Symmetric relations of finite negativity

M. Kaltenbäck, H. Winkler and H. Woracek

Abstract. We construct and investigate a space which is related to a symmetric linear relation S of finite negativity on an almost Pontryagin space. This space is the indefinite generalization of the completion of $\operatorname{dom} S$ with respect to $(S.,.)$ for a strictly positive S on a Hilbert space.

Mathematics Subject Classification (2000). Primary 47B25; Secondary 47B50.

Keywords. Pontryagin space, indefinite scalar product, Friedrichs Extension.

1. Introduction

It is well known that for a symmetric, semibounded and densely defined operator S on a Hilbert space $(\mathfrak{H}, (.,.))$ there exists a distinguished selfadjoint extension, the Friedrichs extension S_F of S. Besides other maximal properties (see e.g. [9],[5]) the Friedrichs extension is distinguished among all semibounded selfadjoint extensions A of S by the fact that $\operatorname{dom}(|A|^{\frac{1}{2}})$ is minimal.

The domain $\operatorname{dom}(|S_F|^{\frac{1}{2}})$ coincides with the closure $\mathfrak{H}_S$ of $\operatorname{dom} S$ with respect to the inner product $h_m^S(.,.) = (S.,.) - m(.,.)$ where $m \in \mathbb{R}$ is sufficiently small. In fact, the usual construction of S_F is done with the help of the space $\mathfrak{H}_S$ (see Section 3).

Later on Friedrichs extensions were generalized for the case of nondensely defined operators or even for the case of symmetric linear relations ([5]). For the concept of linear relations, see for example [1].

The main subject of this note is to generalize the construction of the space $\mathfrak{H}_S$ to the almost Pontryagin space setting and to study the properties of these spaces.

An almost Pontryagin space $(\mathfrak{L}, [.,.], \mathcal{O})$ can be seen as a in general degenerated closed subspace of a Pontryagin space $(\mathfrak{P}, [,.])$, and $\mathcal{O}$ is the subspace topology induced by the Pontryagin space topology of $(\mathfrak{P}, [,.])$ on $\mathfrak{L}$. For an axiomatic treatment of such spaces see [7].

The linear relation S will be assumed to be closed and symmetric on an almost Pontryagin space $(\mathfrak{L}, [.,.], \mathcal{O})$ such that S is contained in its adjoint with finite codimension. Moreover, we assume that the form $h^S[.,.]$, which is $[S.,.]$ for operators S and which is defined accordingly if S is a proper relation, has finitely many negative squares on $\operatorname{dom} S$. Such relations S will be called to be of finite negativity and the resulting space will be denoted by $\mathfrak{L}_S$. We will also provide $\mathfrak{L}_S$ with a Hilbert space topology $\mathcal{O}_S$ such that $(\mathfrak{L}_S, \mathcal{O}_S)$ is continuously embedded in $(\mathfrak{L}, \mathcal{O})$.

In order to construct $\mathfrak{L}_S$ it is not necessary to impose special spectral assumptions on S. In particular, it can happen that S has no points of regular type.

Among other results we will see that $S - \epsilon I$ is of finite negativity for some $\epsilon > 0$ if and only if $(\mathfrak{L}_S, h^S[.,.], \mathcal{O}_S)$ is an almost Pontryagin space. This and other results about symmetries of finite negativity will be of great importance in one of our forthcoming papers about symmetric de Branges spaces ([8]).

In the short Section 2 we will introduce notations used throughout this note in the Hilbert space case as well as in the general almost Pontryagin space setting. In Section 3 we will recall well-known results in the Hilbert space situation and for convenience we will also provide short proofs. In the final section we introduce the proper analogue of the space $\mathfrak{H}_S$ in the almost Pontryagin space case so that we can generalize most of the results from Section 3 to the indefinite case.

2. Symmetric relations on almost Pontryagin spaces

We are going to consider a closed symmetric relation S on an almost Pontryagin space $(\mathfrak{L}, [.,.], \mathcal{O})$, i.e. a closed linear subspace of $\mathfrak{L}^2 = \mathfrak{L} \times \mathfrak{L}$ with the property that

$$[f_1, g_2] - [g_1, f_2] = 0, \ (f_1; g_1), (f_2; g_2) \in S.$$

Remark 2.1. We know from Proposition 3.2 in [7] that any almost Pontryagin space $(\mathfrak{L}, [.,.], \mathcal{O})$ can be viewed as a closed subspace of codimension $\Delta(\mathfrak{L}, [.,.])$ of a Pontryagin space $(\mathfrak{P}, [.,.])$ with degree $\kappa_-(\mathfrak{L}, [.,.]) + \Delta(\mathfrak{L}, [.,.])$ of negativity. Then a linear relation S on $(\mathfrak{L}, [.,.], \mathcal{O})$ is symmetric (closed) if and only if it is symmetric (closed) as a linear relation on $(\mathfrak{P}, [.,.])$.

If, in addition, J is a fundamental symmetry on $(\mathfrak{P}, [.,.])$, then S is symmetric (closed) on $(\mathfrak{L}, [.,.], \mathcal{O})$ if and only if the linear relation JS is a symmetric (closed) relation on the Hilbert space $(\mathfrak{P}, [J.,.])$. This fact is as easily verifiable by the following connection between the adjoint relation $S^{[*]}$ of S in $(\mathfrak{P}, [.,.])$ and the adjoint relation $(JS)^*$ of JS in the Hilbert space $(\mathfrak{P}, [J.,.])$:

$$(JS)^* = JS^{[*]}.$$

Definition 2.2. Let S be a symmetric relation on an almost Pontryagin space $(\mathfrak{L}, [.,.], \mathcal{O})$. We define a scalar product $h^S[.,.]$ on

$$\operatorname{dom} S = \{x \in \mathfrak{L} : (x; y) \in S \text{ for some } y \in \mathfrak{L}\}.$$

For $x, u \in \operatorname{dom} S$ let $y, v \in \mathfrak{L}$ be such that $(x; y), (u; v) \in S$ and set

$$h^S[x, u] = [y, u].$$

This scalar product is well defined and hermitian. In fact, if $\tilde{y} \in \mathfrak{L}$ with $(x;\tilde{y}) \in S$, then the fact that S is symmetric yields

$$h^S[x,u] = [y,u] = [x,v] = [\tilde{y},u],$$

and

$$h^S[x,u] = [y,u] = [x,v] = \overline{[v,x]} = \overline{h^S[u,x]}.$$

Note also that $h^S[x,u] = [Sx,u]$, if S is an operator.

Remark 2.3. If $(\mathfrak{P}, [.,.])$ is a Pontryagin space containing $(\mathfrak{L}, [.,.], \mathcal{O})$ as a closed subspace (see Remark 2.1) and J is a fundamental symmetry on it, then it is straight forward to check that

$$h^S[.,.] = h^{JS}[J.,.]. \tag{2.1}$$

The following little lemma will be of use later on. Hereby an orthogonal projection P in an almost Pontryagin space $(\mathfrak{L}, [.,.], \mathcal{O})$ is an everywhere defined linear operator on $\mathfrak{L}$ which satisfies $P^2 = P$ and $[Px,y] = [x,Py]$ for $x,y \in \mathfrak{L}$.

Lemma 2.4. *Let S be a symmetric relation on an almost Pontryagin space $(\mathfrak{L}, [.,.], \mathcal{O})$. If P is an orthogonal projection in $(\mathfrak{L}, [.,.], \mathcal{O})$ such that $\operatorname{dom}(S) \subseteq P(\mathfrak{L})$, then $h^S[.,.] = h^{PS}[.,.]$.*

Proof. For $(x_1;y_1),(x_2;y_2) \in S$ we have

$$h^S[x_1,x_2] = [y_1,x_2] = [y_1,Px_2] = [Py_1,x_2] = h^{PS}[x_1,x_2].$$

□

3. Semibounded linear relations on Hilbert spaces

In this section we recall some results about semibounded relations on Hilbert spaces which are going to be important for us later on. A symmetric relation S on a Hilbert space is called semibounded if there exists a real number m such that

$$m(x,x) \le h^S(x,x), \text{ for all } x \in \operatorname{dom} S. \tag{3.1}$$

The maximum of all $m \in \mathbb{R}$ such that (3.1) holds true is denoted by $m(S)$ and is called the lower bound of S.

In order to avoid complicated formulas in the sequel we define the scalar product $(m \in \mathbb{R})$

$$h^S_m(.,.) = h^S(.,.) - m(.,.).$$

For $m < m(S)$ the inner product $h^S_m(.,.)$ is a positive definite inner product.

Note further that with S also its closure in $\mathfrak{H}^2$ is semibounded with the same lower bound, i.e. $m(\overline{S}) = m(S)$.

Definition 3.1. Let S be a semibounded relation on a Hilbert space $(\mathfrak{H}, (.,.))$, and let $m < m(S)$. By $\mathfrak{H}_S$ we denote the completion of $\operatorname{dom} S$ with respect to $h^S_m(.,.)$.

The following remarks are more or less explicitly contained in [5].

Remark 3.2. For $m_2 \leq m_1 < m(S)$ and $x \in \operatorname{dom} S$ we have

$$h^S_{m_1}(x,x) = h^S(x,x) - m_1(x,x) \leq h^S(x,x) - m_2(x,x) = h^S_{m_2}(x,x),$$

and

$$\frac{m(S)-m_1}{m(S)-m_2} h^S_{m_2}(x,x) =$$

$$\frac{m(S)-m_1}{m(S)-m_2}(h^S(x,x) - m(S)(x,x)) + (m(S)-m_1)(x,x).$$

As $h^S(x,x) - m(S)(x,x) \geq 0$ and $m(S) - m_1 \leq m(S) - m_2$ this expression is less or equal to

$$(h^S(x,x) - m(S)(x,x)) + (m(S)-m_1)(x,x) = h^S_{m_1}(x,x).$$

Therefore, the topology induced by $h^S_m(.,.)$ on $\operatorname{dom} S$ and, hence, the Hilbert space $\mathfrak{H}_S$ does not depend on the choice of $m < m(s)$.

By Lemma 2.4 with $h^S_m(.,)$ also $\mathfrak{H}_S$ remains unaltered if we switch from S to PS where P is an orthogonal projection onto a subspace of $\mathfrak{H}$ which contains $\operatorname{dom} S$, i.e. $\mathfrak{H}_S = \mathfrak{H}_{PS}$.

Since $((a;b);(x;y)) \mapsto h^S_m(a,x)$ is continuous with resect to the graph norm, we have $\mathfrak{H}_S = \mathfrak{H}_{\overline{S}}$.

Remark 3.3. For $m < m(S)$ and $x \in \operatorname{dom} S$ we have

$$(m(S)-m)(x,x) \leq h^S(x,x) - m(S)(x,x) + (m(S)-m)(x,x) = h^S_m(x,x).$$

Thus by continuity one can extend $(.,.)$ to $\mathfrak{H}_S$. Having done this we can define $h^S_l(.,.)$ on $\mathfrak{H}_S$ for all $l \in \mathbb{R}$ by

$$h^S_l(.,.) = h^S_m(.,.) + (m-l)(.,.).$$

Clearly, $h^S_l(.,.)$ is the unique extension by continuity of the originally on $\operatorname{dom} S$ defined scalar product $h^S_l(.,.)$.

Remark 3.4. From Remark 3.3 we conclude that the embedding

$$\iota : (\operatorname{dom} S, h^S_m(.,.)) \to (\mathfrak{H}, (.,.))$$

is bounded and can therefore be continued to a bounded mapping $\iota : (\mathfrak{H}_S, h^S_m(.,.)) \to (\mathfrak{H}, (.,.))$. The latter operator is in fact an embedding. For if $\iota(x) = 0$, then let $x_n \in \operatorname{dom} S$ converge to x within $\mathfrak{H}_S$. By continuity $\iota(x_n) = x_n \to 0$ within $\mathfrak{H}$. For $(a;b) \in S$ we have

$$h^S_m(a,x) = \lim_{n\to\infty} h^S_m(a,x_n) = \lim_{n\to\infty} (h^S(a,x_n) - m(a,x_n)) =$$

$$\lim_{n\to\infty} ((b,x_n) - m(a,x_n)) = 0,$$

and, hence, x is orthogonal to $\operatorname{dom} S$ within $\mathfrak{H}_S$ which yields $x = 0$.

As a consequence of the injectivity of ι we can consider $\mathfrak{H}_S$ as a linear subspace of $\mathfrak{H}$ where $x \in \mathfrak{H}$ belongs to $\mathfrak{H}_S$ if there exists a sequence $((x_n;y_n))$ in S such that

$$\lim_{n\to\infty}(x-x_n, x-x_n) = 0, \quad \lim_{k,l\to\infty}(x_k - x_l, y_k - y_l) = 0. \tag{3.2}$$

Finally, it is elementary to see that for $x \in \mathfrak{H}_S$ and $(a;b) \in S$ we have

$$h_m^S(a,x) = (b - ma, x).$$

We will use this fact without giving explicit references.

The space $\mathfrak{H}_S$ is used to define the Friedrichs extension of S as defined in [5]. The following way to introduce the Friedrichs extension is slightly different from the conventional access and is closely connected to the constructions given in [10],[11] and [12]. See also [2].

Theorem 3.5. *Let S be a symmetric and semibounded linear relation on the Hilbert space $(\mathfrak{H}, (.,.))$. Let $m < m(S)$ and consider the Hilbert space $(\mathfrak{H}_S, h_m^S(.,.))$ and the embedding*

$$\iota : (\mathfrak{H}_S, h_m^S(.,.)) \to (\mathfrak{H}, (.,.)).$$

Then the linear relation $S_F = (\iota\iota^*)^{-1} + mI$ is a selfadjoint and semibounded extension of S with $m(S_F) = m(S)$. Moreover, it does not depend on the particularly chosen $m < m(S)$. In fact,

$$S_F = \{(x;y) \in S^* : x \in \mathfrak{H}_S\}. \tag{3.3}$$

Proof. Clearly, $\iota\iota^*$ is a selfadjoint and bounded linear operator on $\mathfrak{H}$. Using standard arguments about linear relations we see that $(\iota\iota^*)^{-1}$ is a selfadjoint linear relation. Since for $y \in \operatorname{dom}(\iota\iota^*)^{-1} = \operatorname{ran}\iota\iota^*$ with $\iota\iota^* x = y$ we have

$$h^{(\iota\iota^*)^{-1}}(y,y) = (x,y) = h_m^S(\iota^* x, \iota^* x) \geq 0, \tag{3.4}$$

this relation is semibounded with a non-negative lower bound. With $(\iota\iota^*)^{-1}$ also S_F is selfadjoint and semibounded. If $(a;b) \in S - mI$ and $u \in \operatorname{dom} S$, then $(a; b + ma) \in S$ and $\iota(u) = u$ because we identify $\mathfrak{H}_S$ with a subspace of $\mathfrak{H}$. Therefore

$$h_m^S(a,u) = (b + ma, u) - m(a,u) = (b,u) = (b, \iota(u)) = h_m^S(\iota^* b, u),$$

and we obtain from the density of $\operatorname{dom} S$ in $\mathfrak{H}_S$ that $a = \iota^* b = \iota\iota^* b$. This proves $S \subseteq S_F$, and by the selfadjointness of S_F we see that S_F is contained in the right hand side of (3.3). Conversely, if $(x;y) \in S^* - mI$ and $x \in \mathfrak{H}_S$, let (x_n) be a sequence in $\operatorname{dom} S$ which converges to x within $\mathfrak{H}_S$ and, hence, also within $\mathfrak{H}$. We calculate for $(u;v) \in S$

$$h_m^S(u, \iota^*(y)) = (\iota(u), y) = (u,y) = (v,x) - m(u,x) =$$
$$\lim_{n\to\infty}((v,x_n) - m(u,x_n)) = \lim_{n\to\infty} h_m^S(u,x_n) = h_m^S(u,x),$$

and obtain $\iota\iota^*(y) = x$. Thus we verified (3.3) which, in turn, together with Remark 3.2 implies the independence of S_F from $m < m(S)$.

Finally, from $m((\iota\iota^*)^{-1}) \geq 0$ we get $m(S_F) \geq m$ and from the independence of S_F from $m < m(S)$ the relation $m(S_F) \geq m(S)$. The converse inequality is an immediate consequence of $S \subseteq S_F$. □

Definition 3.6. The selfadjoint linear relation S_F is called the Friedrichs extension of S.

Remark 3.7. It is easy to see that $\mathfrak{H}_{S+rI} = \mathfrak{H}_S$ and $(S+rI)_F = S_F + rI$ for $r \in \mathbb{R}$.

With the notation from the proof of Theorem 3.5 we have

$$S_F(0) = (\iota^*)^{-1}(0) = \ker \iota^* = (\operatorname{dom} S)^\perp.$$

Remark 3.8. First note that since S has a selfadjoint extension any closed, symmetric and semibounded relation has equal defect indices, i.e. the Hilbert space dimension of $\ker(S^* - zI)$ is the same for all $z \in r(S)$ where $r(S)$ $(\supseteq \mathbb{C} \setminus \mathbb{R})$ is the set of all points of regular type for S.

For $m < m(S) = m(S_F)$ and $(x; y) \in S_F$ we have

$$\|x\| \|y - mx\| \geq (y - mx, x) \geq (m(S) - m)(x, x).$$

We conclude $m \in \rho(S_F)$ and

$$\|(S_F - mI)^{-1}\| \leq \frac{1}{m(S) - m}. \tag{3.5}$$

Therefore $\mathbb{C} \setminus [m(S), \infty) \subseteq \rho(S_F)$ and, hence, $\mathbb{C} \setminus [m(S), \infty) \subseteq r(S)$.

The fact that $(-\infty, m(S)) \subseteq \rho(S_F)$ can also be seen from the proof of Theorem 3.5. In fact, if we provide $\mathfrak{H}_S$ with $h_m^S(.,.)$, $m < m(S)$, then we constructed S_F such that $(S_F - mI)^{-1}$ is the bounded operator ι^*.

We are going to consider arbitrary selfadjoint and semibounded extensions H of S in $\mathfrak{H}$ and for $m < m(H)$ the relation between the Hilbert spaces $(\mathfrak{H}_H, h_m^H(.,.))$ and $(\mathfrak{H}_S, h_m^S(.,.))$. This well-known result is strongly connected with the second representation theorem from Kato, [9]. See also Chapter 10 of [3].

Theorem 3.9. *Let S be semibounded on the Hilbert space $(\mathfrak{H}, (.,.))$ and H be a selfadjoint and semibounded extension of S. Moreover, let $H = H_s \oplus H_\infty$ be the decomposition of H into the purely relational part $H_\infty = \{0\} \times H(0)$ and the operator part H_s, which is a selfadjoint operator on $H(0)^\perp$.*

Then the space $\mathfrak{H}_H$ as a subspace of $\mathfrak{H}$ coincides with $\operatorname{dom} |H_s|^{\frac{1}{2}}$, and for $m < m(H)$ the Hilbert space inner product $h_m^H(.,.)$ can be calculated as

$$h_m^H(x, y) = ((H_s - mI)^{\frac{1}{2}} x, (H_s - mI)^{\frac{1}{2}} y), \ x, y \in \mathfrak{H}_H. \tag{3.6}$$

The space $\mathfrak{H}_H$ contains $\mathfrak{H}_S$ as a closed subspace, and on this closed subspace the products $h_m^H(.,.)$ and $h_m^S(.,.)$ coincide. If $\mathfrak{H}_H$ is provided with $h_m^H(.,.)$, then

$$\mathfrak{H}_H \ominus \mathfrak{H}_S = \mathfrak{H}_H \cap \ker(S^* - mI). \tag{3.7}$$

We have $\mathfrak{H}_H = \mathfrak{H}_S$ if and only if $H = S_F$.

Proof. The assumption $S \subseteq H$ immediately yields $h_m^H(.,.) = h_m^S(.,.)$ on $\operatorname{dom} S$. Thus the completion $\mathfrak{H}_S$ of $\operatorname{dom} S$ with respect to $h_m^S(.,.)$ is a closed subspace of $\mathfrak{H}_H$.

Since H is semibounded and $m < m(H)$, the selfadjoint operator $H_s - mI$ is strictly positive on $H(0)^\perp$. Therefore, we can consider the square root of it. For $x, y \in \operatorname{dom} H_s = \operatorname{dom} H$ we have $(x; H_s x), (y; H_s y) \in H_s$, and hence

$$h_m^H(x, y) = (H_s x, y) - m(x, y) = ((H_s - mI)x, y) = ((H_s - m)^{\frac{1}{2}} x, (H_s - m)^{\frac{1}{2}} y).$$

Using the boundedness of $(H_s - mI)^{-1}$ we see that the norm induced by $h_m^H(.,.)$ is equivalent to the graph norm of $(H_s - mI)^{\frac{1}{2}}$ on $\operatorname{dom} H_s$. By the functional calculus for selfadjoint operators $\operatorname{dom}(H_s - mI)^{\frac{1}{2}} = \operatorname{dom}|H_s|^{\frac{1}{2}}$, and $\operatorname{dom} H_s$ is dense in $\operatorname{dom}(H_s - mI)^{\frac{1}{2}}$ with respect to the the graph norm of $(H_s - mI)^{\frac{1}{2}}$. Thus $\mathfrak{H}_H = \operatorname{dom}|H_s|^{\frac{1}{2}}$, and relation (3.6) extends to all $x, y \in \mathfrak{H}_H$.

If $H = S_F$, we obtain from (3.3) that $\operatorname{dom} S_F \subseteq \mathfrak{H}_S$. As we already identified $\mathfrak{H}_S$ as a subspace of $\mathfrak{H}_H$ we get $\mathfrak{H}_H = \mathfrak{H}_S$. Conversely, if we assume $\mathfrak{H}_H = \mathfrak{H}_S$, then by definition $\operatorname{dom} H \subseteq \mathfrak{H}_H$ and hence

$$H \subseteq \{(x; y) \in S^* : x \in \mathfrak{H}_H\} = \{(x; y) \in S^* : x \in \mathfrak{H}_S\} = S_F.$$

As both relations are selfadjoint we obtain $S_F = H$. To verify (3.7) note that for $x \in \mathfrak{H}_H$ and $(a; b) \in S$

$$h_m^H(a, x) = ((H_s - m)^{\frac{1}{2}} a, (H_s - m)^{\frac{1}{2}} x) = ((H_s - m)a, x) = (b - ma, x).$$

The final equality follows from $H_s a - b \in H(0)$ and the fact that

$$\mathfrak{H}_H = \operatorname{dom}|H_s|^{\frac{1}{2}} \perp H(0).$$

Thus $x \in \mathfrak{H}_H \ominus \mathfrak{H}_S$ if and only if $x \in \operatorname{ran}(S - mI)^{\perp} = \ker(S^* - mI)$. □

Remark 3.10. If we choose $H = S_F$ in (3.7), then we see that $\mathfrak{H}_S$ is disjoint to $\ker(S^* - mI)$ for all $m < m(S)$.

Remark 3.11. If S is closed with finite defect indices, then any selfadjoint extension H of S in $\mathfrak{H}$ is a finite dimensional perturbation of S_F. Hence every canonical selfadjoint extension is semibounded. Hereby canonical means that H is a selfadjoint extension within $\mathfrak{H}$.

Moreover, by Theorem 3.9 any space $\mathfrak{H}_H$ contains $\mathfrak{H}_S$ and is contained in $\mathfrak{H}_S \dot{+} \ker(S^* - mI)$. We are going to show that any linear space $\mathfrak{G}$ with $\mathfrak{H}_S \subseteq \mathfrak{G} \subseteq \mathfrak{H}_S \dot{+} \ker(S^* - mI)$ equals a space $\mathfrak{H}_H$ for some H.

From now on we assume that S is a closed, symmetric and semibounded linear relation with finite defect indices.

Remark 3.12. As already mentioned the space $\mathfrak{H}_S \dot{+} \ker(S^* - mI)$ is of particular interest for $m < m(S)$. If $z \in \rho(S_F)$, we have

$$\mathfrak{H}_S \dot{+} \ker(S^* - zI) = \mathfrak{H}_S + \operatorname{dom} S^*. \tag{3.8}$$

As $(-\infty, m(S)) \subseteq \rho(S_F)$ we conclude that $\mathfrak{H}_S \dot{+} \ker(S^* - mI)$ does not depend on $m < m(S)$.

To verify (3.8) recall that for $z, w \in \rho(S_F)$ the operator

$$I + (z - w)(S_F - z)^{-1},$$

maps $\ker(S^* - wI)$ bijectively onto $\ker(S^* - zI)$. Since $\operatorname{dom} S_F \subseteq \mathfrak{H}_S$ (Theorem 3.9), we see that the space on the left hand side of the equality sign in (3.8) is independent from $z \in \rho(S_F)$. The relation (3.8) is now an immediate consequence of the von Neumann formula (see e.g. Theorem 6.1 in [6]).

Definition 3.13. By $\mathfrak{H}^S$ we denote the space in (3.8).

Proposition 3.14. *Assume that S is a closed, symmetric and semibounded linear relation with finite defect indices. Let $\mathfrak{G}$ be a subspace of $\mathfrak{H}^S$ which contains $\mathfrak{H}_S$. Then there exists a canonical selfadjoint extension H of S such that $\mathfrak{H}_H = \mathfrak{G}$.*

Proof. We provide $\mathfrak{G}$ with a Hilbert space inner product $h_m^{\mathfrak{G}}(.,.)$ which extends $h_m^S(.,.)$, $m < m(S)$, such that

$$\mathfrak{G} = \mathfrak{H}_S \oplus_{h_m^{\mathfrak{G}}(.,.)} (\ker(S^* - mI) \cap \mathfrak{G}). \tag{3.9}$$

As $\dim \ker(S^* - mI) < \infty$ the Hilbert space $(\mathfrak{G}, h_m^{\mathfrak{G}}(.,.))$ is continuously embedded in $\mathfrak{H}$, and we denote by $\iota_{\mathfrak{G}}$ the corresponding inclusion map.

Similar as for ι in the proof of Theorem 3.5 we see that $(\iota_{\mathfrak{G}}\iota_{\mathfrak{G}}^*)^{-1}$ is a semibounded selfadjoint linear relation with a non-negative lower bound. Then also $H := (\iota_{\mathfrak{G}}\iota_{\mathfrak{G}}^*)^{-1} + mI$ is a semibounded selfadjoint linear relation.

If $(a;b) \in S - mI$ and $u = u_1 + u_2 \in \operatorname{dom} S \dot{+} (\ker(S^* - mI) \cap \mathfrak{G})$, then $(a; b + ma) \in S$ and $\iota_{\mathfrak{G}}(u) = u$ as we identify $\mathfrak{G}$ with a linear subspace of $\mathfrak{H}$. As $\ker(S^* - mI) = \operatorname{ran}(S - mI)^\perp$

$$h_m^{\mathfrak{G}}(a, u) = h_m^S(a, u_1) = (b + ma, u_1) - m(a, u_1) =$$
$$(b, u_1) = (b, u) = (b, \iota_{\mathfrak{G}}(u)) = h_m^{\mathfrak{G}}(\iota_{\mathfrak{G}}^* b, u),$$

and we obtain from the density of $\operatorname{dom} S \dot{+} (\ker(S^* - mI) \cap \mathfrak{G})$ in $\mathfrak{G}$ that $a = \iota_{\mathfrak{G}}\iota_{\mathfrak{G}}^* b$. Thus we verified $S \subseteq H$.

Since $\iota_{\mathfrak{G}}$ is injective, its adjoint has a dense range in $\mathfrak{G}$. This range clearly coincides with $\operatorname{dom} H$. Moreover,

$$h_m^H(a, x) = (b - ma, x) = (b - ma, \iota_{\mathfrak{G}}\iota_{\mathfrak{G}}^*(y - mx)) =$$
$$h_m^{\mathfrak{G}}(\iota_{\mathfrak{G}}^*(b - ma), \iota_{\mathfrak{G}}^*(y - mx)) = h_m^{\mathfrak{G}}(a, x),$$

for $(a;b), (x;y) \in H$, and hence $\mathfrak{H}_H = \mathfrak{G}$. □

As an immediate consequence of the previous results we obtain

Corollary 3.15. *With the same assumptions as in Proposition* 3.14 *the space $\mathfrak{H}^S$ contains $\mathfrak{H}_H$ for all canonical selfadjoint extensions H of S, and for some canonical selfadjoint extensions H of S we have $\mathfrak{H}^S = \mathfrak{H}_H$.*

Let $\mathfrak{G}$ be such that $\mathfrak{H}_S \subseteq \mathfrak{G} \subseteq \mathfrak{H}^S$, and let $\mathfrak{G}$ be provided with a Hilbert space scalar product $h_m^{\mathfrak{G}}(.,.)$ which coincides with $h_m^S(.,.)$ on $\mathfrak{H}_S$ such that (3.9) holds. We denote by P the orthogonal projection of $\mathfrak{G}$ onto $\mathfrak{H}_S$. Now we set

$$T = S \cap (\mathfrak{G} \times \mathfrak{G}).$$

Proposition 3.16. *Under the above assumptions the linear relation T considered in $(\mathfrak{G}, h_m^{\mathfrak{G}}(.,.))$ is closed, symmetric and semibounded with a lower bound larger than m.*

It is of defect index (r, r) with $r \le n$. If $\mathfrak{H}$ satisfies the minimality condition

$$\mathfrak{H} = \operatorname{cls}(\operatorname{dom} S \cup \operatorname{ran} S), \tag{3.10}$$

then $r = n$.

Proof. The closedness is an immediate consequence of the boundedness of the inclusion map $\iota_{\mathfrak{G}}$. For $(a;b),(x;y)\in T$ we have $Pa=a,\ Px=x$. Using $\ker(S^*-mI)\perp_{(.,.)}\operatorname{ran}(S-mI)$, the fact that T is symmetric follows from

$$h_m^{\mathfrak{G}}(a,y)=h_m^S(a,Py)=(b-ma,Py)=(b-ma,y)=(b,y-mx)=h_m^{\mathfrak{G}}(b,x).$$

For later use we point out that more generally we have for $(a;b)\in S,\ y\in\mathfrak{G}$

$$h_m^{\mathfrak{G}}(a,y)=h_m^S(a,Py)=(b-ma,Py)=(b-ma,y). \tag{3.11}$$

As

$$h_m^{\mathfrak{G}}(a,b)=(b-ma,b)=(b-ma,b-ma)+m(b-ma,a)=\\ (b-ma,b-ma)+mh_m^{\mathfrak{G}}(a,a),$$

T is semibounded with a lower bound larger or equal to m. For $\epsilon>0, m+\epsilon<m(S)$ we obtain from (3.5)

$$(b-ma,b-ma)=(b-(m+\epsilon)a,b-(m+\epsilon)a)+2\epsilon(b-(m+\epsilon)a,a)+\epsilon^2(a,a)=\\ \|b-(m+\epsilon)a\|^2+2\epsilon h_m^{\mathfrak{G}}(a,a)-\epsilon^2(a,a)\geq\\ (m(S)-(m+\epsilon)-\epsilon^2)\|a\|^2+2\epsilon h_m^{\mathfrak{G}}(a,a).$$

For sufficiently small ϵ we get

$$h_m^{\mathfrak{G}}(a,b)\geq(m+2\epsilon)h_m^{\mathfrak{G}}(a,a),$$

and therefore $m(T)>m$.

As $\operatorname{dom}S\subseteq\mathfrak{H}_S\subseteq\mathfrak{G}$ we have for $z\in r(T)$,

$$\operatorname{ran}(T-zI)=\operatorname{ran}(S-zI)\cap\mathfrak{G}=\\ \{x\in\mathfrak{G}:(\iota_{\mathfrak{G}}(x),y)=0,\ y\in\ker(S^*-\bar{z}I)\}=\\ (\iota_{\mathfrak{G}}^*\ker(S^*-\bar{z}I))^{\perp_{h_m^{\mathfrak{G}}(.,.)}}.$$

Therefore, T has defect index (r,r) where $r\leq n$.

If $r<n$, then $\iota_{\mathfrak{G}}^*(y)=0$ for some $y\in\ker(S^*-\bar{z}I),\ y\neq0$. From $y\in\ker\iota_{\mathfrak{G}}^*=(\operatorname{ran}\iota_{\mathfrak{G}})^\perp\subseteq S^*(0)$ we conclude $y\in\ker(S^*)$. Hence, condition (3.10) cannot be satisfied. □

As a consequence of the previous proof note that

$$\iota_{\mathfrak{G}}^*(\ker(S^*-mI))=\ker(T^*-mI),$$

where this correspondence between the defect spaces is bijective if (3.10) holds true. On $\operatorname{ran}(S-mI)=\ker(S^*-mI)^\perp$ we have $(x\in\mathfrak{G})$

$$h_m^{\mathfrak{G}}(\iota_{\mathfrak{G}}^*(b-ma),x)=(b-ma,x)=h_m^{\mathfrak{G}}(a,x).$$

Hence, $\iota_{\mathfrak{G}}^*(b-ma)=a$.

In the following, $h^Th_m^{\mathfrak{G}}(.,.)$ is the scalar product and $\mathfrak{G}_T$ is the space constructed from $\mathfrak{G},h_m^{\mathfrak{G}}(.,.),T$ in the same as $h^S(.,.)$ and $\mathfrak{H}_S$ were constructed from $\mathfrak{H},(.,.),S$.

As already noted we have for $(a;b),(x;y)\in T$

$$h^Th_m^{\mathfrak{G}}(a,x)=h_m^{\mathfrak{G}}(a,y)=h_m^{\mathfrak{G}}(a,Py)=(b-ma,Py)=$$

$$(b-ma,y-mx)+m(b-ma,x)=(b-ma,y-mx)+m(h^{\mathfrak{G}}(a,x)-m(a,x))=$$
$$(b-ma,y-mx)+mh_m^{\mathfrak{G}}(a,x),$$

and hence $h_m^T h_m^{\mathfrak{G}}(a,x)=(b-ma,y-mx)$.

Proposition 3.17. *With the above assumptions and notations $\iota_{\mathfrak{G}}^*$ maps $(\overline{\mathfrak{G}}\cap\operatorname{ran}(S-mI),(.,.))$ unitarily onto $(\mathfrak{G}_T,h_m^T h_m^{\mathfrak{G}}(.,.))$, where $\mathfrak{G}_T$ coincides with $\operatorname{dom}(S\cap(\overline{\mathfrak{G}}\times\overline{\mathfrak{G}}))$ and $h_m^T h_m^{\mathfrak{G}}(.,.)$ induces the graph norm on $\mathfrak{G}_T$.*

If we denote by R the symmetry $T\cap(\mathfrak{G}_T\times\mathfrak{G}_T)$ on $(\mathfrak{G}_T,h_m^T h_m^G(.,.))$, then

$$((\iota_{\mathfrak{G}}^*)^{-1}\times(\iota_{\mathfrak{G}}^*)^{-1})(R)=S\cap((\overline{\mathfrak{G}}\cap\operatorname{ran}(S-mI))\times(\overline{\mathfrak{G}}\cap\operatorname{ran}(S-mI))).$$

Proof. For the proof we first mention that the fact that $\operatorname{ran}(S-mI)$ has finite codimension in $\mathfrak{H}$ ensures

$$\overline{\mathfrak{G}\cap\operatorname{ran}(S-mI)}=\overline{\mathfrak{G}}\cap\operatorname{ran}(S-mI).$$

As

$$h_m^T h_m^G(\iota_{\mathfrak{G}}^*(b-ma),\iota_{\mathfrak{G}}^*(y-mx))=h_m^T h_m^G(a,x)=(b-ma,y-mx), \tag{3.12}$$

we see that $\iota_{\mathfrak{G}}^*|_{\operatorname{ran}(S-mI)}=(S-mI)^{-1}$ maps $\operatorname{ran}(T-mI)$ unitarily onto $\operatorname{dom}T$. By continuity $\iota_{\mathfrak{G}}^*|_{\operatorname{ran}(S-mI)}=(S-mI)^{-1}$ then maps $(\overline{\mathfrak{G}}\cap\operatorname{ran}(S-mI),(.,.))$ unitarily onto $(\mathfrak{G}_T,h_m^T h_m^{\mathfrak{G}}(.,.))$. Thus

$$\mathfrak{G}_T=(S-mI)^{-1}(\overline{\mathfrak{G}}\cap\operatorname{ran}(S-mI))=\operatorname{dom}(S\cap(\overline{\mathfrak{G}}\times\overline{\mathfrak{G}})).$$

The continuity of $(S-mI)^{-1}$ together with (3.12) shows that $h_m^T h_m^{\mathfrak{G}}(.,.)$ induces the graph norm on $\mathfrak{G}_T$.

For $x,y\in\overline{\mathfrak{G}}\cap\operatorname{ran}(S-mI)$ we have $(x;y)\in S$ if and only if $x=(H-m)^{-1}(y-mx)$, where H is the selfadjoint extension $(\iota_{\mathfrak{G}}\iota_{\mathfrak{G}}^*)^{-1}+mI$ of S (see Proposition 3.14). As $\iota_{\mathfrak{G}}(\mathfrak{G})^\perp=\ker\iota_{\mathfrak{G}}^*=H(0)$ this is equivalent to $(H-m)^{-2}(y-mx)=(H-m)^{-1}x$ or because of $(H-m)^{-1}y-m(H-m)^{-1}x=x\in\operatorname{ran}(S-m)$ in turn equivalent to

$$(\iota_{\mathfrak{G}}^*x;\iota_{\mathfrak{G}}^*y)=((H-m)^{-1}x;(H-m)^{-1}y)\in S\cap(\mathfrak{G}_T\times\mathfrak{G}_T)=R.$$

□

Thus we showed that for a closed and semibounded symmetry S with finite defect index (n,n) one can partially reconstruct $\mathfrak{H}$ and S from $\mathfrak{H}^S$ and T by focusing on $\overline{\mathfrak{G}}\cap\operatorname{ran}(S-mI)$.

4. Symmetric relations of finite negativity

Definition 4.1. Let $(\mathfrak{L},[.,.],\mathcal{O})$ be an almost Pontryagin space, and let S be a closed symmetric relation on $\mathfrak{L}$ such that S has finite codimension in

$$S^{[*]}=\{(a;b)\in\mathfrak{L}\times\mathfrak{L}:[a,y]=[b,x]\text{ for all }(x;y)\in S\}.$$

Then S is called to be of finite negativity κ_S in $(\mathfrak{L},[.,.],\mathcal{O})$ if the inner product $h^S[.,.]$ has κ_S negative squares on $\operatorname{dom}S$. If $\kappa_S=0$, we shall call S non-negative.

By well-known results in the theory of inner product spaces (see e.g. [4]) $h^S[.,.]$ has finitely many negative squares if and only if there exists a linear subspace of $\operatorname{dom} S$ of finite codimension such that $h^S[.,.]$ restricted to this subspace is positive semidefinite. Moreover, $h^S[.,.]$ has κ_S negative squares on $\operatorname{dom} S$ if and only if there exists a κ_S-dimensional subspace $\mathfrak{N}$ of $\operatorname{dom} S$ such that $(\mathfrak{N}, -h^S[.,.])$ is a Hilbert space, and there is no higher dimensional subspace of $\operatorname{dom} S$ with this property. In this case we can decompose $\operatorname{dom} S$ as

$$\operatorname{dom} S = \mathfrak{M} \dot{+} \mathfrak{N},$$

where $\mathfrak{M}$ is the orthogonal complement of $\mathfrak{N}$ with respect to $h^S[.,.]$, and $h^S[.,.]$ is non-negative on $\mathfrak{M}$.

Remark 4.2. It is easy to see that S is of finite negativity κ_S in $(\mathfrak{L}, [.,.], \mathcal{O})$, if and only if it is of finite negativity κ_S as a relation on a Pontryagin space $(\mathfrak{P}, [.,.])$ containing $(\mathfrak{L}, [.,.], \mathcal{O})$ as a closed subspace with finite codimension (see Remark 2.1).

If J is a fundamental symmetry of $(\mathfrak{P}, [.,.])$, then we see from (2.1) that S is of finite negativity κ_S in $(\mathfrak{L}, [.,.], \mathcal{O})$ if and only if JS is of finite negativity κ_S in the Hilbert space $(\mathfrak{P}, [J.,.])$.

Thus certain questions related to symmetries with finite negativity can be considered in a Hilbert space setting. There symmetries have the following important property.

Lemma 4.3. *Every symmetric relation of finite negativity on a Hilbert space is semibounded. Moreover,* $\operatorname{ran}(S - mI)$ *is closed and of finite codimension for all* $m < 0$.

Proof. Let S be a symmetry in a Hilbert space $(\mathfrak{H}, (.,.))$ of finite negativity κ_S. Now we consider $\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{H}$ with the symmetric relation $T = S \oplus S^{-1}$ on it. As $T^* = S^* \oplus S^{-1*}$ it is straightforward to check that T is of finite negativity $2\kappa_S$ and that T has finite and equal defect indices.

Let A be a canonical selfadjoint extension of T in $\mathfrak{G}$. Since $\operatorname{dom} S \subseteq \operatorname{dom} A$ with finite codimension, also A is of finite negativity. Using the functional calculus for selfadjoint relations we derive from this fact that $\sigma(A) \cap (-\infty, 0)$ consists of finitely many eigenvalues of finite multiplicity. The proof for this assertion is very similar to the proof of Proposition 2.3 in [7] and is therefore omitted.

So we see that A and with A also its restriction S is semibounded. From the mentioned spectral properties for A we also see that $\operatorname{ran}(A - mI)$ is closed and of finite codimension for $m < 0$. The mapping $(x; y) \mapsto y - mx$ from A onto $\operatorname{ran}(A - mI)$ is continuous and has a finite dimensional kernel. Hence the closed subspace T of A is mapped onto a closed subspace of $\operatorname{ran}(A - mI)$ of finite codimension. The structure of T shows that $\operatorname{ran}(S - mI)$ is closed and of finite codimension. □

Due to the previous lemma we can define a space associated to a symmetry of finite negativity.

Definition 4.4. Let $(\mathfrak{L}, [.,.], \mathcal{O})$ be an almost Pontryagin space, and let S be a symmetric relation of finite negativity on $(\mathfrak{L}, [.,.], \mathcal{O})$. Moreover, let $(\mathfrak{P}, [.,.])$ be a Pontryagin space which contains $(\mathfrak{L}, [.,.], \mathcal{O})$ as a closed subspace of finite codimension, and let J be a fundamental symmetry on this Pontryagin space. Then we define the space $\mathfrak{L}_S$ by

$$\mathfrak{L}_S = \mathfrak{P}_{JS},$$

where $\mathfrak{P}_{JS}$ is the space corresponding to the symmetry JS on the Hilbert space $(\mathfrak{P}, [J.,.])$ defined as in Definition 3.1.

We provide $\mathfrak{L}_S$ with the inner product $h^{JS}[J.,.]$ and denote it by $h^S[.,.]$ (see Remark 3.3). Moreover, let $\mathcal{O}_S$ denote the Hilbert space topology induced by $h_m^{JS}[J.,.],\ m < m(JS)$ on $\mathfrak{L}_S$.

Remark 4.5. By Remark 3.4 $\mathfrak{P}_{JS}$ is continuously embedded in $\mathfrak{P}$. Denoting the inclusion mapping by ι its continuity yields

$$\iota(\mathfrak{P}_{JS}) = \iota(\overline{\operatorname{dom} JS}) = \iota(\overline{\operatorname{dom} S}) \subseteq \overline{\operatorname{dom} S} \subseteq \mathfrak{L}.$$

Hereby the latter closure is taken with respect to the topology $\mathcal{O}$ (which coincides with the topology induced by $[J.,.]$, see [7]) and the others are taken with respect to $\mathcal{O}_S$.

Thus $\mathfrak{L}_S$ is a linear subspace of $\mathfrak{L}$. Moreover, it is independent from the fundamental symmetry J and even from the space $\mathfrak{P}$. For by (3.2) a vector $x \in \mathfrak{L}$ belongs to $\mathfrak{L}_S$ if and only if there exists a sequence $((x_n; y_n))$ in S such that $x_n \to x$ with respect to $\mathcal{O}$ and

$$\lim_{k,l\to\infty} [x_k - x_l, y_k - y_l] = 0.$$

This characterization also shows that $\mathfrak{L}_S = \mathfrak{L}_{S-mI}$ whenever $S - mI$ is of finite negativity.

By the closed graph theorem and by the fact that ι is continuous the topology $\mathcal{O}_S$ is also independent from J and from $\mathfrak{P}$.

Finally, the $\mathcal{O}_S$-continuous scalar product $h^S[.,.]$ (on $\mathfrak{L}_S$) restricted to the the $\mathcal{O}_S$-dense linear subspace $\operatorname{dom} S$ coincides with $h^S[.,.]$ as it was defined in Definition 2.2. Hence $h^S[.,.]$ on $\mathfrak{L}_S$ is the unique continuation of $h^S[.,.]$ on $\operatorname{dom} S$ by continuity. Therefore, also $h^S[.,.]$ is independent from J and from $\mathfrak{P}$.

Remark 4.6. With the same assumptions as in Definition 4.4 let $\mathfrak{M}$ be a closed subspace of $(\mathfrak{L}, [.,.], \mathcal{O})$ such that $S \subseteq \mathfrak{M} \times \mathfrak{M}$. Then $(\mathfrak{M}, [.,.], \mathcal{O} \cap \mathfrak{M})$ is also an almost Pontryagin space (see [7]). By similar arguments as in the previous remark it is easy to verify that the triple $(\mathfrak{L}_S, h^S[.,.], \mathcal{O}_S)$ coincides with $(\mathfrak{M}_S, h^S[.,.], (\mathcal{O} \cap \mathfrak{M})_S)$. The latter is defined as above but just with the use of $(\mathfrak{M}, [.,.], \mathcal{O} \cap \mathfrak{M})$ instead of $(\mathfrak{L}, [.,.], \mathcal{O})$.

Proposition 4.7. *The triple $(\mathfrak{L}_S, h^S[.,.], \mathcal{O}_S)$ is an almost Pontryagin space if and only if there exists an $\epsilon > 0$ such that $S - \epsilon I$ is of finite negativity.*

Proof. Let $(\mathfrak{P}, [.,.])$ be a Pontryagin space which contains $(\mathfrak{L}, [.,.], \mathcal{O})$ as a closed subspace, and let J be a fundamental symmetry on this Pontryagin space. By definition $(\mathfrak{L}_S, h^S[.,.], \mathcal{O}_S) = (\mathfrak{P}_{JS}, h^{JS}[J.,.], \mathcal{O}_{JS})$, where $\mathcal{O}_{JS}$ denotes the Hilbert space topology induced by $h_m^{JS}[J.,.],\ m < m(JS)$, on $\mathfrak{P}_{JS}$.

By Remark 4.2 the symmetric relation $S - \epsilon I$ is of finite negativity on $(\mathfrak{L}, [.,.], \mathcal{O})$ if and only if $JS - \epsilon J$ is of finite negativity on the Hilbert space $(\mathfrak{P}, [J.,.])$. Since the fundamental symmetry operator J is a finite dimensional perturbation of I, the scalar product $h^{JS-\epsilon J}[J.,.]$ is a finite dimensional perturbation of $h^{JS-\epsilon I}[J.,.]$ on $\operatorname{dom} S$. Hence $JS - \epsilon J$ is of finite negativity if and only if $JS - \epsilon I$ has this property.

We just showed that in order to prove the present proposition we may assume that $(\mathfrak{L}, [.,.])$ is a Hilbert space. Under this additional assumption let $(\mathfrak{L}_S, h^S[.,.], \mathcal{O}_S)$ be an almost Pontryagin space. By the definition of almost Pontryagin spaces (see [7]) there exists a closed subspace $\mathfrak{M}_S$ of finite codimension of $(\mathfrak{L}_S, h^S[.,.], \mathcal{O}_S)$ such that $(\mathfrak{M}_S, h^S[.,.])$ is a Hilbert space. Hence, if we choose $m < m(S)$, then there exist $c, d > 0$ such that for all $x \in \mathfrak{M}_S$

$$ch^S[x,x] \le h_m^S[x,x] \le dh^S[x,x]. \tag{4.1}$$

The space $\mathfrak{M}_S \cap \operatorname{dom} S$ has finite codimension in $\operatorname{dom} S$, and for $x \in \mathfrak{M}_S \cap \operatorname{dom} S$ we have

$$dh^{S-\frac{m(S)-m}{d}I}[x,x] \ge h_m^S[x,x] - (m(S)-m)[x,x] = h^S[x,x] - m(S)[x,x] \ge 0.$$

If we set

$$\epsilon = \frac{m(S)-m}{d},$$

then $\epsilon > 0$ and $h^{S-\epsilon I}[.,.]$ has finitely many negative squares, i.e. $S - \epsilon I$ is of finite negativity.

Conversely, if $S - \epsilon I$ is of finite negativity, then we can find a linear subspace $\mathfrak{M}$ of $\operatorname{dom} S$ of finite codimension such that

$$0 \le h^{S-\epsilon I}[x,x] = h^S[x,x] - \epsilon[x,x],$$

for all $x \in \mathfrak{M}$. Since $h^S[.,.]$ and $[.,.]$ are continuous with respect to $\mathcal{O}_S$ on $\mathfrak{L}_S$, we see that $h^S[x,x] \ge \epsilon[x,x]$ for all x belonging to the closure $\mathfrak{M}_S$ of $\mathfrak{M}$ with respect to $\mathcal{O}_S$. Thus $h^S[.,.]$ induces a topology on $\mathfrak{M}_S$ with respect to which $[.,.]$, and hence also $h_m^S[.,.],\ m \in \mathbb{R}$, is continuous. If $m < 0$ and $m < m(S)$, we see that (4.1) holds for $x \in \mathfrak{M}_S$ and for some $c, d > 0$. This means that $\mathcal{O}_S$ is also induced by $h^S[.,.]$ on $\mathfrak{M}_S$, and as this closed subspace has finite codimension in $\mathfrak{L}_S$ the triple $(\mathfrak{L}_S, h^S[.,.], \mathcal{O}_S)$ is an almost Pontryagin space. □

Remark 4.8. As the sum of hermitian scalar products with finitely many negative squares also has this property we see that if $S - \epsilon I,\ \epsilon > 0$ is of finite negativity, then $S - \eta I$ is of finite negativity for all $\eta \le \epsilon$.

Remark 4.9. If the condition from the previous proposition is satisfied, then $\operatorname{ran} S$ is closed and of finite codimension. In fact, this assertion is equivalent to the fact that $\operatorname{ran} JS$ is closed and of finite codimension in the Hilbert space $(\mathfrak{P}, [J.,.])$. We saw in the previous proof that $JS - \epsilon I$ is of finite negativity. Therefore, by Lemma 4.3, $\operatorname{ran} JS$ is closed and of finite codimension.

As $\operatorname{ran} S \perp_{[.,.]} \ker S$ we in particular obtain $\dim \ker S < \infty$.

The following lemma has an interesting consequence.

Lemma 4.10. *Let* $(\mathfrak{L}, [.,.], \mathcal{O})$ *be an almost Pontryagin space, and let* S *be a symmetric relation of finite negativity. Moreover, assume that*

$$\operatorname{dom} S = \operatorname{dom} T + \mathfrak{N},$$

where T *is a closed restriction of* S *such that the adjoint of* T *contains* T *with finite codimension. Moreover, assume* $\dim \mathfrak{N} < \infty$. *Then*

$$\mathfrak{L}_S = \mathfrak{L}_T + \mathfrak{N}.$$

Proof. Let $\mathfrak{P}$ and J be as in Definition 4.4. As $JT \subseteq JS$ it follows from Definition 3.1 that $\mathfrak{P}_{JT}(= \mathfrak{L}_T)$ is a closed subspace of $\mathfrak{P}_{JS}(= \mathfrak{L}_S)$. Since $\mathfrak{N}$ is finite dimensional, $\mathfrak{L}_T + \mathfrak{N}$ is also a closed subspace of $\mathfrak{L}_S$. On the other hand $\operatorname{dom} S = \operatorname{dom} T + \mathfrak{N}(\subseteq \mathfrak{L}_T + \mathfrak{N})$ is dense in $\mathfrak{L}_S$. □

In the following we will consider two scalar products $[.,.]_1$ and $[.,.]$ on $\mathfrak{L}$. Then $[.,.]_1$ is said to be finite dimensional perturbation of $[.,.]$, if for some linear subsapce $\mathfrak{M}$ of $\mathfrak{L}$ of finite codimension one has $[x,y]_1 - [x,y] = 0$ for all $x \in \mathfrak{M},\ y \in \mathfrak{L}$.

Corollary 4.11. *Let* $(\mathfrak{L}, [.,.], \mathcal{O})$ *be an almost Pontryagin space, and let* $[.,.]_1$ *be another scalar product on* $\mathfrak{L}$ *which is continuous with respect to* $\mathcal{O}$ *and which is a finite dimensional perturbation of* $[.,.]$. *Moreover, let* S *be a symmetric relation of finite negativity on* $(\mathfrak{L}, [.,.], \mathcal{O})$ *such that* S *is also symmetric with respect to* $[.,.]_1$.

Under these assumptions $(\mathfrak{L}, [.,.]_1, \mathcal{O})$ *is an almost Pontryagin space. The symmetry* S *is of finite negativity on* $(\mathfrak{L}, [.,.]_1, \mathcal{O})$. *Moreover, the space* $\mathfrak{L}_S$ *and the topology* $\mathcal{O}_S$ *remain the same if they are defined with* $(\mathfrak{L}, [.,.]_1, \mathcal{O})$ *instead of* $(\mathfrak{L}, [.,.], \mathcal{O})$. *Finally,* $(\mathfrak{L}_S, h^S[.,.], \mathcal{O}_S)$ *is an almost Pontryagin space if and only if* $(\mathfrak{L}_S, h^S[.,.]_1, \mathcal{O}_S)$ *is an almost Pontryagin space.*

Proof. By our assumptions there exists a closed subspace $\mathfrak{M}$ of $\mathfrak{L}$ of finite codimension such that

$$[x,y]_1 - [x,y] = 0,\ x \in \mathfrak{M},\ y \in \mathfrak{L}.$$

By the definition of almost Pontryagin spaces there exists a closed subspace $\mathfrak{N}$ of $\mathfrak{L}$ of finite codimension such that $[.,.]$ restricted to $\mathfrak{N}$ is a Hilbert space inner product which induces $\mathcal{O} \cap \mathfrak{N}$ on $\mathfrak{N}$. Hence, $\mathfrak{M} \cap \mathfrak{N}$ is a closed subspace of $\mathfrak{L}$ of finite codimension such that $[.,.]_1$ restricted to $\mathfrak{M} \cap \mathfrak{N}$ is a Hilbert space inner product which induces $\mathcal{O} \cap (\mathfrak{M} \cap \mathfrak{N})$ on $\mathfrak{M} \cap \mathfrak{N}$. This in turn means that $(\mathfrak{L}, [.,.]_1, \mathcal{O})$ is an almost Pontryagin space.

By what was mentioned after Definition 4.1 the finite negativity of S on $(\mathfrak{L}, [.,.], \mathcal{O})$ is equivalent to the fact that $h^S[.,.]$ is positive semidefinite on a linear subspace $\mathfrak{Q}$ of finite codimension of $\operatorname{dom} S$. With $\mathfrak{Q}$ also $\mathfrak{Q} \cap \mathfrak{M}$ is a subspace of finite codimension of $\operatorname{dom} S$, and $h^S[.,.]$ coincides with $h^S[.,.]_1$ on $\mathfrak{Q} \cap \mathfrak{M}$. Hence S is of finite negativity on $(\mathfrak{L}, [.,.]_1, \mathcal{O})$.

Clearly, the almost Pontryagin spaces $(\mathfrak{M}, [.,.], \mathcal{O} \cap \mathfrak{M})$ and $(\mathfrak{M}, [.,.]_1, \mathcal{O} \cap \mathfrak{M})$ coincide. If we set $T = S \cap (\mathfrak{M} \times \mathfrak{M})$, then we obtain from Remark 4.6 that the space $\mathfrak{L}_T$ remains unchanged if we used $(\mathfrak{L}, [.,.]_1, \mathcal{O})$ instead of $(\mathfrak{L}, [.,.], \mathcal{O})$ for its construction. Since $\operatorname{dom} T$ is of finite codimension in $\operatorname{dom} S$, we can apply Lemma 4.10 and see that also $\mathfrak{L}_S$ remains unchanged. Using the fact that the inclusion mapping from $\mathfrak{L}_S$ into $\mathfrak{L}$ is injective and continuous the closed graph theorem implies that the topology $\mathcal{O}_S$ is also independent from the scalar product, which was used for its construction , i.e. $[.,.]$ or $[.,.]_1$.

By what was proved above $S - \epsilon I$ is of finite negativity on $(\mathfrak{L}, [.,.], \mathcal{O})$ if and only if it has this property on $(\mathfrak{L}, [.,.]_1, \mathcal{O})$. Thus the final assertion is an immediate consequence of Proposition 4.7. □

Definition 4.12. Let $(\mathfrak{L}, [.,.], \mathcal{O})$ be an almost Pontryagin space, and let S be a closed symmetric linear relation of finite negativity on $(\mathfrak{L}, [.,.], \mathcal{O})$. Moreover, let $(\mathfrak{P}, [.,.])$ be a Pontryagin space which contains $(\mathfrak{L}, [.,.], \mathcal{O})$ as a closed subspace of finite codimension, and let J be a fundamental symmetry on this Pontryagin space. Then we define the space $\mathfrak{L}^S$ as

$$\mathfrak{L}^S = \mathfrak{P}^{JS} \cap \mathfrak{L},$$

where $\mathfrak{P}^{JS}$ is the space corresponding to the symmetry JS on the Hilbert space $(\mathfrak{P}, [J.,.])$ defined as in Definition 3.13.

Remark 4.13. As $J(JS)^{(*)} = S^{[*]}$ we obtain from (3.8) and Remark 4.5

$$\mathfrak{L}^S = \mathfrak{L}_S + (\operatorname{dom} S^{[*]} \cap \mathfrak{L}).$$

By $S^{[*]}$ we mean here the adjoint relation within $(\mathfrak{P}, [.,.])$.

We can describe $\operatorname{dom} S^{[*]} \cap \mathfrak{L}$ as the set of all $a \in \mathfrak{L}$ such that for $(x; y) \in S$

$$x \mapsto [y, a],$$

is a well defined and $\mathcal{O}$ continuous linear functional on $\operatorname{dom} S$. Hence $\mathfrak{L}^S$ neither depends on J nor on $\mathfrak{P}$.

If $S - mI$ is also of finite negativity, then we immediately see that $\mathfrak{L}^S = \mathfrak{L}^{S-mI}$.

Since we always assume that $\operatorname{codim}_{S^{[*]}} S < \infty$, $\mathfrak{L}^S$ contains $\mathfrak{L}_S$ as a subspace of finite codimension. It therefore carries a unique Hilbert space topology such that $(\mathfrak{L}_S, \mathcal{O}_S)$ is a closed subspace of it. We are going to denote this topology by $\mathcal{O}^S$.

In analogy to Corollary 4.11 we have

Proposition 4.14. *Let* $(\mathfrak{L}, [.,.], \mathcal{O})$*,* $[.,.]_1$ *and* S *be as in Corollary* 4.11*. Moreover, assume that for all* $a \in \mathfrak{L}$ *the mapping*

$$x \mapsto [y, a] - [y, a]_1, \text{ for } (x; y) \in S,$$

is a well defined and $\mathcal{O}$ *continuous linear functional on* $\operatorname{dom} S$*. Then the space* $\mathfrak{L}^S$ *is the same whether it is defined via* $(\mathfrak{L}, [.,.]_1, \mathcal{O})$ *or via* $(\mathfrak{L}, [.,.], \mathcal{O})$*.*

Proof. This result immediately follows from the corresponding invariance property for $\mathfrak{L}_S$ (Corollary 4.11) and from the characterization of $\operatorname{dom} S^{[*]} \cap \mathfrak{L}$ given in Remark 4.13. □

The rest of the paper is devoted to indefinite generalizations of the results in the part of Section 3 which comes after Corollary 3.15. These results will be an essential tool in our forthcoming paper [8].

From now on we will study the case that $(\mathfrak{L}_S, h^S[.,.], \mathcal{O}_S)$ is an almost Pontryagin space. We introduce a linear relation T on any subspace $\mathfrak{G} \subseteq \mathfrak{L}^S$ which contains $\mathfrak{L}_S$:

$$T = S \cap (\mathfrak{G} \times \mathfrak{G}).$$

By $\mathcal{O}_\mathfrak{G}$ we denote the Hilbert space topology $\mathcal{O}^S \cap \mathfrak{G}$.

Definition 4.15. An admissible scalar product $h^\mathfrak{G}[.,.]$ on $\mathfrak{G}$ is a hermitian continuation of $h^S[.,.]$ such that $(\mathfrak{L}_S, h^S[.,.], \mathcal{O}_S)$ is an almost Pontryagin subspace of $(\mathfrak{G}, h^\mathfrak{G}[.,.], \mathcal{O}_\mathfrak{G})$ and such that

$$h^\mathfrak{G}[b, x] = [b, y],$$

for all $b \in \mathfrak{G}, (x; y) \in S$.

Such an admissible product always exists. To see this note that $\mathfrak{L}_S = \mathfrak{P}_{JS} \subseteq \mathfrak{G} \subseteq \mathfrak{L}^S \subseteq \mathfrak{P}^{JS}$. If $(.,.) = [J.,.]$, $m < m(JS)$, and $h_m^\mathfrak{G}(.,.)$ is defined as in Proposition 3.16 with S replaced by JS, then we set

$$h^\mathfrak{G}[.,.] = h_m^\mathfrak{G}(.,.) + m(.,.).$$

This hermitian product is a continuation of $h^{JS}(.,.) = h^S[.,.]$ and for $b \in \mathfrak{G},\ (x; y) \in S$ we obtain from (3.11) that

$$h^\mathfrak{G}[b, x] = h_m^\mathfrak{G}(b, x) + m(b, x) = (b, Jy - mx) + m(b, x) = (b, Jy) = [b, y].$$

Proposition 4.16. *Assume that $(\mathfrak{L}_S, h^S[.,.], \mathcal{O}_S)$ is an almost Pontryagin space and let $h^\mathfrak{G}[.,.]$ be an admissible hermitian inner product on $\mathfrak{G}$ such that $(\mathfrak{L}_S, h^S[.,.], \mathcal{O}_S)$ is an almost Pontryagin subspace of $(\mathfrak{G}, h^\mathfrak{G}[.,.], \mathcal{O}_\mathfrak{G})$.*

Then T considered in $(\mathfrak{G}, h^\mathfrak{G}[.,.], \mathcal{O}_\mathfrak{G})$ is closed, symmetric, of finite codimension in $T^{h^\mathfrak{G}[*]}$ and it is of finite negativity κ_T, where κ_T coincides with the degree of negativity $\kappa_-(\mathrm{ran}(T), [.,.])$ of $(\mathrm{ran}(T), [.,.])$.

Finally, for sufficiently small $\epsilon > 0$ also $T - \epsilon I$ is of finite negativity.

Proof. For $(a; b), (x; y) \in T$ we see from

$$h^\mathfrak{G}[b, x] = [b, y] = h^\mathfrak{G}[a, y], \tag{4.2}$$

that T is symmetric. Moreover, this relation proves that $h^T h^\mathfrak{G}[.,.]$ has as many negative squares as $[.,.]$ on $\mathrm{ran}(T)$.

We see from Proposition 3.16 that $R = (JS) \cap (\mathfrak{G} \times \mathfrak{G})$ is a symmtry with finite defect indices, or equivalently it is contained in its adjoint (with respect to $h_m^\mathfrak{G}(.,.)$) with finite codimension. Let $\mathfrak{M}$ be a $\mathcal{O}_\mathfrak{G}$-closed subspace of $\mathfrak{G}$ on which $J = I$ and such that $h^\mathfrak{G}[.,.]$ is a Hilbert space inner product on $\mathfrak{M}$. With R also $R \cap (\mathfrak{M} \times \mathfrak{M})$ has finite defect index. Clearly,

$$R \cap (\mathfrak{M} \times \mathfrak{M}) = S \cap (\mathfrak{M} \times \mathfrak{M}) = T \cap (\mathfrak{M} \times \mathfrak{M}).$$

It is straightforward to show that also the adjoint of $R \cap (\mathfrak{M} \times \mathfrak{M})$ within $\mathfrak{M}$ with respect to $h^{\mathfrak{G}}[.,.]$ contains $R \cap (\mathfrak{M} \times \mathfrak{M})$ with finite codimension. The same is true for the adjoint of $R \cap (\mathfrak{M} \times \mathfrak{M})$ within $\mathfrak{G}$. Hence also the symmetric extension T of $R \cap (\mathfrak{M} \times \mathfrak{M})$ is contained in $T^{h^{\mathfrak{G}}[*]}$ with finite codimension. Thus according to Definition 4.1 the symmetry T is of finite negativity in $(\mathfrak{G}, h^{\mathfrak{G}}[.,.], \mathcal{O}_{\mathfrak{G}})$.

By Proposition 4.7 $S - \epsilon I$ is of finite negativity for sufficiently small $\epsilon > 0$. For $(a;b), (x;y) \in T$ we have

$$h^{\mathfrak{G}}[b - \epsilon a, x] = [b, y] - h^S[\epsilon a, x] = [b - \epsilon a, y] =$$
$$[b - \epsilon a, y - \epsilon x] + \epsilon h^{S-\epsilon I}[a, x].$$

So we identify $h^{T-\epsilon I}(h^{\mathfrak{G}}[.,.])$ as the sum of two hermitian scalar products with finitely many negative squares. Therefore, it also has finitely many negative squares and $T - \epsilon I$ is of finite negativity. □

By Proposition 4.7 $(\mathfrak{G}_T, h^T h^{\mathfrak{G}}[.,.], (\mathcal{O}_{\mathfrak{G}})_T))$ is an almost Pontryagin space.

Proposition 4.17. *The space $\mathfrak{G}_T$ coincides with the domain of the relation*

$$X = S \cap (\overline{\mathfrak{G}} \times \overline{\mathfrak{G}}),$$

where the closure is taken in $\mathfrak{L}$ with respect to $\mathcal{O}$.

The topology $(\mathcal{O}_{\mathfrak{G}})_T$ coincides with the graph topology of the closed operator

$$\{(x; y + X(0)) : (x; y) \in X\} \subseteq \overline{\mathfrak{G}} \times (\overline{\mathfrak{G}}/X(0)),$$

where $\overline{\mathfrak{G}}$ is provided with $\mathcal{O} \cap \overline{\mathfrak{G}}$ and $\overline{\mathfrak{G}}/X(0)$ with the factor topology $(\mathcal{O} \cap \overline{\mathfrak{G}})/X(0)$.

Proof. From Remark 4.5 we know that $\mathfrak{G}_T$ is the set of all $x \in \mathfrak{G}$ such that there exists a sequence $((x_n; y_n))$ in T which satisfies

$$x_n \to x \text{ w.r.t. } \mathcal{O}_{\mathfrak{G}} \text{ and } [y_n - y_m, y_n - y_m] = h^{\mathfrak{G}}[y_n - y_m, x_n - x_m] \to 0. \quad (4.3)$$

The convergence of x_n with respect to $\mathcal{O}_{\mathfrak{G}}$ implies

$$[y_n - y_m, y] = h^{\mathfrak{G}}[x_n - x_m, y] \to 0,$$

for all $y \in \mathfrak{G}$. Therefore (4.3) is equivalent to $x_n \to x$ with respect to $\mathcal{O}_{\mathfrak{G}}$ and the fact that $(y_n + \overline{\mathfrak{G}}^{[o]})$ is a Cauchy sequence within the Pontryagin space $(\overline{\mathfrak{G}}/\overline{\mathfrak{G}}^{[o]}, [.,.])$ with respect to its Pontryagin space topology.

Using Remark 4.5 once more we see that $x \in \mathfrak{G}_T$ if and only if there exists a sequence $((x_n; y_n))$ in T such that $x_n \to x$ with respect to $\mathcal{O}$,

$$[y_n - y_m, x_n - x_m] \to 0,$$

and $(y_n + \overline{\mathfrak{G}}^{[o]})$ is a Cauchy sequence within the Pontryagin space $(\overline{\mathfrak{G}}/\overline{\mathfrak{G}}^{[o]}, [.,.])$. By the Cauchy-Schwartz inequality here the second condition is a consequence of the remaining two.

Hence, $\mathfrak{G}_T$ is the domain of the linear relation $Q \subseteq \overline{\mathfrak{G}} \times \overline{\mathfrak{G}}/\overline{\mathfrak{G}}^{[o]}$ where Q is the closure of $T + (\{0\} \times \overline{\mathfrak{G}}^{[o]})$. As $\overline{\mathfrak{G}}^{[o]}$ is finite dimensional

$$Q = \overline{T} + \{0\} \times \overline{\mathfrak{G}}^{[o]}.$$

On the other hand as $\operatorname{ran} S$ is closed and of finite codimension (see Remark 4.9) we obtain

$$\overline{\operatorname{ran} T} = \operatorname{ran} S \cap \overline{\mathfrak{G}}.$$

Since the mapping $(x; y) \mapsto y$ from S onto $\operatorname{ran} S$ has a finite dimensional kernel (see Remark 4.9),

$$\overline{\operatorname{ran} T} = \operatorname{ran} \overline{T},$$

and we see that

$$\overline{T} + (\{0\} \times S(0) \cap \overline{\mathfrak{G}}) = S \cap (\overline{\mathfrak{G}} \times \overline{\mathfrak{G}}),$$

and hence

$$\mathfrak{G}_T = \operatorname{dom}(Q) = \operatorname{dom}(\overline{T}) = \operatorname{dom}(S \cap (\overline{\mathfrak{G}} \times \overline{\mathfrak{G}})).$$

The assertion about the topology follows from the closed graph theorem since all involved topologies are Hilbert space topologies. □

Corollary 4.18. *In addition to the assumptions in Proposition* 4.16 *suppose that S is an invertible operator. Then $S^{-1}|_{\operatorname{ran} S \cap \overline{\mathfrak{G}}}$ sets up an isomorphism from the almost Pontryagin space* $(\operatorname{ran} S \cap \overline{\mathfrak{G}}, [.,.], \mathcal{O} \cap \operatorname{ran} S \cap \overline{\mathfrak{G}})$ *onto* $(\mathfrak{G}_T, h^T h^{\mathfrak{G}}[.,.], (\mathcal{O}_{\mathfrak{G}})_T)$.

If we denote by R the symmetry $T \cap (\mathfrak{G}_T \times \mathfrak{G}_T)$ on $(\mathfrak{G}_T, h^T h^{\mathfrak{G}}[.,.], (\mathcal{O}_{\mathfrak{G}})_T)$, then

$$\{(Sx; Sy) : (x; y) \in R\} = S \cap ((\operatorname{ran} S \cap \overline{\mathfrak{G}}) \times (\operatorname{ran} S \cap \overline{\mathfrak{G}})).$$

Proof. Using the notation from Proposition 4.16 and its proof with S also X is an invertible operator. By the proof of Proposition 4.16

$$\operatorname{dom} X = \mathfrak{G}_T, \ \operatorname{ran} X = \operatorname{ran} S \cap \overline{\mathfrak{G}}.$$

Since $\operatorname{ran} X$ is closed, the closed graph theorem implies that X^{-1} is even continuous. Hence, by Proposition 4.17 the topology $(\mathcal{O}_{\mathfrak{G}})_T$ is just the initial topology induced by X.

Because of (4.2) we have

$$[b, y] = h^T h^{\mathfrak{G}}[X^{-1}b, X^{-1}y],$$

for $y \in \operatorname{ran} S \cap \mathfrak{G}$. By continuity we can extend this relation to $\operatorname{ran} S \cap \overline{\mathfrak{G}}$.

For $x, y \in \operatorname{ran} S \cap \overline{\mathfrak{G}}$ we conclude from $(x; y) \in S$ that $S^{-1}y = x = SS^{-1}x$ and $y = SS^{-1}y \in \overline{\mathfrak{G}}$. Hence $(S^{-1}x; S^{-1}y), (S^{-1}y, y) \in X$ (see Proposition 4.17), and further

$$(S^{-1}x; S^{-1}y) \in S \cap (\operatorname{dom}(X) \times \operatorname{dom}(X)) = T \cap (\operatorname{dom}(X) \times \operatorname{dom}(X)) = R.$$

Conversely, if $(S^{-1}x; S^{-1}y) \in R$, then $S^{-1}x \in \mathfrak{G}_T \subseteq \operatorname{dom} S$ and $x = SS^{-1}x = S^{-1}y$, or $(x; y) \in S$. □

References

[1] R. Arens, *Operational calculus of linear relations*, Pacific J. Math., **11** (1961), 9–23

[2] Yu. M. Arlinskiĭ, S. Hassi, Z. Sebestyén, H.S.V. de Snoo, *On the class of extremal extensions of a nonnegative operator*, Recent advances in operator theory and related topics (Szeged, 1999), Operator Theory Adv. Appl. **127**, 41–81, Birkhuser, Basel, 2001

[3] M. Sh. Birman, M. Z. Solomjak, *Spectral theory of selfadjoint operators in Hilbert space*, Mathematics and its Applications (Soviet Series), D. Reidel Publishing Co., Dordrecht 1987

[4] J. Bognar, *Indefinite inner product spaces*, Springer Verlag, Berlin 1974

[5] E.A. Coddington, H. de Snoo, *Positive selfadjoint extensions of positive symmetric subspaces*, Mathematische Zeitschrift **159** (1978), 203–214

[6] A. Dijksma, H. de Snoo, *Self-adjoint extensions of symmetric subspaces*, Pacific J. Math. **54** (1974), 71–100

[7] M. Kaltenbäck, H. Winkler, H. Woracek, *Almost Pontryagin spaces*, to appear in the Proceedings of the IWOTA 2003 in Cagliari, Operator Theory Adv. Appl.

[8] M. Kaltenbäck, H. Winkler, H. Woracek, *Symmetric dB-spaces*, in preparation

[9] T. Kato, *Perturbation Theory for Linear Operator*, Springer Verlag, Berlin 1980

[10] V. Prokaj, Z. Sebestyén, *On Friedrichs extensions of operators*, Acta Sci. Math. (Szeged) **62** (1996), 243–246

[11] V. Prokaj, Z. Sebestyén, *On extremal positive operator extensions*, Acta Sci. Math. (Szeged) **62** (1996), 485–491

[12] Z. Sebestyén, J. Stochel, *Restrictions of positive selfadjoint operators*, Acta Sci. Math. (Szeged) **55** (1991), 149–154

M. Kaltenbäck
Institut für Analysis und Scientific Computing
Technische Universität Wien
Wiedner Hauptstraße 8-10
A 1040 Wien
Austria
e-mail: `michael.kaltenbaeck@tuwien.ac.at`

H. Winkler
Faculteit der Wiskunde en Natuurwetenschappen
Rijksuniversiteit Groningen
9700 AV Groningen
The Netherlands
e-mail: `winkler@math.rug.nl`

H. Woracek
Institut für Analysis und Scientific Computing
Technische Universität Wien
Wiedner Hauptstraße 8-10
A 1040 Wien
Austria
e-mail: `harald.woracek@tuwien.ac.at`