.3 implies the relations (3.9) and (3.10). □

Lemma 3.5 *Let $S[L,\mathfrak{m}]$ be some string and $\hat{S}[\hat{L},\mathfrak{m}]$ be its dual string. Let $\widehat{\varphi}(\hat{\xi},z)$ and $\widehat{\psi}(\hat{\xi},z)$ satisfy the corresponding relations* (2.14) *and* (2.13) *for $\hat{S}[\hat{L},\hat{\mathfrak{m}}]$ instead of $S[L,\mathfrak{m}]$ with the corresponding initial conditions* (2.12). *Then*

$$\widehat{\varphi}(\hat{\xi},z) = \psi'(\hat{m}(\hat{\xi}),z), \qquad \hat{\xi} \in \operatorname{supp}\hat{\mathfrak{m}}, \tag{3.11}$$

$$\widehat{\psi}(\hat{\xi},z) = -z^{-1}\varphi'(\hat{m}(\hat{\xi}),z), \quad \hat{\xi} \in \operatorname{supp}\hat{\mathfrak{m}}, \tag{3.12}$$

and

$$\varphi(\xi, z) = \widehat{\psi}'(m(\xi), z), \qquad \xi \in \operatorname{supp} \mathfrak{m}, \tag{3.13}$$

$$\psi(\xi, z) = -z^{-1}\widehat{\varphi}'(m(\xi), z), \quad \xi \in \operatorname{supp} \mathfrak{m}. \tag{3.14}$$

P r o o f. The main part of the proof is to check that the functions $\psi'(\hat{m}(\hat{\xi}), z)$ and $-z^{-1}\varphi'(\hat{m}(\hat{\xi}), z)$ satisfy the equations (2.13) and (2.14) for $\hat{\mathfrak{m}}$ replaced by $\mathfrak{m}$ and $\hat{\xi} \in \operatorname{supp} \hat{\mathfrak{m}}$: Let $\hat{\xi} \in \operatorname{supp} \hat{\mathfrak{m}}$ such that $\hat{m}(\hat{\xi}+) > \hat{m}(\hat{\xi})$. Then $\mathfrak{m}((\hat{m}(\hat{\xi}), \hat{m}(\hat{\xi}+)]) = 0$ by Lemma 3.1, and the relations (2.15), (2.16), (2.17), and (2.18) imply for $\xi \in (\hat{m}(\hat{\xi}), \hat{m}(\hat{\xi}+))$ that $\varphi'(\hat{m}(\hat{\xi}+)-, z) = \varphi'(\xi, z) = \varphi'(\hat{m}(\hat{\xi}), z)$ and $\psi'(\hat{m}(\hat{\xi}+)-, z) = \psi'(\xi, z) = \psi'(\hat{m}(\hat{\xi}), z)$. Note that the relation

$$\inf\{\hat{s} : \; u < \hat{m}(\hat{s})\} = \sup\{\hat{s} : \; \hat{m}(\hat{s}) \leq u\} = m(u+), \qquad u \in [0, L),$$

together with the Corollary 3.3 and the relation (3.8) imply

$$\int_{[0,\hat{m}(\hat{\xi}+))} \chi_{[0,s)}(u) d\mathfrak{m}(s) = \int_{[0,\hat{\xi})} \chi_{[0,\hat{m}(\hat{s}))}(u) d\hat{s} = \hat{\xi} - m(u+), \; \hat{\xi} \in \operatorname{supp} \hat{\mathfrak{m}}.$$

With the last relation and Corollary 3.4 it follows that

$$\begin{aligned}
\psi'(\hat{m}(\hat{\xi}), z) &= 1 - z \int_{[0,\hat{m}(\hat{\xi})]} \psi(s, z) d\mathfrak{m}(s) \\
&= 1 - z \int_{[0,\hat{m}(\hat{\xi}+))} \psi(s, z) d\mathfrak{m}(s) \\
&= 1 - z \int_{[0,\hat{m}(\hat{\xi}+))^2} \chi_{[0,s)}(u) \psi'(u, z) du \; d\mathfrak{m}(s) \\
&= 1 - z \int_{[0,\hat{m}(\hat{\xi}+))} (\hat{\xi} - m(u+)) \psi'(u, z) du \\
&= 1 - z \int_{[0,\hat{\xi}]} (\hat{\xi} - \hat{u}) \psi'(\hat{m}(\hat{u}), z) d\hat{\mathfrak{m}}(\hat{u}).
\end{aligned}$$

The last equality sign is a consequence of (3.9) and the fact that $m(u+) = m(u)$ on $[0, L)$ with the exception of a countable set. Hence, the function $\psi'(\hat{m}(\hat{\xi}), z)$ satisfies the relation (2.13) for $\hat{\xi} \in \operatorname{supp} \hat{\mathfrak{m}}$. For any $\hat{\xi} \in [0, \hat{L})$, let $\hat{x} = \sup\{u \in \operatorname{supp} \hat{m} : u \leq \hat{\xi}\}$ and define

$$\phi(\hat{\xi}, z) = \psi'(\hat{m}(\hat{x}), z) - z(\hat{\xi} - \hat{x}) \int_{[0,\hat{x})} \psi'(\hat{m}(\hat{u}), z) d\hat{\mathfrak{m}}(\hat{u}).$$

In particular, $\phi(\hat{\xi}, z) = \psi'(\hat{m}(\hat{\xi}), z)$ if $\hat{\xi} \in \operatorname{supp} \hat{\mathfrak{m}}$. It is easy to see that $\phi(\hat{\xi}, z)$ satisfies the relation (2.13) for all $\hat{\xi} \in [0, \hat{L})$. As the solution of the equation (2.13) is unique, one finds that $\phi(\hat{\xi}, z) = \widehat{\varphi}(\hat{\xi}, z)$, and the relation (3.11) follows as a special case. In the same way we find that

$$\begin{aligned}
\varphi'(\hat{m}(\hat{\xi}), z) &= -z \int_{[0,\hat{m}(\hat{\xi})]} \varphi(s, z) d\mathfrak{m}(s) \\
&= -z \int_{[0,\hat{m}(\hat{\xi}+))} \varphi(s, z) d\mathfrak{m}(s) \\
&= -z \int_{[0,\hat{m}(\hat{\xi}+))} \left(1 + \int_{[0,\hat{m}(\hat{\xi}+)} \chi_{[0,s)}(u) \varphi'(u, z) du \right) d\mathfrak{m}(s) \\
&= -z\hat{\xi} - z \int_{[0,\hat{m}(\hat{\xi}+))} (\hat{\xi} - m(u+)) \varphi'(u, z) du \\
&= -z\hat{\xi} - z \int_{[0,\hat{\xi}]} (\hat{\xi} - \hat{u}) \varphi'(\hat{m}(\hat{u}), z) d\hat{\mathfrak{m}}(\hat{u}),
\end{aligned}$$

and the relation (3.12) follows as above from the last relation and the relation (2.14). □

Lemma 3.6 *Let q_S be the principal Titchmarsh-Weyl coefficient of some string $S[L,\mathfrak{m}]$, and let $q_{\hat{S}}$ denote the principal Titchmarsh-Weyl coefficient of the corresponding dual string $\hat{S}[\hat{L},\hat{\mathfrak{m}}]$. Then*

$$q_{\hat{S}}(z) = \frac{-1}{zq_S(z)}. \tag{3.15}$$

Proof. If the string $S[L,\mathfrak{m}]$ is singular, the limit point case prevails at L. Then also (see [16])

$$q_S(z) = \lim_{\xi\to L} \frac{\psi'(\xi,z)}{\varphi'(\xi,z)},\ z \in \mathbb{C}\setminus[0,\infty).$$

As $\hat{S}[\hat{L},\hat{\mathfrak{m}}]$ is also singular, the last relation together with the relations (3.11), (3.12), and $\hat{m}(\hat{L}) = L$ imply that

$$q_{\hat{S}}(z) = \lim_{\hat{\xi}\to\hat{L}} \frac{\widehat{\psi}(\hat{\xi},z)}{\widehat{\varphi}(\hat{\xi},z)} = \frac{-1}{zq_S(z)}.$$

Let $L + m(L) < \infty$, then $\hat{L} = \infty$ and $\hat{S}[\hat{L},\hat{\mathfrak{m}}]$ is regular. The relation (2.21) implies that

$$q_{\hat{S}}(z) = \frac{\widehat{\psi}'(\hat{l},z)}{\widehat{\varphi}'(\hat{l},z)},\ z \in \mathbb{C}\setminus[0,\infty). \tag{3.16}$$

The relations $\hat{l} = m(L)$, $l = \hat{m}(\hat{l})$ and $\hat{\mathfrak{m}}(\{\hat{l}\}) = L - l$ hold. If $m(l) < m(l+) = m(L)$, then $\hat{\mathfrak{m}}((m(l), m(L)) = 0$. The relations (3.11) and (3.13) imply that

$$\varphi'(l,z) = \varphi'(\hat{m}(\hat{l}),z) = -z\widehat{\psi}(\hat{l},z)$$

and

$$\varphi(l,z) = \widehat{\psi}'(m(l),z) = \widehat{\psi}'(\hat{l}-,z).$$

As m is constant on (l,L), it follows that $\varphi'(x,z) = \varphi'(l,z)$ for $x \in (l,L)$, and hence

$$\varphi(L,z) = (L-l)\varphi'(l,z) + \varphi(l,z) = -z\widehat{\psi}(\hat{l},z)\mathfrak{m}(\{\hat{l}\}) + \widehat{\psi}'(\hat{l}-,z) = \widehat{\psi}'(\hat{l},z).$$

In the same way using the relations (3.12) and (3.14) one finds $\psi(L,z) = -z^{-1}\widehat{\varphi}'(\hat{l},z)$. Consequently, the relation (3.16) implies that

$$q_{\hat{S}}(z) = -z^{-1}\frac{\varphi(L,z)}{\psi(L,z)} = \frac{-1}{zq_S(z)}.$$

If the string $S[L,\mathfrak{m}]$ is regular with $L = \infty$, the relation (2.21) implies that

$$q_S(z) = \frac{\psi'(l,z)}{\varphi'(l,z)},\ z \in \mathbb{C}\setminus[0,\infty). \tag{3.17}$$

As above a straightforward calculation using the relations (3.11), (3.12), (3.13), and (3.14) leads to

$$\widehat{\varphi}(\hat{L},z) = \psi'(l,z), \qquad \widehat{\psi}(\hat{L},z) = -z^{-1}\varphi'(l,z),$$

and using (3.17) we obtain

$$q_{\hat{S}}(z) = -z^{-1}\frac{\varphi'(l,z)}{\psi'(l,z)} = \frac{-1}{zq_S(z)}.$$

□

4 Relations between strings and canonical systems

Let a string $S[L, \mathfrak{m}]$ be given. Define $x(t) := t + m(t)$ for $t \in [0, L)$. Then the Lebesgue measure dt and $d\mathfrak{m}(t)$ are absolutely continuous with respect to $dx(t) := dt + d\mathfrak{m}(t)$. Let $I_x := \operatorname{ran} x(\cdot)$ denote the range of the function x, then clearly $[0, \infty) \setminus I_x$ consists of the (at most countable) union of all intervals of the form $(x(t), x(t+))$, where $x(t+) - x(t) = m(t+) - m(t)$. Let

$$h_1(x) := \frac{dt}{dx(t)}, \quad x \in I_x, \qquad h_1(x) = 0, \quad x \in [0, \infty) \setminus I_x, \tag{4.1}$$

$$h_2(x) := \frac{d\mathfrak{m}(t)}{dx(t)}, \quad x \in I_x, \qquad h_2(x) = 1, \quad x \in [0, \infty) \setminus I_x. \tag{4.2}$$

Then $h_1(x) + h_2(x) = 1$ a.e. on $[0, \infty)$, and by

$$H_d(x) := \begin{pmatrix} h_1(x) & 0 \\ 0 & h_2(x) \end{pmatrix}, \qquad x \in [0, \infty). \tag{4.3}$$

a trace normed and diagonal Hamiltonian is defined.

Conversely, let a trace normed Hamiltonian $H_d = \operatorname{diag}(h_1, h_2)$ be given. Define $l_d = c$ if (c, ∞) is a maximal H_d-indivisible interval of type $\pi/2$, and $l_d = \infty$ otherwise, and define $\hat{l}_d = c$ if (c, ∞) is a maximal H_d-indivisible interval of type 0, and $\hat{l}_d = \infty$ otherwise. Let

$$\xi(x) := \int_0^x h_1(t)dt, \qquad L := \int_0^{l_d} h_1(t)dt, \tag{4.4}$$

$$\hat{\xi}(x) := \int_0^x h_2(t)dt, \qquad \hat{L} := \int_0^{\hat{l}_d} h_2(t)dt. \tag{4.5}$$

and

$$m(\xi) := \int_0^{\inf\{x:\, \xi(x) = \xi\}} h_2(t)dt, \ 0 \le \xi \le L, \tag{4.6}$$

$$\hat{m}(\hat{\xi}) := \int_0^{\inf\{x:\, \hat{\xi}(x) = \hat{\xi}\}} h_1(t)dt, \ 0 \le \hat{\xi} \le \hat{L}. \tag{4.7}$$

Then the functions m and $\hat{m}$ are non-decreasing and left-continuous on $[0, L)$ and $[0, \hat{L})$, respectively, and $S[L, \mathfrak{m}]$ is the string associated with H_d, whereas $\hat{S}[\hat{L}, \hat{\mathfrak{m}}]$ is its corresponding dual string. The above indicated mappings $S[L, \mathfrak{m}] \mapsto H_d$ and $H_d \mapsto S[L, \mathfrak{m}]$ are inverse to each other.

Let a string $S[L, \mathfrak{m}]$ and its dual string $\hat{S}[\hat{L}, \hat{\mathfrak{m}}]$ be given. Define

$$H_0(x) := \begin{cases} \begin{pmatrix} 1 & -m(x) \\ -m(x) & m(x)^2 \end{pmatrix} & \text{if } 0 \le x \le L, \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } L + \int_0^L m(t)^2 dt < \infty, \ L < x < \infty. \end{cases} \tag{4.8}$$

$$H_s(x) := \begin{cases} \begin{pmatrix} \hat{m}(x)^2 & \hat{m}(x) \\ \hat{m}(x) & 1 \end{pmatrix} & \text{if } 0 \le x \le \hat{L}, \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \hat{L} + \int_0^{\hat{L}} \hat{m}(t)^2 dt < \infty, \ \hat{L} < x < \infty. \end{cases} \tag{4.9}$$

Lemma 4.1 *The fundamental matrices W_0, W_d, and W_s corresponding to H_0, H_d, and H_s are given by*

$$W_0(x,z) = \begin{pmatrix} zm(x)\psi(x,z) + \psi'(x-,z) & z\psi(x,z) \\ m(x)\varphi(x,z) + z^{-1}\varphi'(x-,z) & \varphi(x,z) \end{pmatrix},\ 0 \le x < L, \tag{4.10}$$

$$W_d(x,z) = \begin{pmatrix} \psi'(\xi-,z^2) & z\psi(\xi,z^2) \\ z^{-1}\varphi'(\xi-,z^2) & \varphi(\xi,z^2) \end{pmatrix},\ x := \xi + m(\xi),\ 0 \le \xi < L, \tag{4.11}$$

$$W_s(x,z) = \begin{pmatrix} \widehat{\varphi}(x,z) & -\hat{m}(x)\widehat{\varphi}(x,z) - z^{-1}\widehat{\varphi}'(x-,z) \\ -z\widehat{\psi}(x,z) & z\hat{m}(x)\widehat{\psi}(x,z) + \widehat{\psi}'(x-,z) \end{pmatrix},\ 0 \le x < \hat{L}. \tag{4.12}$$

In particular,

$$W_s(x,z) = \begin{pmatrix} \psi'(\hat{m}(x),z) & \psi(\hat{m}(x),z) - \hat{m}(x)\psi'(\hat{m}(x),z) \\ \varphi'(\hat{m}(x),z) & \varphi(\hat{m}(x),z) - \hat{m}(x)\varphi'(\hat{m}(x),z) \end{pmatrix},\ x \in \operatorname{supp}\hat{\mathfrak{m}}. \tag{4.13}$$

Proof. First note that

$$W_0(x,z)J = \begin{pmatrix} z\psi(x,z) & -zm(x)\psi(x,z) - \psi'(x-,z) \\ \varphi(x,z) & -m(x)\varphi(x,z) - z^{-1}\varphi'(x-,z) \end{pmatrix},$$

and

$$zW_0(x,z)H_0(x) = \begin{pmatrix} z\psi'(x-,z) & -zm(x)\psi'(x-,z) \\ \varphi'(x-,z) & -m(x)\varphi'(x-,z) \end{pmatrix}.$$

Integration by parts and the relations (2.19) and (2.20) imply that

$$\int_{[0,x)} m(s)\varphi'(s-,z)ds = m(x)\varphi(x,z) - \int_{[0,x)} \varphi(s,z)d\mathfrak{m}(s), \tag{4.14}$$

$$\int_{[0,x)} m(s)\psi'(s-,z)ds = m(x)\psi(x,z) - \int_{[0,x)} \psi(s,z)d\mathfrak{m}(s). \tag{4.15}$$

With the relations (2.16) and (2.17) we obtain that

$$-\int_{[0,x)} m(s)\varphi'(s-,z)ds = -m(x)\varphi(x,z) - z^{-1}\varphi'(x-,z), \tag{4.16}$$

$$-z\int_{[0,x)} m(s)\psi'(s-,z)ds - 1 = -zm(x)\psi(x,z) - \psi'(x-,z). \tag{4.17}$$

The last relations and the relations (2.19) and (2.20) imply that for W_0 and H_0 the equation (2.27) holds, hence W_0 is the fundamental matrix corresponding to the Hamiltonian H_0.

Note that $W_s(x,z) = J\widehat{W_0}(x,z)J^T$, where $\widehat{W_0}(x,z)$ arises from $W_0(x,z)$ if m is replaced by $\hat{m}$ and the functions φ and ψ are replaced by $\widehat{\varphi}$ and $\widehat{\psi}$. Thus $W_s(x,z)$ is the fundamental matrix corresponding to H_s. If $x \in \operatorname{supp}\hat{\mathfrak{m}}$, then $x = m(\hat{m}(x)+)$ by Lemma 3.1. As $\hat{m}(x) \in \operatorname{supp}\mathfrak{m}$, it follows with the relations (3.13) and (3.14) that

$$\begin{aligned} &\widehat{\varphi}'(x-,z) = \widehat{\varphi}'(m(\hat{m}(x)+)-,z) \\ &= \widehat{\varphi}'(m(\hat{m}(x)),z) = -z\psi(\hat{m}(x),z), \end{aligned}$$

and

$$\begin{aligned} &\widehat{\psi}'(x-,z) = \widehat{\psi}'(m(\hat{m}(x)+)-,z) \\ &= \widehat{\psi}'(m(\hat{m}(x)),z) = \varphi(\hat{m}(x),z). \end{aligned}$$

The last relations together with (3.11), (3.12) and (4.12) imply the relation (4.13). For (4.11), note that for $x = \xi + m(\xi)$

$$W_d(x,z)J = \begin{pmatrix} z\psi(\xi,z^2) & -z\psi'(\xi-,z^2) \\ \varphi(\xi,z^2) & -z^{-1}\varphi'(\xi-,z^2), \end{pmatrix}$$

and

$$zW_d(x,z)H_d(x) = z\begin{pmatrix} \psi'(\xi-,z^2)h_1(x) & z\psi(\xi-,z^2)h_2(x) \\ z^{-1}\varphi'(\xi-,z^2)h_1(x) & \varphi(\xi,z^2)h_2(x) \end{pmatrix}.$$

The relations (4.4) and (4.5) imply that

$$\int_0^x \psi'(\xi(s)-,z^2)h_1(s)ds = \int_0^{\xi(x)} \psi'(\tilde{\xi}-,z^2)d\tilde{\xi} = \psi(\xi,z^2),$$

and

$$-1 + z\int_0^x \psi(\xi(s)-,z^2)h_2(s)ds = -1 + z\int_0^{\xi(x)} \psi(\tilde{\xi}-,z^2)d\mathfrak{m}(\tilde{\xi}) = -\psi'(\xi-,z^2).$$

Similar relations hold for $\varphi(\xi,z^2)$ and $\varphi'(\xi-,z^2)$. Let $x = \xi + m(\xi)$ and $x_+ = \xi + m(\xi+)$. If $m(\xi) < m(\xi+)$, then $H_d = \operatorname{diag}(0,1)$ a.e. on the indivisible interval (x, x_+). For $u \in (x, x_+)$ we set

$$W_d(u,z) = W_d(x,z)\begin{pmatrix} 1 & 0 \\ -z(u-x) & 1 \end{pmatrix}. \tag{4.18}$$

Then $W_d(u,z)'J = zW_d(x,z)\operatorname{diag}(0,1) = zW_d(u,z)H_d(u)$ a.e. on (x,x_+), that is, $W_d(u,z)$ satisfies the relation (2.27). Note that the relations (4.18) and (4.11) match, because the relations (2.15), (2.16), and (2.17), (2.18) imply that

$$W_d(x_+,z) = \begin{pmatrix} \psi'(\xi+,z^2) & z\psi(\xi,z^2) \\ z^{-1}\varphi'(\xi,z^2) & \varphi(\xi,z^2) \end{pmatrix} = W_d(x,z)\begin{pmatrix} 1 & 0 \\ -z\mathfrak{m}(\{\xi\}) & 1 \end{pmatrix}.$$

It follows that W_d defined by the relations (4.11) and (4.18) satisfies the relation (2.27) for all $x \in [0, L+m(L))$. Hence, it is the fundamental matrix corresponding to the Hamiltonian H_d. □

Theorem 4.2 *Let q_S be the principal Titchmarsh-Weyl coefficient of some string $S[L,\mathfrak{m}]$, and let Q_0, Q_d and Q_s denote the Titchmarsh-Weyl coefficients corresponding to the Hamiltonians H_0, H_d and H_s, respectively. Then the following relations hold:*

$$Q_0(z) = zq_S(z), \tag{4.19}$$

$$Q_d(z) = zq_S(z^2), \tag{4.20}$$

$$Q_s(z) = q_S(z). \tag{4.21}$$

P r o o f. At first we show the relation (4.19). If $L + \int_0^L m(t)^2 dt = \infty$, it follows easily from the relations (2.21), (2.29) and (4.10). If $L + \int_0^L m(t)^2 dt < \infty$, the relation (4.8) implies that the interval (L,∞) is H_0- indivisible of type $\pi/2$. It follows from the relation (2.27) that the entries $w_{12}(\cdot,z)$ and $w_{22}(\cdot,z)$ of W_0 are constant on $[L,\infty)$, and the relations (2.21), (4.10), and (2.30) imply that

$$Q_0(z) = \lim_{x\to L} z\frac{\psi(x,z)}{\varphi(x,z)} = zq_S(z).$$

Analogously, if $\hat{L} + \int_0^{\hat{L}} \hat{m}(t)^2 dt < \infty$, the relation (4.9) implies that the interval $(\hat{L},\infty)$ is H_s- indivisible of type 0. By relation (2.27), the entries $w_{11}(\cdot,z)$ and $w_{21}(\cdot,z)$ of W_s are constant on $[\hat{L},\infty)$, and the relations (3.15), (4.13), and (2.30) imply that

$$Q_s(z) = \lim_{x\to\hat{L}} -z^{-1}\frac{\widehat{\psi}(x,z)}{\widehat{\varphi}(x,z)} = -z^{-1}(q_{\hat{S}}(z))^{-1} = q_S(z).$$

If $L + \int_0^L m(t)^2 dt = \infty$, the relations (3.15), (2.29) and (4.13) imply the relation (4.21). To show the relation (4.20), note that if $L + m(L) < \infty$, the interval $(L + m(L), \infty)$ is H_d-indivisible of type $\pi/2$, hence the entries $w_{12}(\cdot, z)$ and $w_{22}(\cdot, z)$ of W_d are constant there. Consequently, with the relations (2.21), (4.11), and (2.30) one finds that

$$Q_d(z) = \lim_{\xi \to L} z \frac{\psi(\xi, z^2)}{\varphi(\xi, z^2)} = z q_S(z^2).$$

The case that $L + m(L) = \infty$ is plain. □

The following scheme describes how the principal Titchmarsh-Weyl coefficient q_S of a string $S[L, \mathfrak{m}]$ is related to the Titchmarsh-Weyl coefficients of its dual string $\hat{S}[\hat{L}, \hat{\mathfrak{m}}]$ and the corresponding Hamiltonians H_d, H_0 and H_s given by the relations (4.3), (4.8) and (4.9), and the Hamiltonians $\widehat{H}_d$, $\widehat{H}_0$ and $\widehat{H}_s$ defined by (2.32).

$S \Leftrightarrow q_S(z)$	$H_d \Leftrightarrow z q_S(z^2)$	$H_0 \Leftrightarrow z q_S(z)$	$H_s \Leftrightarrow q_S(z)$
$\hat{S} \Leftrightarrow \frac{-1}{z q_S(z)}$	$\widehat{H}_d \Leftrightarrow \frac{-1}{z q_S(z^2)}$	$\widehat{H}_0 \Leftrightarrow \frac{-1}{z q_S(z)}$	$\widehat{H}_s \Leftrightarrow \frac{-1}{q_S(z)}$

Let $H_d = \operatorname{diag}(h_1, h_2)$ be some given trace normed Hamiltonian of diagonal form. In the following it is assumed that the equivalence class $f \in L^2_{H_d}$ is represented by an element $f = (f_1, f_2)^T$ with $\operatorname{supp} f_1 \subseteq \operatorname{supp} h_1$ and $\operatorname{supp} f_2 \subseteq \operatorname{supp} h_2$, and that f_1 and f_2 are constant on the closure of any maximal H_d-indivisible interval. Let

$$L^2_{H_{d,1}} := \{f \in L^2_{H_d} : \ f_2 = 0\}, \ L^2_{H_{d,2}} := \{f \in L^2_{H_d} : \ f_1 = 0\}, \tag{4.22}$$

and note that

$$L^2_{H_d} = L^2_{H_{d,1}} \oplus L^2_{H_{d,2}}, \ f = (f_1, f_2)^T = (f_1, 0)^T \oplus (0, f_2)^T. \tag{4.23}$$

Let $S[L, \mathfrak{m}]$ be the string associated with H_d, and let $\hat{S}[\hat{L}, \hat{\mathfrak{m}}]$ be its dual string. Recall that $L + m(L) < \infty$ implies that $h_1 = 0$ a.e. on $(L + m(L), \infty)$, and if $\hat{L} + \hat{m}(\hat{L}) < \infty$ then $h_2 = 0$ a.e. on $(\hat{L} + \hat{m}(\hat{L}), \infty)$. Define with $\xi(x)$ and $\hat{\xi}(x)$ given by (4.4) and (4.5) the functions

$$r(\xi) = \inf\{u \geq 0 : \ \xi(u) = \xi\}, \quad \xi \in [0, L], \tag{4.24}$$

$$s(\hat{\xi}) = \inf\{u \geq 0 : \ \hat{\xi}(u) = \hat{\xi}\}, \quad \hat{\xi} \in [0, \hat{L}], \tag{4.25}$$

and note that $\operatorname{ran} r \subseteq \operatorname{supp} h_1$ and $\operatorname{ran} s \subseteq \operatorname{supp} h_2$ such that $\operatorname{supp} h_1 \setminus \operatorname{ran} r$ and $\operatorname{supp} h_2 \setminus \operatorname{ran} s$ are sets of Lebesgue measure zero. Moreover, the functions r and s are injective, and satisfy the relations

$$m(\xi) = \int_0^{r(\xi)} h_2(t) dt, \qquad \hat{m}(\hat{\xi}) = \int_0^{s(\hat{\xi})} h_1(t) dt. \tag{4.26}$$

If $r(\xi+) > r(\xi)$ then $(r(\xi), r(\xi+))$ is a maximal H_d-indivisible interval of type $\pi/2$, and if $s(\hat{\xi}+) > s(\hat{\xi})$ then $(s(\hat{\xi}), s(\hat{\xi}+))$ is a maximal H_d-indivisible interval of type 0. In particular, the relations

$$s(m(\xi+)) = r(\xi+), \qquad r(\hat{m}(\hat{\xi}+)) = s(\xi+),$$

follow. Let $L^2[0, L; m]$ be the subspace of the space of the square integrable functions $L^2[0, L]$ consisting of all functions f with the property that if I_1 is an interval contained in $[0, L] \setminus \operatorname{supp} m$ then f is constant on the closure of I_1. In the same way, let $L^2[0, \hat{L}; \hat{m}]$ be the subspace of $L^2[0, \hat{L}]$ of all functions f with the property that if I_2 is an interval contained in $[0, \hat{L}] \setminus \operatorname{supp} \hat{m}$ then f is constant on the closure of I_2.

Let $f \in L^2_{H_d}$. Define functions

$$f^1(\xi) = f_1(r(\xi)), \quad \xi \in [0, L], \tag{4.27}$$

$$f^2(\hat{\xi}) = f_2(s(\hat{\xi})), \quad \hat{\xi} \in [0, \hat{L}], \tag{4.28}$$

and note that $f^1 \in L^2[0, L; m]$ and $f^2 \in L^2[0, \hat{L}; \hat{m}]$. Conversely, as r and s are injective functions, each $f^1 \in L^2[0, L; m]$ defines via the relation (4.27) a function $f \in L^2_{H_{d_1}}$, and each $f^2 \in L^2[0, \hat{L}; \hat{m}]$ defines via the relation (4.28) a function $f \in L^2_{H_{d_2}}$. Clearly, if $f \in L^2_{H_d}$ then

$$\|f\|^2_{H_d} = \|f^1\|^2_{L^2[0,L;m]} + \|f^2\|^2_{L^2[0,\hat{L};\hat{m}]}.$$

Let $f^0 \in L^2_{H_0}$ and $f^s \in L^2_{H_s}$ be given, and let

$$f^1(\xi) = f^0_1(\xi) - m(\xi) f^0_2(\xi), \quad \xi \in [0, L], \tag{4.29}$$

$$f^2(\hat{\xi}) = \hat{m}(\hat{\xi}) f^s_1(\hat{\xi}) + f^s_2(\hat{\xi}), \quad \hat{\xi} \in [0, \hat{L}]. \tag{4.30}$$

Then $f^1 \in L^2[0, L; m]$ and $f^2 \in L^2[0, \hat{L}; \hat{m}]$, and the relations

$$\|f^1\|_{L^2[0,L;m]} = \|f^0\|_{H_0}, \quad \|f^2\|_{L^2[0,\hat{L};\hat{m}]} = \|f^s\|_{H_s},$$

hold. Conversely, let $f^1 \in L^2[0, L; m]$ and $f^2 \in L^2[0, \hat{L}; \hat{m}]$ be given. Then there is a unique $f^0 \in L^2_{H_0}$ satisfying the relation (4.29) and a unique $f^s \in L^2_{H_s}$ satisfying (4.30). To see this, let

$$f^1(\xi) = f^0_1(\xi) - m(\xi) f^0_2(\xi) = \tilde{f}^0_1(\xi) - m(\xi) \tilde{f}^0_2(\xi).$$

It follows that $f^0, \tilde{f}^0 \in L^2_{H_0}$, and that $\|f^0 - \tilde{f}^0\|_{H_0} = 0$, hence f^0 and $\tilde{f}^0$ are identical in $L^2_{H_0}$. In the same way the uniqueness of f^s in $L^2_{H_s}$ can be shown. In particular, f^0 may be chosen to be equal to $(f^1, 0)^T$ and f^s may be chosen to be equal to $(0, f^2)^T$, as f^1 is constant on the H_0-indivisible intervals and f^2 is constant on the H_s-indivisible intervals. Hence, the following mappings are isometric isomorphisms:

$$L^2[0, L; m] \mapsto L^2_{H_0}: \qquad f \to (f, 0)^T, \tag{4.31}$$

$$L^2[0, \hat{L}; \hat{m}] \mapsto L^2_{H_s}: \qquad f \to (0, f)^T. \tag{4.32}$$

We are going to show that the following spaces are isometrically isomorphic:

$$L^2_{\mathfrak{m}} \cong L^2_{H_{d,2}} \cong L^2[0, \hat{L}; \hat{m}] \cong L^2_{H_s}, \tag{4.33}$$

$$L^2_{\hat{\mathfrak{m}}} \cong L^2_{H_{d,1}} \cong L^2[0, L; m] \cong L^2_{H_0}. \tag{4.34}$$

Let $f \in L^2_{H_d}$, and

$$g(\xi) = f_2(r(\xi)), \quad \xi \in [0, L], \tag{4.35}$$

$$h(\hat{\xi}) = f_1(s(\hat{\xi})), \quad \hat{\xi} \in [0, \hat{L}]. \tag{4.36}$$

Then $g \in L^2_{\mathfrak{m}}$ and $h \in L^2_{\hat{\mathfrak{m}}}$. Conversely, let $g \in L^2_{\mathfrak{m}}$ and $h \in L^2_{\hat{\mathfrak{m}}}$ be given. Put $g = 0$ on $[0, L] \setminus \operatorname{supp} \mathfrak{m}$ and $h = 0$ on $[0, \hat{L}] \setminus \operatorname{supp} \hat{\mathfrak{m}}$, and define functions f_2 and f_1 via the relations (4.35) and (4.36), and the conditions that if $r(\xi)$ is the endpoint of an maximal H_d-indivisible interval $I_{\pi/2}$ then $f_2 = g(\xi)$ on the closure of $I_{\pi/2}$, and if $s(\hat{\xi})$ is the endpoint of an maximal H_d-indivisible interval I_0 then $f_1 = h(\xi)$ on the closure of I_0. These conditions coincide with the general assumption that f_1 and f_2 are constant on the closure of any maximal H_d-indivisible interval, hence $f = (f_1, f_2) \in L^2_{H_d}$.

Moreover, the relations (4.35) and (4.36) establish an isometric isomorphism between $L^2_{H_{d,1}}$ and $L^2_{\hat{\mathfrak{m}}}$, and $L^2_{H_{d,2}}$ and $L^2_{\mathfrak{m}}$. To see this, note that (4.26) implies that $\mathfrak{m}(\{\xi\}) = r(\xi+) - r(\xi)$ and that $d\mathfrak{m}(\xi)$ is absolutely continuous with respect to the measure $dr(\xi)$. Hence $\frac{d\mathfrak{m}(\xi)}{dr(\xi)} = h_2(r(\xi))$ a. e. on $[0,L]$, and with $r(L) = L+m(L)$ it follows that

$$\int_0^L |g(\xi)|^2 d\mathfrak{m}(\xi) = \int_0^L |f_2(r(\xi))|^2 d\mathfrak{m}(\xi) = \int_0^\infty |f_2(x)|^2 h_2(x) dx.$$

The relations (4.7) and (4.25) imply the isometry between $L^2_{H_{d,1}}$ and $L^2_{\hat{\mathfrak{m}}}$ in a similar way.

5 Fourier transformations

Let $f \in L^2_{H_d}$ have compact support. As $f = (f_1, 0)^T \oplus (0, f_2)^T$, corresponding to (2.33), (4.11) and (4.23) the Fourier transformation F_{H_d} can be written as

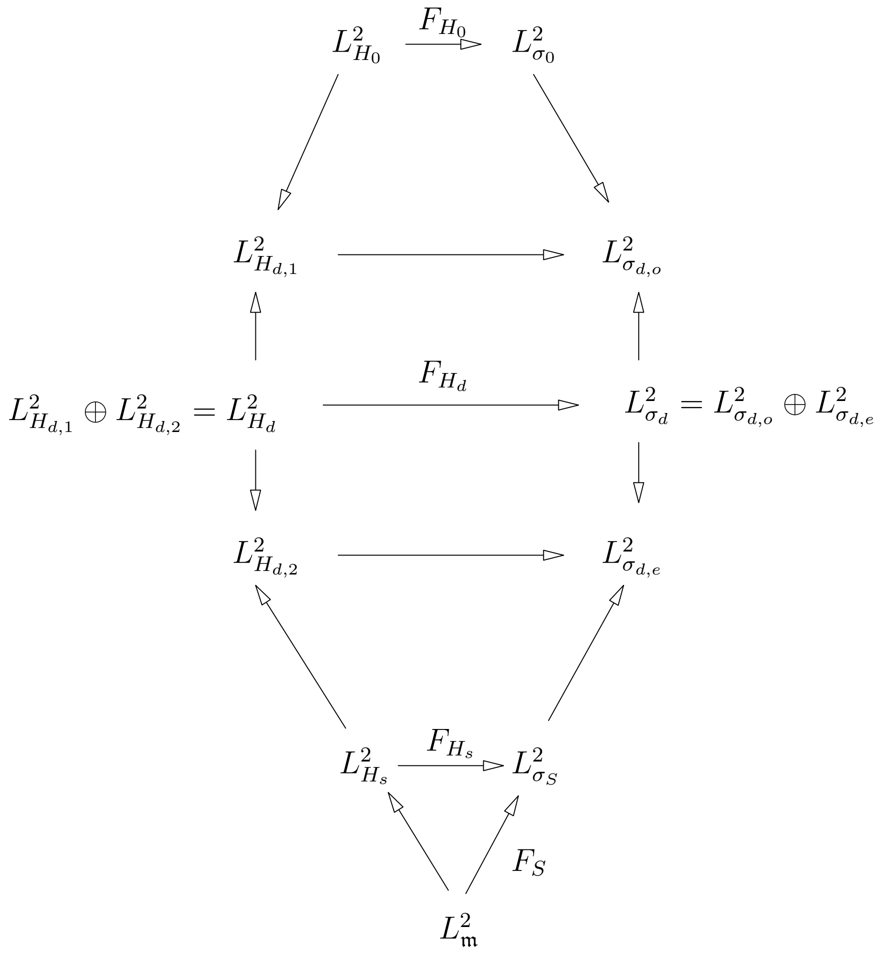
$$F_{H_d}(f,z) = F_{H_{d,1}}(f_1,z) + F_{H_{d,2}}(f_2,z), \tag{5.1}$$

with

$$F_{H_{d,1}}(f_1,z) := \int_0^\infty w_{21}(x,z) f_1(x) h_1(x) dx, \quad F_{H_{d,2}}(f_2,z) := \int_0^\infty w_{22}(x,z) f_2(x) h_2(x) dx. \tag{5.2}$$

The Fourier transformations of a string and its associated canonical systems are related as follows.

Theorem 5.1 *Let $S[L,\mathfrak{m}]$ be some string with related Hamiltonians H_d, H_0, and H_s. The interaction between the corresponding Fourier transformations F_S, F_{H_d}, F_{H_0}, and F_{H_s} is presented in the following commutative diagram of isometric isomorphisms.*



The following two lemmas form the core of the proof of Theorem 5.1.

Lemma 5.2 *The mappings*

$$F_{H_{d,1}}: \quad L^2_{H_{d,1}} \mapsto L^2_{\sigma,o}, \tag{5.3}$$

$$F_{H_{d,2}}: \quad L^2_{H_{d,1}} \mapsto L^2_{\sigma,e}. \tag{5.4}$$

establish isometric isomorphisms.

Proof. With the functions $f^1 \in L^2[0,L;m]$ defined in (4.27) and $g \in L^2_{\mathfrak{m}}$ defined in (4.35) the Fourier transformations $F_{H_{d,1}}$ and $F_{H_{d,2}}$ can be written with the help of (4.11) as

$$F_{H_{d,1}}(f_1,z) = \int_0^L z^{-1}\varphi'(\xi,z^2)f^1(\xi)d\xi, \tag{5.5}$$

$$F_{H_{d,2}}(f_2,z) = \int_0^L \varphi(\xi,z^2)g(\xi)d\mathfrak{m}(\xi). \tag{5.6}$$

It follows that $F_{H_{d,1}}(f_1,\cdot)$ is an odd function and that $F_{H_{d,2}}(f_2,\cdot)$ is an even function. Conversely, as $w_{21}(x,\cdot)$ is odd and $w_{22}(x,\cdot)$ is even, and as the measure σ is symmetric, the relation (2.34) implies that the origins of $L^2_{\sigma,e}$ in L^2_H have representatives with $f_1 = 0$, and the origins of $L^2_{\sigma,o}$ in L^2_H have representatives with $f_2 = 0$. □

Lemma 5.3 *Let $f \in L^2_{H_d}$ have compact support, and let f^0, f^s and g satisfy the relations* (4.29), (4.30) *and* (4.35). *Then*

$$F_S(g,z) = F_{H_s}(f^s,z), \tag{5.7}$$

$$F_{H_d}(f,z) = F_{H_{d,1}}(f_1,z) + F_{H_{d,2}}(f_2,z), \tag{5.8}$$

$$F_{H_{d,2}}(f_2,z) = F_{H_s}(f^s,z^2), \tag{5.9}$$

$$F_{H_{d,1}}(f_1,z) = zF_{H_0}(f^0,z^2). \tag{5.10}$$

Proof. In (5.6) we already saw that $F_{H_{d,2}}(f_2,z) = F_S(g,z^2)$. Let f^0 and f^s be given by (4.29) and (4.30). Corresponding to the relations (2.33) and (4.10), the Fourier transformation F_{H_0} in $L^2_{H_0}$ reads as

$$F_{H_0}(f^0,z) = \int_0^L z^{-1}\varphi'(\xi-,z)(f^0_1(\xi) - m(\xi)f^0_2(\xi))d\xi, \tag{5.11}$$

and the relations (4.29) and (5.5) imply that $F_{H_{d,1}}(f_1,z) = zF_{H_0}(f^0,z^2)$. Corresponding to the relations (2.33) and (4.12), the Fourier transformation F_{H_s} in $L^2_{H_s}$ reads as

$$F_{H_s}(f^s,z) = \int_0^{\hat{L}} \widehat{\psi}'(\hat{\xi}-,z)(\hat{m}(\xi)f^s_1(\hat{\xi}) + f^s_2(\hat{\xi}))d\hat{\xi}. \tag{5.12}$$

Let $\xi \in \operatorname{supp} m$, and put $\hat{\xi} = m(\xi+)$. Then $\widehat{\psi}'(m(\xi+)-,z) = \widehat{\psi}'(m(\xi),z) = \varphi(\xi,z)$ by the relation (3.13), and hence

$$\varphi(\xi,z) = \widehat{\psi}'(\hat{\xi}-,z), \quad \xi \in \operatorname{supp}\mathfrak{m}.$$

Moreover, the relations (4.30), (4.28) and (4.35), and the assumption that the elements of $L^2_{H_d}$ are constant on the closed maximal indivisible intervals imply that

$$\hat{m}(\xi)f^s_1(\hat{\xi}) + f^s_2(\hat{\xi}) = f^2(\hat{\xi}) = f_2(s(m(\xi+))) = f_2(r(\xi+)) = f_2(r(\xi)) = g(\xi).$$

It follows from $d\hat{\xi} = d\mathfrak{m}(\xi)$ and the relation (5.12) that

$$F_{H_s}(f^s, z) = \int_0^L \varphi(\xi, z) g(\xi) d\mathfrak{m}(\xi), \tag{5.13}$$

and the relation (5.6) implies that $F_{H_{d,2}}(f_2, z) = F_{H_s}(f^s, z^2)$. □

All the Fourier transformations we have introduced for functions with compact support are isometries and have dense ranges. Hence they can be extended by continuity to isometric isomorphisms defined on the whole space. All assertions of Theorem 5.1 are proved.

In order to find an isometric isomorphism connecting the Fourier transformation $F_{\hat{S}}$ of the dual string $\hat{S}[\hat{L}, \hat{\mathfrak{m}}]$ with $L^2_{H_{d,1}}$ we have to consider the dual Hamiltonian $\widehat{H}_d$. As $H_d = \operatorname{diag}(h_1, h_2)$, the relation $\widehat{H}_d = \operatorname{diag}(h_2, h_1)$ follows. Let $\hat{\sigma}$ be the spectral measure corresponding to the Titchmarsh-Weyl coefficient $\widehat{Q}(z) = -Q(z)^{-1}$ of $\widehat{H}_d$. If $F_{\widehat{H}_d}$ denotes the Fourier transformation which maps $L^2_{\widehat{H}_d}$ onto $L^2_{\hat{\sigma}}$, the isometry between $L^2_{H_d}$ and $L^2_{\widehat{H}_d}$ establishes an associated Fourier transformation $\widehat{F}_{H_d}$ mapping $L^2_{H_d}$ onto $L^2_{\hat{\sigma}}$ by

$$\widehat{F}_{H_d}(f, \cdot) := F_{\widehat{H}_d}(\hat{f}, \cdot), \qquad f \in L^2_{H_d}. \tag{5.14}$$

Assume that $f \in L^2_{H_d}$ has compact support. Then

$$\widehat{F}_{H_d}(f, z) = \int_0^\infty (w_{11}(x, z) f_1(x) h_1(x) + w_{12}(x, z) f_2(x) h_2(x)) dx.$$

Defining

$$\widehat{F}_{H_{d,1}}(f_1, z) = \int_0^\infty w_{11}(x, z) f_1(x) h_1(x) dx, \tag{5.15}$$

$$\widehat{F}_{H_{d,2}}(f_2, z) = \int_0^\infty w_{12}(x, z) f_2(x) h_2(x) dx, \tag{5.16}$$

it follows as above that the mappings $\widehat{F}_{H_{d,1}} : L^2_{H_{d,1}} \mapsto L^2_{\hat{\sigma},e}$, and $\widehat{F}_{H_{d,2}} : L^2_{H_{d,1}} \mapsto L^2_{\hat{\sigma},o}$, establish isometric isomorphisms. Let $\hat{\xi} \in \operatorname{supp} \hat{\mathfrak{m}}$ and assume that $\xi = \hat{m}(\hat{\xi}+)$. Then

$$\psi'(\xi-, z^2) = \psi'(\hat{m}(\hat{\xi}+)-, z^2) = \psi'(\hat{m}(\hat{\xi}), z^2) = \widehat{\varphi}(\hat{\xi}, z^2),$$

and if h and f_1 are related via (4.36), the last relation in combination with (4.11) and (4.7) implies that

$$\widehat{F}_{H_{d,1}}(f_1, z) = \int_0^{\hat{L}} \widehat{\varphi}(\hat{\xi}, z^2) h(\hat{\xi}) d\hat{\mathfrak{m}}(\hat{\xi}) = F_{\hat{S}}(h, z^2). \tag{5.17}$$

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