Singularities of generalized strings

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Abstract. We investigate the structure of a maximal chain of matrix functions whose Weyl coefficient belongs to $\mathcal{N}_\kappa^+$. It is shown that its singularities must be of a very particular type. As an application we obtain detailed results on the structure of the singularities of a generalized string which are explicitly stated in terms of the mass function and the dipole function. The main tool is a transformation of matrices, the construction of which is based on the theory of symmetric and semibounded de Branges spaces of entire functions. As byproducts we obtain inverse spectral results for the classes of symmetric and essentially positive generalized Nevanlinna functions.

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1. Introduction

A vibrating string $S[L,m]$ with inhomogeneous mass distribution is given by its length $L>0$ and a function $m:[0,L)\to[0,\infty)$ which is nondecreasing and continuous from the left. The function m measures the total mass of the part of the string between 0 and x. In the description of the motion of the string the following boundary value problem appears:

$$y'(x)+z\int_0^x y(t)dm(t)=0\,,$$

$$y'(0)=0,\ y(L)=0 \text{ if } L+m(L)<\infty\,.$$

Thereby z is a complex parameter. The concept of the principal Titchmarsh-Weyl coefficient q_S of a string S was introduced by I.S.Kac and M.G.Krein, cf. [KaK1]. It turned out to be of fundamental importance. The principal Titchmarsh-Weyl coefficient belongs to the Stieltjes class $\mathcal{S}$, i.e. it is analytic in the open upper half plane $\mathbb{C}^+$ and satisfies

$$\operatorname{Im} q_S(z)\geq 0, \operatorname{Im} zq_S(z)\geq 0,\ z\in\mathbb{C}^+\,.$$

A basic inverse result states that to each function $q \in \mathcal{S}$ there exists a unique string $S[L,m]$, such that q is the principal Titchmarsh-Weyl coefficient of $S[L,m]$.

A canonical system of differential equations, or 1-dimensional Hamiltonian system, is a 2×2-system of differential equations of the form

$$y'(x) = zJH(x)y(x),\ x \in [0, l_H)\,, \tag{1.1}$$

where H is a locally integrable 2×2-matrix valued function on $[0, l_H)$ whose values are real and nonnegative matrices. Moreover, J is the matrix

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad .$$

If x is interpreted as time parameter, it models the motion of a particle under the influence of a time-dependent potential. The function H is called the Hamiltonian of the system under consideration and describes its total energy. To a canonical system there is associated its Weyl coefficient q_H, which is a function belonging to the Nevanlinna class $\mathcal{N}$, i.e. is analytic in $\mathbb{C}^+$ and satisifes $\operatorname{Im} q_H(z) \geq 0$, $z \in \mathbb{C}^+$. A basic inverse result of L.de Branges, cf. [dB1] (a proper formulation can be found e.g. in [W1]), states that to each function $q \in \mathcal{N}$ there exists an essentially unique Hamiltonian which has q as its Weyl coefficient.

The notions of strings and canonical systems are closely related. In fact, in view of the above inverse results, we know that to each string $S[L,m]$ there exists a unique Hamiltonian H_s such that $q_S = q_{H_s}$, and that the behavior of the string is completely determined by the behavior of the canonical system with Hamiltonian H_s.

There are also other ways to relate strings and canonical systems. Since $q_S \in \mathcal{S}$ we know that also $zq_S(z) \in \mathcal{N}$, and therefore that there exists a Hamiltonian H_0 such that $q_{H_0} = zq_S(z)$. Moreover, it is known that, if $q \in \mathcal{S}$, then also $zq(z^2) \in \mathcal{N}$. Thus we have naturally associated yet another Hamiltonian H_d, namely such that $q_{H_d} = zq_S(z^2)$. Each of the Hamiltonians H_s, H_0 and H_d fully describes the string, the most natural choice is H_s. Each of H_s, H_0 and H_d can be determined explicitly in terms of the string $S[L,m]$. A detailed exposition of these topics is given in [KWW3].

During the last decades a theory was developed which deals with generalizations of the notions and theorems mentioned above to an indefinite setting. The class $\mathcal{N}_{<\infty}$ of generalized Nevanlinna functions is defined by substituting the positivity condition in the definition of $\mathcal{N}$ by the requirement that a certain kernel function has only a finite number of negative squares. For the exact definition see §2.3. This class of functions was intensively studied, for the basic results we refer to [KL1]. Moreover, a function q is said to belong to the class $\mathcal{N}_{<\infty}^+$, if $q \in \mathcal{N}_{<\infty}$ and $zq(z) \in \mathcal{N}$. This can be viewed as a generalization of the Stieltjes class. For example it is known that, if $q \in \mathcal{N}_{<\infty}^+$, then $zq(z^2) \in \mathcal{N}_{<\infty}$.

The theory of canonical systems and the inverse spectral theorem of L.de Branges was generalized to an indefinite setting in [KW1, KW2, KW3].

Thereby the differential equation $y'(x) = zJH(x)y$ is substituted by the family of its fundamental matrices $\omega(x)$, which form a so-called chain of matrices. The notion of chains of matrices can be axiomatically accessed and generalized to the indefinite situation. It is then proved that to each chain of matrices a function $q_\infty(\omega) \in \mathcal{N}_{<\infty}$ is associated, which is called the Weyl coefficient of the chain, and that conversely to each $q \in \mathcal{N}_{<\infty}$ there exists an essentially unique chain of matrices such that q is its Weyl coefficient. We will recall these notions and results in more detail in §4. The interpretation of an indefinite chain of matrices as the family of fundamental matrices of an indefinite canonical system is work in progress, cf. [KW4]. In contrast to the classical situation, a chain of matrices in the indefinite case has singularities. The peculiarities of indefiniteness are reflected in the structure of these singularities.

A generalization of the notion of a string to the indefinite setting was proposed in [LW]. A generalized string is a triple $S[L, m, D]$ where $L > 0$, m is a locally square integrable function defined on $[0, L)$ which is nondecreasing and continuous from the left with possible exception of a finite number of points, and where D is a stepfunction defined on $[0, L)$ which has only a finite number of points of increase, is nondecreasing and continuous from the left. A point $x_e \in [0, L)$ is called critical, if either $D(x_e+) - D(x_e) > 0$, $m(x_e+) - m(x_e) < 0$ or $\limsup_{x\to x_e} |m(x)| = \infty$. The relation

$$f'(x) + z \int_{[0,x]} f(x)\, dm(x) + z^2 \int_{[0,x]} f(x)\, dD(x) = 0, \quad f'(0-) = 0 .$$

is called the differential equation of the generalized string. Of course, this equation requires an appropriate interpretation. Also to a generalized string $S[L, m, D]$ a function q_S is associated and again called the principal Titchmarsh-Weyl coefficient of $S[L, m, D]$. It belongs to the class $\mathcal{N}^+_{<\infty}$. An inverse theorem is established which states that this notion induces a bijective correspondence between the set of all generalized strings and $\mathcal{N}^+_{<\infty}$.

By the above inverse results we know that, given a generalized string $S[L, m, D]$, there exist chains of matrices ω_s, ω_0 and ω_d, such that $q_\infty(\omega_s)(z) = q_S(z)$, $q_\infty(\omega_0)(z) = zq_S(z)$ and $q_\infty(\omega_d)(z) = zq_S(z^2)$. Thereby ω_0 will be positive definite, since $zq_S(z) \in \mathcal{N}$, whereas ω_s and ω_d will in general be indefinite. The behavior of the generalized string is fully determined by each of these chains.

It is the aim of this paper to describe the chain ω_s, in particular the structure of its singularities, in terms of the generalized string $S[L, m, D]$. It turns out that the singularities of ω_s correspond exactly to the critical points of $S[L, m, D]$, and that their structure can be explicitly read off from the behavior of the mass function m and the possible presence of dipoles. We obtain a noteworthy inverse result, which states that the singularities of a chain whose Weyl coefficient belongs to $\mathcal{N}^+_{<\infty}$ are of a very special kind. In fact, there are just five different types which can occur.

We will obtain these results by explicit construction. Assume that $q \in \mathcal{N}_{<\infty}$ such that also $zq(z), zq(z^2) \in \mathcal{N}_{<\infty}$. Let $\omega_s, \omega_0, \omega_d$ be the chains of matrices with

$$q_\infty(\omega_s)(z) = q(z),\ q_\infty(\omega_0)(z) = zq(z),\ q_\infty(\omega_d)(z) = zq(z^2)\,.$$

We will give a method how to construct all three of the chains $\omega_s, \omega_0, \omega_d$, once one of them is known. From this we deduce results on the structure of singularities of either of these chains. In the situation that $S[L, m, D]$ is a generalized string and $q = q_S$ we can apply this knowledge to obtain what we were aiming for.

We are also led to the conclusion that indeed the principal Titchmarsh-Weyl coefficient q_S, whose definition in [LW] might seem to be a bit 'ad hoc', is the most natural object to describe the structure of a generalized string $S[L, m, D]$. Although, of course, $S[L, m, D]$ is also determined by either of $zq_S(z)$ or $zq_S(z^2)$. However, looking at the chains ω_0 and ω_d, we see that ω_0 does not have any singularities and it will be apparent from our results that the singularities of ω_d can be of a much more complicated type as those of ω_s. Hence the information on $S[L, m, D]$ in ω_0 is somewhat hidden and ω_d is simply to big to describe $S[L, m, D]$ in a neat way.

The present work is divided into three parts. The first part consists of Sections 2 and 3. In Section 2 we introduce some classes of functions which are of importance in our investigations, study the relationship between those classes as well as the reproducing kernel spaces generated by such type of functions. These are first of all the $\mathcal{M}$-classes of 2×2-matrix functions, in particular the subclasses of symmetric and essentially positive matrix functions, cf. Definition 2.1,Definition 2.2. Secondly we recall the notions of generalized Nevanlinna functions and of Hermite-Biehler functions, and the appropriate analogues of symmetry and essential positivity on the level of these functions. Moreover, we recall the definition of a de Branges space of entire functions, cf. Definition 2.12, and the relation of those spaces to the introduced classes of functions. Some of these results are well known, however, we wish to set up these notations in sufficient generality and to collect what is needed in the sequel. In Section 3 we deal with a transformation of matrix functions and its converse. This transformation relates the $\mathcal{M}$-classes of symmetric and essentially positive matrix functions, cf. Theorem 3.2. Although the methods employed in these investigations are mostly elementary, they lead to two striking results on the structure of de Branges spaces, Proposition 3.12 and Proposition 3.14.

The second part of this paper consists of Sections 4, 5 and 6. In this part we lift the above mentioned transformations to the level of chains of matrices and investigate the evolution of singularities. In Section 4 we give the definition of a chain of matrices, cf. Definition 4.1, and of symmetric and essentially positive chains of matrices, cf. Definition 4.3. Moreover, we recall some basic facts concerning these notions. A noteworthy inverse result, which is proved in this section, states that a chain of matrices is symmetric if and only if its Weyl coefficient is symmetric, cf. Proposition 4.4. Section 5 deals with the proper lifting of the transformations of matrix functions to whole chains of matrices. It contains Theorem 5.1 which can be viewed as the core of our present work. It shows explicitly how we can obtain the chain ϖ whose Weyl coefficient is $q(z)$ from the chain ω whose Weyl coefficient is $zq(z^2)$. As a corollary we obtain another inverse result which states that a chain of matrices is essentially positive if and only if its Weyl coefficient has this property, cf. Proposition 5.6. Also a first result on the structure of essentially positive chains is deduced, cf. Corollary 5.8. Moreover, we consider the inverse transformation, cf. Theorem 5.10 where we give an explicit construction of the chain whose Weyl coefficient is $zq(z^2)$ assuming that the chain with Weyl coefficient $q(z)$ is known. These results also lead to a construction of the chain with Weyl coefficient $zq(z)$ out of the one with Weyl coefficient q as indicated in Remark 5.11. The statement of Remark 5.11 is of fundamental importance, since it tells us how to proceed in order to reach our aim stated above. In Section 6 we recall the classification of singularities of a chain of matrices and investigate how singularities appear and transform when switching from the chain with Weyl coefficient q to those whose Weyl coefficients are $zq(z^2)$ and $-(zq(z))^{-1}$, respectively. In view of our needs in the discussion of generalized strings we restrict ourselves to the case that $q \in \mathcal{N}$. However, it is obvious how a more general discussion can be carried out (and how tedious this might be).

Finally, in the third part, Section 7, we prove the results on generalized strings we were aiming for, cf. Theorem 7.3. Following the idea which was made explicit in Remark 5.11, they are deduced from the results of Sections 5 and 6.

2. Some classes of functions and their interrelation

In this preliminary we set up our notation and collect some results on various classes of functions and their interrelation. Only some of these results are new, some are well known, some are taken from previous work. However, we feel that it is a benefit for the reader to have this collection of preliminaries at hand.

2.1. The class $\mathcal{M}$ of matrix functions

Main objects of our studies in the present paper are 2×2-matrix valued entire functions of a particular kind. For a function $f:\mathbb{C}\to\mathbb{C}$ we denote by $f^\#$ the function

$$f^\#(z) := \overline{f(\overline{z})}\,. \tag{2.1}$$

If $f = f^\#$, we call f *real.*

2.1. Definition. Let $\mathcal{M}$ be the set of all 2×2-matrix valued functions

$$W = (w_{ij})_{i,j=1}^2 : \mathbb{C}\to\mathbb{C}^{2\times 2}$$

such that the entries w_{ij} are real and entire functions, $\det W \equiv 1$ and $W(0) = I$. Denote by $\mathcal{M}^{sym}$ the subset of $\mathcal{M}$ which consists of those functions W such that w_{11}, w_{22} are even and w_{12}, w_{21} are odd. Let $\mathcal{M}^{ep}$ be the set of all functions $W\in\mathcal{M}$ which have the property that each of their entries has only finitely many zeros off the positive real axis.

Let J be the matrix

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

2.2. Definition. Let $\kappa \in \mathbb{N} \cup \{0\}$. We write $W \in \mathcal{M}_\kappa$, if W belongs to $\mathcal{M}$ and if the 2×2-matrix valued kernel

$$H_W(w,z) := \frac{W(z)JW(w)^* - J}{z - \overline{w}}$$

has κ negative squares on $\mathbb{C}$.

Throughout this paper we will use the notation

$$\mathcal{M}_{\leq\kappa} := \bigcup_{0\leq\nu\leq\kappa} \mathcal{M}_\nu,\ \mathcal{M}_{<\infty} := \bigcup_{\nu\in\mathbb{N}\cup\{0\}} \mathcal{M}_\nu \, ,$$

and write $\operatorname{ind}_- W = \kappa$ to express the fact that $W \in \mathcal{M}$ belongs to $\mathcal{M}_\kappa$. Moreover,

$$\mathcal{M}_\kappa^{sym} := \mathcal{M}^{sym} \cap \mathcal{M}_\kappa,\ \mathcal{M}_\kappa^{ep} := \mathcal{M}^{ep} \cap \mathcal{M}_\kappa \, ,$$

and $\mathcal{M}_{\leq\kappa}^{sym}$, $\mathcal{M}_{<\infty}^{sym}$, $\mathcal{M}_{\leq\kappa}^{ep}$, $\mathcal{M}_{<\infty}^{ep}$ are defined correspondingly.

Similarly we can define classes ${}^+\mathcal{M}_\kappa$, ${}^+\mathcal{M}_\kappa^{sym}$ etc., by imposing a restriction on the numbers of positive squares, instead of negative squares, of the kernel H_W.

For later reference let us explicitly state the following elementary and mostly well known results. Put

$$V := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

2.3. Lemma. *The classes $\mathcal{M}$, $\mathcal{M}^{sym}$, $\mathcal{M}_{<\infty}$, $\mathcal{M}_{<\infty}^{sym}$, ${}^+\mathcal{M}_{<\infty}$, ${}^+\mathcal{M}_{<\infty}^{sym}$ are closed with respect to multiplication. Each of the following transformations ϕ_j is an involution of $\mathcal{M}$. The subclasses $\mathcal{M}^{sym}$, $\mathcal{M}^{ep}$ remain invariant (symbolized by a ✓) or not and the class $\mathcal{M}_\kappa$ remains invariant or is mapped to the class ${}^+\mathcal{M}_\kappa$ according to the following scheme:*

Transformation	$\mathcal{M}^{sym}$	$\mathcal{M}^{ep}$	$\mathcal{M}_\kappa$
$\phi_1 : W(z) \mapsto W(-z)^{-1}$	✓		✓
$\phi_2 : W(z) \mapsto W(-z)$	✓		${}^+\mathcal{M}_\kappa$
$\phi_3 : W(z) \mapsto W(z)^{-1}$	✓	✓	${}^+\mathcal{M}_\kappa$
$\phi_4 : W(z) \mapsto -JW(z)J$	✓	✓	✓
$\phi_5 : W(z) \mapsto VW(z)^{-1}V$	✓	✓	✓
$\phi_6 : W(z) \mapsto W(z)^T$	✓	✓	${}^+\mathcal{M}_\kappa$

Proof. The fact that $\mathcal{M}$ and $\mathcal{M}^{sym}$ are closed with respect to multiplication is obvious. The kernel relation

$$\frac{(W_1W_2)(z)J(W_1W_2)^*(w) - J}{z - \overline{w}} =$$

$$= W_1(z)\frac{W_2(z)JW_2^*(w) - J}{z - \overline{w}}W_1(w)^* + \frac{W_1(z)JW_1^*(w) - J}{z - \overline{w}}$$

shows that also $\mathcal{M}_{<\infty}$ and ${}^+\mathcal{M}_{<\infty}$ have this property.

Each of the transformations ϕ_j is an involution on $\mathcal{M}$. The facts that $\mathcal{M}^{sym}$ is mapped into itself by all ϕ_j and that $\mathcal{M}^{ep}$ is invariant under $\phi_3,\dots,\phi_6$ is seen by explicitly writing down the matrix $\phi_j(W)$.

The fact that ϕ_1,ϕ_4 and ϕ_5 map $\mathcal{M}_\kappa$ into itself follows from the kernel relations

$$\frac{W(-z)^{-1}JW(-w)^{-*}-J}{z-\overline{w}}=W(-z)^{-1}\frac{-J+W(z)JW(w)^*}{(-z)-(-\overline{w})}W(w)^{-*}\,,$$

$$\frac{(-JW(z)J)J(-JW(w)J)^*-J}{z-\overline{w}}=J\frac{W(z)JW(w)^*-J}{z-\overline{w}}J^*\,,$$

and

$$\frac{(VW^{-1}V)(z)J(VW^{-1}V)^*(w)-J}{z-\overline{w}}=VW(z)^{-1}\frac{W(z)JW^*(w)-J}{z-\overline{w}}W(w)^{-*}V\,.$$

Moreover, $\phi_2(\mathcal{M}_\kappa)={}^+\mathcal{M}_\kappa$ since

$$H_{W(-z)}(w,z)=-H_W(-w,-z)\,.$$

Since $\phi_3=\phi_1\circ\phi_2$ and $\phi_6=\phi_3\circ\phi_4$, we find that also $\phi_3(\mathcal{M}_\kappa)={}^+\mathcal{M}_\kappa$ and $\phi_6(\mathcal{M}_\kappa)={}^+\mathcal{M}_\kappa$.

❑

2.4. *Remark.* The simplest examples of matrices in $\mathcal{M}$, besides the constant I, are linear polynomials. An elementary argument shows that a nonconstant linear polynomial W belongs to $\mathcal{M}$ if and only it is of the form

$$W(z)=W_{(l,\phi)}(z):=\begin{pmatrix}1-lz\sin\phi\cos\phi & lz\cos^2\phi\\ -lz\sin^2\phi & 1+lz\sin\phi\cos\phi\end{pmatrix}\tag{2.2}$$

for some $l\in\mathbb{R}\setminus\{0\}$ and $\phi\in[0,\pi)$. The numbers l and ϕ are uniquely determined by W.

Note that for all $l\in\mathbb{R}\setminus\{0\}$ and $\phi\in[0,\pi)$ we have $W_{(l,\phi)}\in\mathcal{M}^{ep}$, that $W_{(l,\phi)}\in\mathcal{M}^{sym}$ if and only if $\phi=0$ or $\phi=\frac{\pi}{2}$, and that

$$W_{(l,\phi)}\in\begin{cases}\mathcal{M}_0\cap{}^+\mathcal{M}_1 & ,\ l>0\\ \mathcal{M}_1\cap{}^+\mathcal{M}_0 & ,\ l<0\end{cases}$$

Any function $W\in\mathcal{M}_\kappa$ generates a Pontryagin space $\mathfrak{K}(W)$ by means of the reproducing kernel H_W. Recall that this space is obtain as completion of the linear space

$$\operatorname{span}\Big\{H_W(w,.)\begin{pmatrix}x\\y\end{pmatrix}:\ w\in\mathbb{C},\begin{pmatrix}x\\y\end{pmatrix}\in\mathbb{C}^2\Big\}\,,$$

equipped with the inner product

$$\Big[H_W(w_1,.)\begin{pmatrix}x_1\\y_1\end{pmatrix},H_W(w_2,.)\begin{pmatrix}x_2\\y_2\end{pmatrix}\Big]:=\begin{pmatrix}x_2\\y_2\end{pmatrix}^*H_W(w_1,w_2)\begin{pmatrix}x_1\\y_1\end{pmatrix}\,,$$

see for example [ADRS].

Let $W \in \mathcal{M}_{<\infty}$. It follows from the fact that the entries of W are real, i.e. satisfy $w_{ij}^{\#} = w_{ij}$, that the mapping

$$\begin{pmatrix} f \\ g \end{pmatrix} \longmapsto \begin{pmatrix} f \\ g \end{pmatrix}^{\#} := \begin{pmatrix} f^{\#} \\ g^{\#} \end{pmatrix}$$

is an anti-isometry of $\mathfrak{K}(W)$ onto itself. Moreover, cf. [KW1, Proposition 8.3], the space $\mathfrak{K}(W)$ is invariant under the difference quotient operator ($w \in \mathbb{C}$)

$$\mathcal{R}_w : \begin{pmatrix} f \\ g \end{pmatrix} \longmapsto \begin{pmatrix} \frac{f(z)-f(w)}{z-w} \\ \frac{g(z)-g(w)}{z-w} \end{pmatrix}.$$

We put

$$\mathfrak{K}_-(W) := \operatorname{cls}\left\{H_W(w,z)\begin{pmatrix} 0 \\ 1 \end{pmatrix} : w \in \mathbb{C}\right\}, \ ,$$

and, similarly, $\mathfrak{K}_+(W) := \operatorname{cls}\{H_W(w,z)\begin{pmatrix} 1 \\ 0 \end{pmatrix} : w \in \mathbb{C}\}$.

2.2. A Characterization of $\mathcal{M}_{<\infty}^{sym}$

The fact that $W \in \mathcal{M}_\kappa^{sym}$ is reflected in a symmetry property of $\mathfrak{K}(W)$.

Denote by $\mathcal{O}(\mathbb{C})$ the set of all entire functions and consider the map

$$M : \begin{pmatrix} F(z) \\ G(z) \end{pmatrix} \mapsto \begin{pmatrix} -F(-z) \\ G(-z) \end{pmatrix} \tag{2.3}$$

This map is an involution of $\mathcal{O}(\mathbb{C})^2$.

2.5. Proposition. *Let $W \in \mathcal{M}_\kappa$. Then $W \in \mathcal{M}_\kappa^{sym}$ if and only if $M|_{\mathfrak{K}(W)}$ is an isometry of $\mathfrak{K}(W)$ onto itself.*

Proof. Assume that $W \in \mathcal{M}_\kappa^{sym}$. Then

$$W(-z) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} W(z) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and hence

$$H_W(-w,-z) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} H_W(w,z) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since

$$M \begin{pmatrix} F(z) \\ G(z) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F(-z) \\ G(-z) \end{pmatrix},$$

we obtain

$$M\, H_W(w,z) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} H_W(w,-z) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} =$$

$$= H_W(-w,z) \begin{pmatrix} -\alpha \\ \beta \end{pmatrix} \in \mathfrak{K}(W).$$

Moreover,

$$\left[M\, H_W(w_1,z)\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, M\, H_W(w_2,z)\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}\right]_{\mathfrak{K}(W)} =$$

$$= \left[H_W(-w_1,z) \begin{pmatrix} -\alpha_1 \\ \beta_1 \end{pmatrix}, H_W(-w_2,z) \begin{pmatrix} -\alpha_2 \\ \beta_2 \end{pmatrix} \right]_{\mathfrak{K}(W)} =$$

$$= \begin{pmatrix} -\alpha_2 \\ \beta_2 \end{pmatrix}^* H_W(-w_1,-w_2) \begin{pmatrix} -\alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}^* H_W(w_1,w_2) \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} =$$

$$= \left[H_W(w_1,z) \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, H_W(w_2,z) \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right]_{\mathfrak{K}(W)} .$$

Let

$$\mathcal{L} := \operatorname{span} \left\{ H_W(w,z) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} :\ w,\alpha,\beta \in \mathbb{C} \right\} .$$

Then $\mathcal{L}$ is a dense linear subspace of $\mathfrak{K}(W)$ and $M|_{\mathcal{L}}$ maps $\mathcal{L}$ isometrically onto itself. Hence there exists an isometric continuation of $M|_{\mathcal{L}}$ to $\mathfrak{K}(W)$ which must be given by (2.3), since point evaluation is continuous.

Conversely, assume that (2.3) is an isometry of $\mathfrak{K}(W)$ onto itself. As $M^2 = \mathrm{id}$, we obtain

$$\left[\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}, M\, H_W(w,z) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right]_{\mathfrak{K}(W)} = \left[M \begin{pmatrix} F(z) \\ G(z) \end{pmatrix}, H_W(w,z) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right]_{\mathfrak{K}(W)} =$$

$$= \begin{pmatrix} \alpha \\ \beta \end{pmatrix}^* \begin{pmatrix} -F(-w) \\ G(-w) \end{pmatrix} = \begin{pmatrix} -\alpha \\ \beta \end{pmatrix}^* \begin{pmatrix} F(-w) \\ G(-w) \end{pmatrix} =$$

$$= \left[\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}, H_W(-w,z) \begin{pmatrix} -\alpha \\ \beta \end{pmatrix} \right]_{\mathfrak{K}(W)} .$$

It follows that

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} H_W(w,-z) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = M\, H_W(w,z) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} =$$

$$= H_W(-w,z) \begin{pmatrix} -\alpha \\ \beta \end{pmatrix} = H_W(-w,z) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} .$$

Since α, β were arbitrary, we conclude that

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} H_W(w,-z) = H_W(-w,z) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Substituting $w = 0$ in this relation yields

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{W(-z)J - J}{-z} = \frac{W(z)J - J}{z} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

and since

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} J = -J \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

it follows that

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} W(-z) = W(z) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

i.e. that $W \in \mathcal{M}^{sym}$.

❑

This result puts us in position to apply [KWW2, Lemma 2.1], and to obtain a splitting of the space $\mathfrak{K}(W)$.

2.6. Corollary. *Define*

$$\mathfrak{K}(W)_e := \ker(I-M) = \left\{ \begin{pmatrix} F \\ G \end{pmatrix} \in \mathfrak{K}(W) : F \text{ odd}, G \text{ even} \right\},$$

$$\mathfrak{K}(W)_o := \ker(I+M) = \left\{ \begin{pmatrix} F \\ G \end{pmatrix} \in \mathfrak{K}(W) : F \text{ even}, G \text{ odd} \right\}.$$

Then $\mathfrak{K}(W)_e$ and $\mathfrak{K}(W)_o$ are closed subspaces of $\mathfrak{K}(W)$ and $\mathfrak{K}(W) = \mathfrak{K}(W)_e[\dot{+}]\mathfrak{K}(W)_o$. The reproducing kernels of $\mathfrak{K}(W)_e$ and $\mathfrak{K}(W)_o$ are given by $\frac{1}{2}(I+M)H_W(w,z)$ and $\frac{1}{2}(I-M)H_W(w,z)$, respectively.

Of particular interest is the situation when $\mathfrak{K}_-(W) = \mathfrak{K}(W)$.

2.7. Lemma. *Let $W \in \mathcal{M}_{<\infty}^{sym}$, $(F,G)^T \in \mathfrak{K}(W)$, and assume that $\mathfrak{K}_-(W) = \mathfrak{K}(W)$. Then $(F,G)^T \in \mathfrak{K}(W)_e$ if and only if G is even, and $(F,G)^T \in \mathfrak{K}(W)_o$ if and only if G is odd. The analogous assertion holds when $\mathfrak{K}_+(W) = \mathfrak{K}(W)$ and G is replaced by F.*

Proof. Assume that $\mathfrak{K}_-(W) = \mathfrak{K}(W)$ and that $(F,G)^T \in \mathfrak{K}(W)$ is such that G is even. Then

$$(I-M)\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} F(z)+F(-z) \\ 0 \end{pmatrix}$$

and hence $(I-M)(F,G)^T = 0$. The other assertions follow similarly.

❑

2.3. The class $\mathcal{N}_{<\infty}$ of generalized Nevanlinna functions

Let us recall the notion of matrix valued *generalized Nevanlinna functions*: A $n \times n$-matrix valued function Q is said to belong to $\mathcal{N}_\kappa^{n\times n}$, if it is defined and meromorphic on $\mathbb{C} \setminus \mathbb{R}$, satisfies $Q(\overline{z}) = Q(z)^*$, and has the property that the $n \times n$-matrix valued kernel

$$L_Q(w,z) := \frac{Q(z) - Q(w)^*}{z - \overline{w}}$$

has κ negative squares.

The following subclasses of Nevanlinna functions were investigated in [KWW2]. A function $Q \in \mathcal{N}_\kappa^{n\times n}$ is said to be

- (i) *symmetric*, if $Q(-z) = -Q(z)$, i.e. if Q is odd.
- (ii) *essentially positive*, if Q is analytic on $\mathbb{C} \setminus [0,\infty)$ with possible exception of finitely many poles.

The subset of $\mathcal{N}_\kappa^{n\times n}$ which consists of all symmetric (essentially positive) functions will be denoted by $\mathcal{N}_\kappa^{n\times n,sym}$ ($\mathcal{N}_\kappa^{n\times n,ep}$, respectively). If we deal with scalar valued functions, i.e. $n=1$, then the upper index $n \times n$ will be suppressed. Moreover, the scalar function $q(z) \equiv \infty$ will be regarded as an element of $\mathcal{N}_0$. We will freely use selfexplanatory notation like $\mathcal{N}_{\leq\kappa}^{ep}$, $\mathcal{N}_{<\infty}^{n\times n,sym}$ etc.

The $\mathcal{M}$-classes of matrix functions are related to the generalized Nevanlinna classes in several ways, two of them are of importance in the present context.

The first one we deal with is the Potapov-Ginzburg transform. If W is a 2×2-matrix function whose entries are real analytic functions such that $\det W \equiv 1$ and $w_{21} \not\equiv 0$, then the *Potapov-Ginzburg transform* $\Psi(W)$ is defined as (cf. e.g. [Br])

$$\Psi(W)(z) := \begin{pmatrix} \frac{w_{11}(z)}{w_{21}(z)} & \frac{1}{w_{21}(z)} \\ \frac{1}{w_{21}(z)} & \frac{w_{22}(z)}{w_{21}(z)} \end{pmatrix} .$$

2.8. Lemma. *We have $W \in \mathcal{M}_\kappa$ if and only if $\Psi(W) \in \mathcal{N}_\kappa^{2\times 2}$. Moreover, $W \in \mathcal{M}_\kappa^{sym}$ ($W \in \mathcal{M}_\kappa^{ep}$) if and only if $\Psi(W) \in \mathcal{N}_\kappa^{sym}$ ($\Psi(W) \in \mathcal{N}_\kappa^{ep}$, respectively).*

Proof. The assertion follows from the kernel relation

$$H_W(w,z) = \begin{pmatrix} -1 & w_{11}(z) \\ 0 & w_{21}(z) \end{pmatrix} \frac{\Psi(W)(z) - \Psi(W)(w)^*}{z - \overline{w}} \begin{pmatrix} -1 & w_{11}(w) \\ 0 & w_{21}(w) \end{pmatrix}^*$$

❑

Secondly, matrix functions of the class $\mathcal{M}_{<\infty}$ operate on $\mathcal{N}_{<\infty}$ via fractional linear transformations: If $W \in \mathcal{M}$ and $\tau : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ is an analytic function such that $w_{21}(z)\tau(z) + w_{22}(z) \not\equiv 0$, then we define

$$(W \star \tau)(z) := \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_{21}(z)\tau(z) + w_{22}(z)} .$$

Then $W \star \tau$ is a meromorphic function on Ω. If $w_{21}(z)\tau(z) + w_{22}(z) \equiv 0$ we set $W \star \tau \equiv \infty$. Moreover,

$$(W \star \infty)(z) := \frac{w_{11}(z)}{w_{21}(z)} .$$

2.9. Lemma. *If $W \in \mathcal{M}_\kappa$ and $\tau \in \mathcal{N}_\nu$, then $W \star \tau \in \mathcal{N}_{\leq \kappa+\nu}$. If $W \in {}^+\mathcal{M}_\kappa$ and $\tau \in \mathbb{R} \cup \{\infty\}$, then $-(W \star \tau) \in \mathcal{N}_{\leq\kappa}$.*

Proof. This assertion follows from the kernel relation

$$(w_{21}(z)\tau(z) + w_{22}(z)) \frac{(W\star\tau)(z) - (W\star\tau)(\overline{w})}{z - \overline{w}} (w_{21}(\overline{w})\tau(\overline{w}) + w_{22}(\overline{w})) =$$
$$= \begin{pmatrix} -\tau(z) & 1 \end{pmatrix} \frac{(VW^{-1}V)(z)J(VW^{-1}V)^*(w) - J}{z - \overline{w}} \begin{pmatrix} -\tau(\overline{w}) \\ 1 \end{pmatrix} + \frac{\tau(z) - \tau(\overline{w})}{z - \overline{w}} .$$

❑

2.10. Corollary.

(*i*) *If $W \in \mathcal{M}_\kappa$, then each of the functions*

$$\frac{w_{11}(z)}{w_{21}(z)}, \frac{w_{12}(z)}{w_{22}(z)}, \quad -\frac{w_{11}(z)}{w_{12}(z)}, -\frac{w_{21}(z)}{w_{22}(z)} \tag{2.4}$$

belongs to $\mathcal{N}_{\leq\kappa}$.

(*ii*) *A function* $W \in \mathcal{M}_{<\infty}$ *belongs to* $\mathcal{M}^{ep}_{<\infty}$ *if and only if one of its entries has the property to have only finitely many zeros in* $\mathbb{C}\setminus[0,\infty)$*. In fact, if* $W \in \mathcal{M}_\kappa$ *and one entry has* n *zeros in* $\mathbb{C}\setminus[0,\infty)$*, then each other entry has at most* $4n+6+3\kappa$ *zeros in this region.*

Proof. For the first two functions in (2.4) use the first part of Lemma 2.9 with the matrix W and the parameters $\tau = \infty$ and $\tau = 0$, respectively. For the second pair of functions use the second part of Lemma 2.9 with the matrix W^T and the parameters $\tau = 0, \infty$.

The second assertion follows from (2.4) by employing [KWW2, Lemma 4.5]. An inspection of its proof shows the explicit estimate.

❑

The estimate in Corollary 2.10, (*ii*), is very rough, but sufficient for our purposes.

2.4. deBranges spaces of entire functions. The class $\mathcal{HB}_{<\infty}$ of Hermite-Biehler functions

Let us recall the notion of dB-spaces and their connection to the class of Hermite-Biehler functions. To make the present work more self-contained, let us recall the notion of an almost Pontryagin space, [KWW1].

2.11. Definition. Let $\mathfrak{L}$ be a linear space, $[.,.]$ an inner product on $\mathfrak{L}$ and $\mathcal{O}$ a Hilbert space topology on $\mathfrak{L}$. The triplet $(\mathfrak{L}, [.,.], \mathcal{O})$ is called an *almost Pontryagin space*, if

- **(aPS1)** $[.,.]$ is $\mathcal{O}$-continuous.
- **(aPS2)** There exists a $\mathcal{O}$-closed linear subspace $\mathfrak{M}$ of $\mathfrak{L}$ with finite codimension such that $(\mathfrak{M}, [.,.])$ is a Hilbert space.

2.12. Definition. An inner product space $(\mathfrak{P}, [.,.])$ is called a *deBranges space* (*dB-space*, for short), if the following axioms hold true:

- **(dB1)** $(\mathfrak{P}, [.,.])$ is a reproducing kernel almost Pontryagin space on $\mathbb{C}$ whose elements are entire functions.
- **(dB2)** If $F \in \mathfrak{P}$, then also $F^\# \in \mathfrak{P}$. Moreover,
$$[F^\#, G^\#] = [G, F],\ \ F, G \in \mathfrak{P}\,.$$
- **(dB3)** If $F \in \mathfrak{P}$ and $z_0 \in \mathbb{C}\setminus\mathbb{R}$ with $F(z_0) = 0$, then
$$\frac{z-\overline{z_0}}{z-z_0}F(z) \in \mathfrak{P}\,.$$
Moreover, if additionally $G \in \mathfrak{P}$ with $G(z_0) = 0$, then
$$\Big[\frac{z-\overline{z_0}}{z-z_0}F(z), \frac{z-\overline{z_0}}{z-z_0}G(z)\Big] = [F, G]\,.$$

We will assume throughout this paper that also

- **(Z)** For every $t \in \mathbb{R}$ there exists $F \in \mathfrak{P}$ with $F(t) \neq 0$.

Let $\mathfrak{P}$ be a dB-space. We define the set of *associated functions* as

$$\operatorname{Assoc}\mathfrak{P} := z\mathfrak{P} + \mathfrak{P}\,.$$

A function S belongs to $\operatorname{Assoc}\mathfrak{P}$ if and only if $\mathfrak{P}$ is closed with respect to the difference quotient operator ($w \in \mathbb{C}$)

$$F(z) \mapsto \frac{F(z)S(w) - F(w)S(z)}{z - w}\,.$$

The *Hermite-Biehler class* $\mathcal{H}B_\kappa$ with negative index $\kappa \in \mathbb{N} \cup \{0\}$ is defined as the set of all entire functions E, such that E and $E^\#$ have no common nonreal zeros, $E^{-1}E^\#$ is not constant, and the kernel

$$S_{\frac{E^\#}{E}}(w,z) := i\frac{1 - \frac{E^\#(z)}{E(z)}\overline{\left(\frac{E^\#(w)}{E(w)}\right)}}{z - \overline{w}}$$

has κ negative squares on $\mathbb{C}^+$.

The Hermite-Biehler class is related to the notion of dB-Pontryagin spaces, i.e. nondegenerated dB-spaces, by the fact that, if $\mathfrak{P}$ is a dB-Pontryagin space, then its reproducing kernel K is of the form

$$K(w,z) = i\frac{E(z)\overline{E(w)} - E^\#(z)E(\overline{w})}{2\pi(z - \overline{w})}$$

for a (not necessarily unique) Hermite-Biehler function E. Conversely, every Hermite-Biehler E function generates in this way a dB-Pontryagin space which we will denote by $\mathfrak{P}(E)$, cf. [KW1]

The class of Hermite-Biehler functions is also connected with generalized Nevanlinna functions. Let E be an entire function and write $E = A - iB$ with A, B real entire functions. Then

$$E \in \mathcal{H}B_\kappa \iff \frac{A}{B} \in \mathcal{N}_\kappa\,,$$

$$E \in \mathcal{H}B_\kappa^{sym} \iff \frac{A}{B} \in \mathcal{N}_\kappa^{sym},\ E \in \mathcal{H}B_\kappa^{sb} \iff \frac{A}{B} \in \mathcal{N}_\kappa^{ep}\,.$$

The following two special classes of dB-spaces were investigated in [KWW4]. Let $M : \mathcal{O}(\mathbb{C}) \to \mathcal{O}(\mathbb{C})$ be defined as

$$M : \begin{cases} \mathcal{O}(\mathbb{C}) & \to & \mathcal{O}(\mathbb{C}) \\ F(z) & \mapsto & F(-z) \end{cases}$$

The map M is a linear involution of $\mathcal{O}(\mathbb{C})$. A dB-space $(\mathfrak{P}, [.,.])$ is called *symmetric*, if M induces an isometric involution of $\mathfrak{P}$, i.e. if $M(\mathfrak{P}) \subseteq \mathfrak{P}$ and

$$[MF, MG] = [F, G],\ F, G \in \mathfrak{P}\,.$$

Let $(\mathfrak{P}, [.,.])$ be a dB-space, and denote by $\mathcal{S}$ the operator of multiplication with the independent variable in $\mathfrak{P}$. Then $\mathfrak{P}$ is called *semibounded* if the inner product

$$[F, G]_S := [\mathcal{S}F, G],\ F, G \in \operatorname{dom}\mathcal{S}\,,$$

has a finite number of negative squares on $\operatorname{dom}\mathcal{S}$.

The sets of Hermite-Biehler functions which correspond to these classes of dB-Pontryagin spaces are the following: Define $\mathcal{HB}_\kappa^{sym}$ to be the subset of $\mathcal{HB}_\kappa$ consisting of all functions E which have the property that $E^\#(z) = E(-z)$. Then $\mathfrak{P}$ is a symmetric dB-Pontryagin space if and only $\mathfrak{P} = \mathfrak{P}(E)$ for some $E \in \mathcal{HB}_{<\infty}^{sym}$. Moreover, we denote by $\mathcal{HB}_\kappa^{sb}$ the set of all functions $E = A - iB \in \mathcal{HB}_\kappa$ such that B has only finitely many zeros in $\mathbb{C} \setminus [0, \infty)$. Then $\mathfrak{P}$ is a semibounded dB-Pontryagin space if and only $\mathfrak{P} = \mathfrak{P}(E)$ for some $E \in \mathcal{HB}_{<\infty}^{sb}$.

We will need the following result which supplements our discussion of symmetric dB-spaces in [KWW4].

A subspace $\mathfrak{Q}$ of a dB-space $\mathfrak{P}$ is called a *dB-subspace* of $\mathfrak{P}$, if it is with the topology and inner product inherited from $\mathfrak{P}$ a dB-space. This is the case if and only if $F \in \mathfrak{Q}$ implies $F^\# \in \mathfrak{Q}$ and if $F \in \mathfrak{Q}$, $F(w) = 0$, implies $(z-w)^{-1}F(z) \in \mathfrak{Q}$. A main result in the theory of dB-spaces, cf. [dB1], [KW1], is the ordering theorem for subspaces of $\mathfrak{P}$. It states that the set of all dB-subspaces of a given dB-space $\mathfrak{P}$ is totally ordered with respect to inclusion.

2.13. Lemma. *Let $\mathfrak{P}$ be a symmetric a dB-space and let $\mathfrak{Q}$ be a dB-subspace of $\mathfrak{P}$. Then $\mathfrak{Q}$ is symmetric.*

Proof. Put $\tilde{\mathfrak{Q}} := \mathfrak{Q}$, we have to show that $\tilde{\mathfrak{Q}} \subseteq \mathfrak{Q}$. It is straightforward to check that $\tilde{\mathfrak{Q}}$ is a dB-subspace of $\mathfrak{P}$. By the ordering theorem for subspaces of $\mathfrak{P}$ we have either $\tilde{\mathfrak{Q}} \subseteq \mathfrak{Q}$ or $\mathfrak{Q} \subseteq \tilde{\mathfrak{Q}}$. In the first case we are already done. In the second case we have $M\tilde{\mathfrak{Q}} = \mathfrak{Q} \subseteq \tilde{\mathfrak{Q}}$. Since M is an involution, this implies $M\tilde{\mathfrak{Q}} = \tilde{\mathfrak{Q}}$.

❑

Matrices of the class $\mathcal{M}_{<\infty}$ give rise to Hermite-Biehler functions. If $W \in \mathcal{M}_\kappa$, then the function $E_W := w_{22} + iw_{21}$ belongs to $\mathcal{HB}_{\leq\kappa}$. It was shown in [KW1] that the projection $\pi_- : (F, G)^T \mapsto G$ is an isometric isomorphism of $\mathfrak{K}_-(W)/\mathfrak{K}_-(W)^\circ$ onto $\mathfrak{P}(E_W)$.

Similarly, if we put $\tilde{E}_W := w_{11} - iw_{12}$, then $\tilde{E}_W \in \mathcal{HB}_{\leq\kappa}$ and the projection $\pi_+ : (F, G)^T \mapsto F$ is an isometric isomorphism of $\mathfrak{K}_+(W)/\mathfrak{K}_+(W)^\circ$ onto $\mathfrak{P}(\tilde{E}_W)$.

It is an important result, cf. [KW1], that for a nondegenerated dB-space $\mathfrak{P}$ we have $1 \in \operatorname{Assoc}\mathfrak{P}$ if and only if there exists a matrix $W \in \mathcal{M}_{<\infty}$, $\mathfrak{K}_-(W) = \mathfrak{K}(W)$ such that $\mathfrak{P} = \mathfrak{P}(E_W)$.

If $\mathfrak{Q}$ is a dB-subspace of $\mathfrak{P}$, then trivially $\operatorname{Assoc}\mathfrak{Q} \subseteq \operatorname{Assoc}\mathfrak{P}$. It follows from [dB1] that, if $1 \in \operatorname{Assoc}\mathfrak{P}$, then also $1 \in \operatorname{Assoc}\mathfrak{Q}$.

The dB-subspaces of a $\mathfrak{P}(E_W)$ can be obtained from certain subspaces of $\mathfrak{K}(W)$. Note that, if we assume that $\mathfrak{K}_-(W) = \mathfrak{K}(W)$, then the projection π_- onto the second component is an isometry of $\mathfrak{K}(W)$ onto $\mathfrak{P}(E_W)$.

2.14. Lemma. *Let $W \in \mathcal{M}_{<\infty}$ and assume that $\mathfrak{K}_-(W) = \mathfrak{K}(W)$. Then the projection π_- induces an order-preserving bijection of the set of all closed subspaces $\mathcal{L}$ of $\mathfrak{K}(W)$ which are invariant under the mappings $.^\#$ (cf. (2.1)) and $\mathcal{R}_w$, $w \in \mathbb{C}$, and the set of all dB-subspaces of $\mathfrak{P}(E_W)$. Thereby $\operatorname{ind}_- \pi_-(\mathcal{L}) = \operatorname{ind}_- \mathcal{L}$ and $\dim \pi_-(\mathcal{L})^\circ = \dim \mathcal{L}^\circ$.*

Under the assumption that $\mathfrak{K}_+(W) = \mathfrak{K}(W)$ *the same assertion holds with* π_+ *and* $\mathfrak{P}(\tilde{E}_W)$.

Proof. Since $\mathfrak{K}_-(W) = \mathfrak{K}(W)$, the mapping π_- is an isometric isomorphism of $\mathfrak{K}(W)$ onto $\mathfrak{P}(E_W)$. Hence it induces an order preserving bijection of the set of all closed subspaces of $\mathfrak{K}(W)$ onto the set of all closed subspaces of $\mathfrak{P}(E_W)$, and hence leaves negative indices and degree of degeneracy invariant.

Assume that $\mathcal{L}$ is a closed subspace of $\mathfrak{K}(W)$ which is closed with respect to $.^\#$ and $\mathcal{R}_w$. Since $\pi_- \circ .^\# = .^\# \circ \pi_-$ and $\pi_- \circ \mathcal{R}_w = \mathcal{R}_w \circ \pi_-$, also $\pi(\mathcal{L})$ has this property. We see that $F \in \pi_-(\mathcal{L})$ implies $F^\# \in \pi_-(\mathcal{L})$ and that, if $F \in \pi_-(\mathcal{L})$ and $F(w) = 0$, then $(z-w)^{-1}F(z) \in \pi_-(\mathcal{L})$. Thus $\pi_-(\mathcal{L})$ is a dB-subspace of $\mathfrak{P}(E_W)$.

Conversely, let $\mathfrak{Q}$ be a dB-subspace of $\mathfrak{P}(E_W)$ and put $\mathcal{L} := \pi_-^{-1}(\mathfrak{Q})$. Then $F \in \mathfrak{Q}$ implies $F^\# \in \mathfrak{Q}$. Assume that $F = \pi_-(G,F)^T$, $F^\# = \pi_-(H,F^\#)^T$. It follows that $(G^\# - H, 0)^T \in \mathfrak{K}(W)$ and thus that $G^\# = H$. Thus also $\mathcal{L}$ is closed under $.^\#$. As $1 \in \operatorname{Assoc}\mathfrak{P}(E_W)$, also $1 \in \operatorname{Assoc}\mathfrak{Q}$, i.e. $\mathfrak{Q}$ is invariant under $\mathcal{R}_w$. Again using that $\mathfrak{K}_-(W) = \mathfrak{K}(W)$, we see that also $\mathcal{L}$ has this property.

❑

3. The square root transformation

In this section we investigate a transformation $\mathcal{T}_{\surd}$ which assigns to each matrix $W \in \mathcal{M}^{sym}$ an element of $\mathcal{M}$ and its converse transformations $\mathcal{T}_{2,\gamma}$. These results have consequences on the structure of symmetric and semibounded dB-spaces. Moreover, they are the basic tool for the subsequent sections.

3.1. The transformations $\mathcal{T}_{\surd}$ and $\mathcal{T}_{2,\gamma}$

3.1. Definition. Define a transformation $\mathcal{T}_{\surd} : \mathcal{M}^{sym} \to \mathcal{M}$ by

$$\mathcal{T}_{\surd}(W)(z^2) := \begin{pmatrix} w_{11}(z) & \frac{w_{12}(z)}{z} - w_{12}'(0)w_{11}(z) \\ zw_{21}(z) & w_{22}(z) - w_{12}'(0)zw_{21}(z) \end{pmatrix}.$$

Let $\gamma \in \mathbb{R}$. Define a transformation $\mathcal{T}_{2,\gamma} : \mathcal{M} \to \mathcal{M}^{sym}$ by

$$\mathcal{T}_{2,\gamma}(W)(z) := \begin{pmatrix} w_{11}(z^2) & z(w_{12}(z^2) + \gamma w_{11}(z^2)) \\ \frac{w_{21}(z^2)}{z} & w_{22}(z^2) + \gamma w_{21}(z^2) \end{pmatrix}$$

The facts that $\mathcal{T}_{\surd}(W)$ is well-defined and belongs to $\mathcal{M}$, and that $\mathcal{T}_{2,\gamma}(W) \in \mathcal{M}^{sym}$ follow on inspecting the defining formulas.

3.2. Theorem. *The transformation* $\mathcal{T}_{\surd}$ *maps* $\mathcal{M}^{sym}$ *surjectively onto* $\mathcal{M}$. *For each* $W \in \mathcal{M}$ *we have*

$$\mathcal{T}_{\surd}^{-1}\big(\{W\}\big) = \big\{\mathcal{T}_{2,\gamma}(W) : \gamma \in \mathbb{R}\big\}.$$

By $\mathcal{T}_{\surd}$ *the class* $\mathcal{M}^{sym}_{<\infty}$ *is mapped onto* $\mathcal{M}^{ep}_{<\infty}$. *In fact,* $\operatorname{ind}_- \mathcal{T}_{\surd}(W) \leq \operatorname{ind}_- W$. *Moreover, the map*

$$\Phi : \begin{pmatrix} f(z) \\ g(z) \end{pmatrix} \mapsto \begin{pmatrix} zf(z^2) \\ g(z^2) \end{pmatrix}$$

is an isometry of $\mathfrak{K}(\mathcal{T}_{\sqrt{}}(W))$ *onto* $\mathfrak{K}(W)_e$.

The proof of this theorem needs some preparation. First we list some elementary properties of $\mathcal{T}_{\sqrt{}}$ and $\mathcal{T}_{2,\gamma}$, and show that these transformations are in a way inverse to each other.

3.3. Lemma.

(*i*) *Let* $W \in \mathcal{M}^{sym}$, *then*

$$\mathcal{T}_{\sqrt{}}(W)(z^2) = \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & 1 \end{pmatrix} W(z) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -w_{12}'(0) \\ 0 & 1 \end{pmatrix} . \tag{3.1}$$

If additionally $\hat{W} = (\hat{w}_{ij})_{i,j=1}^2 \in \mathcal{M}^{sym}$, *then*

$$\begin{gathered} \mathcal{T}_{\sqrt{}}(W)^{-1}(z^2) \cdot \mathcal{T}_{\sqrt{}}(\hat{W})(z^2) = \\ = \begin{pmatrix} 1 & w_{12}'(0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & 1 \end{pmatrix} W^{-1}(z)\hat{W}(z) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\hat{w}_{12}'(0) \\ 0 & 1 \end{pmatrix} . \end{gathered} \tag{3.2}$$

Moreover,

$$\operatorname{tr}\left(\mathcal{T}_{\sqrt{}}(W)'(0)J\right) = -w_{21}'(0) + \frac{w_{12}'''(0)}{6} - w_{12}'(0)\frac{w_{11}''(0)}{2} , \tag{3.3}$$

and we have

$$\begin{gathered} z\left(\mathcal{T}_{\sqrt{}}(W) \star \infty\right)(z^2) = (W \star \infty)(z) , \\ \frac{\mathcal{T}_{\sqrt{}}(W)_{22}(z^2)}{\mathcal{T}_{\sqrt{}}(W)_{21}(z^2)} = \frac{w_{22}(z)}{w_{21}(z)} \cdot \frac{1}{z} - w_{12}'(0) . \end{gathered} \tag{3.4}$$

(*ii*) *Let* $W \in \mathcal{M}$, *then*

$$\mathcal{T}_{2,\gamma}(W)(z) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} W(z^2) \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & 1 \end{pmatrix} . \tag{3.5}$$

If additionally $\hat{W} \in \mathcal{M}$, *then*

$$\begin{gathered} \mathcal{T}_{2,\gamma}(W)^{-1}(z)\mathcal{T}_{2,\hat{\gamma}}(\hat{W})(z) = \\ = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix} W(z^2)^{-1}\hat{W}(z^2) \begin{pmatrix} 1 & \hat{\gamma} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & 1 \end{pmatrix} . \end{gathered} \tag{3.6}$$

Moreover,

$$\operatorname{tr}\left(\mathcal{T}_{2,\gamma}(W)'(0)J\right) = \gamma - w_{21}'(0) . \tag{3.7}$$

(*iii*) *We have*

$$\left(\mathcal{T}_{\sqrt{}} \circ \mathcal{T}_{2,\gamma}\right)(W) = W, \ W \in \mathcal{M} ,$$

$$\left(\mathcal{T}_{2,w_{12}'(0)} \circ \mathcal{T}_{\sqrt{}}\right)(W) = W, \ W \in \mathcal{M}^{sym} .$$

Proof. The formulas (3.1), (3.5) and (3.7) are verified by straightforward computation. The relation (3.3) is proved by comparison of the Taylor coefficients in

$$\mathcal{T}_{\surd}(W)'(z^2)\cdot 2z=\begin{pmatrix} w_{11}'(z) & \frac{w_{12}'(z)}{z}-\frac{w_{12}(z)}{z^2}-w_{12}'(0)w_{11}'(z)\\ w_{21}(z)+zw_{21}'(z) & w_{22}'(z)-w_{12}'(0)w_{21}(z)-w_{12}'(0)zw_{21}'(z)\end{pmatrix}$$

The relation (3.2) is established by substituting the expression (3.1) for $\mathcal{T}_{\surd}(W)(z^2)$ and $\mathcal{T}_{\surd}(\hat{W})(z^2)$, respectively. The relation (3.6) follows from (3.5). Finally, (3.4) follows from the definition of $\mathcal{T}_{\surd}(W)$.

The second relation in *(iii)* follows from (3.5) and (3.1). Since $\mathcal{T}_{2,\gamma}(W)_{12}'(0)=\gamma$, the same source implies the validity of the first relation in *(iii)*.

❑

The kernel of the map $\mathcal{T}_{\surd}$ can be determined explicitly.

3.4. Lemma. *Let $W,\hat{W}\in\mathcal{M}^{sym}$. Then $\mathcal{T}_{\surd}(W)=\mathcal{T}_{\surd}(\hat{W})$ if and only if*

$$\hat{W}=WW_{(l,0)}\,,$$

for some $l\in\mathbb{R}$, where $W_{(l,0)}$ is as in (2.2).

Proof. Assume that $\hat{W}=WW_{(l,0)}$. Then

$$\hat{W}(z)=W(z)\begin{pmatrix}1 & lz\\ 0 & 1\end{pmatrix},\ \hat{w}_{12}'(0)=w_{12}'(0)+l\,,$$

and hence, by (3.2),

$$\mathcal{T}_{\surd}(W)^{-1}(z^2)\mathcal{T}_{\surd}(\hat{W})(z^2)=$$

$$=\begin{pmatrix}1 & w_{12}'(0)\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{z} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & lz\\ 0 & 1\end{pmatrix}\begin{pmatrix}z & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -w_{12}'(0)-l\\ 0 & 1\end{pmatrix}=$$

$$=\begin{pmatrix}1 & w_{12}'(0)\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & l\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -w_{12}'(0)-l\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Conversely, assume that $\mathcal{T}_{\surd}(W)^{-1}\mathcal{T}_{\surd}(\hat{W})=I$. Then we obtain from (3.2) that

$$W(z)^{-1}\hat{W}(z)=\begin{pmatrix}z & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & \hat{w}_{12}'(0)-w_{12}'(0)\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{z} & 0\\ 0 & 1\end{pmatrix}=$$

$$=\begin{pmatrix}1 & (\hat{w}_{12}'(0)-w_{12}'(0))z\\ 0 & 1\end{pmatrix}.$$

❑

The relationship between the spaces $\mathfrak{K}(W)$ and $\mathfrak{K}(\mathcal{T}_{\surd}(W))$ is expressed by the following kernel relation.

3.5. Lemma. *For each $W \in \mathcal{M}^{sym}$ we have*

$$\begin{gathered} 2 \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} H_{\mathcal{T}_{\surd}(W)}(w^2, z^2) \begin{pmatrix} \overline{w} & 0 \\ 0 & 1 \end{pmatrix} = \\ = H_W(w,z) + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} H_W(w,-z) = (I+M) H_W(w,z) \,. \end{gathered} \tag{3.8}$$

Proof. Since $w_{12}'(0) \in \mathbb{R}$ we have

$$\begin{pmatrix} 1 & -w_{12}'(0) \\ 0 & 1 \end{pmatrix} J \begin{pmatrix} 1 & -w_{12}'(0) \\ 0 & 1 \end{pmatrix}^* = J \,,$$

and hence we obtain, using (3.1),

$$\begin{gathered} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \frac{\mathcal{T}_{\surd}(W)(z^2) J \mathcal{T}_{\surd}(W)(w^2)^* - J}{z^2 - \overline{w}^2} \begin{pmatrix} \overline{w} & 0 \\ 0 & 1 \end{pmatrix} = \\ = \frac{1}{z^2 - \overline{w}^2} \Big[W(z) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} J \begin{pmatrix} \overline{w} & 0 \\ 0 & 1 \end{pmatrix} W(w)^* - \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} J \begin{pmatrix} \overline{w} & 0 \\ 0 & 1 \end{pmatrix} \Big] = \\ = \frac{1}{z^2 - \overline{w}^2} \Big[W(z) \begin{pmatrix} 0 & -z \\ \overline{w} & 0 \end{pmatrix} W(w)^* - \begin{pmatrix} 0 & -z \\ \overline{w} & 0 \end{pmatrix} \Big] \,. \end{gathered}$$

We compute the expression on the right hand side of (3.8).

$$\begin{gathered} H_W(w,z) + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} H_W(w,-z) = \\ = \frac{W(z) J W(w)^* - J}{z - \overline{w}} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{W(-z) J W(w)^* - J}{-z - \overline{w}} = \\ = \frac{1}{z^2 - \overline{w}^2} \Big[W(z) \Big((z+\overline{w}) J - (z-\overline{w}) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} J \Big) W(w)^* - \\ - \Big((z+\overline{w}) J - (z-\overline{w}) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} J \Big) \Big] \,. \end{gathered}$$

Since

$$(z+\overline{w}) J - (z-\overline{w}) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} J = \begin{pmatrix} 0 & -2z \\ 2\overline{w} & 0 \end{pmatrix} ,$$

the equality (3.8) follows.

❑

We are now in position to prove Theorem 3.2.

Proof. (of Theorem 3.2*)* By the first relation in Lemma 3.3, (*iii*), it follows that $\mathcal{T}_{\surd} : \mathcal{M}^{sym} \to \mathcal{M}$ is surjective. Moreover, for each $W \in \mathcal{M}$ and all $\gamma \in \mathbb{R}$, we have $\mathcal{T}_{2,\gamma}(W) \in \mathcal{T}_{\surd}^{-1}(\{W\})$. The relation (3.6) shows that

$$\mathcal{T}_{2,\hat{\gamma}}(W) = \mathcal{T}_{2,\gamma}(W) W_{(\hat{\gamma}-\gamma,0)}$$

and hence we conclude from Lemma 3.4 that the matrices $\mathcal{T}_{2,\gamma}(W)$ exhaust $\mathcal{T}_{\surd}^{-1}(\{W\})$.

Let $W \in \mathcal{M}_\kappa^{sym}$. Then, by the kernel relation (3.8), we have $\mathcal{T}_{\surd}(W) \in \mathcal{M}_{\leq\kappa}$. Corollary 2.6 together with (3.8) yields that the map Φ is an isometry of $\mathfrak{K}(\mathcal{T}_{\surd}(W))$ onto $\mathfrak{K}(W)_e$.

By (3.4) and Corollary 2.10, (i), the function $q(z) = \mathcal{T}_{\surd}(W) \star \infty$ has the property that $zq(z^2)$ belongs to $\mathcal{N}_{\leq\kappa}$. It follows from [KWW2, Theorem 4.1] that $q \in \mathcal{N}_{\leq\kappa}^{ep}$. Since the entries of the first column of $\mathcal{T}_{\surd}(W)$ cannot have common zeros, we see that the entry $\mathcal{T}_{\surd}(W)_{21}$ has only finitely many zeros in $\mathbb{C} \setminus [0,\infty)$. Thus, by Corollary 2.10, (ii), we have $\mathcal{T}_{\surd}(W) \in \mathcal{M}_{\leq\kappa}^{ep}$.

Conversely, assume that $W \in \mathcal{M}_{<\infty}^{ep}$. We show that $\mathcal{T}_{2,\gamma}(W) \in \mathcal{M}_{<\infty}^{sym}$. Since $\mathcal{T}_{2,\gamma}(W) = \mathcal{T}_{2,0}(W)W_{(\gamma,0)}$, it suffices to consider the particular case $\gamma = 0$. In this case

$$\mathcal{T}_{2,0}(W)(z) = \begin{pmatrix} w_{11}(z^2) & zw_{12}(z^2) \\ \frac{w_{21}(z^2)}{z} & w_{22}(z^2) \end{pmatrix}.$$

Hence the Potapov-Ginzburg transform computes as

$$\Psi\big(\mathcal{T}_{2,0}(W)\big)(z) = \begin{pmatrix} z\frac{w_{11}(z^2)}{w_{21}(z^2)} & \frac{z}{w_{21}(z^2)} \\ \frac{z}{w_{21}(z^2)} & z\frac{w_{22}(z^2)}{w_{21}(z^2)} \end{pmatrix} =$$

$$= z\Psi(W)(z^2).$$

Since $W \in \mathcal{M}_{<\infty}^{ep}$ we have $\Psi(W) \in \mathcal{N}_{<\infty}^{2\times2,ep}$ and hence, by [KWW2, Theorem 4.1],

$$\Psi\big(\mathcal{T}_{2,0}(W)\big)(z) \in \mathcal{N}_{<\infty}^{2\times2,sym}.$$

This shows $\mathcal{T}_{2,0}(W) \in \mathcal{M}_{<\infty}^{sym}$.

❑

The subject of the next proposition is to clarify how linear polynomials are transformed when either of $\mathcal{T}_{\surd}$ or $\mathcal{T}_{2,\gamma}$ is performed. This result is an important tool for our later investigation of transformations of matrix chains.

3.6. Proposition.

(i) *Let $W, \hat{W} \in \mathcal{M}^{sym}$ and assume that $W^{-1}\hat{W} = W_{(l,\alpha)}$. Then $\alpha \in \{0, \frac{\pi}{2}\}$. If $\alpha = 0$, we have $\mathcal{T}_{\surd}(W)^{-1}\mathcal{T}_{\surd}(\hat{W}) = I$. If $\alpha = \frac{\pi}{2}$, then*

$$\mathcal{T}_{\surd}(W)^{-1}\mathcal{T}_{\surd}(\hat{W}) = W_{(l',\phi)},$$

where

$$l' = l(1 + w_{12}'(0)^2),\ \phi = \operatorname{Arccot} w_{12}'(0).$$

(ii) *Let $W, \hat{W} \in \mathcal{M}$, $\gamma, \hat{\gamma} \in \mathbb{R}$. If $W^{-1}\hat{W} = W_{(l,\phi)}$ for some $l \in \mathbb{R}$, $\phi \in [0,\pi)$, cf.* (2.2)*, then*

$$\mathcal{T}_{2,\gamma}(W)^{-1}(z)\mathcal{T}_{2,\hat{\gamma}}(\hat{W})(z) = \tag{3.9}$$

$$= \begin{pmatrix} 1 - z^2 l \sin\phi(\cos\phi - \gamma\sin\phi) & \begin{matrix} z(\hat{\gamma}-\gamma)+ \\ +z^3 l(\cos\phi-\gamma\sin\phi)(\cos\phi-\hat{\gamma}\sin\phi) \end{matrix} \\ -zl\sin^2\phi & 1 + z^2 l\sin\phi(\cos\phi - \hat{\gamma}\sin\phi) \end{pmatrix}.$$

We have $\mathcal{T}_{2,\gamma}(W)^{-1}\mathcal{T}_{2,\hat\gamma}(\hat W) = W_{(L,\alpha)}$ *if and only if* $W^{-1}\hat W = W_{(l,\phi)}$ *and either*

$$l=0,\ \textit{in which case } L=\hat\gamma-\gamma \textit{ and } \alpha=0\,,$$

or

$$l\neq 0, \phi\neq 0, \gamma=\hat\gamma=\cot\phi, \textit{in which case } L=l\sin^2\phi, \alpha=\frac{\pi}{2}\,.$$

Proof. The matrix function $W_{(l,\alpha)}$ belongs to $\mathcal{M}^{sym}$ if and only if $\alpha\in\{0,\frac{\pi}{2}\}$. This proves the first assertion in (i).

The case that $\alpha=0$ was already treated in Lemma 3.4. Assume that $\alpha=\frac{\pi}{2}$. Then

$$\hat W(z)=W(z)\begin{pmatrix}1 & 0\\ -lz & 1\end{pmatrix},\ \hat w_{12}'(0)=w_{12}'(0)\,.$$

It follows that

$$\mathcal{T}_{\surd}(W)^{-1}(z^2)\mathcal{T}_{\surd}(\hat W)(z^2)=$$

$$=\begin{pmatrix}1 & w_{12}'(0)\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{z} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -lz & 1\end{pmatrix}\begin{pmatrix}z & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -w_{12}'(0)-l\\ 0 & 1\end{pmatrix}=$$

$$=\begin{pmatrix}1 & w_{12}'(0)\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -lz^2 & 1\end{pmatrix}\begin{pmatrix}1 & -w_{12}'(0)\\ 0 & 1\end{pmatrix}=$$

$$=\begin{pmatrix}1-lw_{12}'(0)z^2 & lw_{12}'(0)^2z^2\\ -lz^2 & 1+lw_{12}'(0)z^2\end{pmatrix}=W_{(l',\phi)}(z^2)$$

when $l'=l(1+w_{12}'(0))$ and $\cot\phi=w_{12}'(0)$.

We come to the proof of (ii). Assume that $W^{-1}\hat W=W_{(l,\phi)}$. Then

$$\mathcal{T}_{2,\gamma}(W)^{-1}(z)\mathcal{T}_{2,\hat\gamma}(\hat W)(z)=$$

$$=\begin{pmatrix}z & -\gamma z\\ 0 & 1\end{pmatrix}\begin{pmatrix}1-lz^2\sin\phi\cos\phi & lz^2\cos^2\phi\\ -lz^2\sin^2\phi & 1+lz^2\sin\phi\cos\phi\end{pmatrix}\begin{pmatrix}\frac{1}{z} & \hat\gamma\\ 0 & 1\end{pmatrix}=$$

$$=\begin{pmatrix}1-z^2l\sin\phi(\cos\phi-\gamma\sin\phi) & \substack{z(\hat\gamma-\gamma)+\\ +z^3l(\cos^2\phi-\gamma\hat\gamma\sin^2\phi-\hat\gamma\sin\phi\cos\phi-\gamma\sin\phi\cos\phi)}\\ -zl\sin^2\phi & 1+z^2l\sin\phi(\cos\phi-\hat\gamma\sin\phi)\end{pmatrix}.$$

Assume that $\mathcal{T}_{2,\gamma}(W)^{-1}(z)\mathcal{T}_{2,\hat\gamma}(\hat W)(z)=W_{(L,\alpha)}$. By the already proved part (i) of the present proposition, we must have $\alpha\in\{0,\frac{\pi}{2}\}$ and $W^{-1}\hat W=W_{(l,\phi)}$ for certain $l\in\mathbb{R}$, $\phi\in[0,\pi)$. Considering the just proved formula (3.9) we see that either (i) or (ii) holds. The converse follows from (3.9).

❑

3.7. *Remark.* Note that in case $\phi=0$ the matrix (3.9) is equal to

$$\begin{pmatrix}1 & z(\hat\gamma-\gamma)+lz^3\\ 0 & 1\end{pmatrix}.$$

If $\phi \neq 0, l \neq 0$ we can decompose (3.9) as

$$\begin{pmatrix} 1 & (\cot\phi - \gamma)z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -lz\sin^2\phi & 1 \end{pmatrix} \begin{pmatrix} 1 & -(\cot\phi - \hat{\gamma})z \\ 0 & 1 \end{pmatrix}.$$

Then its Potapov-Ginzburg transform is equal to

$$-\frac{1}{z}\frac{1}{l\sin^2\phi}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + z\begin{pmatrix} \cot\phi - \gamma & 0 \\ 0 & -(\cot\phi - \hat{\gamma}) \end{pmatrix}.$$

From the isomorphy of $\mathfrak{K}(W)_e$ and $\mathfrak{K}(\mathcal{T}_{\checkmark}(W))$ we also obtain a relation between the spaces $\mathfrak{K}_-(W)$ and $\mathfrak{K}_-(\mathcal{T}_{\checkmark}(W))$.

3.8. Proposition. *Let $W \in \mathcal{M}_\kappa^{sym}$, then*

$$\dim\mathfrak{K}_-\big(\mathcal{T}_{\checkmark}(W)\big)^\perp = \Big[\frac{1}{2}\dim\mathfrak{K}_-(W)^\perp\Big],$$

and

$$\dim\mathfrak{K}_-\big(\mathcal{T}_{\checkmark}(W)\big)^\circ = \Big[\frac{1}{2}\dim\mathfrak{K}_-(W)^\circ\Big].$$

Proof. By [KW1, Corollary 9.7,Proposition 8.3] we have for any matrix W

$$\mathfrak{K}_-(W)^\perp = \Big\{\begin{pmatrix} p(z) \\ 0 \end{pmatrix} \in \mathfrak{K}(W) : p(z) \text{ polynomial}\Big\} =$$

$$= \Big\{\begin{pmatrix} p(z) \\ 0 \end{pmatrix} : p(z) \text{ polynomial}, \deg p < \dim\mathfrak{K}_-(W)^\perp\Big\}.$$

Let Φ be defined as in Theorem 3.2. Then we conclude that Φ maps $\mathfrak{K}_-(\mathcal{T}_{\checkmark}(W))^\perp$ into $\mathfrak{K}_-(W)^\perp$, and

$$\dim\mathfrak{K}_-(W)^\perp \geq 2\dim\mathfrak{K}_-\big(\mathcal{T}_{\checkmark}(W)\big)^\perp.$$

Conversely, if l is the largest odd number $\leq \dim\mathfrak{K}_-(W)^\perp - 1$, then

$$\begin{pmatrix} z^l \\ 0 \end{pmatrix} \in \mathfrak{K}_-(W)^\perp,$$

and hence

$$\begin{pmatrix} z^{\frac{l-1}{2}} \\ 0 \end{pmatrix} \in \mathfrak{K}_-\big(\mathcal{T}_{\checkmark}(W)\big)^\perp.$$

If $\dim\mathfrak{K}_-(W)^\perp \equiv 0 \mod 2$, then

$$\frac{l-1}{2} = \frac{(\dim\mathfrak{K}_-(W)^\perp - 1) - 1}{2} = \frac{\dim\mathfrak{K}_-(W)^\perp}{2} - 1,$$

and hence $\dim\mathfrak{K}_-(\mathcal{T}_{\checkmark}(W))^\perp \geq \frac{1}{2}\dim\mathfrak{K}_-(W)^\perp$. If $\dim\mathfrak{K}_-(W)^\perp \equiv 1 \mod 2$, then

$$\frac{l-1}{2} = \frac{(\dim\mathfrak{K}_-(W)^\perp - 2) - 1}{2} = \frac{\dim\mathfrak{K}_-(W)^\perp - 1}{2} - 1,$$

and hence $\dim\mathfrak{K}_-(\mathcal{T}_{\checkmark}(W))^\perp \geq \frac{1}{2}(\dim\mathfrak{K}_-(W)^\perp - 1)$. This yields the first equality.

For any matrix W we have

$$\mathfrak{K}_-(W)^\circ = \big(\mathfrak{K}_-(W)^\perp\big)^\circ = \Big\{ \begin{pmatrix} p(z) \\ 0 \end{pmatrix} \in \mathfrak{K}(W) : p(z) \text{ polynomial} \Big\}^\circ ,$$

and it follows from [KW1, Proposition 8.3] that the space $\mathfrak{K}_-(W)^\circ$ is invariant with respect the difference quotient operator. Hence

$$\mathfrak{K}_-(W)^\circ = \Big\{ \begin{pmatrix} p(z) \\ 0 \end{pmatrix} : p(z) \text{ polynomial}, \deg p < \dim \mathfrak{K}_-(W)^\circ \Big\} .$$

Using the fact that

$$\mathfrak{K}_-(W)^\perp = \big(\mathfrak{K}_-(W)^\perp \cap \mathfrak{K}_-(W)_e\big)[\dot+]\big(\mathfrak{K}_-(W)^\perp \cap \mathfrak{K}_-(W)_o\big) =$$

$$= \operatorname{span} \Big\{ \begin{pmatrix} z^l \\ 0 \end{pmatrix} : l \text{ odd}, l < \dim \mathfrak{K}_-(W)^\perp \Big\} [\dot+]$$

$$[\dot+] \operatorname{span} \Big\{ \begin{pmatrix} z^l \\ 0 \end{pmatrix} : l \text{ even}, l < \dim \mathfrak{K}_-(W)^\perp \Big\} ,$$

the same argument as in the previous paragraph shows that

$$\dim \mathfrak{K}_-\big(\mathcal{T}_{\surd}(W)\big)^\circ = \Big[\frac{1}{2} \dim \mathfrak{K}_-(W)^\circ\Big] .$$

❑

3.2. Two characteristic values of a matrix $W \in \mathcal{M}^{ep}_{<\infty}$

It was shown in [KWW2, Proposition 4.9] that for a function $q \in \mathcal{N}^{ep}_{<\infty}$ the limit $\lim_{t\to-\infty} q(t)$ exists in $\mathbb{R}\cup\{\pm\infty\}$. If $W \in \mathcal{M}^{ep}_{<\infty}$, we obtain two essentially positive generalized Nevanlinna functions, namely $W \star \infty = w_{21}^{-1} w_{11}$ and $w_{21}^{-1} w_{22}$. We denote the respective limits by

$$m(W) := \lim_{t\to-\infty} (W \star \infty)(t), \ \lambda(W) := \lim_{t\to-\infty} \frac{w_{22}(t)}{w_{21}(t)} . \tag{3.10}$$

These numbers have interpretations in terms of the transformation $\mathcal{T}_{2,\gamma}$ and the corresponding Pontryagin spaces.

3.9. Proposition. *Let $W \in \mathcal{M}^{ep}_{<\infty}$ and $\gamma \in \mathbb{R}$. Then $m(W) \in \mathbb{R}$ if and only if $\mathfrak{K}_-(W) = \mathfrak{K}(W)$ and $\mathfrak{K}_-(\mathcal{T}_{2,\gamma}(W))^\circ = \{0\}$. Thereby $\mathfrak{K}_-(\mathcal{T}_{2,\gamma}(W)) = \mathfrak{K}(\mathcal{T}_{2,\gamma}(W))$ if and only if $m(W) = 0$. Otherwise, if $m(W) \in \mathbb{R} \setminus \{0\}$, we have $\dim \mathfrak{K}_-(\mathcal{T}_{2,\gamma}(W))^\perp = 1$.*

Proof. Since $W \in \mathcal{M}^{ep}_{<\infty}$ we have $\hat W := \mathcal{T}_{2,\gamma}(W) \in \mathcal{M}^{sym}_{<\infty}$, and, hence, may consider the space $\mathfrak{K}(\hat W)$. Keep in mind that $W = \mathcal{T}_{\surd}(\hat W)$.

Assume first that $m(W) \in \mathbb{R}$. By (3.4) we have

$$m(W) = \lim_{y\to+\infty} \big(W \star \infty\big)\big((iy)^2\big) = \lim_{y\to+\infty} \frac{1}{iy} \big(\hat W \star \infty\big)(iy) .$$

We conclude from [KW2, Theorem 5.7] that

$$\tilde{W}(z) := \begin{pmatrix} 1 & -m(W)z \\ 0 & 1 \end{pmatrix} \hat{W}(z) .$$

satisfies $\mathfrak{K}_-(\tilde{W}) = \mathfrak{K}(\tilde{W})$. In case $m(W) \neq 0$ the space generated by the linear matrix in the above relation is one-dimensional and spanned by the constant $(1,0)^T$, in fact $\mathfrak{K}(\hat{W}) = \mathfrak{K}(\tilde{W}) \oplus \operatorname{span}\{(1,0)^T\}$. Hence $\mathfrak{K}_-(\hat{W})$ is orthocomplemented in $\mathfrak{K}(\hat{W})$ and $\dim \mathfrak{K}_-(\hat{W})^\perp = 1$. We conclude from Proposition 3.8 that $\mathfrak{K}_-(W) = \mathfrak{K}(W)$. In the case $m(W) = 0$ we have $\dim \mathfrak{K}_-(\hat{W})^\perp = 0$.

Conversely, assume that $\mathfrak{K}_-(W) = \mathfrak{K}(W)$ and $\mathfrak{K}(\hat{W})^\circ = \{0\}$. Again appealing to [KW2, Theorem 5.7] we find that there exists a polynomial $p(z)$ such that

$$\tilde{W}(z) := \begin{pmatrix} 1 & p(z) \\ 0 & 1 \end{pmatrix} \hat{W}(z)$$

satisfies $\mathfrak{K}_-(\tilde{W}) = \mathfrak{K}(\tilde{W})$. Thereby $\dim \mathfrak{K}_-(\hat{W})^\perp = \deg p$. Since, by Proposition 3.8, $\dim \mathfrak{K}_-(\hat{W})^\perp \leq 1$ we can choose $p(z) = az$ and thus

$$\lim_{y\to+\infty} \big(W \star \infty\big)\big((iy)^2\big) = \lim_{y\to+\infty} \frac{1}{iy}\big(\hat{W} \star \infty\big)(iy) =$$

$$= \lim_{y\to+\infty} \frac{1}{iy}\big(\tilde{W} \star \infty\big)(iy) - a = -a .$$

❑

3.10. Proposition. *Let $W \in \mathcal{M}^{ep}_{<\infty}$ and assume that $m(W) = 0$. Then $\lambda(W) \notin \mathbb{R}$ if and only if $\operatorname{ind}_- \mathcal{T}_{2,\gamma}(W) \star \infty < \operatorname{ind}_- \mathcal{T}_{2,\gamma}(W)$ for all $\gamma \in \mathbb{R}$. If $\lambda(W) \in \mathbb{R}$, we have*

$$\lambda(W) = -\sup\big\{\gamma \in \mathbb{R} : \operatorname{ind}_- \mathcal{T}_{2,\gamma}(W) \star \infty < \operatorname{ind}_- \mathcal{T}_{2,\gamma}(W)\big\} .$$

If $\gamma, \hat{\gamma} \in \mathbb{R}$, then we have

$$\operatorname{ind}_- \mathcal{T}_{2,\hat{\gamma}}(W) = \operatorname{ind}_- \mathcal{T}_{2,\gamma}(W) + \begin{cases} -1 & , \ \gamma < -\lambda(W) \leq \hat{\gamma} \\ 0 & , \ \gamma, \hat{\gamma} \geq -\lambda(W) \ \text{or}\ \gamma, \hat{\gamma} < -\lambda(W) \\ 1 & , \ \hat{\gamma} < -\lambda(W) \leq \gamma \end{cases} .$$

Proof. According to Proposition 3.9 our assumption $m(W) = 0$ implies that $\mathfrak{K}_-(W) = \mathfrak{K}(W)$ and that $\mathfrak{K}_-(\mathcal{T}_{2,\gamma}(W)) = \mathfrak{K}(\mathcal{T}_{2,\gamma}(W))$ for all $\gamma \in \mathbb{R}$. Put $W_\gamma := \mathcal{T}_{2,\gamma}(W)$ and $E_\gamma := w_{\gamma,22} + i w_{\gamma,21}$. Note that the function

$$q(z) := \big(W_\gamma \star \infty\big)(z) = z\big(W \star \infty\big)(z^2)$$

does not depend on γ. By (3.4) we have

$$\frac{w_{22}(-y^2)}{w_{21}(-y^2)} = \frac{1}{iy}\frac{w_{\gamma,22}(iy)}{w_{\gamma,21}(iy)} - \gamma .$$

There are four possibilities:

$$\lambda(W)+\gamma=\lim_{y\to+\infty}\frac{1}{iy}\frac{w_{\gamma,22}(iy)}{w_{\gamma,21}(iy)}=$$
$$=\begin{cases}=0 & ,\ w_{\gamma,21}\notin\mathfrak{P}(E_\gamma)\\ >0 & ,\ w_{\gamma,21}\in\mathfrak{P}(E_\gamma),\ [w_{\gamma,21},w_{\gamma,21}]>0\\ <0 & ,\ w_{\gamma,21}\in\mathfrak{P}(E_\gamma),\ [w_{\gamma,21},w_{\gamma,21}]<0\\ \nexists & ,\ w_{\gamma,21}\in\mathfrak{P}(E_\gamma),\ [w_{\gamma,21},w_{\gamma,21}]=0\end{cases}\tag{3.11}$$

In the first two cases (cf. [KW2, Lemma 5.12])

$$\operatorname{ind}_- W_\gamma=\operatorname{ind}_- W_\gamma\star\infty=\operatorname{ind}_- q\,,$$

and in the third

$$\operatorname{ind}_- W_\gamma=\operatorname{ind}_- W_\gamma\star\infty+1=\operatorname{ind}_- q+1\,.$$

Assume now that $\lambda(W)\in\mathbb{R}$, i.e. we are in one of the first three cases of (3.11). Then

$$\operatorname{ind}_- W_\gamma=\begin{cases}\operatorname{ind}_- q & ,\ \gamma\geq-\lambda(W)\\ \operatorname{ind}_- q+1 & ,\ \gamma<-\lambda(W)\end{cases}$$

Hence, in this case, the assertion of the lemma follows.

Consider the case that $\lambda(W)\notin\mathbb{R}$, i.e. that $w_{\gamma,21}\in\mathfrak{P}(E_\gamma)$ and $[w_{\gamma,21},w_{\gamma,21}]=0$. Then, by the proof of [KW2, Theorem 7.1], for all $l<0$

$$\operatorname{ind}_- W_\gamma W_{(l,0)}=\operatorname{ind}_- W_\gamma\,.$$

It follows that $\operatorname{ind}_- W_\gamma=\operatorname{ind}_- W_{\hat\gamma}$ and that $\operatorname{ind}_- W_\gamma\star\infty<\operatorname{ind}_- W_\gamma$ for all $\gamma,\hat\gamma\in\mathbb{R}$ (cf. [KW3]).

❑

3.3. Structure of symmetric and semibounded dB-spaces

The above Proposition 3.9 has two consequences on the structure of symmetric and semibounded dB-Pontryagin spaces, which we shall elaborate in the following. The first one gives a growth restriction on the elements of a semibounded dB-space. For the proof we use a lemma which supplements [KW5, Theorem 3.17, Corollary 3.18].

3.11. Lemma. *Let $q\in\mathcal{N}_{<\infty}$ and let B be an entire function of finite order ρ. If $A(z):=q(z)B(z)$ is entire, then the order of A is also equal to ρ. Moreover, B is of finite type if and only if A is. If ρ is not an integer, then B being of minimal type is equivalent to A possessing the same property.*

Proof. By [DLLS] we can write $q(z)=r(z)\cdot q_1(z)$ with $q_1\in\mathcal{N}_0$ and a rational function r. This shows that it suffices to prove the assertion for the case $\kappa=0$.

By our assumption the function q must be meromorphic in the whole plane and hence we may write according to [L, VII.Lehrsatz 1]

$$q(z) = c\frac{z-a_0}{z-b_0}\prod_{k\neq 0}\Big(1-\frac{z}{a_k}\Big)\Big(1-\frac{z}{b_k}\Big)^{-1}. \tag{3.12}$$

Denote the zeros of B by x_k. Since $q(z)\cdot B(z)$ is entire, the zero set of B splits up as $\{x_k\}=\{b_k\}\cup X$. Here and for the rest of this proof we understand that a zero is listed as often as its multiplicity states, and also understand set theoretic notations as including multiplicities. It follows that the zeros $\{y_k\}$ of A are given by $\{y_k\}=\{a_k\}\cup X$. From [KWW2, Lemma 4.5] we conclude that the convergence exponents of $\{x_k\}$ and $\{y_k\}$ are equal and that the upper densities of these sequences coincide:

$$\limsup_{r\to\infty}\frac{1}{r^\rho}\big|\{x_k:\,|x_k|\le r\}\big| = \limsup_{r\to\infty}\frac{1}{r^\rho}\big|\{y_k:\,|y_k|\le r\}\big|.$$

Moreover, the values

$$\limsup_{r\to\infty}\sum_{|x_k|\le r}\frac{1}{x_k^\rho},\ \limsup_{r\to\infty}\sum_{|x_k|\le r}\frac{1}{y_k^\rho}$$

are together finite or infinite. If we arrange the sequences (a_k) and (b_k) so that $a_k<b_k<a_{k+1}$, for all $l\in\mathbb{N}$ the series

$$\sum_k\Big[\Big(\frac{1}{a_k}\Big)^l-\Big(\frac{1}{b_k}\Big)^l\Big] \tag{3.13}$$

converges. Consider the Hadamard factorization of B:

$$B(z)=e^{P(z)}\prod\Big(1-\frac{z}{x_k}\Big)\exp\Big[\sum_{l=1}^p\frac{1}{l}\big(\frac{z}{x_k}\big)^l\Big]=$$

$$=e^{P(z)}\prod\Big(1-\frac{z}{b_k}\Big)\exp\Big[\sum_{l=1}^p\frac{1}{l}\big(\frac{z}{b_k}\big)^l\Big]\prod_X\Big(1-\frac{z}{x_k}\Big)\exp\Big[\sum_{l=1}^p\frac{1}{l}\big(\frac{z}{x_k}\big)^l\Big].$$

Here P is a polynomial of degree at most ρ and p is the genus of the zeros of B. From $A=qB$, (3.12) and the convergence of the series (3.13) it follows that the entire function $\tilde{P}$ in the product representation

$$A(z)=e^{\tilde{P}(z)}\prod\Big(1-\frac{z}{a_k}\Big)\exp\Big[\sum_{l=1}^p\frac{1}{l}\big(\frac{z}{a_k}\big)^l\Big]\prod_X\Big(1-\frac{z}{x_k}\Big)\exp\Big[\sum_{l=1}^p\frac{1}{l}\big(\frac{z}{x_k}\big)^l\Big]$$

of A must be equal to

$$\tilde{P}(z)=\log\big(c\frac{a_0}{b_0}\big)\cdot P(z)\cdot\sum_{l=1}^p\frac{z^l}{l}\sum_k\Big[\Big(\frac{1}{b_k}\Big)^l-\Big(\frac{1}{a_k}\Big)^l\Big].$$

Hence $\tilde{P}$ is in fact a polynomial of degree at most ρ. It follows that A is of finite order at most ρ. Since in this argument the roles of A and B can be exchanged, we conclude that the order of A actually equals ρ. The assertion of the lemma now follows from what was said above by Lindelöf's Theorem, see e.g. [L, I.Lehrsatz

14,15].

❑

Let us remark that the assumption $\rho \notin \mathbb{Z}$ in the last part of Lemma 3.11 cannot be dropped.

3.12. Proposition. *Let $W \in \mathcal{M}^{ep}_{<\infty}$. Then every entry w_{ij} is an entire function of growth at most order $\frac{1}{2}$, finite type. In particular, if $\mathfrak{P}$ is a semibounded dB-space such that $1 \in \operatorname{Assoc}\mathfrak{P} := \mathfrak{P} + z\mathfrak{P}$, then*

$$\sup_{F\in\operatorname{Assoc}\mathfrak{P}} \limsup_{r\to\infty} \frac{\log \max_{|z|=r} |F(z)|}{\sqrt{r}} < \infty \tag{3.14}$$

Proof. Consider the matrix $\hat{W} := \mathcal{T}_{2,0}(W) \in \mathcal{M}^{sym}_{<\infty}$. Since at most one of $\mathfrak{K}_-(\hat{W})$ and $\mathfrak{K}_+(\hat{W})$ is degenerated, we conclude that $1 \in \operatorname{Assoc}\mathfrak{P}(E)$ where E either is $\hat{w}_{11} - i\hat{w}_{12}$ or $\hat{w}_{22} + i\hat{w}_{21}$. In any case we conclude from [KWW2, Theorem 3.10] that E and, hence, one entry of $\hat{W}$ is of exponential type. Lemma 3.11 now implies that all entries of $\hat{W}$ are of exponential type. Therefore the entries of W are of growth at most order $\frac{1}{2}$, finite type.

Let $\mathfrak{P}$ be a semibounded dB-space and choose an inner product $(.,.)$ on $\mathfrak{P}$ such that $(\mathfrak{P}, (.,.)) = \mathfrak{P}(E)$ is a semibounded dB-Pontryagin space, this is possible by [KWW4]. Since $\mathfrak{P}$ and $\mathfrak{P}(E)$ coincide as sets, we have $\operatorname{Assoc}\mathfrak{P} = \operatorname{Assoc}\mathfrak{P}(E)$. If $1 \in \operatorname{Assoc}\mathfrak{P}$ hence also $1 \in \operatorname{Assoc}\mathfrak{P}(E)$, and we conclude that there exists a matrix $W \in \mathcal{M}^{ep}_{<\infty}$ such that $(0,1)W = (-B, A)$. By what was proved in the first paragraph the function E is of growth at most order $\frac{1}{2}$, finite type. The relation (3.14) now follows from [KW5, Theorem 3.4].

❑

In order to give the second promised structure result on dB-spaces, we need to recall a construction introduced in [KWW4]: If $\mathfrak{P}$ is a symmetric dB-space, then define

$$\mathfrak{P}_+ := \{F \in \mathcal{O}(\mathbb{C}) : F(z^2) \in \mathfrak{P}\}.$$

If $\mathfrak{P}_+$ is endowed with a topology and inner product so that the map $F(z) \mapsto F(z^2)$ becomes an isometric homeomorphism, this space is a semibounded dB-space. The main result of [KWW4] states that every semibounded dB-space can be obtained in this way and determines the kernel of the assignment $\Upsilon : \mathfrak{P} \mapsto \mathfrak{P}_+$. Moreover, if $\mathfrak{P}$ and hence also $\mathfrak{P}_+$ is a dB-Pontryagin space, the action of the assignment Υ is explicitly determined in terms of the respective generating Hermite-Biehler functions.

This construction on the level of dB-spaces is the exact analogue of the transformations $\mathcal{T}_{\surd}$, $\mathcal{T}_{2,\gamma}$ on the level of matrix functions. This follows by comparing the definition of $\mathcal{T}_{\surd}$ and $\mathcal{T}_{2,\gamma}$ with [KWW4, Theorem 4.5].

3.13. Lemma. *Let $W \in \mathcal{M}^{sym}_{<\infty}$. Then $E_W \in \mathcal{HB}^{sym}$, $E_{\mathcal{T}_{\surd}(W)} \in \mathcal{HB}^{ep}$, and*

$$\mathfrak{P}(E_W)_+ = \mathfrak{P}(E_{\mathcal{T}_{\surd}(W)}).$$

Conversely, let $W \in \mathcal{M}^{ep}_{<\infty}$ *and* $\gamma \in \mathbb{R}$. *Then* $E_W \in \mathcal{HB}^{ep}$, $E_{\mathcal{T}_{2,\gamma}(W)} \in \mathcal{HB}^{sym}$, *and*

$$\mathfrak{P}(E_{\mathcal{T}_{2,\gamma}(W)})_+ = \mathfrak{P}(E_W)\,.$$

Each two spaces $\mathfrak{P}(E_{\mathcal{T}_{2,\gamma}(W)})$ *are not isometrically equal and every dB-Pontryagin space* $\mathfrak{P}$ *with* $\mathfrak{P}_+ = \mathfrak{P}(E_W)$ *is of this form.*

It was shown in [KWW4, Proposition 2.6] that every even function $F \in \operatorname{Assoc}\mathfrak{P}$ can be obtained as $F(z) = G(z^2)$ with $G \in \operatorname{Assoc}\mathfrak{P}_+$. In particular, if $1 \in \operatorname{Assoc}\mathfrak{P}$, then also $1 \in \operatorname{Assoc}\mathfrak{P}_+$. The converse does not hold in general.

3.14. Proposition. *Let* $E = A - iB \in \mathcal{HB}^{sym}_{<\infty}$ *and put* $E_+ := A_+ - iB_+$ *with*

$$A_+(z^2) = A(z),\ B_+(z^2) = zB(z)\,,$$

so that $\mathfrak{P}(E)_+ = \mathfrak{P}(E_+)$. *Assume that* $1 \in \operatorname{Assoc}\mathfrak{P}(E)_+$ *and let* $W_+ \in \mathcal{M}^{ep}_{<\infty}$ *be such that* $\mathfrak{K}_-(W_+) = \mathfrak{K}(W_+)$ *and* $(0,1)W_+ = (-B_+, A_+)$. *Then* $1 \in \operatorname{Assoc}\mathfrak{P}(E)$ *if and only if* $m(W_+) \in \mathbb{R}$. *In this case there exists* $W \in \mathcal{M}^{sym}_{<\infty}$, $\mathfrak{K}_-(W) = \mathfrak{K}(W)$, *such that* $(0,1)W = (-B, A)$.

Proof. By its definition the function B_+ has only finitely many zeros in $\mathbb{C} \setminus [0,\infty)$. Hence $W_+ \in \mathcal{M}^{ep}_{<\infty}$ and thus $\hat{W} := \mathcal{T}_{2,0}(W_+) \in \mathcal{M}^{sym}_{<\infty}$. By the definition of $\mathcal{T}_{2,0}$ the matrix $\hat{W}$ satisfies $(0,1)\hat{W} = (-B, A)$. We have $1 \in \operatorname{Assoc}\mathfrak{P}(E)$ if and only if $\mathfrak{K}_-(\hat{W})^\circ = \{0\}$, cf. [KW1, Proposition 10.3], [KW2, Lemma 5.11]. This, however, is in view of Proposition 3.9 equivalent to $m(W_+) \in \mathbb{R}$.

❑

4. Chains of matrix functions

In this section we investigate chains of matrix functions and introduce the appropriate analogues of the notion of symmetry and semiboundedness on the level of chains of matrices.

Let us recall the notion of a maximal chain of matrices as introduced in [KW3]. For a matrix $W \in \mathcal{M}$ denote by $\mathfrak{t}(W)$ the *trace function* $\mathfrak{t}(W) := \operatorname{tr}(W'(0)J)$.

4.1. Definition. A mapping $\omega : \mathcal{I} \to \mathcal{M}_{<\infty}$ is called a *maximal chain of matrices* if the following axioms are satisfied:

- **(W1)** The set $\mathcal{I}$ equals $(0, M)$, $0 < M < \infty$, with possible exception of finitely many points.
- **(W2)** The function ω is not constant on any interval contained in $\mathcal{I}$.
- **(W3)** For all $s, t \in \mathcal{I}$, $s \leq t$, we have $\omega(s)^{-1}\omega(t) \in \mathcal{M}_{<\infty}$ and
$$\operatorname{ind}_- \omega(t) = \operatorname{ind}_- \omega(s) + \operatorname{ind}_- \omega(s)^{-1}\omega(t)\,.$$
- **(W4)** If $t \in \mathcal{I}$ and for some $W \in \mathcal{M}_{<\infty}$, $W \neq I$, we have $W^{-1}\omega(t) \in \mathcal{M}_{<\infty}$ and $\operatorname{ind}_- \omega(t) = \operatorname{ind}_- W + \operatorname{ind}_- W^{-1}\omega(t)$, then there exists a number $s \in \mathcal{I}$ such that $W = \omega(s)$.

(W5) We have $\lim_{t\nearrow M}\mathfrak{t}(\omega(t))=+\infty$. If $\mathcal{I}$ is not connected, there exist numbers $s<t$, both contained in the last connected component $\mathcal{I}_\infty$ of $\mathcal{I}$ (that is $\sup\mathcal{I}_\infty=M$), such that $\omega(s)^{-1}\omega(t)$ is not a linear polynomial.

It is proved in [KW3, Lemma 3.5] that the function $\operatorname{ind}_-\omega(t)$ is constant on each connected component of $\mathcal{I}$ and takes different values on different components. Moreover, by (W3), it is nondecreasing. In particular, it is bounded and attains its maximum on $\mathcal{I}_\infty$. Let us define $\operatorname{ind}_-\omega:=\max_{t\in\mathcal{I}}\operatorname{ind}_-\omega(t)$. The set of all maximal chains ω with $\operatorname{ind}_-\omega=\kappa$ will be denoted by $\mathfrak{M}_\kappa$. Moreover,

$$\mathfrak{M}_{\leq\kappa}:=\bigcup_{\nu\leq\kappa}\mathfrak{M}_\nu,\ \mathfrak{M}_{<\infty}:=\bigcup_{\nu\in\mathbb{N}\cup\{0\}}\mathfrak{M}_\nu\,.$$

It is already seen from the axiom (W5) that points s,t where the transfer matrix $\omega(s)^{-1}\omega(t)$ is a linear polynomial play a special role. This is formalized by the notion of indivisible intervals. Let $\omega:\mathcal{I}\to\mathcal{M}_{<\infty}$ be a maximal chain of matrices. An interval $(s,t)\subseteq\mathcal{I}$ is called *indivisible of type* $\phi\in[0,\pi)$ if for all $s',t'\in(s,t)$

$$\omega(s')^{-1}\omega(t')=W_{(l(s',t'),\phi)}$$

The number $L:=\sup\{l(s',t'):\ s'\leq t',s',t'\in(s,t)\}$ is called the *length* of the indivisible interval (s,t).

If (s_1,t_1) and (s_2,t_2) are indivisible intervals of types ϕ_1 and ϕ_2, respectively, which have nonempty intersection, then $\phi_1=\phi_2$ and $(\min\{s_1,s_2\},\max\{t_1,t_2\})$ is again indivisible of the same type. Hence every indivisible interval is contained in a maximal indivisible interval.

It can happen that for some $s,t\in\mathcal{I}$, $s\leq t$, we have $\omega(s)^{-1}\omega(t)=W_{(l,\phi)}$ for some $l<0$. Although in this case $(s,t)\notin\mathcal{I}$ we shall speak of an *indivisible interval of negative length.*

Chains which can be obtained out of each other by a change of variable will share their important properties. More precisely: Let $\mathcal{J}_1,\mathcal{J}_2$ be open subsets of $\mathbb{R}$ and let $\omega_i:\mathcal{J}_i\to\mathcal{M}_{<\infty}$ be functions. Then we say that ω_2 is a *reparameterization* of ω_1 if there exists an increasing and bijective map $\alpha:\mathcal{J}_2\to\mathcal{J}_1$ such that $\omega_2=\omega_1\circ\alpha$. In this case we write $\omega_2\sim\omega_1$. It is obvious that $\sim$ is an equivalence relation.

A central role in the theory of maximal chains of matrices is played by the Weyl coefficient associated to a maximal chain. It is proved in [KW2] that for all functions $\tau\in\mathcal{N}_0$ the limit

$$q_\infty(\omega)(z):=\lim_{t\nearrow\sup\mathcal{J}}(\omega(t)\star\tau)(z)$$

exists locally uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$ with respect to the chordal metric and does not depend on the particular choice of τ. It is called the *Weyl coefficient* of the chain ω. The main result of [KW2] states that the set $\mathfrak{M}_\kappa/\sim$ bijectively corresponds to $\mathcal{N}_\kappa$ via

$$\omega/\sim\ \longmapsto\ q_\infty(\omega)\,.$$

The elements of maximal chains can be characterized by means of decompositions of the Weyl coefficient. Recall the following fact from [KW3]: Let $\omega : \mathcal{I} \to \mathcal{M}_{<\infty}$ be a maximal chain and let $W \in \mathcal{M}_{<\infty}$. Then $W \in \omega(\mathcal{I})$ if and only if $\operatorname{ind}_- W \leq \operatorname{ind}_- \omega$ and there exists a function $\tau \in \mathcal{N}_{\operatorname{ind}_- \omega - \operatorname{ind}_- W}$ such that $q_\infty(\omega) = W \star \tau$.

Let a function $\upsilon : \mathcal{J} \to \mathcal{M}_{<\infty}$ be given. In the next lemma we give conditions, adapted to our needs, under which υ can be extended to a maximal chain. This result is an immediate consequence of [KW3, Lemma 3.7] and [KW2, Lemma 8.5] with its proof.

4.2. Lemma. *Let $\upsilon : \mathcal{J} \to \mathcal{M}_{<\infty}$ be given and assume that υ satisfies (W3) and*

(C1) *We have $\kappa_m := \sup_{t\in\mathcal{J}} \operatorname{ind}_- \upsilon(t) < \infty$.*

(C2) *The following two implications hold true.*

(a) If $\limsup_{t\nearrow\sup\mathcal{J}} \mathfrak{t}(\omega(t)) < +\infty$, then $\lim_{t\nearrow\sup\mathcal{J}} \omega(t) \in \mathcal{M}_\kappa$.

(b) If $\limsup_{t\nearrow\sup\mathcal{J}} \mathfrak{t}(\omega(t)) = +\infty$ and if there is a number $\phi \in [0,\pi)$ such that

$$\omega(s)^{-1}\omega(t) = W_{(l(s,t),\phi)},\ \ s,t \in \omega^{-1}\big(\mathcal{M}_{\kappa_m}\big)\,,$$

then for one (and hence for all) $t \in \omega^{-1}\big(\mathcal{M}_{\kappa_m}\big)$

$$\omega(t) \star \cot\phi \in \mathcal{N}_{\kappa_m}\,.$$

Then there exists a maximal chain $\omega \in \mathfrak{M}_{\kappa_m}$ and a nondecreasing function $\lambda : \mathcal{J} \to \mathcal{I}$ such that $\upsilon = \omega \circ \lambda$. If $\limsup_{t\nearrow\sup\mathcal{J}} \mathfrak{t}(\omega(t)) = +\infty$, the chain ω is unique and we have

$$q_\infty(\omega) = \lim_{t\nearrow\sup\mathcal{J}} \big[\upsilon(t) \star \tau\big],\ \ \tau \in \mathcal{N}_0\,.$$

Otherwise, the set of all extensions in $\mathfrak{M}_{\kappa_m}$ is parameterized by the set of all functions $\tau \in \mathcal{N}_0$ with the property that

$$\operatorname{ind}_- \Big[\lim_{t\nearrow\sup\mathcal{J}} \upsilon(t)\Big] \star \tau = \operatorname{ind}_- \lim_{t\nearrow\sup\mathcal{J}} \upsilon(t)\,.$$

This correspondence is established via the relation

$$q_\infty(\omega) = \Big[\lim_{t\nearrow\sup\mathcal{J}} \upsilon(t)\Big] \star \tau\,.$$

For the purposes of the present paper two particular kinds of chains of matrices are of interest.

4.3. Definition. Let $\omega \in \mathfrak{M}_\kappa$. We write $\omega \in \mathfrak{M}_\kappa^{sym}$ if $\omega(t) \in \mathcal{M}_{<\infty}^{sym}$ for all $t \in \mathcal{I}$. Moreover, we write $\omega \in \mathfrak{M}_\kappa^{ep}$ if $\omega(t) \in \mathcal{M}_{<\infty}^{ep}$ for all $t \in \mathcal{I}$ and if the number of zeros which an entry of $\omega(t)$ possesses in $\mathbb{C} \setminus [0,\infty)$ is bounded independently of $t \in \mathcal{I}$.

The following inverse result gives a connection between the classes $\mathfrak{M}_{<\infty}^{sym}$ and $\mathcal{N}_{<\infty}^{sym}$. The corresponding result for the class $\mathfrak{M}_{<\infty}^{ep}$ will be seen later, cf. Proposition 5.6.

4.4. Proposition. *Let ω be a maximal chain of matrices. Then $\omega \in \mathfrak{M}_{<\infty}^{sym}$ if and only if $q_\infty(\omega) \in \mathcal{N}_{<\infty}^{sym}$.*

Proof. Assume that $\omega \in \mathfrak{M}_{<\infty}^{sym}$. Then for all $t \in \mathcal{I}$ the function $\omega(t) \star \infty$ is odd. Since

$$\lim_{t \to \sup \mathcal{I}} \big(\omega(t) \star \infty\big)(z) = q_\infty(\omega)$$

we conclude that also $q_\infty(\omega)$ belongs to $\mathcal{N}_{<\infty}^{sym}$.

Assume conversely that $q_\infty(\omega)$ is odd and let $t \in \mathcal{I}$ be given. Then there exists a function $\tau \in \mathcal{N}_{<\infty}$, $\operatorname{ind}_- \tau = \operatorname{ind}_- \omega - \operatorname{ind}_- \omega(t)$, such that

$$q_\infty(\omega)(z) = \big(\omega(t) \star \tau\big)(z) = \frac{\omega(t)_{11}(z)\tau(z) + \omega(t)_{12}(z)}{\omega(t)_{21}(z)\tau(z) + \omega(t)_{22}(z)} \,.$$

It follows that

$$q_\infty(\omega)(z) = -q_\infty(\omega)(-z) = \frac{\omega(t)_{11}(-z)[-\tau(-z)] - \omega(t)_{12}(-z)}{-\omega(t)_{21}(-z)[-\tau(-z)] + \omega(t)_{22}(-z)} =$$

$$= \big(- JV\omega(t)^{-1}VJ\big)(-z) \star [-\tau(-z)] \,.$$

Hereby the constant matrices J and V are defined as in Lemma 2.3. With $\omega(t)$ and τ we also have $(-JV\omega(t)^{-1}VJ)(-z) \in \mathcal{M}_{<\infty}$ and $-\tau(-z) \in \mathcal{N}_{<\infty}$, respectively. In fact

$$\operatorname{ind}_- \big(- JV\omega(t)^{-1}VJ\big)(-z) = \operatorname{ind}_- \omega(t)(z),\ \operatorname{ind}_- \big(- \tau(-z)\big) = \operatorname{ind}_- \tau(z) \,.$$

Hence

$$\operatorname{ind}_- q_\infty(\omega)(z) = \operatorname{ind}_- \omega(t) + \operatorname{ind}_- \tau =$$

$$= \operatorname{ind}_- \big(- JV\omega(t)^{-1}VJ\big)(-z) + \operatorname{ind}_- \big(- \tau(-z)\big) \,.$$

We conclude that both, $\omega(t)$ and $(-JV\omega(t)^{-1}VJ)(-z)$, are members of the maximal chain of matrices having $q_\infty(\omega)$ as its Weyl coefficient. Since $\operatorname{ind}_-(-JV\omega(t)^{-1}VJ)(-z) = \operatorname{ind}_- \omega(t)$ and $\mathfrak{t}((-JV\omega(t)^{-1}VJ)(-z)) = \mathfrak{t}(\omega(t))$ we conclude that $(-JV\omega(t)^{-1}VJ)(-z) = \omega(t)$, cf. [KW3, Lemma 3.5]. This means that $\omega(t) \in \mathcal{M}_{<\infty}^{sym}$.

❑

The following technical condition on a maximal chain ω will appear frequently:

(K₋) For all $t \in \mathcal{I}$ we have $\mathfrak{K}_-(\omega(t)) = \mathfrak{K}(\omega(t))$.

Recall from [KW2] that ω satisfies (K$_-$) if and only if for one $t \in \mathcal{I}$ the equality $\mathfrak{K}_-(\omega(t)) = \mathfrak{K}(\omega(t))$ holds. Moreover, the inverse result [KW2, Theorem 5.7] shows that ω satisfies (K$_-$) if and only if $\lim_{y\to+\infty} y^{-1}q_\infty(\omega)(iy) = 0$.

We need two general constructions which can be made with chains of matrices. The first one formalizes the intuitive idea of *linking of chains*, compare the discussion after [KW2, Theorem 7.1]. Let υ_1 and υ_2 be functions into $\mathcal{M}_{<\infty}$ defined on open subsets $\mathcal{J}_1, \mathcal{J}_2$ of $\mathbb{R}$. If both, $\mathcal{J}_1$ and $\mathcal{J}_2$ are nonempty we define a function $\upsilon_1 \uplus \upsilon_2$ as follows: Choose increasing bijections α_i of $\mathcal{J}_i$ onto open sets with the property that

$$\inf \alpha_i(\mathcal{J}_i) = i-1,\ \sup \alpha_i(\mathcal{J}_i) = i,\ i = 1,2 \,,$$

and define $\upsilon_1 \uplus \upsilon_2 : \alpha_1(\mathcal{J}_1) \cup \alpha_2(\mathcal{J}_2) \to \mathcal{M}_{<\infty}$ by

$$\upsilon_1 \uplus \upsilon_2(t) := \begin{cases} (\upsilon_1 \circ \alpha_1^{-1})(t) & , \ t \in \alpha_1(\mathcal{J}_1) \\ (\upsilon_2 \circ \alpha_2^{-1})(t) & , \ t \in \alpha_2(\mathcal{J}_2) \end{cases}$$

Note that this definition is independent of the choice of α_1, α_2 if we identify functions which are equal up to reparameterization. For the function ε with empty domain we set

$$\upsilon \uplus \varepsilon = \varepsilon \uplus \upsilon = \upsilon \,.$$

Up to reparameterization the operation $\uplus$ is associative.

The second construction is simply *extension by continuity*: Assume that the function υ is defined on a set $\mathcal{J}$ of the form $(a,b) \setminus \{x_1, \ldots, x_n\}$. Let L be the set of all those points x_i such that the limit $\lim_{x \to x_i} \upsilon(x)$ exists. Then we can define $\mathsf{C}\upsilon : \mathcal{J} \cup L \to \mathcal{M}_{<\infty}$ by

$$\mathsf{C}\upsilon(t) := \begin{cases} \upsilon(t) & , \ t \in \mathcal{I} \\ \lim_{x \to x_i} \upsilon(x) & , \ t = x_i \in L \end{cases}$$

5. Transformation of matrix chains

The transformation $\mathcal{T}_{\surd}$ can be applied pointwise to a maximal chain of matrices. The outcome will almost be a maximal chain.

5.1. The square root transformation

Consider a chain $\omega : \mathcal{I} \to \mathcal{M}_{<\infty}$ in $\mathfrak{M}_\kappa^{sym}$. Write the index set $\mathcal{I}$ as

$$\mathcal{I} = (0, \sigma_1) \cup (\sigma_1, \sigma_2) \cup \ldots \cup (\sigma_n, M) \,,$$

and put $\sigma_0 := 0, \sigma_{n+1} := M$. To each point σ_i, $i = 1, \ldots, n+1$, we associate a function ς_i according to the following table:

<table>
<tr><th>$\lim_{t\nearrow\sigma_i} \mathcal{T}_{\surd}(\omega(t))$</th><th>$\lim_{t\searrow\sigma_i} \mathcal{T}_{\surd}(\omega(t))$</th><th colspan="2">Definition of $\varsigma_{\mathbf{i}}$
$i = 1, \dots, n:$
$\alpha := \operatorname{ind}_- \lim_{t\searrow\sigma_i} \mathcal{T}_{\surd}(\omega(t)) - \operatorname{ind}_- \lim_{t\nearrow\sigma_i} \mathcal{T}_{\surd}(\omega(t))$</th></tr>
<tr><td rowspan="2">$\exists$</td><td rowspan="2">$\exists$</td><td>$\alpha = 0$</td><td>$l \mapsto \lim_{t\nearrow\sigma_i} \mathcal{T}_{\surd}(\omega(t)) \cdot W_{(l,0)}, \quad l \in (0, L)$
$L := \mathfrak{t}(\lim_{t\searrow\sigma_i} \mathcal{T}_{\surd}(\omega(t))) - \mathfrak{t}(\lim_{t\nearrow\sigma_i} \mathcal{T}_{\surd}(\omega(t)))$</td></tr>
<tr><td>$\alpha \neq 0$</td><td>$\left[\begin{array}{c} l \mapsto \lim\limits_{t\nearrow\sigma_i} \mathcal{T}_{\surd}(\omega(t)) \cdot W_{(l,0)} \\ l\in(0,\infty) \end{array} \right] \uplus \left[\begin{array}{c} l \mapsto \lim\limits_{t\searrow\sigma_i} \mathcal{T}_{\surd}(\omega(t)) \cdot W_{(l,0)} \\ l\in(-\infty,0) \end{array} \right]$</td></tr>
<tr><td>$\nexists$</td><td>$\exists$</td><td colspan="2">$l \mapsto \lim_{t\searrow\sigma_i} \mathcal{T}_{\surd}(\omega(t)) \cdot W_{(l,0)}, \quad l \in (-\infty, 0)$</td></tr>
<tr><td>$\exists$</td><td>$\nexists$</td><td colspan="2">$l \mapsto \lim_{t\nearrow\sigma_i} \mathcal{T}_{\surd}(\omega(t)) \cdot W_{(l,0)}, \quad l \in (0, \infty)$</td></tr>
<tr><td>$\nexists$</td><td>$\nexists$</td><td colspan="2">ε</td></tr>
<tr><td colspan="4">$i = n + 1:$</td></tr>
<tr><td>$\exists$</td><td></td><td colspan="2">$l \mapsto \lim_{t\nearrow\sigma_i} \mathcal{T}_{\surd}(\omega(t)) \cdot W_{(l,0)}, \quad l \in (0, \infty)$</td></tr>
<tr><td>$\nexists$</td><td></td><td colspan="2">ε</td></tr>
</table>

Define a function $\omega_{\surd}$ as

$$\omega_{\surd} := \mathsf{C}\big(\mathcal{T}_{\surd} \circ \omega|_{(0,\sigma_1)} \uplus \varsigma_1 \uplus \mathcal{T}_{\surd} \circ \omega|_{(\sigma_1,\sigma_2)} \uplus \varsigma_2 \uplus \dots \uplus \varsigma_n \uplus \mathcal{T}_{\surd} \circ \omega|_{(\sigma_n,M)} \uplus \varsigma_{n+1}\big)$$

5.1. Theorem. *Let $\omega \in \mathfrak{M}_\kappa^{sym}$ and assume that ω satisfies (K_-). Let ϖ be the maximal chain with $zq_\infty(\varpi)(z^2) = q_\infty(\omega)(z)$. Then there exist functions $\lambda : \operatorname{dom}\varpi \to \operatorname{dom}\omega_{\surd}$, $\mu : \operatorname{dom}\omega_{\surd} \to \operatorname{dom}\varpi$ such that*

$$\varpi = \omega_{\surd} \circ \lambda, \ \omega_{\surd} = \varpi \circ \mu\,.$$

The proof of this theorem will be carried out in four steps. Before we come to the first step, we need to provide two lemmata. The first of which is implicitly contained in [dB1], the second one follows from the considerations in [KW2]. We shall however provide complete proofs.

5.2. Lemma. *Let $\omega = (W_t)_{t\in\mathcal{I}} \in \mathfrak{M}_{<\infty}$ and let $\mathcal{I}_1$ be a connected component of $\mathcal{I}$. Assume that for all $s, t \in \mathcal{I}_1$ the transfer matrix $W_{st} = \omega(s)^{-1}\omega(t)$ is a polynomial and that*

$$n := \sup_{s,t\in\mathcal{I}_1} \deg W_{st} < \infty\,.$$

Then

$$\mathcal{I}_1 = (m_0, m_1] \cup [m_1, m_2] \cup \dots \cup [m_{n-1}, m_n)\,,$$

where

$$\inf \mathcal{I}_1 = m_0 < m_1 < \ldots < m_{n-1} < m_n = \sup \mathcal{I}_1 \,,$$

and where the intervals (m_{i-1}, m_i) *are indivisible in* ω *of certain types* $\phi_i \in [0,\pi)$ *with* $\phi_i \neq \phi_{i+1}$.

Proof. Let us start with the following remark: If $E \in \mathcal{H}B_{<\infty}$ is a polynomial, then the chain of dB-subspaces of $\mathfrak{P}(E)$ is given by

$$\{0\} \subsetneq \operatorname{span}\{1\} \subsetneq \operatorname{span}\{1,z\} \subsetneq \ldots \subsetneq \operatorname{span}\{1,z,\ldots,z^{\deg E-1}\} = \mathfrak{P}(E)\,.$$

If $\operatorname{ind}_- \mathfrak{P}(E) = 0$, every dB-subspace is also positive definite and in particular nondegenerated.

It follows from [dB1] that a polynomial matrix $W \in \mathcal{M}_0$ can be factorized uniquely as

$$W = W_{(l_1,\phi_1)} \cdot \ldots \cdot W_{(l_n,\phi_n)}$$

with $n = \deg W$, $l_i > 0$, $\phi_i \in [0,\pi)$, $\phi_i \neq \phi_{i+1}$.

Let us come to the proof of the present assertion. Choose $s_0, t_0 \in \mathcal{I}_1$, $s_0 < t_0$, such that $\deg W_{s_0t_0} = n$. Since $\operatorname{ind}_- W_{s_0t_0} = 0$, we can factorize $W_{s_0t_0}$ as

$$W_{s_0t_0} = W_{(l_1,\phi_1)} \cdot \ldots \cdot W_{(l_n,\phi_n)}\,.$$

If $t \in \mathcal{I}_1$, $t > t_0$, we have

$$W_{s_0t} = W_{s_0t_0} W_{t_0t} = W_{(l_1,\phi_1)} \cdot \ldots \cdot W_{(l_n,\phi_n)} W_{t_0t}\,.$$

On the other hand we have the factorization

$$W_{s_0t} = W_{(l_1',\phi_1')} \cdot \ldots \cdot W_{(l_k',\phi_k')}\,.$$

By uniqueness of the factorization we obtain $k = n$, $l_i = l_i'$, $\phi_i = \phi_i'$ for $i = 1,\ldots,n-1$ and

$$W_{(l_n',\phi_n')} = W_{(l_n,\phi_n)} W_{t_0t}\,.$$

This implies $\phi_n = \phi_n'$ and $W_{t_0t} = W_{(l,\phi_n)}$ for some $l > 0$.

The same argument shows that for all $s \in \mathcal{I}_1$, $s < s_0$, the transfer matrix W_{ss_0} is of the form $W_{(l,\phi_1)}$.

If we choose $m_i \in \mathcal{I}_1$, $i = 1,\ldots,n-1$, such that

$$W_{m_i} = W_{s_0} \prod_{j=1}^{i} W_{(l_j,\phi_j)}$$

we obtain the desired result.

❑

Note that in the situation of the previous lemma the types ϕ_i in the assertion are exactly the types occurring in the factorization of any transfer matrix of maximal degree.

5.3. Lemma. *If $W \in \mathcal{M}_{<\infty}$ and $W \star \infty = \infty$, then*

$$W = \begin{pmatrix} 1 & p(z) \\ 0 & 1 \end{pmatrix} \tag{5.1}$$

for some polynomial p. The matrix (5.1) *can not be decomposed as $W_1 W_2$ with $W_1, W_2 \in \mathcal{M}_{<\infty}$ and* $\operatorname{ind}_- W = \operatorname{ind}_- W_1 + \operatorname{ind}_- W_2$ *differently than in the form $W = [W W_{(-l,0)}] W_{(l,0)}$ or $W = W_{(l,0)}[W_{(-l,0)} W]$.*

Proof. The assumption $W \star \infty = \infty$ just means that $W_{21} \equiv 0$. As $\det W = 1$, the functions W_{11} and W_{22} are zerofree and hence equal to e^{v_1} and e^{v_2}, respectively. Thereby $v_i^\# = v_i$ and $v_i(0) = 0$. Every entry of W is of finite exponential type, and therefore $v_1(z) = az$ (and thus $v_2(z) = -az$) for some $a \in \mathbb{R}$. Since every entry of W is of bounded type in $\mathbb{C}^+$, it follows that $a = 0$.

Since $W \star 0 \in \mathcal{N}_{<\infty}$, the function W_{12} can have only finitely many zeros. The same argument as above shows that W_{12} must be a polynomial.

If p is linear, the assertion is clear. Assume that the degree of p is at least 2. Then $\operatorname{ind}_- W > 0$. Consider the chain $(W_t)_{t \in \mathcal{J}}$ which goes downwards from W as constructed in [KW2, Theorem 7.1]. Its domain is of the form $(c_-, \xi_1) \cup (\xi_1, \xi_2) \cup \ldots \cup (\xi_m, 0]$. The interval (c_-, ξ_1) must be indivisible of type 0 and infinite length since $(1,0)^T$ belongs to $\mathfrak{K}(W)$ and is neutral. Also $(\xi_m, 0]$ must be indivisible of type 0 and infinite length since $\operatorname{ind}_- W \star \infty < \operatorname{ind}_- W$. Since $\lim_{t \nearrow \xi_1} W_t \circ = \lim_{t \searrow \xi_n} W_t \circ$ it follows from the results of [KW3, §5] on intermediate Weyl coefficients that $\xi_1 = \xi_n$.

❑

In the proof of Theorem 5.1 we mainly deal with the function $\mathcal{T}_{\surd} \circ \omega : \mathcal{I} \to \mathcal{M}_{<\infty}$.

Step 1: The function $\mathcal{T}_{\surd} \circ \omega$ satisfies (C1). This follows immediately from:

5.4. Lemma. *Let $\omega \in \mathfrak{M}^{sym}_{<\infty}$ and assume that ω satisfies (K_-). Then* $\operatorname{ind}_-(\mathcal{T}_{\surd} \circ \omega)(t)$ *is constant on each connected component of $\mathcal{I}$.*

Proof. Put $E_t := E_{\omega(t)}$, $t \in \mathcal{I}$. If $E_{t,+}$ and $E_{t,-}$ are defined by

$$B_{t,+}(z^2) := z B_t(z),\ A_{t,+}(z^2) := A_t(z)\,,$$

$$B_{t,-}(z^2) := \frac{B_t(z)}{z},\ A_{t,-}(z^2) := A_t(z)\,,$$

then, cf. Lemma 3.13, [KWW4, Proposition 4.9],

$$\mathfrak{P}(E_t)_e \cong \mathfrak{P}(E_{t,+}) \cong \mathfrak{K}((\mathcal{T}_{\surd} \circ \omega)(t)),\ \mathfrak{P}(E_t)_o \cong \mathfrak{P}(E_{t,-})\,.$$

In particular, we get

$$\operatorname{ind}_- \mathfrak{P}(E_t) = \operatorname{ind}_- \mathfrak{P}(E_{t,+}) + \operatorname{ind}_- \mathfrak{P}(E_{t,-})\,.$$

Recall that $\operatorname{ind}_- \mathfrak{P}(E_t)$ is constant on $\mathcal{I}_1$, say $\operatorname{ind}_- \mathfrak{P}(E_t) = \kappa_1$, $t \in \mathcal{I}_1$. We shall show that the set

$$M_\nu := \left\{ t \in \mathcal{I}_1 : \operatorname{ind}_-(\mathcal{T}_{\surd} \circ \omega)(t) = \nu \right\}$$

is closed in $\mathcal{I}_1$. Let $(t_n)_{n\in\mathbb{N}}$, $t_n \in M_\nu$, and assume that $t_n \to t_0 \in \mathcal{I}_1$. Then

$$\operatorname{ind}_- \mathfrak{P}(E_{t_n,+}) = \operatorname{ind}_-(\mathcal{T}_{\checkmark} \circ \omega)(t) = \nu,\ n \in \mathbb{N}$$

and hence

$$\operatorname{ind}_- \mathfrak{P}(E_{t_n,-}) = \kappa_1 - \nu,\ n \in \mathbb{N}\,.$$

Recall that each set $\mathcal{H}B_{\leq\kappa}$ is closed with respect to locally uniform convergence. Since $\lim_{n\to\infty} \omega(t_n) = \omega(t_0)$ locally uniformly, it follows that also $\lim_{n\to\infty} E_{t_n,\pm} = E_{t_0,\pm}$. Therefore

$$\operatorname{ind}_- \mathfrak{P}(E_{t_0,+}) \leq \nu,\ \operatorname{ind}_- \mathfrak{P}(E_{t_0,-}) \leq \kappa_1 - \nu\,. \tag{5.2}$$

However, we must have

$$\operatorname{ind}_- \mathfrak{P}(E_{t_0,+}) + \operatorname{ind}_- \mathfrak{P}(E_{t_0,-}) = \kappa_1\,.$$

Hence in both inequalities (5.2) equality must hold, and we conclude that $t_0 \in M_\nu$.

Since $\mathcal{I}_1 = \bigcup_{\nu=0}^{\kappa_1} M_\nu$ and $\mathcal{I}_1$ is connected, it follows that all but one of the sets M_ν is empty, i.e. that $\operatorname{ind}_-(\mathcal{T}_{\checkmark} \circ \omega)(t)$ is constant on $\mathcal{I}_1$.

❑

Step 2: We show that $\mathcal{T}_{\checkmark} \circ \omega$ satisfies (W3).

Proof. (of Step 2) Let $t, s \in \mathcal{I}$, $t \leq s$, be given. Put $W_t := \omega(t)$, $\hat{W}_t := (\mathcal{T}_{\checkmark} \circ \omega)(t)$.

Case: Assume first that t is not contained in the interior of an indivisible interval of the chain ω. Then $\mathfrak{P}(E_{W_t}) \subseteq \mathfrak{P}(E_{W_s})$ isometrically, and hence also $\mathfrak{P}(E_{W_t})_e \subseteq \mathfrak{P}(E_{W_s})_e$. It follows that

$$\mathfrak{P}(E_{\hat{W}_t}) = \mathfrak{P}(E_{W_t})_+ \subseteq \mathfrak{P}(E_{W_s})_+ = \mathfrak{P}(E_{\hat{W}_s})$$

isometrically. By [KW1, Theorem 12.2] there exists a matrix $\hat{W}_{ts} \in \mathcal{M}_{<\infty}$, $\operatorname{ind}_- \hat{W}_{ts} = \operatorname{ind}_- \mathfrak{P}(E_{\hat{W}_s}) - \operatorname{ind}_- \mathfrak{P}(E_{\hat{W}_t})$ such that

$$(-B_{\hat{W}_s}, A_{\hat{W}_s}) = (-B_{\hat{W}_t}, A_{\hat{W}_t})\, \hat{W}_{ts}\,.$$

From our assumption $\mathfrak{K}_-(W_t) = \mathfrak{K}(W_t)$, [KW1, Corollary 10.4] and [KWW2, Proposition 2.6] we know that $1 \in \operatorname{Assoc}\mathfrak{P}(E_{\hat{W}_t}), \operatorname{Assoc}\mathfrak{P}(E_{\hat{W}_s})$. The matrices $\hat{W}_t$ and $\hat{W}_s$ are the unique matrices belonging to $\mathcal{M}_{\operatorname{ind}_- \mathfrak{P}(E_{\hat{W}_t})}$ and $\mathcal{M}_{\operatorname{ind}_- \mathfrak{P}(E_{\hat{W}_s})}$, respectively, with (cf. [KW1, Corollary 10.4])

$$\mathfrak{K}_-(\hat{W}_t) = \mathfrak{K}(\hat{W}_t),\ \mathfrak{K}_-(\hat{W}_s) = \mathfrak{K}(\hat{W}_s)\,, \tag{5.3}$$

and

$$(-B_{\hat{W}_t}, A_{\hat{W}_t}) = (0,1)\hat{W}_t,\ (-B_{\hat{W}_s}, A_{\hat{W}_s}) = (0,1)\hat{W}_s\,.$$

It follows that (cf. proof of [KW2, Theorem 7.1])

$$\hat{W}_s = \hat{W}_t\, \hat{W}_{ts}\,.$$

By (5.3) we have $\operatorname{ind}_- \mathfrak{P}(E_{\hat{W}_t}) = \operatorname{ind}_- \hat{W}_t$ and $\operatorname{ind}_- \mathfrak{P}(E_{\hat{W}_s}) = \operatorname{ind}_- \hat{W}_s$, respectively, i.e.

$$\operatorname{ind}_- \hat{W}_{ts} = \operatorname{ind}_- \hat{W}_s - \operatorname{ind}_- W_t\,,$$

and we have proved the requirement of (W3) in the considered case.

Case: Next assume that t belongs to the interior of a maximal indivisible interval (t_-, t_+) of type 0. Then, by [KW3, Proposition 3.16], at least one of t_- and t_+ belongs to $\mathcal{I}$, say $t_- \in \mathcal{I}$. By Proposition 3.6 we have $\hat{W}_t = \hat{W}_{t_-}$ and hence the assertion follows from what we already proved.

Case: Finally consider the case that t belongs to a maximal indivisible interval (t_-, t_+) of type $\frac{\pi}{2}$ (regardless whether this indivisible interval is of positive, negative or infinite length). We divide the proof of this case into three subcases.

$s \le t_+$: In this case we have $W_s = W_t W_{(l,\frac{\pi}{2})}$ for some $l \in \mathbb{R} \setminus \{0\}$. It follows from Proposition 3.6 that $\hat{W}_s = \hat{W}_t W_{(l',\phi)}$, i.e. $\hat{W}_{ts} = W_{(l',\phi)}$. Thereby $\phi \in (0,\pi)$ and $\operatorname{sgn} l' = \operatorname{sgn} l$.

Consider the space $\mathfrak{P}(E_{W_s})$. Since s is the right endpoint of an indivisible interval of type $\frac{\pi}{2}$ we have $A_{W_s} \in \mathfrak{P}(E_{W_s})$. It follows from the fact $\overline{\operatorname{dom} S} = \operatorname{span}\{A\}^\perp$ that $\mathfrak{P}(E_{W_s})_o \subseteq \overline{\operatorname{dom} S}$. The spaces $\mathfrak{P}(E_{W_s})$ and $\mathfrak{P}(E_{W_t})$ are equal as sets but not isometrically. However, on the subspace $\overline{\operatorname{dom} S}$ the inner products of $\mathfrak{P}(E_{W_t})$ and $\mathfrak{P}(E_{W_s})$ coincide. Hence $\mathfrak{P}(E_{W_t})_o = \mathfrak{P}(E_{W_s})_o$ isometrically. We have

$$\operatorname{ind}_- \mathfrak{P}(E_{W_s})_e + \operatorname{ind}_- \mathfrak{P}(E_{W_s})_o = \operatorname{ind}_- \mathfrak{P}(E_{W_s}) =$$

$$= \operatorname{ind}_- \mathfrak{P}(E_{W_t}) + \begin{cases} 0 & , \ l > 0 \\ 1 & , \ l < 0 \end{cases} = \operatorname{ind}_- \mathfrak{P}(E_{W_t})_e + \operatorname{ind}_- \mathfrak{P}(E_{W_t})_o + \begin{cases} 0 & , \ l > 0 \\ 1 & , \ l < 0 \end{cases}.$$

Since $\operatorname{sgn} l' = \operatorname{sgn} l$, $\operatorname{ind}_- \mathfrak{P}(E_{W_t})_o = \operatorname{ind}_- \mathfrak{P}(E_{W_s})_o$ and $\mathfrak{P}(E_{\hat{W}_t}) \cong \mathfrak{P}(E_{W_t})_e$, $\mathfrak{P}(E_{\hat{W}_s}) \cong \mathfrak{P}(E_{W_s})_e$, it follows that

$$\operatorname{ind}_- \mathfrak{P}(E_{\hat{W}_s}) = \operatorname{ind}_- \mathfrak{P}(E_{\hat{W}_s}) + \operatorname{ind}_- \hat{W}_{ts}.$$

$s > t_+, t_+ \in \mathcal{I}$: We decompose $\hat{W}_s$ as $\hat{W}_s = \hat{W}_{t_+} \hat{W}_{t_+ s}$. Since t_+ is not contained in the interior of an indivisible interval, it follows from what we already proved that $\operatorname{ind}_- \hat{W}_s = \operatorname{ind}_- \hat{W}_{t_+} + \operatorname{ind}_- \hat{W}_{t_+ s}$. Moreover, we decompose $\hat{W}_{t_+} = \hat{W}_t \hat{W}_{tt_+}$. By the above treated case also in this relation negative indices add up. Since $\operatorname{ind}_-(\hat{W}_{tt_+} \hat{W}_{t_+ s}) \le \operatorname{ind}_- \hat{W}_{tt_+} + \operatorname{ind}_- \hat{W}_{t_+ s}$, it follows that also in the factorization $\hat{W}_s = \hat{W}_t (\hat{W}_{tt_+} \hat{W}_{t_+ s})$ negative indices add up.

$s > t_+, t_+ \notin \mathcal{I}$: In this case we must have $t_- \in \mathcal{I}$. To shorten notation put $E_t := E_{W_t}$ and let $\hat{E}_t$, A_t, etc. be defined correspondingly.

The interval $[t_-, t]$ is in ω indivisible of type $\frac{\pi}{2}$ and positive length $l > 0$. Hence $A_{t_-} = A_t \in \mathfrak{P}(E_t)$. Since $t_+ \notin \mathcal{I}$, we have $[A_t, A_t]_{\mathfrak{P}(E_s)} = 0$. It follows that

$$\mathfrak{P}(E_t)_e \supsetneq \mathfrak{P}(E_{t_-})_e$$

with codimension 1 and hence also $\mathfrak{P}(\hat{E}_t) \supsetneq \mathfrak{P}(\hat{E}_{t_-})$ with codimension 1. The set $\mathfrak{P}(\hat{E}_t)$ endowed with the inner product inherited from $\mathfrak{P}(\hat{E}_s)$ is a degenerated dB-subspace $\mathfrak{P}$ of $\mathfrak{P}(\hat{E}_s)$, and we have

$$\mathfrak{P} = \mathfrak{P}(\hat{E}_{t_-})[\dot{+}] \operatorname{span}\left\{ A_{t_-}(\sqrt{z}) \right\},$$

$$\mathfrak{P}^\circ = \operatorname{span}\left\{ A_{t_-}(\sqrt{z}) \right\}.$$

Consider the space $\mathfrak{K}(\hat{W}_{t_-s})$. By [KW2, Lemma 7.6] it contains a constant $(\cos\psi, \sin\psi)^T$. This constant has the property that the space

$$\mathfrak{P}(\hat{E}_{t_-})[\dot{+}]\,\text{span}\big\{-\hat{B}_{t_-}\cos\psi + \hat{A}_{t_-}\sin\psi\big\}$$

where

$$\big[-\hat{B}_{t_-}\cos\psi + \hat{A}_{t_-}\sin\psi, -\hat{B}_{t_-}\cos\psi + \hat{A}_{t_-}\sin\psi\big] = \Big[\begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix}, \begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix}\Big]_{\mathfrak{K}(\hat{W}_{t_-s})}$$

is a dB-subspace of $\mathfrak{P}(\hat{E}_s)$. It follows that

$$\mathfrak{P}(\hat{E}_{t_-})[\dot{+}]\,\text{span}\big\{-\hat{B}_{t_-}\cos\psi + \hat{A}_{t_-}\sin\psi\big\} = \mathfrak{P}\,,$$

and hence that

$$\text{span}\big\{-\hat{B}_{t_-}\cos\psi + \hat{A}_{t_-}\sin\psi\big\} = \text{span}\big\{A_{t_-}(\sqrt{z})\big\}\,,$$

i.e.

$$A_{t_-}(\sqrt{z}) = \lambda\big(-\hat{B}_{t_-}(z)\cos\psi + \hat{A}_{t_-}(z)\sin\psi\big) \tag{5.4}$$

for some $\lambda \in \mathbb{C}$, and that

$$\Big[\begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix}, \begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix}\Big]_{\mathfrak{K}(\hat{W}_{t_-s})} = 0\,.$$

From the definition of the transformation $\mathcal{T}_{\surd}$ we obtain

$$\hat{B}_{t_-}(z^2) = zB_{t_-}(z),\ \hat{A}_{t_-}(z^2) = A_{t_-}(z) + w'_{t_-,12}(0)zB_{t_-}(z)\,.$$

It follows that

$$A_{t_-}(\sqrt{z}) = \hat{A}_{t_-}(z) - w'_{t_-,12}(0)\hat{B}_{t_-}(z)\,.$$

Comparing this with (5.4) we obtain $\psi = \text{Arccot}\, w'_{t_-,12}(0)$.

Proposition 3.6 shows that $\hat{W}_{t_-t} = W_{(l',\psi)}$ with some $l' > 0$. Hence the constant $(\cos\psi, \sin\psi)^T$ belongs to $\mathfrak{K}(\hat{W}_{t_-s})$ as well as to $\mathfrak{K}(\hat{W}_{t_-t}^{-1})$. In the first space it is neutral, in the second (one-dimensional) space it is negative. It follows (cf. [ADRS]) that

$$\text{ind}_-\,\hat{W}_{t_-t}^{-1}\hat{W}_{t_-s} = \text{ind}_-\,\hat{W}_{t_-s}\,.$$

We have $\text{ind}_-\,\hat{W}_t = \text{ind}_-\,\hat{W}_{t_-}$ and hence ($\hat{W}_{ts} = W_t^{-1}W_s = W_{t_-t}^{-1}W_{t_-s}$)

$$\text{ind}_-\,\hat{W}_s = \text{ind}_-\,\hat{W}_{t_-} + \text{ind}_-\,\hat{W}_{t_-s} = \text{ind}_-\,\hat{W}_t + \text{ind}_-\,\hat{W}_{ts}\,.$$

We have seen that in any case in the relation $W_s = W_tW_{ts}$ negative indices add.

❑

Step 3: We show that $\mathcal{T}_{\surd}\circ\omega$ satisfies (C2).

Proof. (of Step 3) Also this proof is divided into several cases. We use again the notation $\omega(t) = W_t$ and $(\mathcal{T}_{\surd}\circ\omega)(t) = \hat{W}_t$.

Case: ω ends with an indivisible interval $(m, \sup\mathcal{I})$ of type 0.

Then $\hat{W}_t = \hat{W}_m$, $t \in (m, \sup\mathcal{I})$, and hence

$$\limsup_{t\nearrow\sup\mathcal{I}} \mathfrak{t}(\hat{W}_t) = \mathfrak{t}(\hat{W}_m) < +\infty$$

and

$$\lim_{t\nearrow\sup\mathcal{I}} \hat{W}_t = \hat{W}_m .$$

We are therefore in case (a) of (C2). By the already proved property (W3) we have

$$\operatorname{ind}_- \hat{W}_t \leq \operatorname{ind}_- \hat{W}_m,\ t \in \mathcal{I}, t \leq m .$$

For $t \in (m, \sup\mathcal{I})$ trivially $\operatorname{ind}_- \hat{W}_t = \operatorname{ind}_- \hat{W}_m$. Hence the implication (a) of (C2) holds true.

Case: ω ends neither with an indivisible interval of type zero nor is the last component $\mathcal{I}_\infty$ of ω of the form $(m_-, m] \cup [m, \sup\mathcal{I})$ with an indivisible interval (m_-, m) of type 0 and an indivisible interval $(m, \sup\mathcal{I})$ of type $\frac{\pi}{2}$.

We claim that there exist $s, t \in \hat{\mathcal{I}}_\infty := \{u \in \mathcal{I} : \operatorname{ind}_- \hat{W}_u = \max_{v\in\mathcal{I}} \operatorname{ind}_- \hat{W}_v\}$ such that $\hat{W}_{st}$ $(:= \hat{W}_s^{-1}\hat{W}_t)$ is not a linear polynomial. This excludes the occurrence of case (b) of (C2). In case (a) we appeal to [KW3, Lemma 3.4] to obtain the validity of the desired implication.

In order to establish our claim assume on the contrary that for some $\phi \in [0, \pi)$ we have $\hat{W}_{st} = W_{(l(s,t),\phi)}$ whenever $s, t \in \hat{\mathcal{I}}_\infty$. By Lemma 5.4 this relation holds especially for $s, t \in \mathcal{I}_\infty = \{u \in \mathcal{I} : \operatorname{ind}_- W_u = \max_{v\in\mathcal{I}} \operatorname{ind}_- W_v\}$.

$\phi = 0$: Choose $s, t \in \mathcal{I}_\infty$, $s < t$, such that W_{st} is not a linear polynomial. By Remark 3.7 we have for some $\alpha \in \mathbb{R}$

$$W_{st} = \begin{pmatrix} 1 & \alpha z + l(s,t)z^3 \\ 0 & 1 \end{pmatrix} .$$

Since W_{st} is not a linear polynomial we must have $l(s,t) \neq 0$. Thus $\operatorname{ind}_- W_{st} > 0$, a contradiction.

$\phi \neq 0$: We conclude from Remark 3.7 that for all $s, t \in \mathcal{I}_\infty$, $s < t$,

$$W_{st} = W_{(l_1,0)} W_{(l_2,\frac{\pi}{2})} W_{(l_3,0)}$$

with some $l_i \geq 0$. By Lemma 5.2 we must be in one of the following situations:

$$\mathcal{I}_\infty = \begin{cases} (m_0, m_1] \cup [m_1, m_2] \cup [m_2, m_3) & \text{of types } 0, \frac{\pi}{2}, 0 \\ (m_0, m_1] \cup [m_1, m_2) & \text{of types } \frac{\pi}{2}, 0 \\ (m_0, m_1] \cup [m_1, m_2) & \text{of types } 0, \frac{\pi}{2} \end{cases}$$

Again a contradiction since these are exactly those cases we do not consider in the present step of the proof.

Case: The last component $\mathcal{I}_\infty$ of ω is equal to $(m_0, m_1] \cup [m_1, m_2)$ with an indivisible interval $(m_0, m_1]$ of type 0 and an indivisible interval $[m_1, m_2)$ of type $\frac{\pi}{2}$.

In this case the interval (m_0, m_2) is indivisible in $(\hat{W}_t)_{t\in\mathcal{I}}$ of some type $\psi \in (0,\pi)$. It follows that $\lim_{t\nearrow\sup\mathcal{I}} \mathfrak{t}(\hat{W}_t) = +\infty$, which rules out the occurrence of case (a) in (C2).

Fix $t_0 \in (m_1, m_2)$. Since $[m_1, t_0]$ is indivisible of type $\frac{\pi}{2}$ in $(W_t)_{t\in\mathcal{I}}$ and has positive length, we obtain that

$$A_{m_1} \in \mathfrak{P}(E_{t_0}),\ [A_{m_1}, A_{m_1}]_{\mathfrak{P}(E_{t_0})} > 0\,.$$

As we saw in the proof of (W3), the linear combination $\hat{S}_\psi$ of $\hat{A}_{t_0}$ and $\hat{B}_{t_0}$ which belongs to the space $\mathfrak{P}(\hat{E}_{t_0})$ is linearly dependent with $A_{m_1}(\sqrt{z})$ and hence

$$[\hat{S}_\psi, \hat{S}_\psi]_{\mathfrak{P}(\hat{E}_{t_0})} > 0\,.$$

It follows from [KW2, Lemma 5.12] that

$$\operatorname{ind}_- \hat{W}_{t_0} \star \cot\psi = \operatorname{ind}_- \hat{W}_{t_0}\,.$$

We conclude that the implication (b) of (C2) holds true.

❑

We have established that $\mathcal{T}_{\surd} \circ \omega$ can be extended to a maximal chain ϖ by means of Lemma 4.2. If $\lim_{t\nearrow M} \mathfrak{t}((\mathcal{T}_{\surd} \circ \omega)(t)) = +\infty$, this extension is unique and satisfies

$$q_\infty(\varpi) = \lim_{t\nearrow M} (\mathcal{T}_{\surd} \circ \omega(t)) \star \infty\,.$$

By (3.4) this yields

$$zq_\infty(\varpi)(z^2) = q_\infty(\omega)(z)\,. \tag{5.5}$$

If $\lim_{t\nearrow M} \mathfrak{t}((\mathcal{T}_{\surd} \circ \omega)(t)) < +\infty$, so that the extension of $\mathcal{T}_{\surd} \circ \omega$ is not unique, we should choose the parameter $\tau = \infty$ in Lemma 4.2 in order to achieve the relation (5.5). The fact that this choice is permitted needs justification. However, we saw in the proof of Step 3 that there exist $s, t \in \mathcal{I}_\infty$ such that $\hat{W}_{st}$ is not of the form $W_{(l,0)}$. As indicated in the proof of [KW2, Lemma 8.5] this implies that $\operatorname{ind}_-[\lim_{t\nearrow M} \mathcal{T}_{\surd} \circ \omega(t)] \star \infty = \operatorname{ind}_- \lim_{t\nearrow M} \mathcal{T}_{\surd} \circ \omega(t)$.

Let ϖ be the maximal chain with $zq_\infty(\varpi)(z^2) = q_\infty(\omega)(z)$. Due to the previous steps there exists a nondecreasing function $\hat{\mu} : \mathcal{I} \to \operatorname{dom}\varpi$ such that $\mathcal{T}_{\surd} \circ \omega = \varpi \circ \hat{\mu}$.

Step 4: There exists a surjective function $\mu : \operatorname{dom}\omega_{\surd} \to \operatorname{dom}\varpi$ such that $\omega_{\surd} = \varpi \circ \mu$.

Proof. (of Step 4) Write $\mathcal{I} = (\sigma_0, \sigma_1) \cup \ldots \cup (\sigma_n, \sigma_{n+1})$. By Lemma 5.4 for each i the set $\hat{\mu}((\sigma_i, \sigma_{i+1}))$ is contained in one connected component of $\operatorname{dom}\varpi$. As it is seen from (3.3), the function $(\mathfrak{t} \circ \varpi)(\hat{\mu}(t))$ depends continuously on $t \in (\sigma_i, \sigma_{i+1})$. Thus $\hat{\mu}$ is continuous, and it follows that $\hat{\mu}((\sigma_i, \sigma_{i+1}))$ is an interval. Put

$$\xi_{i,-} := \lim_{t\to\sigma_i-} \hat{\mu}(t),\ \xi_{i,+} := \lim_{t\to\sigma_i+} \hat{\mu}(t)\,.$$

Since $\mathcal{T}_{\surd}(W)$ depends continuously on W and $\mathcal{T}_{\surd}(I) = I$, we have $\xi_{0,+} = 0$.

We have $\xi_{n+1,-}=\sigma_{n+1}$ if and only if $\lim_{t\nearrow\sigma_{n+1}}\mathfrak{t}((\mathcal{T}_{\checkmark}\circ\omega)(t))=+\infty$. Otherwise, by (5.5) and the definition of ϖ, we have

$$q_\infty(\varpi)=\lim_{t\nearrow\sigma_{n+1}}(\mathcal{T}_{\checkmark}\circ\omega)(t)\star\infty=\varpi(\xi_{n+1,-})\star\infty\,.$$

This implies that the interval $(\xi_{n+1,-},\sup\operatorname{dom}\varpi)$ is indivisible of type 0 in ϖ.

Consider a point σ_i, $i\in\{1,\dots,n\}$. The essential observation for the following proof is that, by (3.4) and the existence of intermediate Weyl-coefficients (see [KW3, §5]), the function $(\mathcal{T}_{\checkmark}\circ\omega)\star\infty$ has a continuous extension to $[\sigma_0,\sigma_{n+1}]$.

We divide cases similar as in the definition of ς_i.

$\xi_{i,-},\xi_{i,+}\notin\operatorname{dom}\varpi$: Denote by $q_{\xi_{i,\pm}}$ the intermediate Weyl coefficient of ϖ at the singularity $\xi_{i,\pm}$. We have

$$q_{\xi_{i,-}}=\lim_{s\to\xi_{i,-}-}\varpi(s)\star\infty=\lim_{t\to\sigma_i-}(\mathcal{T}_{\checkmark}\circ\omega)(t)\star\infty=\lim_{t\to\sigma_i+}(\mathcal{T}_{\checkmark}\circ\omega)(t)\star\infty=$$

$$=\lim_{s\to\xi_{i,+}+}\varpi(s)\star\infty=q_{\xi_{i,+}}\,,$$

and hence $\xi_{i,-}=\xi_{i,+}$.

$\xi_{i,-}\in\operatorname{dom}\varpi,\xi_{i,+}\notin\operatorname{dom}\varpi$: We have

$$q_{\xi_{i,+}}=\lim_{t\to\sigma_i+}(\mathcal{T}_{\checkmark}\circ\omega)(t)\star\infty=\lim_{t\to\sigma_i-}(\mathcal{T}_{\checkmark}\circ\omega)(t)\star\infty=\varpi(\xi_{i,-})\star\infty\,.$$

Since $q_{\xi_{i,+}}$ is the Weyl-coefficient of the maximal chain $\varpi|_{\operatorname{dom}\varpi\cap(0,\xi_{i,+})}$, it follows that $(\xi_{i,-},\xi_{i,+})$ is indivisible of type 0 and infinite length.

$\xi_{i,-}\notin\operatorname{dom}\varpi,\xi_{i,+}\in\operatorname{dom}\varpi$: The same argument as above yields $q_{\xi_{i,-}}=\varpi(\xi_{i,+})\star\infty$ and hence that $\operatorname{ind}_-\varpi(\xi_{i,+})\star\infty<\operatorname{ind}_-\varpi(\xi_{i,+})$. This implies that the interval $(\xi_{i,-},\xi_{i,+})$ is indivisible of type 0 and infinite length.

$\xi_{i,-}\in\operatorname{dom}\varpi,\xi_{i,+}\in\operatorname{dom}\varpi$: In this case we find $\varpi(\xi_{i,-})\star\infty=\varpi(\xi_{i,+})\star\infty$ and hence, by Lemma 5.3,

$$\varpi(\xi_{i,-})^{-1}\varpi(\xi_{i,+})=\begin{pmatrix}1 & p(z)\\ 0 & 1\end{pmatrix}$$

If $\operatorname{ind}_-\varpi(\xi_{i,-})=\operatorname{ind}_-\varpi(\xi_{i,+})$, the interval $(\xi_{i,-},\xi_{i,+})$ is indivisible of type 0 and positive length

$$\mathfrak{t}(\lim_{t\searrow\sigma_i}\mathcal{T}_{\checkmark}(\omega(t)))-\mathfrak{t}(\lim_{t\nearrow\sigma_i}\mathcal{T}_{\checkmark}(\omega(t)))\,.$$

If $\operatorname{ind}_-\varpi(\xi_{i,-})<\operatorname{ind}_-\varpi(\xi_{i,+})$, there exists a singularity ξ of ϖ in the interval $(\xi_{i,-},\xi_{i,+})$. By Lemma 5.3 the intervals $(\xi_{i,-},\xi)$ and $(\xi,\xi_{i,+})$ are indivisible of type 0 and infinite length.

The case $\operatorname{ind}_-\varpi(\xi_{i,-})>\operatorname{ind}_-\varpi(\xi_{i,+})$ cannot occur.

We saw that in any case $\varpi|_{(\xi_{i,-},\xi_{i,+})}\sim\varsigma_i$. The required function μ is now defined in the obvious way.

❑

We have constructed the function μ required in Theorem 5.1. To complete the proof of Theorem 5.1 choose for λ a right inverse of μ. ✌

5.5. Corollary. *Let $\omega \in \mathfrak{M}_\kappa^{sym}$ and assume that ω satisfies (K_-). Then the maximal chain ϖ with $zq_\infty(\varpi)(z^2) = q_\infty(\omega)(z)$ belongs to $\mathfrak{M}_{\leq\kappa}^{ep}$ and also satisfies (K_-).*

Proof. We know from Theorem 5.1 that $\varpi = \omega_{\surd} \circ \lambda$. By (3.4) the number of zeros of $\mathcal{T}_{\surd}(\omega(t))_{21}$ in $\mathbb{C} \setminus [0,\infty)$ equals the number of nonreal zeros of $\omega(t)_{21}$ and is therefore, by Corollary 2.10, bounded uniformly by $\operatorname{ind}_- \omega$. On indivisible intervals of type 0 the entry $\varpi(s)_{21}$ is constant. Hence the number of zeros of $\varpi(s)_{21}$ in $\mathbb{C} \setminus [0,\infty)$ is bounded independently of s. By Corollary 2.10, (ii), the maximum number of zeros that any entry of $\varpi(s)$ can have in $\mathbb{C}\setminus[0,\infty)$ is bounded independently of s.

The fact that ϖ satisfies (K_-) readily follows from the relation of Weyl coefficients and [KW2, Theorem 5.7].

❑

We deduce an inverse result for the class $\mathfrak{M}_{<\infty}^{ep}$, the analogue to Proposition 4.4.

5.6. Proposition. *Let ϖ be a maximal chain. Then $\varpi \in \mathfrak{M}_{<\infty}^{ep}$ if and only if $q_\infty(\varpi) \in \mathcal{N}_{<\infty}^{ep}$.*

Proof. Assume first that $\varpi \in \mathfrak{M}_{<\infty}^{ep}$. For every $t \in \operatorname{dom}\varpi$ the function $\varpi(t) \star \infty$ belongs to $\mathcal{N}_{\leq \operatorname{ind}_- \varpi}^{ep}$. Moreover, the number of poles $\gamma(\varpi(t) \star \infty)$ of $\varpi(t) \star \infty$ which are located in $\mathbb{C} \setminus [0,\infty)$ is equal to the number of zeros of $\varpi(t)_{21}$. Hence it is bounded independently of $t \in \operatorname{dom}\varpi$. We have $\lim_{t\to\sup\operatorname{dom}\varpi} \varpi(t)\star\infty = q_\infty(\varpi)$ and hence [KWW2, Proposition 4.10] implies that $q_\infty(\varpi) \in \mathcal{N}_{<\infty}^{ep}$.

Conversely, let ϖ be given such that $q_\infty(\varpi) \in \mathcal{N}_{<\infty}^{ep}$ and assume first that additionally $\lim_{x\to-\infty} q_\infty(\varpi)(x) = 0$. By [KWW2, Theorem 4.1] the function $zq_\infty(\varpi)(z^2)$ belongs to $\mathcal{N}_{<\infty}^{sym}$. Let $\omega \in \mathfrak{M}_{<\infty}^{sym}$ be the maximal chain with $q_\infty(\omega) = zq_\infty(\omega)(z^2)$. Since

$$\lim_{y\to\infty} \frac{1}{iy} q_\infty(\omega)(iy) = \lim_{y\to\infty} q_\infty(\varpi)(-y^2) = 0\,,$$

we know that ω satisfies (K_-). Corollary 5.5 implies that $\varpi \in \mathfrak{M}_{<\infty}^{ep}$.

The general case is reduced to the already proved particular instance by a possible application of one of the transforms $\mathcal{T}_J$ or $\mathcal{T}_\alpha$ of [KW2, Section 10].

❑

5.7. *Remark.* Note that Proposition 5.6 could most likely also be deduced without help of the transformation $\mathcal{T}_{\surd}$ by employing the the theory of isometric embeddings of dB-spaces into 'L^2-spaces' induced by distributions associated to integral representations of generalized Nevanlinna functions. For this not yet fully developed theory see [KW2, §4,§6], [KWW2, §5].

5.8. Corollary. *Let $\varpi \in \mathfrak{M}_{<\infty}^{ep}$ and assume that $\lim_{x\to-\infty} q_\infty(\varpi)(x) = 0$. Then there exist only finitely many indivisible intervals of type 0 in ϖ. In fact, the number of such intervals is bounded by $2\operatorname{ind}_- \varpi + 1$.*

Proof. We know that $\varpi = \omega_{\surd} \circ \lambda$ where $\omega : \mathcal{I} \to \mathcal{M}_{<\infty}$, $\mathcal{I} = (\sigma_0, \sigma_1) \cup \ldots \cup (\sigma_n, \sigma_{n+1})$, is the maximal chain with $q_\infty(\omega)(z) = z q_\infty(\varpi)(z^2)$. By Remark 3.7 the parts $(\mathcal{T}_{\surd} \circ \omega)|_{(\sigma_i, \sigma_{i+1})}$ of $\omega_{\surd}$ cannot contain indivisible intervals of type 0. Thus ϖ contains at most $2n+1$ indivisible intervals of type 0.

❑

5.2. The inverse transformation

We employ the rather detailed discussion of Theorem 5.1 to define and investigate the inverse of the square root transformation.

Let $\varpi \in \mathfrak{M}^{ep}_{<\infty}$. Then by [KWW2, Proposition 4.9] a function $\Lambda : \operatorname{dom} \varpi \to \mathbb{R} \cup \{\pm\infty\}$ is well-defined by

$$\Lambda(t) := - \lim_{x \to -\infty} \frac{\varpi(t)_{22}(x)}{\varpi(t)_{21}(x)} .$$

The following lemma will play a central role.

5.9. Lemma. *Let $\varpi \in \mathfrak{M}^{ep}_{<\infty}$ and assume that $\lim_{x\to\infty} q_\infty(\varpi)(x) = 0$. Then there exist pairwise disjoint open intervals (a_k, b_k), $k = 0, \ldots, n$, with $a_k < b_k \le a_{k+1}$, such that*

$$\operatorname{dom} \varpi \setminus \bigcup \big\{ [t_-, t_+] : (t_-, t_+) \textit{ maximal indivisible of type } 0 \textit{ in } \varpi \big\} =$$

$$= \bigcup_{k=0}^{n} (a_k, b_k) \cup \big\{ a_k : a_k \in \operatorname{dom} \varpi, a_k = b_{k-1} \big\}$$

and that the function Λ has the following properties:

- *(i)* $\Lambda|_{(a_k, b_k)}$ *is finite, nondecreasing and continuous from the left.*
- *(ii) If $\alpha \in \{a_k : a_k \in \operatorname{dom}\varpi, a_k = b_{k-1}\}$, then either $\lim_{t\to\alpha\pm} \Lambda(t) \in \mathbb{R}$ and $\lim_{t\to\alpha+} \Lambda(t) - \lim_{t\to\alpha-} \Lambda(t) < 0$, or $\limsup_{t\to\alpha} |\Lambda(t)| = \infty$.*

The intervals (a_k, b_k) are uniquely determined by these properties.

For each k the function $\tau_k(t) := \Lambda(t) - \varpi(t)'_{21}(0)$, $t \in (a_k, b_k)$, is strictly increasing and continuous from the left.

The proof of this lemma will also show how the inverse of the square root transformation acts. Let us describe this action precisely.

Let $\varpi \in \mathfrak{M}^{ep}_{<\infty}$, assume that $\lim_{x\to\infty} q_\infty(\varpi)(x) = 0$, and let (a_k, b_k) be the intervals from the above lemma. For each $k \in \{1, \ldots, n\}$ define $\varpi_k : \mathbb{R} \to \mathcal{M}_{<\infty}$ as follows: If $t \in \operatorname{ran} \tau_k$, put

$$\varpi_k(t) := \big(\mathcal{T}_{2,\Lambda(t)} \circ \varpi\big)(\tau_k^{-1}(t)) .$$

Since τ_k is strictly increasing and continuous from the left, we have

$$\big(\inf \operatorname{ran} \tau_k, \sup \operatorname{ran} \tau_k\big) = \operatorname{ran} \tau_k \dot\cup \bigcup_l (\alpha_l, \beta_l] ,$$

with at most countably many pairwise disjoint intervals $(\alpha_l, \beta_l]$. If $t \in (\alpha_l, \beta_l]$ define

$$\varpi_k(t) := \begin{cases} \big[\lim_{\substack{t \nearrow \alpha_l \\ t \in \operatorname{ran} \tau_k}} \varpi_k(t)\big] \cdot W_{(t-\alpha_l, 0)} & , \ t \in (\alpha_l, \beta_l] \\ \big[\lim_{\substack{t \nearrow \sup \operatorname{ran} \tau_k \\ t \in \operatorname{ran} \tau_k}} \varpi_k(t)\big] \cdot W_{(t-\sup \operatorname{ran} \tau_k, 0)} & , \ t \in (\sup \operatorname{ran} \tau_k, \infty) \\ \big[\lim_{\substack{t \searrow \inf \operatorname{ran} \tau_k \\ t \in \operatorname{ran} \tau_k}} \varpi_k(t)\big] \cdot W_{(t-\inf \operatorname{ran} \tau_k, 0)} & , \ t \in (-\infty, t, \inf \operatorname{ran} \tau_k) \end{cases}$$

For $k = 0$ we define a function $\varpi_0 : (0, \infty) \to \mathcal{M}_{<\infty}$ in exactly the same manner.

5.10. Theorem. *Let* $\varpi \in \mathfrak{M}^{ep}_{<\infty}$ *and assume that* $\lim_{x\to\infty} q_\infty(\varpi)(x) = 0$. *Then*

$$\omega := \varpi_0 \uplus \varpi_1 \uplus \ldots \uplus \varpi_n \in \mathfrak{M}^{sym}_{<\infty} ,$$

and

$$q_\infty(\omega)(z) = z q_\infty(\varpi)(z^2) .$$

Proof. (of Lemma 5.9 *and Theorem* 5.10*)* We use the same notation as in the first paragraph of the proof of Step 4. Consider the continuous and nondecreasing function $\hat{\mu} : \operatorname{dom} \omega \to \operatorname{dom} \varpi$. Put

$$\mathcal{L} := \operatorname{dom} \omega \setminus \bigcup \big\{(t_-, t_+] : (t_-, t_+) \text{ maximal indivisible of type } 0\big\} .$$

Then, by Lemma 3.4, $\hat{\mu}|_{\mathcal{L}}$ is injective. The set $\mathcal{L}$ has the property that it is closed in $\operatorname{dom} \omega$ with respect to monotonically increasing limits. We show that the function $(\hat{\mu}|_{\mathcal{L}})^{-1}$ is continuous from the left: Assume that $s_n \nearrow s$, $s_n, s \in \hat{\mu}(\mathcal{L})$, and put $t_n := (\hat{\mu}|_{\mathcal{L}})^{-1}, t := (\hat{\mu}|_{\mathcal{L}})^{-1}$. Then $(t_n)_{n\in\mathbb{N}}$ is increasing and bounded above by t. Hence $\lim_{n\to\infty} t_n =: t_0$ exists and belongs to $\operatorname{dom} \omega$. Therefore it belongs to $\mathcal{L}$. We have

$$\hat{\mu}(t_0) = \lim_{n\to\infty} \hat{\mu}(t_n) = s = \hat{\mu}(t) ,$$

and therefore $t_0 = t$.

For $k = 1, \ldots, n$ we define numbers $a_k := \xi_{k,+}$ and $b_k := \xi_{k+1,-}$. Then

$$(a_k, b_k) \subseteq \hat{\mu}\big((\sigma_k, \sigma_{k+1})\big) \subseteq [a_k, b_k] .$$

By Remark 3.7 (a_k, b_k) does not contain any indivisible interval of type 0. Moreover, by Step 4, every interval in $\operatorname{dom} \omega$ which has empty intersection with $\bigcup_{k=0}^n (a_k, b_k)$ is indivisible of type 0. It follows that

$$\operatorname{dom} \varpi = \bigcup_{k=0}^{n} (a_k, b_k) \dot\cup \big\{a_k : a_k \in \operatorname{dom} \varpi, a_k = b_{k-1}\big\} \dot\cup$$

$$\dot\cup \bigcup \big\{[t_-, t_+] : (t_-, t_+) \text{ maximal indivisible of type } 0\big\} ,$$

in particular $\bigcup_{k=0}^n (a_k, b_k) \subseteq \mathcal{L}$.

Let $t \in \mathcal{L}$. Then t is not the right endpoint of an indivisible interval of type 0 and hence $\omega(t)_{21} \notin \mathfrak{P}(E_{\omega(t)})$. Hence

$$\lim_{y\to+\infty} \frac{1}{iy} \frac{\omega(t)_{22}(iy)}{\omega(t)_{21}(iy)} = 0 ,$$

and it follows from (3.4) that

$$\omega(t)'_{12}(0) = (\Lambda \circ \hat{\mu})(t),\ t \in \mathcal{L}.$$

It readily follows that Λ is real and nondecreasing on (a_k, b_k). Moreover, since $\omega(t)'_{12}(0)$ depends continuously on t, it follows from

$$\Lambda(t) = \omega\big((\hat{\mu}|_{\mathcal{L}})^{-1}\big)'_{12}(0),\ t \in (a_k, b_k),$$

that Λ is continuous from the left. By Lemma 3.3 we also obtain

$$\big(\mathcal{T}_{2,\Lambda(t)} \circ \varpi\big)(t) = \big(\omega \circ (\hat{\mu}|_{\mathcal{L}}^{-1})\big)(t),\ t \in (a_k, b_k).$$

Since for all $t \in \operatorname{dom}\omega$ the relation $\omega(t)'_{21}(0) = \varpi(t)'_{21}(0)$ holds, we see that

$$\tau_k(t) = \big(\mathfrak{t} \circ \omega \circ (\hat{\mu}|_{\mathcal{L}}^{-1})\big)(t),\ t \in (a_k, b_k).$$

Thus τ_k is nondecreasing, injective and continuous from the left.

We conclude from the above discussion that, what is missing from $\varpi_k|_{\operatorname{ran}\tau_k}$ to all of ω are just indivisible intervals of type 0. However, the definition of ϖ_k on $\mathbb{R} \setminus \operatorname{ran}\tau_k$ just fills in indivisible intervals of type 0. Note that

$$\mathfrak{t}\big[\big(\mathcal{T}_{2,\Lambda(t)} \circ \varpi\big)(\tau_k^{-1}(t))\big] = t,\ t \in \operatorname{ran}\tau_k.$$

Thus the filled in indivisible intervals have the proper length.

❑

5.11. *Remark.* We know from [KWW2, Corollary 3.4, Proposition 4.7] that, if $q \in \mathcal{N}_{<\infty}$, then $zq(z) \in \mathcal{N}_{<\infty}$ if and only if $q \in \mathcal{N}_{<\infty}^{ep}$. Under an additional regularity condition on q, the previous results lead to an explicit method to construct the maximal chain with Weyl-coefficient $zq(z)$ out of the one with Weyl-coefficient q.

To see this recall from [KW2, Lemma 10.1] that, if $\omega \in \mathfrak{M}_\kappa$ and if we define $\upsilon(t) := -J\omega(t)J$, then $\upsilon \in \mathfrak{M}_\kappa$ and the respective Weyl-coefficients are related by $q_\infty(\upsilon) = -q_\infty(\omega)^{-1}$. Denote this transformation by $\mathcal{T}_J$, let $\mathcal{T}_{\surd}$ be the square root transformation and $\mathcal{T}_2$ its inverse as studied above. Then for each $q \in \mathcal{N}_{<\infty}^{ep}$ we have

$$q(z) \overset{\mathcal{T}_2}{\rightsquigarrow} zq(z^2) \overset{\mathcal{T}_J}{\rightsquigarrow} \frac{-1}{zq(z^2)} \overset{\mathcal{T}_{\surd}}{\rightsquigarrow} \frac{-1}{zq(z)} \overset{\mathcal{T}_J}{\rightsquigarrow} zq(z)$$

In order to justify the application of Theorem 5.1 and Theorem 5.10, we have to assume that

$$\lim_{x\to-\infty} q(x) = \lim_{x\to-\infty} \frac{1}{xq(x)} = 0.$$

This just means that the chain ω whose Weyl-coefficient is $zq(z^2)$ does not start with an indivisible interval.

The same proceedure can be applied in order to construct the chain with Weyl-coefficient $z^{-1}q(z)$ out of the chain corresponding to $q(z) \in \mathcal{N}_{<\infty}^{ep}$. This is exactly what will be needed in the discussion of generalized strings. In fact, under the assumption that

$$\lim_{x\to-\infty} \frac{1}{q(x)} = \lim_{x\to-\infty} \frac{q(x)}{x} = 0,$$

we have

$$q(z) \overset{\mathcal{T}_J}{\rightsquigarrow} \frac{-1}{q(z)} \overset{\mathcal{T}_2}{\rightsquigarrow} \frac{-z}{q(z^2)} \overset{\mathcal{T}_J}{\rightsquigarrow} \frac{q(z^2)}{z} \overset{\mathcal{T}_{\surd}}{\rightsquigarrow} \frac{q(z)}{z}$$

In both cases the essential part of the sequence is $\mathcal{T}_{\surd} \circ \mathcal{T}_J \circ \mathcal{T}_2$.

6. Evolution of singularities

In the following discussion we recall some facts and notions on singularities of maximal chains from [KW3]. If $\omega \in \mathfrak{M}_{<\infty}$, $\operatorname{dom}\omega = (0,\sigma_1)\cup(\sigma_1,\sigma_2)\cup\ldots\cup(\sigma_n,M)$, then the numbers σ_i are called the *singularities of the chain* ω. One reason is that $\lim_{t\to\sigma_i} |\mathfrak{t}(\omega(t))| = \infty$, another one is that the σ_i are exactly the points of increase of $\operatorname{ind}_-\omega(t)$. A first characteristic value attached to a singularity is

$$\kappa(\sigma_i) := \lim_{t\searrow\sigma_i} \operatorname{ind}_-\omega(t) - \lim_{t\nearrow\sigma_i} \operatorname{ind}_-\omega(t) \in \mathbb{N}\,.$$

Another characteristic of a singularity is whether or not there is an indivisible interval to the left or to the right of it. Put

$$\sigma_i^+ := \sup\big(\{t\in\mathcal{I} :\, (\sigma_i,t) \text{ indivisible}\}\cup\{\sigma_i\}\big),$$

$$\sigma_i^- := \inf\big(\{t\in\mathcal{I} :\, (t,\sigma_i) \text{ indivisible}\}\cup\{\sigma_i\}\big)\,.$$

We call σ_i of *polynomial type*, if $\sigma_i^- < \sigma_i < \sigma_i^+$. Moreover, σ_i is called *left dense*, *right dense* or *dense*, if $\sigma_i^- = \sigma_i < \sigma_i^+$, $\sigma_i^- < \sigma_i = \sigma_i^+$ or $\sigma_i^- = \sigma_i = \sigma_i^+$, respectively.

A deeper insight in the structure of a singularity is obtained by considering the chain of dB-spaces associated with the chain ω. If ω satisfies (K_-), then $\mathfrak{K}(\omega(t)) \cong \mathfrak{K}_-(\omega(t)) \cong \mathfrak{P}(E_{\omega(t)})$. Moreover, if $s,t\in\operatorname{dom}\omega$, $s\le t$, then $\mathfrak{P}(E_{\omega(s)}) \subseteq \mathfrak{P}(E_{\omega(t)})$ and this inclusion is isometric unless for some $\epsilon>0$ the interval $(s-\epsilon,t)$ is indivisible. Conversely, if $t\in\mathcal{I}$, then every nondegenerated dB-subspace of $\mathfrak{P}(E_{\omega(t)})$ is of the form $\mathfrak{P}(E_{\omega(s)})$. It is an important observation that the singularities of ω correspond to the degenerated dB-spaces in this chain. In fact, if we put

$$\mathfrak{P}_{\sigma_i^-} := \operatorname{cls}\big\{\mathfrak{P}(E_{\omega(t)}) :\, t<\sigma_i^-\big\},\ \mathfrak{P}_{\sigma_i^+} := \bigcup\big\{\mathfrak{P}(E_{\omega(t)}) :\, t>\sigma_i^+\big\}\,,$$

then every dB-space $\mathfrak{P}_{\sigma_i^-} \subsetneq \mathfrak{P} \subsetneq \mathfrak{P}_{\sigma_i^+}$ is degenerated. Conversely, unless (σ_i^-,σ_i^+) is indivisible with negative length, there exists a degenerated dB-space $\mathfrak{P}_{\sigma_i^-} \subseteq \mathfrak{P} \subseteq \mathfrak{P}_{\sigma_i^+}$. More exactly: Put $\delta := \dim \mathfrak{P}_{\sigma_i^+}/\mathfrak{P}_{\sigma_i^-} \in \mathbb{N}\cup\{0\}$, and $\delta_- := \dim \mathfrak{P}^\circ_{\sigma_i^-}$, $\delta_+ := \dim \mathfrak{P}^\circ_{\sigma_i^+}$. If $\delta>1$, then there exist degenerated dB-spaces $\mathfrak{P}_1,\ldots,\mathfrak{P}_{\delta-1}$ with

$$\mathfrak{P}_{\sigma_i^-} \subsetneq \mathfrak{P}_1 \subsetneq \mathfrak{P}_2 \subsetneq \ldots \subsetneq \mathfrak{P}_{\delta-1} \subsetneq \mathfrak{P}_{\sigma_i^+}\,.$$

It was proved in [KW3] that the isotropic parts of the members of a chain of subsequent degenerated dB-spaces show a very particular behaviour. If $\mathfrak{Q}_1,\ldots,\mathfrak{Q}_n$

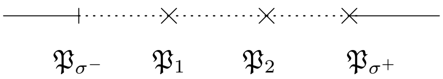
are degenerated dB-spaces, $\mathfrak{Q}_i \subsetneq \mathfrak{Q}_{i+1}$ with codimension 1, then there exists an index $i_{max} \in \{1, \ldots, n\}$ such that

$$\mathfrak{Q}_1^\circ \subsetneq \mathfrak{Q}_2^\circ \subsetneq \ldots \subsetneq \mathfrak{Q}_{i_{max}}^\circ \supseteq \mathfrak{Q}_{i_{max}+1}^\circ \supsetneq \ldots \supsetneq \mathfrak{Q}_n^\circ ,$$

where in each inclusion the codimension is at most 1 and only in the middle inclusion equality can hold.

The singularity σ_i is of polynomial type, left dense, right dense or dense, if and only if $\delta_- = 0 = \delta_+$, $\delta_- > 0 = \delta_+$, $\delta_- = 0 < \delta_+$ or $\delta_- > 0 < \delta_+$, respectively. We can have $\delta = 0$ only if σ_i is dense. The case $\delta = 1$, $\delta_- = \delta_+ = 0$, just means that (σ_i^-, σ_i^+) is indivisible with negative length.

We will visualize this inner structure of a singularity in the following way: For example



should describe a singularity which is right dense with $\delta = 3$.

It is our aim in this section to describe the evolution of singularities when performing the transformation $\mathcal{T}_{\surd} \circ \mathcal{T}_J \circ \mathcal{T}_2$. In view of our needs in the investigation of generalized strings we will content ourselves with a sound discussion of the case that this transformation is applied to a chain ϖ with $\operatorname{ind}_- \varpi = 0$.

In the first step we deal with the transformation $\mathcal{T}_2$. To this end let $\varpi \in \mathfrak{M}_0^{ep}$ be given, assume that $\lim_{x\to-\infty} q_\infty(\varpi)(x) = 0$, and let $\omega \in \mathfrak{M}_{<\infty}^{sym}$ be such that $q_\infty(\omega)(z) = zq_\infty(\varpi)(z^2)$.

We have to investigate the structure of the chain of dB-spaces arising from $\mathfrak{P}(E_{\omega(t)})$. By Lemma 2.13 all these spaces $\mathfrak{P}$ are symmetric. Moreover, since $\operatorname{ind}_- \varpi = 0$, we know that always $\mathfrak{P}_+$ is a Hilbert space, see Lemma 3.13. In particular, by [KWW4, Lemma 2.4] for every dB-space $\mathfrak{P}$ in this chain we have $\dim \mathfrak{P}^\circ \leq 1$. By the structure theory of degenerated dB-spaces developed in [KW3] this knowledge already has a big influence on the kind of singularities that may appear.

6.1. Proposition. *Let $\varpi \in \mathfrak{M}_0^{ep}$ and assume that $\lim_{x\to-\infty} q_\infty(\varpi)(x) = 0$. Moreover, let $\omega \in \mathfrak{M}_{<\infty}^{sym}$ be such that $q_\infty(\omega)(z) = zq_\infty(\varpi)(z^2)$ and write $\operatorname{dom}\omega = (\sigma_0, \sigma_1) \cup \ldots \cup (\sigma_n, \sigma_{n+1})$. We have for every $k \in \{1, \ldots, n\}$*

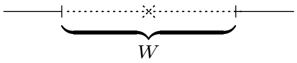
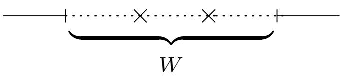
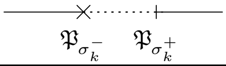
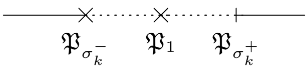
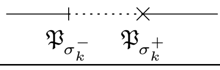
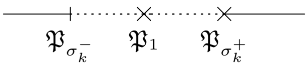
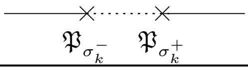
$$\kappa(\sigma_k) = 1,\ \delta(\sigma_k) \leq 3,\ \delta_\pm(\sigma_k) \leq 1 .$$

Let a_k, b_k, $k = 0, \ldots, n$, and Λ be as in Lemma 5.9. According to the following table the structure of σ_k can be read off the behaviour of Λ at a_k and b_{k-1} and the fact whether or not there is an indivisible interval between b_{k-1} and a_k. Thereby we set

$$l := \mathfrak{t}(\varpi(a_k)) - \mathfrak{t}(\varpi(b_{k-1})) ,$$

$$\Lambda(b_{k-1}-) := \lim_{t\to b_{k-1}-} \Lambda(t),\ \Lambda(a_k+) := \lim_{t\to a_k+} \Lambda(t) ,$$

and, in case both of these limits are finite, $m := \Lambda(a_k+) - \Lambda(b_{k-1}-)$.

$\Lambda(b_{k-1}-)$	$\Lambda(a_k+)$	$b_{k-1}=a_k$	$b_{k-1}<a_k$	Structure of σ_k	
$\in\mathbb{R}$	$\in\mathbb{R}$	✓ $m<0$			$W=W_{(m,0)}$
$\in\mathbb{R}$	$\in\mathbb{R}$		✓		$W=\begin{pmatrix}1 & mz+lz^3\\ 0 & 1\end{pmatrix}$
$+\infty$	$\in\mathbb{R}$	✓			$\mathfrak{P}_{\sigma_k^-,e}=\mathfrak{P}_{\sigma_k^+,e}$
$+\infty$	$\in\mathbb{R}$		✓		$\mathfrak{P}_{\sigma_k^-,e}\neq\mathfrak{P}_{1,e}=\mathfrak{P}_{\sigma_k^+,e}$
$\in\mathbb{R}$	$-\infty$	✓			$\mathfrak{P}_{\sigma_k^-,e}=\mathfrak{P}_{\sigma_k^+,e}$
$\in\mathbb{R}$	$-\infty$		✓		$\mathfrak{P}_{\sigma_k^-,e}=\mathfrak{P}_{1,e}\neq\mathfrak{P}_{\sigma_k^+,e}$
$+\infty$	$-\infty$	✓			
$+\infty$	$-\infty$		✓		$\mathfrak{P}_{\sigma_k^-,e}\neq\mathfrak{P}_{\sigma_k^+,e}$

Proof. As we already noted for every dB-space $\mathfrak{P}$ in the chain arising from $\mathfrak{P}(E_{\omega(t)})$, we must have $\dim\mathfrak{P}^\circ\leq 1$. From this it is immediate that $\delta_\pm(\sigma_k)\leq 1$. The fact that $\delta(\sigma_k)\leq 3$ is clear from the structure of the isotropic parts of a chain of subsequent degenerated dB-spaces.

The proof of the remaining assertions of the present proposition is based on the following observations:

(i) The limit $\Lambda(a_k+)$ is finite if and only if $\mathfrak{P}_{\sigma_k^+}$ is nondegenerated. Similarly, the limit $\Lambda(b_{k-1}-)$ is finite if and only if $\mathfrak{P}_{\sigma_k^-}$ is nondegenerated.

(ii) We have $\mathfrak{P}_{\sigma_k^+,+}=\mathfrak{P}(E_{\varpi(a_k)})$ and $\mathfrak{P}_{\sigma_k^-,+}=\mathfrak{P}(E_{\varpi(b_{k-1})})$.

(iii) If $\mathfrak{P}_1\subsetneq\mathfrak{P}_2$ are two degenerated symmetric dB-spaces such that $\mathfrak{P}_{1,+}$ and $\mathfrak{P}_{2,+}$ are nondegenerated, then $\mathfrak{P}_{1,+}\subsetneq\mathfrak{P}_{2,+}$.

(iv) If $b_{k-1}=a_k$, then $\delta(\sigma_k)\leq 1$.

(v) There exist at most two degenerated dB-spaces $\mathfrak{P}$ with $\mathfrak{P}_{\sigma_k^-}\subseteq\mathfrak{P}\subseteq\mathfrak{P}_{\sigma_k^+}$.

ad(i): We have $\Lambda(a_k+)>-\infty$ if and only if $\lim_{t\searrow a_k}\tau_k(t)>-\infty$. This implies that $\zeta_+:=\lim_{t\searrow a_k}(\hat\mu|_{\mathcal{L}})^{-1}>\sigma_k$. Since in this case (σ_k,ζ_+) is indivisible, we have $\mathfrak{P}_{\sigma_k^+}=\mathfrak{P}(E_{\omega(\zeta_+)})$. The same argument applies to $\lambda(b_{k-1}-)$.

`ad`(ii): We have

$$\mathfrak{P}_{\sigma_k^+,e} = \Big(\bigcap_{t>\sigma_k^+} \mathfrak{P}(E_{\omega(t)}) \Big)_e = \bigcap_{t>\sigma_k^+} \mathfrak{P}(E_{\omega(t)})_e \,.$$

However,

$$\mathfrak{P}(E_{\omega(t)})_e \cong \mathfrak{P}(E_{\omega(t)})_+ = \mathfrak{P}(E_{\varpi(\hat\mu(t))})\,,$$

and, since $\hat\mu(\sigma_k^+) = a_k$,

$$\bigcap_{t>\sigma_k^+} \mathfrak{P}(E_{\varpi(\hat\mu(t))}) = \mathfrak{P}(E_{\varpi(a_k)})\,.$$

The same argument applies to $\mathfrak{P}(E_{\varpi(b_{k-1})})$.

`ad`(iii): Assume that $\mathfrak{P}_{1,+} = \mathfrak{P}_{2,+} =: \mathfrak{P}$. By [KWW4, Proposition 4.7] there exists exactly one degenerated dB-space $\mathfrak{Q}$ with $\mathfrak{Q}_+ = \mathfrak{P}$, a contradiction.

`ad`(iv): If $b_{k-1} = a_k$, then, by (ii), $\mathfrak{P}_{\sigma_k^+,+} = \mathfrak{P}_{\sigma_k^-,+}$. By [KWW4, Theorem 3.11] this implies that $\dim \mathfrak{P}_{\sigma_k^+}/\mathfrak{P}_{\sigma_k^-} \leq 1$.

`ad`(v): Since $\dim \mathfrak{P}^\circ \leq 1$, this follows from the structure of a chain of subsequent degenerated dB-spaces.

We will now go through the cases listed in the above table.

$\Lambda(b_{k-1}-), \Lambda(a_k+) \in \mathbb{R},\ b_{k-1} = a_k$: We know that $\delta_- = \delta_+ = 0$ and that $\mathfrak{P}_{\sigma_k^-,+} = \mathfrak{P}_{\sigma_k^+,+}$ It follows from Lemma 3.4 that $\omega(\sigma_k^-)^{-1}\omega(\sigma_k^+) = W_{(\alpha,0)}$ for some α. However, in the present case we have

$$\alpha = \lim_{t\searrow a_k} \tau_k(t) - \lim_{t\nearrow b_{k-1}} \tau_{k-1}(t) = \Lambda(a_k+) - \Lambda(b_{k-1}-) = m\,.$$

$\Lambda(b_{k-1}-), \Lambda(a_k+) \in \mathbb{R},\ b_{k-1} < a_k$: Again $\delta_- = \delta_+ = 0$. We have $\mathcal{T}_{\surd}(\omega(\sigma_k^-))^{-1}\mathcal{T}_{\surd}(\omega(\sigma_k^+)) = W_{(l,0)}$. Since $\omega(\sigma_k^-) = \mathcal{T}_{2,\Lambda(b_{k-1}-)}(\varpi(b_{k-1}))$ and $\omega(\sigma_k^+) = \mathcal{T}_{2,\Lambda(a_k+)}(\varpi(a_k))$, the assertion follows from Remark 3.7.

$\Lambda(b_{k-1}-) = +\infty, \Lambda(a_k+) \in \mathbb{R},\ b_{k-1} = a_k$: It follows from (iv) that $\delta \leq 1$. Moreover, we know from (i) that $\delta_- = 1$ and $\delta_+ = 0$, which implies that $\delta \neq 0$.

$\Lambda(b_{k-1}-) = +\infty, \Lambda(a_k+) \in \mathbb{R},\ b_{k-1} < a_k$: We have $\delta_- = 1$, $\delta_+ = 0$, and know that $\mathfrak{P}_{\sigma_k^-,+} \neq \mathfrak{P}_{\sigma_k^+,e}$. Since (σ_k, σ_k^+) is indivisible of type 0, it follows that $\omega(\sigma_k^+)_{21} \in \mathfrak{P}(E_{\omega(\sigma_k^+)})$, and hence that $\mathfrak{P}_{\sigma_k^+,e} \subseteq \overline{\operatorname{dom} \mathcal{S}_{\mathfrak{P}_{\sigma_k^+}}}$. This space is a dB-subspace with codimension 1 in $\mathfrak{P}_{\sigma_k^+}$ and is degenerated since $\delta_- > 0$. However, it contains the same even functions than $\mathfrak{P}_{\sigma_k^+}$, and thus cannot be equal to $\mathfrak{P}_{\sigma_k^-}$. By (v) it must be the only space which lies strictly between $\mathfrak{P}_{\sigma_k^+}$ and $\mathfrak{P}_{\sigma_k^-}$.

$\Lambda(b_{k-1}-) \in \mathbb{R}, \Lambda(a_k+) = -\infty,\ b_{k-1} = a_k$: It follows from (iv) that $\delta \leq 1$. We know, moreover, from (i) that $\delta_- = 0$ and $\delta_+ = 1$, which also implies that $\delta \neq 0$.

$\Lambda(b_{k-1}-) \in \mathbb{R}, \Lambda(a_k+) = -\infty,\ b_{k-1} < a_k$: We have $\delta_- = 0$, $\delta_+ = 1$, and $\mathfrak{P}_{\sigma_k^+,+} \neq \mathfrak{P}_{\sigma_k^-,+}$. Let $\mathfrak{P}_1$ be the smallest degenerated dB-space which contains $\mathfrak{P}_{\sigma_k^-}$. Since (σ_k^-, σ_k) is indivisible of type 0, we have $\mathfrak{P}_1 = \operatorname{span}(\mathfrak{P}_{\sigma_k,-} \cup \{\omega(\sigma_k^-)_{21}\})$, and therefore $\mathfrak{P}_{1,e} = \mathfrak{P}_{\sigma_k^-,e}$. This rules out the possibility that $\delta = 1$.

$\Lambda(b_{k-1}-) = +\infty, \Lambda(a_k+) = -\infty,\ b_{k-1} = a_k$: We have $\delta_- = \delta_+ = 1$ and $\mathfrak{P}_{\sigma_k^-,+} = \mathfrak{P}_{\sigma_k^+,+}$. Thus (iii) implies that $\mathfrak{P}_{\sigma_k^-} = \mathfrak{P}_{\sigma_k^+}$.

$\Lambda(b_{k-1}-) = +\infty, \Lambda(a_k+) = -\infty,\ b_{k-1} < a_k$: We have $\delta_- = \delta_+ = 1$ and $\mathfrak{P}_{\sigma_k^-,+} \neq \mathfrak{P}_{\sigma_k^+,+}$. Thus $\delta > 0$, and by (v) it follows that $\delta = 1$.

It remains to show that $\kappa(\sigma_k) = 1$. To this end recall that $\operatorname{ind}_- \mathfrak{P}_{\sigma_k^-} = \max_{t<\sigma_k} \operatorname{ind}_- \omega(t)$ and $\operatorname{ind}_- \mathfrak{P}_{\sigma_k^+} + \delta_+ = \min_{t>\sigma_k} \operatorname{ind}_- \omega(t)$, cf. [KW3].

Consider the case that $\delta_- = \delta_+ = 0$. If $\delta = 1$ the assertion is clear. If $\delta = 3$ it follows since $l > 0$. Assume that not both of $\delta_\pm$ are equal to 0. If $\delta = 1$, we have $\operatorname{ind}_- \mathfrak{P}_{\sigma_k^+} = \operatorname{ind}_- \mathfrak{P}_{\sigma_k^-}$, and again $\kappa(\sigma_k) = 1$. Consider the fourth of the cases in the table. Then $\operatorname{ind}_- \mathfrak{P}_{\sigma_k^-} = \operatorname{ind}_- \mathfrak{P}_1$ since $\mathfrak{P}_{\sigma_k^-,e}$ and $\mathfrak{P}_{1,e}$ have the same negative index. The increase of negative squares from $\mathfrak{P}_1$ to $\mathfrak{P}_{\sigma_k^+}$ is then 1. In the last remaining case, the same argument shows that $\operatorname{ind}_- \mathfrak{P}_1 = \operatorname{ind}_- \mathfrak{P}_{\sigma_k^+}$, and thus that $\kappa(\sigma_k) = 1$.

❑

Next we deal with $\mathcal{T}_J$. It is a consequence of Lemma 2.14 that an application of this transformation does not change the structure of singularities.

6.2. Corollary. *Let $\omega \in \mathfrak{M}_{<\infty}$ and assume that $\mathfrak{K}_-(\omega(t)) = \mathfrak{K}(\omega(t))$ and $\mathfrak{K}_+(\omega(t)) = \mathfrak{K}(\omega(t))$. Moreover, let $\upsilon := \mathcal{T}_J(\omega)$. Let π_+, π_- be the projections of $\mathfrak{K}(\omega(t))$ onto the first and second component, respectively, and put $\phi := \pi_+ \circ \pi_-^{-1}$. Then ϕ induces an order preserving bijection of the chain of all dB-subspaces of $\mathfrak{P}(E_{\omega(t)})$ onto the chain of all dB-subspaces of $\mathfrak{P}(E_{\upsilon(t)})$. Thereby for all dB-subspaces $\mathfrak{Q}$ of $\mathfrak{P}(E_{\omega(t)})$*

$$\operatorname{ind}_- \phi(\mathfrak{Q}) = \operatorname{ind}_- \mathfrak{Q},\ \dim \phi(\mathfrak{Q})^\circ = \dim \mathfrak{Q}^\circ .$$

If $\omega \in \mathfrak{M}_{<\infty}^{sym}$, then also υ is symmetric, and we have

$$\phi(\mathfrak{Q}_e) = \phi(\mathfrak{Q})_o,\ \phi(\mathfrak{Q}_o) = \phi(\mathfrak{Q})_e . \tag{6.1}$$

Proof. With the notation of Lemma 2.14 we have $E_{\upsilon(t)} = \tilde{E}_{\omega(t)}$. Hence the first assertion is an immediate corollary of Lemma 2.14. It remains to investigate the symmetric situation. However, by Lemma 2.7, $\phi = \pi_+ \circ \pi_-^{-1}$ maps even to odd functions and odd to even functions. Since ϕ^{-1} has the same property, the relation (6.1) follows.

❑

We can now deduce which singularities are created by an application of $\mathcal{T}_{\surd} \circ \mathcal{T}_J \circ \mathcal{T}_2$.

6.3. Proposition. *Let $\varpi \in \mathfrak{M}_0^{ep}$ and assume that*

$$\lim_{x\to-\infty} q_\infty(\varpi)(x) = \lim_{x\to-\infty} \frac{1}{q_\infty(\varpi)(x)} = 0 .$$

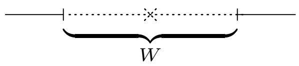
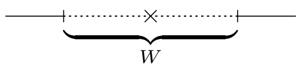
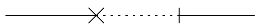
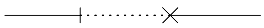
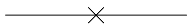
Moreover, let $\upsilon \in \mathfrak{M}_{<\infty}^{ep}$ be such that $q_\infty(\upsilon)(z) = -(z q_\infty(\varpi)(z))^{-1}$. Let a_k, b_k, $k = 0, \dots, n$, and Λ be as in Lemma 5.9. Then υ has exactly n singularities, say

$\gamma_1 < \ldots < \gamma_n$. We have for every $k \in \{1, \ldots, n\}$

$$\kappa(\gamma_k) = 1,\ \delta(\gamma_k) \le 2,\ \delta_\pm(\gamma_k) \le 1\,.$$

The structure of γ_k can be read off the behaviour of Λ at a_k and b_{k-1} and the fact whether or not there is an indivisible interval between b_{k-1} and a_k. Let $l, \Lambda(b_{k-1}-), \Lambda(a_k+)$ and m be as in Proposition 6.1. Moreover, put

$$\alpha := \varpi(b_{k-1})'_{21}(0),\ p(z) := -(mz + lz^2),\ m' := m(1+\alpha^2),\ \phi := -\operatorname{Arccot}\alpha\,.$$

$\Lambda(b_{k-1}-)$	$\Lambda(a_k+)$	Structure of σ_k		
$\in \mathbb{R}$	$\in \mathbb{R}$	$b_{k-1} = a_k$		$W = W_{(m',\phi)}$
		$b_{k-1} < a_k$		$W = \begin{pmatrix} 1-\alpha p(z) & -\alpha^2 p(z) \\ p(z) & 1+\alpha p(z) \end{pmatrix}$
$+\infty$	$\in \mathbb{R}$			
$\in \mathbb{R}$	$-\infty$			
$+\infty$	$-\infty$			

Proof. Let $\omega := \mathcal{T}_2(\varpi)$ and $\tilde{\omega} := \mathcal{T}_J(\omega)$, so that $\upsilon = \mathcal{T}_{\checkmark}(\tilde{\omega})$. By Theorem 5.1 and Corollary 6.2 singularities of υ can only occur at singularities of ω. We shall go through the different possibilities of singularities of ω and show that each of them gives rise to a singularity of υ with the asserted structure. We will use the same notation as in the previous proofs.

The first two cases require explicit computation.

$\Lambda(b_{k-1}-) \in \mathbb{R}, \Lambda(a_k+) \in \mathbb{R},\ b_{k-1} = a_k$: Then $\omega(\sigma_k^-)^{-1}\omega(\sigma_k^+) = W_{(m,0)}$, thus $\tilde{\omega}(\sigma_k^-)^{-1}\tilde{\omega}(\sigma_k^+) = W_{(m,\frac{\pi}{2})}$. Moreover, $\tilde{\omega}(\sigma_k^-)'_{21}(0) = -\omega(b_{k-1})'_{21}(0) = -\varpi(b_{k-1})'_{21}(0) = -\alpha$. We obtain from Proposition 3.6 that

$$\upsilon(\hat{\mu}(\sigma_k^-))^{-1}\upsilon(\hat{\mu}(\sigma_k^+)) = W_{(m',\phi)}\,.$$

$\Lambda(b_{k-1}-) \in \mathbb{R}, \Lambda(a_k+) \in \mathbb{R},\ b_{k-1} < a_k$: In this case

$$\tilde{\omega}(\sigma_k^-)^{-1}\tilde{\omega}(\sigma_k^+) = \begin{pmatrix} 1 & 0 \\ -p(z) & 1 \end{pmatrix}$$

and $\tilde{\omega}(\sigma_k^-)'_{12}(0) = \tilde{\omega}(\sigma_k^+)'_{12}(0) = -\alpha$. By (3.2) it follows that

$$\upsilon(\hat{\mu}(\sigma_k^-))^{-1}\upsilon(\hat{\mu}(\sigma_k^+)) = \begin{pmatrix} 1-\alpha p(z) & -\alpha^2 p(z) \\ p(z) & 1+\alpha p(z) \end{pmatrix}$$

For the investigation of the remaining cases note that, by Corollary 6.2, we just have to inspect the behaviour of the odd parts of the dB-spaces arising from the chain ω.

$\Lambda(b_{k-1}-) = +\infty, \Lambda(a_k+) \in \mathbb{R},\ b_{k-1} = a_k$: By Proposition 6.1 $\mathfrak{P}_{\sigma_k^-,o}$ is degenerated, $\mathfrak{P}_{\sigma_k^+,o}$ is nondegenerated and $\dim \mathfrak{P}_{\sigma_k^+,o}/\mathfrak{P}_{\sigma_k^-,o} = 1$. Hence we have a singularity with $\delta_- = 1$, $\delta_+ = 0$ and $\delta = 1$.

$\Lambda(b_{k-1}-) = +\infty, \Lambda(a_k+) \in \mathbb{R},\ b_{k-1} < a_k$: We have $\mathfrak{P}_{\sigma_k^-,o} = \mathfrak{P}_{1,o}$ and this space is degenerated. The space $\mathfrak{P}_{\sigma_k^+,o}$ in nondegenerated. Thus we also have a singularity with $\delta_- = 1$, $\delta_+ = 0$ and $\delta = 1$.

The remaining cases are treated exactly in the same manner. This knowledge on the structure of the singularities implies that in any case $\delta \leq 2$, $\delta_\pm \leq 1$ and, cf. [KW3, Corollary 2.12] and its proof, that the increase of the negative index is equal to 1.

❑

7. On generalized strings

Recall [KaK1] that a *string* $S[L, m]$ is given by its length L, $0 \leq L \leq \infty$, and a nonnegative and nondecreasing function m defined on $[0, L)$ which may be chosen to be left-continuous. This concept can be generalized as follows: Let m be a function defined on $[0, L)$ which is nondecreasing and left-continuous except a finite number of points from $[0, L)$, and let D be a nondecreasing and left-continuous step function defined on $[0, L)$ which is constant except a finite number of growth points. Note that D corresponds to the so-called dipoles in [KL2].

Some point $x_e \in [0, L)$ is called a *dipole* if $D(x_e+) - D(x_e) > 0$, a *negative jump,* if $m(x_e+) - m(x_e) < 0$, and a *singularity,* if $m(x) \to +\infty$ for $x \nearrow x_e$, or $m(x) \to -\infty$ for $x \searrow x_e$, and $\int_{(x_e-\epsilon, x_e+\epsilon)} m(t)^2 dt < \infty$. The point $x_e \in [0, L)$ is called *critical* if it is a dipole, a singularity or a negative jump. The point 0 is critical if it is a dipole or if $-\infty \leq m(0+) < 0$, the point L is never a critical point. Observe that at a critical point can be both a dipole and a singularity or a negative jump. The tripel $S[L, m, D]$ is called *generalized string.* The relation

$$f'(x) + z \int_{[0,x]} f(x) dm(x) + z^2 \int_{[0,x]} f(x) dD(x) = 0, \quad f'(0-) = 0. \tag{7.1}$$

is the differential equation of a generalized string. Of course, this equation requires some explanation if $S[L, m, D]$ has singularities among its critical points. The appropriate interpretation is given by means of canonical systems, and for the sake of completeness we continue to recall some basic facts about these systems.

Let H be a real, symmetric and non-negative 2×2–matrix function on the interval $[0, l_H)$:

$$H(x) = \begin{pmatrix} h_1(x) & h_3(x) \\ h_3(x) & h_2(x) \end{pmatrix}, \quad x \in [0, l_H),$$

with locally integrable functions h_1, h_2 and h_3. A *canonical system* is a boundary value problem of the form

$$Jf'(x) = -zH(x)f(x),\ x \in [0, l_H), \quad f_1(0) = 0, \tag{7.2}$$

with $f(x) = (f_1(x)\ f_2(x))^T$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and a complex parameter z. Here the differential equation in (7.2) is considered to hold almost everywhere on $[0, l_H)$. Weyl's limit point case prevails at the point l_H for the canonical system (7.2) if and only if

$$\int_0^{l_H} \operatorname{tr} H(x)dx = \infty, \tag{7.3}$$

and from now on we assume that for each Hamiltonian H the relation (7.3) holds. The *fundamental matrix function*

$$W(x, z) = \begin{pmatrix} w_{11}(x, z) & w_{12}(x, z) \\ w_{21}(x, z) & w_{22}(x, z) \end{pmatrix}$$

of a canonical system (7.2) with Hamiltonian H is the unique solution of the integral equation

$$W(x, z)J - J = z \int_0^x W(s, z)H(s)ds. \tag{7.4}$$

Note that $W(0, z) = I$, and that $x \to W(x, z)$, $0 \le x < l_H$ is a maximal chain of matrix functions belonging to $\mathfrak{M}_0$.

For each $\omega \in \widetilde{\mathcal{N}_0}$ and $z \in \mathbb{C}^+$ the limit

$$Q(z) := \lim_{x \to l_H} \frac{w_{11}(x, z)\omega(z) + w_{12}(x, z)}{w_{21}(x, z)\omega(z) + w_{22}(x, z)} \tag{7.5}$$

exists, is independent of ω, and, as a function of z, belongs to the set of Nevanlinna functions $\mathcal{N}_0$, cf. [dB2]. The function Q is called the *Titchmarsh-Weyl coefficient* of the canonical system (7.2) or of the Hamiltonian H. Note that $W(\cdot, z)$ is a maximal matrix chain, and that Q coincides with its Weyl coefficient.

Let $\xi_\phi := (\cos\phi, \sin\phi)^T$ for some $\phi \in [0, \pi)$. The open interval $I_\phi \subset [0, l_H)$ is called H-indivisible of type ϕ if the relation

$$\xi_\phi^T JH = 0, \text{ a.e. on } I_\phi, \tag{7.6}$$

holds, see [Ka], [dB1]. This notion is the same as introduced for maximal chains in Section 4. In particular, $\det H = 0$ a.e. on I_ϕ. If (x_1, x_2) is a H-indivisible of type ϕ and lenght l, the fundamental matrix W satisfies the relation $W(x_2, z) = W(x_1, z)W_{(l,\phi)}(z)$, where the factor $W_{(l,\phi)}(z)$ is defined in relation (2.2), that is, H-indivisible intervals are also indivisible.

A Hamiltonian H is called *trace normed* if $h_1 + h_2 = 1$ a.e. on $[0, \infty)$. For the class of trace normed Hamiltonians a basic inverse result in [dB1] can be formulated as follows (see [W1]): Each function $Q \in \mathcal{N}_0$ is the Titchmarsh-Weyl coefficient of a canonical system with a trace normed Hamiltonian H on $[0,\infty)$. This correspondence is bijective if two Hamiltonians which coincide almost everywhere are identified.

Let Q_H denote the Titchmarsh-Weyl coefficient corresponding to some Hamiltonian H, and let

$$\widehat{H} = JHJ^T. \tag{7.7}$$

Then

$$\widehat{W}(x,z) = JW(x,z)J^T \tag{7.8}$$

is the fundamental matrix corresponding to $\widehat{H}$, and the relation (7.5) implies that

$$Q_{\widehat{H}}(z) = -(Q_H(z))^{-1} \tag{7.9}$$

If H is of diagonal form, that is $H = \operatorname{diag}(h_1,h_2)$, then $Q_H \in \mathcal{N}_0^{sym}$. The following proposition will be of use in what follows.

For the following we need the fact that any $Q \in \mathcal{N}_0 \setminus \{\infty\}$ admits a unique integral representation:

$$a + bz + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, d\sigma(t)\,, \tag{7.10}$$

where $a,b \in \mathbb{R}$, $b \geq 0$, σ is a nonnegative Borel measure on $\mathbb{R}$ with $\int_{\mathbb{R}} \frac{1}{1+t^2}\, d\sigma(t) < \infty$. σ is called the spectral measure of $Q(z)$.

7.1. Proposition. *Let $Q \neq \infty$ be some Nevanlinna function with a semibounded spectral measure,* $\operatorname{supp}\sigma \subset [c,\infty)$*, $c \in \mathbb{R}$. Let H be some Hamiltonian corresponding to Q with left-continuous components h_i, $1 \leq i \leq 3$ and* $\det H = 0$*. If W denotes the corresponding fundamental matrix, the relations*

$$\frac{h_3(x)}{h_2(x)} = \lim_{z\to-\infty} -\frac{w_{12}(x,z)}{w_{11}(x,z)} = \lim_{z\to-\infty} -\frac{w_{22}(x,z)}{w_{21}(x,z)}, \qquad x>0,\ h_2(x)>0, \tag{7.11}$$

$$\frac{h_3(x)}{h_1(x)} = \lim_{z\to-\infty} -\frac{w_{11}(x,z)}{w_{12}(x,z)} = \lim_{z\to-\infty} -\frac{w_{21}(x,z)}{w_{22}(x,z)}, \qquad x>0,\ h_1(x)>0, \tag{7.12}$$

hold.

Proof. The existence of a Hamiltonian H corresponding to Q with the required properties was shown in [W3], Theorem 3.1. We continue in two steps

Step 1: First we assume that $\operatorname{supp}\sigma \subseteq (0,\infty)$ and that $b = \lim_{y\to\infty} Q(iy)/iy = 0$. According to Corollary 3.2 of [W3], the function

$$v(x) = \frac{h_3(x)}{h_2(x)}, \qquad x \in (0, l_H)$$

is nondecreasing with and left-continuous. A rescaling allows to assume without loss of generality that the Hamiltonian H is of the form

$$H(x) = \begin{pmatrix} v(x)^2 & v(x) \\ v(x) & 1 \end{pmatrix}. \tag{7.13}$$

We are going to show that

$$v(x) = \lim_{z\to-\infty} -\frac{w_{22}(x,z)}{w_{21}(x,z)}, \qquad x>0. \tag{7.14}$$

If $v(x) = v(x_0)$ for some $x_0 < x$, the interval (x_0, x) is H-indivisible with constant v. It follows with $l = x - x_0$ that

$$\begin{pmatrix} w_{21}(x,z) \\ w_{22}(x,z) \end{pmatrix} = \begin{pmatrix} 1-zlv & -zl \\ zlv^2 & 1+zlv \end{pmatrix} \begin{pmatrix} w_{21}(x_0,z) \\ w_{22}(x_0,z) \end{pmatrix},$$

which leads to the relation

$$-\frac{w_{22}(x,z)}{w_{21}(x,z)} = v + \frac{1}{zl+N(z)}, \quad N(z) = -\left(\frac{w_{22}(x_0,z)}{w_{21}(x_0,z)} + v\right)^{-1}. \tag{7.15}$$

As $w_{22}(x_0,z)/w_{21}(x_0,z)$ is a Nevanlinna function, also $N(z)$ is a Nevanlinna function, and the relation (7.14) follows from the relation (7.15). Now we assume that $v(x) > v(t)$ for all $0 < t < x$. The relation (7.4) implies that

$$\begin{pmatrix} w_{21}'(x,z) \\ w_{22}'(x,z) \end{pmatrix} = -z(v(x)w_{21}(x,z) + w_{22}(x,z)) \begin{pmatrix} 1 \\ -v(x) \end{pmatrix}. \tag{7.16}$$

It follows that $-w_{22}'(x,z) = v(x)w_{21}'(x,z)$, which implies that

$$w_{22}(x,z) = 1 - v(x)w_{21}(x,z) + \int_0^x w_{21}(t,z)dv(t). \tag{7.17}$$

We will show that

$$\lim_{z\to-\infty} w_{21}(x,z) = \infty \tag{7.18}$$

and

$$\lim_{z\to-\infty} \frac{w_{21}(t,z)}{w_{21}(x,z)} = 0, \quad 0 \le t < x, \tag{7.19}$$

then the relation (7.17) implies the relation (7.14). Because of

$$w_{21}'(x,z) = -z(v(x)w_{21}(x,z)+w_{22}(x,z)) = -zv(x)w_{21}(x,z)-z+z\int_0^x v(t)w_{21}'(t,z)dt,$$

one finds with integration by parts that

$$w_{21}'(x,z) = -z\left(1 + \int_0^x w_{21}(t,z)dv(t)\right), \tag{7.20}$$

and it follows that

$$w_{21}(x,z) = -zx - z\int_0^x (x-u)w_{21}(u,z)dv(u). \tag{7.21}$$

Let $z < 0$. The relation (7.20) implies that the function $w_{21}'(\cdot,z)$ is positive in a neighborhood of 0. Then the relation (7.21) implies that $w_{21}(\cdot,z)$ is positive and nondecreasing, and the relations (7.18) and

$$w_{21}(x,z) \ge -zw_{21}(t,z)\int_t^x (x-u)dv(u) \tag{7.22}$$

follow. As $\int_t^x (x-u)dv(u) > 0$, the relation (7.22) implies the relation (7.19).

Step 2: Now we assume that $\operatorname{supp}\sigma \subset (c,\infty)$ for some $c<0$. Recall from [W2] that then the Hamiltonian $\tilde{H}$ defined by

$$\tilde{H}(x) = W(x,-c)H(x)W(x,-c)^T \tag{7.23}$$

corresponds to the Titchmarsh-Weyl coefficient $\tilde{Q}(z)$ given by

$$\tilde{Q}(z) = Q(z-c),$$

and the fundamental matrix

$$\tilde{W}(x,z) = W(x,z-c)W(x,-c)^{-1}.$$

In particular, $\operatorname{supp}\tilde{\sigma} \subset (0,\infty)$, and it follows that there exists a nondecreasing function $\tilde{v}$ such that the Hamiltonian $\tilde{H}$ is of the form (7.13). Moreover, the relation (7.14) is satisfied. It follows that

$$\frac{h_3(x)}{h_2(x)} = \frac{w_{22}(x,-c)\tilde{v}(x) - w_{12}(x,-c)}{-w_{21}(x,-c)\tilde{v}(x) + w_{11}(x,-c)},$$

and that

$$\begin{pmatrix} w_{21}(x,z-c) \\ w_{22}(x,z-c) \end{pmatrix} = \begin{pmatrix} \tilde{w}_{21}(x,z)w_{11}(x,-c) + \tilde{w}_{22}(x,z)w_{21}(x,-c) \\ \tilde{w}_{21}(x,z)w_{12}(x,-c) + \tilde{w}_{22}(x,z)w_{22}(x,-c) \end{pmatrix}.$$

Together with (7.14) the last relation implies

$$\lim_{z\to-\infty} -\frac{w_{22}(x,z)}{w_{21}(x,z)} = \frac{h_3(x)}{h_2(x)}.$$

The second relation in (7.11) follows from

$$-\frac{w_{12}(x,z)}{w_{11}(x,z)} + \frac{w_{22}(x,z)}{w_{21}(x,z)} = \frac{1}{w_{11}(x,z)w_{21}(x,z)} \to 0 \qquad (z\to-\infty),$$

where the relation $\det W(x,z) = 1$ has been used. The relations (7.12) can be shown in a similar way. If $\lim_{y\to\infty} Q(iy)/iy = b > 0$, the Hamiltonian is of the form $\operatorname{diag}(1,0)$ on the interval $(0,b)$, and it is easy to see that the relation (7.12) for $x \in (0,b)$ holds.

❑

If a generalized string $S[L,m,D]$ is given, a Hamiltonian H_0 of a canonical system can be constructed as follows: Define a new scale by

$$x(t) = t + \int_{[0,t)} dD(u),\, L_0 = L + \int_{[0,L)} dD(u). \tag{7.24}$$

Let $x(t+) = x(t) + D(t) - D(t-)$. If t_e is a dipole of $S[L,m,D]$, the interval $(x(t_e), x(t_e+))$ is assumed to be maximal H_0-indivisible of type $\pi/2$, that is, $H_0 = \operatorname{diag}(0,1)$ on $(x(t_e), x(t_e+))$. Define

$$m_0(x(t)) = m(t), \quad H_0(x(t)) = \begin{pmatrix} 1 & -m_0(x(t)) \\ -m_0(x(t)) & m_0(x(t))^2 \end{pmatrix}, \tag{7.25}$$

and put $H_0(x(t_e)) = \operatorname{diag}(0,1)$ if the point t_e is a singularity of $S[L,m,D]$. If $L + \int_{[0,L)} m(t)^2 dt < \infty$, it is assumed that $H_0 = \operatorname{diag}(0,1)$ on (L_0,∞). Summing up, there exists a (possibly empty) finite sequence of n maximal H_0-indivisible intervals $\mathfrak{D}_k$ of type $\pi/2$ such that $\mathfrak{D}_k < \mathfrak{D}_{k+1}$, and with $\mathfrak{D} = \bigcup_{k=1}^n \mathfrak{D}_k$ and $\mathfrak{I} := [0, L_0) \setminus \mathfrak{D}$ the Hamiltonian H_0 is given as follows.

$$H_0(x) = \begin{cases} \begin{pmatrix} 1 & -m_0(x) \\ -m_0(x) & m_0(x)^2 \end{pmatrix} & \text{if } x \in \mathfrak{I}, \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } x \in \mathfrak{D}. \end{cases} \tag{7.26}$$

Moreover, the construction of H_0 implies that the limit point case prevails. Conversely, if a Hamiltonian H_0 of the form (7.26) is given, the corresponding generalized string $S[L, m, D]$ can be recovered as follows: For $x \in \mathfrak{I}$, let

$$t(x) := \int_0^x \chi_{\{h_1 \neq 0\}}(u)du, \ L := t(L_0), \tag{7.27}$$

and

$$m(t) := m_0(x), \ D(t) := \int_0^t \chi_{\{h_1 = 0\}}(u)du. \tag{7.28}$$

If $Q \in \mathcal{N}_0$ with spectral measure σ has the property that $\operatorname{supp} \sigma \cap (-\infty, 0]$ consists of finitely many isolated points, there exists a Hamiltonian H of the form (7.26) such that Q is its Titchmarsh-Weyl coefficient, see [W3].

Let f be the solution of the relation (7.2) with Hamiltonian H_0. For $t(x)$ given by the relation (7.27), the function

$$f(t) := f_2(x) \tag{7.29}$$

is the solution of the relation (7.1). To justify the relation (7.29), note that if 0 is a dipole or a singularity the relations $Jf' = -zHf$, $f_1(0) = 0$, and $H_0(0) = \operatorname{diag}(0, 1)$ imply that $f_2'(0) = 0$. Otherwise, the relation $f'(0) = f_2'(0) = zm(0)f(0)$ holds and matches with the equation (7.1). On intervals where m_0 is defined and bounded, the relation $Jf' = -zHf$ implies that f_2 satisfies the differential equation

$$df_2' = -zf_2 dm_0. \tag{7.30}$$

Now consider a maximal H-indivisible interval $\mathfrak{D}_k = (a, b)$ of type $\pi/2$. Then

$$f_2'(x) = 0, \ f_1'(x) = -zf_2(x) \text{ if } x \in (a, b),$$

which yields

$$f_2(a) = f_2(b), \ f_1(b) - f_1(a) = -z(b - a)f_2(a). \tag{7.31}$$

The relation $f_2'(x) = zf_1(x) - zm_0(x)f_2(x)$ for $x = a$ and $x = b$ and the relations (7.31) yield

$$f_2'(b) - f_2'(a) + z(m_0(b) - m_0(a))f_2(a) + z^2(b - a)f_2(a) = 0. \tag{7.32}$$

In particular, as f_2 is constant on (a, b), the function f is well-defined by the relation (7.29).

Recall [LW] that the solution f of (7.1) can also be characterized in terms of m. Namely, if x_e is singularity of m, for a solution f of (7.1) on $[0, x_e)$ with $f'(0-) = 0$ the limits $f(x_e-)$ and $\int_0^{x_e-} m(t)f'(t)dt$ exist. Moreover, if $x > x_e$, there is exactly one solution f on (x_e, x) of (7.1) such that $f(x_e-) = f(x_e+)$ and $\int_{x_e-}^{x} m(t)f'(t)dt$ is finite. This solution f coincides with the function defined in (7.29).

Let Q_0 be the Titchmarsh-Weyl coefficient of the canonical system with the Hamiltonian (7.26). The function

$$Q_S(z) := z^{-1}Q_0(z) \tag{7.33}$$

is called the *principal Titchmarsh-Weyl coefficient of the generalized string* $S[L, m, D]$, see [LW]. Let $\mathcal{N}_\kappa^+$ be the set of all functions $q \in \mathcal{N}_\kappa$ such that $zq(z) \in \mathcal{N}_0$ is a Nevanlinna function. The basic inverse result from [LW] can be formulated as follows:

If $S[L, m, D]$ is a generalized string with κ critical points, then its principal Titchmarsh-Weyl coefficient Q_S belongs to the class $\mathcal{N}_\kappa^+$. Conversely, each function $Q \in \mathcal{N}_\kappa^+$ is the principal Titchmarsh-Weyl coefficient of a generalized string with κ critical points, which is uniquely determined by Q.

Let a generalized string $S[L, m, D]$ with principal Titchmarsh-Weyl coefficient $Q_S \in \mathcal{N}_\kappa^+$ be given. Then Q_S is also the Weyl coefficient of some matrix chain υ, and now the problem arises how the singularities of ϖ which are characterized in Proposition 6.3 can be described in terms of the generalized string $S[L, m, D]$.

As $Q_0 \in \mathcal{N}_0^{ep}$ implies in particular that the corresponding spectral measure σ is semibounded, Proposition 7.1 can be applied to the Hamiltonian H_0 of (7.26):

7.2. Corollary. *Let W_0 be the fundamental matrix of the canonical system with Hamiltonian H_0 from the relation* (7.26). *Then*

$$m_0(x) = \lim_{z\to-\infty} \frac{w_{12}^0(x,z)}{w_{11}^0(x,z)}, \qquad x \in \mathfrak{I}. \tag{7.34}$$

Moreover,

$$\lim_{z\to-\infty} \frac{w_{12}^0(x,z)}{w_{11}^0(x,z)} = \infty, \qquad x \in \mathfrak{D}.$$

7.3. Theorem. *Let $Q_S \in \mathcal{N}_\kappa^+$ be the principal Titchmarsh - Weyl coefficient of some generalized string $S[L, m, D]$, and assume that υ is the maximal chain with Weyl coefficient Q_S. Then the function m_0 from the relation* (7.25) *is equal to the function Λ from Lemma* 5.9*, and the maximal chain υ has κ singularities, which correspond to the critical points of $S[L, m, D]$. The five possible cases concerning the structure of a singularity of the chain υ which are described in Proposition* 6.3 *correspond to a negative jump in case 1, a dipole in case 2, and to a singularity in the last 3 cases.*

Proof. Let ϖ be the maximal chain with Weyl coefficient

$$q_\infty(\varpi)(z) = -\frac{1}{zQ_S(z)}.$$

Then $q_\infty(\varpi) \in \mathcal{N}_0$, and the representation formulas for $\mathcal{N}_\kappa^+$ functions from [KL1] imply that $\lim_{z\to\infty} q_\infty(\varpi)(z) = 0$. One finds from the relations (7.9) and (7.33) that the the matrix chain $\widehat{W_0}$ is equal to the chain ϖ from Proposition 6.3, and the relations (7.34) and (7.8) imply that m_0 is equal to the function Λ from Lemma 5.9. Note that the intervals (a_k, b_k) from Lemma 5.9 are the maximal intervals of $\mathfrak{I}$ which contain no critical point.

❑

If W_0^D denotes the factor in the chain W_0 which corresponds to a maximal H_0-indivisible interval (x_1, x_2) of type $\pi/2$ and length d, that is, $W_0(x_2, z) = W_0(x_1, z)W_0^D(z)$, then

$$W_0^D(z) = \begin{pmatrix} 1 & 0 \\ -zd & 1 \end{pmatrix}. \tag{7.35}$$

Roughly speaking, W_0^D corresponds to a dipole interval of length d. If $\Delta m_0 = m_0(x_2+) - m_0(x_1)$, the corresponding factor in the maximal chain with Titchmarsh-Weyl coefficient $Q_d(z) = zQ_S(z^2)$ is equal to

$$W_d^{D,\Delta}(z) = \begin{pmatrix} 1 & 0 \\ -z\Delta m_0 - z^3 d & 1 \end{pmatrix}, \tag{7.36}$$

and factor in the chain with Titchmarsh-Weyl coefficient $Q_S(z)$ is equal to

$$W_s^{D,\Delta}(z) = I - (z^2 d + z\Delta m_0)(t(x_1), 1)^T (t(x_1), 1)J. \tag{7.37}$$

Note that both matrix functions generate 1 negative square if $d > 0$ or if $d = 0$ and $\Delta m_0 < 0$.

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