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Schmidt-representation of difference quotient operators

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Abstract. We consider difference quotient operators in de Branges Hilbert spaces of entire functions. We give a description of the spectrum and a formula for the spectral subspaces. The question of completeness of the system of eigenvectors and generalized eigenvectors is discussed. For certain cases the s-numbers and the Schmidt-representation of the operator under discussion is explicitly determined.

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1. Introduction and Preliminaries

Let $\mathcal{H}$ be a Hilbert space whose elements are entire functions. We call $\mathcal{H}$ a *de Branges Hilbert space*, or *dB-space* for short, if it satisfies the following axioms (cf. [dB]):

(dB1) For each $w \in \mathbb{C}$ the functional $F \mapsto F(w)$ is continuous.

(dB2) If $F \in \mathcal{H}$, then also $F^\#(z) := \overline{F(\overline{z})}$ belongs to $\mathcal{H}$. For all $F, G \in \mathcal{H}$

$$(F^\#, G^\#) = (G, F)\,.$$

(dB3) If $w \in \mathbb{C} \setminus \mathbb{R}$ and $F \in \mathcal{H}$ with $F(w) = 0$, then also

$$\frac{z - \overline{w}}{z - w} F(z) \in \mathcal{H}\,.$$

For all $F, G \in \mathcal{H}$ with $F(w) = G(w) = 0$

$$\left(\frac{z - \overline{w}}{z - w} F(z), \frac{z - \overline{w}}{z - w} G(z)\right) = \big(F, G\big)\,.$$

For a dB-space $\mathcal{H}$ the linear space of *associated functions* can be defined as

$$\operatorname{Assoc} \mathcal{H} := \mathcal{H} + z\mathcal{H}.$$

Consider the operator $\mathcal{S}$ of multiplication by the independent variable in the dB-space $\mathcal{H}$:

$$\operatorname{dom}\mathcal{S} := \big\{F \in \mathcal{H} :\ zF(z) \in \mathcal{H}\big\}$$
$$(\mathcal{S}F)(z) := zF(z),\ F \in \operatorname{dom}\mathcal{S}.$$

By **(dB1)-(dB3)** the operator $\mathcal{S}$ is a closed symmetric operator with defect index $(1,1)$, is real with respect to the involution $F \mapsto F^{\#}$, and the set of regular points of $\mathcal{S}$ equals $\mathbb{C}$.

There is a natural bijection between the set $\operatorname{Assoc}\mathcal{H}$ and the set of all relational extensions $\mathcal{A}$ of $\mathcal{S}$ with nonempty resolvent set, see e.g. [KW1, Proposition 4.6]. It is established by the formula

$$(\mathcal{A} - w)^{-1}F(z) = \frac{F(z) - \frac{S(z)}{S(w)}F(w)}{z - w},\ w \in \mathbb{C}, S(w) \neq 0. \tag{1.1}$$

Throughout this paper we will denote the relation corresponding to a function $S \in \operatorname{Assoc}\mathcal{H}$ via (1.1) by $\mathcal{A}_S$ and will put $\mathcal{R}_S := \mathcal{A}_S^{-1}$. Note that

$$\ker(\mathcal{A}_S - w)^{-1} = \operatorname{span}\{S\} \cap \mathcal{H}\,,$$

and therefore $\mathcal{A}_S$ is a proper relation if and only if $S \in \mathcal{H}$. Moreover, it is a consequence of the formula (1.1) that the finite spectrum of the relation $\mathcal{A}_S$ is given by the zeros of the function S. Hence

$$\sigma(\mathcal{R}_S) \setminus \{0\} = \big\{\lambda \in \mathbb{C} :\ S\big(\frac{1}{\lambda}\big) = 0\big\}.$$

If $S(0) \neq 0$ the relation $\mathcal{R}_S$ has no multivalued part, i.e. is an operator, and is given by (1.1) with $w = 0$. As is seen by a perturbation argument (cf. Lemma 2.1) it is in fact a compact operator.

In this note we give some results on the completeness of the system of eigenvalues and generalized eigenvalues (Theorem 3.3) and determine the s-numbers and the Schmidt-representation of $\mathcal{R}_S$ (Theorem 4.5) when S belongs to a certain subclass of $\operatorname{Assoc}\mathcal{H}$. In fact, we are mainly interested in the operator $\mathcal{R}_E$ where $\mathcal{H} = \mathcal{H}(E)$ in the sense explained further below in this introduction. However, our results are valid, and thus stated, for a slightly bigger subclass of $\operatorname{Assoc}\mathcal{H}$. As a preliminary result, in Section 2, we give a self-contained proof of the explicit form of spectral subspaces of $\mathcal{R}_S$ at nonzero eigenvalues.

In the final Section 5 we add a discussion of the operator $\mathcal{R}_E$ in the case of a space which is symmetric about the origin, for the definition see (5.1). This notion was introduced by de Branges, and originates in the classical theory of Fourier transforms, i.e. the theory of Paley-Wiener spaces. In our context it turns out that in this case $\mathcal{R}_E$ is selfadjoint with respect to a canonical Krein space inner product on $\mathcal{H}(E)$. An investigation of the case of symmetry is of particular interest for several reasons. Firstly, in spaces symmetric about the origin a rich structure theory is available and thus much stronger results can be expected, cf. [KWW2], [B]. Secondly, it appears in many applications, as for example in the spectral theory of strings, cf. [LW], [KWW1], the theory of Hamiltonian systems with semibounded spectrum, cf. [W], or in the theory of functions of classical analysis like Gauss' hypergeometric functions, cf. [dB], or the Riemann Zeta-function, cf. [KW2]. Moreover, by comparing Lemma 4.3 with Theorem 1 of [OS] more operator theoretic methods could be brought into the theory of sampling sequences in de Branges spaces symmetric about the origin. It is a possible future direction of research to investigate these subjects.

The present note should also be viewed as a possible starting point with connections to several areas of research. For example in Corollary 4.6 we actually apply the present results to the field of growth of entire function. The proofs given are often elementary, which is due to the fact that we (basically) deal with those operators $\mathcal{R}_S$ having one-dimensional imaginary part. Thus many notions are accessible to explicit computation.

Let us collect some necessary preliminaries. A function f analytic in the open upper half plane $\mathbb{C}^+$ is said to be of *bounded type*, $f \in N(\mathbb{C}^+)$, if it can be written as a quotient of two bounded analytic functions. If the assumption that f is analytic is weakened to f being merely meromorphic, we speak of functions of *bounded characteristic*, $f \in \tilde{N}(\mathbb{C}^+)$. If $f \in \tilde{N}(\mathbb{C}^+)$ there exists a real number $\operatorname{mt} f$, the *mean type* of f, such that for all $\theta \in (0, \pi)$ with possible exception of a set of measure zero

$$\lim_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r} = \operatorname{mt} f \cdot \sin\theta \,.$$

For $f \in N(\mathbb{C}^+)$ the mean type can be obtained as

$$\operatorname{mt} f = \limsup_{y\to+\infty} \frac{\log|f(iy)|}{y} \,.$$

An entire function E is said to belong to the *Hermite-Biehler class*, $E \in \mathcal{H}B$, if it has no zeros in $\mathbb{C}^+$ and satisfies

$$\big|E^\#(z)\big| \leq \big|E(z)\big|,\ z \in \mathbb{C}^+ \,.$$

If, additionally, E has no real zeros we shall write $E \in \mathcal{H}B^\times$.

Recall that the notions of dB-spaces and Hermite-Biehler functions are intimately related: For given $E \in \mathcal{H}B$ define $\mathcal{H}(E)$ to be the set of all entire functions F such that $E^{-1}F$ and $E^{-1}F^\#$ are of bounded type and nonpositive mean type in $\mathbb{C}^+$ and, moreover, belong to $L^2(\mathbb{R})$. If $\mathcal{H}(E)$ is equipped with the inner product

$$\big(F, G\big) := \int_{\mathbb{R}} F(t)\overline{G(t)}\,\frac{dt}{|E(t)|^2} \,,$$

it becomes a dB-space. Conversely, for any given dB-space $\mathcal{H}$ there exists a function $E \in \mathcal{H}B$ such that $\mathcal{H} = \mathcal{H}(E)$. In fact the function E is in essence uniquely determined by $\mathcal{H}$: Let $E_1, E_2 \in \mathcal{H}B$ and write $E_1 = A_1 - iB_1$, $E_2 = A_2 - iB_2$, with $A_1 = A_1^\#$, $A_2 = A_2^\#$, etc. Then we have $\mathcal{H}(E_1) = \mathcal{H}(E_2)$ if and only if there exists a 2×2-matrix M whose entries are real numbers and which has determinant 1 such that $(A_2, B_2) = (A_1, B_1)M$.

By its definition a dB-space $\mathcal{H}$ is a reproducing kernel Hilbert space; denote its reproducing kernel by $K(w,z)$, i.e.

$$F(w) = \big(F(.), K(w,.)\big),\ F \in \mathcal{H}, w \in \mathbb{C}\,.$$

If $\mathcal{H}$ is written as $\mathcal{H} = \mathcal{H}(E)$, the kernel $K(w,z)$ can be represented in terms of E:

$$K(w,z) = \frac{E(z)E^{\#}(\overline{w}) - E(\overline{w})E^{\#}(z)}{2\pi i(\overline{w} - z)},\ z \neq \overline{w}\,,$$

$$K(\overline{z},z) = \frac{-1}{2\pi i}\big(E'(z)E^{\#}(z) - E(z)E^{\#}(z)'\big)\,.$$

The function E, and thus also $E^{\#}$ as well as any linear combination of those functions, belongs to $\operatorname{Assoc}\mathcal{H}$ and henceforth gives rise to an extension of the operator $\mathcal{S}$. Thereby the functions ($E = A - iB$)

$$S_\phi(z) := \frac{1}{2}e^{i(\phi-\frac{\pi}{2})}E(z) + \frac{1}{2}e^{-i(\phi-\frac{\pi}{2})}E^{\#}(z) = \sin\phi\, A(z) - \cos\phi\, B(z),\ \phi \in \mathbb{R},$$

play a special role: The set

$$\big\{\mathcal{A}_{S_\phi} : \phi \in [0,\pi)\big\}$$

is equal to the set of selfadjoint extensions of $\mathcal{S}$, cf. [KW1, Proposition 6.1]. Note that there exists $\phi \in [0,\pi)$ such that $S_\phi \in \mathcal{H}$ if and only if $\overline{\operatorname{dom}\mathcal{S}} \neq \mathcal{H}$ in which case ϕ is unique and $\overline{\operatorname{dom}\mathcal{S}} \oplus \operatorname{span}\{S_\phi\} = \mathcal{H}$, cf. [dB, Theorem 29, Problem 46]. Let us note for later reference that the reproducing kernel K can be expressed in terms of the functions S_ϕ as

$$\begin{aligned} K(w,z) &= \frac{S_\phi(z)S_{\phi+\frac{\pi}{2}}(\overline{w}) - S_{\phi+\frac{\pi}{2}}(z)S_\phi(z)}{\pi i(\overline{w} - z)},\ z \neq \overline{w}\,, \\ K(\overline{z},z) &= \frac{1}{\pi}\big(S_\phi(z)S'_{\phi+\frac{\pi}{2}}(z) - S'_\phi(z)S_{\phi+\frac{\pi}{2}}(z)\big)\,. \end{aligned} \tag{1.2}$$

If f is analytic at a point w we denote by $\operatorname{Ord}_w f \in \mathbb{N}\cup\{0\}$ the order of w as a zero of f. Note that by the definition of $\mathcal{H}(E)$ we have

$$(\mathfrak{d}\mathcal{H})(w) := \min_{F\in\mathcal{H}} \operatorname{Ord}_w F = \begin{cases} \operatorname{Ord}_w E & ,\ w \in \mathbb{R} \\ 0 & ,\ w \notin \mathbb{R} \end{cases}\,. \tag{1.3}$$

We will confine our attention to dB-spaces $\mathcal{H}$ with $\mathfrak{d}\mathcal{H} = 0$ which means, by virtue of (1.3), to restrict to dB-spaces that can be written as $\mathcal{H} = \mathcal{H}(E)$ with $E \in \mathcal{H}B^\times$. This assumption is no essential restriction since, if $E \in \mathcal{H}B$ and C denotes a Weierstraß product formed with the real zeros of E, we have $C^{-1}E \in \mathcal{H}B$ and the mapping $F \mapsto C^{-1}F$ is an isometry of $\mathcal{H}(E)$ onto $\mathcal{H}(C^{-1}E)$, cf. [KW3, Lemma 2.4].

2. Spectral subspaces

Let us start with the following observation:

2.1. Lemma. *Let $S \in \operatorname{Assoc}\mathcal{H}$, $S(0) \neq 0$. Then the operator $\mathcal{R}_S$ is compact.*

Proof. Let $S, T \in \operatorname{Assoc}\mathcal{H}$, $S(0), T(0) \neq 0$, then $\mathcal{R}_S$ and $\mathcal{R}_T$ differ only by a one-dimensional operator:

$$\begin{aligned}\big(\mathcal{R}_S - \mathcal{R}_T\big)F(z) &= \frac{F(z) - \frac{S(z)}{S(0)}F(0)}{z} - \frac{F(z) - \frac{T(z)}{T(0)}F(0)}{z} = \\ &= \frac{1}{z}\Big[\frac{T(z)}{T(0)} - \frac{S(z)}{S(0)}\Big]\big(F(.), K(0,.)\big)\,.\end{aligned} \tag{2.1}$$

Choose $T = S_\phi$ where ϕ is such that $S_\phi(0) \neq 0$. For this choice the operator $\mathcal{R}_T$ is a bounded selfadjoint operator whose nonzero spectrum is discrete, cf. [KW1, Proposition 4.6] and hence $\mathcal{R}_T$ is compact. It follows that $\mathcal{R}_S$ is compact for any $S \in \operatorname{Assoc}\mathcal{H}$, $S(0) \neq 0$.

❑

2.2. *Remark.* Assume that $\mathcal{H} = \mathcal{H}(E)$ with a function $E \in \mathcal{H}B$ of finite order ρ. Let $S \in \operatorname{Assoc}\mathcal{H}$, $S(0) \neq 0$, be given. Then for any $\rho' > \rho$ the operator $\mathcal{R}_S$ belongs to the symmetrically-normed ideal $\mathfrak{S}_{\rho'}$ (cf. [GK]).

To see this recall, e.g. from [dB], that the nonzero spectrum of the selfadjoint operator $\mathcal{R}_{S_\phi}$ consists of the simple eigenvalues $\{\lambda \in \mathbb{R} : S_\phi(\frac{1}{\lambda}) = 0\}$. Since E is of order ρ, also every function S_ϕ possesses this growth, cf. [KW3, Theorem 3.4]. Thus, for every $\rho' > \rho$, its zeros μ_k satisfy

$$\sum \frac{1}{{\mu_k}^{\rho'}} < \infty\,.$$

We start with determining the spectral subspaces of $\mathcal{R}_S$. The following result is standard, however, since it is a basic tool for the following and no explicit reference is known to us, we provide a complete proof.

If $\sigma(\mathcal{R}_S) \cap M$ is an isolated component of the spectrum denote by $\mathcal{P}_M$ the corresponding Riesz-projection.

2.3. Proposition. *Let $\mathcal{H}$ be a dB-space, $\mathfrak{d}\mathcal{H} = 0$, and let $S \in \operatorname{Assoc}\mathcal{H}$, $S(0) \neq 0$, be given. Unless $\mathcal{H}$ is finite dimensional and $S \in (\operatorname{Assoc}\mathcal{H}) \setminus \mathcal{H}$, we have*

$$\sigma(\mathcal{R}_S) = \big\{\lambda \in \mathbb{C} :\, S\big(\frac{1}{\lambda}\big) = 0\big\} \cup \{0\}.$$

In the case $\dim\mathcal{H} < \infty$, $S \in (\operatorname{Assoc}\mathcal{H}) \setminus \mathcal{H}$,

$$\sigma(\mathcal{R}_S) = \big\{\lambda \in \mathbb{C} :\, S\big(\frac{1}{\lambda}\big) = 0\big\}.$$

If $\lambda \in \sigma(\mathcal{R}_S) \setminus \{0\}$, the Riesz-projection $\mathcal{P}_{\{\lambda\}}$ is given as ($n := \operatorname{Ord}_{\lambda^{-1}} S$)

$$\big(\mathcal{P}_{\{\lambda\}}F\big)(z) = \sum_{l=1}^{n} \frac{1}{(n-l)!}\frac{d^{n-l}}{dz^{n-l}}\left[\big(z - \frac{1}{\lambda}\big)^n \frac{F(z)}{S(z)}\right]_{z=\frac{1}{\lambda}} \frac{S(z)}{(z - \frac{1}{\lambda})^l}\,. \tag{2.2}$$

The spectral subspace $\operatorname{ran}\mathcal{P}_{\{\lambda\}}$ *is spanned by the Jordan chain*

$$\left\{ \begin{pmatrix} \frac{S(z)}{1-\lambda z} \\ \vdots \\ \frac{S(z)}{(1-\lambda z)^n} \end{pmatrix}^T (\lambda M)^k \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \ k = 0, \dots, n-1 \right\} \tag{2.3}$$

where we have put

$$M := \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 0 & 0 & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

If $S \in \operatorname{dom}(\mathcal{S}^k)$ *for some* $k \in \mathbb{N} \cup \{0\}$*, then*

$$\{\, S(z), (1+z)S(z), \dots, (1+\dots+z^k)S(z) \,\} \tag{2.4}$$

is a Jordan chain of $\mathcal{R}_S$ *at* 0*. Moreover, any Jordan-chain at* 0 *is of the form* $\{p(z)S(z), \mathcal{R}_S(p(z)S(z)), \mathcal{R}_S^2(p(z)S(z)), \dots, \alpha S(z)\}$ *where* p *is a polynomial of degree at most* k *and* α *is constant.*

Proof. Assume that $0 \notin \sigma(\mathcal{R}_S)$. Then $|\sigma(\mathcal{R}_S)| < \infty$ and since every nonzero spectral point is an eigenvalue of finite type we conclude that $\dim \mathcal{H} < \infty$. Since

$$\ker \mathcal{R}_S = \operatorname{span}\{S\} \cap \mathcal{H}\,, \tag{2.5}$$

we must have $S \notin \mathcal{H}$. Conversely assume that $\dim \mathcal{H} < \infty$, $S \notin \mathcal{H}$. Then $\sigma(\mathcal{R}_S) = \sigma_p(\mathcal{R}_S)$ and by (2.5) we have $0 \notin \sigma(\mathcal{R}_S)$.

Let $\lambda \in \sigma(\mathcal{R}_S) \setminus \{0\}$ be given. In order to compute the Riesz-projection $\mathcal{P}_{\{\lambda\}}$ we use the relation ($\mu \in \rho(\mathcal{R}_S)$, $\mu \neq 0$)

$$\big(\mathcal{R}_S - \mu\big)^{-1} = -\frac{1}{\mu} - \frac{1}{\mu^2}\big(\mathcal{A}_S - \frac{1}{\mu}\big)^{-1}.$$

Choose a sufficiently small circle Γ around λ so that neither 0 nor any spectral point other than λ is contained in the interior of Γ. Then

$$
\begin{aligned}
\big(\mathcal{P}_{\{\lambda\}}F\big)(z) &= \frac{-1}{2\pi i}\oint_{\Gamma}\big((\mathcal{R}_S-\mu)^{-1}F\big)(z)\,d\mu = \\
&= \frac{-1}{2\pi i}\oint_{\Gamma}\Big(\Big(-\frac{1}{\mu}-\frac{1}{\mu^2}\big(\mathcal{A}_S-\frac{1}{\mu}\big)^{-1}\Big)F\Big)(z)\,d\mu = \\
&= \frac{-1}{2\pi i}\oint_{\Gamma}\big(\big(\mathcal{A}_S-\frac{1}{\mu}\big)^{-1}F\big)(z)\,\big(-\frac{1}{\mu^2}d\mu\big) = \\
&= \frac{-1}{2\pi i}\oint_{\frac{1}{\Gamma}}\big((\mathcal{A}_S-\nu)^{-1}F\big)(z)\,d\nu = \frac{-1}{2\pi i}\oint_{\frac{1}{\Gamma}}\frac{F(z)-\frac{S(z)}{S(\nu)}F(\nu)}{z-\nu}\,d\nu = \\
&= \frac{F(z)}{2\pi i}\oint_{\frac{1}{\Gamma}}\frac{d\nu}{\nu-z}-\frac{S(z)}{2\pi i}\oint_{\frac{1}{\Gamma}}\frac{F(\nu)}{(\nu-z)S(\nu)}\,d\nu\,. \qquad (2.6)
\end{aligned}
$$

Assume that z is located in the exterior of the circle Γ^{-1}, put $\xi:=\lambda^{-1}$ and let $n:=\operatorname{Ord}_\xi S$. Then the first integral in (2.6) vanishes and the integrand in the second term is analytic with exception of a pole at ξ with order n. Thus

$$
\begin{aligned}
\big(\mathcal{P}_{\{\lambda\}}F\big)(z) &= -S(z)\operatorname{Res}_{\mu=\xi}\frac{F(\mu)}{(\mu-z)S(\mu)} = \\
&= -\frac{S(z)}{(n-1)!}\frac{d^{n-1}}{d\mu^{n-1}}\left[\frac{(\mu-\xi)^nF(\mu)}{(\mu-z)S(\mu)}\right]_{\mu=\xi}
\end{aligned}
$$

and we compute

$$
\begin{aligned}
&\frac{d^{n-1}}{d\mu^{n-1}}\left[\frac{(\mu-\xi)^nF(\mu)}{(\mu-z)S(\mu)}\right]_{\mu=\xi} = \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}\frac{d^k}{d\mu^k}\left[\frac{1}{\mu-z}\right]\cdot\frac{d^{n-1-k}}{d\mu^{n-1-k}}\left[\frac{(\mu-\xi)^nF(\mu)}{S(\mu)}\right]_{\mu=\xi} = \\
&= \sum_{k=0}^{n-1}\frac{(n-1)!}{(n-1-k)!}\frac{d^{n-1-k}}{d\mu^{n-1-k}}\left[\frac{(\mu-\xi)^nF(\mu)}{S(\mu)}\right]_{\mu=\xi}\frac{(-1)^k}{(\mu-z)^{k+1}}\,.
\end{aligned}
$$

Then $\mathcal{P}_{\{\lambda\}}F$ coincides for z in the exterior of Γ^{-1} with the function on the right hand side of (2.2). Since both functions are entire this establishes (2.2).

Put $\Phi_l(z) := (1-\lambda z)^{-l} S(z)$, $l = 1, 2, \ldots, n$. Then, by the already proved formula (2.2), we have $\operatorname{ran} \mathcal{P}_S \subseteq \operatorname{span}\{\Phi_1, \ldots, \Phi_n\}$. We compute

$$\big(\mathcal{R}_S - \lambda\big)\Phi_k(z) = \frac{1}{z}\Big[\frac{S(z)}{(1-\lambda z)^k} - \frac{S(z)}{S(0)}\Phi_k(0)\Big] - \lambda\frac{S(z)}{(1-\lambda z)^k} =$$

$$= \frac{S(z) - (1-\lambda z)^k S(z) - \lambda z S(z)}{z(1-\lambda z)^k} = \frac{S(z)}{(1-\lambda z)^{k-1}} \cdot \frac{1-(1-\lambda z)^{k-1}}{z} =$$

$$= \frac{S(z)}{(1-\lambda z)^{k-1}}\lambda\sum_{l=0}^{k-2}(1-\lambda z)^l = \sum_{j=1}^{k-1}\lambda\Phi_j(z)\,.$$

Hence $\operatorname{span}\{\Phi_1, \ldots, \Phi_n\}$ is an invariant subspace for $\mathcal{R}_S$ and with respect to the basis $\{\Phi_1, \ldots, \Phi_n\}$ the operator $\mathcal{R}_S - \lambda$ has the matrix representation

$$\mathcal{R}_S - \lambda = \lambda M\,.$$

The only eigenvalue of this matrix is 0 and therefore $\operatorname{ran}\mathcal{P}_S = \operatorname{span}\{\Phi_1, \ldots, \Phi_n\}$. Moreover, this space is spanned by the Jordan chain (2.3).

Assume that $S \in \operatorname{dom}(\mathcal{S}^k)$ and put $\tau_l(z) := (1 + z + \ldots + z^l)S(z)$, $l = 0, 1, \ldots, k$. Then

$$\mathcal{R}_S\,\tau_0(z) = \mathcal{R}_S\,S(z) = 0\,,$$

and for $l \geq 1$

$$\mathcal{R}_S\,\tau_l(z) = \frac{1}{z}\Big[(1 + z + \ldots + z^l)S(z) - \frac{S(z)}{S(0)}\cdot S(0)\Big] = \tau_{l-1}(z).$$

We see that (2.4) is a Jordan chain at 0.

❑

3. Completeness of eigenvectors

In general the system $\mathcal{E}$ of eigenvectors and generalized eigenvectors of $\mathcal{R}_S$ need not be complete. For example consider a dB-space $\mathcal{H}$ with $1 \in (\operatorname{Assoc}\mathcal{H}) \setminus \mathcal{H}$. By Proposition 2.3 the operator $\mathcal{R}_1$ has no eigenvectors. However, in two special situations a completeness result holds. The following statements answer the question on completeness of eigenvectors in our particular situation. They complement classical results on completeness of eigenvectors such as [KL, K, L, M].

The first case is easily explained, it follows immediately from Proposition 2.3. Denote by $\mathbb{C}[z]$ the set of all polynomials with complex coefficients.

3.1. Proposition. *Let $\mathcal{H}$ be a dB-space. Assume that $\mathbb{C}[z] \subseteq \mathcal{H}$ and that $S \in \mathbb{C}[z]$, $S(0) \neq 0$. Then*

$$\begin{aligned} &\operatorname{span}\mathcal{E} = \mathbb{C}[z]\,,\\ &\operatorname{ran}\mathcal{P}_{\{0\}} \supseteq S(z)\mathbb{C}[z]\,, \qquad (3.1)\\ &\operatorname{ran}\mathcal{P}_{\mathbb{C}\setminus\{0\}} = \big\{\, p \in \mathbb{C}[z] :\ \deg p < \deg S\,\big\}\,. \end{aligned}$$

The following are equivalent:

(*i*) $\operatorname{cls}\mathcal{E} = \mathcal{H}$,
(*ii*) $\overline{\mathbb{C}[z]} = \mathcal{H}$,
(*iii*) $\operatorname{ran}\mathcal{P}_{\{0\}} = \operatorname{cls}\{z^k S(z) : k = 0, 1, 2, \dots\}$.

Proof. Let $n := \deg S$. Since S has exactly n zeros (taking into account multiplicities) and $S(0) \neq 0$, we conclude from Proposition 2.3 that

$$\operatorname{span} \bigcup_{\{\lambda:\, S(\lambda)=0\}} \operatorname{ran}\mathcal{P}_{\{\lambda\}}$$

is an n-dimensional linear space which contains only polynomials with degree less than n. Hence

$$\operatorname{ran}\mathcal{P}_{\mathbb{C}\setminus\{0\}} = \big\{p \in \mathbb{C}[z] : \deg p < \deg S\big\}.$$

Moreover, since $S \in \operatorname{dom}(\mathcal{S}^k)$ for all $k \in \mathbb{N}$, Proposition 2.3 implies that

$$S(z), (1+z)S(z), (1+z+z^2)S(z), \dots$$

is a Jordan chain of infinite length of $\mathcal{R}_S$ at 0. We have

$$\operatorname{span}\{S(z), (1+z)S(z), (1+z+z^2)S(z), \dots\} = S(z)\mathbb{C}[z].$$

We have proved all relations (3.1) and henceforth also the equivalence of (*i*), (*ii*) and (*iii*).

❑

Let $\mathcal{H}$ be a dB-space and write $\mathcal{H} = \mathcal{H}(E)$ for some $E \in \mathcal{H}B$. The next result, Theorem 3.3, which is the first main result of this note, deals with functions $S \in \operatorname{span}\{E, E^\#\} =: \mathcal{D}$. It will be proved that generically for such S a completeness result holds true. However, let us first clarify the meaning of the (two dimensional) space $\mathcal{D}$.

3.2. Lemma. *Let a dB-space $\mathcal{H}$ be given and write $\mathcal{H} = \mathcal{H}(E)$ for some $E \in \mathcal{H}B$. The space $\mathcal{D} = \operatorname{span}\{E, E^\#\}$ does not depend on the choice of E. We can write $\mathcal{D}$ as the disjoint union*

$$\mathcal{D} = \mathcal{G} \cup \mathcal{C} \cup \mathcal{G}^\#\,,$$

with

$$\mathcal{C} := \big\{\, \alpha T : \alpha \in \mathbb{C}, T \in \operatorname{Assoc}\mathcal{H}, \mathcal{A}_T \textit{ selfadjoint} \,\big\}\,,$$

$$\mathcal{G} := \big\{\, \rho H : \rho > 0, H \in \mathcal{H}B, \mathcal{H}(H) = \mathcal{H} \,\big\}\,.$$

We have $\mathcal{D} \cap \mathcal{H} \subseteq \mathcal{C}$, $\dim(\mathcal{D} \cap \mathcal{H}) \leq 1$, and $\{S \in \mathcal{D} : \operatorname{Ord}_0 S > \operatorname{Ord}_0 E\} = \operatorname{span}\{S_\phi\}$ for an appropriate $\phi \in [0, \pi)$.

Proof. Let $E, H \in \mathcal{H}B$ and write $E = A - iB$, $H = K - iL$, with A, B, K, L real. Then, by [KW1, Corollary 6.2], we have $\mathcal{H}(E) = \mathcal{H}(H)$ if and only if for some 2×2-matrix M with real entries and $\det M = 1$ the relation $(K, L) = (A, B)M$ holds. Hence

$$\operatorname{span}\{H, H^\#\} = \operatorname{span}\{K, L\} = \operatorname{span}\{A, B\} = \operatorname{span}\{E, E^\#\}\,.$$

Choose $E \in \mathcal{H}B$ such that $\mathcal{H} = \mathcal{H}(E)$ and write $E = A - iB$. Let $S \in \mathcal{D}$, then $S = uA + vB$ for some appropriate $u, v \in \mathbb{C}$. Then

$$\Big(\frac{S+S^\#}{2}, i\frac{S-S^\#}{2}\Big) = (A,B)\begin{pmatrix}\operatorname{Re} u & -\operatorname{Im} u\\ \operatorname{Re} v & -\operatorname{Im} v\end{pmatrix}. \tag{3.2}$$

Consider the determinant Δ of the matrix on the right hand side of (3.2). We have $\Delta = 0$ if and only if $u\overline{v} \in \mathbb{R}$ and hence $S = \alpha S_\phi$ for certain $\alpha \in \mathbb{C}$ and $\phi \in [0,\pi)$. If $\Delta > 0$, the function

$$H(z) := \frac{1}{\sqrt{\Delta}} S(z)$$

belongs to $\mathcal{H}B$ and $\mathcal{H}(H) = \mathcal{H}(E)$. Thus $S \in \mathcal{G}$. In case $\Delta < 0$ apply this argument to $S^\#$ instead of S to conclude that $S \in \mathcal{G}^\#$.

The fact that $\mathcal{D} \cap \mathcal{H}$ is at most one dimensional and is a subset of $\mathcal{C}$ was proved in [dB, Problem 85]. Since we have $E \in \mathcal{D}$, the set of all functions of $\mathcal{D}$ which vanish at the origin with higher order than E is a at most one-dimensional subspace of $\mathcal{D}$. The present assertion follows since there exists a (unique) value $\phi \in [0,\pi)$ such that $\operatorname{Ord}_0 S_\phi > \operatorname{Ord}_0 E$.

❑

For a dB-space $\mathcal{H}$ and numbers $\alpha, \beta \leq 0$ denote by $\mathcal{H}_{(\alpha,\beta)}$ the closed linear subspace (cf. [KW3, Lemma 2.6])

$$\mathcal{H}_{(\alpha,\beta)} := \Big\{ F \in \mathcal{H} :\ \operatorname{mt}\frac{F}{E} \leq \alpha, \operatorname{mt}\frac{F^\#}{E} \leq \beta \Big\}.$$

3.3. Theorem. *Let $\mathcal{H}$ be a dB-space, $\mathfrak{d}\mathcal{H} = 0$. Assume that $S \in \mathcal{D}$, $S(0) \neq 0$, and put $\tau := \frac{1}{2}\operatorname{mt} S^{-1}S^\#$. Then*

$$\mathcal{E}^\perp = \big\{ F \in \mathcal{H} :\ \operatorname{Ord}_w F \geq \operatorname{Ord}_w S^\# \textit{ for all } w \in \mathbb{C} \big\}. \tag{3.3}$$

We have $\operatorname{cls}\mathcal{E} = \mathcal{H}$ if and only if $\tau = 0$. In the case $\tau \neq 0$

$$\operatorname{cls}\mathcal{E} = \begin{cases}\mathcal{H}_{(0,2\tau)} & ,\ \tau < 0\\ \mathcal{H}_{(-2\tau,0)} & ,\ \tau > 0\end{cases},$$

and

$$\mathcal{E}^\perp = S^\#(z)e^{i\tau z}\mathcal{H}(e^{-i|\tau|z}).$$

Proof. If $S \in \mathcal{C}$, we have $\tau = 0$ and $\operatorname{cls}\mathcal{E} = \mathcal{H}$ by [dB, Theorem 22]. In the case $S \in \mathcal{G}$ we may assume without loss of generality that $\mathcal{H} = \mathcal{H}(S)$.

Let $\mathcal{H} = \mathcal{H}(E)$ be given, then Proposition 2.3 implies that ($\lambda \in \mathbb{C}$, $E(\lambda^{-1}) = 0$)

$$\operatorname{ran}\mathcal{P}_{\{\lambda\}} = \operatorname{span}\Big\{ \frac{\partial^k}{\partial z^k} K\big(\frac{1}{\lambda}, z\big) :\ 0 \leq k < \operatorname{Ord}_{\frac{1}{\lambda}} E \Big\},$$

and we conclude that (3.3) holds.

Let $F \in \mathcal{H}(E)$ and consider the inner-outer factorizations of $E^{-1}F, E^{-1}E^\# \in N(\mathbb{C}^+)$:

$$\frac{F}{E}(z) = B(z)U(z),\ \ \frac{E^\#}{E}(z) = B_1(z)e^{-iz\operatorname{mt}(E^{-1}E^\#)},$$

where B and B_1 denote the Blaschke products to the zeros of F and $E^{\#}$, respectively, and U is an outer function. By the already proved relation (3.3) we have $F \perp \operatorname{cls}\mathcal{E}$ if and only if $B_1|B$ in $H^2(\mathbb{C}^+)$. Under the hypothesis $\tau = 0$ this tells us that $F(E^{\#})^{-1}$ belongs to $H^2(\mathbb{C}^+)$, and hence that $F = 0$ since

$$\frac{1}{E}\mathcal{H}(E) = H^2(\mathbb{C}^+) \ominus \frac{E^{\#}}{E} H^2(\mathbb{C}^+).$$

Next assume that $\tau < 0$ and put $E_0(z) := E(z)\exp[-i\tau z]$. Then $\mathcal{H}(E_0) = \exp[-i\tau z]\mathcal{H}_{(2\tau,0)}$ and (cf. [KW3, Theorem 2.7])

$$\mathcal{H}(E) = \mathcal{H}_{(2\tau,0)} \oplus E_0(z)\mathcal{H}(e^{i\tau z}).$$

Hence also

$$\mathcal{H}(E) = \mathcal{H}_{(0,2\tau)} \oplus E^{\#}(z)e^{i\tau z}\mathcal{H}(e^{i\tau z}).$$

From what we have proved in the previous paragraph ($\operatorname{mt}(E_0^{-1}E_0^{\#}) = 0$, $\operatorname{Ord}_w E_0 = \operatorname{Ord}_w E$), we know that a function $F \in \mathcal{H}_{(0,2\tau)} = (\mathcal{H}(E_0)\exp[i\tau z])^{\#}$ with $\operatorname{Ord}_w F \geq \operatorname{Ord}_w E^{\#}$ for all $w \in \mathbb{C}$ must vanish identically. Hence $\mathcal{E}^{\perp} = E^{\#}(z)\exp[i\tau z]\mathcal{H}(\exp[i\tau z])$ and therefore $\operatorname{cls}\mathcal{E} = \mathcal{H}_{(0,2\tau)}$.

Finally let us turn to the case that $S \in \mathcal{G}^{\#}$. Applying the already proved result to the function $S^{\#}$ we obtain the assertion of the theorem also in this case.

❑

3.4. *Remark.* We would like to mention that the part of Theorem 3.3 which states that $\operatorname{cls}\mathcal{E} = \mathcal{H}$ if and only if $\tau = 0$, could also be approached differently. In fact it can be deduced from a statement which is asserted without a proof in [GT]. However, the approach chosen here is self-contained and gives a more detailed result.

The case $\tau \neq 0$ in Theorem 3.3 is actually exceptional: Put $\mathcal{D}_0 := \{S \in \mathcal{D} : \operatorname{mt}(S^{-1}S^{\#}) \neq 0\} \cup \{0\}$, then

3.5. Lemma. *We have either* $\mathcal{D}_0 = \{0\}$ *or*

$$\mathcal{D}_0 = \operatorname{span}\{H_0\} \cup \operatorname{span}\{H_0^{\#}\},$$

for some $H_0 \in \mathcal{G}$.

Proof. Assume that $H_0 \in \mathcal{H}B$ generates the space $\mathcal{H}$ and satisfies $\operatorname{mt}(H_0^{-1}H_0^{\#}) < 0$. All other functions $H \in \mathcal{H}B$ with $\mathcal{H}(H) = \mathcal{H}$ are obtained as ($H = K - iL$, $H_0 = K_0 - iL_0$)

$$(K, L) = (K_0, L_0)M,$$

where M runs through the group of all real 2×2-matrices with determinant 1.

Every such matrix M can be factorized uniquely as

$$M = \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} \lambda & t \\ 0 & \frac{1}{\lambda} \end{pmatrix}, \tag{3.4}$$

with $\gamma \in [0, 2\pi)$, $\lambda > 0$ and $t \in \mathbb{R}$.

If the second factor in (3.4) is not present, i.e. if $(\lambda, t) = (1, 0)$, we have $H(z) = H_0(z)e^{-i\gamma}$. We see that $H \in \operatorname{span}\{H_0\}$ and thus as well $\operatorname{mt}(H^{-1}H^{\#}) < 0$.

Assume that $(\lambda, t) \neq (1, 0)$. Put $H_1 := K_1 - iL_1$ where

$$(K_1, L_1) := (K_0, L_0) \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix} .$$

We already saw that $\operatorname{mt}(H_1^{-1}H_1^{\#}) < 0$. The functions H_1 and H are connected by

$$(K, L) = (K_1, L_1) \begin{pmatrix} \lambda & t \\ 0 & \frac{1}{\lambda} \end{pmatrix} .$$

We compute

$$\frac{H^{\#}}{H_1^{\#}} = \frac{\lambda K_1 + i(\frac{1}{\lambda}L_1 + tK_1)}{K_1 + iL_1} = \lambda + i\frac{(\frac{1}{\lambda} - \lambda)L_1 + tK_1}{K_1 + iL_1} =$$

$$= \lambda + \frac{i}{\sqrt{(\frac{1}{\lambda} - \lambda)^2 + t^2}} \cdot \frac{S_{\phi,1}}{H_1^{\#}} ,$$

for a certain $\phi \in [0, 2\pi)$. Since

$$\operatorname{mt} \frac{S_{\phi,1}}{H_1^{\#}} = \operatorname{mt} \frac{H_1}{H_1^{\#}} > 0,$$

it follows that

$$\operatorname{mt} \frac{H^{\#}}{H_1^{\#}} = \operatorname{mt} \frac{H_1}{H_1^{\#}} .$$

Since both, H and H_1, generate the same dB-Hilbert space, we must have $\operatorname{mt}(H_1^{-1}H) = 0$ and conclude that

$$\operatorname{mt} \frac{H^{\#}}{H} = \operatorname{mt} \Big[\frac{H^{\#}}{H_1^{\#}} \cdot \frac{H_1^{\#}}{H_1} \cdot \frac{H_1}{H}\Big] = 0.$$

We have proved that the set of all functions $H \in \mathcal{HB}$, $\mathcal{H}(H) = \mathcal{H}$, with $\operatorname{mt}(H^{-1}H^{\#}) < 0$ is either empty or of the form $H_0(z)e^{-i\gamma}$, $\gamma \in [0, 2\pi)$. From this knowledge the assertion of the lemma can be easily deduced: First note that, by $S_\phi = S_\phi^{\#}$, we have $\operatorname{mt}(S^{-1}S^{\#}) = 0$ for all functions $S \in \mathcal{C} \setminus \{0\}$ and hence $\mathcal{D}_0 \cap \mathcal{C} = \{0\}$. Moreover, by $\operatorname{mt}(S^{-1}S^{\#}) = -\operatorname{mt}[(S^{\#})^{-1}S]$, it suffices to determine $\mathcal{D}_0 \cap \mathcal{G}$. Finally, since for $\rho > 0$ we have $\operatorname{mt}[(\rho S)^{-1}(\rho S)^{\#}] = \operatorname{mt}(S^{-1}S^{\#})$, we are in the case $H \in \mathcal{HB}$, $\mathcal{H}(H) = \mathcal{H}$.

❑

4. s-numbers

In this section we investigate more closely the operators $\mathcal{R}_S$ for $S \in \mathcal{D}$. The next lemma is immediate from [S, §4], and will therefore stated without a proof:

4.1. Lemma. *Assume that $S\in\mathcal{D}$, $S(0)\neq 0$. Then*

$$\big[(\mathcal{A}_S-\overline{w})^{-1}\big]^*=(\mathcal{A}_{S^\#}-w)^{-1},\ w\in\mathbb{C},\ S(\overline{w})\neq 0. \tag{4.1}$$

In the case $S\in\mathcal{C}\setminus\{0\}$, $S(0)\neq 0$, the operator $\mathcal{R}_S$ is selfadjoint. Hence the Schmidt-representation of $\mathcal{R}_S$ can be read off its spectral representation:

$$\mathcal{R}_S=\sum_{\{\lambda\in\mathbb{C}:\,S(\lambda)=0\}}\frac{1}{\lambda}(.,\Phi_\lambda)\Phi_\lambda\,,$$

with $\Phi_\lambda=K(\lambda,z)K(\lambda,\lambda)^{-\frac{1}{2}}$. If $S\in\mathcal{D}\setminus\mathcal{C}$, then $\mathcal{R}_S$ is no longer selfadjoint. However, let us remark the following (cf. [dB, Theorem 27])

4.2. Lemma. *Assume that $S\in\mathcal{D}\setminus\mathcal{C}$. Then either $\mathcal{R}_S$ or $-\mathcal{R}_S$ is dissipative, depending on whether $S\in\mathcal{G}$ or $S\in\mathcal{G}^\#$.*

Proof. It suffices to prove that $\mathcal{R}_E$ is dissipative whenever $\mathcal{H}=\mathcal{H}(E)$. We compute (cf. (2.1))

$$\begin{aligned}\operatorname{Im}\big(\mathcal{R}_EF,F\big)&=\Big(\frac{\mathcal{R}_E-\mathcal{R}_E^*}{2i}F,F\Big)=\Big(\frac{\mathcal{R}_E-\mathcal{R}_{E^\#}}{2i}F,F\Big)=\\&=\Big(\frac{E^\#(z)E(0)-E(z)E^\#(0)}{2iE(0)E^\#(0)z}F(0),F(z)\Big)=\\&=\pi\frac{F(0)}{|E(0)|^2}\big(K(0,z),F(z)\big)=\pi\frac{|F(0)|^2}{|E(0)|^2}\,.\end{aligned}$$

❑

In order to compute the Schmidt-representation of $\mathcal{R}_S$ we need some information about the spectrum of $\mathcal{R}_S^*\mathcal{R}_S$.

4.3. Lemma. *Let $S,T\in(\operatorname{Assoc}\mathcal{H})\setminus\mathcal{H}$, $S(0),T(0)\neq 0$, and put*

$$U_{S,T}(z):=T(z)S(-z)+T(-z)S(z)\,.$$

Then

$$\sigma(\mathcal{R}_S\mathcal{R}_T)\setminus\{0\}=\big\{\,\lambda\in\mathbb{C}:\,U_{S,T}(\frac{1}{\sqrt{\lambda}})=0\,\big\}\,. \tag{4.2}$$

We have $0\notin\sigma_p(\mathcal{R}_S\mathcal{R}_T)$. Denote by $\mathcal{E}_\lambda$ the geometric eigenspace at a nonzero eigenvalue $\lambda\in\sigma(\mathcal{R}_S\mathcal{R}_T)\setminus\{0\}$. Then $\dim\mathcal{E}_\lambda=1,2$ where the latter case appears if and only if

$$T\big(\pm\frac{1}{\sqrt{\lambda}}\big)=S\big(\pm\frac{1}{\sqrt{\lambda}}\big)=0\,.$$

If $\dim\mathcal{E}_\lambda=2$, then

$$\mathcal{E}_\lambda=\operatorname{span}\Big\{\frac{T(z)}{1-\lambda z^2},\frac{zS(z)}{1-\lambda z^2}\Big\}\,. \tag{4.3}$$

Let $\dim\mathcal{E}_\lambda=1$. If $\big(T(\frac{1}{\sqrt{\lambda}}),S(\frac{1}{\sqrt{\lambda}})\big)\neq(0,0)$, then

$$\mathcal{E}_\lambda=\operatorname{span}\Big\{\frac{1}{1-\lambda z^2}\big[T(z)\frac{1}{\sqrt{\lambda}}S(\frac{1}{\sqrt{\lambda}})-zS(z)T(\frac{1}{\sqrt{\lambda}})\big]\Big\}\,,$$

if $\big(T(-\frac{1}{\sqrt{\lambda}}),S(-\frac{1}{\sqrt{\lambda}})\big)\neq(0,0)$, *then*

$$\mathcal{E}_\lambda=\operatorname{span}\Big\{\frac{1}{1-\lambda z^2}\big[T(z)\frac{1}{\sqrt{\lambda}}S(-\frac{1}{\sqrt{\lambda}})-zS(z)T(-\frac{1}{\sqrt{\lambda}})\big]\Big\}\,. \tag{4.4}$$

Proof. Since for any nonzero constants a,b we have $\mathcal{R}_{aS}=\mathcal{R}_S$, $\mathcal{R}_{bT}=\mathcal{R}_T$, and $U_{aS,bT}(z)=abU_{S,T}(z)$, we can assume without loss of generality that $S(0)=T(0)=1$. We compute $\mathcal{R}_S\mathcal{R}_T$:

$$\begin{aligned}\mathcal{R}_S\mathcal{R}_TF(z)&=\frac{1}{z}\Big[\frac{F(z)-T(z)F(0)}{z}-S(z)\Big(F'(0)-T'(0)F(0)\Big)\Big]=\\&=\frac{1}{z^2}\big[F(z)-T(z)F(0)-zS(z)\big(F'(0)-T'(0)F(0)\big)\big]\,.\end{aligned}$$

It is readily seen from this formula that $\ker\mathcal{R}_S\mathcal{R}_T=\{0\}$, since the hypothesis $S,T\in(\operatorname{Assoc}\mathcal{H})\setminus\mathcal{H}$ implies that the functions $zS(z)$ and $T(z)$ are linearly independent and $\operatorname{span}\{zS(z),T(z)\}\cap\mathcal{H}=\{0\}$.

Let $\mu\in\mathbb{C}$ and assume that $F\in\ker(\mathcal{R}_S\mathcal{R}_T-\mu)$, $F\not\equiv 0$. Then we must have

$$\begin{aligned}F(z)(1-\mu z^2)&=T(z)F(0)+zS(z)\big(F'(0)-T'(0)F(0)\big)=\\&=T(z)\phi_0+zS(z)\phi_1\,,\end{aligned} \tag{4.5}$$

with certain $\phi_0,\phi_1\in\mathbb{C}$, not both zero. In the sequel always put $\nu:=(\sqrt{\mu})^{-1}$ where we choose e.g. the square root lying in the right half plane. From (4.5) we see that

$$\begin{array}{rcrl}T(\nu)\phi_0&+&\nu S(\nu)\phi_1&=0\\T(-\nu)\phi_0&-&\nu S(-\nu)\phi_1&=0\end{array}\,. \tag{4.6}$$

Hence,

$$0=\det\begin{pmatrix}T(\nu)&\nu S(\nu)\\T(-\nu)&-\nu S(-\nu)\end{pmatrix}=(-\nu)U_{S,T}(\nu)\,,$$

and we conclude that the inclusion "$\subseteq$" in (4.2) holds.

Conversely, assume that $\mu\in\mathbb{C}\setminus\{0\}$ and that $U_{S,T}(\nu)=0$ where again $\nu=(\sqrt{\mu})^{-1}$. Then the system (4.6) of linear equations has nontrivial solutions. If (ϕ_0,ϕ_1) is any such nontrivial solution of (4.6), the function

$$F(z):=\frac{1}{1-\mu z^2}\big[T(z)\phi_0+zS(z)\phi_1\big]$$

is entire and belongs to the space $\mathcal{H}$. We have $F(0)=\phi_0$ and

$$\begin{aligned}F'(0)=\lim_{z\to 0}\frac{F(z)-\phi_0}{z}&=\lim_{z\to 0}\frac{1}{1-\mu z^2}\Big[\frac{T(z)-1}{z}\phi_0+S(z)\phi_1\Big]=\\&=T'(0)\phi_0+\phi_1\,.\end{aligned}$$

Hence F satisfies the first equality in (4.5) and thus belongs to $\ker(\mathcal{R}_S\mathcal{R}_T-\mu)$.

The formulas (4.3) and (4.4) for $\mathcal{E}_\lambda$ now follow on solving the linear system (4.6).

❑

4.4. Lemma. *Assume that $E \in \mathcal{H}B$, $E(0) = 1$, and denote by $K(w,z)$ the reproducing kernel of the space $\mathcal{H}(E)$. Let $z, w \in \mathbb{C}$, $w \neq 0$, $z \neq \overline{w}$. Then*

$$\mathcal{R}_E K(w,z) = \frac{1}{2\pi i z \overline{w}(\overline{w}-z)} \Big[\overline{w}E(\overline{w})\big(E(z) - E^\#(z)\big) - zE(z)\big(E(\overline{w}) - E^\#(\overline{w})\big)\Big] .$$

Proof. We substitute in the definition of $\mathcal{R}_E$:

$$\mathcal{R}_E K(w,z) = \frac{1}{z}\Big[\frac{E(z)E^\#(\overline{w}) - E(\overline{w})E^\#(z)}{2\pi i(\overline{w}-z)} - E(z)\frac{E(\overline{w}) - E^\#(\overline{w})}{2\pi i \overline{w}}\Big] =$$

$$= \frac{1}{2\pi i z\overline{w}(\overline{w}-z)}\Big[\overline{w}E(z)E^\#(\overline{w}) - \overline{w}E(\overline{w})E^\#(z) -$$

$$-(\overline{w}-z)\big(E(z)E^\#(\overline{w}) - E(z)E(\overline{w})\big)\Big] =$$

$$= \frac{1}{2\pi i z\overline{w}(\overline{w}-z)}\Big[- \overline{w}E(\overline{w})E^\#(z) + \overline{w}E(z)E(\overline{w}) + zE(z)E^\#(\overline{w}) - zE(z)E(\overline{w})\Big] =$$

$$= \frac{1}{2\pi i z\overline{w}(\overline{w}-z)}\Big[\overline{w}E(\overline{w})\big(E(z) - E^\#(z)\big) - zE(z)\big(E(\overline{w}) - E^\#(\overline{w})\big)\Big] .$$

❑

4.5. Theorem. *Let $\mathcal{H}$ be a dB-space, $\mathfrak{d}\mathcal{H} = 0$, let $S \in \mathcal{D} \setminus \mathcal{C}$ and let $\rho > 0$ be such that $\mathcal{H} = \mathcal{H}(\rho^{-1}S)$ ($\mathcal{H} = \mathcal{H}(\rho^{-1}S^\#)$, respectively). The zeros of the function*

$$U_{S,S^\#}(z) = S(z)S^\#(-z) + S(-z)S^\#(z)$$

are real, simple, nonzero, and symmetric with respect to 0. Denote the sequence of positive zeros of $U_{S,S^\#}$ by

$$0 < \mu_1 < \mu_2 < \dots .$$

Then the s-numbers of the operator $\mathcal{R}_S$ are given as

$$s_j(\mathcal{R}_S) = \frac{1}{\mu_j},\ j = 1, 2, \dots$$

In the Schmidt-representation $\mathcal{R}_S = \sum s_j(.,\phi_j)\psi_j$ we have

$$\phi_j(z) = \rho\sqrt{2\pi}\frac{S(-\mu_j)K(\mu_j,z) + S(\mu_j)K(-\mu_j,z)}{|S(\mu_j)S^\#(-\mu_j)U'_{S,S^\#}(\mu_j)|^{\frac{1}{2}}} , \tag{4.7}$$

$$\psi_j(z) = \rho\sqrt{2\pi}\frac{S(-\mu_j)K(\mu_j,z) - S(\mu_j)K(-\mu_j,z)}{|S(\mu_j)S^\#(-\mu_j)U'_{S,S^\#}(\mu_j)|^{\frac{1}{2}}} . \tag{4.8}$$

Proof. Let us first assume that $\mathcal{H} = \mathcal{H}(E)$ and determine the Schmidt-representation of $\mathcal{R}_E$. By Lemma 4.3 the zeros of $U := U_{E,E^\#}$ ($= U_{E^\#,E}$) are exactly the eigenvalues of the selfadjoint operator $\mathcal{R}_E\mathcal{R}_{E^\#}$ and, hence, are all real. The fact that all zeros of U are simple will follow from the following relation which holds for all λ with $U(\lambda) = 0$:

$$\frac{1}{2\pi i}U'(\lambda)E(\lambda)E^\#(-\lambda) = \big\|E(-\lambda)K(\lambda,z) \pm E(\lambda)K(-\lambda,z)\big\|^2 . \tag{4.9}$$

In order to establish this formula we compute the norm on the right hand side of (4.9):

$$\begin{aligned}
&\big\|E(-\lambda)K(\lambda,z) \pm E(\lambda)K(-\lambda,z)\big\|^2 = \\
&= |E(-\lambda)|^2 K(\lambda,\lambda) + |E(\lambda)|^2 K(-\lambda,-\lambda)\pm \\
&\pm\big[E(-\lambda)\overline{E(\lambda)}K(\lambda,-\lambda) + E(\lambda)\overline{E(-\lambda)}K(-\lambda,\lambda)\big]\,.
\end{aligned}$$

Since $\lambda \in \mathbb{R}$ and $E(\lambda)E^{\#}(-\lambda) = -E(-\lambda)E^{\#}(\lambda)$ the term in the square bracket vanishes:

$$\begin{aligned}
&E(-\lambda)E^{\#}(\lambda)\Big(\frac{E(-\lambda)E^{\#}(\lambda) - E(\lambda)E^{\#}(-\lambda)}{2\pi i(2\lambda)}\Big)+ \\
&+E(\lambda)E^{\#}(-\lambda)\Big(\frac{E(\lambda)E^{\#}(-\lambda) - E(-\lambda)E^{\#}(\lambda)}{2\pi i(-2\lambda)}\Big) = \\
&= \frac{1}{2\pi i\lambda}\Big[\big(E(-\lambda)E^{\#}(\lambda)\big)^2 - \big(E(\lambda)E^{\#}(-\lambda)\big)^2\Big] = 0\,.
\end{aligned}$$

The first two summands compute as

$$\begin{aligned}
&E(-\lambda)E^{\#}(-\lambda)\Big(\frac{E'(\lambda)E^{\#}(\lambda) - E(\lambda)E'(\lambda)^{\#}}{-2\pi i}\Big)+ \\
&+E(\lambda)E^{\#}(\lambda)\Big(\frac{E'(-\lambda)E^{\#}(-\lambda) - E(-\lambda)E'(-\lambda)^{\#}}{-2\pi i}\Big) = \\
&= \frac{E(-\lambda)E^{\#}(\lambda)}{-2\pi i}\Big(E'(\lambda)E^{\#}(-\lambda) - E(\lambda)E'(-\lambda)^{\#}\Big)+ \\
&+\frac{E(\lambda)E^{\#}(-\lambda)}{-2\pi i}\Big(-E(-\lambda)E'(\lambda)^{\#} + E'(-\lambda)E^{\#}(\lambda)\Big) = \\
&= \frac{E(-\lambda)E^{\#}(\lambda)}{-2\pi i}\Big[E'(\lambda)E^{\#}(-\lambda) - E(\lambda)E'(-\lambda)^{\#}+ \\
&+E(-\lambda)E'(\lambda)^{\#} - E'(-\lambda)E^{\#}(\lambda)\Big] = \frac{E(-\lambda)E^{\#}(\lambda)}{-2\pi i}U'(\lambda)\,.
\end{aligned}$$

By definition the singular values of $\mathcal{R}_E$ are the positive roots of the eigenvalues of $\mathcal{R}_E^*\mathcal{R}_E = \mathcal{R}_{E^\#}\mathcal{R}_E$ and hence $s_j(\mathcal{R}_E) = \mu_j^{-1}$.

The elements ϕ_j in the Schmidt-representation for $\mathcal{R}_E$ are the members of the orthogonal system of eigenvectors of $\mathcal{R}_E^*\mathcal{R}_E$. Note here that by Lemma 4.3 the operator $\mathcal{R}_E^*\mathcal{R}_E$ has only simple eigenvalues. Hence ϕ_j must be a scalar multiple of the function

$$\frac{1}{\mu_j^2 - z^2}\big[E(z)\mu_j E^{\#}(\mu_j) - E^{\#}(z)zE(\mu_j)\big]\,.$$

We compute $(U(\lambda)=0)$

$$E(-\lambda)K(\lambda,z)+E(\lambda)K(-\lambda,z)=$$
$$=E(-\lambda)\frac{E(z)E^{\#}(\lambda)-E(\lambda)E^{\#}(z)}{2\pi i(\lambda-z)}+E(\lambda)\frac{E(z)E^{\#}(-\lambda)-E(-\lambda)E^{\#}(z)}{2\pi i(-\lambda-z)}=$$
$$=\frac{E(-\lambda)}{2\pi i}\Big[\frac{E(z)E^{\#}(\lambda)-E(\lambda)E^{\#}(z)}{\lambda-z}+\frac{E(z)E^{\#}(\lambda)+E(\lambda)E^{\#}(z)}{\lambda+z}\Big]=$$
$$=\frac{E(-\lambda)}{\pi i}\frac{E(z)E^{\#}(\lambda)\lambda-E(\lambda)E^{\#}(z)z}{\lambda^2-z^2}\,.$$

By virtue of (4.9) we may take ϕ_j as stated in (4.7).

It remains to identify the elements ψ_j. First note that they must form an orthonormal system of eigenvalues of $\mathcal{R}_E\mathcal{R}_E^*$. Hence ψ_j is a scalar multiple of

$$\frac{1}{\mu_j^2-z^2}\big[E(z)zE^{\#}(\mu_j)-E^{\#}(z)\mu_jE(\mu_j)\big]\,.$$

A similar computation as in the previous paragraph shows that ψ_j henceforth is a scalar multiple of the function

$$E(-\lambda)K(\lambda,z)-E(\lambda)K(-\lambda,z)\,.$$

In order to prove (4.8) it is therefore sufficient to evaluate at a point $z=z_0$ with $\psi_j(z_0)\neq 0$ in the equation

$$\mathcal{R}_E\phi_j=\frac{1}{\mu_j}\psi_j\,. \tag{4.10}$$

We have $(U(\lambda)=0)$

$$E(-\lambda)K(\lambda,0)-E(\lambda)K(-\lambda,0)=$$
$$=E(-\lambda)\frac{E^{\#}(\lambda)-E(\lambda)}{2\pi i\lambda}-E(\lambda)\frac{E^{\#}(-\lambda)-E(-\lambda)}{2\pi i(-\lambda)}=\frac{-E(-\lambda)E(\lambda)}{\pi i\lambda}\,.$$

Note that this value is nonzero. In order to evaluate the left hand side of (4.10) we use Lemma 4.4:

$$\mathcal{R}_E\big(E(-\lambda)K(\lambda,z)+E(\lambda)K(-\lambda,z)\big)=$$
$$=\frac{E(-\lambda)}{2\pi i\lambda(\lambda-z)}\Big[\lambda E(\lambda)\frac{E(z)-E^{\#}(z)}{z}-E(z)\big(E(\lambda)-E^{\#}(\lambda)\big)\Big]+$$
$$\frac{E(\lambda)}{2\pi i\lambda(\lambda+z)}\Big[-\lambda E(-\lambda)\frac{E(z)-E^{\#}(z)}{z}-E(z)\big(E(-\lambda)-E^{\#}(-\lambda)\big)\Big]\,.$$

Letting z tend to 0 we see that the first summands of each square bracket cancel. For λ with $U(\lambda)=0$, therefore,

$$\mathcal{R}_E\big(E(-\lambda)K(\lambda,0)+E(\lambda)K(-\lambda,0)\big)=\frac{-E(-\lambda)E(\lambda)}{\pi\lambda^2}\,,$$

and we obtain the desired representation of $\mathcal{R}_E$.

Let $S\in\mathcal{D}\setminus\mathcal{C}$ be given. We reduce the assertion to the already proved case.

Assume that $S \in \mathcal{G}$ and write $S = \rho E$ with $\rho > 0$ and $\mathcal{H} = \mathcal{H}(E)$. We saw that

$$\mathcal{R}_S = \mathcal{R}_E = \sum s_j\big(.,\phi_j(E)\big)\psi_j(E)\,,$$

where $\phi_j(E)$ and $\psi_j(E)$ denote the elements (4.7) and (4.8) with E instead of S. Then

$$\phi_j(E) = \sqrt{2\pi}\frac{E(-\mu_j)K(\mu_j,z)+E(\mu_j)K(-\mu_j,z)}{|E(\mu_j)E^\#(-\mu_j)U'_{E,E^\#}(\mu_j)|^{\frac12}} =$$

$$= \sqrt{2\pi}\frac{\rho^{-1}\big(S(-\mu_j)K(\mu_j,z)+S(\mu_j)K(-\mu_j,z)\big)}{\rho^{-2}|S(\mu_j)S^\#(-\mu_j)U'_{S,S^\#}(\mu_j)|^{\frac12}}\,.$$

The same computation applies to $\psi_j(E)$ and hence the assertion of the theorem follows for $S \in \mathcal{G}$.

Assume finally that $S \in \mathcal{G}^\#$ and write $S = \rho E^\#$ with $\rho > 0$ and $\mathcal{H} = \mathcal{H}(E)$. We have

$$\mathcal{R}_S = \mathcal{R}_{E^\#} = \big(\mathcal{R}_E\big)^* = \sum s_j\big(.,\psi_j(E)\big)\phi_j(E)\,.$$

The element $\phi_j(E)$ computes as

$$\phi_j(E) = \rho\sqrt{2\pi}\frac{\overline{S(-\mu_j)}K(\mu_j,z)+\overline{S(\mu_j)}K(-\mu_j,z)}{|S^\#(\mu_j)S(-\mu_j)U'_{S^\#,S}(\mu_j)|^{\frac12}}\,.$$

Since $U_{S^\#,S}(\mu_j) = 0$, it follows that this expression is equal to

$$\rho\sqrt{2\pi}\frac{S(-\mu_j)K(\mu_j,z)-S(\mu_j)K(-\mu_j,z)}{|S(\mu_j)S^\#(-\mu_j)U'_{S,S^\#}(\mu_j)|^{\frac12}}\cdot\frac{\overline{S(-\mu_j)}}{S(-\mu_j)}\,.$$

Analogously

$$\psi_j(E) = \rho\sqrt{2\pi}\frac{S(-\mu_j)K(\mu_j,z)+S(\mu_j)K(-\mu_j,z)}{|S(\mu_j)S^\#(-\mu_j)U'_{S,S^\#}(\mu_j)|^{\frac12}}\cdot\frac{\overline{S(-\mu_j)}}{S(-\mu_j)}\,,$$

and henceforth also in the case $S \in \mathcal{G}^\#$ the assertion of the theorem follows.

❑

4.6. Corollary. *Let E be a Hermite-Biehler function of finite order $\rho > 1$ and assume that E can be written as*

$$E(z) = e^{-iaz}\prod_{n\in\mathbb{N}}\big(1-\frac{z}{z_n}\big)\exp\left[z\,\mathrm{Re}\,\frac{1}{z_n}+\ldots+\frac{z^p}{p}\,\mathrm{Re}\,\frac{1}{z_n^p}\right]\,, \qquad (4.11)$$

with $a \geq 0$ and $z_n \in \mathbb{C}^-$, compare [KW3, Lemma 3.12]. *Then the order of the function*

$$F(z) := E(z)E^\#(-z) + E(-z)E^\#(z)$$

is also equal to ρ.

Proof. First of all let us note that, since E is of the form (4.11) and we assume that $\rho > 1$, the order of the product $\prod_{n\in\mathbb{N}}(1-\frac{z}{z_n})\exp[z\operatorname{Re}\frac{1}{z_n}+\ldots+\frac{z^p}{p}\operatorname{Re}\frac{1}{z_n^p}]$ must also be equal to ρ.

The fact that the order of F does not exceed the order of E is clear. Assume that the order of F is $\rho' < \rho$ and choose $\epsilon > 0$ such that $1 < \rho' + \epsilon < \rho' + 2\epsilon < \rho$.

Denote by μ_k the sequence of zeros of F, then the series $\sum_k |\mu_k|^{-(\rho'+\epsilon)}$ is convergent. By the above theorem the operator $\mathcal{R}_E$ in the space $\mathcal{H}(E)$ belongs to the class $\mathfrak{S}_{\rho'+\epsilon}$. By the proof of Lemma 2.1 for every $S \in \operatorname{Assoc}\mathcal{H}(E)$ we have $\mathcal{R}_S \in \mathfrak{S}_{\rho'+\epsilon}$.

Consider the function $A(z) := \frac{1}{2}(E(z) + E^{\#}(z))$, and denote by (λ_k) the sequence of its zeros. Since $\mathcal{R}_A$ is a selfadjoint operator of the class $\mathfrak{S}_{\rho'+\epsilon}$ and its spectrum coincides with $\{\lambda_k\}$, we know that $\sum_k |\lambda_k|^{-(\rho'+\epsilon)}$ converges. By [KW3, Theorem 3.17], applied with the growth function $\lambda(r) := r^{\rho'+2\epsilon}$, we obtain that there exists a Hermite-Biehler function E_1 of order $\rho'' \leq \rho + 2\epsilon$ and a real and zero free function C such that

$$\mathcal{H}(E) = C \cdot \mathcal{H}(E_1)\,.$$

By [KW3, Lemma 2.4], it follows that

$$\mathcal{H}(E_1) = \frac{1}{C}\mathcal{H}(E) = \mathcal{H}(\frac{1}{C}E)\,,$$

and in particular $C^{-1}E \in \operatorname{Assoc}\mathcal{H}(E_1)$. Thus the order of $C^{-1}E$ cannot exceed the order of ρ'' of E_1. However, since C is zero free and E is of the form (4.11), certainly the order of $C^{-1}E$ is at least equal to ρ and we have reached a contradiction.

❑

The sequence of zeros of the function $U_{S,S^{\#}}$ can be obtained from the knowledge of a phase function. Recall from [dB, Problem 48] that, if $E \in \mathcal{H}B$, a *phase function* is a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$E(t)e^{i\varphi(t)} \in \mathbb{R}.$$

By this relation the function φ is uniquely determined up to integer multiples of π.

4.7. Lemma. *Let $E \in \mathcal{H}B$ be given and let φ be a phase function of E. Then $U_{E,E^{\#}}(t) = 0$ if and only if*

$$\varphi(t) - \varphi(-t) \equiv \frac{\pi}{2} \mod \pi\,.$$

Proof. We have for $t \in \mathbb{R}$

$$U_{E,E^{\#}}(t) = E^{\#}(-t)E^{\#}(t)\Big[\frac{E(t)}{E^{\#}(t)} + \frac{E(-t)}{E^{\#}(-t)}\Big].$$

Both summands in the square bracket are complex numbers of modulus 1. Hence their sum vanishes if and only if their arguments differ by an odd multiple of π, i.e.

$$\arg\frac{E(t)}{E^{\#}(t)} \equiv \arg\frac{E(-t)}{E^{\#}(-t)} + \pi \mod 2\pi.$$

Since

$$\arg \frac{E(t)}{E^{\#}(t)} \equiv -2\varphi(t) \mod 2\pi, \quad \arg \frac{E(-t)}{E^{\#}(-t)} \equiv -2\varphi(-t) \mod 2\pi,$$

we obtain

$$-2\varphi(t) \equiv -2\varphi(-t) + \pi \mod 2\pi.$$

❑

4.8. *Remark.* Assume that E satisfies the functional equation $E^{\#}(z) = E(-z)$. Then $E(0) \in \mathbb{R}$, hence we may choose a phase function φ such that $\varphi(0) = 0$. Then φ is an odd function and hence $U_{E,E^{\#}}(t) = 0$ if and only if

$$\varphi(t) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}.$$

This observation is explained by the fact that in the present case the function $U_{E,E^{\#}}$ can be factorized as

$$U_{E,E^{\#}}(z) = 4S_{\frac{\pi}{4}}(z)S_{\frac{3\pi}{4}}(z).$$

5. Spaces symmetric about the origin

Let us consider the situation that the dB-space $\mathcal{H}$ is *symmetric with respect to the origin* (cf. [dB]), i.e. has the property that the mapping $F(z) \mapsto F(-z)$ is an isometry of $\mathcal{H}$ into itself. By [dB, Theorem 47] an equivalent property is that $\mathcal{H}$ can be written as $\mathcal{H} = \mathcal{H}(E)$ with some $E \in \mathcal{H}B$ satisfying

$$E^{\#}(z) = E(-z), \ z \in \mathbb{C}. \tag{5.1}$$

This symmetry property can also be read off the reproducing kernel $K(w, z)$ of the space $\mathcal{H}$: In order that $\mathcal{H}$ is symmetric about the origin it is necessary and sufficient that

$$K(w, z) = K(-w, -z), \ w, z \in \mathbb{C}\,. \tag{5.2}$$

A space $\mathcal{H}$ being symmetric about the origin can be decomposed orthogonally as

$$\mathcal{H} = \mathcal{H}^g \oplus \mathcal{H}^u\,, \tag{5.3}$$

where

$$\mathcal{H}^g := \big\{\, F \in \mathcal{H} : \ F(-z) = F(z) \,\big\}\,,$$

$$\mathcal{H}^u := \big\{\, F \in \mathcal{H} : \ F(-z) = -F(z) \,\big\}\,.$$

The orthogonal projections $P^g : \mathcal{H} \to \mathcal{H}^g$ and $P^u : \mathcal{H} \to \mathcal{H}^u$ are given by

$$\big(P^g F\big)(z) = \frac{F(z) + F(-z)}{2}, \ \big(P^u F\big)(z) = \frac{F(z) - F(-z)}{2}\,. \tag{5.4}$$

In particular the reproducing kernel K^g of $\mathcal{H}^g$ (K^u of $\mathcal{H}^u$, respectively) is given by

$$\begin{aligned} K^g(w,z) &= \frac{1}{2}\Big(K(w,z)+K(w,-z)\Big) = \frac{1}{2}\Big(K(w,z)+K(-w,z)\Big)\,, \\ K^u(w,z) &= \frac{1}{2}\Big(K(w,z)-K(w,-z)\Big) = \frac{1}{2}\Big(K(w,z)-K(-w,z)\Big)\,. \end{aligned} \tag{5.5}$$

Taking (5.3) as a fundamental decomposition $\mathcal{H}$ can be regarded as a Krein space $\langle\mathcal{H},[.,.]\rangle$, $[.,.]=(\mathcal{J}.,.)$ where the fundamental symmetry $\mathcal{J}$ is given by

$$\mathcal{J} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} : \begin{matrix}\mathcal{H}^g \\ \oplus \\ \mathcal{H}^u\end{matrix} \longrightarrow \begin{matrix}\mathcal{H}^g \\ \oplus \\ \mathcal{H}^u\end{matrix} \quad .$$

Note that, by the formulas (5.4), $\mathcal{J}$ is nothing else but the isometry $(\mathcal{J}F)(z) = F(-z)$.

5.1. Theorem. *Assume that $\mathcal{H}$, $\mathfrak{d}\mathcal{H}=0$, is symmetric with respect to the origin and write $\mathcal{H}=\mathcal{H}(E)$ where $E\in\mathcal{H}B^\times$ satisfies (5.1) and $E(0)>0$. Then the operator $-i\mathcal{R}_E$ is selfadjoint in the Krein space $\langle\mathcal{H},[.,.]\rangle$. In fact,*

$$-i\mathcal{J}\mathcal{R}_E = \sum_j (-1)^{j+1} s_j\big(.,\phi_j\big)\phi_j\,, \tag{5.6}$$

where

$$\big\{(-1)^{j+1}s_j : j=1,2,\dots\big\} = \big\{w\in\mathbb{C} : S_{\frac{\pi}{2}}(\frac{1}{w})=0\big\}\,.$$

Let $\lambda\in\sigma(-i\mathcal{R}_E)\setminus\{0\}$ and let e_λ be a corresponding eigenvector. Then e_λ is neutral if and only if either $\lambda\notin\mathbb{R}$ or $-i\lambda^{-1}\in i\mathbb{R}^-$ is a multiple root of E. Denote by $\hat{n}(t)$ the number of zeros of E lying in $[0,-it]$ counted according to their multiplicities. If $-i\lambda^{-1}\in i\mathbb{R}^-$ is a simple root of E, then

$$\operatorname{sgn}[e_\lambda,e_\lambda] = (-1)^{\hat{n}(\lambda^{-1})+1}. \tag{5.7}$$

Proof. Let $\mathcal{R}_E=\sum_j s_j(.,\phi_j)\phi_j$ be the Schmidt-representation of $\mathcal{R}_E$. Then

$$-i\mathcal{J}\mathcal{R}_E = \sum_j (-i)s_j(.,\phi_j)\mathcal{J}\psi_j\,,$$

and hence establishing (5.6) amounts to show that

$$\mathcal{J}\psi_j = i(-1)^{j+1}\phi_j\,. \tag{5.8}$$

From (5.6) selfadjointness with respect to $[.,.]$ follows immediately.

The function ψ_j is given by (4.8) and we obtain from (5.2)

$$\begin{aligned} &\frac{1}{\sqrt{\pi}}\big|E(\mu_j)\big|\cdot\big|U'_{E,E^\#}(\mu_j)\big|^{\frac{1}{2}}\cdot\mathcal{J}\psi_j = \mathcal{J}\left[E(-\mu_j)K(\mu_j,z)-E(\mu_j)K(-\mu_j,z)\right] = \\ &= E(-\mu_j)K(-\mu_j,z)-E(\mu_j)K(\mu_j,z) = \overline{E(\mu_j)}K(\mu_j,z)-\overline{E(\mu_j)}K(-\mu_j,z)\,. \end{aligned}$$

Let φ be the phase function with $\varphi(0)=0$, then by Remark 4.8 the numbers $\mu_j = s_j^{-1}$ are such that

$$\varphi(\mu_j) = \frac{\pi}{4} + (j-1)\frac{\pi}{2}\,, \tag{5.9}$$

which means that

$$\arg E(\mu_j) = i\Big(\frac{\pi}{4} - j\frac{\pi}{2}\Big)\,.$$

From this we obtain

$$\overline{E(\mu_j)} = \big|E(\mu_j)\big| e^{-i(\frac{\pi}{4}-j\frac{\pi}{2})} = E(\mu_j)e^{-2i(\frac{\pi}{4}-j\frac{\pi}{2})} = E(\mu_j)(-i)(-1)^j\,,$$

and by symmetry $\overline{E(-\mu_j)} = E(-\mu_j)i(-1)^j$. Thus we have

$$\overline{E(\mu_j)}K(\mu_j,z) - \overline{E(\mu_j)}K(-\mu_j,z) = \\ = i(-1)^{j+1}\big[E(-\mu_j)K(\mu_j,z) + E(\mu_j)K(-\mu_j,z)\big]\,,$$

and (5.8) follows. Next note that, since φ is odd, (5.9) implies

$$\varphi\big((-1)^{j+1}\mu_j\big) = \begin{cases} \frac{\pi}{4} + \frac{j-1}{2}\pi & , j \text{ odd} \\ \frac{\pi}{4} - \frac{j}{2}\pi & , j \text{ even} \end{cases}\,,$$

and hence $(-1)^{j+1}\mu_j$, $j = 1,3,5,\ldots$, enumerates the positive zeros of $S_{\frac{\pi}{4}}$ and $(-1)^{j+1}\mu_j$, $j = 2,4,6,\ldots$, the negative zeros of this function.

Let $\lambda \in \sigma(-i\mathcal{R}_E)$ be given. By Proposition 2.3 the geometric eigenspace at λ is one-dimensional and spanned by $e_\lambda := K(i\overline{\lambda}^{-1}, z)$. The assertion concerning neutrality of eigenvectors follows from the selfadjointness of $\mathcal{R}_E$: If $\lambda \notin \mathbb{R}$ the eigenvector must be neutral. If $\lambda \in \mathbb{R}$ and $-i\lambda^{-1}$ is a multiple root of E then by Proposition 2.3 there exists a Jordan chain at λ and, hence, the eigenvector must be neutral. It remains to consider the case that $-i\lambda^{-1}$ is a simple root of E. To this end we compute (put $w := i\overline{\lambda}^{-1}$)

$$\big[e_\lambda, e_\lambda\big] = \big[K(w,z), K(w,z)\big] = \big(K(-w,z), K(w,z)\big) = K(-w,w)\,.$$

Since $w \in i\mathbb{R}^-$, we have

$$K(-w,w) = K(\overline{w},w) = \frac{i}{2\pi}\big(E'(w)E^\#(w) - E(w)E^\#(w)'\big) = iE'(w)\frac{E^\#(w)}{2\pi}\,.$$

From the symmetry relation (5.1) we find that $E(i\mathbb{R}) \subseteq \mathbb{R}$. Since $E(0) > 0$ and E has no zeros in $\mathbb{C}^+$, we have $E(i\mathbb{R}^+) \subseteq \mathbb{R}^+$ and therefore

$$\operatorname{sgn}[e_\lambda, e_\lambda] = \operatorname{sgn} iE'(w)\,.$$

Consider the function $f(t) := E(-it) : [0,\infty) \to \mathbb{R}$. Then $f(0) > 0$ and hence at a simple zero t_0 of f we have

$$\operatorname{sgn} f'(t_0) = (-1)^{\hat{n}(f,t_0)}\,,$$

where $\hat{n}(f,t_0)$ denotes the number of zeros of f in $[0,t_0]$ counted according to their multiplicities. Since $f'(t) = -iE'(-it)$ the relation (5.7) follows.

❏

Let us conclude with giving the matrix representation of $\mathcal{R}_E$ with respect to the decomposition (5.3).

5.2. Lemma. *With respect to* (5.3) *the operator* $\mathcal{R}_E$ *has the representation*

$$\mathcal{R}_E = \sum_j \frac{4\pi i(-1)^{j+1}s_j}{|U'_{E,E^\#}(\mu_j)|} M_j \,,$$

with

$$M_j = \begin{pmatrix} \big(\,.\,, K^g(\mu_j,z)\big)K^g(\mu_j,z) & i(-1)^j\big(\,.\,, K^u(\mu_j,z)\big)K^g(\mu_j,z) \\ i(-1)^j\big(\,.\,, K^g(\mu_j,z)\big)K^u(\mu_j,z) & -\big(\,.\,, K^u(\mu_j,z)\big)K^u(\mu_j,z) \end{pmatrix} \,.$$

Proof. We determine the decomposition of ϕ_j with respect to (5.3). Using (5.9) we obtain

$$\phi_j = \frac{\sqrt{2\pi}}{|E(\mu_j)|\cdot|U'_{E,E^\#}(\mu_j)|^{\frac12}}\cdot\frac{|E(\mu_j)|}{\sqrt2}\cdot$$

$$\cdot\Big[\big((-1)^{[\frac{j}{2}]} - i(-1)^{[\frac{j+1}{2}]}\big)K(\mu_j,z) + \big((-1)^{[\frac{j}{2}]} + i(-1)^{[\frac{j+1}{2}]}\big)K(-\mu_j,z)\Big] =$$

$$= \frac{2\sqrt{\pi}}{|U'_{E,E^\#}(\mu_j)|^{\frac12}}\Big[(-1)^{[\frac{j}{2}]}K^g(\mu_j,z) - i(-1)^{[\frac{j+1}{2}]}K^u(\mu_j,z)\Big]\,.$$

Substituting into (5.6) the assertion of the lemma follows.

❑

5.3. *Remark.* The value of $|U'_{E,E^\#}(\mu_j)|$ can be determined from A and B: Making use of (5.9) we obtain from $U_{E,E^\#}(z) = E(z)^2 + E^\#(z)^2$ that

$$U'_{E,E^\#}(\mu_j) = 2\sqrt2|E(\mu_j)|(-1)^{[\frac{j}{2}]}\big(A'(\mu_j) + (-1)^jB'(\mu_j)\big)\,.$$

Since $\operatorname{sgn}(-i)E(\mu_j)^2 = (-1)^j$, we obtain from (4.9) that $\operatorname{sgn} U'_{E,E^\#}(\mu_j) = (-1)^j$ and, hence, conclude that

$$\big|U'_{E,E^\#}(\mu_j)\big| = 2\sqrt2|E(\mu_j)|(-1)^{[\frac{j+1}{2}]}\big(A'(\mu_j) + (-1)^jB'(\mu_j)\big)\,.$$

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