Shifted Hermite-Biehler functions and their applications

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Abstract. We investigate a particular subclass of so-called symmetric indefinite Hermite-Biehler functions and give a characterization of functions of this class in terms of the location of their zeros. For the proof we employ the theory of de Branges Pontryagin spaces of entire functions. We apply our results to obtain information on the eigenvalues of some boundary value problems.

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1. Introduction

The Hermite-Biehler class is the set of all entire functions E which have no zeros in the open upper half-plane $\mathbb{C}^+$ and satisfy

$$|E(\overline{z})| \leq |E(z)|,\ z \in \mathbb{C}^+ , \tag{1.1}$$

An indefinite generalization of this notion is obtained when these conditions are substituted by the conditions that $E(z)$ and $E^\#(z) := \overline{E(\overline{z})}$ have no common nonreal zeros and that the kernel

$$S(w,z) := \frac{i}{z-\overline{w}}\Big[1 - \frac{E^\#(z)}{E(z)}\overline{\Big(\frac{E^\#(w)}{E(w)}\Big)}\Big] \tag{1.2}$$

has a finite number of negative squares. The fact that positive definiteness of the kernel (1.2) coincides with the condition (1.1) is thereby a classical result, cf. [Pi].

Functions of the Hermite-Biehler class appear in several contexts of complex– and functional analysis, see for example [dB], [B] or [L1], and are a classical object of analysis. The origin of this notion goes back to the investigation of polynomials

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and their zeros, cf. [H]. The various definitions found in the literature often differ in some unessential details; for a comparison see Remark 2.2.

In the present note we deal with two subclasses of indefinite Hermite-Biehler functions. The first one is defined by the requirement that the function E satisfies the functional equation

$$E(-z) = E^{\#}(z),\ z \in \mathbb{C}\,, \tag{1.3}$$

and we speak of symmetric indefinite Hermite-Biehler function, cf. Definition 2.7. The other one, the class of semibounded indefinite Hermite-Biehler functions, is defined by the requirement that the function $A(z) := \frac{1}{2}(E(z) + E^{\#}(z))$ has only finitely many zeros off the positive real axis, cf. Definition 2.9. These two classes are related via the transformation

$$\mathfrak{T} : E(z) \ \longmapsto \ A(z^2) - izB(z^2)\,, \tag{1.4}$$

where $A(z) := \frac{1}{2}(E(z) + E^{\#}(z))$ and $B(z) := \frac{i}{2}(E(z) - E^{\#}(z))$, cf. Proposition 2.10.

In the main result of this paper we characterize, in terms of the location of their zeros, those symmetric indefinite Hermite-Biehler functions which are $\mathfrak{T}$-transforms of positive definite semibounded Hermite-Biehler functions. It turns out that all zeros in the upper half-plane must be simple, lie on the imaginary axis, and that their location restricts the freedom of zeros on the negative imaginary axis, cf. Theorem 3.1. Our method of proof relies heavily on the theory of symmetric and semibounded de Branges spaces as developed in [KWW3]. For the particular case of polynomials (note that $\mathfrak{T}$ maps the set of all polynomials onto the set of all polynomials satisfying (1.3)) an analogous characterization was obtained by different methods in [P3].

Our motivation to investigate functions of this particular kind, namely $\mathfrak{T}$-transforms of positive definite semibounded Hermite-Biehler functions, and to study the location of their zeros stems from two sources. First, the classes of semibounded and symmetric indefinite Hermite-Biehler functions readily appeared in several contexts, where also the connection (1.4) between them played a prominent role. For example in the theory of de Branges spaces of entire functions, cf. [dB, Theorems 47,54], [KW2], [KWW3], and in the study of strings and their indefinite generalizations, cf. [KK], [LW], [KWW2]. It turned out in [KWW2] that the $\mathfrak{T}$-transforms of semibounded positive definite Hermite-Biehler functions correspond to what is called a generalized string in [LW]. From the viewpoint of complex analysis it is natural to ask for product representations and distribution as well as location of zeros. Secondly, $\mathfrak{T}$-transforms of positive definite semibounded Hermite-Biehler functions appear in the study of various boundary value problems and there describe the eigenvalues of the problem. Hence, our results can be employed to describe the location of the eigenvalues of such problems, in particular one obtains information on the eigenvalues lying on the imaginary axis. Knowledge on their location has turned out to be of importance for solving the corresponding inverse problems. In concrete cases results of this type where obtained separately, e.g. in [MP1], [MP2], [Si], [PM] or [MoP]. Let us point out that our aim in the study of the presently treated boundary value problems was not to obtain new knowledge on their resonances, but to show that the presented general results provide a structural and unified approach to the study of such questions.

Let us summarize the contents of the present note. After this introduction, in Section 2, we set up some notation and provide some preliminary results on the mentioned classes of Hermite-Biehler functions. We deal with their product representations and the structure of the generated de Branges spaces of entire functions. Section 3 is devoted to the formulation and proof of our main result, namely Theorem 3.1. Finally, in Section 4, we apply Theorem 3.1 in four concrete cases; the following boundary value problems are investigated:

I.The Regge problem:

$$-y'' + q(x)y = \lambda^2 y\,,$$
$$y(0) = 0\,,$$
$$y'(a) - i\lambda y(a) = 0\,.$$

Here λ is the spectral parameter and the potential q is real-valued and belongs to $L_2(0,a)$.

This problem occurs in the theory of scattering when the potential is supposed to have finite support. It is certainly the most popular among the boundary value problems we deal with and was well-studied by a variety of authors, see e.g. [R1], [R2], [Kr], [Ko], [S], [Hr], [IP], [Si], [KaKo].

II. The generalized Regge problem, cf. [GP], [PM]:

$$-y'' + q(x)y = \lambda^2 y\,,$$
$$y(0) = 0\,,$$
$$y'(a) - i\alpha\lambda y(a) + \beta y(a) = 0\,.$$

with $\alpha > 0$ and $\beta \in \mathbb{R}$.

III. Vibrations of a damped string. The problem of small transversal vibrations of a damped smooth inhomogeneous string with fixed left endpoint whose right end carries a point mass able to move with damping in the direction orthogonal to the equilibrium position of the string can be reduced to the following spectral problem, cf. [P2], [MP1], [MP2]:

$$-y'' - ip\lambda y + q(x)y = \lambda^2 y\,,$$
$$y(0) = 0\,,$$
$$y'(a) + (\beta - i\alpha\lambda - m\lambda^2)y(a) = 0\,.$$

Thereby the coefficient $p > 0$ is proportional to the damping along the length of the string, $\alpha > 0$ is proportional to the coefficient of damping of the point mass $m > 0$ at the right end, and β is a real parameter.

IV. A fourth order problem which describes small transversal vibrations of an elastic beam, cf. [MoP]:

$$\begin{aligned} y^{(4)} - (g(x)y')' &= \lambda^2 y\,, \\ y(0) = y''(0) &= 0\,, \\ y(a) &= 0\,, \\ y''(a) - i\alpha\lambda y'(a) &= 0\,. \end{aligned}$$

Hereby $g(x)$ is a continuously differentiable function describing the distributed stretching or compressing force. The left end of the beam is hinge connected and the right end is hinge connected with damping.

2. Some preliminaries on indefinite Hermite-Biehler functions

If $\Omega\subseteq\mathbb{C}$ is a domain and $K(w,z)$ is a function defined on $\Omega\times\Omega$, which is analytic in the variables z and $\overline{w}$ and has the property that $K(w,z)=\overline{K(z,w)}$, then K is called an *analytic symmetric kernel* (shortly *kernel*) on Ω. Let $\kappa\in\mathbb{N}\cup\{0\}$. We say that the kernel K has κ *negative squares*, if for each choice of $n\in\mathbb{N}$ and $z_1,\dots,z_n\in\Omega$ the quadratic form

$$Q_K(\xi_1,\dots,\xi_n):=\sum_{i,j=1}^n K(z_j,z_i)\xi_i\overline{\xi_j}$$

has at most κ negative squares, and if for some choice of $n,z_1,\dots,z_n$ this upper bound is actually attained.

Recall that every kernel K with a finite number κ of negative squares on a domain Ω generates a reproducing kernel Pontryagin space $\mathfrak{P}(K)$ whose elements are analytic function on Ω, cf. [ADRS]. In fact, $\mathfrak{P}(K)$ is obtained as the Pontryagin space completion of $\operatorname{span}\{K(w,.):\,w\in\Omega\}$ with respect to the inner product given by $[K(w,.),K(w',.)]=K(w,w')$.

In the present note kernels of a particular form play a crucial role. If Θ is a meromorphic function on the open upper half-plane $\mathbb{C}^+$ and Ω denotes its domain of holomorphy, define

$$S_\Theta(w,z):=i\frac{1-\Theta(z)\overline{\Theta(w)}}{z-\overline{w}},\ \ z,w\in\Omega\,.$$

Clearly, S_Θ is a kernel on Ω in the above sense.

For a function F, we denote by $F^\#$ the function $F^\#(z):=\overline{F(\overline{z})}$. We call F *real*, if $F=F^\#$. Let $\kappa\in\mathbb{N}\cup\{0\}$. If E is meromorphic on the whole plane $\mathbb{C}$, we write $\operatorname{ind}_- E=\kappa$ in order to express the fact that the kernel $S_{\frac{E^\#}{E}|_{\mathbb{C}^+}}$ has κ negative squares.

2.1. Definition. Let $\kappa\in\mathbb{N}\cup\{0\}$. The set $\mathcal{HB}_\kappa$ of *Hermite-Biehler functions with κ negative squares* is defined to be the set of all entire functions E which satisfy

$\operatorname{ind}_- E = \kappa$ and are such that E and $E^\#$ have no common nonreal zeros. Moreover, we define the set of *indefinite Hermite-Biehler functions* as

$$\mathcal{HB}_{<\infty} := \bigcup_{\kappa\in\mathbb{N}\cup\{0\}} \mathcal{HB}_\kappa \,.$$

2.2. *Remark.* The notion of Hermite-Biehler functions appeared frequently in the literature. However, the existing definitions do not completely coincide with each other. We would like to make this more explicit, and to state the relationship with the present Definition 2.1.

(i) In the book [L1], one finds the following definitions: An entire function E belongs to HB, if it has no zeros in the closed lower half-plane $\mathbb{C}^-\cup\mathbb{R}$ and if

$$|E(z)| < |E^\#(z)|,\ \operatorname{Im} z > 0\,.$$

Moreover, E is said to belong to $\overline{\text{HB}}$, if E has no zeros in the open lower half-plane $\mathbb{C}^-$ and $|E(z)|\leq|E^\#(z)|$, $\operatorname{Im} z>0$.

(ii) In [B] (originally in [L2]) a class P is defined as the set of all entire functions E of exponential type which have no zeros in $\mathbb{C}^-$ and satisfy $|E(z)|\leq|E^\#(z)|$, $\operatorname{Im} z>0$.

(iii) In the book [dB] the class of all entire functions is considered which satisfy $|E^\#(z)|<|E(z)|$, $\operatorname{Im} z>0$.

(iv) In the series [KWW1]-[KWW3] as well as in [KW1], [KW2], classes of indefinite Hermite-Biehler functions are defined by requiring the conditions of Definition 2.1 and, additionally, that $\frac{E^\#}{E}$ is not constant.

The relationships among these various, in their essence equivalent but in their details different, notions can now be formulated as follows.

- An entire function E belongs to P as in (ii) if and only if it belongs to $\overline{\text{HB}}$, as in (i) and is of exponential type.
- An entire function belongs to HB as in (i) if and only if it has no real zeros and the function $E^\#(z)$ possesses the property stated in (iii). Thus (iii) in comparison to (i) allows real zeros and exchanges the roles of upper and lower half-plane.
- We have $E\in\mathcal{HB}_0$ as in Definition 2.1 if and only if $E^\#\in\overline{\text{HB}}$. This follows from a classical result of G.Pick, cf. [Pi].
- We have $\frac{E^\#}{E}=\text{const}$ if and only if there exists a constant $\lambda\in\mathbb{C}$ with $|\lambda|=1$ such that λE is real. Hence the definition mentioned in (iv) differs from Definition 2.1 only by, in our context, somewhat trivial functions.

a. Zeros, product representation and limits

In order to study the distribution of zeros of entire functions it is practical to use the language of divisors, see e.g. [R]: Put

$$\mathcal{D} := \big\{\mathfrak{d}:\mathbb{C}\to\mathbb{Z} : \operatorname{supp}\mathfrak{d} \text{ has no accumulation point in } \mathbb{C}\big\}\,,$$

where $\operatorname{supp}\mathfrak{d}$ denotes the set of all points w where $\mathfrak{d}$ assumes a nonzero value. To a function F which is meromorphic in the whole plane, there is associated a divisor $\mathfrak{d}(F)$ which assigns to each point w the multiplicity of w as a zero of F. For example, for the function $F(z) := z + \frac{1}{z}$ we have

$$\mathfrak{d}(F)(w) = \begin{cases} +1 & , \ w = \pm i \\ -1 & , \ w = 0 \\ 0 & , \ \text{otherwise} \end{cases} .$$

Clearly, a meromorphic function F is entire if and only $\mathfrak{d}(F) \geq 0$.

In the particular instance of Hermite-Biehler functions, the following subset of $\mathcal{D}$ plays a distinguished role:

$$\mathcal{D}_{\mathcal{HB}} := \Big\{ \mathfrak{d} \in \mathcal{D}, \mathfrak{d} \geq 0 : \ \#(\operatorname{supp} \mathfrak{d} \cap \mathbb{C}^+) < \infty, \sum_{w \neq 0} \mathfrak{d}(w) \operatorname{Im} \frac{1}{w} < \infty \Big\} .$$

2.3. *Remark.*

(*i*) The following is an immediate corollary of [KL, Satz 6.4] and basic H^∞-theory, see e.g. [RR]: Let $E \in \mathcal{HB}_{<\infty}$. Then

$$\frac{E^\#(z)}{E(z)} = \gamma e^{-iaz} B(z) \cdot B_1(z) ,$$

where B is the Blaschke product associated with the zeros of E in the open lower half-plane $\mathbb{C}^-$, B_1 is the (finite) Blaschke product associated with the zeros of E in $\mathbb{C}^+$, the real number a is the mean type

$$a = \operatorname{mt} \frac{E^\#}{E} := \limsup_{y \to +\infty} \frac{1}{y} \log \Big| \frac{E^\#}{E} \Big| ,$$

and $\gamma = e^{-2i \arg E(0)}$. Moreover, $\mathfrak{d}(E) \in \mathcal{D}_{\mathcal{HB}}$.

(*ii*) We have $E \in \mathcal{HB}_{<\infty}$ if and only if E can be factorized as $E = p \cdot E_0$ with a function $E_0 \in \mathcal{HB}_0$ and a polynomial p whose zeros lie in $\mathbb{C}^+$ and are different from the conjugates of the zeros of E_0. Thereby $\operatorname{ind}_- E$ is equal to the number of zeros of p counted according to their multiplicities, i.e.

$$\operatorname{ind}_- E = \sum_{w \in \mathbb{C}} \mathfrak{d}(p)(w) = \sum_{w \in \mathbb{C}^+} \mathfrak{d}(E)(w) .$$

(*iii*) If $\operatorname{ind}_- E_1 < \infty$ and $\operatorname{ind}_- E_2 < \infty$, then also $\operatorname{ind}_-(E_1 E_2) < \infty$. In fact, the inequality

$$\operatorname{ind}_-(E_1 E_2) \leq \operatorname{ind}_- E_1 + \operatorname{ind}_- E_2 \tag{2.1}$$

holds. Moreover, if $E_1, E_2, E_1 E_2 \in \mathcal{HB}_{<\infty}$, then in (2.1) actually equality prevails.

(*iv*) Let E be an entire function, let D be meromorphic in the whole plane and assume that $D = D^\#$. Then $\frac{(DE)^\#}{DE} = \frac{E^\#}{E}$, and hence $\operatorname{ind}_- DE = \operatorname{ind}_- E$. If we assume moreover that $\operatorname{supp} \mathfrak{d}(D) \subseteq \mathbb{R}$ and $\mathfrak{d}(D)(w) + \mathfrak{d}(E)(w) \geq 0$, $w \in \mathbb{C}$, then $E \in \mathcal{HB}_\kappa$ if and only if $DE \in \mathcal{HB}_\kappa$.

A particular instance of this situation is frequently of good use: Let P be a real Weierstraß product to the real zeros of E and choose $D = P^{-1}$ above.

This shows that in the investigation of indefinite Hermite-Biehler functions one can often restrict without loss of generality that no real zeros are present.

From these facts we see that Krein's factorization theorem, cf. [K] or [L1, Lehrsatz VII.3.6], immediately extends to the indefinite case. Its indefinite version can be formulated as follows.

Krein's Factorization Theorem: *If $E \in \mathcal{HB}_{<\infty}$, then $\mathfrak{d}(E) \in \mathcal{D}_{\mathcal{HB}}$ and*

$$\mathfrak{d}(E)(w) > 0 \Rightarrow \mathfrak{d}(E)(\overline{w}) = 0,\ w \in \mathbb{C} \setminus \mathbb{R}. \tag{2.2}$$

The function E admits a locally uniformly convergent product representation of the form

$$E(z) = \gamma D(z) e^{-iaz} \prod_{w \notin \mathbb{R}} \big(1 - \frac{z}{w}\big)^{\mathfrak{d}(E)(w)} \exp\Big(\mathfrak{d}(E)(w) \sum_{n=1}^{p(w)} \frac{z^n}{n} \operatorname{Re} \frac{1}{w^n}\Big), \tag{2.3}$$

where D is real, $\operatorname{supp}\mathfrak{d}(D) \subseteq \mathbb{R}$, *$|\gamma| = 1$, $a \geq 0$ and $p : \operatorname{supp}\mathfrak{d}(E) \setminus \mathbb{R} \to \mathbb{N} \cup \{0\}$ satisfies*

$$\sum_{w \notin \mathbb{R}} \frac{\mathfrak{d}(E)(w)}{|w|^{p(w)+1}} < \infty .$$

Conversely, every function of this form belongs to $\mathcal{HB}_{<\infty}$.

2.4. *Remark.* The above discussion shows that from the viewpoint of the asymptotic distribution of zeros, and hence also from the function theoretic viewpoint of growth etc., the classes of indefinite and positive definite Hermite-Biehler functions behave the same. However, in questions concerning the exact location of zeros, in the indefinite case essentially new questions and phenomena appear.

The set $\mathcal{HB}_0$ is closed with respect to locally uniform convergence. This follows e.g. from the fact that $\mathcal{HB}_0 = \{E^\# : E \in \overline{\mathrm{HB}}\}$, cf. Remark 2.2, and that, clearly, $\overline{\mathrm{HB}}$ is closed. However, note that none of the sets $\mathcal{HB}_\kappa$ for $\kappa > 0$, $\bigcup_{\kappa \leq N} \mathcal{HB}_\kappa$ or $\mathcal{HB}_{<\infty}$ is closed with respect to locally uniform convergence. In this context we would like to mention the following result:

2.5. Proposition. *Let E be an entire function.*

(i) *We have* $\operatorname{ind}_- E = \kappa$ *if and only if there exists a real entire function D such that $D^{-1}E \in \mathcal{HB}_\kappa$.*

(ii) *Let $\overline{\mathcal{HB}_\kappa}$ denote the closure of $\mathcal{HB}_\kappa$ with respect to locally uniform convergence. Assume that E does not vanish identically. Then $E \in \bigcup_{\kappa \in \mathbb{N}\cup\{0\}} \overline{\mathcal{HB}_\kappa}$ if and only if*

$$\operatorname{ind}_- E < \infty,\ \#(\operatorname{supp}\mathfrak{d}(E) \cap \mathbb{C}^+) < \infty .$$

Proof. The necessity in (i) follows since for real functions D we have $\operatorname{ind}_- DE = \operatorname{ind}_- E$. To see the sufficiency assume that $\operatorname{ind}_- E = \kappa < \infty$. Let $S := \{w \in \mathbb{C} \setminus \mathbb{R} :$

$E(w) = E(\overline{w}) = 0\}$ and let D be a real entire function with

$$\mathfrak{d}(D)(w) = \begin{cases} \min\{\mathfrak{d}(E)(w), \mathfrak{d}(E)(\overline{w})\} & , \ w \in S \\ 0 & , \ \text{otherwise} \end{cases}.$$

Put $E_1 := \frac{E}{D}$. Then E_1 is entire, $\operatorname{ind}_- E_1 = \operatorname{ind}_- E$ and E_1 and $E_1^\#$ have no common nonreal zeros. Thus $E_1 \in \mathcal{H}B_\kappa$.

We come to the proof of (ii). For sufficiency assume that E is the locally uniform limit of a sequence $(E_n)_{n\in\mathbb{N}}$ of functions $E_n \in \mathcal{H}B_\kappa$. Then, by the continuity of the kernel $S_{\frac{E^\#}{E}}$,

$$\operatorname{ind}_- E \leq \liminf_{n\to\infty} \operatorname{ind}_- E_n = \kappa\,,$$

and, by the Theorem of Logarithmic Residues,

$$\#(\operatorname{supp}\mathfrak{d}(E) \cap \mathbb{C}^+) \leq \liminf_{n\to\infty} \#(\operatorname{supp}\mathfrak{d}(E_n) \cap \mathbb{C}^+) = \kappa\,.$$

To establish necessity in (ii), assume that $\operatorname{ind}_- E < \infty$ and $\#(\operatorname{supp}\mathfrak{d}(E) \cap \mathbb{C}^+) < \infty$. Let p be a polynomial with

$$\mathfrak{d}(p)(w) = \begin{cases} \mathfrak{d}(E)(w) & , \ w \in \mathbb{C}^+ \\ 0 & , \ \text{otherwise} \end{cases},$$

and put $E_1 := \frac{E}{p}$. Then $\operatorname{ind}_- E_1 \leq \operatorname{ind}_- E + \deg p < \infty$. Since, E_1 and $E_1^\#$ have no common nonreal zeros, by [KL, Satz 6.4] the number $\operatorname{ind}_- E_1$ coincides with the number of zeros of E_1 in $\mathbb{C}^+$. We see that $\operatorname{ind}_- E_1 = 0$, and hence $E_1 \in \mathcal{H}B_0$. Let $S' := \{w \in \mathbb{C}^- : E_1(w) = p(\overline{w}) = 0\}$. Then, for $a > 0$,

$$E_1(z) \prod_{w\in S'} \frac{z - w + ia}{z - w} \in \mathcal{H}B_0\,.$$

Since this function has for sufficiently small values of $a > 0$ no common zeros with $p^\#$, we conclude that

$$E^a(z) := p(z)E_1(z) \prod_{w\in S'} \frac{z - w + ia}{z - w} \in \mathcal{H}B_{\deg p}\,.$$

Clearly, $\lim_{a\searrow 0} E^a = E$. □

b. The de Branges space associated to $E \in \mathcal{H}B_{<\infty}$

A function $E \in \mathcal{H}B_{<\infty}$ generates a reproducing kernel Pontryagin space whose elements are entire functions. To this end consider the kernel

$$\begin{aligned} K(w,z) &:= \frac{i}{2\pi} \frac{E(z)\overline{E(w)} - E^\#(z)\overline{E^\#(w)}}{z - \overline{w}}, \ z, w \in \mathbb{C}, z \neq \overline{w}\,, \\ K(\overline{z}, z) &:= \frac{i}{2\pi}\Big(\frac{\partial E}{\partial z}(z)E^\#(z) - E(z)\frac{\partial E^\#}{\partial z}(z)\Big), \ z \in \mathbb{C}\,. \end{aligned} \tag{2.4}$$

The reproducing kernel Pontryagin space generated by this kernel K will be called the *de Branges space* induced by E and denoted by $\mathfrak{P}(E)$, cf. [KW1].

Let $E \in \mathcal{HB}_{<\infty}$. A maximal negative subspace of $\mathfrak{P}(E)$ can be determined explicitly. We do not show this in full generality; we restrict ourselves to the case of simple zeros, since this is enough for what is needed later on.

2.6. Lemma. *Let $E \in \mathcal{HB}_\kappa$ and assume that all zeros of E which lie in the open upper half-plane are simple. Denote these zeros by $w_1, \ldots, w_\kappa$. Then*

$$\mathcal{L} := \operatorname{span}\big\{K(w_k,.) : \, k = 1, \ldots, \kappa\big\}$$

is a maximal negative subspace of $\mathfrak{P}(E)$.

Proof. Since each w_k is a zero of E, we have

$$K(w_k, z) = -\frac{i}{2\pi}\frac{E^\#(z)E(\overline{w_k})}{z-\overline{w_k}}, \ k = 1, \ldots, \kappa .$$

It follows that for $x, y \in \mathcal{L}$, $x = \sum_{k=1}^\kappa \lambda_k K(w_k, .)$, $y = \sum_{j=1}^\kappa \mu_j K(w_j, .)$,

$$[x, y] = \sum_{k,j=1}^{\kappa} K(w_k, w_j)\lambda_k\overline{\mu_j} =$$

$$= -\frac{1}{2\pi}\sum_{k,j=1}^{\kappa} \frac{i}{w_j-\overline{w_k}}\big(E(\overline{w_k})\lambda_k\big)\overline{\big(E(\overline{w_j})\mu_j\big)} .$$

Since $E(\overline{w_k}) \neq 0$, $k = 1, \ldots, \kappa$, and the matrix

$$P := \Big(\frac{i}{w_j-\overline{w_k}}\Big)_{j,k=1}^{\kappa}$$

is positive definite (e.g. as the Pick-matrix of the constant function $f(z) := i$, $z \in \mathbb{C}^+$), it follows that for $(\lambda_1, \ldots, \lambda_\kappa) \neq 0$,

$$[x, x] = -\frac{1}{2\pi}\begin{pmatrix} E(\overline{w_1})\lambda_1 \\ \vdots \\ E(\overline{w_\kappa})\lambda_\kappa \end{pmatrix}^* \Big(\frac{i}{w_j-\overline{w_k}}\Big)_{j,k=1}^{\kappa} \begin{pmatrix} E(\overline{w_1})\lambda_1 \\ \vdots \\ E(\overline{w_\kappa})\lambda_\kappa \end{pmatrix} < 0 .$$

We conclude that $\{K(w_1, .), \ldots, K(w_\kappa, .)\}$ is linearly independent and that $\mathcal{L}$ is a negative subspace. Since $\operatorname{ind}_- E = \kappa$, it is a maximal negative subspace. □

c. Symmetric and semibounded Hermite-Biehler functions

We are interested in the study of indefinite Hermite-Biehler functions which satisfy a certain functional equation:

2.7. Definition. Let $\kappa \in \mathbb{N}\cup\{0\}$. We define the set of all *symmetric Hermite-Biehler functions with κ negative squares* as

$$\mathcal{HB}_\kappa^{sym} := \big\{E \in \mathcal{HB}_\kappa : \, E^\#(z) = E(-z)\big\} .$$

Again let us put

$$\mathcal{HB}_{<\infty}^{sym} := \bigcup_{\kappa\in\mathbb{N}\cup\{0\}} \mathcal{HB}_\kappa^{sym} = \big\{E \in \mathcal{HB}_{<\infty} : \, E^\#(z) = E(-z)\big\} .$$

A product representation of symmetric indefinite Hermite-Biehler functions can be deduced from Krein's factorization theorem.

Product Representation for $\mathcal{HB}^{sym}_{<\infty}$: *If $E \in \mathcal{HB}^{sym}_{<\infty}$, then $\mathfrak{d}(E) \in \mathcal{D}_{\mathcal{HB}}$, satisfies* (2.2) *and*

$$\mathfrak{d}(E)(-\overline{w}) = \mathfrak{d}(E)(w),\ w \in \mathbb{C}\,.$$

The function E admits a locally uniformly convergent product representation of the form

$$E(z) = \gamma D(z) e^{-iaz} \prod_{w \in i\mathbb{R}\setminus\{0\}} \big(1 - \frac{z}{w}\big)^{\mathfrak{d}(E)(w)} \cdot$$

$$\cdot \prod_{\substack{\operatorname{Re} w>0\\ w\notin\mathbb{R}}} \big(1 + 2iz\frac{\operatorname{Im} w}{|w^2|} - \frac{z^2}{|w|^2}\big)^{\mathfrak{d}(E)(w)} \exp\big(2\mathfrak{d}(E)(w) \sum_{n=1}^{[\frac{p(w)}{2}]} \frac{z^{2n}}{2n} \operatorname{Re}\frac{1}{w^{2n}}\big)\,,$$

where D is real and even, i.e. $D(z) = D^{\#}(z) = D(-z)$, $\operatorname{supp}\mathfrak{d}(D) \subseteq \mathbb{R}$*, $\gamma = \pm 1$, $a \geq 0$ and*

$$p : \{w \in \operatorname{supp}\mathfrak{d}(E) :\ \operatorname{Re} w > 0, w \notin \mathbb{R}\} \to \mathbb{N} \cup \{0\}$$

satisfies

$$\sum_{\substack{\operatorname{Re} w>0\\ w\notin\mathbb{R}}} \frac{\mathfrak{d}(E)(w)}{|w|^{p(w)+1}} < \infty\,.$$

Conversely, every function of this form belongs to $\mathcal{HB}^{sym}_{<\infty}$.

Let us recall from [KWW3] that the de Branges space generated by a function $E \in \mathcal{HB}^{sym}_{<\infty}$ has an important symmetry property. In fact, the map

$$M : F(z) \mapsto F(-z)$$

induces an isometric involution of $\mathfrak{P}(E)$ onto itself. Thus the space $\mathfrak{P}(E)$ can be decomposed as the direct and orthogonal sum

$$\mathfrak{P}(E) = \mathfrak{P}(E)_e [\dot{+}] \mathfrak{P}(E)_o\,,$$

where

$$\mathfrak{P}(E)_e = \ker(I - M) = \big\{F \in \mathfrak{P}(E) :\ F \text{ even}\big\}\,,$$
$$\mathfrak{P}(E)_o = \ker(I + M) = \big\{F \in \mathfrak{P}(E) :\ F \text{ odd}\big\}\,.$$

The orthogonal projections P_e and P_o of $\mathfrak{P}(E)$ onto $\mathfrak{P}(E)_e$ or $\mathfrak{P}(E)_o$, respectively, are given by

$$P_e = \frac{I+M}{2},\ P_o = \frac{I-M}{2}\,.$$

For later use, let us recall that indefinite Hermite-Biehler functions are related to generalized Nevanlinna functions. Denote by $\mathcal{N}_\kappa$ the set of all functions q which are meromorphic in $\mathbb{C}\setminus\mathbb{R}$, satisfy $q^{\#} = q$, and are such that the Nevanlinna-kernel

$$Q_q(w,z) := \frac{q(z) - \overline{q(w)}}{z - \overline{w}},\ z, w \in \mathbb{C}^+$$

has κ negative squares. Moreover, let us put

$$\mathcal{N}_\kappa^{sym} := \big\{ q \in \mathcal{N}_\kappa : q(-z) = -q(z) \big\} .$$

If E is entire, write $E = A - iB$ with A, B real, i.e.

$$A := \frac{E + E^\#}{2}, \ B := i\frac{E - E^\#}{2} . \tag{2.5}$$

Let us make the notational convention that, whenever E is an entire function, we denote by A and B the corresponding functions (2.5).

2.8. *Remark.* Let E be an entire function. Then the following hold:

(i) The common zeros of E and $E^\#$ correspond to the common zeros of A and B. Since, for real points t, $E(t) = 0$ if and only if $E^\#(t) = 0$, we obtain in particular that real zeros of E correspond to real common zeros of A and B.

(ii) The classes $\mathcal{HB}_\kappa$ and $\mathcal{N}_\kappa$ ($\mathcal{HB}_\kappa^{sym}$ and $\mathcal{N}_\kappa^{sym}$, respectively) are closely related: If E and $E^\#$ do not have common nonreal zeros, then $\operatorname{ind}_- E = \kappa$ if and only if $\operatorname{ind}_- \frac{B}{A} = \kappa$. Hence, $E \in \mathcal{HB}_\kappa$ if and only if E and $E^\#$ have no common noreal zeros and $\frac{B}{A} \in \mathcal{N}_\kappa^{sym}$. Moreover, we have $E^\#(z) = E(-z)$, $z \in \mathbb{C}$, if and only if A is even and B is odd. Thus, if $E \in \mathcal{HB}_\kappa^{sym}$, we have $\frac{B}{A} \in \mathcal{N}_\kappa^{sym}$.

The class of symmetric Hermite-Biehler functions is related to another subclass of indefinite Hermite-Biehler functions as we shall now explain, cf. [KWW3].

2.9. Definition. Let $\kappa \in \mathbb{N} \cup \{0\}$ and let E be an entire function. Then E is called a *semibounded Hermite-Biehler functions with κ negative squares*, if $E \in \mathcal{HB}_\kappa$ and the meromorphic function $\frac{B}{A}$ has only finitely many poles in $\mathbb{C} \setminus [0, \infty)$. The set of all semibounded Hermite-Biehler functions with κ negative squares will be denoted by $\mathcal{HB}_\kappa^{sb}$, and again we put

$$\mathcal{HB}_{<\infty}^{sb} := \bigcup_{\kappa \in \mathbb{N} \cup \{0\}} \mathcal{HB}_\kappa^{sb} .$$

Let us mention that, in terms of Nevanlinna functions, the corresponding notion is the following, cf. [KWW1]: We write $q \in \mathcal{N}_\kappa^{ep}$, if $q \in \mathcal{N}_\kappa$ and is analytic on $\mathbb{C} \setminus [0, \infty)$ with possible exception of finitely many poles. Clearly, $E \in \mathcal{HB}_\kappa^{sb}$ means that $\frac{B}{A} \in \mathcal{N}_\kappa^{ep}$.

The relation between $\mathcal{HB}_{<\infty}^{sym}$ and $\mathcal{HB}_{<\infty}^{sb}$ can now be stated as follows. Consider the transformation $\mathfrak{T}$ which is defined by

$$(\mathfrak{T}E)(z) := A(z^2) - izB(z^2) ,$$

and define the sets $\mathbb{B}_\kappa$, $\mathbb{B}_{<\infty}$, as

$$\mathbb{B}_\kappa := \{ E \in \mathcal{HB}_\kappa^{sb} : \mathfrak{d}(E)|_{(-\infty,0)} = 0 \}, \ \kappa \in \mathbb{N} \cup \{0\} ,$$

$$\mathbb{B}_{<\infty} := \bigcup_{\kappa \in \mathbb{N} \cup \{0\}} \mathbb{B}_\kappa = \{ E \in \mathcal{HB}_{<\infty}^{sb} : \mathfrak{d}(E)|_{(-\infty,0)} = 0 \} .$$

2.10. Proposition. *The transformation $\mathfrak{T}$ induces a bijection of $\mathbb{B}_{<\infty}$ onto $\mathcal{HB}_{<\infty}^{sym}$. Thereby* $\operatorname{ind}_- \mathfrak{T}E \geq \operatorname{ind}_- E$.

Proof. Assume that $E = A - iB \in \mathcal{HB}^{sb}_{<\infty}$, so that $q := \frac{B}{A}$ belongs to $\mathcal{N}^{ep}_{<\infty}$. Then, by [KWW1, Theorem 4.1], the function

$$zq(z^2) = \frac{zB(z^2)}{A(z^2)}$$

belongs to $\mathcal{N}^{sym}_{<\infty}$. Assume now moreover that $E(t) \neq 0$ for $t \in (-\infty, 0)$. Then A and B have no common zeros in $\mathbb{C} \setminus [0,\infty)$, and hence the functions $A_1(z) := A(z^2)$ and $B_1(z) := zB(z^2)$ have no common nonreal zeros. Clearly, A_1 is even and B_1 is odd. We conclude that $\mathfrak{T}E \in \mathcal{HB}^{sym}_{<\infty}$.

The map $\mathfrak{T}$ is clearly injective. To show that $\mathfrak{T}(\mathbb{B}_{<\infty}) = \mathcal{HB}^{sym}_{<\infty}$, let $E = A - iB \in \mathcal{HB}^{sym}_{\kappa}$ be given. We define entire functions A_- and B_- by

$$A_-(z^2) := A(z),\; B_-(z^2) := \frac{B(z)}{z}\,. \tag{2.6}$$

Then A_- and B_- have no common zeros in $\mathbb{C} \setminus [0,\infty)$ since A and B have no common nonreal zeros. Put $E_- := A_- - iB_-$, then, clearly, $\mathfrak{T}E_- = E$. Again appealing to [KWW1, Theorem 4.1] we obtain $E_- \in \mathcal{HB}^{sb}_{\nu}$ for some $\nu \leq \kappa$. □

2.11. *Remark.* The transformation $\mathfrak{T}$ is compatible with multiplication with real functions: Let E be entire and D meromorphic and real, then

$$\mathfrak{T}(DE)(z) = D(z^2)(\mathfrak{T}E)(z)\,. \tag{2.7}$$

This shows that the requirement in Proposition 2.10 that E has no real negative zeros is no essential restriction of generality, since we can always split off a real Weierstraß product to the real negative zeros of E.

2.12. *Remark.* A function $E \in \mathcal{HB}^{sym}_{<\infty}$ satisfies $\frac{E^\#}{E} = \text{const}$ if and only if E either satisfies $E^\#(z) = E(z) = E(-z)$ or $E^\#(z) = -E(z) = E(-z)$, i.e.

$$\big\{E \in \mathcal{HB}^{sym}_{<\infty} : \frac{E^\#}{E} = \text{const}\big\} = \big\{E \in \mathcal{HB}^{sym}_{<\infty} : \frac{E^\#}{E} \in \{+1,-1\}\big\}.$$

As a short computation shows, we have

$$\mathfrak{T}\big(\big\{E \in \mathbb{B}_{<\infty} : \frac{E^\#}{E} \in \{+1,-1\}\big\}\big) = \big\{E \in \mathcal{HB}^{sym}_{<\infty} : \frac{E^\#}{E} \in \{+1,-1\}\big\},$$

whereas

$$\mathfrak{T}E_-(z) = A_-(z^2)\big(1 + \frac{c-1}{c+1}z\big)$$

when $E_- \in \mathbb{B}_{<\infty}$, $\frac{E_-^\#}{E_-} = c$, $c \neq \pm 1$.

3. Zeros of symmetric Hermite-Biehler functions

In the main result of this paper we give a characterization of the subset $\mathfrak{T}(\mathbb{B}_0)$ of $\mathcal{HB}^{sym}_{<\infty}$ in terms of the location of their zeros.

3.1. Theorem. *Let $E \in \mathcal{HB}^{sym}_{<\infty}$. Then $E \in \mathfrak{T}(\mathbb{B}_0)$ if and only if its zeros are distributed according to the following two rules:*

(*i*) *All zeros of* E *in* $\mathbb{C}^+$ *are simple and are located on the imaginary axis.*
(*ii*) *Let* $\operatorname{supp}\mathfrak{d}(E)\cap\mathbb{C}^+ = \{iy_1,\dots,iy_\kappa\}$ *with* $0 < y_1 < \dots < y_\kappa$*, where* $\kappa := \operatorname{ind}_- E$*. Then*

$$\sum_{w\in[-iy_{k-1},-iy_k]} \mathfrak{d}(E)(w) \quad \textit{is odd},\ k = 2,\dots,\kappa\,,$$

$$\sum_{w\in(0,-iy_1]} \mathfrak{d}(E)(w) \quad \textit{is} \begin{cases} \textit{even}\ , & \mathfrak{d}(E)(0)\ \textit{is even} \\ \textit{odd}\ , & \mathfrak{d}(E)(0)\ \textit{is odd} \end{cases}$$

The proof of this result will be carried out in several steps. The core of the result is to establish that, if E has the described distribution of zeros, it belongs to $\mathfrak{T}(\mathbb{B}_0)$.

Step 1: Removing real zeros. Let us show that without loss of generality we can make the assumption that E has no real zeros with possible exception of a simple zero at the origin. To see this note that, by the symmetry of the zeros of E, there exists an even and real entire function C which satisfies

$$\mathfrak{d}(C)(w) = \begin{cases} \mathfrak{d}(E)(w) & ,\ w\in\mathbb{R}\setminus\{0\} \\ 2\cdot\left[\frac{\mathfrak{d}(E)(0)}{2}\right] & ,\ w = 0 \\ 0 & ,\ w\in\mathbb{C}\setminus\mathbb{R} \end{cases}.$$

Since $C^\#(z) = C(z) = C(-z)$, this function assumes real values on $\mathbb{R}\cup i\mathbb{R}$. Define an entire function C_- by $C_-(z^2) := C(z)$. Then $C_-(\mathbb{R})\subseteq\mathbb{R}$ and thus $C_-^\# = C_-$. Moreover,

$$\mathfrak{d}(C_-)(w) = \begin{cases} \mathfrak{d}(E)(\sqrt{w}) & ,\ w > 0 \\ \left[\frac{\mathfrak{d}(E)(0)}{2}\right] & ,\ w = 0 \\ 0 & ,\ w\in\mathbb{C}\setminus[0,\infty) \end{cases}.$$

The conditions (*i*) and (*ii*) on the distribution of zeros clearly hold for E if and only if they hold for the function $\tilde{E} := C^{-1}E$. Also, E belongs to $\mathfrak{T}(\mathbb{B}_0)$ if and only if $\tilde{E}$ does: To see this assume first that $\tilde{E} = \mathfrak{T}\tilde{E}_-$ for some $\tilde{E}_-\in\mathbb{B}_0$. Then $C_-\tilde{E}_-$ also belongs to $\mathbb{B}_0$, and, by (2.7), $\mathfrak{T}(C_-\tilde{E}_-) = C\tilde{E} = E$. Conversely, assume that $E = \mathfrak{T}E_-$ for some $E_-\in\mathbb{B}_0$. Then

$$\mathfrak{d}(E_-)(t) = \left\{ \begin{array}{r} \mathfrak{d}(E)(\sqrt{t}),\ t > 0 \\ \left[\frac{\mathfrak{d}(E)(0)}{2}\right],\ t = 0 \\ 0,\ t < 0 \end{array} \right\} = \mathfrak{d}(C_-)(t),\ t\in\mathbb{R}\,,$$

and it follows that $C_-^{-1}E_-\in\mathbb{B}_0$. Moreover, again by (2.7), we have $\mathfrak{T}(C_-^{-1}E_-) = C^{-1}E = \tilde{E}$.

Hence, in all future steps of the proof of Theorem 3.1, we can assume that E has no real zeros with possible exception of a simple zero at the origin.

Step 2: Proof of necessity, absence of zeros in $\mathbb{C}^+ \setminus i\mathbb{R}$. Assume that $E = \mathfrak{T}(E_-)$ for some $E_- \in \mathbb{B}_0$. Note that we can write ($E_- = A_- - iB_-$)

$$E(z) = A_-(z^2) - izB_-(z^2) = \frac{E_-(z^2) + E_-^{\#}(z^2)}{2} - iz \cdot i \frac{E_-(z^2) - E_-^{\#}(z^2)}{2} =$$

$$= \frac{E_-(z^2)}{2}(1+z) + \frac{E_-^{\#}(z^2)}{2}(1-z)\,.$$

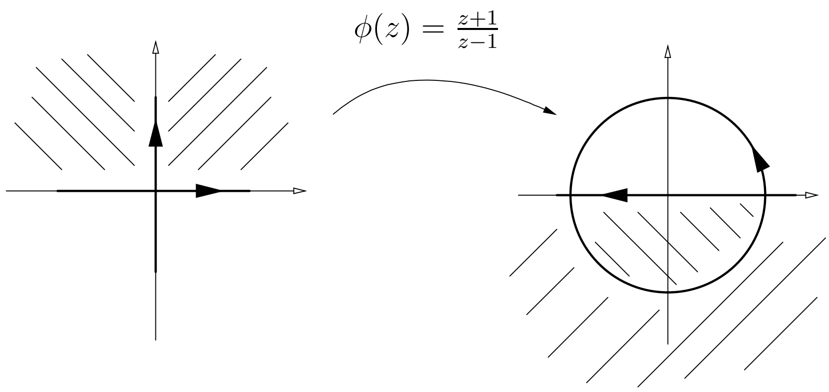
Hence, if $z \in \mathbb{C}$ is such that $\operatorname{Re} z > 0$ and $\operatorname{Im} z > 0$, so that $z^2 \in \mathbb{C}^+$ and thus $E_-(z^2) \neq 0$, we have $E(z) = 0$ if and only if

$$\frac{E_-^{\#}(z^2)}{E_-(z^2)} = \frac{z+1}{z-1}\,.$$

However, the function $\frac{E_-^{\#}}{E_-}$ maps the open upper half-plane into the open unit disk, whereas the fractional linear transformation

$$\phi(z) := \frac{z+1}{z-1}$$

maps this region as indicated in the below picture:



Thus E cannot have zeros with $\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$, and, by symmetry, therefore also no zeros with $\operatorname{Re} z < 0$, $\operatorname{Im} z > 0$.

Step 3: Proof of necessity, zeros on $i\mathbb{R}$. Assume again that $E \in \mathfrak{T}(\mathbb{B}_0)$, that is $E(z) = A_-(z^2) - izB_-(z^2)$ with $E_- := A_- - iB_- \in \mathbb{B}_0$. We use a geometric idea similar, but more refined, as in the previous step.

We show that the functions $A_-(z)$ and $B_-(z)$ do not have common zeros in $\mathbb{C}$: If $t \in \mathbb{R} \setminus \{0\}$ were a common zero of A_- and B_-, then $\sqrt{t} \in (\mathbb{R} \cup i\mathbb{R}) \setminus \{0\}$ would be a common zero of $A(z) := A_-(z^2)$ and $B(z) := zB_-(z^2)$, a contradiction. If $A_-(0) = B_-(0) = 0$, the function E would have a zero at the origin of order at least 2, again a contradiction. Finally, since $E_- \in \mathcal{H}B_{<\infty}$, A_- and B_- cannot have common nonreal zeros.

The fact that A_- and B_- have no common zeros shows that the zeros of E, with exception of a possible zero at the origin, coincide with the zeros of the function

$$Q(z) := \frac{B_-(z^2)}{A_-(z^2)} + \frac{i}{z}\,.$$

We shall deduce the desired distribution of zeros on the imaginary axis from the following statement.

3.2. Lemma. *Let $q \in \mathcal{N}_0^{ep}$ be meromorphic in the whole plane and denote by $\xi_n < \xi_{n-1} < \ldots < \xi_1 < 0$ its poles on the negative real axis. Put*

$$\xi_0 := 0 \text{ and } \eta_j := \sqrt{|\xi_j|},\ j = 0, \ldots, n\,.$$

Consider the function

$$\hat{q}(t) := q(-t^2) + \frac{1}{t}\,.$$

Then we have

$$\sum_{w \in (\eta_{j-1}, \eta_j)} \mathfrak{d}(\hat{q})(w) = 1,\ j = 1, \ldots, n\,, \tag{3.1}$$

and

$$\sum_{w \in (\eta_n, \infty)} \mathfrak{d}(\hat{q})(w) = \begin{cases} 0\ ,\ \lim_{t \to -\infty} q(t) \geq 0 \\ 1\ ,\ \lim_{t \to -\infty} q(t) < 0 \end{cases} \tag{3.2}$$

Denote by $0 < y_1 < \ldots < y_m$ the zeros of of $\hat{q}$ on the positive real axis. Then $\hat{q}(-y_j) \neq 0$, $j = 1, \ldots, m$, and the formulas

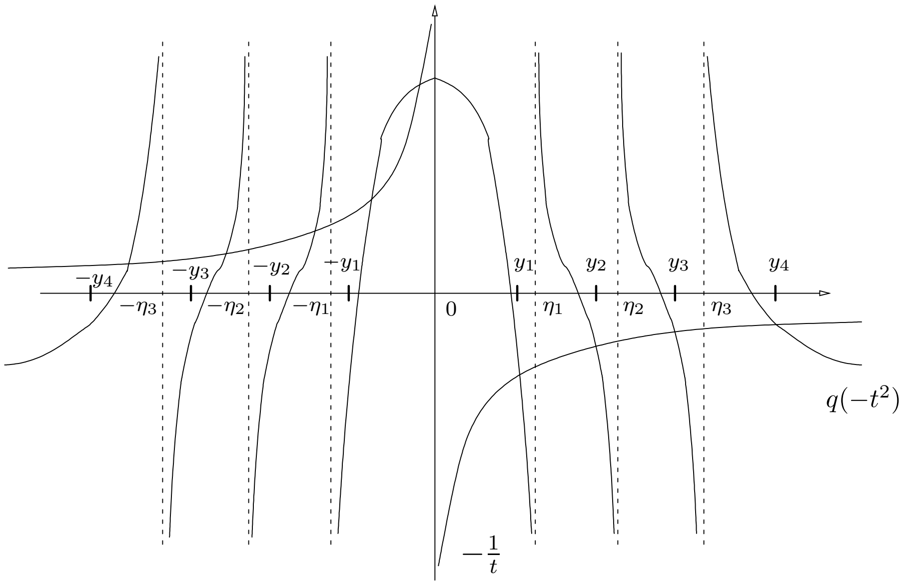
$$\sum_{w \in [-y_{j+1}, -y_j]} \mathfrak{d}(\hat{q})(w) \text{ is odd},\ y_{j+1} < \eta_n\,, \tag{3.3}$$

and

$$\sum_{w \in [-y_1, 0)} \mathfrak{d}(\hat{q})(w) \text{ is } \begin{cases} \text{even}\ ,\ \mathfrak{d}(q)(0) \geq 0 \\ \text{odd}\ \ ,\ \mathfrak{d}(q)(0) < 0 \end{cases} \tag{3.4}$$

hold true.

Proof. The proof of this lemma follows by elementary calculus inspecting the graphs of $q(-t^2)$ and $-\frac{1}{t}$:



First of all note that the set of poles of $\hat{q}$ on $\mathbb{R}$ equals $\{\pm\eta_1, \dots \pm\eta_n\} \cup \{0\}$. Thereby

$$\mathfrak{d}(\hat{q})(\pm\eta_j) = -1, \ j = 1, \dots, n,$$

$$\mathfrak{d}(\hat{q})(0) = \begin{cases} -1 \ , \ \mathfrak{d}(q)(0) \geq 0 \\ -2 \ , \ \mathfrak{d}(q)(0) < 0 \end{cases}$$

Since $q|_{\mathbb{R}}$ is nondecreasing between each pair of consecutive poles, the function $q(-t^2)$ is nonincreasing on each of the intervals

$$(0, \eta_1), (\eta_1, \eta_2), \dots, (\eta_{n-1}, \eta_n), (\eta_n, \infty).$$

Since $\frac{1}{t}$ is strictly decreasing on $\mathbb{R}^+$, the function $\hat{q}$ will be strictly decreasing on each of the above intervals. Since it tends to $\pm\infty$ at the poles $0, \eta_1, \dots, \eta_n$, the formula (3.1) follows. Moreover, there is at most one zero on (η_n, ∞). Clearly, such a zero exists if and only if $\lim_{t\to\infty} \hat{q}(t) < 0$, which is nothing else than $\lim_{u\to-\infty} q(u) < 0$. This proves (3.2).

For $t \in [y_i, \eta_i)$, $i = 1, \dots, n$, we have $\hat{q}(t) \leq 0$. Since

$$\hat{q}(-t) = q(-t^2) - \frac{1}{t} = \hat{q}(t) - \frac{2}{t},$$

it follows that

$$\hat{q}(u) < 0, \ u \in (-\eta_i, -y_i], \ i = 1, \dots, n.$$

Since $\lim_{u\nearrow-\eta_j} \hat{q}(u) = +\infty$, $j = 1, \dots, n$, it follows that (taking into account multiplicities) there is an odd number of zeros on the interval $[-y_i, -\eta_{i-1})$, $i =$

$2,\dots,n$. This shows (3.3). To establish (3.4) it is enough to note that

$$\lim_{u\nearrow 0} \hat{q}(u) = \begin{cases} -\infty & , \ \mathfrak{d}(q)(0) \geq 0 \\ +\infty & , \ \mathfrak{d}(q)(0) < 0 \end{cases}$$

□

An application of this lemma with the function $q := \frac{B_-}{A_-}$ now gives the desired information on the zeros of E of the form $z = it$, $t \in \mathbb{R}$.

Step 4: Proof of sufficiency ($E(0) \neq 0$), geometric reformulation. Let $E \in \mathcal{HB}^{sym}_{<\infty}$ and assume that E has no real zeros. Since $E(0) \in \mathbb{R}\setminus\{0\}$ and multiplication with a real constant changes neither of the two conditions in the present theorem, we can assume without loss of generality that $E(0) = 1$.

Let $E_- \in \mathbb{B}_{<\infty}$ be such that $\mathfrak{T}E_- = E$, i.e. let $E_- = A_- - iB_-$ with A_-, B_- as in (2.6). Then, by [KWW3, Proposition 4.9], we have $\mathfrak{P}(E)_o \cong \mathfrak{P}(E_-)$. Thus we have to prove that $\operatorname{ind}_- \mathfrak{P}(E)_o = 0$.

Step 5: Proof of sufficiency ($E(0) \neq 0$), employing the hypothesis. Let $E \in \mathcal{HB}^{sym}_{<\infty}$, $E(0) = 1$, have a zero distribution as described in (i) and (ii). If $E(w) = 0$, we see from (2.4) that

$$K(w,\overline{w}) = -\frac{i}{2\pi} E(\overline{w}) \frac{\partial E^{\#}}{\partial z}(\overline{w}) \,.$$

Since $E^{\#}(z) = E(-z)$, we have $\frac{\partial E^{\#}}{\partial z}(z) = -\frac{\partial E}{\partial z}(-z)$, and hence

$$K(w,\overline{w}) = \frac{i}{2\pi} E(\overline{w}) \frac{\partial E}{\partial z}(-\overline{w}) \,.$$

If $w = iy$, this formula reads as

$$K(iy,-iy) = \frac{i}{2\pi} E(-iy) \frac{\partial E}{\partial z}(iy) \,.$$

Consider the real-analytic function

$$e : \begin{cases} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & E(ix) \end{cases} .$$

Then we can further rewrite $K(iy,-iy) = \frac{1}{2\pi} e(-y) e'(y)$.

Let $0 < y_1 < \ldots < y_\kappa$ be such that $w_k := iy_k$, $k = 1,\dots,\kappa$, are the zeros of E in the upper half-plane. Since these zeros are all simple and since $e(0) = 1 > 0$, we must have $\operatorname{sgn} e'(y_k) = (-1)^k$, $k = 1,\dots,\kappa$. Since the (total) number of zeros of e on the interval $(-y_1, 0)$ is even and on each of the intervals $(-y_k, -y_{k-1})$, $k = 2,\dots,\kappa$, odd, we must have $\operatorname{sgn} e(-y_k) = (-1)^{k+1}$, $k = 1,\dots,\kappa$. Alltogether it follows that

$$K(iy_k, -iy_k) < 0, \ k = 1,\dots,\kappa \,.$$

Step 6: Proof of sufficiency ($E(0) \neq 0$), conclusion. Let $w_1, \dots, w_\kappa$ be the zeros of E in the upper half-plane. Consider the map $\Phi := \sqrt{2} P_e|_{\mathcal{L}}$ where $\mathcal{L} := \operatorname{span}\{K(w_k, .) : k = 1, \dots, \kappa\}$. Since $K(w, -z) = K(-w, z)$, we have

$$\Phi K(w_k, .) = \frac{1}{\sqrt{2}} \big(K(w_k, .) + K(-w_k, .) \big) ,$$

and hence can compute

$$\big[\Phi K(w_k, .), \Phi K(w_j, .)\big] = \frac{1}{2} \big[K(w_k, .) + K(-w_k, .), K(w_j, .) + K(-w_j, .)\big] =$$

$$= K(w_k, w_j) + K(w_k, -w_j) = [K(w_k, .), K(w_j, .)] + K(w_k, -w_j) .$$

Note that, since $w_j \in i\mathbb{R}$, we have $-w_j = \overline{w_j}$. For $j \neq k$, we obtain

$$K(w_k, -w_j) = K(w_k, \overline{w_j}) = \frac{i}{2\pi} \frac{E(\overline{w_j})\overline{E(w_k)} - E^\#(\overline{w_j})E(\overline{w_k})}{\overline{w_j} - \overline{w_k}} = 0 .$$

For $j = k$, we have proved in the previous step that $K(w_k, -w_k) < 0$. It follows that for $x \in \mathcal{L} \setminus \{0\}$, $x = \sum_{k=1}^{\kappa} \lambda_k K(w_k, .)$, we have

$$[\Phi x, \Phi x] = [x, x] + \sum_{k=1}^{\kappa} |\lambda_k|^2 K(w_k, -w_k) < [x, x] ,$$

i.e. that $\Phi|_{\mathcal{L}}$ is a contraction in the Pontryagin space sense. Since $\mathcal{L}$ is a maximal negative subspace, Φ is injective and $\Phi(\mathcal{L})$ is again a maximal negative subspace of $\mathfrak{P}(E)$.

We conclude that $\mathfrak{P}(E)_e$ contains a maximal negative subspace, and thus that its orthogonal complement $\mathfrak{P}(E)_o$ is positive definite.

Step 7: Proof of sufficiency ($E(0) = 0$). Let $E = A - iB \in \mathcal{HB}_{<\infty}^{sym}$ have the prescribed distribution of zeros and assume that $E(0) = 0$. Since, cf. Step 1, E has a simple zero at the origin and since A is even and B is odd, we must have $B'(0) \neq 0$. Thus we can assume without loss of generality that $B'(0) = 1$. Again define A_- and B_- by (2.6), so that $A_- - iB_- \in \mathbb{B}_{<\infty}$ and

$$E(z) = (\mathfrak{T} E_-)(z) = A_-(z^2) - izB_-(z^2) .$$

Note that $A_-(0) = 0$ and $B_-(0) = 1$.

Our aim is to show that $\operatorname{ind}_- E_- = 0$. Consider the function $E_+ := iE_- = B_- + iA_-$, then $E_+ \in \mathbb{B}_{<\infty}$ and $\operatorname{ind}_- E_+ = \operatorname{ind}_- E_-$. By [KWW3, Theorem 4.5] the function

$$\tilde{E}(z) := B_-(z^2) + i\frac{A_-(z^2)}{z}$$

belongs to $\mathcal{HB}_{<\infty}^{sym}$ and we have $\mathfrak{P}(\tilde{E})_e \cong \mathfrak{P}(E_+)$.

Since $\tilde{E}(z) = \frac{i}{z} E(z)$, the function $\tilde{E}$ has the same distribution of zeros as E with the exception that $\tilde{E}(0) = 1$. The same reasoning as in the above Steps 5 and 6 will show that $\operatorname{ind}_- \mathfrak{P}(\tilde{E})_e = 0$, since this time the number of zeros between 0 and $-iy_1$ is odd, hence $K(iy,-iy)$ will be positive and hence the map $\sqrt{2}P_o$ will be a contraction of $\mathcal{L}$ into $\mathfrak{P}(\tilde{E})_o$. It follows that

$$\operatorname{ind}_- E_- = \operatorname{ind}_- E_+ = \operatorname{ind}_- \mathfrak{P}(E_+) = \operatorname{ind}_- \mathfrak{P}(\tilde{E})_e = 0\,.$$

All assertions of Theorem 3.1 are proved. ✌

4. Application to some boundary value problems

In this final section we apply Theorem 3.1 in order to describe the location of the pure imaginary eigenvalues of the boundary problems I–IV listed in the introduction. For the problems II–IV their location has been found separately in each case, see [PM] for II, [MP1] for III ([MP2] for a detailed version), and [MoP] for the problem IV.

Let us note that in these papers the roles of the upper and lower half-planes were exchanged in comparison to the present paper, cf. the remark we made in the third paragraph of the introduction.

I. The Regge problem.

This is the problem

$$-y'' + q(x)y = \lambda^2 y\,, \tag{4.1}$$

$$y(0) = 0\,, \tag{4.2}$$

$$y'(a) - i\lambda y(a) = 0\,, \tag{4.3}$$

where λ is the spectral parameter and the potential q is real-valued and belongs to $L_2(0,a)$.

Denote $s(\zeta,x)$ the solution of equation

$$-y'' + q(x)y = \zeta y, \tag{4.4}$$

which satisfies the conditions $s(\zeta,0) = s'(\zeta,0) - 1 = 0$. Let us show following [KK] that the function $\frac{s(\zeta,a)}{s'(\zeta,a)}$ belongs to the Nevanlinna class and has only a finite number of poles on $(-\infty,0)$, i.e. $\frac{s(\zeta,a)}{s'(\zeta,a)} \in \mathcal{N}_0^{ep}$.

Let us multiply (4.4) by $\overline{y}$ and subtract the equation complex-conjugate to (4.4) multiplied by y. Then we obtain

$$\overline{y}''y - y''\overline{y} = 2i\operatorname{Im}\zeta|y|^2,$$

or

$$(\overline{y}'y - y'\overline{y})' = 2i\operatorname{Im}\zeta|y|^2,$$

Integrating this equation we obtain

$$\overline{y}'y - y'\overline{y}|_0^a = 2i\operatorname{Im}\zeta\int_0^a |y|^2\,dx,$$

If we take $y = s(\zeta,x)$ then we obtain

$$\operatorname{Im}\frac{s(\zeta,a)}{s'(\zeta,a)}|s'(\zeta,a)|^2 = 2\operatorname{Im}\zeta\int_0^a |s(\zeta,x)|^2\,dx, \quad \operatorname{Im}\zeta > 0.$$

This means that the function $\frac{s(\zeta,a)}{s'(\zeta,a)}$ belongs to the Nevanlinna class. It is well known (see e.g. [A]) that the set of zeros of $s'(\zeta,a)$ is bounded below. That means $\frac{s(\zeta,a)}{s'(\zeta,a)}$ is meromorphic and has finitely many poles in $\mathbb{C}\backslash[0,\infty)$. It is also known that $s'(\zeta,a)$ and $s(\zeta,a)$ cannot be zero simultanously. Therefore, the function

$$\varphi(\zeta) = s'(\zeta,a) - is(\zeta,a)$$

belongs to $\mathbb{B}_0$.

On the other hand, the set of zeros of the function

$$\varphi(\lambda) = s'(\lambda^2,a) - i\lambda s(\lambda^2,a)$$

coincides with the spectrum of problem (4.1)–(4.3) and the function $\tilde{\varphi}(\lambda)$ belongs to $\mathfrak{T}(\mathbb{B}_0)$.

Therefore, we can apply Theorem 3.1 and obtain that

(1) all (if any) eigenvalues of problem (4.1)–(4.3) in the closed upper half-plane are pure imaginary and simple (we denote them $\{i\gamma_j\}$, $j = 1,2,...,\kappa$ in increasing order: $\gamma_j < \gamma_{j+1}$);
(2) all the points $-i\gamma_j$, $j = 1,2,...,\kappa$ do not belong to the spectrum of the problem except for $i\gamma_1$ if $\gamma_1 = 0$;
(3) each interval $(-i\gamma_{j+1}, -i\gamma_j)$, $j = 1,2,...,\kappa-1$ contains an odd number (with account of multiplicities) of the eigenvalues;
(4) if $\gamma_1 \neq 0$, then the interval $(-i\gamma_1, 0)$ contains an even number (with account of multiplicities) of the eigenvalues.

II. The generalized Regge problem.

This is the problem

$$-y'' + q(x)y = \lambda^2 y\,,$$
$$y(0) = 0\,,$$
$$y'(a) - i\alpha\lambda y(a) + \beta y(a) = 0\,,$$

with $\alpha > 0$ and $\beta \in \mathbb{R}$.

Clearly, multiplying $s(\zeta,a)$ by a positive constant α does not change anything (the function $\alpha\frac{s(\zeta,a)}{s'(\zeta,a)}$ is also Nevanlinna). To prove that $\frac{\alpha s(\zeta,a)}{s'(\zeta,a)+\beta s(\zeta,a)}$ belongs also to the Nevanlinna class, let us consider its imaginary part

$$\operatorname{Im}\frac{\alpha s(\zeta,a)}{s'(\zeta,a)+\beta s(\zeta,a)} = \alpha\left|\frac{s'(\zeta,a)}{s(\zeta,a)}+\beta)\right|^{-2}\operatorname{Im}\left(\overline{\left(\frac{s'(\zeta,a)}{s(\zeta,a)}\right)}+\beta\right).$$

If $\operatorname{Im}\zeta > 0$, then $\operatorname{Im}\overline{\left(\frac{s'(\zeta,a)}{s(\zeta,a)}\right)} > 0$ and consequently

$$\operatorname{Im}\frac{\alpha s(\zeta,a)}{s'(\zeta,a)+\beta s(\zeta,a)} > 0, \quad \operatorname{Im}\zeta > 0.$$

That means that $\frac{\alpha s(\zeta,a)}{s'(\zeta,a)+\beta s(\zeta,a)}$ is a Nevanlinna function. The set of zeros of its denominator is bounded below, see [A]. In the same way as for the Regge problem we obtain that the above statements (1)-(4) are true, i.e. we obtain an alternative proof of [PM, Lemma 2.2, 3)] and [PM, Theorem 3.1, 1)-3)].

III. Vibrations of a damped string.

We consider the spectral problem

$$-y'' - ip\lambda y + q(x)y = \lambda^2 y\,, \tag{4.5}$$

$$y(0) = 0\,, \tag{4.6}$$

$$y'(a) + (\beta - i\alpha\lambda - m\lambda^2)y(a) = 0\,, \tag{4.7}$$

where $p > 0$, $\alpha > 0$, $m > 0$ and β is a real parameter. From the physical motivation of the problem it follows that $q(x)$ and β are such that the spectrum of problem (4.5)–(4.7) lies in the open lower half-plane, see [P2].

Let us change the spectral parameter: $z = \lambda + \frac{ip}{2}$. Then we obtain

$$y'' + z^2 y + \left(\frac{p^2}{4} - q(x)\right) y = 0, \tag{4.8}$$

$$y(0) = 0, \tag{4.9}$$

$$y'(a) + \left(-mz^2 - i(\alpha - mp)z + \frac{mp^2}{4} - \frac{\alpha p}{2} + \beta\right) y(a) = 0. \tag{4.10}$$

If we denote by $s(\zeta, x)$ the solution of the equation

$$y'' + \zeta y + \left(\frac{p^2}{4} - q(x)\right) y = 0,$$

which satisfies the conditions $s(\zeta, 0) = s'(\zeta, 0) - 1 = 0$ then we obtain that $\frac{s(\zeta,a)}{s'(\zeta,a)}$ belongs to $\mathcal{N}_0^{ep}$.

We distinguish three cases:

Case 1: $mp - \alpha = 0$. In this case problem (4.8)–(4.10) is selfadjoint and its spectrum lies on the real axis and on a finite subinterval of the imaginary axis. The spectrum is symmetric with respect to the real axis and to the imaginary axis. Respectively, the spectrum of the problem (4.5)–(4.7) lies on the axis $\operatorname{Im}\lambda = -\frac{p}{2}$ and on a finite subinterval of the imaginary axis. The spectrum is symmetric with respect to the imaginary axis and to the axis $\operatorname{Im}\lambda = -\frac{p}{2}$.

Case 2: $\alpha - mp > 0$. Then again $\frac{(\alpha - mp)s(\zeta,a)}{s'(\zeta,a)}$ is a Nevanlinna function. Let us show that $\frac{(\alpha - mp)s(\zeta,a)}{s'(\zeta,a)+(\beta_1 - m\zeta)s(\zeta,a)}$ where $\beta_1 = \beta + \frac{mp^2}{4} - \frac{\alpha p}{2}$ is also a Nevanlinna function. First of all we notice that

$$\operatorname{Im} \frac{(\alpha - mp)s(\zeta, a)}{s'(\zeta, a) + (\beta_1 - m\zeta)s(\zeta, a)} = (\alpha - mp) \operatorname{Im} \left(\frac{s'(\zeta, a)}{s(\zeta, a)} + (\beta_1 - m\zeta)\right)^{-1} =$$

$$= \left|\frac{s'(\zeta, a)}{s(\zeta, a)} + (\beta_1 - m\zeta)\right|^{-2} \operatorname{Im} \left(\overline{\left(\frac{s'(\zeta, a)}{s(\zeta, a)}\right)} + \beta_1 - m\overline{\zeta}\right).$$

If $\operatorname{Im}\zeta > 0$, then $\operatorname{Im}\overline{\left(\frac{s'(\zeta,a)}{s(\zeta,a)}\right)} > 0$ and consequently

$$\operatorname{Im}\frac{(\alpha - mp)s(\zeta,a)}{s'(\zeta,a)+(\beta_1 - m\zeta)s(\zeta,a)} > 0, \quad \operatorname{Im}\zeta > 0.$$

Let us now consider the denominator $s'(\zeta,a)+(\beta_1 - m\zeta)s(\zeta,a)$. The set of its zeros coincides with the spectrum of the boundary value problem

$$-y'' + q(x)y = \zeta y,$$
$$y(0) = y'(a) + (\beta_1 - m\zeta)y = 0.$$

The eigenvalues of this problem are real and bounded below, see e.g. [P1]. This means that the function $s'(z^2,a)+(\beta_1 - mz^2)s(z^2,a) - i(\alpha - mp)zs(z^2,a)$ belongs to $\mathfrak{T}(\mathbb{B}_0)$. Taking into account that $\lambda = z - \frac{ip}{2}$ and using Theorem 3.1 we obtain that

(1) all (if any) eigenvalues of problem (4.5)–(4.7) in the closed half-plane $\operatorname{Im}\lambda \geq -\frac{p}{2}$ (they all lie in the strip $-\frac{p}{2} \leq \operatorname{Im}\lambda < 0$) are pure imaginary and simple (we denote them $\{i\gamma_j\}$, $j = 1,2,...,\kappa$ in increasing order: $\gamma_j < \gamma_{j+1}$);
(2) all the points $-ip + i|\gamma_j|$, $j = 1,2,...,\kappa$ do not belong to the spectrum of the problem except of $i\gamma_1$ if $\gamma_1 = -\frac{ip}{2}$;
(3) each interval $(-ip + i|\gamma_{j+1}|, -ip + i|\gamma_j|)$, $j = 1,2,...,\kappa - 1$ contains an odd number (with account of multiplicities) of the eigenvalues;
(4) if $\gamma_1 \neq -\frac{ip}{2}$, then the interval $\left(-\frac{ip}{2}, -ip + i|\gamma_1|\right)$ contains an even number (with account of multiplicities) of the eigenvalues.

Case 3: $mp - \alpha > 0$. Then again $\frac{(mp-\alpha)s(\zeta,a)}{s'(\zeta,a)}$ is a Nevanlinna function and $\frac{(mp-\alpha)s(\zeta,a)}{s'(\zeta,a)+(\beta_1 - m\zeta)s(\zeta,a)}$ where $\beta_1 = \beta + \frac{mp^2}{4} - \frac{\alpha p}{2}$ is also a Nevanlinna function.

It means that, if we set $\xi = -z$, then the function

$$s'(\xi^2,a)+(\beta_1 - m\xi^2)s(\xi^2,a) - i(mp - \alpha)\xi s(\xi^2,a)$$

belongs to $\mathfrak{T}(\mathbb{B}_0)$. Taking into account that $\lambda = -\xi - \frac{ip}{2}$ and using Theorem 3.1 we obtain that

(1) all (if any) eigenvalues of problem (4.5)–(4.7) in the closed half-plane $\operatorname{Im}\lambda \leq -\frac{p}{2}$ are pure imaginary and simple (we denote them $\{i\gamma_j\}$, $j = 1,2,...,\kappa$ in increasing order: $|\gamma_j| < |\gamma_{j+1}|$);
(2) all the points $-ip + i|\gamma_j|$, $j = 1,2,...,\kappa$ do not belong to the spectrum of the problem except of $i\gamma_1$ if $\gamma_1 = -\frac{ip}{2}$;
(3) each interval $(-ip + i|\gamma_j|, -ip + i|\gamma_{j+1}|)$, $j = 1,2,...,\kappa - 1$ contains an odd number (with account of multiplicities) of the eigenvalues;
(4) if $\gamma_1 \neq -\frac{ip}{2}$, then the interval $\left(-\frac{ip}{2}, -ip + i|\gamma_1|\right)$ contains an even number (with account of multiplicities) of the eigenvalues.

Thus we have deduced [MP1, Theorem 4, 2] and [MP1, Theorem 5, 2'].

IV. A fourth order problem.

We consider the problem

$$y^{(4)} - (g(x)y')' = \lambda^2 y, \tag{4.11}$$

$$y(0) = y''(0) = 0, \tag{4.12}$$

$$y(a) = 0, \tag{4.13}$$

$$y''(a) - i\alpha\lambda y'(a) = 0 \tag{4.14}$$

where $g(x)$ is a continuously differentiable function.

Denote by $s_j(\lambda^2, x)$ $(j = 0, 1, 2, 3)$ the solution of the equation (4.11) which satisfies the conditions

$$s_j^{(k)}(\lambda^2, 0) = \delta_{k,j}, \quad j = 0, 1, 2, 3,$$

where δ_{kj} is the Kronecker delta. Let us introduce the function

$$\varphi(\lambda^2, x) = s_3(\lambda^2, a)s_1(\lambda^2, x) - s_1(\lambda^2, a)s_3(\lambda^2, x)$$

This function automatically satisfies the conditions (4.12), (4.13). It satisfies (4.14) if

$$\varphi''(\lambda^2 a) - i\alpha\lambda\varphi'(\lambda^2 a) = 0.$$

Let us prove that the function $\frac{\varphi'(\zeta,a)}{\varphi''(\zeta,a)}$ is a Nevanlinna function. To do it we multiply the equation

$$y^{(4)} - (g(x)y')' = \zeta y, \tag{4.15}$$

by $\overline{y}$ and subtract the equation complex-conjugate to (4.15) multiplied by y. Then we obtain

$$y^{(4)}\overline{y} - \overline{y}^{(4)}y - ((g(x)y')'\overline{y} - (g(x)\overline{y}')'y) = 2i \operatorname{Im} \zeta |y|^2$$

or

$$(y^{(3)}\overline{y} - \overline{y}^{(3)}y)' - (y^{(2)}\overline{y}' - \overline{y}^{(2)}y')' - (g(x)y'\overline{y} - g(x)\overline{y}'y)' = 2i \operatorname{Im} \zeta |y|^2.$$

Integrating this equation we obtain

$$\begin{aligned} (y^{(3)}\overline{y} - \overline{y}^{(3)}y)\Big|_0^a - (y^{(2)}\overline{y}' - \overline{y}^{(2)}y')\Big|_0^a - (g(x)y'\overline{y} - g(x)\overline{y}'y)|_0^a = \\ = 2i \operatorname{Im} \zeta \int_0^a |y|^2 \, dx \,. \end{aligned} \tag{4.16}$$

Let us apply this result to $\varphi(\zeta, x)$. This solution satisfies the conditions (4.12), (4.13), and therefore we obtain from (4.16):

$$-(\varphi^{(2)}(\zeta, a)\overline{\varphi}'(\zeta, a) - \overline{\varphi}^{(2)}(\zeta, a)\varphi'(\zeta, a)) = 2i \operatorname{Im} \zeta \int_0^a |\varphi(\zeta, x)|^2 \, dx$$

or

$$\operatorname{Im} \frac{\varphi'(\zeta, a)}{\varphi^{(2)}(\zeta, a)} |\varphi^{(2)}(\zeta, a)|^2 = 2 \operatorname{Im} \zeta \int_0^a |\varphi(\zeta, x)|^2 \, dx, \quad \operatorname{Im} \zeta > 0.$$

That means that after reducing the ratio $\frac{\varphi'(\zeta,a)}{\varphi''(\zeta,a)}$ is a Nevanlinna function. Moreover, it belongs to $\mathcal{N}_0^{ep}$ because the set of zeros of the denominator $\varphi''(\zeta,a)$, being nothing else but the spectrum of the problem

$$y^{(4)} - (g(x)y')' = \zeta y,$$
$$y(0) = y''(0) = y(a) = y''(a) = 0\,,$$

is real and bounded below.

This in turn means that

$$\varphi^{(2)}(\lambda^2) - i\lambda\varphi'(\lambda^2) = c(\lambda^2)\psi(\lambda)$$

where $c(\zeta)$ is a real function bounded below and $\psi(\lambda) \in \mathfrak{T}(\mathbb{B}_0)$. Therefore, we can apply Theorem 3.1 and obtain that the spectrum of problem (4.11)–(4.14) consists of two subsequences which can intersect. One of them consists of real and pure imaginary eigenvalue which are located symmetrically with respect to the real and to the imaginary axis. The eigenvalues of the second subsequence lie in the open lower half-plane and on the positive imaginary half-axis. The pure imaginary eigenvalues of the second subsequence satisfy the conditions (1)-(4) of I. Thus we have obtained [MoP, Theorem 6.5, 1-6].

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