Canonical differential equations of Hilbert-Schmidt type

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Abstract. A canonical system of differential equations, or Hamiltonian system, is a system of order two of the form $Jy'(x) = -zH(x)y(x)$, $x \in \mathbb{R}^+$. We characterize the property that the selfadjoint operators associated to a canonical system have resolvents of Hilbert-Schmidt type in terms of the Hamiltonian H as well as in terms of the associated Titchmarsh-Weyl coefficient.

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1. Introduction

A canonical system of differential equations is an equation of the form

$$Jy'(x) = -zH(x)y(x),\ x \in \mathbb{R}^+, \tag{1.1}$$

where $y(x)$ is a $\mathbb{C}^2$-valued function on $\mathbb{R}^+$,

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and $H(x)$ is a $\mathbb{R}^{2\times 2}$-valued function on $\mathbb{R}^+$ such that $H(x) \geq 0$ and $H|_{(0,x]} \in L^1$ for all $x \in \mathbb{R}^+$. The function $H(x)$ is called the Hamiltonian corresponding to the canonical differential equation (1.1). We always assume that moreover $\operatorname{tr}(H(x)) = 1,\ x \in \mathbb{R}^+$.

Canonical differential equations are usually studied with operator theoretic methods. A very good and detailed account on the operator model associated with the equation (1.1) can be found in [5]. Let us recall the basic notions: Denote by $L^2(H, \mathbb{R}^+)$ the Hilbert space of all measurable $\mathbb{C}^2$-valued functions $f(x)$ on $\mathbb{R}^+$ such that

$$\|f\|^2 = \int_0^{+\infty} f(x)^* H(x) f(x) dx < +\infty.$$

One considers the closed subspace $L^2_s(H,\mathbb{R}^+)$ of $L^2(H,\mathbb{R}^+)$ consisting of all $f \in L^2(H,\mathbb{R}^+)$ such that $H(x)f(x)$ is constant on H-indivisible intervals. Thereby an interval $I \subseteq \mathbb{R}^+$ is called H-indivisible if for some $\varphi \in \mathbb{R}$

$$H(x) = \begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}^T, \ x \in I \text{ a.e.}$$

Moreover one considers the linear relation

$$T_{max,s} = \{(f;g) \in L^2_s(H,\mathbb{R}^+)^2 : f \text{ is loc. abs. cont.}, \ Jf' = -Hg\}, \tag{1.2}$$

and its restriction

$$T_{min,s} = \{(f;g) \in T_{max,s} : f(0+) = 0\}, \tag{1.3}$$

which turns out to be a symmetric operator such that

$$T^*_{min,s} = T_{max,s}.$$

All the selfadjoint extensions of $T_{min,s}$ are given by

$$A(\nu) = \{(f;g) \in T_{max,s} : \sin\nu f_1(0+) = \cos\nu f_2(0+)\}, \ \nu \in \mathbb{R}, \tag{1.4}$$

and thereby we have $A(\nu_1) = A(\nu_2)$ if and only if $\nu_1 - \nu_2 \in \pi\mathbb{Z}$. For a detailed discussion of linear relations in Hilbert spaces see [2].

A fundamental notion in the theory of canonical systems is the Titchmarsh-Weyl coefficient associated with the equation (1.1). Let $W(x,z) = (w_{ij}(x,z))_{i,j=1,2}$, $x \in \mathbb{R}^+, z \in \mathbb{C}$, be the 2×2-matrix valued solution of the initial value problem

$$\frac{dW(x,z)}{dx}J = zW(x,z)H(x), \ x > 0, \ W(0+,z) = I,$$

and define

$$q_H(z) = \lim_{x\to+\infty} \frac{w_{11}(x,z)\tau + w_{12}(x,z)}{w_{21}(x,z)\tau + w_{22}(x,z)}. \tag{1.5}$$

This limit exists for $z \in \mathbb{C}\setminus\mathbb{R}$ and does not depend on $\tau \in \mathbb{R}$. The function q_H is called the Titchmarsh-Weyl coefficient of (1.1). It belongs to the Nevanlinna class $\mathcal{N}_0$, i.e. is holomorphic, satisfies $q_H(\bar z) = \overline{q_H(z)}, \ z \in \mathbb{C}\setminus\mathbb{R}$, and has the property that the kernel

$$L_{q_H}(w,z) = \frac{q_H(z) - \overline{q_H(w)}}{z - \bar w}$$

is positive semidefinite.

The inverse spectral theorem, a deep result due to L.de Branges (see [1]), shows that (1.5) sets up a bijective correspondence between the set of all trace normed Hamiltonians and the Nevanlinna class $\mathcal{N}_0$.

In a generalization to the indefinite setting one allows the Titchmarsh-Weyl coefficient q to belong to the generalized Nevanlinna class $\mathcal{N}_{<\infty}$, i.e. allows the kernel L_q to have a finite number of negative squares, and asks for an equation of similar type as (1.1), or for an operator model similar to $L^2_s(H,\mathbb{R}^+)$, $T_{max,s}$, such that again there is a bijective correspondence between $\mathcal{N}_{<\infty}$ and the resective indefinite Hamiltonians.

It turns out that an indefinite Hamiltonian is composed out of a finite number $H_1, \dots, H_n$ of positive Hamiltonians and a finite number of real parameters. Thereby the occuring Hamiltonians and parameters are subject to certain conditions. One among them is that selfadjoint extensions of the symmetry $T_{min,s}$ in $L^2_s(H_i, \mathbb{R}^+)$ have resolvents of Hilbert-Schmidt type.

It is the aim of this note to characterize this property explicitly in terms of the Hamiltonian as well as in terms of the spectral measure of the associated Titchmarsh-Weyl coefficient. In §2 we prove our main result Theorem 2.4 which gives a necessary and sufficient integrability condition on H in order that selfadjoint extensions have resolvents of Hilbert-Schmidt type. To this end we reduce the problem to a specific boundary condition and apply classical criteria for integral operators which can be found e.g. in [3]. It is the subject of §3 to reformulate these conditions in terms of the spectral measure of the Titchmarsh-Weyl coefficient q_H, see Theorem 3.1. Thereby we use the theory of integral representations and operator models for Nevanlinna functions as developed e.g. in [4]. In general our exposition relies on the material presented in [5] and the classical theory of integral operators, otherwise is fairly elementary.

2. Integral Operators of Hilbert-Schmidt Type

In the subsequent lemma we are going to use a well known criterion on whether an integral operator is of Hilbert-Schmidt type. Concerning this lemma note that the square root of $H(t)$ exists because $H(t)$ is assumed to be positive semidefinite.

Lemma 2.1. *Let $H(t) = (h_{ij}(t))_{i,j=1,2}$ be a trace normed Hamiltonian on $\mathbb{R}^+$ such that*

$$\int_0^{+\infty} h_{11}(t)dt < +\infty. \tag{2.1}$$

Consider the space $L^2(\mathbb{R}^+)^2$ of all $\mathbb{C}^2$-valued functions on $\mathbb{R}^+$ which are square integrable with respect to the Lebesgue-measure, and consider the kernel

$$K(x,y) = H(x)^{\frac{1}{2}} \begin{pmatrix} 0 & -\chi_{\{y<x\}} \\ -\chi_{\{y>x\}} & 0 \end{pmatrix} H(y)^{\frac{1}{2}}. \tag{2.2}$$

On $L^2(\mathbb{R}^+)^2$ define the integral operator

$$(Cg)(x) = \int_0^{+\infty} K(x,y)g(y)dy \tag{2.3}$$

with $g \in \operatorname{dom}(C)$ if the integral exists for almost every x and $Cg \in L^2(\mathbb{R}^+)^2$. Moreover, we set

$$M(x) = (m_{ij}(x))_{i,j=1,2} = \int_0^x H(t)dt.$$

Then C is a continuous, everywhere defined operator which belongs to the Hilbert-Schmidt class if and only if

$$2\int_0^{+\infty} m_{22}(t)h_{11}(t)dt < +\infty. \tag{2.4}$$

In this case the Hilbert-Schmidt norm of C coincides with this number.

Proof. Its well known (see for example [3]) that C is a continuous, everywhere defined operator which belongs to the Hilbert-Schmidt class if and only if

$$\int_0^{+\infty}\int_0^{+\infty} \operatorname{tr}(K(x,y)^*K(x,y))dxdy < +\infty.$$

In this case $\|C\|_2^2$ coincides with the value of integral. We calculate

$$K(x,y)^*K(x,y) =$$

$$H(y)^{\frac12}\begin{pmatrix} 0 & -\chi_{\{y>x\}} \\ -\chi_{\{y<x\}} & 0 \end{pmatrix} H(x) \begin{pmatrix} 0 & -\chi_{\{y<x\}} \\ -\chi_{\{y>x\}} & 0 \end{pmatrix} H(y)^{\frac12} =$$

$$H(y)^{\frac12}\left(\chi_{\{y>x\}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} H(x) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \chi_{\{y<x\}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} H(x) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) H(y)^{\frac12} =$$

$$H(y)^{\frac12}\left(\chi_{\{y>x\}}h_{22}(x)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \chi_{\{y<x\}}h_{11}(x)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) H(y)^{\frac12}. \tag{2.5}$$

Set $(k_{ij}(y))_{i,j=1,2} = H(y)^{\frac12}$, and note that $k_{12} = k_{21}$. We then rewrite (2.5) as

$$\chi_{\{y>x\}}h_{22}(x)\begin{pmatrix} k_{11}(y)k_{11}(y) & k_{11}(y)k_{12}(y) \\ k_{11}(y)k_{12}(y) & k_{12}(y)k_{12}(y) \end{pmatrix} +$$

$$+\chi_{\{y<x\}}h_{11}(x)\begin{pmatrix} k_{12}(y)k_{12}(y) & k_{12}(y)k_{22}(y) \\ k_{12}(y)k_{22}(y) & k_{22}(y)k_{22}(y) \end{pmatrix}.$$

As $k_{11}^2(y) + k_{12}^2(y) = h_{11}(y)$ and $k_{12}^2(y) + k_{22}^2(y) = h_{22}(y)$ we obtain

$$\operatorname{tr}(K(x,y)^*K(x,y)) = \chi_{\{y>x\}}h_{22}(x)h_{11}(y) + \chi_{\{y<x\}}h_{11}(x)h_{22}(y),$$

and by Fubini's theorem

$$\int_0^{+\infty}\int_0^{+\infty} \operatorname{tr}(K(x,y)^*K(x,y))dxdy =$$

$$\int_0^{+\infty} h_{11}(y)\int_0^{y} h_{22}(x)dxdy + \int_0^{+\infty} h_{22}(y)\int_y^{+\infty} h_{11}(x)dxdy =$$

$$2\int_0^{+\infty} h_{11}(y)\int_0^{y} h_{22}(x)dxdy = 2\int_0^{+\infty} m_{22}(y)h_{11}(y)dy.$$

□

Lemma 2.2. *Let $H(t) = (h_{ij}(t))_{i,j=1,2}$ be a trace normed Hamiltonian on $\mathbb{R}^+$ such that (2.1) holds. Moreover, let B be the operator defined in $L^2_s(H, \mathbb{R}^+)$ by*

$$(Bf)(x) = \int_0^x JH(t)f(t)dt - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \int_0^{+\infty} JH(t)f(t)dt,$$

with $f \in \operatorname{dom}(B)$ if $f \in L^2_s(H, \mathbb{R}^+)$ such that $(Bf)(x)$ exists for almost every $x \in \mathbb{R}^+$ and $Bf \in L^2_s(H, \mathbb{R}^+)$.

Then B is a continuous, everywhere defined operator which belongs to the Hilbert-Schmidt class if and only if (2.4) holds true.

Proof. The integral operator B can be written as

$$(Bf)(x) = \int_0^{+\infty} k(x,y)f(y)dy,$$

where $k(x,y)$ is the kernel

$$k(x,y) = \begin{pmatrix} \chi_{\{y<x\}} & 0 \\ 0 & -\chi_{\{y>x\}} \end{pmatrix} JH(y).$$

Now we consider the mapping ϕ from $L^2_s(H, \mathbb{R}^+)$ into $L^2(\mathbb{R}^+)^2$ given by

$$\phi(f)(t) = H(t)^{\frac{1}{2}} f(t).$$

It is straightforward to show that ϕ is an isometry. Let $K(x,y)$ be as in (2.2) and the operator C defined as in (2.3). By Lemma 2.1 the operator C is a continuous, everywhere defined operator which belongs to the Hilbert-Schmidt class if and only if (2.4) holds true.

We have $C\phi = \phi B$. In fact, $\phi B \subseteq C\phi$ is obvious from the definitions of the respective kernels. For the other inclusion let $f \in \operatorname{dom}(C\phi)$. From (2.2) we see that $C(\phi(f))(x)$ is of the form $H(x)^{\frac{1}{2}} g(x)$ for $g \in L^2(H, \mathbb{R}^+)$ with

$$g(x) = \int_0^{+\infty} k(x,y)f(y)dy =$$

$$\int_0^x JH(t)f(t)dt - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \int_0^{+\infty} JH(t)f(t)dt.$$

It remains to show that $g \in L^2_s(H, \mathbb{R}^+)$. So let (a,b) be an indivisible interval of type $\varphi \in \mathbb{R}$ and calculate for $x \in (a,b)$

$$\frac{d}{dx} \xi_\varphi^T g(x) = \xi_\varphi^T JH(x)f(x) = \xi_\varphi^T J \xi_\varphi \xi_\varphi^T f(x) = 0.$$

Hence $g \in L^2_s(H, \mathbb{R}^+)$, and we proved that $C\phi = \phi B$.

If $g \in L^2(\mathbb{R}^+)^2 \ominus \phi(L^2_s(H, \mathbb{R}^+))$, then

$$(Cg)(x) = H(x)^{\frac{1}{2}} \left(\int_0^x JH(y)^{\frac{1}{2}} g(y)dy - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \int_0^{+\infty} \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T H(y)^{\frac{1}{2}} g(y)dy \right).$$

The second integral vanishes because

$$H(y)^{\frac{1}{2}}\begin{pmatrix}1\\0\end{pmatrix} \in \phi(L^2_s(H,\mathbb{R}^+)).$$

If x is contained in an indivisible interval (a,b) of type $\varphi \in \mathbb{R}$, we assume that (a,b) is maximal, i.e. (a,b) is not contained in a larger indivisible interval. For $a < y < x$ we have

$$H(x)^{\frac{1}{2}}JH(y)^{\frac{1}{2}} = \xi_\varphi\xi_\varphi^T J\xi_\varphi\xi_\varphi^T = 0.$$

Therefore,

$$(Cg)(x) = H(x)^{\frac{1}{2}}\int_0^a JH(y)^{\frac{1}{2}}g(y)dy.$$

Since the columns of the matrix function $\chi_{\{y\leq a\}}H(y)^{\frac{1}{2}}J^T$ belong to $\phi(L^2_s(H,\mathbb{R}^+))$, we see that $Cg = 0$.

We conclude that $\text{dom}(C) = \phi(\text{dom}(B)) \oplus \phi(L^2_s(H,\mathbb{R}^+))^\perp$ and $C = \phi B\phi^{-1}P$, where P is the orthogonal projection onto $\phi(L^2_s(H,\mathbb{R}^+))$. Thus C is a Hilbert-Schmidt operator if and only if B is. □

The previous lemma is of importance if we consider the operator theoretical background of canonical differential equations. Using the notation from the introduction we obtain the subsequent result.

Corollary 2.3. *Let $H(t) = (h_{ij}(t))_{i,j=1,2}$ be a trace normed Hamiltonian on $\mathbb{R}^+$ such that (2.1) holds true. Then the selfadjoint extensions $A(\nu)$, $\nu \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ of $T_{min,s}$ have a resolvent*

$$(A(\nu) - z)^{-1},\ z \in \rho(A(\nu))$$

which is of Hilbert-Schmidt type if and only if (2.4) holds.

Proof. Because of the resolvent identity and since the product of a bounded operator and a Hilbert-Schmidt operator is of Hilbert-Schmidt type, the selfadjoint extension $A(\nu)$ of $T_{min,s}$ has a resolvent which is of Hilbert-Schmidt type if and only if $(A(\nu) - z)^{-1}$ is of Hilbert-Schmidt type for one $z \in \rho(A(\nu))$.

Moreover, by Krein's formula (see Proposition 4.4 in [5]) the resolvents of different selfadjoint extensions are one-dimensional perturbations of each other. Thus, the resolvents of $A(\nu)$, $\nu \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ being of Hilbert-Schmidt type is equivalent to the fact that

$$(A(\frac{\pi}{2}) - z)^{-1} \tag{2.6}$$

is a Hilbert-Schmidt operator for some $z \in \rho(A(\frac{\pi}{2}))$.

Note that by our assumption (2.1)

$$\text{span}\{\begin{pmatrix}1\\0\end{pmatrix}\} = \ker T_{max,s}, \tag{2.7}$$

and hence $\ker A(\frac{\pi}{2}) = \{0\}$. This means that $A(\frac{\pi}{2})^{-1}$ is a densely defined selfadjoint operator on $L^2_s(H,\mathbb{R}^+)$. Moreover, as the elements $f \in \text{dom}\, A(\frac{\pi}{2})$ vanish at 0 in the upper entry we see that $B \subseteq A(\frac{\pi}{2})^{-1}$, where B is defined as in Lemma 2.2.

If (2.4) is satisfied, then B is an everywhere defined, bounded operator belonging to the Hilbert-Schmidt class. We conclude $B = A(\frac{\pi}{2})^{-1}$, and $0 \in \rho(A(\frac{\pi}{2}))$.

For the converse direction assume that (2.6) is of Hilbert-Schmidt type. We are going to show that $A(\frac{\pi}{2})^{-1}$ is an everywhere defined, bounded operator belonging to the Hilbert-Schmidt class. If $z = 0$, we are done. Otherwise, besides zero the spectrum of (2.6) consists of eigenvalues. Since $\ker A(\frac{\pi}{2}) = \{0\}$, the number $-z^{-1}$ cannot be an eigenvalue of (2.6). Because of

$$A(\frac{\pi}{2})^{-1} = \frac{1}{z}(A(\frac{\pi}{2}) - z)^{-1}(\frac{1}{z} + (A(\frac{\pi}{2}) - z)^{-1})^{-1}, \tag{2.8}$$

$A(\frac{\pi}{2})^{-1}$ is an everywhere defined, bounded operator belonging to the Hilbert-Schmidt class.

If $g \in L^2_s(H, \mathbb{R}^+)$ and $f = A(\frac{\pi}{2})^{-1}g$, then $f' = JHg$. It follows from Lemma 7.8, [5] and from (2.7) that $f(x)$ tends to zero in the lower component for $x \to +\infty$. Moreover, $f(0+)$ is zero in the upper component. Thus Bg exists and coincides with $f = A(\frac{\pi}{2})^{-1}g$, and we proved that $B = A(\frac{\pi}{2})^{-1}$. By Lemma 2.2 condition (2.4) holds true. □

The above assertions yield a criterion for a canonical differential equation to have a compact resolvent of Hilbert-Schmidt type. Put

$$\xi_\varphi := \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} .$$

Theorem 2.4. *Let $H(t) = (h_{ij}(t))_{i,j=1,2}$ be a trace normed Hamiltonian on $\mathbb{R}^+$. Then the selfadjoint extensions $A(\nu)$, $\nu \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ of $T_{min,s}$ have a resolvent*

$$(A(\nu) - z)^{-1}, \; z \in \rho(A(\nu)),$$

which is of Hilbert-Schmidt type if and only if there exists a real φ such that

$$\int_0^{+\infty} \xi_\varphi^T H(t)\xi_\varphi dt < +\infty, \tag{2.9}$$

and

$$\int_0^{+\infty} \xi^T_{\varphi+\frac{\pi}{2}} M(t)\xi_{\varphi+\frac{\pi}{2}} \; \xi_\varphi^T H(t)\xi_\varphi dt < +\infty. \tag{2.10}$$

Here $M(x)$ is defined by

$$M(x) = (m_{ij}(x))_{i,j=1,2} = \int_0^x H(t)dt.$$

Proof. First we consider a transformation of the Hamiltonian $H(t)$. For $\mu \in \mathbb{R}$ set

$$N_\mu = \begin{pmatrix} \sin\mu & -\cos\mu \\ \cos\mu & \sin\mu \end{pmatrix},$$

and $H_\mu(t) = N_\mu^* H(t) N_\mu$ (see [5]). It is straightforward to verify that $\psi : f \mapsto N_\mu f$ is an isomorphism from $L^2_s(H, \mathbb{R}^+)$ onto $L^2_s(H_\mu, \mathbb{R}^+)$.

If we denote by $T^\mu_{min,s}$ ($T^\mu_{max,s}$) the respective differential operators on the space $L^2_s(H_\mu,\mathbb{R}^+)$ and by $A^\mu(\nu)$ the corresponding selfadjoint extensions of $T^\mu_{min,s}$, then

$$\psi T_{min,s} = T^\mu_{min,s}\psi,\ \psi T_{max,s} = T^\mu_{max,s}\psi,$$

and $\psi A(\nu) = A^\mu(\frac{\pi}{2}+\nu-\mu)\psi$. Thus, the $A(\nu)$'s have resolvents of Hilbert-Schmidt type if and only if the $A^\mu(\nu)$'s do so.

Condition (2.9) for the given $H(t)$ is equivalent to condition (2.1) when $h_{11}(t)$ now denotes the left upper entry of $H_\mu(t)$ with $\mu = \frac{\pi}{2} - \varphi$. Similarly condition (2.10) for $H(t)$ is equivalent to condition (2.4) when $h_{11}(t), h_{22}(t)$ ($m_{11}(t), m_{22}(t)$) now denote the diagonal entries of $H_\mu(t)$ (the primitive of $H_\mu(t)$) with $\mu = \frac{\pi}{2} - \varphi$.

If there exists a real φ such that (2.9) and (2.10) hold, then by Corollary 2.3 the resolvents of the $A^\mu(\nu)$'s are Hilbert-Schmidt type for $\mu = \frac{\pi}{2} - \varphi$. As we mentioned above this is equivalent to the $A(\nu)$'s having resolvents of Hilbert-Schmidt type.

Conversely, assume that $A(\nu)$, $\nu \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, have resolvents of Hilbert-Schmidt type. If there exists a $\varphi \in \mathbb{R}$ such that $\xi_\varphi \in L^2_s(H,\mathbb{R}^+)$, then we set $\mu = \frac{\pi}{2} - \varphi$ and see that (2.9) holds. We just mentioned that this is the same as saying that (2.1) holds true when $h_{11}(t)$ denotes the left upper entry of $H_\mu(t)$. Since also the $A^\mu(\nu)$'s have resolvents of Hilbert-Schmidt type, Corollary 2.3 shows that (2.4) holds for $H_\mu(t)$ which in turn yields (2.10).

It remains to exclude the case that none of the constant functions ξ_φ belong to $L^2_s(H,\mathbb{R}^+)$. In fact, if we were in this situation, then all of the selfadjoint relations $A(\nu)$ would have a trivial kernel because the scalar multiples of ξ_φ are the only possible candidates for elements of $L^2_s(H,\mathbb{R}^+)$ to belong to $\ker A(\varphi)$.

Thus all $A(\nu)^{-1}$ are operators, and by (2.8) with $A(\frac{\pi}{2})$ replaced by $A(\nu)$ obtain that the operators $A(\nu)^{-1}$ are everywhere defined, bounded and of Hilbert-Schmidt type. Hence the closed one-dimensional restriction $(T_{min,s})^{-1}$ is a bounded, symmetric operator with a closed domain of co-dimension one. It is easy to check that

$$D = (T_{min,s})^{-1} \oplus (\{0\} \times \mathrm{dom}((T_{min,s})^{-1})^\perp)$$

is a selfadjoint extension of $(T_{min,s})^{-1}$. Its inverse would then be a selfadjoint extension of $T_{min,s}$, and would, therefore, coincide with $A(\nu)$ for some ν. But this contradicts the fact that D^{-1} has a non-trivial kernel. □

3. Titchmarsh-Weyl Coefficients and the Hilbert-Schmidt property

Recall that a Nevanlinna function has a unique integral representation

$$a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1}\right) d\sigma(t), \tag{3.1}$$

where $a, b \in \mathbb{R},\ b \geq 0$ and σ is a non-negative Borel measure on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \frac{1}{t^2+1} d\sigma(t) < +\infty.$$

Theorem 3.1. *Let* $H(t) = (h_{ij}(t))_{i,j=1,2}$ *be a trace normed Hamiltonian on* $\mathbb{R}^+$*, and let* $q_H(z)$ *be the Titchmarsh-Weyl coefficient of the canonical differential equation (*1.1*). Assume that (*3.1*) is the integral representation of* $q_H(z)$*.*

Then $H(x)$ *satisfies (*2.9*) and (*2.10*) if and only of* σ *is a discrete measure*

$$\sigma = \sum_n \alpha_n \delta_{t_n}, \tag{3.2}$$

such that

$$\sum_n \frac{1}{t_n^2} < +\infty. \tag{3.3}$$

Proof. Corollary 4.2, [5] shows that $T_{min,s}$ is a completely non-selfadjoint symmetric relation with defect index $(1,1)$, and by Theorem 4.3, [5] the Titchmarsh-Weyl coefficient $q_H(z)$ is the Q-function of $A(\frac{\pi}{2})$ and $T_{min,s}$.

Thus according to Theorem 2.5, [4] the triplet $(L_s^2(H,\mathbb{R}^+), T_{min,s}, A(\frac{\pi}{2}))$ is unitarily equivalent to $(\mathcal{H}, S, A)$, where

$$\mathcal{H} = L^2(\sigma),$$
$$A = \{(\varphi(t); t\varphi(t)) : \varphi(t), t\varphi(t) \in L^2(\sigma)\},$$
$$S = \{(\varphi(t); t\varphi(t)) : \varphi(t), t\varphi(t) \in L^2(\sigma),\ \int_{\mathbb{R}} \varphi(t) d\sigma(t) = 0\},$$

in the case that $b = 0$ in the representation (3.1) of $q_H(z)$ and

$$\mathcal{H} = L^2(\sigma) \oplus \mathbb{C},$$
$$A = \left\{ \left(\begin{pmatrix} \varphi(t) \\ 0 \end{pmatrix}; \begin{pmatrix} t\varphi(t) \\ c \end{pmatrix} \right) : \varphi(t), t\varphi(t) \in L^2(\sigma),\ c \in \mathbb{C} \right\},$$
$$S = \left\{ \left(\begin{pmatrix} \varphi(t) \\ 0 \end{pmatrix}; \begin{pmatrix} t\varphi(t) \\ c \end{pmatrix} \right) : \varphi(t), t\varphi(t) \in L^2(\sigma),\ bc + \int_{\mathbb{R}} \varphi(t) d\sigma(t) = 0 \right\},$$

otherwise. We conclude that $A(\frac{\pi}{2} - z)^{-1}$ is of Hilbert-Schmidt type if and only if the operator

$$\varphi(t) \mapsto \frac{\varphi(t)}{t - z},$$

on $L^2(\sigma)$ is of Hilbert-Schmidt type. This in turn is equivalent to the fact that σ is a discrete measure (3.2) such that (3.3) holds. □

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