Pontryagin spaces of entire functions IV

Michael Kaltenbäck, Harald Woracek

Abstract

A canonical differential equation is a system $y' = zJHy$ with a real, nonnegative and locally integrable 2×2-matrix valued function H. The theory of a canonical system is closely related to the spectral theory of a symmetric operator $T_{min}(H)$ which acts in a Hilbert space $L^2(H)$, and, moreover, is closely related to the theory of positive definite Nevanlinna functions by means of the Titchmarsh-Weyl coefficient associated to it.

In the present paper we define an indefinite analogue of canonical systems, construct an operator model which now acts in a Pontryagin space, and show that the spectral theory of the indefinite model is the perfect analogue of the classical theory of $T_{min}(H)$.

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1 Introduction

A canonical system, or Hamiltonian system, of differential equations is a system of the form

$$y'(x) = zJH(x)y(x), \ x \in [0, L)\,, \tag{1.1}$$

where H is a locally integrable, real and nonnegative 2×2-matrix valued function, where

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,,$$

and where z is a complex parameter. The function H is called the Hamiltonian of the system (1.1). Canonical systems are intensively analyzed via various approaches, see e.g. [AD], [dB], [GK], [HSW], [K1]–[K3], [KL3], [O], [S1], [S2], for approaches by means of operator methods.

Equations of the form (1.1) frequently appear in natural sciences, for example in Hamiltonian mechanics, cf. [Ar], [F], or as natural generalizations of Sturm-Liouville equations, cf. [R], or in the study of a vibrating string with non-homogeneous mass distribution, cf. [At], [KK].

In the theory of canonical systems an operator model is associated to the equation (1.1). It consists of a Hilbert space $L^2(H)$, a linear operator $T_{max}(H)$ and a boundary value map $\Gamma(H)$. The operator theory of the symmetry $T_{min}(H) := T_{max}(H)^*$ and its selfadjoint extensions governs the behaviour of the system (1.1), and is of outstanding importance for the investigation of its spectral theory.

Canonical systems are intimitely related to Nevanlinna functions. Assume that in the equation (1.1) Weyl's limit point case prevails, this means that

$$\int_0^L \operatorname{tr} H(x)\,dx = \infty\,,$$

and denote by $W(x,z) = (w_{ij}(x,z))_{i,j=1,2}$ the (transposed of the) fundamental solution of (1.1):

$$\frac{d}{dx}W(x,z)J = zW(x,z)H(x),\ x \in [0,L), \quad W(0,z) = I\,. \tag{1.2}$$

Then, for each $\tau \in \mathbb{R} \cup \{\infty\}$, the limit

$$\lim_{x \nearrow L} \frac{w_{11}(x,z)\tau + w_{12}(x,z)}{w_{21}(x,z)\tau + w_{22}(x,z)} =: q_H(z)$$

exists locally uniformly on $\mathbb{C} \setminus \mathbb{R}$ and does not depend on τ. The function q_H is called the Titchmarsh-Weyl coefficient associated to the Hamiltonian H. It belongs to the Nevanlinna class $\mathcal{N}_0$, which means that

$$q_H \text{ is analytic on } \mathbb{C} \setminus \mathbb{R}, \quad q_H(\overline{z}) = \overline{q_H(z)},\ z \in \mathbb{C} \setminus \mathbb{R}\,,$$

$$\operatorname{Im} q_H(z) \geq 0 \text{ for } \operatorname{Im} z > 0\,.$$

The Inverse Spectral Theorem of L.de Branges states that to every function $q \in \mathcal{N}_0$ there exists (up to reparameterization) one and only one Hamiltonian H such that $q = q_H$, cf. [dB], [W1]. This result tells us that the properties of (1.1) must be fully reflected in properties of its Titchmarsh-Weyl coefficient. Actually, there is a broad variety of results on the correspondence between H and q_H, see e.g. [GK], [K6], [W2], [WW].

The notion of the Nevanlinna class $\mathcal{N}_0$ admits a generalization to an indefinite setting. Let the generalized Nevanlinna class $\mathcal{N}_\kappa$, $\kappa \in \mathbb{N} \cup \{0\}$, be defined as the set of all functions q which are meromorphic in $\mathbb{C} \setminus \mathbb{R}$, satisfy $q_H(\overline{z}) = \overline{q_H(z)}$, $z \in \mathbb{C} \setminus \mathbb{R}$, and have the property that the kernel

$$K_q(w,z) := \frac{q(z) - \overline{q(w)}}{z - \overline{w}}$$

has κ negative squares.

It is a long standing open problem to find an indefinite generalization of

(1) the notion of a Hamiltonian,

(2) the operator theoretic interpretation of the equation (1.1),

and

(3) the Inverse Spectral Theorem.

In some situations partial answers or examples were obtained e.g. in [KL1], [LLS], [LW], [RS2] or [RS4]. In the present paper we settle the problems (1) and (2) in full generality, i.e. we define the notion of an indefinite Hamiltonian, we construct an operator model associated to it, and we investigate its operator theoretic properties. The solution of the problem (3), which is based on the present work and on our previous work on Pontryagin spaces of entire functions [KW2]–[KW5], will be presented in the forthcoming Part V of this series of papers.

Our notion of indefinite Hamiltonian carries its name with full right. This is shown by the following three facts: Firstly, the complete analogy of the spectral theory of the model relation in comparison to the positive definite case, secondly, the validity of the Inverse Spectral Theorem, and, finally, the coincidence with known particular cases.

The principal motivation of our present work is the intrinsic mathematical interest of the problems (1)–(3). Motivation for the study of these problems, however, can also be drawn from some investigations of differential operators with singularities appearing in mathematical physics, or from the investigation of indefinite versions of some problems of classical analysis. In this context let us mention

– The continuation problem for positive definite functions as treated in [KL2], [Ka], [KW1].

– The Stieltjes- or Hamburger- Moment problem, cf. [A], [K4], [K5], [KL1], [RS1].

– Sturm Liouville operators with discrete singularities, arising e.g. from the study of point interactions in quantum mechanics, cf. [AlKu], [GeS], [AGHH].

– Sturm-Liouville operators with non-integrable potential, cf. [G, Ku, SS, BDL].

– Differential operators with floating singularities which depend nonlinearly on the eigenvalue parameter, arising in magnetohydrodynamics, astrophysics, or polymerization chemistry, cf. [Ad], [Ko], [L].

For some of the mentioned differential operators Pontryagin space models were constructed, cf. [vDT], [P], [Sh], [DL], [DLSZ], in other contexts Pontryagin spaces anyway appear in an immediate and natural way.

We would like to indicate our intuition which led to the present work. From the very beginning one has a picture of an indefinite canonical system as an equation of the form (1.1) where H has some -finitely many- singularities. These singularities may be of different types. In the simplest situation they can be a kind of negative point-mass or a derivation of it. In more complicated situations, they can also be a kind of non-integrable singularity of the function H, or a combination of both. This, very rough, picture stems from

– inspecting existing examples: [KL1] deals with the indefinite moment problem. In [LLS] one example of a Hamiltonian with an inner (non-integrable) singularity appears. In [LW] and [KWW2] generalized strings, a particular instance of indefinite canonical systems, are studied.

– the structure theory of maximal chains of matrices, which are the indefinite analogue of the fundamental solution (1.2) of (1.1), as provided in [KW5].

– inspecting the distributional model for selfadjoint operators in Pontryagin spaces, and the corresponding representation of generalized Nevanlinna functions, as given in [JLT].

The present construction is modelled after the construction given in the mentioned paper [JLT], taking into account the desired variety and structure of singularities. The reader will maybe recognize this in our choice of notation, for instance the appearance of elements $\delta_0, \delta_1, \dots$ which one should think of as Dirac-distribution and its derivatives. However, our construction is carried out explicitly. We did not yet succeed in giving a proper interpretation of the present model in a distributional context. This will be the task of future work.

In the remaining part of this introduction we shall describe the contents of the present paper a bit more detailed. It is divided into several chapters:

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In Chapter 2 we deal with classical, i.e. positive definite, canonical systems. We formulate the basic construction of $L^2(H)$ and $T_{max}(H)$ in a setting which is sufficiently general for our later needs. For this purpose we have to introduce an adopted version of the notion of a boundary triplet. The classical results on the structure of $T_{max}(H)$ are recalled in Theorem 2.18 and Theorem 2.19. We give some supplements to the classical theory, for instance we deal with the multivalued part of powers of $T_{max}(H)$. Moreover, we are concerned with a compactness property of $T_{max}(H)$.

In Chapter 3 we introduce and investigate the notion of H-polynomials. These functions are buildt with help of 'H-integration',

$$f \mapsto \int JH(t)f(t)\,dt\,,$$

in a similar way as polynomials can be buildt with help of normal integration. Although on first sight a bit technical, H-polynomials are not only an indispensable tool for the later construction of indefinite Hamiltonians, but also reflect a portion of the inner structure of a singularity. They will be used to measure the growth of a Hamiltonian towards a singularity in the right way.

After these chapters of preliminary character we proceed in Chapters 4 and 5 to the definition and analysis of elementary indefinite Hamiltonians. This part forms the core of the present paper.

In Chapter 4 we introduce elementary indefinite Hamiltonians, cf. Definition 4.1. They are composed of a Hamiltonian which has an inner singularity and is of limited growth towards this singularity, a data-part which is concentrated at the singularity, and a set of interface conditions. The purpose is to model the simplest situation of an indefinite canonical system: a 'regular indefinite system with only one singularity'. Later on elementary indefinite Hamiltonians will be used as building blocks for modelling the most general situation. Due to the presence of different types of singularities which are in their nature essentially different, we have to define three kinds of elementary indefinite Hamiltonians.

To an elementary indefinite Hamiltonian $\mathfrak{h}$ an operator model is associated. It consists of a Pontryagin space $\mathcal{P}(\mathfrak{h})$, a linear relation $T(\mathfrak{h})$, and a boundary relation $\Gamma(\mathfrak{h})$. Moreover, there is a map $\psi(\mathfrak{h})$ which allows us to associate a space of functions with $\mathcal{P}(\mathfrak{h})$. Thereby $(\mathcal{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h}))$ is the indefinite analogue of $(L^2(H), T_{max}(H), \Gamma(H))$. The map $\psi(\mathfrak{h})$ realizes the idea that, as long as one stays away from the singularity, the indefinite system behaves like a positive definite one. After the actual definition of the model we provide some basic results on the geometry of the model.

Chapter 5 is devoted to the proof of the operator theoretic properties of the model $(\mathcal{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h}))$, cf. Theorem 5.1. This result already shows the analogy to the classical theory of canonical systems as stated in Theorem 2.18. Cornerstones on the way are the verification of the abstract Green's identity, Proposition 5.2, and the fact that $\ker \Gamma(\mathfrak{h}) = T(\mathfrak{h})^*$, Proposition 5.4. Besides this, we investigate reparameterizations, i.e. changes of the scale, in the setting of elementary indefinite Hamiltonians.

The Chapters 6 and 7 are of intermediate character. In Chapter 6 we provide a general construction of pasting of boundary value triplets, cf. Definition 6.1, Proposition 6.2. It formalizes the, frequently appearing, idea of gluing together boundary value problems by means of continuous boundary values, as for example in [dSW1], [dSW2]. The machinery provided in this chapter is used later on to paste elementary indefinite Hamiltonians; this is the way we wil obtain a model for the general situation.

The first application of the operation of pasting can be found in Chapter 7, where we obtain a result on the further splitting up of elementary indefinite Hamiltonians. This generalizes the natural splitting of a space $L^2(H)$ into $L^2(H_1) \oplus L^2(H_2)$, when H is split into $H_1 := H|_{[0,a)}$ and $H_2 := H|_{[a,L)}$. Although proofs are pretty technical, the results of this chapter are natural. They play an important role in the forthcoming investigation of Titchmarsh-Weyl coefficients, since they will enable us to split indefinite Hamiltonians at any point.

Finally, in Chapter 8, we come to the definition of an indefinite Hamiltonian and the model associated to it. An indefinite Hamiltonian consists of a Hamiltonian which has a finite number of inner singularities and is of restricted growth towards them, a data-part concentrated at these singularities, and a set of interface conditions at each of them, cf. Definition 8.1, Remark 8.3. The construction of the model is done by partitioning the Hamiltonian into finitely many elementary indefinite Hamiltonians, and plugging together their models with help of the operation of pasting provided in Chapter 6. In a similar way the results Theorem 8.6 and Theorem 8.7 are deduced. These theorems settle the operator theory of indefinite canonical systems and can be viewed as the main results of the present work. They are the perfect indefinite analogues of the classical Theorem 2.18 and Theorem 2.19.

The paper closes with a discussion of the dependency on the choice of partitioning, and a short investigation of reparameterizations.

2 Positive definite canonical systems

In this section we consider classical, i.e. positive definite, canonical systems. We recall the notion of a Hamiltonian and the associated model space in a sufficiently general setting and state some well known facts concerning the model relation. Moreover, we give some supplements to these results and deal with a compactness property of the model relation.

2.1 The model associated to a Hamiltonian

a. Definition of a Hamiltonian

Let $I = (s_-, s_+)$ be an interval on the real axis where $s_- < s_+$, $s_-, s_+ \in \mathbb{R} \cup \{\pm\infty\}$. A Hamiltonian on I is a measurable function H defined on I which takes real and nonnegative 2×2-matrices as values, is locally integrable on I, and does not vanish on any set of positive measure.

An important role is played by the primitive $\mathfrak{t}$ of $\operatorname{tr} H$. It is determined up to an additive constant. Since $\operatorname{tr} H$ is nonnegative, locally integrable, and does not vanish on any set of positive measure, $\mathfrak{t}$ is absolutely continuous and strictly increasing. Thus $\mathfrak{t}$ maps I bijectively onto some interval (L_-, L_+). Note that, since $\operatorname{tr} H$ does not vanish on any set of positive measure, also the inverse function $\mathfrak{t}^{-1}$ is absolutely continuous.

Since $\mathfrak{t}$ is determined up to an additive constant, it is meaningful to call H regular at the endpoint s_- (at s_+), if $L_- > -\infty$ ($L_+ < \infty$, respectively). If H is not regular at s_- or s_+, it is called singular at the respective endpoint.

Intervals where H is of a particularly simple form play a special role. For $\phi \in \mathbb{R}$ denote by ξ_ϕ the vector

$$\xi_\phi := \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}.$$

Note that ξ_{ϕ_1} and ξ_{ϕ_2} are linearly dependent if and only if $\phi_1 - \phi_2 \in \pi\mathbb{Z}$. An interval $(\alpha_-, \alpha_+) \subseteq I$, $\alpha_- < \alpha_+$, is called H-indivisible of type $\phi \in [0, \pi)$ if

$$\operatorname{ran} H(t) = \operatorname{span}\{\xi_\phi\},\ t \in (\alpha_-, \alpha_+) \text{ a.e.}$$

In this case we have, with an appropriate measurable, scalar and a.e. positive function $h(t)$,

$$H(t) = h(t)\xi_\phi\xi_\phi^T,\ t \in (\alpha_-, \alpha_+) \text{ a.e.}$$

If (α_-, α_+) is H-indivisible, the difference $\mathfrak{t}(\alpha_+) - \mathfrak{t}(\alpha_-) \in (0, \infty]$ is called the length of this H-indivisible interval.

It is clear that, if (α_-, α_+) and (α'_-, α'_+) are H-indivisible intervals with nonempty intersection, then their types must coincide and their union is again H-indivisible. Hence every H-indivisible interval is contained in a maximal H-indivisible interval.

Given a Hamiltonian H, we define numbers $\alpha_k^-(H)$ inductively for $k \in \mathbb{N} \cup \{0\}$ by

(i) $\alpha_0^-(H) := s_-$.

and

(ii_a) If $\alpha_k^-(H)$ is the left endpoint of an H-indivisible interval, let $\alpha_{k+1}^-(H)$ be the right endpoint of the maximal H-indivisible interval with left endpoint $\alpha_k^-(H)$.

(ii_b) If $\alpha_k^-(H)$ is not the left endpoint of an H-indivisible interval, put $\alpha_{k+1}^-(H) := \alpha_k^-(H)$.

Note that no point $\alpha_k^-(H)$ can be contained in an H-indivisible interval.

Numbers $\alpha_k^+(H)$, $k \in \mathbb{N} \cup \{0\}$, are defined in the same way starting from $\alpha_0^+(H) := s_+$ and proceeding downwards.

If it is clear from the context from which Hamiltonian H the points $\alpha_k^\pm(H)$ were constructed, we shall drop the argument H and just write $\alpha_k^\pm$. Similarly, if no confusion can occur, we shall just speak of indivisible intervals instead of H-indivisible intervals.

Let us remark that the whole interval (s_-, s_+) is indivisible if and only if one of the following three equivalent statements hold true: $\alpha_1^- = s_+$, $\alpha_1^+ = s_-$, $\alpha_1^- > \alpha_1^+$. In general, we have $\alpha_n^- = s_+$ if and only if $\alpha_n^+ = s_-$ if and only if the Hamiltonian consists of n maximal indivisible intervals.

b. The space $L^2(H)$

To a Hamiltonian H an inner product space $L^2(H)$ is associated. Denote by $\mathcal{M}(I)$ the set of all measurable functions on I with values in $\mathbb{C}^2$ which possess the property

(C) If (α_-, α_+) is indivisible of type ϕ, then $\xi_\phi^T f$ is constant a.e. on (α_-, α_+).

Moreover, denote by $\mathrm{AC}(I)$ the subset of $\mathcal{M}(I)$, which consists of all locally absolutely continuous functions which satisfy (C).

Let $L^2(H)$ be the subset of $\mathcal{M}(I)$ containing all functions $f \in \mathcal{M}(I)$ which satisfy

(L2) $\int_I f^* H f < \infty$.

An inner product is defined on $L^2(H)$ by

$$(f, g)_{L^2(H)} := \int_I g^* H f \,.$$

Here, and throughout this paper, integration is understood with respect to the Lebesgue measure, unless explicitly indicated differently.

Let us remark that, if H is singular at s_+, then for every $f \in L^2(H)$ we have $(Hf)(t) = 0$, $t \in (\alpha_1^+, s_+)$ a.e. The analogous statement holds for the endpoint s_-.

2.1 Remark. Our standard reference concerning the classical theory of canonical systems will be [HSW]. But note that there the authors always assume that $\operatorname{tr} H = 1$ a.e. on $\mathbb{R}$. Moreover, the space $L^2(H)$ in the present work is in their notation denoted by $L_s^2(H)$. What is called $L^2(H)$ in [HSW] is the space defined by requiring (L2) but dropping the condition (C).

On $\mathcal{M}(I)$ we can define an equivalence relation $=_H$ by

$$f =_H g \quad :\Longleftrightarrow \quad H(f - g) = 0 \text{ a.e.}$$

Clearly $L^2(H) \subseteq \mathcal{M}(I)$ is saturated with respect to this equivalence relation.

Denote by $\pi : \mathcal{M}(I) \to \mathcal{M}(I)/_{=_H}$ the canonical projection. On $L^2(H)/_{=_H}$ an inner product is well-defined by

$$\big(\pi f, \pi g\big)_{L^2(H)/_{=_H}} := (f, g)_{L^2(H)}, \; f, g \in L^2(H) \,.$$

We will use the following notational convention: The space $L^2(H)/_{=_H}$ of equivalence classes will again be denoted by $L^2(H)$, and its inner product again by $(.,.)_{L^2(H)}$. In general, this abuse of language will not cause any confusion. However, if the distinction between single functions and equivalence classes of functions is essential, which will indeed be the case in several of our later discussions, we will write $f \in L^2(H) \subseteq \mathcal{M}(I)$ or $f \in L^2(H) \subseteq \mathcal{M}(I)/_{=_H}$. When $f \in \mathcal{M}(I)/_{=_H}$ we will also use the notation Hf meaning the function $H\hat{f}$ where $\hat{f}$ is any representant of f.

It is fundamental that $L^2(H) \subseteq \mathcal{M}(I)/_{=_H}$ is a Hilbert space, cf. [K1], [K2]. It is finite-dimensional if and only if I consists of finitely many maximal indivisible intervals. In this case its dimension coincides with the number of those maximal indivisible intervals which are of finite length.

The space $L^2(H)$ carries a conjugate linear and anti-isometric involution. In fact, the complex conjugation

$$\bar{.} : f(t) \mapsto \overline{f(t)},$$

is a conjugate linear involution on $\mathcal{M}(I)$, and induces a conjugate linear and anti-isometric involution on $L^2(H)$.

c. The relation $T_{max}(H)$

Let J be the matrix

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

We define a linear relation $T_{max}(H)$ on $L^2(H) \subseteq \mathcal{M}(I)$ as the set of all pairs $(f;g) \in L^2(H) \times L^2(H)$ where f is locally absolutely continuous and

$$f' = JHg, \quad \text{a.e. on } I .$$

Let us note at this point that, if $f : I \to \mathbb{C}^2$ is locally absolutely continuous and $f'(t) \in \operatorname{ran} JH(t)$ a.e., then f automatically satisfies (C), cf. [HSW, Lemma 3.3].

The linear relation $T_{max}(H)$ is also projected to $L^2(H)/_{=_H}$:

$$(\pi \times \pi)T_{max}(H) = \big\{(\pi f; \pi g) : (f;g) \in T_{max}(H)\big\} \subseteq (L^2(H)/_{=_H})^2 .$$

Following our general abuse of language, the projected relation will again be denoted by $T_{max}(H)$.

In many places the following statement, given in [HSW] as Lemma 3.5, is of importance:

2.2 Remark. Assume that (s_-, s_+) is not indivisible. Then for each pair $(f;g) \in T_{max}(H) \subseteq (\mathcal{M}(I)/_{=_H})^2$ there exists a unique representant $\hat{f} \in \mathrm{AC}(I)$ of the equivalence class $f \in \mathcal{M}(I)/_{=_H}$ which satisfies $\hat{f}' = JHg$.

d. Boundary values

If H is regular at the endpoint s_- and $(f;g) \in T_{max}(H) \subseteq \mathcal{M}(I)^2$, then f has a continuous extension to this endpoint, in fact $f \in \mathrm{AC}([s_-, s_+))$, where $\mathrm{AC}([s_-, s_+))$ denotes the set of all locally absolutely continuous functions on $[s_-, s_+)$ with values in $\mathbb{C}^2$ which satisfy the condition (C). Clearly, we can consider $\mathrm{AC}([s_-, s_+))$ as a linear subspace of $\mathcal{M}(I)$. The analogous statement holds for the endpoint s_+.

This fact allows us to define boundary values $\Gamma(H) \subseteq T_{max}(H) \times (\mathbb{C}^2 \times \mathbb{C}^2)$ where $T_{max}(H)$ is understood as a subspace of $(\mathcal{M}(I)/_{=_H})^2$: A pair

$((f;g);(a;b))$ belongs to $\Gamma(H)$ by definition if and only if there exists a representant $\hat{f} \in \mathrm{AC}(I)$, $\hat{f}/_{=_H} = f$, with $\hat{f}' = JHg$, such that

$$a = \begin{cases} \hat{f}(s_-) & , H \text{ regular at } s_- \\ 0 & , H \text{ singular at } s_- \end{cases} , \quad b = \begin{cases} \hat{f}(s_+) & , H \text{ regular at } s_+ \\ 0 & , H \text{ singular at } s_+ \end{cases}$$

2.3 Remark. Assume that (s_-, s_+) is not indivisible. Then by Remark 2.2 the representant $\hat{f}$ is unique. Hence in this case $\operatorname{mul}\Gamma(H) = \{0\}$. Thereby $\operatorname{mul}\Gamma(H)$ denotes the multivalued part of the relation $\Gamma(H)$, that is

$$\operatorname{mul}\Gamma(H) := \big\{(a;b) \in \mathbb{C} \times \mathbb{C} : \big(0;(a;b)\big) \in \Gamma(H)\big\},$$

cf. [DS2].

e. The case that (s_-, s_+) is indivisible

This case will often play an exceptional, though mostly trivial, role. It will usually be given an explicit treatment based on the following considerations.

If (s_-, s_+) is indivisible, then for some $\phi \in [0, \pi)$ and an appropriate scalar function h we have

$$H(t) = h(t)\xi_\phi \xi_\phi^T, \ t \in (s_-, s_+) \text{ a.e.}$$

Then $f \in \mathcal{M}(I)$, if and only if f is measureable and $\xi_\phi^T f$ is constant on I. If $f \in \mathcal{M}(I)$, we have

$$Hf = h\xi_\phi \xi_\phi^T f = (\xi_\phi^T f)h \cdot \xi_\phi .$$

We see that $f =_H g$ if and only if $\xi_\phi^T f = \xi_\phi^T g$, and thus we can view $\mathcal{M}(I)/_{=_H}$ as a linear subspace of $\mathbb{C}$.

Assume first that H is regular at both endpoints, i.e. that $\int_{s_-}^{s_+} h < \infty$. Then, as a set of functions $L^2(H) = \mathcal{M}(I)$, and thus, if $L^2(H)$ is considered as set of equivalence classes, $\dim L^2(H) = 1$. The relation $T_{max}(H) \subseteq \mathrm{AC}(I) \times \mathcal{M}(I)$ is given as

$$T_{max}(H) = \Big\{(f;g) \in \mathrm{AC}(I) \times \mathcal{M}(I) : \exists\, a \in \mathbb{C}^2 : f(x) = \xi_\phi^T g \cdot (\int_{s_-}^x h) \cdot J\xi_\phi + a\Big\}.$$

It follows that $T_{max}(H)$, as a set of equivalence classes, is equal to $L^2(H) \times L^2(H)$. In order to compute the relation $\Gamma(H)$, note that for $(f;g) \in T_{max}(H) \subseteq \mathrm{AC}(I) \times \mathcal{M}(I)$ we have $f/_{=_H} = f(s_-)/_{=_H}$. Hence, if $(f;g) \in T_{max}(H) \subseteq \mathrm{AC}(I)/_{=_H} \times \mathcal{M}(I)/_{=_H}$ is given, the set of all possible locally absolutely continuous representants $\hat{f}$ of f with $\hat{f}' = JHg$ is equal to

$$\Big\{(\xi_\phi^T f) \cdot \xi_\phi + (\xi_\phi^T g)(\int_{s_-}^x h) \cdot J\xi_\phi + \gamma J\xi_\phi : \gamma \in \mathbb{C}\Big\}. \tag{2.1}$$

It follows that

$$\Gamma(H)(f;g) = \Big\{\big((\xi_\phi^T f)\xi_\phi + \gamma J\xi_\phi; (\xi_\phi^T f)\xi_\phi + \big[\gamma + (\xi_\phi^T g)\int_{s_-}^{s_+} h\big] J\xi_\phi\big) : \gamma \in \mathbb{C}\Big\}. \tag{2.2}$$

We see that $\operatorname{mul}\Gamma(H) = \operatorname{span}\{(J\xi_\phi; J\xi_\phi)\}$.

Assume now that H is regular at s_- and singular at s_+, i.e. $\int_{s_-}^t h < \infty$ for $t \in (s_-, s_+)$ but $\int_{s_-}^{s_+} h = \infty$. Then as a set of functions $L^2(H) = \{f \in \mathcal{M}(I) : \xi_\phi^T f = 0\}$, and hence as a set of equivalence classes $L^2(H) = \{0\}$. Moreover, we have

$$T_{max}(H) = \left\{(\gamma J\xi_\phi; g) \in \mathrm{AC}(I) \times \mathcal{M}(I) : \gamma \in \mathbb{C}, \xi_\phi^T g = 0\right\},$$

and hence $\operatorname{mul}\Gamma(H) = \operatorname{span}\{(J\xi_\phi; 0)\}$.

f. Reparameterization

It is important to identify Hamiltonians which arise from each other by reparameterization.

2.4 Lemma. *Let H be a Hamiltonian defined on the interval (s_-, s_+). Let $(\tilde{s}_-, \tilde{s}_+)$ be another interval in $\mathbb{R}$ and let $\varphi : (\tilde{s}_-, \tilde{s}_+) \to (s_-, s_+)$ be an absolutely continuous and increasing bijection such that also φ^{-1} is absolutely continuous. Define*

$$\tilde{H}(t) := H(\varphi(t)) \cdot \varphi'(t),\ t \in (\tilde{s}_-, \tilde{s}_+)\,.$$

Then $\tilde{H}$ is a Hamiltonian on $(\tilde{s}_-, \tilde{s}_+)$. It is regular or singular at $\tilde{s}_-$ (or $\tilde{s}_+$) if and only if H is regular or singular at s_- (or s_+, respectively). The map

$$C_\varphi : f \mapsto f \circ \varphi$$

induces an isometric isomorphism of $L^2(H)$ onto $L^2(\tilde{H})$ which is compatible with conjugation, i.e.

$$C_\varphi \overline{f} = \overline{C_\varphi f},\ f \in L^2(H)\,.$$

We have

$$\begin{aligned} (C_\varphi \times C_\varphi) T_{max}(H) &= T_{max}(\tilde{H}) \\ \Gamma(\tilde{H}) \circ (C_\varphi \times C_\varphi) &= \Gamma(H)\,, \end{aligned} \tag{2.3}$$

and

$$\int_{s_1}^{s_2} \operatorname{tr} H(s)\, ds = \int_{\varphi^{-1}(s_1)}^{\varphi^{-1}(s_2)} \operatorname{tr} \tilde{H}(t)\, dt\ . \tag{2.4}$$

Proof. The fact that $\tilde{H}$ is a Hamiltonian is obvious. We show that C_φ is an isometry from $L^2(H)$ onto $L^2(\tilde{H})$:

$$\int_{s_-}^{s_+} f(s)^* H(s) g(s)\, ds = \int_{\varphi^{-1}(s_-)}^{\varphi^{-1}(s_+)} f(\varphi(t))^* H(\varphi(t)) g(\varphi(t)) \varphi'(t)\, dt =$$

$$= \int_{\varphi^{-1}(s_-)}^{\varphi^{-1}(s_+)} (f \circ \varphi)(t)^* \tilde{H}(t) (g \circ \varphi)(t)\, dt =$$

$$= \int_{\tilde{s}_-}^{\tilde{s}_+} (C_\varphi f)(t)^* \tilde{H}(t) (C_\varphi g)(t)\, dt\,.$$

Formula (2.4) can be verified in the same way. Let $(f; g) \in T_{max}(H)$, then

$$(C_\varphi f)' = (f' \circ \varphi) \cdot \varphi' = J(H \circ \varphi)(g \circ \varphi) \cdot \varphi' = J\tilde{H}(C_\varphi g),\ \text{a.e.}$$

Since the same argument can be applied with φ^{-1}, the first relation in (2.3) follows. The second relation is clear since $(C_\varphi f)(\tilde{s}_\pm) = f(s_\pm)$.

❑

By means of the above lemma an equivalence relation "equal up to reparameterization" is defined on the set of all Hamiltonians: We say that H and $\tilde{H}$ are equivalent up to reparametrization, if and only if there exists an increasing bijection φ between the respective domains $(\tilde{s}_-, \tilde{s}_+)$ and (s_-, s_+) of $\tilde{H}$ and H, such that φ and φ^{-1} are absolutely continuous, and such that $\tilde{H}(t) = H(\varphi(t)) \cdot \varphi'(t)$.

Particular reparameterizations of a Hamiltonian are obtained by using the primitive of $\operatorname{tr} H$. If we put $\varphi = \mathfrak{t}^{-1}$ in Lemma 2.4, then, by (2.4), we have $\operatorname{tr} \tilde{H} = 1$ a.e.

2.5 Remark. A Hamiltonian with the property that $\operatorname{tr} H = 1$ a.e., is called trace normed. We see from the above consideration that every equivalence class of Hamiltonians modulo reparameterization contains trace normed Hamiltonians. For this reason it is common to restrict to the case that H is trace normed. However, in our context it is necessary to work with the general notion of a Hamiltonian.

A statement similar to Lemma 2.4 holds true if the bijection φ is decreasing instead of increasing. We shall, in this case, speak of an order-reversing reparameterization. However, an essential difference is that in this case the relations $T_{max}(H)$ and $T_{max}(\tilde{H})$ are not anymore unitarily equivalent.

2.6 Lemma. *Let H be a Hamiltonian defined on the interval (s_-, s_+). Let $(\tilde{s}_-, \tilde{s}_+)$ be another interval in $\mathbb{R}$ and let $\varphi : (\tilde{s}_-, \tilde{s}_+) \to (s_-, s_+)$ be an absolutely continuous and decreasing bijection such that also φ^{-1} is absolutely continuous. Define*

$$\tilde{H}(t) := -H(\varphi(t)) \cdot \varphi'(t), \; t \in (\tilde{s}_-, \tilde{s}_+) \, .$$

Then $\tilde{H}$ is a Hamiltonian on $(\tilde{s}_-, \tilde{s}_+)$. It is regular or singular at $\tilde{s}_-$ (or $\tilde{s}_+$) if and only if H is regular or singular at s_+ (or s_-, respectively). The map C_φ induces an isometric isomorphism of $L^2(H)$ onto $L^2(\tilde{H})$. We have

$$(C_\varphi \times C_\varphi) T_{max}(H) = -T_{max}(\tilde{H}) \, ,$$

$$\Gamma(\tilde{H}) \circ ((-C_\varphi) \times C_\varphi) = \psi \circ \Gamma(H) \, ,$$

where

$$\psi : \left\{ \begin{array}{ccc} \mathbb{C}^2 \times \mathbb{C}^2 & \to & \mathbb{C}^2 \times \mathbb{C}^2 \\ (a; b) & \mapsto & (b; a) \end{array} \right.$$

Moreover,

$$\int_{s_1}^{s_2} \operatorname{tr} H(s) \, ds = \int_{\varphi^{-1}(s_2)}^{\varphi^{-1}(s_1)} \operatorname{tr} \tilde{H}(t) \, dt \; .$$

Proof. The proof of this assertion is completely similar to the one of Lemma 2.4 and will therefore not be carried out explicitly.

❑

g. Restriction

We will frequently encounter the situation that a Hamiltonian H is restricted to a smaller interval. If H is a Hamiltonian on $I = (s_-, s_+)$ and $L = (\sigma_-, \sigma_+) \subseteq I$, then $H|_L$ is a Hamiltonian on L. For the moment let us just note the following statements which are immediate from the definition:

(*i*) $H|_L$ is singular at σ_+ or σ_- if and only if $\sigma_+ = s_+$ ($\sigma_- = s_-$) and H is singular at the respective endpoint.

(*ii*) If $\sigma_- \in [\alpha_l^-(H), \alpha_{l+1}^-(H))$, then $\alpha_n^-(H|_L) = \alpha_{n+l}^-(H)$. A similar statement holds for σ_+.

(*iii*) If $f \in L^2(H)$, then $f|_L \in L^2(H|_L)$.

2.2 On the structure of the relation T_{max}

First we wish to recall some classical and fundamental results on the operator theory of T_{max}. We will thereby include the case that (s_-, s_+) is indivisible. In order to formulate these results in a concise way, we need to give a properly adapted definition of boundary triplets and some of their spectral properties.

a. Boundary triplets of defect 2 **and** 1

The following notion turns out to be useful.

2.7 Definition. A triple $(\mathcal{P}, T, \Gamma)$ is called a boundary triplet, if $\mathcal{P}$ is a Pontryagin space which carries a conjugate linear and anti-isometric involution $\bar{.} : \mathcal{P} \to \mathcal{P}$, if $T \subseteq \mathcal{P} \times \mathcal{P}$ is a closed linear relation which is real, i.e.

$$(f; g) \in T \iff (\overline{f}; \overline{g}) \in T \,,$$

if $\Gamma \subseteq T \times (\mathbb{C}^2 \times \mathbb{C}^2)$ is a closed linear relation with $\operatorname{dom}\Gamma = T$ which is compatible with the involution $\bar{.}$ in the sense that

$$\big((f; g); (a; b)\big) \in \Gamma \iff \big((\overline{f}; \overline{g}); (\overline{a}; \overline{b})\big) \in \Gamma \,, \tag{2.5}$$

and if the following conditions are satisfied:

(*i*) The abstract Green's identity holds:

$$\begin{aligned} [g, h] - [f, k] = & \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \\ & \qquad ((f; g); (x_1; x_2)), ((h; k); (y_1; y_2)) \in \Gamma \,. \end{aligned} \tag{2.6}$$

(*ii*) $\ker\Gamma = T^*$.

There is a vast literature on the notion of boundary triplets, for more details see e.g. [DHMS] or [D]. However, it is not the purpose of this paper to further develop the theory of boundary triplets, and hence we content ourselves with what is needed in the later sections of this paper.

We also give an adopted definition of the defect of a boundary triplet. The need for this will become clear in the subsequently stated Theorem 2.18 and Theorem 2.19, as well as in the later Section 6, cf. Remark 6.10.

2.8 Definition. Let $(\mathcal{P}, T, \Gamma)$ be a boundary triplet. We say that $(\mathcal{P}, T, \Gamma)$ has defect 2, if the following condition holds:

(Def 2) If $\operatorname{mul}\Gamma = \{0\}$, then $\dim T/T^* = 4$. If $\operatorname{mul}\Gamma \neq \{0\}$, then $\dim T/T^* = 2$ and $\operatorname{mul}\Gamma$ is of the form $\operatorname{span}\{(m; m)\}$ for some $m \in \mathbb{C}^2$.

We speak of a boundary triplet of defect 1, if the following condition holds:

(Def 1) We have $\operatorname{ran}\Gamma \subseteq \mathbb{C}^2 \times \{0\}$. If $\operatorname{mul}\Gamma = \{0\}$, then $\dim T/T^* = 2$. If $\operatorname{mul}\Gamma \neq \{0\}$, then $\dim T/T^* = 0$ and $\operatorname{mul}\Gamma$ is of the form $\operatorname{span}\{(m;0)\}$ for some $m \in \mathbb{C}^2$.

2.9 Remark. Let $(\mathcal{P}, T, \Gamma)$ be a boundary triplet of defect 2 (or 1).

(*i*) Write $\operatorname{mul}\Gamma = \operatorname{span}\{(m;m)\}$ (or $\operatorname{mul}\Gamma = \operatorname{span}\{(m;0)\}$, respectively). Then the element m can be choosen real, since T is invariant with respect to $\bar{.}$.

(*ii*) Assume $\operatorname{mul}\Gamma = \{0\}$. Then $\operatorname{ran}\Gamma = \mathbb{C}^2 \times \mathbb{C}^2$ or $\operatorname{ran}\Gamma = \mathbb{C}^2 \times \{0\}$, respectively. This follows since in this case

$$T/T^* = T/\ker\Gamma \cong \operatorname{ran}\Gamma\,.$$

2.10 Remark. One important feature of boundary value maps is that they allow to describe the selfadjoint extensions of the symmetry $S := T^*$. Let $(\mathcal{P}, T, \Gamma)$ be a boundary triplet of defect 2 and assume that $\operatorname{mul}\Gamma = \{0\}$. Then for any linear relation A with $S \subseteq A \subseteq T$ we have

$$A^* = \Gamma^{-1}\big(\Gamma(A)^{\perp_{\mathfrak{J}}}\big)$$

where $\perp_{\mathfrak{J}}$ refers to the inner product on $\mathbb{C}^2 \times \mathbb{C}^2$ defined by the Gram-matrix

$$\mathfrak{J} := \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}.$$

In particular, A is symmetric if and only if $\Gamma(A)$ is neutral, and A is selfadjoint if and only if $\Gamma(A)$ is hypermaximal neutral. Therefore examples of selfadjoint extensions of S are obtained by

$$A(\phi_-;\phi_+) := \big\{(f;g) \in T : (\xi_{\phi_-}^T, \xi_{\phi_+}^T)\Gamma(f;g) = 0\big\},\ \phi_-, \phi_+ \in [0,\pi)\,.$$

If we deal with a boundary triplet of defect 1, $\operatorname{mul}\Gamma = \{0\}$, the situation is similar. Only we have to cancel the second component of Γ which is anyway always zero and use the inner product $(J.,.)$ on $\mathbb{C}^2$:

$$A^* = \Gamma^{-1}\big(\{(a;0) \in \mathbb{C}^2 \times \mathbb{C}^2 : a \perp_J \Gamma(x)_1, x \in A\}\big)\,,$$

where $\Gamma(x)_1$ denotes the first component of $\Gamma(x) \in \mathbb{C}^2 \times \mathbb{C}^2$, i.e. $\Gamma(x) = (\Gamma(x)_1; 0)$. Selfadjoint extensions of $S = T^*$ are given by

$$A(\phi) := \big\{(f;g) \in T : \xi_\phi^T \Gamma(f;g)_1 = 0\big\},\ \phi \in [0,\pi)\,. \tag{2.7}$$

In this case actually all selfadjoint extensions are obtained in this way.

2.11 Remark. Let us provide an example. The situation $\operatorname{mul}\Gamma \neq \{0\}$ will appear in our context from blowing up an ordinary boundary map. Assume that T is a closed linear relation $\dim T/T^* = 2$ and let $\Lambda : T \to \mathbb{C}^2$ be surjective and satisfy the Green's identity

$$[g,h] - [f,k] = \Lambda(h;k)^* J \Lambda(f;g),\ (f;g),(h;k) \in T\,. \tag{2.8}$$

A linear relation Γ is now defined as (if $v \in \mathbb{C}^2$, write $v = (v_1, v_2)^T$)

$$\begin{aligned}\Gamma := \big\{((f;g);(x_1;x_2)) \in T \times (\mathbb{C}^2 \times \mathbb{C}^2) :& \\ x_{1,1} - x_{2,1} = \Lambda(f;g)_1, x_{1,2} = x_{2,2} = \Lambda(f;g)_2\big\}&\,.\end{aligned} \tag{2.9}$$

Then $(\mathcal{P}, T, \Gamma)$ is a boundary triplet of defect 2, and

$$\operatorname{mul}\Gamma = \operatorname{span}\Big\{\big(\begin{pmatrix}1\\0\end{pmatrix};\begin{pmatrix}1\\0\end{pmatrix}\big)\Big\}\,.$$

In fact, let $((f;g);(x_1;x_2)),((h;k);(y_1;y_2)) \in \Gamma$ and write $x_1 = (x_{1,1}, x_{1,2})^T$ etc., then

$$\begin{pmatrix}y_1\\y_2\end{pmatrix}^* \mathfrak{J}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \big(x_{1,1}\overline{y_{1,2}} - x_{1,2}\overline{y_{1,1}}\big) - \big(x_{2,1}\overline{y_{2,2}} - x_{2,2}\overline{y_{2,1}}\big) =$$

$$\underbrace{(x_{1,1} - x_{2,1})}_{\Lambda(f;g)_1}\overline{\Lambda(h;k)_2} - \underbrace{(\overline{y_{1,1}} - \overline{y_{2,1}})}_{\overline{\Lambda(h;k)_1}}\Lambda(f;g)_2 = \Lambda(h;k)^* J \Lambda(f;g)\,.$$

2.12 Definition. Let $(\mathcal{P}, T, \Gamma)$ and $(\tilde{\mathcal{P}}, \tilde{T}, \tilde{\Gamma})$ be boundary triplets. A pair (ϖ, ϕ) is called an isomorphism of $(\mathcal{P}, T, \Gamma)$ to $(\tilde{\mathcal{P}}, \tilde{T}, \tilde{\Gamma})$ if:

- (i) ϖ is an isometric isomorphism of $\mathcal{P}$ onto $\tilde{\mathcal{P}}$ which is compatible with the respective involutions in the sense that $\varpi(\overline{x}) = \overline{\varpi(x)}$, $x \in \mathcal{P}$. ϕ is an isometric isomorphism of $(\mathbb{C}^2 \times \mathbb{C}^2, (\mathfrak{J}.,.))$ onto itself.
- (ii) $(\varpi \times \varpi)(T) = \tilde{T}$.
- (iii) $\tilde{\Gamma} \circ (\varpi \times \varpi)|_T = \phi \circ \Gamma$.

Examples of isomorphisms between boundary triplets already appeared in Lemma 2.4 when we studied reparameterizations. In this situation, and with the notation of Lemma 2.4, the pair (C_φ, id) is an isomorphism of $(L^2(H), T_{max}(H), \Gamma(H))$ and $(L^2(\tilde{H}), T_{max}(\tilde{H}), \Gamma(\tilde{H}))$.

2.13 Remark.

- (i) The condition (ii) of Definition 2.12 is equivalent to $\varpi \circ T = \tilde{T} \circ \varpi$.
- (ii) If (ϖ_1, ϕ_1) is an isomorphism of $(\mathcal{P}_1, T_1, \Gamma_1)$ to $(\mathcal{P}_2, T_2, \Gamma_2)$, and (ϖ_2, ϕ_2) is an isomorphism of $(\mathcal{P}_2, T_2, \Gamma_2)$ to $(\mathcal{P}_3, T_3, \Gamma_3)$, then $(\varpi_2 \circ \varpi_1, \phi_2 \circ \phi_1)$ is an isomorphism of $(\mathcal{P}_1, T_1, \Gamma_1)$ to $(\mathcal{P}_3, T_3, \Gamma_3)$.
- (iii) If ϕ is of the special form $\phi = \hat{\phi} \times \hat{\phi}$ with an isometric isomorphism $\hat{\phi}$ of $(\mathbb{C}^2, (J.,.))$ satisfying $\hat{\phi}(\overline{x}) = \overline{\hat{\phi}(x)}$, the property (Def 2) as well as (Def 1) is inherited.

2.14 Remark. Using isomorphisms is also a way of constructing boundary triplets: Assume that $(\mathcal{P}, T, \Gamma)$ is a boundary triplet and that $\tilde{\mathcal{P}}$ is another Pontryagin space which carries a conjugate linear and anti-isometric involution. Moreover, let ϖ be an isometric isomorphism of $\mathcal{P}$ onto $\tilde{\mathcal{P}}$ which is compatible with the respective involutions, and that ϕ is an isometric isomorphism of

$(\mathbb{C}^2 \times \mathbb{C}^2, (\mathfrak{J}.,.))$ onto itself. Define $\tilde{T} := (\varpi \times \varpi)(T)$ and $\tilde{\Gamma} := \varphi \circ \Gamma \circ (\varpi \times \varpi)|_T^{-1}$. Then, as a straightforward argument shows, $(\tilde{\mathcal{P}}, \tilde{T}, \tilde{\Gamma})$ is a boundary triplet and (ϖ, ϕ) is an isomorphism of $(\mathcal{P}, T, \Gamma)$ to $(\tilde{\mathcal{P}}, \tilde{T}, \tilde{\Gamma})$.

If ϕ is again of the special form $\phi = \hat{\phi} \times \hat{\phi}$ as in Remark 2.13, (*iii*), then $(\mathcal{P}, T, \Gamma)$ being of defect 1 or 2 implies $(\tilde{\mathcal{P}}, \tilde{T}, \tilde{\Gamma})$ having the respective property.

We will deal with two spectral properties of boundary triplets.

2.15 Definition. We say that a symmetric relation S has compact resolvents, if

(CR) Whenever $S \subseteq A \subseteq S^*$ and $z \in \rho(A)$, then $(A - z)^{-1}$ is a compact operator.

Note that by the resolvent identity it is equivalent to require that for every A, $S \subseteq A \subseteq S^*$, there exists $z \in \rho(A)$ such that $(A-z)^{-1}$ is compact. Moreover, if S has finite defect index, i.e. $\dim S^*/S < \infty$, then in order to ensure (CR) it is enough to find one extension A and one number $z \in \rho(A)$ such that $(A-z)^{-1}$ is compact.

2.16 Definition. For a boundary triplet $(\mathcal{P}, T, \Gamma)$ of defect 2 (or defect 1) we consider the following condition:

(E) If $z \in \mathbb{C}$, $(f; zf) \in T$, $f \neq 0$, and $((f; zf); (a; b)) \in \Gamma$, then $a \neq 0$ and $b \neq 0$ (in case of defect 1: $a \neq 0$, $b = 0$).

Note that this condition implies that

$$\ker(T^* - z) = \{0\}, \ z \in \mathbb{C}.$$

In case of defect 1 it is even equivalent to this property.

2.17 Remark. Assume that $(\mathcal{P}, T, \Gamma)$ and $(\tilde{\mathcal{P}}, \tilde{T}, \tilde{\Gamma})$ are boundary triplets and that (ϖ, ϕ) is an isomorphism betweem them. Then T^* satisfies (CR) if and only if $\tilde{T}^*$ does.

If ϕ is again of the form $\phi = \hat{\phi} \times \hat{\phi}$ as in Remark 2.13, (*iii*), then they also together do or do not satisfy the condition (E).

b. Operator theory of canonical systems

We assume that the reader is familiar with the basic notions of operator theory in general and the theory of symmetric and selfadjoint operators in particular, see e.g. [GGK], [AG].

The operator theory of the relation T_{max} is settled by the following classical results.

2.18 Theorem (see [GK]). *Let H be a Hamiltonian on the interval $I = (s_-, s_+)$ and assume that H is regular at both endpoints. Then $(L^2(H), T_{max}(H), \Gamma(H))$ is a boundary triplet of defect 2 which satisfies the condition (E). The adjoint $T_{min}(H) := T_{max}(H)^*$ is a completely nonselfadjoint symmetric operator. Its defect index is $(2, 2)$ unless (s_-, s_+) is indivisible, in which case it is $(1, 1)$. Moreover, the set $r(T_{min}(H))$ of regular points of $T_{min}(H)$ equals $\mathbb{C}$ and T_{min} possesses the property (CR).*

2.19 Theorem (see [HSW, K1]). *Let H be a Hamiltonian on the interval $I = (s_-, s_+)$ and assume that H is regular at s_- and singular at s_+. Then $(L^2(H), T_{max}(H), \Gamma(H))$ is a boundary triplet of defect 1 which satisfies the condition (E). The adjoint $T_{min}(H) = T_{max}(H)^*$ is a completely nonselfadjoint symmetric operator. Its defect index is $(1,1)$ unless (s_-, s_+) is indivisible, in which case it is selfadjoint.*

Clearly, a result similar to Theorem 2.19 holds true if H is singular at s_- and regular at s_+.

2.20 Remark. In Theorem 2.18 the abstract Green's identity (2.6) reads as

$$(g,h)_{L^2(H)} - (f,k)_{L^2(H)} = \hat{h}(s_-)^* J \hat{f}(s_-) - \hat{h}(s_+)^* J \hat{f}(s_+)\,,$$

whenever $(f;g),(h;k) \in T_{max}(H)$, $\hat{f}, \hat{h} \in \mathrm{AC}([s_-, s_+])$ with $\hat{f}/_{=_H} = f$, $\hat{h}/_{=_H} = h$, and $\hat{f}' = JHg$, $\hat{h}' = JHk$. In Theorem 2.19 it takes the form

$$(g,h)_{L^2(H)} - (f,k)_{L^2(H)} = \hat{h}(s_-)^* J \hat{f}(s_-)\,. \tag{2.10}$$

Let us explicitly state the following result which is a cornerstone in the proof of Theorem 2.19.

2.21 Theorem (see [HSW]). *Let H be a Hamiltonian on the interval $I = (s_-, s_+)$ and assume that H is regular at s_- and singular at s_+. Then for each two pairs $(f;g),(h;k) \in T_{max}$ and respective representants $\hat{f}, \hat{h}$ with $\hat{f}' = JHg$, $\hat{h}' = JHk$, we have*

$$\lim_{t \to s_+} \hat{h}(t)^* J \hat{f}(t) = 0\,.$$

Here we also included the case that (s_-, s_+) is indivisible. Then the above result is a consequence of following elementary observation: Assume that H is a Hamiltonian on (s_-, s_+), that H is singular at s_+ and that $\alpha_1^+ < s_+$. Then, for each two functions $f, h \in L^2(H) \subseteq \mathcal{M}(I)$ we have

$$h(t)^* J f(t) = 0,\ t \in (\alpha_1^+, s_+) \text{ a.e.} \tag{2.11}$$

To see this note that, if ϕ is the type of the indivisible interval (α_1^+, s_+), $\xi_\phi^T f(t) = \xi_\phi^T h(t) = 0$ on (α_1^+, s_+) a.e. Hence we have $f(t), h(t) \in \operatorname{span}\{J\xi_\phi\}$. The relation (2.11) follows from $\xi_\phi^T J \cdot J \cdot J \xi_\phi = 0$.

2.22 Remark. Let us note that for every $\Delta \in \mathbb{N}$,

$$\overline{\operatorname{dom} T_{max}^\Delta} = L^2(H)\,. \tag{2.12}$$

This follows e.g. since, by [HSW, Corollary 3.11], $\operatorname{dom} T_{max}$ is dense and thus T_{min} has densely defined selfadjoint extensions.

c. The multivalued part of T_{max}^n

The multivalued part of powers of T_{max} can be determined explicitly, a result which supplements [HSW, Proposition 3.10]. It will be deduced with help of the next lemma. Before that let us note that, copying the proof of [HSW, Lemma 3.8] word by word, we obtain the following statement: Let f be locally absolutely continuous on I, $f'(t) \in \operatorname{ran} JH(t)$ a.e., and assume that $f =_H 0$. Then $f(s) \neq 0$ implies that s is contained in an H-indivisible interval.

The crucial step in the proof of the following lemma is taken from [HSW, Proposition 3.10].

2.23 Lemma. *Let H be an arbitrary Hamiltonian on the interval (s_-, s_+), let $f_0, \dots, f_{n-1} \in \mathrm{AC}(I)$, $f_n \in \mathcal{M}(I)$, and assume that*

$$f_0 =_H 0,\ f_k' = JHf_{k+1},\ k = 0, \dots, n-1\,,$$

almost everywhere on I. Then

$$\begin{aligned} f_k(t) = 0,\ t \in [\alpha_{k+1}^-, \alpha_{k+1}^+] \cap (s_-, s_+),\ k = 0, \dots, n-1, \\ (Hf_n)(t) = 0,\ t \in (\alpha_n^-, \alpha_n^+)\ \text{a.e.} \end{aligned} \tag{2.13}$$

If, additionally, $f_0, \dots, f_n \in L^2(H)$ and H is singular at s_+, then in the relations (2.13) *we can write s_+ instead of α_{k+1}^+, α_n^+. The analogous assertion holds true for the left endpoint.*

Proof. We use induction on n. Consider the case $n = 1$, so that $f_0 \in \mathrm{AC}(I)$, $f_1 \in \mathcal{M}(I)$ with $f_0 =_H 0$, $f_0' = JHf_1$. Assume that $f_0(s) \neq 0$ at a point $s \in (\alpha_1^-, \alpha_1^+)$. Then s is contained in an H-indivisible interval. Let (α, β) be the maximal H-indivisible interval containing s. Then $\alpha \neq s_-$ and $\beta \neq s_+$ since $s \in (\alpha_1^-, \alpha_1^+)$. For $t \in (\alpha, \beta)$ a.e. we have $H(t) = h(t)\xi_\phi\xi_\phi^T$ where ϕ is the type of (α, β) and $h(t)$ is an appropriate scalar function. Moreover, since α, β are not contained in an H-indivisible interval, we know that $f_0(\alpha) = f_0(\beta) = 0$. It follows that

$$0 = f_0(\beta) - f_0(\alpha) = \int_\alpha^\beta JH(t)f_1(t)\,dt = J\int_\alpha^\beta h(t)\xi_\phi\xi_\phi^T f_1(t)\,dt\,.$$

Since $f_1 \in \mathcal{M}(I)$, we have $\xi_\phi^T f_1(t) = c$, $t \in (\alpha, \beta)$ a.e., for some constant c. Thus

$$0 = c\underbrace{\int_\alpha^\beta h(t)\,dt}_{>0} \cdot J\xi_\phi\,,$$

and we conclude that $c = 0$. Hence $(Hf_1)(t) = 0$, $t \in (\alpha, \beta)$ a.e. and by continuity f_0 is constant on $[\alpha, \beta]$. Because of $f_0(\alpha) = 0$, we obtain $f_0(t) = 0$, $t \in [\alpha, \beta]$, and arrive at the contradiction $f_0(s) = 0$. We conclude that $f_0(s) = 0$, $s \in (\alpha_1^-, \alpha_1^+)$, and hence by continuity on $[\alpha_1^-, \alpha_1^+] \cap (s_-, s_+)$ in case $\alpha_1^- < \alpha_1^+$. In the case $\alpha_1^- = \alpha_1^+$, we get $f_0(\alpha_1^-) = 0$ from $f_0 =_H 0$, the continuity of f_0 and $f_0' \in \operatorname{ran} JH$. The fact that $(Hf_1)(t) = 0$, $t \in (\alpha_1^-, \alpha_1^+)$ a.e., follows immediately.

Assume that the assertion has been proved for all $n \leq n_0$, and let $f_0, \dots, f_{n_0+1}$ be given and be subject to the hypothesis of the lemma. The inductive hypothesis applied to the functions $f_0, \dots, f_{n_0}$ gives the desired relations (2.13) for $k = 0, \dots, n_0 - 1$. Moreover, it implies that $(Hf_{n_0})(t) = 0$, $t \in (\alpha_{n_0}^-, \alpha_{n_0}^+)$ a.e. An application of the inductive hypothesis to the functions $f_{n_0}|_{(\alpha_{n_0}^-, \alpha_{n_0}^+)}$, $f_{n_0+1}|_{(\alpha_{n_0}^-, \alpha_{n_0}^+)}$ and the Hamiltonian $H|_{(\alpha_{n_0}^-, \alpha_{n_0}^+)}$ yields the desired relations (2.13) for $k = n_0$ and $n_0 + 1$. Note here that in case $\alpha_{n_0}^- \geq \alpha_{n_0}^+$ there is nothing to prove.

In order to prove the additional assertion we investigate the above proof under the additional hypothesis that $f_0, \dots, f_n \in L^2(H)$ and H is singular at s_+. If $\alpha_1^+ = s_+$, then also $\alpha_k^+ = s_+$ and hence there is nothing to prove. Thus, assume moreover that $\alpha_1^+ < s_+$.

In the case $n=1$ we had concluded that $f_0(t)=0$, $t\in[\alpha_1^-,\alpha_1^+]\cap(s_-,s_+)$, and $(Hf_1)(t)=0$, $t\in(\alpha_1^-,\alpha_1^+)$ a.e. Since $f_1\in L^2(H)$ we have in any case $(Hf_1)(t)=0$, $t\in(\alpha_1^+,s_+)$ a.e., so that $(Hf_1)(t)=0$, $t\in(\alpha_1^-,s_+)$ a.e. Thus f_0 is constant on $[\alpha_1^-,s_+)\cap(s_-,s_+)$. If $\alpha_1^-=s_+$, there is nothing to prove, otherwise $[\alpha_1^-,\alpha_1^+]\cap(s_-,s_+)\neq\emptyset$ and on this set f_0 vanishes. It follows that f_0 vanishes on $[\alpha_1^-,s_+)\cap(s_-,s_+)$.

In the inductive step we proceed similar as above, only applying the inductive hypothesis with the Hamiltonian $H|_{(\alpha_{n_0}^-,s_+)}$ instead of $H|_{(\alpha_{n_0}^-,\alpha_{n_0}^+)}$.

The case that $f_0,\dots,f_n\in L^2(H)$ and that H is singular at s_- is treated in a similar way.

❑

2.24 Proposition. *Let H be a Hamiltonian on the interval (s_-,s_+) and let $\Delta\in\mathbb{N}$. Put*

$$\beta_\Delta^+ := \begin{cases}\alpha_\Delta^+ & , H \text{ regular at } s_+\\ s_+ & , H \text{ singular at } s_+\end{cases},\qquad \beta_\Delta^- := \begin{cases}\alpha_\Delta^- & , H \text{ regular at } s_-\\ s_- & , H \text{ singular at } s_-\end{cases}.$$

Then the multivalued part of T_{max}^Δ is given by

$$\operatorname{mul} T_{max}^\Delta = \left\{g\in L^2(H):\ (Hg)(t)=0, t\in(\beta_\Delta^-,\beta_\Delta^+)\ \text{a.e.}\right\}. \tag{2.14}$$

Proof. Assume that $f\in\operatorname{mul}T_{max}^\Delta$, so that there exist $f_1,\dots,f_{\Delta-1}\in L^2(H)\subseteq\mathcal{M}(I)/_{=_H}$ with

$$(0;f_1),(f_1;f_2),\dots,(f_{\Delta-1};f)\in T_{max}\subseteq(\mathcal{M}(I)/_{=_H})^2\,.$$

Let $\hat f_0,\dots,\hat f_{\Delta-1}\in\mathrm{AC}(I)$ be the unique representants of $0,f_1,\dots,f_{\Delta-1}\in\mathcal{M}(I)/_{=_H}$ such that (for arbitrary representants $\tilde f_{k+1}$ of f_{k+1})

$$\hat f_k' = JH\tilde f_{k+1},\ k=0,\dots,\Delta-1\,.$$

We see that we can apply Lemma 2.23 to the functions $\hat f_0,\dots,\hat f_{\Delta-1}\in L^2(H)\cap\mathrm{AC}(I)$ and $\tilde f_\Delta\in L^2(H)\subseteq\mathcal{M}(I)$ and obtain the inclusion '$\subseteq$' in (2.14).

We come to the proof of the converse inclusion. Assume that H is regular at s_- and let $g\in L^2(H)$ be given such that $(Hg)(t)=0$, $t\in(\alpha_\Delta^-,s_+)$ a.e. If $\alpha_\Delta^-=s_-$, we have $g=_H 0\in\operatorname{mul}T_{max}^\Delta$, hence assume that $\alpha_\Delta^->s_-$. Choose $\Delta_0\in\mathbb{N}$ minimal with $\alpha_{\Delta_0}^-=\alpha_\Delta^-$, so that we have

$$s_-=\alpha_0^-<\alpha_1^-<\dots<\alpha_{\Delta_0}^-=\dots=\alpha_\Delta^-\le\dots\le s_+\,.$$

We define functions g_k, $k=0,\dots,\Delta_0$, recursively by

$$g_0:=g,\ g_{k+1}(t):=\int_{\alpha_{\Delta_0-k}^-}^t JHg_k,\ k=0,\dots,\Delta_0-1\,.$$

Note that g_k, $k=1,\dots,\Delta_0$, belongs to $\mathrm{AC}(I)$. We use induction on k to prove that

$$(Hg_k)(t)=0,\ t\in(\alpha_{\Delta_0-k}^-,s_+)\ \text{a.e.},\ k=0,\dots,\Delta_0\,. \tag{2.15}$$

The case $k = 0$ is just our assumption on g. Assume that (2.15) holds for some k, $0 \leq k < \Delta_0$. Then g_{k+1} is constant on $[\alpha^-_{\Delta_0-k}, s_+)$. As $g_{k+1}(\alpha^-_{\Delta_0-k}) = 0$, the function g_{k+1} and, hence, also Hg_{k+1} vanishes on this interval. Since g_{k+1} satisfies (C), the function $\xi_\phi^T g_{k+1}$ is constant on $[\alpha^-_{\Delta_0-k-1}, \alpha^-_{\Delta_0-k}]$ and thus, again by $g_{k+1}(\alpha^-_{\Delta_0-k}) = 0$, equal to 0 on this interval. We see that (2.15) holds for $k+1$.

In particular, (2.15) shows that $g_{\Delta_0} =_H 0$. Since g_k, $k = 1, \ldots, \Delta_0$, belongs to $\mathrm{AC}(I)$, Hg_k vanishes in a neighbourghood of s_+, and H is regular at s_-, certainly $g_k \in L^2(H)$. Clearly also $(g_k; g_{k-1}) \in T_{max}$, and we conclude that $g \in \operatorname{mul} T_{max}^{\Delta_0} \subseteq \operatorname{mul} T_{max}^{\Delta}$.

The same construction can be done for the right endpoint, and thus the inclusion '$\supseteq$' in (2.14) follows.

❑

In some future statements we will, for a relation R, denote by $R(x)$ the set $\{y : (x; y) \in R\}$.

2.25 Corollary. *Let $g \in \operatorname{mul} T_{max}(H)$ and $(a; b) \in \Gamma(H)(0; g)$. Then*

$$a \in \begin{cases} \in \operatorname{span}\{J\xi_\phi\} & , \ \beta_1^-(H) > s_-, \phi \text{ type of } (s_-, \beta_1^-(H)) \\ = 0 & , \ \beta_1^-(H) = s_- \end{cases}$$

$$b \in \begin{cases} \in \operatorname{span}\{J\xi_\phi\} & , \ \beta_1^+(H) < s_+, \phi \text{ type of } (\beta_1^+(H), s_+) \\ = 0 & , \ \beta_1^+(H) = s_+ \end{cases}$$

Proof. This follows from the construction of $\operatorname{mul} T_{max}(H)$ in the proof of the above proposition and the considerations in Section 2.1.e in the case that (s_-, s_+) is indivisible.

❑

d. Formula on integration by parts

Next we give a formula which can be viewed as a supplement to the Green's identity (2.10).

2.26 Proposition. *Let H be a Hamiltonian on $I = (s_-, s_+)$ and assume that H is regular at s_-. Moreover, let $\Delta \in \mathbb{N}$. Let $f_0, \ldots, f_{\Delta-1}, g_0, \ldots, g_{\Delta-1} \in \mathrm{AC}([s_-, s_+))$ and let $f_\Delta, g_\Delta \in \mathcal{M}(I)$ be such that*

$$f_k' = JHf_{k+1}, \ g_k' = JHg_{k+1}, \ k = 0, \ldots, \Delta - 1 \,.$$

Assume that one of the following conditions is satisfied:

(i) $f_k, g_k \in L^2(H)$, $k = 0, \ldots, \Delta$, *and H is singular at s_+.*

(ii) $f_k, g_k \in L^2(H)$, $k = 0, \ldots, \Delta - 1$, *$H$ is singular at s_+ and $\alpha_1^+(H) < s_+$.*

(iii) There exists $\beta \in (s_-, s_+)$ such that $(Hf_0)(t) = 0$, $t \in (\beta, s_+)$ a.e., and $\alpha_\Delta^-(H|_{(\beta,s_+)}) < s_+$, $f_0, \ldots, f_\Delta \in L^2(H)$, H is singular at s_+.

(iv) There exists $\beta \in (s_-, s_+)$ such that $(Hf_0)(t) = 0$, $t \in (\beta, s_+)$ a.e., and $\alpha_1^+ = s_+$.

Then

$$\int_I g_\Delta^* H f_0 = \int_I g_0^* H f_\Delta - \sum_{j=0}^{\Delta-1} g_j(s_-)^* J f_{\Delta-1-j}(s_-)\,. \tag{2.16}$$

Proof. In the case (i) we have $(f_k; f_{k+1}), (g_k; g_{k+1}) \in T_{max}$, $k = 0, \dots, \Delta - 1$, and hence (2.16) follows by repeated application of (2.10).

Let $t \in (s_-, s_+)$ be fixed. Then repeated integration by parts gives

$$\int_{s_-}^t g_\Delta^* H f_0 = \int_{s_-}^t (JHg_\Delta)^* J f_0 =$$
$$= g_{\Delta-1}^* J f_0|_{s_-}^t - \int_{s_-}^t g_{\Delta-1}^* J(JHf_1) = g_{\Delta-1}^* J f_0|_{s_-}^t + \int_{s_-}^t g_{\Delta-1}^* H f_1 = \dots$$
$$\dots = g_{\Delta-1}^* J f_0|_{s_-}^t + \dots + g_0^* J f_{\Delta-1}|_{s_-}^t + \int_{s_-}^t g_0^* H f_\Delta . \tag{2.17}$$

Thus we have to show that under either of the assumptions $(ii) - (iv)$,

$$\lim_{t \to s_+} g_j^*(t) J f_{\Delta-1-j}(t) = 0,\ j = 0, \dots, \Delta - 1\,. \tag{2.18}$$

Under the assumption (ii) this follows from (2.11). Assume that $\beta \in (s_-, s_+)$ is such that $(Hf_0)(t) = 0$, $t \in (\beta, s_+)$ a.e. Apply Lemma 2.23 with the functions $f_k|_{(\beta,s_+)}$, $k = 0, \dots, \Delta$ and the Hamiltonian $H|_{(\beta,s_+)}$. Under either of the assumptions (iii) and (iv) we obtain $f_k(t) = 0$, $t \in [\alpha_\Delta^-(H|_{(\beta,s_+)}), s_+)$, $k = 0, \dots, \Delta - 1$. Since $\alpha_1^+ = s_+$ implies that $\alpha_\Delta^-(H|_{(\beta,s_+)}) < s_+$, we see that in any case f_k vanishes in a neighbourhood of s_+. Hence, the limit relation (2.18) holds true.

❑

2.3 Compactness property of canonical systems

a. The Hilbert-Schmidt condition

Let H be a Hamiltonian on $I = (s_-, s_+)$ which is regular at s_- and singular at s_+. The following condition on H will play a crucial role:

(HS) There exists a selfadjoint extension $A \subseteq L^2(H) \times L^2(H)$ of the symmetric relation T_{min} and $z \in \rho(A)$ such that $(A - z)^{-1}$ is a Hilbert-Schmidt operator.

Note that (HS) implies that for every extension $\tilde{A}$ of T_{min} in $L^2(H)$ and every $z \in \rho(\tilde{A})$, the resolvent $(\tilde{A} - z)^{-1}$ is a Hilbert-Schmidt operator. Moreover, it follows from Lemma 2.4 that two Hamiltonians H_1 and H_2 which are reparameterizations of each other together do or do not satisfy (HS). Finally, note that (HS) trivially holds if (s_-, s_+) is indivisible.

Let us recall that the validity of (HS) can be expressed explicitly in terms of H. For the sake of simplicity we assume that H is trace normed and defined on $I = (0, \infty)$. For general Hamiltonians one has to use an appropriate reparameterization to obtain the analogous statement.

2.27 Theorem (see [KW6]). *Let H be a trace normed Hamiltonian on $I = (0,\infty)$. Then H satisfies (HS) if and only if there exists $\phi = \phi(H) \in [0,\pi)$ such that*

$$\int_I \xi_\phi^T H \xi_\phi < \infty, \tag{2.19}$$

and

$$\int_I \xi_{\phi+\frac{\pi}{2}}^T M \xi_{\phi+\frac{\pi}{2}} \xi_\phi^T H \xi_\phi < +\infty\,, \tag{2.20}$$

where

$$M(x) := \int_{s_-}^x H\,.$$

Note that for any Hamiltonian H which is singular at one endpoint there exists at most one number $\phi(H)$ such that (2.19) holds. We will often restrict ourselves 'without loss of generality' to the case that $\phi(H) = 0$ in the preceeding theorem. This means that we will often assume that, besides (HS), also the condition

(I) The constant function $\binom{1}{0}$ belongs to $L^2(H)$

is satisfied.

Note that if H satisfies (I), then every constant which is linearly independent from $\binom{1}{0}$ does not belong to $L^2(H)$.

2.28 Remark. Let us explain in more detail what it means to restrict 'without loss of generality' to the case that $\phi(H) = 0$. If we put

$$N_\alpha := \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \tag{2.21}$$

then N_α is unitary and J-unitary, i.e. satisfies

$$N_\alpha^{-1} = N_\alpha^*,\ N_\alpha J N_\alpha^* = J\,.$$

Moreover, $N_\alpha^* = N_{-\alpha}$.

If H is any Hamiltonian given on an interval (s_-, s_+) and $\alpha \in [0, 2\pi)$, then $\hat{H} := N_\alpha H N_\alpha^*$ is again a Hamiltonian on (s_-, s_+). In fact, if

$$\varpi : f \mapsto N_\alpha f,\ \hat{\phi} := N_\alpha\,,$$

and $\phi := \hat{\phi} \times \hat{\phi}$, then (ϖ, ϕ) is an isomorphism of the boundary triplets $(L^2(H), T_{max}(H), \Gamma(H))$ and $(L^2(\hat{H}), T_{max}(\hat{H}), \Gamma(\hat{H}))$. Clearly, $\hat{H}$ is regular or singular at an endpoint if and only if H has the respective property and H and $\hat{H}$ together do or do not satisfy (HS). A straightforward argument shows that, in the case (HS) holds,

$$\phi(\hat{H}) = \phi(H) - \alpha\,.$$

2.29 Remark. If H satisifes (HS), then, in particular, all the resolvents mentioned in (HS) are compact operators. The compactness of these resolvents is equivalent to the fact that $\sigma(A)$ is a discrete subset of $\mathbb{R}$ for any selfadjoint extension A of T_{min} in $L^2(H)$. In this case any real number belongs to the spectrum of exactly one selfadjoint extension of T_{min} in $L^2(H)$.

b. The operator B

Assume that a Hamiltonian H, such that (s_-, s_+) is not indivisible, satisifes (HS) and the normalization condition (I), and consider the selfadjoint extension

$$A(0) = \{(f;g) \in T_{max} : (1,0)\Gamma_1(f;g) = 0\}$$

of T_{min}, cf. (2.7). Then $0 \in \rho(A(0))$ and

$$B := A(0)^{-1} \subseteq L^2(H)/_{=_H} \times L^2(H)/_{=_H} \tag{2.22}$$

is a Hilbert-Schmidt operator.

Since the boundary condition defining $A(0)$ is real, clearly B is real with respect to the involution $\bar{.}$, i.e. $B(\overline{f}) = \overline{B(f)}$.

We shall relate $\operatorname{dom} T_{max}^n$ with $\operatorname{ran} B^n$. To this end let us mention the following elementary result.

2.30 Lemma. *Let $\mathcal{H}$ be a linear space and let A, T be linear subspaces of $\mathcal{H}^2$. Assume that there exists a sequence $a_0, a_1, a_2, \ldots$ such that*

$$a_0 \in \ker T,\ (a_n; a_{n-1}) \in A,\ n = 1, 2, 3, \ldots, \tag{2.23}$$

$$T = A + \operatorname{span}\big\{(a_0; 0)\big\}. \tag{2.24}$$

Then for all $\Delta \in \mathbb{N}$ we have

$$\operatorname{dom} T^{\Delta} = \operatorname{dom} A^{\Delta} + \operatorname{span}\big\{a_0, \ldots, a_{\Delta-1}\big\}. \tag{2.25}$$

Proof. We use induction on Δ. Consider the case $\Delta = 1$. By (2.24) we have $\operatorname{dom} T = \operatorname{dom} A + \operatorname{span}\{a_0\}$. Assume next that the relation (2.25) holds true for some $\Delta \in \mathbb{N}$ and let $x \in \operatorname{dom} T^{\Delta+1}$. Then there exist $y, z \in \mathcal{H}$ such that $(x; y) \in T$, $(y; z) \in T^{\Delta}$. By the inductive hypothesis there exist $\lambda_i \in \mathbb{C}$, $i = 0, \ldots \Delta - 1$, $y_1 \in \operatorname{dom} A^{\Delta}$, and by (2.24) there exists $\lambda \in \mathbb{C}$, $(x_1; y) \in A$, such that

$$y = y_1 + \sum_{i=0}^{\Delta-1} \lambda_i a_i,\ (x; y) = (x_1; y) + \lambda(a_0; 0)\,.$$

Let $z_1 \in \mathcal{H}$ be such that $(y_1; z_1) \in A^{\Delta}$. From (2.23) we conclude that

$$(x_1; y) - \sum_{i=0}^{\Delta-1} \lambda_i (a_{i+1}; a_i) \in A.$$

However, this expression can be rewritten as

$$\big(x - \lambda a_0 - \sum_{i=0}^{\Delta-1} \lambda_i a_{i+1}; y - \sum_{i=0}^{\Delta-1} \lambda_i a_i\big) = \big(x - \lambda a_0 - \sum_{i=0}^{\Delta-1} \lambda_i a_{i+1}; y_1\big).$$

Hence $(x - \lambda a_0 - \sum_{i=0}^{\Delta-1} \lambda_i a_{i+1}; z_1) \in A^{\Delta+1}$. We obtain that

$$x = \big(x - \lambda a_0 - \sum_{i=0}^{\Delta-1} \lambda_i a_{i+1}\big) + \lambda a_0 + \sum_{i=0}^{\Delta-1} \lambda_i a_{i+1} \in \operatorname{dom} A^{\Delta+1} + \operatorname{span}\{a_0, \ldots, a_{\Delta}\}\,.$$

The inclusion '$\supseteq$' is clear.

❑

2.31 Lemma. *Assume that H is a Hamiltonian on $I = (s_-, s_+)$ which is regular at s_-, singular at s_+, and which satisfies (HS) and (I). Moreover, let $n \in \mathbb{N}$. Then*

$$\operatorname{dom} T_{max}^n = \operatorname{ran} B^n + \operatorname{span}\Big\{B^k \begin{pmatrix}1\\0\end{pmatrix} : k = 0, \dots, n-1\Big\}. \tag{2.26}$$

If H does not start with an indivisible interval, i.e. if $\alpha_1^- = s_-$, then this sum is direct:

$$\operatorname{ran} B^n \cap \operatorname{span}\Big\{B^k \begin{pmatrix}1\\0\end{pmatrix} : k = 0, \dots, n-1\Big\} = \{0\}.$$

If H starts with an indivisible interval of type $\phi \neq 0$, then $\binom{1}{0} \in \operatorname{ran} B$. Hence, in this case, (2.26) *is for no $n \in \mathbb{N}$ a direct sum.*

Proof. The relation (2.26) is immediate from the above Lemma 2.30 applied with $T = T_{max}$ and $A = A(0) = B^{-1}$.

Assume that H does not start with an indivisible interval. Then $\operatorname{mul} T_{max} = \{0\}$ and thus also $\operatorname{mul} T_{max}^n = \{0\}$ for all $n \in \mathbb{N}$. Assume that for some $\lambda_k \in \mathbb{C}$, and $g \in L^2(H)$,

$$\sum_{k=0}^{n-1} \lambda_k B^k \begin{pmatrix}1\\0\end{pmatrix} =_H B^n g.$$

Since $(B^n g; g), (\sum_{k=0}^{n-1} \lambda_k B^k \binom{1}{0}; 0) \in T_{max}^n$, we obtain $g \in \operatorname{mul} T_{max}^n = \{0\}$, and,hence, $B^n g =_H 0$. Thus (2.26) is a direct sum.

Assume that H starts with an indivisible interval of type $\phi \neq 0$, so that $\alpha_1^- > s_-$ and $H(t) = h(t)\xi_\phi \xi_\phi^T$, $t \in (s_-, \alpha_1^-)$. Consider the function

$$g(t) := \begin{cases} -\Big(\sin\phi \int_{s_-}^{\alpha_1^-} h\Big)^{-1} \cdot \xi_\phi & , \ t \leq \alpha_1^- \\ 0 & , \ t > \alpha_1^- \end{cases}.$$

Then certainly $g \in L^2(H)$. We have for some $c \in \operatorname{span}\{\binom{0}{1}\}$,

$$(Bg)(t) = \int_{s_-}^t JHg + c =$$

$$= \begin{cases} -\int_{s_-}^t Jh(\xi_\phi \xi_\phi^T)\Big(\sin\phi \int_{s_-}^{\alpha_1^-} h\Big)^{-1} \xi_\phi + c & , \ t \leq \alpha_1^- \\ \int_{s_-}^{\alpha_1^-} JHg + c & , \ t > \alpha_1^- \end{cases},$$

and $(t \leq \alpha_1^-)$

$$\int_{s_-}^t Jh(\xi_\phi \xi_\phi^T)\Big(\sin\phi \int_{s_-}^{\alpha_1^-} h\Big)^{-1} \xi_\phi = \int_{s_-}^t h\Big(\sin\phi \int_{s_-}^{\alpha_1^-} h\Big)^{-1} J\xi_\phi.$$

In particular,

$$\int_{s_-}^{\alpha_1^-} JHg = \begin{pmatrix}1\\ -\cot\phi\end{pmatrix}.$$

As $Bg \in L^2(H)$ it follows that

$$c = \begin{pmatrix}0\\ \cot\phi\end{pmatrix}.$$

Alltogether we obtain

$$(Bg)(t) = \begin{cases} -\int_{s_-}^t h\Big(\sin\phi \int_{s_-}^{\alpha_1^-} h\Big)^{-1} J\xi_\phi + \binom{0}{\cot\phi} & , \ t \le \alpha_1^- \\ \binom{1}{0} & , \ t > \alpha_1^- \end{cases},$$

and thus

$$H(t)(Bg)(t) = \begin{cases} h(t)\cdot \xi_\phi \cos\phi & , \ t \le \alpha_1^- \\ H(t)\binom{1}{0} & , \ t > \alpha_1^- \end{cases}.$$

Since

$$H(t)\begin{pmatrix}1\\0\end{pmatrix} = \begin{cases} h(t)\cdot \xi_\phi \cos\phi & , \ t \le \alpha_1^- \\ H(t)\binom{1}{0} & , \ t > \alpha_1^- \end{cases},$$

$Bg =_H \binom{1}{0}$ and, therefore, $\binom{1}{0} \in \operatorname{ran} B$.

❑

c. Restriction

As it is seen from Theorem 2.27, the condition (HS) is stable with respect to restriction: Let H be a Hamiltonian on the interval $I = (s_-, s_+)$ which is regular at s_- and singular at s_+. Moreover, let $s \in [s_-, \alpha_1^+)$ and put $L = (s, s_+)$, so that $\hat{H} := H|_L$ is regular at s and singular at s_+. Then

(i) H satisfies (HS) if and only if $\hat{H}$ does. In this case $\phi(H) = \phi(\hat{H})$.

(ii) H satisfies (I) if and only if $\hat{H}$ does.

(iii) Assume that H satisfies (HS) and (I), and let B and $\hat{B}$ be defined correspondingly. If $f \in L^2(H)$, then there exists a unique number $\lambda(f) \in \mathbb{C}$ such that

$$\hat{B}(f|_L) = (Bf)|_L + \begin{pmatrix}\lambda(f)\\0\end{pmatrix}. \tag{2.27}$$

An inductive application of (2.27) yields that for every $N \in \mathbb{N} \cup \{0\}$,

$$\operatorname{span}\big\{\hat{B}^k(f|_L) : k = 0,\dots,N\big\} \subseteq \operatorname{span}\big\{(B^k f)|_L : k = 0,\dots,N\big\} +$$

$$+ \operatorname{span}\Big\{(B^j\begin{pmatrix}1\\0\end{pmatrix})\Big|_L : j = 0,\dots,N-1\Big\}$$

and

$$\operatorname{span}\Big\{\hat{B}^k(\begin{pmatrix}1\\0\end{pmatrix}\Big|_L) : k = 0,\dots,N\Big\} = \operatorname{span}\Big\{(B^k\begin{pmatrix}1\\0\end{pmatrix})\Big|_L : k = 0,\dots,N\Big\}.$$

3 H-polynomials

Throughout this section let H be a Hamiltonian on $I = (s_-, s_+)$ which is regular at s_-.

Define an operator $\mathcal{I}$ by

$$\mathcal{I} : \begin{cases} \operatorname{dom}\mathcal{I} & \to & \mathrm{AC}([s_-, s_+)) \subseteq \mathcal{M}(I) \\ f & \mapsto & (\mathcal{I}f)(x) := \int_{s_-}^{x} JHf \end{cases} \tag{3.1}$$

$$\operatorname{dom}\mathcal{I} := \big\{ f : [s_-, s_+) \to \mathbb{C}^2 : f \text{ measureable}, Hf \in L^1_{loc}([s_-, s_+)) \big\}$$

To see that $\mathcal{I}$ is well-defined, note that $\mathcal{I}f$ satisfies (C) because $(\mathcal{I}f)'(x) \in \operatorname{ran} JH(x)$ a.e. Moreover, note that $L^2(H) \subseteq \operatorname{dom}\mathcal{I}$.

Consider the linear space $\mathbb{C}^2[z]$ of all polynomials in the variable z with coefficients in $\mathbb{C}^2$ and denote by $\mathbb{C}^2[z]_n$ the linear subspace of all polynomials of degree at most n. We shall identify $\mathbb{C}^2[z]$ in the natural way with $\mathbb{C}[z] \times \mathbb{C}[z]$.

A linear map γ is defined by

$$\gamma : \begin{cases} \mathbb{C}^2[z] & \to & \mathrm{AC}([s_-, s_+)) \subseteq \mathcal{M}(I) \\ \sum_{k=0}^{n} a_k z^k & \mapsto & \sum_{k=0}^{n} \mathcal{I}^k a_k \end{cases} .$$

Note that, since $\mathrm{AC}([s_-, s_+)) \subseteq \operatorname{dom}\mathcal{I}$, the iterates of $\mathcal{I}$ are well-defined. Moreover, note that γ transforms multiplication by z into application of $\mathcal{I}$. The following subspaces of $\mathbb{C}^2[z]$ will be of importance:

$$\mathrm{HPol} := \gamma^{-1}(L^2(H)),\ \mathrm{HPol}_n := \mathrm{HPol} \cap \mathbb{C}^2[z]_n .$$

We shall refer to HPol as the space of H-polynomials.

For $k \in \mathbb{N}_0$ denote by $\pi_k : \mathbb{C}^2[z] \to \mathbb{C}^2$ the linear map which assigns to a polynomial its coefficient of z^k. Then $\pi_k(\mathrm{HPol}_k)$ is a linear subspace of $\mathbb{C}^2$, and as such has dimension 0, 1 or 2.

3.1 Definition. Define a number $\Delta(H) \in \mathbb{N}_0 \cup \{\infty\}$ as

$$\Delta(H) := \inf \big\{ k \in \mathbb{N}_0 : \dim \pi_k(\mathrm{HPol}_k) = 2 \big\} .$$

Note that H is regular at s_+ if and only if $\Delta(H) = 0$. There is another instance when $\Delta(H)$ can be determined.

3.2 Lemma. *Assume that H is singular at s_+ and that $\alpha_1^+ < s_+$. Then we have $\Delta(H) = 1$.*

Proof. Since H is singular at s_+, we have $\Delta(H) \geq 1$. Let $v \in \mathbb{C}^2$ be given and consider the polynomial

$$p(z) := vz - \int_{s_-}^{\alpha_1^+} JHv \in \mathbb{C}^2[z]_1 .$$

Then $\gamma(p)(\alpha_1^+) = 0$ and, since $\gamma(p) \in \mathrm{AC}([s_-, s_+)$ and thus satisfies (C), this implies that $H(x)\gamma(p)(x) = 0$ for $x \in (\alpha_1^+, s_+)$ a.e. It follows that $\gamma(p) \in L^2(H)$, and we see that $v \in \pi_1(\mathrm{HPol}_1)$.

❑

We investigate these notions under some additional hypothesis on H: For the remainder of this section assume that H is singular at s_+ and satisfies (HS) and (I). Moreover, let B always denote the Hilbert-Schmidt operator (2.22). The fact that B is everywhere defined yields the following statement, which also has a consequence on the structure of HPol.

3.3 Lemma. *For every $f \in L^2(H) \subseteq \mathcal{M}(I)$ there exists a unique number $\lambda(f) \in \mathbb{C}$, such that*

$$g := \mathcal{I}f + \begin{pmatrix} 0 \\ \lambda(f) \end{pmatrix} \in L^2(H)\,.$$

We have $g/_{=_H} = B(f/_{=_H})$.

Proof. First assume that $\alpha_1^- < s_+$ and let $f \in L^2(H) \subseteq \mathcal{M}(I)$ be given. Then the unique representant $g \in \mathrm{AC}([s_-, s_+)) \cap L^2(H)$ of the equivalence class $B(f/_{=_H})$ which satisfies $g' = JHf$, can be written as

$$g(x) = g(s_-) + (\mathcal{I}f)(x)\,.$$

Since $A(0)$ is defined by the boundary condition $(1,0)\Gamma(g;f) = 0$, we have $g(s_-) \in \operatorname{span}\{\binom{0}{1}\}$. The relation $g/_{=_H} = B(f/_{=_H})$ holds by the definition of g. The fact that the constant $\lambda(f)$ is unique follows since $\binom{0}{1}$ does not belong to $L^2(H)$.

Secondly, let us assume that $\alpha_1^- = s_+$, so that

$$H(t) = h(t)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\,.$$

Then, as a subset of $\mathcal{M}(I)$,

$$L^2(H) = \left\{ \begin{pmatrix} f_1 \\ 0 \end{pmatrix} : f \text{ measurable} \right\},$$

and it follows that $L^2(H) \subseteq \ker \mathcal{I}$. Since $(0,1)^T \notin L^2(H)$, the unique choice is $\lambda(f) = 0$ so that in total $g = 0$. The relation $g/_{=_H} = B(f/_{=_H})$ is trivially satisfied.

❑

3.4 Corollary. *Let $p \in \mathrm{HPol}$. Then there exists a unique number $\lambda(p) \in \mathbb{C}$, such that*

$$q(z) := zp(z) + \begin{pmatrix} 0 \\ \lambda(p) \end{pmatrix} \in \mathrm{HPol}\,.$$

We have $\gamma(q)/_{=_H} = B(\gamma(p)/_{=_H})$.

Since, by (I), we have $(1,0)^T \in \mathrm{HPol}$, an inductive application of this result leads to the following statement:

3.5 Corollary. *There exist unique numbers $\rho_1, \rho_2, \ldots \in \mathbb{C}$, such that*

$$r_n(z) := \begin{pmatrix} 1 \\ 0 \end{pmatrix} z^n + \sum_{k=0}^{n-1} \begin{pmatrix} 0 \\ \rho_{n-k} \end{pmatrix} z^k \in \mathrm{HPol},\ n \in \mathbb{N}_0\,.$$

We have $\gamma(r_n)/_{=_H} = B^n(\binom{1}{0}/_{=_H})$, $n \in \mathbb{N}_0$.

By virtue of this result we can often 'reduce to the lower component' as, for example, in the following lemma.

3.6 Lemma. *The number $\Delta(H)$ is given by*

$$\Delta(H) = \inf\left\{k \in \mathbb{N}_0 : \operatorname{HPol}_k \cap(\{0\}\times\mathbb{C}[z]) \neq \{0\}\right\}.$$

Proof. Since, by Corollary 3.5, for every $n \in \mathbb{N}_0$

$$\begin{pmatrix}1\\0\end{pmatrix} \in \pi_n(\operatorname{HPol}_n),$$

we have $\dim \pi_n(\operatorname{HPol}_n) = 2$ if and only if there exists a H-polynomial of the form

$$p(z) = \begin{pmatrix}\alpha_n\\1\end{pmatrix} z^n + \sum_{k=0}^{n-1} \begin{pmatrix}\alpha_k\\\beta_k\end{pmatrix} z^k .$$

If $\operatorname{HPol}_n \cap(\{0\}\times\mathbb{C}[z]) \neq \{0\}$, then there exists a H-polynomial of this form. Conversely, if $p \in \operatorname{HPol}$ is of this form, then

$$0 \neq p(z) - \sum_{k=0}^{n} \alpha_k r_k(z) \in \operatorname{HPol}_n \cap(\{0\}\times\mathbb{C}[z]).$$

❑

Assume that $\Delta := \Delta(H) < \infty$. Then Lemma 3.6 implies that there exist unique numbers $\omega_1, \ldots, \omega_\Delta \in \mathbb{C}$, such that

$$w_\Delta(z) := \begin{pmatrix}0\\1\end{pmatrix} z^\Delta + \sum_{k=0}^{\Delta-1} \begin{pmatrix}0\\\omega_{\Delta-k}\end{pmatrix} z^k \in \operatorname{HPol}.$$

An inductive application of Corollary 3.4 yields, moreover, that there exist unique numbers $\omega_{\Delta+1}, \omega_{\Delta+2}, \ldots \in \mathbb{C}$, such that

$$w_l(z) := \begin{pmatrix}0\\1\end{pmatrix} z^l + \sum_{k=0}^{l-1} \begin{pmatrix}0\\\omega_{l-k}\end{pmatrix} z^k \in \operatorname{HPol},\ l > \Delta .$$

Thereby we have $\gamma(w_{l+1}) = B\gamma(w_l)$, $l \geq \Delta$. We use the same formula to define w_l for $l < \Delta$:

$$w_l(z) := \begin{pmatrix}0\\1\end{pmatrix} z^l + \sum_{k=0}^{l-1} \begin{pmatrix}0\\\omega_{l-k}\end{pmatrix} z^k,\ l = 0, \ldots, \Delta - 1 .$$

Clearly, $w_l \notin \operatorname{HPol}$ for $l < \Delta$. It is now obvious that

$$\operatorname{HPol}_n \cap(\{0\}\times\mathbb{C}[z]) = \begin{cases} \{0\} & , n < \Delta \\ \operatorname{span}\{w_l : l = \Delta, \ldots, n\} & , n \geq \Delta \end{cases},$$

and

$$\operatorname{HPol}_n = \begin{cases} \operatorname{span}\left\{r_l : l = 0, \ldots, n\right\} & , n < \Delta \\ \operatorname{span}\left(\{r_l : l = 0, \ldots, n\} \cup \{w_k : k = \Delta, \ldots, n\}\right) & , n \geq \Delta \end{cases}.$$

3.7 Definition. Assume that $\Delta := \Delta(H) < \infty$. Define functions $\mathfrak{w}_l \in \mathrm{AC}([s_-, s_+))$, $l \in \mathbb{N}_0$, by

$$\mathfrak{w}_l := \gamma(w_l),\ l \in \mathbb{N}_0 .$$

Note that, by Lemma 3.6, the set $\{\mathfrak{w}_0, \dots, \mathfrak{w}_{\Delta-1}\}$ is linearly independent modulo $L^2(H)$.

3.8 Remark. Let us explicitly consider the case that $\alpha_1^- = s_+$. Then we have

$$\mathcal{I}\begin{pmatrix}0\\1\end{pmatrix}(x) = \int_{s_-}^x Jh(t)\begin{pmatrix}0&0\\0&1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\int_{s_-}^x h\\0\end{pmatrix},$$

$$\mathcal{I}^k\begin{pmatrix}0\\1\end{pmatrix} = 0,\ k = 2, 3, \dots$$

and therefore $\omega_k = 0$, $k = 1, 2, \dots$. It follows that

$$\mathfrak{w}_0(x) = \begin{pmatrix}0\\1\end{pmatrix},\ \mathfrak{w}_1(x) = \begin{pmatrix}-\int_{s_-}^x h\\0\end{pmatrix},\ \mathfrak{w}_2 = \mathfrak{w}_3 = \dots = 0 .$$

3.9 Lemma. *For all $l \in \mathbb{N}$ we have $\omega_l \in \mathbb{R}$ and the functions $\mathfrak{w}_l$, $l = 0, 1, \dots$ are real valued.*

Proof. If $\alpha_1^- = s_+$, this is obvious from Remark 3.8. Thus assume that $\alpha_1^- < s_+$. We use induction on $l \geq \Delta$.

Consider first the case $l = \Delta$. Since $\gamma(p) \in L^2(H)$ if and only if $\gamma(\overline{p}) \in L^2(H)$, where $\bar{\cdot}$ is understood to act on polynomials by coefficientwise conjugation, we see that $w_\Delta - \overline{w_\Delta} \in \mathrm{HPol}$. However, the degree of $w_\Delta - \overline{w_\Delta}$ is less than Δ, and hence $w_\Delta - \overline{w_\Delta} = 0$. It follows that $\omega_k \in \mathbb{R}$ for $k = 1, \dots, \Delta$.

Assume it is readily proved that $\omega_k \in \mathbb{R}$, $k \leq l$. We have $\mathfrak{w}_{l+1} = B\mathfrak{w}_l$. We use (2.10) with $(\mathfrak{w}_{l+1}; \mathfrak{w}_l)$ and $(\binom{1}{0}; 0)$ to obtain

$$(\mathfrak{w}_{l+1}, 0) - (\mathfrak{w}_l, \begin{pmatrix}1\\0\end{pmatrix}) = \begin{pmatrix}1\\0\end{pmatrix}^* J\mathfrak{w}_{l+1}(s_-) = -\mathfrak{w}_{l+1}(s_-)_2 .$$

It follows that

$$\mathfrak{w}_{l+1}(s_-)_2 = (\mathfrak{w}_l, \begin{pmatrix}1\\0\end{pmatrix}) = \int_{s_-}^{s_+} \mathfrak{w}_l^* H \begin{pmatrix}1\\0\end{pmatrix}$$

and this number belongs by our inductive hypothesis to $\mathbb{R}$. However, $\mathfrak{w}_{l+1}(s_-)_2 = \omega_{l+1}$.

❑

It is important to note that the functions $\mathfrak{w}_l$ are unique.

3.10 Lemma. *Assume that $\Delta := \Delta(H) < \infty$. The functions $(\mathfrak{w}_k)_{k \in \mathbb{N}_0}$ satisfy*

$$\begin{gathered}\mathfrak{w}_0' = 0,\ \mathfrak{w}_k' = JH\mathfrak{w}_{k-1},\ k \in \mathbb{N},\\ \mathfrak{w}_k(s_-) \in \operatorname{span}\Big\{\begin{pmatrix}0\\1\end{pmatrix}\Big\},\ k \in \mathbb{N}_0,\\ \mathfrak{w}_k \in L^2(H),\ k \geq \Delta.\end{gathered} \tag{3.2}$$

Conversely, if for some $n \geq \Delta$ functions $f_0, \dots, f_n \in \mathrm{AC}([s_-, s_+))$ are given which satisfy the formulas (3.2) for $k \leq n$, then there exists $\beta \in \mathbb{C}$ such that $f_k = \beta\mathfrak{w}_k$ for all $k = 0, \dots, n$.

Proof. The fact that the functions $\mathfrak{w}_k$ satisfy (3.2) is clear from their definition. Let $n \geq \Delta$ and $f_1, \ldots, f_n$ be given such that (3.2) holds. Reading the differential equations in (3.2) backwards, we obtain

$$f_\Delta = f_\Delta(s_-) + \mathcal{I} f_{\Delta-1}(s_-) + \ldots + \mathcal{I}^{\Delta-1} f_1(s_-) + \mathcal{I}^\Delta f_0 \,. \tag{3.3}$$

Write $f_k(s_-) = \binom{0}{\beta_k}$, $k = 1, \ldots, \Delta$, $f_0 = \binom{0}{\beta_0}$, then

$$f_\Delta = \gamma\Big(\sum_{k=0}^{\Delta} \begin{pmatrix} 0 \\ \beta_{\Delta-k} \end{pmatrix} z^k \Big) \,.$$

Since $f_\Delta \in L^2(H)$, we conclude that $\beta_{\Delta-k} = \beta_0 \omega_{\Delta-k}$, $k = 0, \ldots, \Delta - 1$, and hence that $f_\Delta = \beta_0 \mathfrak{w}_\Delta$.

The same argument that led us to (3.3) shows that for every $l = 0, \ldots, \Delta - 1$,

$$f_l = f_l(s_-) + \mathcal{I} f_{l-1}(s_-) + \ldots + \mathcal{I}^{l-1} f_1(s_-) + \mathcal{I}^l f_0 =$$

$$= \gamma\Big(\sum_{k=0}^{l} \begin{pmatrix} 0 \\ \beta_{l-k} \end{pmatrix} z^k \Big) = \beta_0 \mathfrak{w}_l \,.$$

We show inductively that $f_l = \beta_0 \mathfrak{w}_l$ for all $l \geq \Delta$. Assume that $f_{l-1} = \beta_0 \mathfrak{w}_{l-1}$, then by (3.2) we have $f_l' = \beta_0 \mathfrak{w}_l'$, and hence $f_l - \beta_0 \mathfrak{w}_l = a \in \mathbb{C}^2$. Since $f_l(s_-), \mathfrak{w}_l(s_-) \in \operatorname{span}\{\binom{0}{1}\}$, we must have $a = \binom{0}{\alpha}$, and since $f_l, \mathfrak{w}_l \in L^2(H)$, it follows that $\alpha = 0$. Thus $f_l = \beta_0 \mathfrak{w}_l$.

❑

3.11 Lemma. *The following assertions hold true:*

(*i*) *Assume that* $\Delta := \Delta(H) < \infty$ *and that* $\alpha_1^+ = s_+$. *Then*

$$\mathfrak{w}_{\Delta+l} \in \operatorname{dom} T_{max}^l \setminus \operatorname{dom} T_{max}^{l+1}, \;\; l = 0, 1, 2, \ldots \tag{3.4}$$

In particular, the set $\{\mathfrak{w}_{\Delta+l} : l = 0, \ldots, n\}$ *is linearly independent modulo* $\operatorname{dom} T_{max}^{n+1}$.

(*ii*) *Assume that* $\alpha_1^+ < s_+$ *so that, in particular,* $\Delta = 1$. *Then* $\mathfrak{w}_1 \in \operatorname{dom} T_{max}$.

Proof.
ad(i): By (3.2) we have $(\mathfrak{w}_{\Delta+l}; \mathfrak{w}_\Delta) \in T_{max}^l$ and hence $\mathfrak{w}_{\Delta+l} \in \operatorname{dom} T_{max}^l$.

Assume that $\mathfrak{w}_{\Delta+l} \in \operatorname{dom} T_{max}^{l+1}$, so that there exist $g_0, \ldots, g_l \in \mathrm{AC}(I)$, $g_{l+1} \in \mathcal{M}(I)$, with

$$g_0, \ldots, g_{l+1} \in L^2(H),$$
$$g_0 =_H \mathfrak{w}_{\Delta+l}, \;\; g_k' = JHg_{k+1}, \;\; k = 0, \ldots, l.$$

Apply Lemma 2.23 with

$$f_k := g_k - \mathfrak{w}_{\Delta+l-k}, \;\; k = 0, \ldots, l+1 \,.$$

Since we assume that $\alpha_1^+ = s_+$, it follows that $H(g_{l+1} - \mathfrak{w}_{\Delta-1})(t) = 0$, $t \in (\alpha_{l+1}^-, s_+)$ a.e. Again since $\alpha_1^+ = s_+$, we surely have $\alpha_{l+1}^- < s_+$, i.e. $\mathfrak{w}_{\Delta-1}$ is H-equal to $g_{l+1} \in L^2(H)$ in a neighbourhood of s_+. Since $\mathfrak{w}_{\Delta-1}$ is continuous, it follows that $\mathfrak{w}_{\Delta-1} \in L^2(H)$, a contradiction.

$\mathtt{ad}\,(ii)$: First note that, since H satisfies (I), the type of the indivisible interval (α_1^+, s_+) must be equal to $\frac{\pi}{2}$. We have $\mathfrak{w}_1(t) = \int_{s_-}^t JH\binom{0}{1} + \binom{0}{\omega_1}$. Put

$$f(t) := \begin{cases} \mathfrak{w}_1(t) & , \ t \in (s_-, \alpha_1^+) \\ \mathfrak{w}_1(\alpha_1^+) & , \ t \in (\alpha_1^+, s_+) \end{cases}.$$

Then, for $t \in (\alpha_1^+, s_+)$, we have

$$\mathfrak{w}_1(t) - f(t) = \int_{\alpha_1^+}^t JH\begin{pmatrix}0\\1\end{pmatrix} \in \operatorname{span}\{\begin{pmatrix}1\\0\end{pmatrix}\}.$$

Thus $H(\mathfrak{w}_1 - f)(t) = 0$, $t \in (\alpha_1^+, s_+)$, and it follows that $\mathfrak{w}_1 =_H f$.

We have $f' = JHg$ where

$$g(t) := \begin{cases} \binom{0}{1} & , \ t \in (s_-, \alpha_1^+) \\ 0 & , \ t \in (\alpha_1^+, s_+) \end{cases},$$

and this function belongs to $L^2(H)$. We see that $\mathfrak{w}_1 \in \operatorname{dom} T_{max}$.

❑

Let us investigate how the functions $\mathfrak{w}_l$ transform when the Hamiltonian is restricted. Let $s \in [s_-, \alpha_1^+)$ and put $L = (s, s_+)$, so that $\hat{H} := H|_L$ is regular at s, singular at s_+, satisfies $\alpha_1^-(\hat{H}) < s_+$ and (HS) and (I). In the following let $\hat{\mathcal{I}}$ be the integral operator defined as in (3.1) with s_- replaced by s. Let $\hat{B}$ be defined similarly (see (2.27)).

3.12 Lemma. *The following assertions hold true:*

(*i*) *For every $f \in \operatorname{dom}\mathcal{I}$ there exist unique constants $b_k(f) \in \mathbb{C}^2$, $k \in \mathbb{N}_0$, such that*

$$\hat{\mathcal{I}}^n(f|_L) = (\mathcal{I}^n f)|_L + \sum_{k=0}^{n-1} (\mathcal{I}^k b_{n-k}(f))|_L .$$

(*ii*) $\Delta(H) = \Delta(\hat{H})$.

Assume that $\Delta(H) < \infty$ and let functions $\hat{\mathfrak{w}}_l$ be defined correspondingly. Then there exist unique constants λ_k, $\hat{\lambda}_k$, $k \in \mathbb{N}_0$, such that

$$\hat{\mathfrak{w}}_l = \Big(\mathfrak{w}_l - \sum_{k=0}^{l-1} \lambda_{l-k} B^k \begin{pmatrix}1\\0\end{pmatrix}\Big)\Big|_L = \mathfrak{w}_l\Big|_L - \sum_{k=0}^{l-1} \hat{\lambda}_{l-k} \hat{B}^k \begin{pmatrix}1\\0\end{pmatrix}.$$

Proof. The first assertion follows by a straightforward inductive argument. We come to the proof of (*ii*). Assume that $p(z) = \binom{0}{1} z^n + \sum_{k=0}^{n-1} a_k z^k \in \operatorname{HPol}(\hat{H})$. Then

$$\hat{\gamma}(p) = \hat{\mathcal{I}}^n \begin{pmatrix}0\\1\end{pmatrix} + \sum_{k=0}^{n-1} \hat{\mathcal{I}}^k a_k \in L^2(\hat{H}).$$

By (*i*) we have for appropriate constants $c_k \in \mathbb{C}^2$,

$$\hat{\gamma}(p) = \Big(\mathcal{I}^n \begin{pmatrix}0\\1\end{pmatrix} + \sum_{k=0}^{n-1} \mathcal{I}^k c_k\Big)\Big|_L,$$

and we see that $\binom{0}{1}z^n + \sum_{k=0}^{n-1} c_k z^k \in \operatorname{HPol}(H)$. It follows that $\Delta(\hat{H}) \geq \Delta(H)$. The converse inequality is seen similarly.

Let us establish the first equality sign in the remaining assertion. To this end note that there is a unique choice of numbers $\lambda_k \in \mathbb{C}$, $k \in \mathbb{N}$, such that

$$\Big(\mathfrak{w}_l - \sum_{k=0}^{l-1} \lambda_{l-k} B^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big)(s) \in \operatorname{span}\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\},\ l \in \mathbb{N}\,.$$

We apply Lemma 3.10 with the functions

$$f_l := \big(\mathfrak{w}_l - \sum_{k=0}^{l-1} \lambda_{l-k} B^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}\big)\big|_L,$$

and obtain the desired assertion.

The proof of the second equality sign is similar, and is therefore omitted.

❑

We will need the following corollary of the formula in Proposition 2.26 on integration by parts.

3.13 Corollary. *Assume that $\Delta(H) < \infty$. If $\lambda_0, \dots, \lambda_{\Delta-1} \in \mathbb{C}$ and $g \in L^2(H)$ are such that*

$$\sum_{k=0}^{\Delta-1} \lambda_k B^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} =_H B^\Delta g\,,$$

then for all $n \in \mathbb{N}_0$,

$$(g, \mathfrak{w}_{n+\Delta})_{L^2(H)} = -\sum_{k=0}^{\Delta-1} \lambda_k \mathfrak{w}_{n+1+k}(s_-)^* \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.5}$$

Proof. Let us first settle the case that $\alpha_1^- = s_+$. Then $L^2(H)$ consists just of the zero element, and hence on the left hand side of (3.5) we always have 0. By Remark 3.8 also the right hand side of (3.5) is always equal to 0.

Assume now that $\alpha_1^- < s_+$. Consider the functions

$$f_l := B^{\Delta-l} g - \sum_{k=0}^{\Delta-1-l} \lambda_{k+l} B^k \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ l = 0, \dots, \Delta\,,$$

$$g_l := \mathfrak{w}_{n+\Delta-l},\ l = 0, \dots, \Delta\,.$$

Then $f_l \in L^2(H)$ for all $l = 0, \dots, \Delta$ and $(Hf_0)(t) = 0$ for $t \in (s_-, s_+)$ a.e. Moreover, $f_k' = JHf_{k+1}$ and $g_k' = JHg_{k+1}$. If $\alpha_1^+ = s_+$, then certainly $\alpha_\Delta^- < s_+$. If $\alpha_1^+ < s_+$, then $\Delta(H) = 1$ (see Lemma 3.2), and again $\alpha_\Delta^- < s_+$. Hence we can apply Proposition 2.26, (iii), and obtain

$$0 = \int_I g_\Delta^* H f_0 = (g, \mathfrak{w}_{n+\Delta})_{L^2(H)} - \sum_{j=0}^{\Delta-1} \mathfrak{w}_{n+\Delta-j}(s_-)^* J f_{\Delta-1-j}(s_-)\,.$$

We have

$$f_{\Delta-1-j}(s_-) = \underbrace{(B^{1+j}g)(s_-) - \sum_{k=1}^{j} \lambda_{k+\Delta-1-j}(B^k \begin{pmatrix}1\\0\end{pmatrix})(s_-)}_{\in \operatorname{span}\{\binom{0}{1}\}} - \lambda_{\Delta-1-j}\begin{pmatrix}1\\0\end{pmatrix}.$$

Since $\mathfrak{w}_{n+\Delta-j}(s_-) \in \operatorname{span}\{\binom{0}{1}\}$, we obtain

$$\mathfrak{w}_{n+\Delta-j}(s_-)^* J f_{\Delta-1-j}(s_-) = -\mathfrak{w}_{n+\Delta-j}(s_-)^* J \lambda_{\Delta-1-j}\begin{pmatrix}1\\0\end{pmatrix}.$$

Formula (3.5) follows.

❑

Let $z \in \mathbb{C}$ be fixed and consider the differential equation $f'(t) = zJH(t)f(t)$, $t \in (s_-, s_+)$ a.e. For each $c \in \mathbb{C}^2$ the initial value problem $f(s_-) = c$,

$$f'(t) = zJH(t)f(t), \tag{3.6}$$

has a unique solution in $\mathrm{AC}([s_-, s_+))$, see e.g. [HSW]. Hence the space

$$N_z := \big\{ f \in \mathrm{AC}([s_-, s_+)) : \, f' = zJHf \big\}$$

has dimension 2.

Assume that $\alpha_1^-(H) < s_+$. By Theorem 2.19 the defect index of T_{min} is $(1, 1)$. Since T_{min} is completely nonselfadjoint and has compact resolvents by the validity of (HS), we have $r(T_{min}) = \mathbb{C}$. It follows that for all $z \in \mathbb{C}$

$$\dim \big(N_z \cap L^2(H) \big) = 1\,.$$

3.14 Lemma. *Assume that* $\Delta = \Delta(H) < \infty$ *and* $\alpha_1^- < s_+$. *Let* $z \in \mathbb{C}$ *and choose* $g_1 \in \mathrm{AC}([s_-, s_+))$ *such that* $\operatorname{span}\{g_1\} = N_z \cap L^2(H)$. *Let* $\phi \in [0, \pi)$ *satisfy* $\xi_\phi^T \Gamma(g_1; zg_1) \neq 0$, *so that we have* $z \in \rho(A(\phi))$. *Finally, let* $h \in \mathrm{AC}([s_-, s_+))$ *be the unique representant of the equivalence class* $(A(\phi)-z)^{-1}\mathfrak{w}_\Delta$ *with*

$$h' = JH(\mathfrak{w}_\Delta + zh)\,,$$

which exists because of $((A(\phi) - z)^{-1}\mathfrak{w}_\Delta; \mathfrak{w}_\Delta + z(A(\phi) - z)^{-1}\mathfrak{w}_\Delta) \in T_{max}$, *and define*

$$g_2 := \sum_{k=0}^{\Delta} z^k \mathfrak{w}_k + z^{\Delta+1} h\,. \tag{3.7}$$

Then

$$N_z = \operatorname{span}\{g_1, g_2\}\,.$$

Proof. Since $\{\mathfrak{w}_0, \ldots, \mathfrak{w}_{\Delta-1}\}$ is linearly independent modulo $L^2(H)$, we have $g_2 \notin L^2(H)$ and thus, in particular, $\{g_1, g_2\}$ is linearly independent. Thus it is enough to show that $g_2 \in N_z$. This, however, follows since

$$g_2' = \sum_{k=0}^{\Delta} z^k \mathfrak{w}_k' + z^{\Delta+1} h' = \sum_{k=0}^{\Delta} z^k JH\mathfrak{w}_{k-1} + z^{\Delta+1} JH(\mathfrak{w}_\Delta + zh) =$$

$$= zJH\Big(\sum_{k=0}^{\Delta-1} z^k\mathfrak{w}_k + z^\Delta\mathfrak{w}_\Delta + z^{\Delta+1}h\Big) = zJHg_2\,.$$

❑

3.15 Corollary. *If, in the situation of the above lemma, $(1,0)g_1(s_-)\neq 0$, then with the choice $\phi=0$ we have $(1,0)g_2(s_-)=0$. On the other hand, if $(1,0)g_1(s_-)=0$, then for every solution f of (3.6) with $f\notin L^2(H)$, we have $(1,0)f(s_-)\neq 0$.*

Proof. We have $h\in\operatorname{dom}A(\phi)$ and thus $\xi_\phi^T h(s_-)=0$. If $\phi=0$, this means that $(1,0)h(s_-)=0$. The functions $\mathfrak{w}_k$ anyway have this property, so that alltogether $(1,0)g_2(s_-)=0$.

If $(1,0)g_1(s_-)=0$, then by the uniqueness of the solution of (3.6) any solution f with $(1,0)f(s_-)=0$ is linearly dependent with g_1 and, thus, contained in $L^2(H)$.

❑

3.16 Remark. In order to cover all cases note that if $\alpha_1^-=s_+$ we have

$$N_z=\operatorname{span}\Big\{\begin{pmatrix}1\\0\end{pmatrix},\mathfrak{w}_0+z\mathfrak{w}_1\Big\}\,,$$

and therefore $N_z\cap L^2(H)=\operatorname{span}\{(1,0)^T\}$.

This follows by explicit computation, cf. Remark 3.8: We have

$$\mathfrak{w}_0+z\mathfrak{w}_1=\begin{pmatrix}0\\1\end{pmatrix}-z\int_{s_-}^x h\cdot\begin{pmatrix}1\\0\end{pmatrix}\,.$$

Thus $\binom{1}{0}$ and $\mathfrak{w}_0+z\mathfrak{w}_1$ are linearly independent and

$$(\mathfrak{w}_0+z\mathfrak{w}_1)'=-zh\begin{pmatrix}1\\0\end{pmatrix}=zhJ\begin{pmatrix}0\\1\end{pmatrix}=zJH\begin{pmatrix}0\\1\end{pmatrix}=zJH(\mathfrak{w}_0+z\mathfrak{w}_1)\,.$$

Finally we are going to prove a lemma which will be of importance later on. Assume that $\Delta=\Delta(H)<\infty$. Then, by Corollary 3.13, a linear functional ϕ_0 is well-defined on the linear space

$$\mathcal{L}:=\operatorname{ran}B^\Delta+\operatorname{span}\Big\{B^i\begin{pmatrix}1\\0\end{pmatrix}:\,i=0,\dots,\Delta-1\Big\}$$

by the requirements that

$$\phi_0(B^\Delta f)=(f,\mathfrak{w}_\Delta)\,,$$

$$\phi_0\Big(B^i\begin{pmatrix}1\\0\end{pmatrix}\Big)=-\mathfrak{w}_{i+\Delta}^*(s_-)\begin{pmatrix}0\\1\end{pmatrix}\,.$$

Note that in case $\alpha_1^-(H)=s_+$ we have $\mathcal{L}=\{0\}$ and thus also $\phi_0=0$.

3.17 Lemma. *Assume that $\Delta=\Delta(H)<\infty$. Let $z\in\mathbb{C}$ and let $g_1\in\mathrm{AC}([s_-,s_+))$ be such that $\operatorname{span}\{g_1\}=N_z\cap L^2(H)$. Then $g_1\in\mathcal{L}$ and $z\phi_0(g_1)\neq(1,0)g_1(s_-)$.*

Proof. The case that $\alpha_1^-(H) = s_+$ is clear since then $g_1 = (\alpha, 0)^T$. The same argument applies if $\alpha_1^-(H) < s_+$ and $z = 0$. Hence assume that $\alpha_1^-(H) < s_+$ and $z \neq 0$.

Consider first the case that $(1,0)g_1(s_-) = 0$, so that $(g_1; zg_1) \in A(0)$. It follows that $z \in \sigma(A(0)) \subseteq \mathbb{R}$ and that $g_1 = z^\Delta B^\Delta g_1$. This shows that $g_1 \in \mathcal{L}$ and $\phi_0(g_1) = z^\Delta (g_1, \mathfrak{w}_\Delta)$. Assume now on the contrary to our assertion that $z\phi_0(g_1) = 0$. Then $\mathfrak{w}_\Delta \perp g_1$, and hence $\mathfrak{w}_\Delta \in \operatorname{ran}(T_{min} - z)$. Choose $\phi \in [0, \pi)$ as in Lemma 3.14. Then $(A(\phi) - z)^{-1}\mathfrak{w}_\Delta \in \operatorname{dom} T_{min}$, and we obtain (with h also as in Lemma 3.14) that $h(s_-) = 0$. Now it follows that $(1,0)g_2(s_-) = 0$, a contradiction to the second part of Corollary 3.15. The assertion follows.

Next consider the case that $(1,0)g_1(s_-) \neq 0$, so that $z \in \rho(A(0))$. Without loss of generality let us assume that $(1,0)g_1(s_-) = 1$. In order to compute $\phi_0(g_1)$, note that since $0, z \in \rho(A(0))$ the operator

$$I + z(A(0) - z)^{-1} = I + zB(I - zB)^{-1}$$

maps $\ker(T_{max})$ bijectively onto $\ker(T_{max} - z)$, and thus

$$g_1 = (I + zB(I - zB)^{-1})\begin{pmatrix}\alpha\\0\end{pmatrix}.$$

Comparing the boundary values at s_- yields $\alpha = 1$. We obtain by an inductive application of $(I - zB)^{-1} = I + zB(I - zB)^{-1}$ that

$$g_1 = \big(I + zB + \cdots + z^{\Delta-1}B^{\Delta-1} + z^\Delta B^\Delta (I - zB)^{-1}\big)\begin{pmatrix}1\\0\end{pmatrix}.$$

This shows that $g_1 \in \mathcal{L}$. As the $\mathfrak{w}_j$ are real valued, the definition of ϕ_0 gives

$$\phi_0(g_1) = -(0,1)\mathfrak{w}_1(s_-) - \ldots - z^{\Delta-1}(0,1)\mathfrak{w}_\Delta(s_-) + z^\Delta\Big((I - zB)^{-1}\begin{pmatrix}1\\0\end{pmatrix}, w_\Delta\Big)$$

Since conjugation commutes with B, the inner product in the last summand is equal to $((I - zB)^{-1})\mathfrak{w}_\Delta, (1,0)^T)$, and hence by the Green's identity applied with

$$\big(B(I - zB)^{-1}\mathfrak{w}_\Delta, (I - zB)^{-1}\mathfrak{w}_\Delta\big), \Big(\begin{pmatrix}1\\0\end{pmatrix}; 0\Big) \in T_{max},$$

equal to $(1,0)Jh(s_-) = (0,1)h(s_-)$, where h is as in Lemma 3.14. Note that $B(I - zB)^{-1} = (A(0) - z)^{-1}$.

By the first part of Corollary 3.15 and the uniqueness of the solution of the inital value problem (3.6), we have $(0,1)g_1(s_-) \neq 0$. However,

$$(0,1)g_2(s_-) = 1 + \sum_{k=1}^{\Delta-1} z^k (0,1)\mathfrak{w}_k(s_-) + z^{\Delta+1}(0,1)h(s_-) = (1,0)g_1(s_-) - z\phi_0(g_1),$$

and the assertion of the lemma follows also in the present case.

❑

We will often have to apply the results of the present section to a Hamiltonian which is regular at s_+ and singular at s_-. To this end let us state the following

3.18 Remark. If H is a Hamiltonian on $I = (s_-, s_+)$ which is regular at s_+, singular at s_-, satisfies (HS) and (I), similar considerations can be carried out. These, corresponding, results can be obtained either by repeating the above given proofs or by using an order-reversing reparameterization, cf. Lemma 2.6.

For future reference let us, as an example, state that in this case formula (3.5) will change to

$$(g, \mathfrak{w}_{n+\Delta})_{L^2(H)} = \sum_{k=0}^{\Delta-1} \lambda_k \mathfrak{w}_{n+1+k}(s_+)^* \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.8}$$

Finally note that the introduced concepts are compatible with reparameterizations:

3.19 Remark. Let H be a Hamiltonian on $I = (s_-, s_+)$ which is regular at s_-, singular at s_+ and satisfies (HS) and (I). Consider a reparameterization $\tilde{H} := (H \circ \varphi) \cdot \varphi'$ of H. Then $\tilde{H}$ is regular at $\tilde{s}_-$, singular at $\tilde{s}_+$ and satisfies (HS) and (I). We have (cf. Lemma 2.4)

$$\tilde{\mathfrak{w}}_j = A_\varphi \mathfrak{w}_j,\ j \in \mathbb{N} \cup \{0\},\ \Delta(\tilde{H}) = \Delta(H).$$

The operators $\tilde{B}$ and B are unitarily equivalent via A_φ.

4 Elementary indefinite Hamiltonians: The model

In this section we introduce the notion of elementary indefinite Hamiltonians. They model the simplest situation of indefinite canonical systems; in fact they can be viewed as "regular indefinite Hamiltonians with only one singularity". Later on, in Chapter 8, we will use them as building blocks for general indefinite canonical systems. Roughly speaking, an elementary indefinite Hamiltonian $\mathfrak{h}$ will consist of a Hamiltonian $H_\mathfrak{h}$ which has a singularity s and is of not too fast growth towards this singularity, some interface conditions at s, and a data part which is concentrated at s.

To an elementary indefinite Hamiltonian $\mathfrak{h}$ we will associate a model. This model consists of a Pontryagin space $\mathcal{P}(\mathfrak{h})$ together with a conjugate linear and anti-isometric involution $\bar{.} : \mathcal{P}(\mathfrak{h}) \to \mathcal{P}(\mathfrak{h})$, a linear map $\psi(\mathfrak{h}) : \mathcal{P}(\mathfrak{h}) \to \mathcal{M}(I)/_{=_H}$, a linear relation $T(\mathfrak{h}) \subseteq \mathcal{P}(\mathfrak{h}) \times \mathcal{P}(\mathfrak{h})$ and a linear relation $\Gamma(\mathfrak{h}) \subseteq T(\mathfrak{h}) \times (\mathbb{C}^2 \times \mathbb{C}^2)$. Moreover, we will distinguish a closed subspace $\mathcal{C}(\mathfrak{h})$ and a finite dimensional subspace X^δ of $\mathcal{P}(\mathfrak{h})$.

The space $\mathcal{P}(\mathfrak{h})$, the relation $T(\mathfrak{h})$ and the relation $\Gamma(\mathfrak{h})$ are the indefinite analogues of the space $L^2(H)$ and the relations $T_{max}(H)$ and $\Gamma(H)$ in the positive definite case. The map $\psi(\mathfrak{h})$ allows us to associate $\mathcal{P}(\mathfrak{h})$ with a space of functions. It formalizes the tight connection of $\mathcal{P}(\mathfrak{h})$ with the Hilbert space $L^2(H_\mathfrak{h})$ and clarifies the meaning of the distinguished space $\mathcal{C}(\mathfrak{h})$, in fact we will see that $\psi(\mathfrak{h})(\mathcal{C}\mathfrak{h}) = L^2(H_\mathfrak{h})$. The space X^δ models the part of the space $\mathcal{P}(\mathfrak{h})$ which is concentrated at the singularity.

4.1 Definition of elementary indefinite Hamiltonians

We will introduce three kinds of elementary indefinite Hamiltonians, which model different types of singularities. Each of them will involve the following data:

(i) Hamiltonians H_- and H_+ defined on intervals $I_- = (s_-, s)$ and $I_+ = (s, s_+)$, respectively, which satisfy

$$\begin{aligned} &H_\pm(x) \text{ is regular at } s_\pm \text{ and singular at } s, \\ &\text{(HS) and (I) hold for } H_-(x) \text{ and } H_+(-x), \\ &\Delta := \max\{\Delta(H_-(x)), \Delta(H_+(-x))\} < \infty. \end{aligned} \tag{4.1}$$

(ii) A number $\ddot{o} \in \mathbb{N} \cup \{0\}$, and numbers $b_j \in \mathbb{R},\ j = 1, \ldots, \ddot{o}+1$ with $b_1 \neq 0$ in the case that $\ddot{o} \geq 1$.

(iii) Numbers $d_0, \ldots, d_{2\Delta-1} \in \mathbb{R}$.

4.1 Definition. An elementary indefinite Hamiltonian $\mathfrak{h}$ of kind (A) ((B) or (C), respectively) is the collection of data $H_+, H_-, \ddot{o}, b_1, \ldots, b_{\ddot{o}+1}, d_0, \ldots, d_{2\Delta-1}$ as in (i)-(iii), which satisfies the respective of the following additional conditions:

(A) $\alpha_1^+(H_-) = s$ or $\alpha_1^-(H_+) = s$.

(B) $\alpha_1^+(H_+) = s$ and $\alpha_1^-(H_-) = s$, $d_1 = 0$, and $b_1 \neq 0$.

(C) $\alpha_1^+(H_+) = s$ and $\alpha_1^-(H_-) = s$, $d_1 = 0$, $d_0 < 0$, and $\ddot{o} = 0, b_1 = 0$.

4.2 Remark. We see that the kinds (A), (B) and (C) exclude each other.

- If $\mathfrak{h}$ is of kind (A) it may happen that one of $\alpha_1^+(H_+) = s$ and $\alpha_1^-(H_-) = s$ holds, however, not both of these equalities can hold at the same time since this would certainly violate our extra condition in (A).
- The condition "$\alpha_1^+(H_-) = s$ or $\alpha_1^-(H_+) = s$" in (A) says that at least one of H_- and H_+ does not end with an indivisible interval of infinite length towards the singularity s. On the other hand, the condition "$\alpha_1^+(H_+) = s$ and $\alpha_1^-(H_-) = s$" in (B) and (C) says that both of H_+ and H_- are just one indivisible interval of infinite length.
- The case that $\alpha_1^+(H_-) < s$, $\alpha_1^-(H_+) > s$, and either $\alpha_1^+(H_+) > s$ or $\alpha_1^-(H_-) < s$, is not covered in the above definition. However, this case will be reduced to a combination of the cases (A), (B) and (C) by pasting together the respective model spaces, cf. Sections 6, 8.
- If $\mathfrak{h}$ is of kind (B) or (C) we have $\Delta = 1$. If $\mathfrak{h}$ is of kind (A), Δ can take any value ≥ 1.

With the data (i) of Definition 4.1 we have associated spaces $\mathcal{M}(I_\pm)$, equivalence relations $=_{H_\pm}$, spaces $L^2(H_\pm)$, the correspondingly defined linear relations $T_{max,\pm}$, $T_{min,\pm}$, operators $B_\pm$ and elements $\mathfrak{w}_{k,\pm}$. Put $I := I_- \cup I_+$,

$$H(t) := \begin{cases} H_-(t) & , \ t \in (s_-, s) \\ H_+(t) & , \ t \in (s, s_+) \end{cases},$$

$$\mathcal{M}(I) := \mathcal{M}(I_-) \oplus \mathcal{M}(I_+),$$

$$(f_-;f_+) =_H (g_-;g_+) \ :\Longleftrightarrow \ \ f_- =_{H_-} g_-, f_+ =_{H_+} g_+ \,,$$
$$L^2(H) := L^2(H_-) \oplus L^2(H_+)\,,$$
$$T_{max} := T_{max,-} \oplus T_{max,+},\ T_{min} := T_{min,-} \oplus T_{min,+}\,,$$
$$B := B_- \oplus B_+\,.$$

We identify an element $(f_-;f_+)$ of $\mathcal{M}(I)$ with the function

$$f(t) := \begin{cases} f_-(t) & ,\ t \in (s_-,s) \\ f_+(t) & ,\ t \in (s,s_+) \end{cases} \quad .$$

In particular

$$\mathfrak{w}_k(t) = \begin{cases} \mathfrak{w}_{k,-}(t) & ,\ t \in (s_-,s) \\ \mathfrak{w}_{k,+}(t) & ,\ t \in (s,s_+) \end{cases} \quad .$$

Moreover, we set $\chi_- := \chi_{(s_-,s)}$, $\chi_+ := \chi_{(s,s_+)}$.

4.2 Definition of the model

We come to the actual definition of the model associated with an elementary indefinite Hamiltonian. This construction is quite elaborate and involved for Hamiltonians of kind (A). Therefore, we first settle the more elementary cases of an elementary indefinite Hamiltonian of kind (B) or (C). Before doing this we define in the case $\ddot{o} \geq 1$ numbers $c_1,\ldots,c_{\ddot{o}}$ uniquely by the equation

$$(c_1,\ldots,c_{\ddot{o}}) \begin{pmatrix} b_1 & \cdots & b_{\ddot{o}} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_1 \end{pmatrix} = (-1,0,\ldots,0)\,. \tag{4.2}$$

Note that $c_1 \neq 0$.

4.3 Definition. Assume that $\mathfrak{h}$ is an elementary indefinite Hamiltonian of kind (C). Let the model space $\mathcal{P}(\mathfrak{h})$ be a one-dimensional linear space spanned by an element denoted by p_0 and equipped with an inner product defined by

$$[p_0,p_0] := d_0\,.$$

Define $\psi : \mathcal{P}(\mathfrak{h}) \to \mathcal{M}(I)/_{=H}$ by $p_0 \mapsto \mathfrak{w}_0/_{=H}$ and linearity, and $\bar{\cdot} : \mathcal{P}(\mathfrak{h}) \to \mathcal{P}(\mathfrak{h})$ by $p_0 \mapsto p_0$ and conjugate linearity. The model relation is defined as $T(\mathfrak{h}) := \mathcal{P}(\mathfrak{h}) \times \mathcal{P}(\mathfrak{h})$. The linear relation $\Gamma(\mathfrak{h})$ is defined by applying the construction of Remark 2.11 with the linear map $\Lambda : T(\mathfrak{h}) \to \mathbb{C}^2$ given by

$$(p_0;0) \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ (0;p_0) \mapsto \begin{pmatrix} [p_0,p_0] \\ 0 \end{pmatrix}. \tag{4.3}$$

Explicitly, that means

$$\Gamma(\mathfrak{h}) := \big\{((f;g);(x_1;x_2)) \in T(\mathfrak{h}) \times (\mathbb{C}^2 \times \mathbb{C}^2) : \\ x_{1,1} - x_{2,1} = \Lambda(f;g)_1, x_{1,2} = x_{2,2} = \Lambda(f;g)_2\big\}\,.$$

Moreover, put $\mathcal{C}(\mathfrak{h}) := \{0\}$ and $X^\delta := \{0\}$.

4.4 Remark. If in the above definition we would choose $d_0 > 0$, we would end up with the space $L^2(H)$ for the Hamiltonian

$$H(t) := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ t \in (0, d_0).$$

This is seen by comparing the notions introduced in Definition 4.3 and Section 2.1.e. We can therefore think of an elementary indefinite Hamiltonian of kind (C) as an 'indivisible interval of negative length'.

4.5 Definition. Let $\mathfrak{h}$ be an elementary indefinite Hamiltonian of kind (B). Let $\mathcal{P}_c$ be a two-dimensional linear space spanned by elements denoted by δ_0 and p_0, equipped with an inner product defined by

$$[\delta_0, \delta_0] := 0, [p_0, p_0] := d_0, [\delta_0, p_0] := -1.$$

Let X^δ be an $\ddot{o}$-dimensional linear space spanned by elements $\delta_1, \dots, \delta_{\ddot{o}}$, equipped with an inner product defined by ($c_i := 0$ for $i \leq 0$)

$$[\delta_k, \delta_l] := c_{k+l-\ddot{o}}, \ k, l = 1, \dots, \ddot{o}.$$

Define the model space $\mathcal{P}(\mathfrak{h})$ as the direct and orthogonal sum

$$\mathcal{P}(\mathfrak{h}) := \mathcal{P}_c[\dot{+}]X^\delta.$$

Let $\psi : \mathcal{P}(\mathfrak{h}) \to \mathcal{M}(I)/_{=_H}$ be given by

$$p_0 \mapsto \mathfrak{w}_0/_{=_H}, \ \delta_j \mapsto 0, j = 0, 1, \dots, \ddot{o}$$

and linearity, and let $\bar{.} : \mathcal{P}(\mathfrak{h}) \to \mathcal{P}(\mathfrak{h})$ be defined by

$$p_0 \mapsto p_0, \ \delta_j \mapsto \delta_j, j = 0, 1, \dots, \ddot{o}$$

and conjugate linearity. The model relation $T(\mathfrak{h}) \subseteq \mathcal{P}(\mathfrak{h}) \times \mathcal{P}(\mathfrak{h})$ is defined as

$$\begin{aligned} T(\mathfrak{h}) := \operatorname{span}\big\{(0; \delta_0)\big\} + \operatorname{span}\big\{(p_0; 0)\big\} + \operatorname{span}\big\{(\delta_{k-1}; \delta_k) : \ k = 1, \dots, \ddot{o}\big\} + \\ + \operatorname{span}\big\{(\mathfrak{b}; p_0 + [p_0, p_0]\delta_0)\big\}, \end{aligned} \tag{4.4}$$

where $\mathfrak{b} := \sum_{l=1}^{\ddot{o}+1} b_l \delta_{1+\ddot{o}-l}$. By the definition of case (B) we always have $\mathfrak{b} \neq 0$. We define a linear map $\Lambda : T(\mathfrak{h}) \to \mathbb{C}^2$ by

$$\begin{gathered} (0; \delta_0) \mapsto \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \ (\delta_{k-1}; \delta_k) \mapsto \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ k = 1, \dots, \ddot{o}, \ (p_0; 0) \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ (\mathfrak{b}; p_0 + [p_0, p_0]\delta_0) \mapsto \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{gathered} \tag{4.5}$$

The relation $\Gamma(\mathfrak{h})$ is now again defined by applying the construction of Remark 2.11. Finally, put $\mathcal{C}(\mathfrak{h}) := \operatorname{span}\{\delta_0\}$.

4.6 Remark. Note that the choice of $\mathfrak{b}$ is made such that

$$[\mathfrak{b}, \delta_j] = \begin{cases} -1 & , \ j = 1 \\ 0 & , \ j = 0, 2, \dots, \ddot{o} \end{cases}.$$

This is seen by computation: Trivially $[\mathfrak{b},\delta_0]=0$. If $j\geq 1$ we have

$$[\mathfrak{b},\delta_j]=\Big[\sum_{l=1}^{\ddot{o}+1} b_l\delta_{1+\ddot{o}-l},\delta_j\Big]=\sum_{l=1}^{\ddot{o}} b_l c_{(1+\ddot{o}-l)+j-\ddot{o}}=$$

$$=b_1c_j+b_2c_{j-1}+\ldots+b_jc_1=\begin{cases}-1 & ,\ j=1\\ 0 & ,\ j=2,\ldots,\ddot{o}\end{cases}\quad .$$

Throughout the remainder of this subsection assume that $\mathfrak{h}$ is an elementary indefinite Hamiltonian of kind (A).

We denote by X_w and X^w, respectively, the spaces

$$X_w:=\operatorname{span}\{\mathfrak{w}_0,\ldots,\mathfrak{w}_{\Delta-1}\},\ X^w:=\operatorname{span}\{\mathfrak{w}_\Delta,\ldots,\mathfrak{w}_{2\Delta-1}\}\,,$$

where X_w is understood as a subspace of $\mathcal{M}(I)/_{=_H}$ and X^w as a subspace of $L^2(H)\subseteq\mathcal{M}(I)/_{=_H}$. Moreover,

$$X_1:=\operatorname{ran}B^\Delta\subseteq L^2(H)\,,$$

$$X_2:=\operatorname{span}\Big\{B^k\chi_-\begin{pmatrix}1\\0\end{pmatrix},B^k\chi_+\begin{pmatrix}1\\0\end{pmatrix}:\ k=0,\ldots,\Delta-1\Big\}\subseteq L^2(H)\,.$$

By Lemma 3.6 the set $\{\mathfrak{w}_0,\ldots,\mathfrak{w}_{\Delta-1}\}$ is linearly independent modulo $L^2(H)$. In particular,

$$(\operatorname{dom}T_{max}^\Delta+X^w)\cap X_w=\{0\}\,.$$

Since either $\alpha_1^+(H_-)=s$ or $\alpha_1^-(H_+)=s$, we know from Lemma 3.11, (i), that $\{\mathfrak{w}_\Delta,\ldots,\mathfrak{w}_{2\Delta-1}\}$ is linearly independent modulo $\operatorname{dom}T_{max}^\Delta$, and thus

$$\operatorname{dom}T_{max}^\Delta\cap X^w=\{0\}\,. \tag{4.6}$$

Finally, by Lemma 2.31, $\operatorname{dom}T_{max}^\Delta=X_1+X_2$. Let us put

$$d_{2\Delta+k}:=(\mathfrak{w}_\Delta,\mathfrak{w}_{\Delta+k}),\ k\in\mathbb{N}\cup\{0\}\,,$$

so that a number d_l is defined for all $l\in\mathbb{N}\cup\{0\}$. Note that, by the selfadjointness of B, these numbers d_l are all real.

We shall define linear functionals ϕ_j, $j=0,\ldots,\Delta-1$, on the linear space

$$X_L:=\operatorname{dom}T_{max}^\Delta+X^w=X_1+X_2+X^w\subseteq L^2(H)\,.$$

4.7 Proposition. *Let $j\in\{0,\ldots,\Delta-1\}$. By the requirements*

$$\phi_j:\begin{cases}B^\Delta f\mapsto(f,\mathfrak{w}_{\Delta+j}) & ,\ f\in L^2(H)\\ B^i\chi_-\binom{1}{0}\mapsto-\mathfrak{w}_{i+j+1}(s_-)^*\binom{0}{1} & ,\ i=0,\ldots,\Delta-1\\ B^i\chi_+\binom{1}{0}\mapsto\mathfrak{w}_{i+j+1}(s_+)^*\binom{0}{1} & ,\ i=0,\ldots,\Delta-1\\ \mathfrak{w}_i\mapsto d_{i+j} & ,\ i=\Delta,\ldots,2\Delta-1\end{cases}$$

a linear functional $\phi_j:X_L\to\mathbb{C}$ is well-defined. No nontrivial linear combination of $\phi_0,\ldots,\phi_{\Delta-1}$ is continuous with respect to the norm $\|.\|_{L^2(H)}$.

Proof. Assume that

$$B^{\Delta} f + \sum_{k=0}^{\Delta-1} \big(\lambda_{k,-} B^k \chi_- \binom{1}{0} + \lambda_{k,+} B^k \chi_+ \binom{1}{0} \big) + \sum_{k=\Delta}^{2\Delta-1} \mu_k \mathfrak{w}_k = 0 \,.$$

It follows from (4.6) that $\mu_k = 0$, $k = \Delta, \dots, 2\Delta - 1$, and hence

$$B_-^{\Delta} f = - \sum_{k=0}^{\Delta-1} \lambda_{k,-} B_-^k \chi_- \binom{1}{0},$$

and

$$B_+^{\Delta} f = - \sum_{k=0}^{\Delta-1} \lambda_{k,+} B_+^k \chi_+ \binom{1}{0} \,.$$

From Corollary 3.13 and Remark 3.18 it follows that

$$(f, \mathfrak{w}_{\Delta+j}) + \sum_{k=0}^{\Delta-1} \big(- \lambda_{k,-} \mathfrak{w}_{k+j+1}(s_-)^* \binom{0}{1} + \lambda_{k,+} \mathfrak{w}_{k+j+1}(s_+)^* \binom{0}{1} \big) = 0 \,,$$

and thus ϕ_j is well-defined.

Assume that

$$\phi = \sum_{j=0}^{\Delta-1} \lambda_j \phi_j : X_L \to \mathbb{C},$$

is continuous with respect to $\|.\|_{L^2(H)}$. Then there exists $g \in L^2(H)$ such that

$$\phi(h) = (h, g), \;\; h \in X_L \,. \tag{4.7}$$

Let $f \in L^2(H)$. By the definition of ϕ_j,

$$\phi(B^{\Delta} f) = \big(f, \sum_{j=0}^{\Delta-1} \overline{\lambda_j} \mathfrak{w}_{\Delta+j} \big) \,.$$

On the other hand, by (4.7),

$$\phi(B^{\Delta} f) = (B^{\Delta} f, g) = (f, B^{\Delta} g) \,.$$

We conclude that

$$\sum_{j=0}^{\Delta-1} \overline{\lambda_j} \mathfrak{w}_{\Delta+j} = B^{\Delta} g \in \operatorname{dom} T_{max}^{\Delta} \,,$$

and hence, by (4.6), $\lambda_j = 0$, $j = 0, \dots, \Delta - 1$.

❑

4.8 Remark. If $\alpha_1^+(H_+) = s$, then by Remark 3.8 we have

$$\phi_j (B^i \chi_+ \binom{1}{0}) = 0, \;\; i, j = 0, \dots, \Delta - 1 \,.$$

The analogous assertion holds for H_-.

Denote by $(\mathcal{C}(\mathfrak{h}), [.,.], \mathcal{O})$ the almost Pontryagin space completion of $(X_L, (.,.)_{L^2(H)}, (\phi_j)_{j=0}^{\Delta-1})$, cf. [KWW1]. That is the completion of X_L with respect to the norm defined by $\|.\|_\phi^2 = \|.\|_{L^2(H)}^2 + \sum_{j=0}^{\Delta-1} |\phi_j(.)|^2$.

Note that, by (2.12), X_L is a dense subspace of $(L^2(H), \|.\|_{L^2(H)})$. Recall from [KWW1] that the completion $(\mathcal{C}(\mathfrak{h}), [.,.], \mathcal{O})$ can be viewed as the space $(L^2(H) \oplus \mathbb{C}^\Delta, [.,.], \mathcal{T})$ where

$$[(x;\xi),(y;\eta)] = (x,y)_{L^2(H)} \,,$$

and $\mathcal{T}$ is the Hilbert space topology induced by the inner product $((x;\xi),(y;\eta)) = (x,y)_{L^2(H)} + \eta^*\xi$. The required embedding of X_L into this space is given by

$$x \mapsto \big(x; (\phi_j(x))_{j=0}^{\Delta-1}\big) \,. \tag{4.8}$$

We will think of $\mathcal{C}(\mathfrak{h})$ as an abstract object containing X_L as a dense subspace, and of $L^2(H) \oplus \mathbb{C}^\Delta$ as the realization of this completion which contains X_L via the embedding (4.8).

Moreover, recall from [KWW1] that $\dim \mathcal{C}(\mathfrak{h})^\circ = \Delta$, and $\mathcal{C}(\mathfrak{h})/\mathcal{C}(\mathfrak{h})^\circ$ is isometrically isomorphic to $L^2(H)$. In fact the canonical embedding $f \mapsto f + \mathcal{C}(\mathfrak{h})^\circ$ of X_L into $\mathcal{C}(\mathfrak{h})/\mathcal{C}(\mathfrak{h})^\circ$ extends to an isometric isomorphism of $L^2(H)$ onto $\mathcal{C}(\mathfrak{h})/\mathcal{C}(\mathfrak{h})^\circ$. By composition of its inverse with the canonical projection $\pi : \mathcal{C}(\mathfrak{h}) \to \mathcal{C}(\mathfrak{h})/\mathcal{C}(\mathfrak{h})^\circ$, we obtain an isometric and continuous surjection $\psi_0 : \mathcal{C}(\mathfrak{h}) \to L^2(H)$. Note that the topology of $L^2(H)$ coincides with the final topology with respect to ψ_0. Since $\dim \mathcal{C}(\mathfrak{h})^\circ < \infty$, it follows that ψ_0 maps closed subspaces onto closed subspaces. Note that in the realization $L^2(H) \oplus \mathbb{C}^\Delta$ of $\mathcal{C}(\mathfrak{h})$ the map ψ_0 is nothing else but the projection onto the first component. This follows from the continuity of ψ_0 since it holds by the form of (4.8) on the dense subset X_L.

Let $\mathcal{P}_c$ be a Pontryagin space which contains $\mathcal{C}(\mathfrak{h})$ as a closed subspace of codimension Δ. The map (4.8) can be extended to an isometric isomorphism

$$\iota : \mathcal{P}_c \to L^2(H)[\dot{+}]\big(\mathbb{C}^\Delta \dot{+} \mathbb{C}^\Delta\big) \tag{4.9}$$

where the inner product on $\mathbb{C}^\Delta \dot{+} \mathbb{C}^\Delta$ is defined such that each of the two copies of $\mathbb{C}^\Delta$ is neutral and that, if $\{e_0, \dots, e_{\Delta-1}\}$ and $\{f_0, \dots, f_{\Delta-1}\}$ denote the respective canonical bases,

$$[e_i, f_j] = \begin{cases} 0 & , \ i \neq j \\ 1 & , \ i = j \end{cases} \,.$$

Thereby $\mathcal{C}(\mathfrak{h})$ corresponds to $L^2(H)[\dot{+}]\operatorname{span}\{e_0, \dots, e_{\Delta-1}\}$ and $\mathcal{C}(\mathfrak{h})^\circ$ to $\operatorname{span}\{e_0, \dots, e_{\Delta-1}\}$.

We wish to extend the functionals ϕ_j to $\mathcal{P}_c$. First note that, since $\phi_j : \mathcal{C}(\mathfrak{h}) \to \mathbb{C}$ is continuous, there exist continuous extensions $\tilde{\phi}_j : \mathcal{P}_c \to \mathbb{C}$.

If $(\tilde{\phi}_j)_{j=0}^{\Delta-1}$ is an extension of $(\phi_j)_{j=0}^{\Delta-1}$, i.e. $\tilde{\phi}_j : \mathcal{P}_c \to \mathbb{C}$ are continuous and $\tilde{\phi}_j|_{\mathcal{C}} = \phi_j$, we assign to it a matrix $G[(\tilde{\phi}_j)_{j=0}^{\Delta-1}]$ as follows: Let $p_j \in \mathcal{P}_c$ be such that $\tilde{\phi}_j(x) = [x, p_j]$, $j = 0, \dots, \Delta-1$, and define

$$G[(\tilde{\phi}_j)_{j=0}^{\Delta-1}] := \big([p_i, p_j]\big)_{i,j=0}^{\Delta-1} \,.$$

We need the following elementary observation:

4.9 Lemma. *There is a bijective correspondence between the set of all extensions* $(\tilde{\phi}_j)_{j=0}^{\Delta-1}$ *of* $(\phi_j)_{j=0}^{\Delta-1}$ *and the set of all* $\Delta\times\Delta$*-matrices* M*. Thereby we have* $G[(\tilde{\phi}_j)_{j=0}^{\Delta-1}] = 2\operatorname{Re} M$.

Proof. Consider the space $\mathcal{P}_c$ in its realization (4.9) as $L^2(H)\oplus(\mathbb{C}^\Delta\dot{+}\mathbb{C}^\Delta)$ and again let $\{e_0,\dots,e_{\Delta-1}\}$ and $\{f_0,\dots,f_{\Delta-1}\}$ be the respective canonical bases. We have

$$\mathcal{C}(\mathfrak{h})^{[\perp]_{\mathcal{P}_c}} = \operatorname{span}\{e_0,\dots,e_{\Delta-1}\}\,.$$

Hence, the extensions $\tilde{\phi}_j$ of ϕ_j bijectively correspond to the elements $p_j = f_j + \sum_{k=0}^{\Delta-1}\lambda_{jk}e_k$, $\lambda_{jk}\in\mathbb{C}$. We compute

$$[p_i,p_j] = \Big[f_i + \sum_{k=0}^{\Delta-1}\lambda_{ik}e_k, f_j + \sum_{k=0}^{\Delta-1}\lambda_{jk}e_k\Big] = \overline{\lambda_{ji}} + \lambda_{ij}\,.$$

If we set $M = (\lambda_{ij})_{i,j=0}^{\Delta-1}$, we have $G[(\tilde{\phi}_j)_{j=0}^{\Delta-1}] = 2\operatorname{Re} M$.

❑

Note that the elements $p_j\in\mathcal{P}_c$ representing $\tilde{\phi}_j$ as $\tilde{\phi}_j = [.,p_j]$ are linearly independent, satisfy

$$\mathcal{C}(\mathfrak{h})\dot{+}\operatorname{span}\{p_0,\dots,p_{\Delta-1}\} = \mathcal{P}_c\,,$$

and $\operatorname{span}(\mathcal{C}(\mathfrak{h})^\circ\cup\{p_0,\dots,p_{\Delta-1}\})$ is nondegenerated. Moreover, if we set $\delta_j := -\iota^{-1}(e_j)$, $j=0,\dots,\Delta-1$, then $\{\delta_0,\dots,\delta_{\Delta-1}\}$ is a basis of $\mathcal{C}(\mathfrak{h})^\circ$ and by (4.8)

$$[\delta_k,p_j] = \begin{cases} 0 & ,\ k\neq j \\ -1 & ,\ k=j \end{cases}.$$

Let $(\tilde{\phi}_j)_{j=0}^{\Delta-1}$ be the extension of $(\phi_j)_{j=0}^{\Delta-1}$ which corresponds to $M = \frac{1}{2}(d_{i+k})_{i,k=0}^{\Delta-1}$, so that

$$G[(\tilde{\phi}_j)_{j=0}^{\Delta-1}] = (d_{i+k})_{i,k=0}^{\Delta-1}\,.$$

We are now ready for the definition of the model space $\mathcal{P}(\mathfrak{h})$, the map $\psi(\mathfrak{h})$, and conjugation.

4.10 Definition. Let $\mathfrak{h}$ be an elementary indefinite Hamiltonian of kind (A). Let X^δ be an $\ddot{o}$-dimensional linear space spanned by elements $\{\delta_\Delta,\dots,\delta_{\Delta+\ddot{o}-1}\}$, equipped with an inner product defined by ($c_i := 0$ for $i\leq 0$)

$$[\delta_k,\delta_l] := c_{k+l+2-2\Delta-\ddot{o}},\ \ k,l=\Delta,\dots,\Delta+\ddot{o}-1\,.$$

Define $\mathcal{P}(\mathfrak{h})$ as the direct and orthogonal sum

$$\mathcal{P}(\mathfrak{h}) := \mathcal{P}_c[\dot{+}]X^\delta\,.$$

The canonical extension of the isomorphism (4.9) to all of $\mathcal{P}(\mathfrak{h})$ by means of $\delta_j\mapsto\delta_j$, $j=\Delta,\dots\Delta+\ddot{o}-1$, yields an isomorphism (again be denoted by ι):

$$\iota:\mathcal{P}(\mathfrak{h})\to L^2(H)[\dot{+}]\big(\mathbb{C}^\Delta\dot{+}\mathbb{C}^\Delta\big)[\dot{+}]X^\delta\,. \tag{4.10}$$

Let $\psi(\mathfrak{h}) : \mathcal{P}(\mathfrak{h}) \to \mathcal{M}(I)/_{=_H}$ be the extension of ψ_0 to all of $\mathcal{P}(\mathfrak{h})$ which is given by linearity and the requirements that

$$p_j \mapsto \mathfrak{w}_j/_{=_H}, j = 0, \ldots, \Delta - 1, \ \delta_k \mapsto 0, k = \Delta, \ldots, \Delta + \ddot{o} - 1\,.$$

Moreover, let $\bar{.} : \mathcal{P}(\mathfrak{h}) \to \mathcal{P}(\mathfrak{h})$ be defined by

$$\iota \circ \bar{.} \circ \iota^{-1} : \begin{cases} L^2(H)[\dot{+}]\big(\mathbb{C}^\Delta \dot{+} \mathbb{C}^\Delta\big)[\dot{+}]X^\delta & \to \quad L^2(H)[\dot{+}]\big(\mathbb{C}^\Delta \dot{+} \mathbb{C}^\Delta\big)[\dot{+}]X^\delta \\ \big(x, \xi, \eta, \sum_{j=\Delta}^{\Delta+\ddot{o}-1} \alpha_j \delta_j\big) & \mapsto \quad \big(\overline{x}, \overline{\xi}, \overline{\eta}, \sum_{j=\Delta}^{\Delta+\ddot{o}-1} \overline{\alpha_j} \delta_j\big) \end{cases} \tag{4.11}$$

Note that in the realization (4.10) of $\mathcal{P}(\mathfrak{h})$ the map $\psi(\mathfrak{h})$ can be described as follows:

$$\psi(\mathfrak{h}) \circ \iota^{-1} : \begin{cases} L^2(H)[\dot{+}]\big(\mathbb{C}^\Delta \dot{+} \mathbb{C}^\Delta\big)[\dot{+}]X^\delta \to \mathcal{M}(I)/_{=_H} \\ \big(x, \xi, \eta, \sum_{j=\Delta}^{\Delta+\ddot{o}-1} \alpha_j \delta_j\big) \mapsto x + \sum_{j=0}^{\Delta-1} \eta_j \mathfrak{w}_j/_{=_H}, \ \eta = (\eta_j)_{j=0}^{\Delta-1} \end{cases} \tag{4.12}$$

Let us come to the definition of the model relation.

4.11 Definition. Let $\mathfrak{h}$ be an elementary indefinite Hamiltonian of kind (A). Then we define $T(\mathfrak{h}) \subseteq \mathcal{P}(\mathfrak{h}) \times \mathcal{P}(\mathfrak{h})$ as

$$T(\mathfrak{h}) := \overline{\operatorname{span}\big\{(Bh; h + [h, p_0]\delta_0) : \ h \in X_L\big\}} + \tag{4.13}$$

$$+ \operatorname{span}\Big\{(\chi_- \begin{pmatrix} 1 \\ 0 \end{pmatrix}; -\delta_0), (\chi_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \delta_0)\Big\} + \tag{4.14}$$

$$\begin{gathered} + \operatorname{span}\big\{(p_0; 0)\big\} + \\ + \operatorname{span}\big\{(p_k; p_{k-1} + [p_{k-1}, p_0]\delta_0) : \ k = 1, \ldots, \Delta - 1\big\} + \\ + \operatorname{span}\big\{(\mathfrak{w}_\Delta + \mathfrak{b}; p_{\Delta-1} + [p_{\Delta-1}, p_0]\delta_0)\big\} + \end{gathered} \tag{4.15}$$

$$+ \operatorname{span}\big\{(\delta_{k-1}; \delta_k) : \ k = \Delta, \ldots, \Delta + \ddot{o} - 1\big\}\,, \tag{4.16}$$

where $\mathfrak{b} := \sum_{l=1}^{\ddot{o}+1} b_l \delta_{\Delta+\ddot{o}-l}$.

In order to increase the readability we use '(4.13)' - '(4.16)' not just as references, but also as variables standing for the expressions in the corresponding lines. Note that, similar as in Remark 4.6, it is seen that the choice of $\mathfrak{b}$ is made to ensure that

$$[\mathfrak{b}, \delta_j] = \begin{cases} -1 & , \ j = \Delta \\ 0 & , \ j = 0, \ldots, \Delta - 1, \Delta + 1 \ldots, \Delta + \ddot{o} - 1 \end{cases}$$

In order to introduce the relation $\Gamma(\mathfrak{h})$ note that, if the map $\Psi := \psi \times \psi$ is restricted to $T(\mathfrak{h})$, it can be factorized. Let us define $\Psi^{ac} : T(\mathfrak{h}) \to \mathrm{AC}(I) \times \mathcal{M}(I)/_{=_H}$ by linearity,

$$(p_0; 0) \mapsto (\mathfrak{w}_0; 0), \quad (p_k; p_{k-1} + [p_{k-1}, p_0]\delta_0) \mapsto (\mathfrak{w}_k; \mathfrak{w}_{k-1}), \ k = 1, \ldots, \Delta - 1,$$

$$(\mathfrak{w}_\Delta + \mathfrak{b}; p_{\Delta-1} + [\mathfrak{p}_{\Delta-1}, p_0]\delta_0) \mapsto (\mathfrak{w}_\Delta; \mathfrak{w}_{\Delta-1}),$$

$$(\delta_{k-1};\delta_k)\mapsto(0;0),\ k=\Delta,\ldots,\Delta+\ddot{o}-1,$$

$$(\chi_-\begin{pmatrix}1\\0\end{pmatrix};-\delta_0)\mapsto(\chi_-\begin{pmatrix}1\\0\end{pmatrix};0),\quad(\chi_+\begin{pmatrix}1\\0\end{pmatrix};\delta_0)\mapsto(\chi_+\begin{pmatrix}1\\0\end{pmatrix};0),$$

and the following procedure: If $(f;g)$ belongs to (4.13), then $\Psi(f;g)\in B^{-1}$ and, hence, there exists a unique function $\hat{f}\in\mathrm{AC}(I)$ with $\hat{f}(s_-)_1=\hat{f}(s_+)_1=0$ such that $\hat{f}=_H f$ and $\hat{f}'=JHg$. Note that the condition $\hat{f}(s_-)_1=0$ has to be added only if $\alpha_1^-(H_-)=s$, and $\hat{f}(s_+)_1=0$ only if $\alpha_1^+(H_+)=s$. Set $\Psi^{ac}(f;g):=(\hat{f};g)$.

It is immediate from this definition that the following diagram commutes:

$$\begin{tikzcd}
T(\mathfrak{h}) \arrow[rr, "\Psi"] \arrow[dr, "\Psi^{ac}"'] & & (\mathcal{M}(I)/_{=_H})^2 \\
& \mathrm{AC}(I)\times\mathcal{M}(I)/_{=_H} \arrow[ur] &
\end{tikzcd}\tag{4.17}$$

where the right lower map is the canonical one. Moreover, we have

$$\Psi^{ac}(T(\mathfrak{h}))\subseteq\big\{(f;g)\in\mathrm{AC}(I)\times\mathcal{M}(I)/_{=_H}:\ f'=JHg\big\}.\tag{4.18}$$

4.12 Definition. Let $\mathfrak{h}$ be an elementary indefinite Hamiltonian of kind (A). Define $\Gamma(\mathfrak{h})\subseteq T(\mathfrak{h})\times(\mathbb{C}^2\times\mathbb{C}^2)$ by

$$\Gamma(\mathfrak{h}):=\big\{\big((f;g);(\Psi^{ac}(f;g)_1(s_-);\Psi^{ac}(f;g)_1(s_+))\big):\ (f;g)\in T(\mathfrak{h})\big\},$$

where $\Psi^{ac}(f;g)_1$ denotes the first component of $\Psi^{ac}(f;g)\in\mathrm{AC}(I)\times\mathcal{M}(I)/_{=_H}$.

4.3 Geometry of $\mathcal{P}$, T and Γ

In this subsection we will establish some basic properties of the introduced notions. First we discuss the geometry of $\mathcal{P}(\mathfrak{h})$. In order to shorten notation we will, if no confusion can occur, frequently drop the argument $\mathfrak{h}$ from $\mathcal{P}(\mathfrak{h})$, $\psi(\mathfrak{h})$, etc.

4.13 Proposition. *Let $\mathfrak{h}$ be an elementary indefinite Hamiltonian. Then*

(i) *$\mathcal{P}(\mathfrak{h})$ is a Pontryagin space with negative index*

$$\mathrm{ind}_-\,\mathcal{P}(\mathfrak{h})=\Delta+[\frac{\ddot{o}}{2}]+\begin{cases}1 & ,\ \ddot{o}\in\{1,3,5,\ldots\},c_1<0\\ 0 & ,\ \textit{otherwise}\end{cases}$$

(ii) *The map $\bar{.}$ is a conjugate linear and anti-isometric involution of $\mathcal{P}(\mathfrak{h})$. All elements δ_j and p_j are real, i.e. $\overline{\delta_j}=\delta_j,\overline{p_j}=p_j$.*

If $\mathfrak{h}$ is of kind (A) the involution $\bar{.}$ acts as complex conjugation on the subset X_L.

(iii) *The map ψ is real with respect to $\bar{.}$, i.e. $\psi(\overline{f})=\overline{\psi(f)}$, $f\in\mathcal{P}(\mathfrak{h})$. We have*

$$\ker\psi=\mathcal{C}^\circ+X^\delta,\ \ \psi^{-1}(L^2(H))=\mathcal{C}+\ker\psi.\tag{4.19}$$

Moreover, $\psi|_{\mathcal{C}}:(\mathcal{C},[.,.])\to(L^2(H),(.,.)_{L^2(H)})$ is isometric.

Proof. We establish the formula for the negative index of $\mathcal{P}(\mathfrak{h})$. In the case that $\mathfrak{h}$ is of kind (C), trivially $\operatorname{ind}_- \mathcal{P}(\mathfrak{h}) = 1$. Since in this case $\Delta = 1$ and $\ddot{o} = \frac{1}{2}$, the desired formula holds. Assume that $\mathfrak{h}$ is of kind (B). Then $\operatorname{ind}_- \mathcal{P}_c = 1 = \Delta$. Consider the space X^δ. With respect to the basis $\{\delta_1, \ldots, \delta_{\ddot{o}}\}$ the inner product is given by the Gram-matrix

$$\begin{pmatrix} 0 & \cdots & c_1 \\ \vdots & \iddots & \vdots \\ c_1 & \cdots & c_{\ddot{o}} \end{pmatrix}$$

Hence $\operatorname{ind}_- X^\delta = [\frac{\ddot{o}}{2}] + 1$ or $[\frac{\ddot{o}}{2}]$ depending whether $\ddot{o}$ is odd and $c_1 < 0$ or not. We see that also in this case the desired formula holds true. In order to compute $\operatorname{ind}_- \mathcal{P}(\mathfrak{h})$ in the case that $\mathfrak{h}$ is of kind (A), we consider the realization (4.10). The first summand is positive definite, the second one has Δ negative squares, and the last one has by the above argument negative index $[\frac{\ddot{o}}{2}] + 1$ or $[\frac{\ddot{o}}{2}]$ depending whether $\ddot{o}$ is odd and $c_1 < 0$ or not.

The fact that in any case $\bar{.}$ is a conjugate linear involution is obvious. Moreover, it follows immediately from the definitions that all elements δ_j are real. If $\mathfrak{h}$ is of kind (C) or (B), by definition all elements p_j are real. If $\mathfrak{h}$ is of kind (A) this follows on inspecting the construction in Lemma 4.9 from the fact that all numbers d_l are real.

Assume that $\mathfrak{h}$ is of kind (A). Denote by $\bar{.}^r$ the canonical conjugation in the realization (4.10) which was used for the definition of $\bar{.}$, i.e. $\iota \circ \bar{.} = \bar{.}^r \circ \iota$. Moreover, denote by $\bar{.}^c$ complex conjugation on X_L. By Lemma 3.9 and their definition, the functionals ϕ_j satisfy

$$\phi_j(\overline{h}^c) = \overline{\phi_j(h)},\ h \in X_L\,.$$

Hence, since ι extends (4.8), we have $\iota(\overline{h}^c) = \overline{\iota(h)}^r$ for all $h \in X_L$. We conclude that $\overline{h}^c = \overline{h}$.

The fact that ψ is real with respect to $\bar{.}$ follows in case of kinds (C),(B) immediately from the definitions, in case of kind (A) it is a consequence of the description of ψ in the realization (4.10).

In order to establish the relation (4.19) assume first that $\mathfrak{h}$ is of kind (C). Then $\ker\psi = \{0\}$ since $\mathfrak{w}_0/_{=H} \neq 0$, and we have $\mathcal{C} = \mathcal{C}^\circ = X^\delta = \{0\}$ and $L^2(H) = \{0\}$. Next assume that $\mathfrak{h}$ is of kind (B). Then, again since $\mathfrak{w}_0/_{=H} \neq 0$, we have $\ker\psi = \operatorname{span}\{\delta_0, \ldots, \delta_{\ddot{o}}\}$. Moreover, $\mathcal{C} = \mathcal{C}^\circ = \operatorname{span}\{\delta_0\}$, $X^\delta = \operatorname{span}\{\delta_1, \ldots, \delta_{\ddot{o}}\}$ and $L^2(H) = \{0\}$. Finally assume that $\mathfrak{h}$ is of kind (A). From the description of ψ in the realization (4.10) we obtain that $\ker\psi = \operatorname{span}\{\delta_0, \ldots, \delta_{\Delta-1}\} + X^\delta = \mathcal{C}^\circ + X^\delta$. Moreover, from the same source, we see that certainly $\psi(\mathcal{C} + X^\delta) = \psi(\mathcal{C} + \ker\psi) = L^2(H)$. Since the elements $\mathfrak{w}_0, \ldots, \mathfrak{w}_{\Delta-1}$ are linearly independent modulo $L^2(H)$, also $\psi^{-1}(L^2(H)) = \mathcal{C} + X^\delta$.

If $\mathfrak{h}$ is of kind (A) the isometry property of ψ holds by definition, since then $\psi|_{\mathcal{C}} = \psi_0$. If $\mathfrak{h}$ is of kind (B) or (C) this is trivial.

❑

Next we will take a closer look at the relation between $\mathcal{P}(\mathfrak{h})$ and $L^2(H)$. If $\mathfrak{h}$ is of kind (B) or (C) this question is not too intriguing, since then anyway $L^2(H) = \{0\}$. Thus let us consider the case that $\mathfrak{h}$ is of kind (A).

The map ψ projects $\mathcal{C}$ isometrically and continuously onto $L^2(H)$. On the other hand it is in general not possible to embed $L^2(H)$ into $\mathcal{P}(\mathfrak{h})$ in a proper way. Note in this place that the embedding which is immediate from the realization (4.10) is not properly adopted to the situation, since it does not take care of the second components of (4.8). However, if we only consider elements of $L^2(H)$ whose support stays away from the singularity s, we can find an appropriate embedding.

For $s^- \in [s_-, s)$ and $s^+ \in (s, s_+]$ which are not inner points of indivisible intervals put

$$J := (s_-, s^-) \cup (s^+, s_+)\,, \tag{4.20}$$
$$L^2(H|_J) := L^2(H_-|_{(s_-,s^-)}) \oplus L^2(H_+|_{(s^+,s_+)})\,.$$

4.14 Proposition. *Let $\mathfrak{h}$ be an elementary indefinite Hamiltonian of kind (A). For every set J of the form (4.20) there exists an isometric and bicontinuous embedding ι_J of $L^2(H|_J)$ into $\mathcal{C} \subseteq \mathcal{P}(\mathfrak{h})$. It satisfies*

$$[\iota_J x, y]_{\mathcal{P}(\mathfrak{h})} = \int_J (\psi y)^* H x,\ \ x \in L^2(H|_J), y \in \mathcal{P}(\mathfrak{h})\,, \tag{4.21}$$

and is compatible with conjugation in the sense that

$$\iota_J(\overline{x}) = \overline{\iota_J(x)},\ \ x \in L^2(H|_J)\,. \tag{4.22}$$

Moreover, whenever J and J' are of the form (4.20) and $J \subseteq J'$, then the following diagram commutes:

$$\begin{tikzcd}
 & \mathcal{C} \arrow[rr, "\psi"] & & L^2(H) \\
L^2(H|_J) \arrow[ur, "\iota_J"] \arrow[rr, "\subseteq"'] & & L^2(H|_{J'}) \arrow[ul, "\iota_{J'}"'] \arrow[ur, "\subseteq"'] &
\end{tikzcd}$$

Proof. For any set J of the form (4.20) denote by ρ_J the restriction map

$$\rho_J : \begin{cases} L^2(H) & \to & L^2(H|_J) \\ f & \mapsto & f|_J \end{cases}\,.$$

This map is surjective and a contraction, in particular, continuous. The map $\psi|_{\mathcal{C}}$ maps $\mathcal{C}$ continuously onto $L^2(H)$, hence $\rho_J\psi|_{\mathcal{C}}$ maps $\mathcal{C}$ continuously onto $L^2(H|_J)$. Since $\psi(p_j) = \mathfrak{w}_j/_{=H} \in \mathrm{AC}(I)/_{=H}$, $j = 0, \dots, \Delta - 1$, and $\psi(\delta_j) = 0$, $j = \Delta, \dots, \Delta + \ddot{o} - 1$, since

$$\mathcal{P}(\mathfrak{h}) = \mathcal{C} \dot{+} \operatorname{span}\{p_j : j = 0, \dots, \Delta - 1\} \dot{+} \operatorname{span}\{\delta_j : j = \Delta, \dots, \Delta + \ddot{o} - 1\}\,,$$

and since $\mathcal{C}$ is a closed subspace of $\mathcal{P}(\mathfrak{h})$, it follows that $\rho_J\psi$ maps $\mathcal{P}(\mathfrak{h})$ continuously onto $L^2(H|_J)$. Define

$$\iota_J := (\rho_J\psi)^* : L^2(H|_J) \to \mathcal{P}(\mathfrak{h})\,.$$

Then ι_J is continuous, injective, and satisfies (4.21). By the closed range theorem $\operatorname{ran}\iota_J$ is a closed subspace of $\mathcal{P}(\mathfrak{h})$. Hence, by the open mapping theorem, ι_J is bicontinuous. Since $\mathcal{C}^\circ + X^\delta = \ker\psi \subseteq \ker\rho_J\psi$, it follows that

$$\operatorname{ran}\iota_J \subseteq (\mathcal{C}^\circ + X^\delta)^\perp = \mathcal{C}\,.$$

We compute ι_J in the realization (4.9) of $\mathcal{C}$. Let $x \in L^2(H|_J)$ and write

$$\iota_J x = \big(\hat{x}; (\xi_j)_{j=0}^{\Delta-1}\big) .$$

By (4.21) we have

$$\xi_j = [\iota_J x, p_j]_{\mathcal{P}(\mathfrak{h})} = \int_J \mathfrak{w}_j^* H x, \;\; j = 0, \ldots, \Delta - 1 .$$

Moreover, since in the realization (4.10) the action of ψ on $\mathcal{C}$ is just projecting onto the first component, we have for every $y \in L^2(H)$

$$(\hat{x}, y)_{L^2(H)} = [(\hat{x}; (\xi_j)_{j=0}^{\Delta-1}), (y; 0, \ldots, 0)]_{\mathcal{P}(\mathfrak{h})} = \int_J y^* H x = (x, y)_{L^2(H)} .$$

It follows that $\hat{x} = x$. Alltogether we see that

$$\iota_J x = \big(x; (\int_J \mathfrak{w}_j^* H x)_{j=0}^{\Delta-1}\big) .$$

We conclude that $(\psi \circ \iota_J)(x) = x$, $x \in L^2(H|_J)$, that ι_J is isometric, and that

$$\iota_J x = \iota_{J'} x, \;\; x \in L^2(H|_J), \;\; J \subseteq J' .$$

Moreover, the relation (4.22) follows, since the functions $\mathfrak{w}_j$ are real-valued.

❏

¿From the fact that $\psi \circ \iota_J$ acts as the identity on $L^2(H|_J)$ and that ι_J is an embedding, we obtain the following

4.15 Corollary. *We have* $\psi(\iota_J(L^2(H|_J))) \subseteq L^2(H|_J)$ *and the map* $\psi|_{\iota_J(L^2(H|_J))} : \iota_J(L^2(H|_J)) \to L^2(H|_J)$ *is the inverse of* ι_J.

Note that if, say, $s^- := \alpha_1^+(H|_{(s_-,s)}) < s$, then $L^2(H|_{(s_-,s)}) = L^2(H|_{(s_-,s^-)})$. Hence, in this case, $L^2(H|_{(s_-,s)})$ is properly embedded in $\mathcal{P}(\mathfrak{h})$.

Our next task is to establish some properties of the linear relation $T(\mathfrak{h})$. For this we need another lemma.

4.16 Lemma. *Let* $\mathfrak{h}$ *be of kind (A). Then we have* $\phi_j \circ B = \phi_{j+1}$, $j = 0, \ldots, \Delta - 2$, *and* $\phi_{\Delta-1} \circ B = (., \mathfrak{w}_\Delta)$.

Proof. First of all note that $B(X_L) \subseteq X_L$. We shall formulate the proof of the present assertion for the case that $\alpha_1^-(H_-) < s$ and $\alpha_1^+(H_+) > s$. If in one of these relations equality holds, then in those steps marked by a star one has to employ Remark 3.8 instead of the mentioned argument.

Let $j \in \{0, \ldots, \Delta - 2\}$. We have

$$\phi_j(B(B^\Delta f)) = \phi_j(B^\Delta(Bf)) = (Bf, \mathfrak{w}_{\Delta+j}) =$$

$$= (f, B\mathfrak{w}_{\Delta+j}) = (f, \mathfrak{w}_{\Delta+j+1}) = \phi_{j+1}(B^\Delta f) .$$

Let $i \in \{0, \ldots, \Delta - 2\}$, then

$$\phi_j(B(B^i \chi_\pm \begin{pmatrix} 1 \\ 0 \end{pmatrix})) = \phi_j(B^{i+1} \chi_\pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \mathfrak{w}_{i+j+2}(s_\pm)^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \phi_{j+1}(B^i \chi_\pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}) .$$

Moreover, an application of (2.10) with $(\chi_-\binom{1}{0};0),(\mathfrak{w}_{\Delta+j+1,-};\mathfrak{w}_{\Delta+j,-})\in T_{max,-}$ yields

$$\phi_j(B(B^{\Delta-1}\chi_-\begin{pmatrix}1\\0\end{pmatrix}))=\phi_j(B^{\Delta}\chi_-\begin{pmatrix}1\\0\end{pmatrix})=(\chi_-\begin{pmatrix}1\\0\end{pmatrix},\mathfrak{w}_{\Delta+j,-})\overset{*}{=}$$

$$=-\mathfrak{w}_{\Delta+j+1,-}(s_-)^*J\begin{pmatrix}1\\0\end{pmatrix}=\phi_{j+1}(B^{\Delta-1}\chi_-\begin{pmatrix}1\\0\end{pmatrix}).$$

The case of $\chi_+\binom{1}{0}$ is treated similarly (cf. Remark 3.18).

Let $i\in\{\Delta,\dots,2\Delta-2\}$, then

$$\phi_j(B\mathfrak{w}_i)=\phi_j(\mathfrak{w}_{i+1})=d_{j+i+1}=\phi_{j+1}(\mathfrak{w}_i).$$

Moreover,

$$\phi_j(B\mathfrak{w}_{2\Delta-1})=\phi_j(B^{\Delta}\mathfrak{w}_{\Delta})=(\mathfrak{w}_{\Delta},\mathfrak{w}_{\Delta+j})=d_{2\Delta+j}=\phi_{j+1}(\mathfrak{w}_{2\Delta-1}).$$

Next consider the functional $\phi_{\Delta-1}$. We have

$$\phi_{\Delta-1}(B(B^{\Delta}f))=\phi_{\Delta-1}(B^{\Delta}(Bf))=(Bf,\mathfrak{w}_{2\Delta-1})=(B^{\Delta}f,\mathfrak{w}_{\Delta}).$$

Let $i\in\{0,\dots,\Delta-2\}$, then

$$\phi_{\Delta-1}(B(B^i\chi_\pm\begin{pmatrix}1\\0\end{pmatrix}))=\pm\mathfrak{w}_{i+\Delta+1}(s_\pm)^*\begin{pmatrix}0\\1\end{pmatrix}.$$

On the other hand, by (2.10),

$$(B^i\chi_\pm\begin{pmatrix}1\\0\end{pmatrix},\mathfrak{w}_{\Delta})=(\chi_\pm\begin{pmatrix}1\\0\end{pmatrix},\mathfrak{w}_{\Delta+i})\overset{*}{=}\pm\mathfrak{w}_{\Delta+i+1}(s_\pm)^*\begin{pmatrix}0\\1\end{pmatrix}.$$

Moreover,

$$\phi_{\Delta-1}(B(B^{\Delta-1}\chi_\pm\begin{pmatrix}1\\0\end{pmatrix}))=\phi_{\Delta-1}(B^{\Delta}\chi_\pm\begin{pmatrix}1\\0\end{pmatrix})=(\chi_\pm\begin{pmatrix}1\\0\end{pmatrix},\mathfrak{w}_{2\Delta-1})=$$

$$=(B^{\Delta-1}\chi_\pm\begin{pmatrix}1\\0\end{pmatrix},\mathfrak{w}_{\Delta}).$$

Let $i\in\{\Delta,\dots,2\Delta-2\}$, then

$$\phi_{\Delta-1}(B\mathfrak{w}_i)=\phi_{\Delta-1}(\mathfrak{w}_{i+1})=d_{\Delta+i}=(\mathfrak{w}_i,\mathfrak{w}_{\Delta}).$$

Finally,

$$\phi_{\Delta-1}(B\mathfrak{w}_{2\Delta-1})=\phi_{\Delta-1}(B^{\Delta}\mathfrak{w}_{\Delta})=(\mathfrak{w}_{\Delta},\mathfrak{w}_{2\Delta-1})=(\mathfrak{w}_{2\Delta-1},\mathfrak{w}_{\Delta}).$$

❑

4.17 Proposition. *Let $\mathfrak{h}$ be an elementary indefinite Hamiltonian. Then*

(i) $T(\mathfrak{h})$ is closed.

(*ii*) *$T(\mathfrak{h})$ is real with respect to $\bar{\cdot}$, i.e. we have $(f;g) \in T(\mathfrak{h})$ if and only if $(\overline{f};\overline{g}) \in T(\mathfrak{h})$.*

(*iii*) *Put $\Psi := \psi \times \psi : \mathcal{P}(\mathfrak{h})^2 \to (\mathcal{M}(I)/_{=_H})^2$. Then $\Psi(T(\mathfrak{h}) \cap \mathcal{C}^2) = T_{max}(H)$.*

(*iv*) *We have $(\delta_{k-1};\delta_k) \in T(\mathfrak{h})$, $k \in \{1,\dots,\Delta-1\}$.*

(*v*) *We have $\overline{\operatorname{dom} T(\mathfrak{h})} = \mathcal{P}(\mathfrak{h})$.*

Two somewhat technical properties are:

(*vi*) *All sums in the definition of $T(\mathfrak{h})$ are direct sums. The set of generators written in* (4.14)-(4.16) *or* (4.4)*, respectively, is linearly independent.*

(*vii*) *Assume that $\mathfrak{h}$ is of kind (A). Then $T \cap \mathcal{C}^2 = (4.13) + (4.14)$ and $\Psi(\,(4.13)\,) = B^{-1}$.*

Proof.
`ad`(*i*): The cases of kind (C), (B) are trivial since all spaces involved are finite-dimensional. If $\mathfrak{h}$ is of kind (A) the fact that $T(\mathfrak{h})$ is closed follows since the span of (4.14)-(4.16) is finite dimensional.

`ad`(*ii*): If $\mathfrak{h}$ is of kind (C), this is trivial. If $\mathfrak{h}$ is of kind (B) it is enough to note that each of the generators written in (4.4) is real. Assume that $\mathfrak{h}$ is of kind (A). Each of the generators written in (4.14)-(4.16) is real. Moreover, since $\bar{\cdot}$ acts as complex conjugation on X_L, with h also $\overline{h}$ belongs to X_L and we have $B\overline{h} = \overline{Bh}$. Thus

$$\big(\overline{Bh};\overline{h+\phi_0(h)\delta_0}\big) = \big(B\overline{h};\overline{h}+\phi_0(\overline{h})\delta_0\big) \in T(\mathfrak{h})\,.$$

By the continuity of $\bar{\cdot}$, the assertion follows.

`ad`(*iii*): If $\mathfrak{h}$ is of kind (C) this is trivial, if it is of kind (B) it follows from $\mathcal{C} = \operatorname{span}\{\delta_0\} \subseteq \ker\psi$.

Assume that $\mathfrak{h}$ is of kind (A). First of all note that the product topology on $(L^2(H))^2$ is the final topology with respect to the map $\Psi|_{\mathcal{C}^2}$. Since the kernel of $\Psi|_{\mathcal{C}^2}$ is finite-dimensional, $\Psi|_{\mathcal{C}^2}$ maps closed subspaces onto closed subspaces. We have

$$\Psi(Bh; h + [h,p_0]\delta_0) = (Bh;h) \in B^{-1}\,.$$

As $\Psi|_{\mathcal{C}^2}$ is continuous $\Psi(\,(4.13)\,) \subseteq B^{-1}$. Since X_L is dense in $L^2(H)$, cf. (2.12), and $B \times \mathrm{id}$ is a continuous map of $L^2(H)$ onto B^{-1}, it follows that in this inclusion equality holds.

To see the desired assertion it suffices to recall that

$$B + \operatorname{span}\{(\chi_-\begin{pmatrix}1\\0\end{pmatrix};0), (\chi_+\begin{pmatrix}1\\0\end{pmatrix};0)\} = T_{max}$$

and

$$\big(\chi_\pm\begin{pmatrix}1\\0\end{pmatrix};0\big) = \Psi\big(\chi_\pm\begin{pmatrix}1\\0\end{pmatrix};\pm\delta_0\big)\,.$$

Note that on the way we also proved the second assertion of (*vii*).

$\texttt{ad}\,(iv)$: This assertion is only nonvoid if $\Delta > 1$. Thus we may assume without loss of generality that $\mathfrak{h}$ is of kind (A). Fix $k \in \{1, \ldots, \Delta - 1\}$ and choose $h_n \in X_L$ such that $h_n \to \delta_k$ in $\mathcal{C}$. It follows that

$$(h_n, h_n) \to 0,\ \phi_j(h_n) \to \begin{cases} -1 & , \ j = k \\ 0 & , \ j \neq k \end{cases}.$$

With h_n also $h_n + [h_n, p_0]\delta_0 \to \delta_k$. Since $\|h_n\|_{L^2(H)} \to 0$, also $\|Bh_n\|_{L^2(H)} \to 0$. By Lemma 4.16 we have

$$\phi_j(Bh_n) \to \begin{cases} -1 & , \ j = k-1 \\ 0 & , \ j \neq k-1 \end{cases} \quad , \ j = 0, \ldots, \Delta - 1.$$

Hence, $(Bh_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in $(X_L, \|.\|_\phi)$ and thus convergent to some element $x \in \mathcal{C}$ in the norm of $\mathcal{P}(\mathfrak{h})$. Since $[x - \delta_{k-1}, x - \delta_{k-1}] = 0$ and $\phi_j(x) = \phi_j(\delta_{k-1})$, it follows that $x = \delta_{k-1}$. We see that $(\delta_{k-1}; \delta_k) \in (4.13)$.

$\texttt{ad}\,(v)$: If $\mathfrak{h}$ is of kind (C) or (B), we have by the definition of $T(\mathfrak{h})$ that $\operatorname{dom} T(\mathfrak{h}) = \mathcal{P}(\mathfrak{h})$. Hence assume that $\mathfrak{h}$ is of kind (A).

Let $f \in \mathcal{P}(\mathfrak{h})$, $f \perp \operatorname{dom} T(\mathfrak{h})$, be given. Our first aim is to show that $f \perp \mathcal{C}^\circ$. If $\ddot{o} > 0$, then we have

$$\delta_k \in \operatorname{dom} T(\mathfrak{h}),\ k = 0, \ldots \Delta + \ddot{o} - 2\,.$$

Since $\Delta + \ddot{o} - 2 \geq \Delta - 1$, we see that $f \perp \mathcal{C}^\circ$.

Consider the case that $\ddot{o} = 0$. Then we have

$$\delta_k \in \operatorname{dom} T(\mathfrak{h}),\ k = 0, \ldots, \Delta - 2\,.$$

We show that $\delta_{\Delta-1} \in \overline{\operatorname{dom} T(\mathfrak{h})}$. To this end consider the functional (see Proposition 4.7)

$$\phi_{\Delta-1}|_{\operatorname{ran} B^{\Delta+1}} : \operatorname{ran} B^{\Delta+1} \to \mathbb{C},$$

and assume that it were continuous in the norm of $L^2(H)$. Then there exists $g \in L^2(H)$ with

$$\phi_{\Delta-1}(B^{\Delta+1}x) = (B^{\Delta+1}x, g),\ x \in L^2(H)\,.$$

We obtain from Lemma 4.16 and from the selfadjointness of B that

$$(x, \mathfrak{w}_{2\Delta}) = (x, B^\Delta \mathfrak{w}_\Delta) = (B^\Delta x, \mathfrak{w}_\Delta) = \phi_{\Delta-1}(B^{\Delta+1}x) =$$

$$= (B^{\Delta+1}x, g) = (x, B^{\Delta+1}g),\ x \in L^2(H)\,.$$

Hence $\mathfrak{w}_{2\Delta} = B^{\Delta+1}g \in \operatorname{ran} B^{\Delta+1}$, a contradiction to Lemma 3.11, since $\mathfrak{h}$ satisfies (A).

We conclude that there exists a sequence $h_n \in L^2(H)$ such that

$$B^{\Delta+1}h_n \overset{L^2(H)}{\to} 0, \quad \phi_{\Delta-1}(B^{\Delta+1}h_n) \to -1\,.$$

Put $y_n := B(B^\Delta h_n)$. Since $B^\Delta h_n \in X_L$, we have $y_n \in \operatorname{dom} T(\mathfrak{h})$ and $[y_n, y_n] = \|y_n\|_{L^2(H)} \to 0$, $\phi_{\Delta-1}(y_n) \to -1 = \phi_{\Delta-1}(\delta_{\Delta-1})$. By adding appropriate linear combinations of $\delta_0,\dots,\delta_{\Delta-2}$ to y_n, we can produce a sequence $z_n\in\mathcal{C}\cap\operatorname{dom}T(\mathfrak{h})$ with

$$[z_n,z_n]\to 0,\ \phi_0(z_n),\dots,\phi_{\Delta-2}(z_n)=0,\ \phi_{\Delta-1}(z_n)\to -1\,.$$

It follows that $[z_n,x]\to[\delta_{\Delta-1},x]$ for every $x\in\mathcal{P}(\mathfrak{h})$, i.e. that $z_n\xrightarrow{w}\delta_{\Delta-1}$. We conclude that $\delta_{\Delta-1}\in\overline{\operatorname{dom}T}$. Thus $f\perp\mathcal{C}^\circ$.

Since, in any case $f\perp\mathcal{C}^\circ$, we can write $f=f_1+\sum_{k=\Delta}^{\Delta+\ddot{o}-1}\alpha_k\delta_k$ with some $f_1\in\mathcal{C}$. If $\ddot{o}>0$, then we get from $f\perp\delta_\Delta,\dots,\delta_{\Delta+\ddot{o}-2}$ that $\alpha_{\Delta+\ddot{o}-1}=\dots=\alpha_{\Delta+1}=0$. Moreover, $f\perp\mathfrak{w}_\Delta+\mathfrak{b}$, i.e.

$$0=[f_1+\alpha_\Delta\delta_\Delta,\mathfrak{w}_\Delta+\mathfrak{b}]=[f_1,\mathfrak{w}_\Delta]+\alpha_\Delta\underbrace{[\delta_\Delta,\mathfrak{b}]}_{=-1},$$

and it follows that $\alpha_\Delta=[f_1,\mathfrak{w}_\Delta]$.

We have $f\perp\operatorname{dom}(T\cap\mathcal{C}^2)$. Thus also f_1 has this property and hence

$$\psi f_1\perp\operatorname{dom}\Psi(T\cap\mathcal{C}^2)=\operatorname{dom}T_{max}\,.$$

Thus $\psi f_1=0$, and it follows that $f_1\in\ker\psi\cap\mathcal{C}=\mathcal{C}^\circ$. It readily follows that $\alpha_\Delta=0$. Moreover, we know that $f\perp p_0,\dots,p_{\Delta-1}$, and this now implies that $f=0$.

`ad` (vii): Clearly, the span of (4.13) and (4.14) is contained in $\mathcal{C}^2$. Assume that

$$\alpha_0(p_0;0)+\sum_{k=1}^{\Delta-1}\alpha_k(p_k;p_{k-1}+[p_{k-1},p_0]\delta_0)+$$

$$+\alpha_\Delta(\mathfrak{w}_\Delta+\mathfrak{b};p_{\Delta-1}+[p_{\Delta-1},p_0]\delta_0)+\sum_{k=\Delta}^{\Delta+\ddot{o}-1}\beta_k(\delta_{k-1};\delta_k)\in\mathcal{C}^2\,.$$

Looking at the second component, we obtain $\beta_k=0$, $k=\Delta,\dots,\Delta+\ddot{o}-1$, and $\alpha_k=0$, $k=1,\dots,\Delta$. The first component now reads as $\alpha_0p_0\in\mathcal{C}$, and hence $\alpha_0=0$. We see that the set of generators written in (4.15) and (4.16) is linearly independent modulo $\mathcal{C}^2$. In particular, $T\cap\mathcal{C}^2=(4.13)+(4.14)$.

The second assertion was already proved, cf. proof of (iii).

`ad` (vi): In case of kind (C) there is nothing to prove. In case of kind (B) assume that

$$\alpha(0;\delta_0)+\beta(p_0;0)+\sum_{k=1}^{\ddot{o}}\lambda_k(\delta_{k-1};\delta_k)+\gamma(\mathfrak{b};p_0+[p_0,p_0]\delta_0)=0\,.$$

Looking at the second component of this equation yields $\gamma=0$ and then $\alpha=0$. Now looking at the first component gives us $\beta=0$ and $\lambda_k=0$ for all k.

Assume that $\mathfrak{h}$ is of kind (A). Note that at least one of the functions $\chi_-\binom{1}{0}$ and $\chi_+\binom{1}{0}$ is not equal to 0 in $L^2(H_\pm)$. In fact, these functions are zero only if $\alpha_1^-(H_-)=s$ or $\alpha_1^+(H_+)=s$, respectively, and these equalities cannot hold at the same time.

We conclude that the pairs $(\chi_\pm\binom{1}{0};\pm\delta_0)$ are linearly independent. Together with the facts already established in the proof of (vi), we conclude that the set of generators written in (4.14)-(4.16) is linearly independent.

It remains to show that $(4.13) \cap (4.14) = \{0\}$. Assume that

$$(\beta_- \chi_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta_+ \chi_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix}; (\beta_+ - \beta_-)\delta_0),$$

belongs to (4.13). Then

$$\Psi(\beta_- \chi_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta_+ \chi_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix}; (\beta_+ - \beta_-)\delta_0) \in B^{-1}.$$

If $\alpha_1^-(H_-) < s$ and $\alpha_1^+(H_+) > s$, then $\chi_\pm \binom{1}{0} \neq_H 0$ and, hence we get $\beta_- = \beta_+ = 0$. If, say $\alpha_1^-(H_-) = s$, then $\alpha_1^+(H_+) > s$, and we conclude $\beta_+ = 0$. Since $[g, p_0] = 0$ for all $(f; g)$ which belong to (4.13), we also get $\beta_- = 0$.

❑

Finally we state some elementary properties of Ψ^{ac} and $\Gamma(\mathfrak{h})$.

4.18 Lemma. *If $\alpha_1^-(H_-) < s$ and $\alpha_1^+(H_+) > s$, then*

$$\ker \Psi^{ac} = \ker \Psi|_{T(\mathfrak{h})} \quad \big(= T(\mathfrak{h}) \cap (\mathcal{C}^\circ + X^\delta)^2 \big) . \tag{4.23}$$

If $\alpha_1^+(H_+) = s$, then

$$\ker \Psi^{ac} \dot{+} \operatorname{span} (\chi_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix}; +\delta_0) = \ker \Psi|_{T(\mathfrak{h})} . \tag{4.24}$$

The similar assertion holds with '$-$' instead of '$+$'.

Proof. The respective inclusions '$\subseteq$' are immediate from (4.17), and, in the case $\alpha_1^\pm(H_\pm) = s$, from $\chi_\pm(1\ 0)^T =_H 0$. To establish the converse inclusion assume that $(f; g) \in \ker \Psi|_{T(\mathfrak{h})}$. This means that $\Psi^{ac}(f; g) = (\tilde{f}; 0)$ where $\tilde{f} =_H 0$. Write

$$(f; g) = (f_1; g_1) + \alpha_0(p_0; 0) + \sum_{k=1}^{\Delta-1} \alpha_k (p_k; p_{k-1} + [p_{k-1}, p_0]\delta_0) +$$

$$+ \alpha_\Delta(\mathfrak{w}_\Delta + \mathfrak{b}; p_{\Delta-1} + [p_{\Delta-1}, p_0]\delta_0) + \sum_{j=\Delta}^{\Delta+\ddot{o}-1} \beta_j (\delta_{j-1}; \delta_j)$$

where $(f_1; g_1) \in T(\mathfrak{h}) \cap \mathcal{C}^2$. Since the last sum is anyway contained in $\ker \Psi^{ac}$, we can assume without loss of generality that $\beta_j = 0$, $j = \Delta, \ldots, \Delta + \ddot{o} - 1$. Hence

$$0 =_H \tilde{f} = \hat{f}_1 + \sum_{k=0}^{\Delta} \alpha_k \mathfrak{w}_k ,$$

where $\hat{f}_1 \in \mathrm{AC}(I)$ is the unique representant of f_1 with $\hat{f}_1' = JH\psi(g_1)$. It follows from Lemma 3.6 and Lemma 3.11 that $\alpha_k = 0$, $k = 0, \ldots, \Delta$. Hence $\tilde{f} = \hat{f}_1 =_H 0$. If $\alpha_1^-(H_-) < s$ and $\alpha_1^+(H_+) > s$, then the continuity of $\tilde{f}$ implies $\tilde{f} = 0$.

If $\alpha_1^-(H_-) = s$, then $\alpha_1^+(H_+) > s$, and we conclude from $\tilde{f} =_H 0$ that $\tilde{f}_{I_+} = 0$. From the definition of Ψ^{ac} we see that $\tilde{f}|_{I_-}$ is collinear with $\chi_-(1\ 0)^T$. Thus

$$(f; g) - \alpha(\chi_-(1\ 0)^T; -\delta_0) \in \ker \Psi^{ac},$$

for a properly chosen $\alpha\in\mathbb{C}$. The case $\alpha_1^+(H_+)=s$ is treated similarly.

❑

4.19 Lemma. *The relation* $\Gamma(\mathfrak{h})$ *is closed and* $\operatorname{dom}\Gamma(\mathfrak{h})=T(\mathfrak{h})$. *We have*

$$\operatorname{mul}\Gamma(\mathfrak{h})=\begin{cases}\{0\} & ,\ \mathfrak{h} \text{ of kind (A)}\\ \operatorname{span}\{(\binom{1}{0};\binom{1}{0})\} & ,\ \mathfrak{h} \text{ of kind (B),(C)}\end{cases}$$

Moreover, $\Gamma(\mathfrak{h})$ *is compatible with the involution* $\bar{.}$ *in the sense of* (2.5).

If $\mathfrak{h}$ *is of kind (A), the map* $\Gamma(\mathfrak{h})$ *is surjective.*

Proof. To shorten notation we write $\mathcal{P}(\mathfrak{h})=\mathcal{P}$, $\Gamma(\mathfrak{h})=\Gamma$ and $T(\mathfrak{h})=T$.

The assertions that $\operatorname{dom}\Gamma=T$ and that $\operatorname{mul}\Gamma$ has the desired form are immediate from the respective definitions.

We show that Γ is closed. If $\mathfrak{h}$ is of kind (B) or (C) this is trivial by finite-dimensionality, hence assume that $\mathfrak{h}$ is of kind (A). In this case we can consider Γ as the map

$$\Gamma:(f;g)\mapsto\big(\Psi^{ac}(f;g)_1(s_-);\Psi^{ac}(f;g)_1(s_+)\big)\,,$$

where $\Psi^{ac}(f;g)_1$ denotes the first component of $\Psi^{ac}(f;g)\in\mathrm{AC}(I)\times\mathcal{M}(I)/_{=_H}$. We show that this map Γ is continuous.

If $\alpha_1^-(H_-)<s$ and $\alpha_1^+(H_+)>s$, the restriction $\Psi|_{T\cap\mathcal{C}^2}$ is by its definition compatible with boundary values: We have

$$\Gamma(f;g)=\Gamma_{L^2(H)}\circ\Psi(f;g),\ (f;g)\in T\cap\mathcal{C}^2\,. \tag{4.25}$$

This relation implies that Γ is continuous. In fact, because $\Gamma_{L^2(H)}$ is continuous on $L^2(H)$, Γ is continuous as a mapping of $T\cap\mathcal{C}^2$ into $\mathbb{C}^2\times\mathbb{C}^2$. However, $T\cap\mathcal{C}^2$ is a closed subspace of T with finite codimension. Thus Γ is continuous as a mapping from T into $\mathbb{C}^2\times\mathbb{C}^2$.

If $\alpha_1^\pm(H_\pm)=s$, then let R be the span of (4.13) and $(\chi_\mp(1\ 0)^T;\mp\delta_0)$. The restriction $\Gamma|_R$ coincides with $\Gamma_{L^2(H)}\circ\Psi|_R$. As in the previous step we see also here that Γ is continuous as a mapping from T into $\mathbb{C}^2\times\mathbb{C}^2$.

We show that Γ is compatible with conjugation. Assume that $\mathfrak{h}$ is of kind (A), then we have

$$\Gamma(\overline{f};\overline{g})=\overline{\Gamma(f;g)},\ (f;g)\in T\,.$$

This follows since $\Gamma_{L^2(H)}$ has this property and since the generators written in (4.14), (4.15) and (4.16) are real and mapped by Γ to real vectors. In the cases that $\mathfrak{h}$ is of kind (B) or (C), the desired relation is immediate from the fact that Λ maps real elements of T to real vectors.

We come to the proof of the last assertion. From the definition of $\Gamma(\mathfrak{h})$ it is clear that

$$(\begin{pmatrix}0\\1\end{pmatrix};\begin{pmatrix}0\\1\end{pmatrix}),(\begin{pmatrix}1\\0\end{pmatrix};0),(0;\begin{pmatrix}1\\0\end{pmatrix})\in\operatorname{ran}\Gamma(\mathfrak{h})\,.$$

Since $\mathfrak{h}$ is of kind (A), one of (s_-,s) and (s,s_+) is not indivisible. Let us assume that (s_-,s) is not indivisible, then by the definition of $\Gamma(\mathfrak{h})$ its first component can be obtained as

$$\Gamma(\mathfrak{h})_1(f;g)=\Gamma(H_-)\big((\chi_-\times\chi_-)\Psi^{ac}(f;g)\big),\ (f;g)\in T(\mathfrak{h})\cap\mathcal{C}^2\,.$$

By Proposition 4.17, (iii), $(\chi_- \times \chi_-)\Psi^{ac}$ maps $T(\mathfrak{h}) \cap \mathcal{C}^2$ onto $T_{max}(H_-)$. Since $\Gamma(H_-)$ is surjective, it follows from Proposition 4.17, (vii), that there exist $(f_-; g_-) \in B_-^{-1}$ such that $\Gamma(H_-)(f_-; g_-) = \binom{0}{1}$. Set $f_+ = g_+ = 0$, then $(f_- + f_+; g_- + g_+) \in B^{-1}$. From the definition of $\Gamma(\mathfrak{h})$ we obtain

$$\Gamma(\mathfrak{h})(Bh; h + [h, p_0]\delta_0) = (\begin{pmatrix} 0 \\ 1 \end{pmatrix}; 0) .$$

❑

5 Operator theory of elementary indefinite systems

In this section we build up the operator theory for the model associated with an elementary indefinite Hamiltonian. It is our aim to prove the following result, which is the indefinite analogue of Theorem 2.18 for elementary indefinite Hamiltonians.

5.1 Theorem. *Let $\mathfrak{h}$ be an elementary indefinite Hamiltonian. Then $(\mathcal{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h}))$ is a boundary triplet of defect 2 which has the property (E). The adjoint*

$$S(\mathfrak{h}) := T(\mathfrak{h})^*$$

is a completely nonselfadjoint symmetric operator. Its defect index is $(2, 2)$ if $\mathfrak{h}$ is of kind (A), and $(1, 1)$ otherwise. It satisfies the condition (CR) and $r(S(\mathfrak{h})) = \mathbb{C}$.

5.1 The abstract Green's identity

The proof of the following proposition requires some elementary but tedious work.

5.2 Proposition. *Let $\mathfrak{h}$ be an elementary indefinite Hamiltonian. Then $\Gamma(\mathfrak{h})$ is a boundary relation for $T(\mathfrak{h})$, i.e. satisfies the identity (2.6).*

Proof. We write $\mathcal{P} := \mathcal{P}(\mathfrak{h})$, $\Gamma := \Gamma(\mathfrak{h})$ and $T := T(\mathfrak{h})$. First let us settle the case that $\mathfrak{h}$ is of kind (C). By Remark 2.11 we have to show that the map Λ defined by (4.3) satisfies (2.8). If $(f; g) = (h; k) = (0; p_0)$ or $(f; g) = (h; k) = (p_0; 0)$, both sides of (2.8) vanish. Let $(f; g) = (0; p_0)$ and $(h; k) = (p_0; 0)$. Then

$$[g, h] - [f, k] = [p_0, p_0]$$

and

$$\Lambda(h; k)^* J \Lambda(f; g) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* J \begin{pmatrix} [p_0, p_0] \\ 0 \end{pmatrix} = [p_0, p_0] .$$

Assume next that $\mathfrak{h}$ is of kind (B). We have to show that the map Λ defined by (4.5) satisfies (2.8).

Case $(f; g) = (0; \delta_0)$:

$\cdot)\,(h;k) = (0;\delta_0)$, $(h;k) = (\delta_{j-1};\delta_j), j = 1,\dots,\ddot{o}$, $(h;k) = (\mathfrak{b};p_0 + [p_0,p_0]\delta_0)$: Both sides of (2.8) vanish.

$\cdot)(h;k) = (p_0;0)$: We have $[g,h] - [f,k] = -1$ and

$$\Lambda(h;k)^* J\Lambda(f;g) = \begin{pmatrix}0\\1\end{pmatrix}^* J\begin{pmatrix}-1\\0\end{pmatrix} = -1\,.$$

Case $(f;g) = (\delta_{j-1};\delta_j), j = 1,\dots,\ddot{o}$: The right hand side of (2.8) is equal to zero.

$\cdot)(h;k) = (\delta_{l-1};\delta_l), l = 1,\dots,\ddot{o}$: By the definition of the inner product on X^δ we have

$$[g,h] - [f,k] = c_{j+(l-1)-\ddot{o}} - c_{(j-1)+l-\ddot{o}} = 0\,.$$

$\cdot)(h;k) = (p_0;0)$: We have $[g,h] - [f,k] = [\delta_l,p_0] = 0$.

$\cdot)(h;k) = (\mathfrak{b};p_0 + [p_0,p_0]\delta_0)$: If $j = 1$, then by the choice of $\mathfrak{b}$

$$[g,h] - [f,k] = [\delta_1,\mathfrak{b}] - [\delta_0,p_0 + d_0\delta_0] = (-1) - (-1) = 0\,.$$

If $j > 1$, we have $[g,h] - [f,k] = 0$.

Case $(f;g) = (p_0;0)$: In both cases $(h;k) = (p_0;0)$ and $(h;k) = (\mathfrak{b};p_0 + [p_0,p_0]\delta_0)$ both sides of (2.8) vanish.

Case $(f;g) = (\mathfrak{b};p_0 + [p_0,p_0]\delta_0), (h;k) = (\mathfrak{b};p_0 + [p_0,p_0]\delta_0)$: The right side of (2.8) is zero. On the other hand

$$[g,h] - [f,k] = (-b_{\ddot{o}+1}) - (-b_{\ddot{o}+1}) = 0\,.$$

In all cases the relation (2.8) holds.

Let us now assume that $\mathfrak{h}$ is of kind (A). Then $\operatorname{mul}\Gamma = \{0\}$, and thus (2.6) writes as

$$\begin{gathered}[g,h] - [f,k] = \Gamma(h;k)^* \begin{pmatrix}J & 0\\0 & -J\end{pmatrix}\Gamma(f;g) = -h^* Jf|_{s_-}^{s_+},\\ (f;g),(h;k) \in T\,,\end{gathered} \tag{5.1}$$

where we use the notational convention $\Gamma(f;g) = (f(s_-);f(s_+))$, $\Gamma(h;k) = (h(s_-);h(s_+))$. In the case that $\alpha_1^-(H_-) < s$ and $\alpha_1^+(H_+) > s$, if $(f;g)$ and $(h;k)$ both belong to $T \cap \mathcal{C}^2$, the relation (5.1) follows from (4.25) and (2.10).

In the case that $\alpha_1^\pm(H_\pm) = s$ let R be the span of (4.13) and $(\chi_\mp(1\ 0)^T;\mp\delta_0)$. For $(f;g),(h;k) \in R$ the relation (5.1) follows from (2.10) and the fact $\Gamma|_R = \Gamma_{L^2(H)} \circ \Psi^{ac}$. If $(f;g) \in R$ and $(h;k) = \lambda(\chi_\mp(1\ 0)^T;\pm\delta_0)$, $\lambda \in \mathbb{C}$, then (5.1) holds true, because both the left and the right hand side of the equality sign are zero. The same reasoning verifies (5.1) if both $(f;g)$ and $(h;k)$ are scalar multiples of $(\chi_\mp(1\ 0)^T;\pm\delta_0)$.

We have proved in any case that (5.1) holds true for $(f;g),(h;k) \in T\cap\mathcal{C}^2$. By the continuity of Γ it is enough to check the remaining cases for the generators written in (4.13)-(4.16).

Case $(f;g) = (Bx;x + [x,p_0]\delta_0); x \in X_L$:

$\cdot)(h;k) = (\delta_{j-1};\delta_j), j = \Delta,\dots,\Delta + \ddot{o} - 1$: Both sides of (5.1) are equal to zero.

$\cdot)\,(h;k) = (p_0;0)$: $[f,k] - [g,h] = -[x + [x,p_0]\delta_0,p_0] = 0$ and $h^*Jf|_{s_-}^{s_+} = \mathfrak{w}_0^* JBx|_{s_-}^{s_+} = 0$ since $\mathfrak{w}_0(s_\pm),(Bx)(s_\pm) \in \operatorname{span}\{(0\ 1)^T\}$.

$\cdot)(h;k) = (p_j; p_{j-1} + [p_{j-1}, p_0]\delta_0)$, $j = 1, \dots, \Delta - 1$: Write x as

$$x = B^\Delta y + \sum_{l=0}^{\Delta-1} \left(\lambda_{l,-} B^l \chi_- \begin{pmatrix}1\\0\end{pmatrix} + \lambda_{l,+} B^l \chi_+ \begin{pmatrix}1\\0\end{pmatrix}\right) + \sum_{i=\Delta}^{2\Delta-1} \mu_i \mathfrak{w}_i \,.$$

Then

$$Bx = B^\Delta By + \sum_{l=0}^{\Delta-1} \left(\lambda_{l,-} B^{l+1} \chi_- \begin{pmatrix}1\\0\end{pmatrix} + \lambda_{l,+} B^{l+1} \chi_+ \begin{pmatrix}1\\0\end{pmatrix}\right) + \sum_{i=\Delta}^{2\Delta-1} \mu_i \mathfrak{w}_{i+1} \,.$$

We compute (see Proposition 4.7 and Lemma 4.16)

$$[f,k] = [Bx, p_{j-1}] = (By, \mathfrak{w}_{\Delta+j-1})_{L^2(H)} +$$

$$\sum_{l=0}^{\Delta-1} \left(\lambda_{l,-}[B^{l+1}\chi_- \begin{pmatrix}1\\0\end{pmatrix}, p_{j-1}] + \lambda_{l,+}[B^{l+1}\chi_+ \begin{pmatrix}1\\0\end{pmatrix}, p_{j-1}]\right) + \sum_{i=\Delta}^{2\Delta-1} \mu_i [\mathfrak{w}_{i+1}, p_{j-1}] =$$

$$= (y, \mathfrak{w}_{\Delta+j})_{L^2(H)} + \sum_{l=0}^{\Delta-1} \left(\lambda_{l,-}[B^l\chi_- \begin{pmatrix}1\\0\end{pmatrix}, p_j] + \lambda_{l,+}[B^l\chi_+ \begin{pmatrix}1\\0\end{pmatrix}, p_j]\right) +$$

$$+ \sum_{i=\Delta}^{2\Delta-1} \mu_i [\mathfrak{w}_i, p_j] = [g,h],$$

so that the left hand side of (5.1) is equal to 0. On the other hand $h^* Jf|_{s_-}^{s_+} = \mathfrak{w}_j^* JBx|_{s_-}^{s_+} = 0$ since $\mathfrak{w}_j(s_\pm), (Bx)(s_\pm) \in \operatorname{span}\{(0\ 1)^T\}$.
$\cdot)$ $(h;k) = (\mathfrak{w}_\Delta + \mathfrak{b}; p_{\Delta-1} + [p_{\Delta-1}, p_0]\delta_0)$: Since $[g, \mathfrak{b}] = 0$ and $\Gamma(h;k) = (\mathfrak{w}_\Delta(s_-); \mathfrak{w}_\Delta(s_+))^T$ the same computation as in the previous case gives the desired result.

Case $(f;g) = (\chi_- \binom{1}{0}; -\delta_0)$:
$\cdot)(h;k) = (\delta_{j-1}; \delta_j)$, $j = 1, \dots, \Delta + \ddot{o} - 1$:

$$[f,k] - [g,h] = 0 = h^* Jf|_{s_-}^{s_+} \,.$$

$\cdot)(h;k) = (p_0; 0)$: $[f,k] - [g,h] = -[-\delta_0, p_0] = -1$,

$$h(s_+)^* Jf(s_+) - h(s_-)^* Jf(s_-) = -\mathfrak{w}_0(s_-)^* J \chi_- \begin{pmatrix}1\\0\end{pmatrix}(s_-) = -1 \,.$$

$\cdot)(h;k) = (p_j; p_{j-1} + [p_{j-1}, p_0]\delta_0)$, $j = 1, \dots, \Delta - 1$:

$$[f,k] - [g,h] = [\chi_- \begin{pmatrix}1\\0\end{pmatrix}, p_{j-1}] = -\mathfrak{w}_j(s_-)^* \begin{pmatrix}0\\1\end{pmatrix},$$

$$h^* Jf|_{s_-}^{s_+} = -\mathfrak{w}_j(s_-)^* J \begin{pmatrix}1\\0\end{pmatrix} = -\mathfrak{w}_j(s_-)^* \begin{pmatrix}0\\1\end{pmatrix}.$$

$\cdot)(h;k) = (\mathfrak{w}_\Delta + \mathfrak{b}; p_{\Delta-1} + [p_{\Delta-1}, p_0]\delta_0)$:

$$[f,k] - [g,h] = [\chi_- \begin{pmatrix}1\\0\end{pmatrix}, p_{\Delta-1}] = -\mathfrak{w}_\Delta(s_-)^* \begin{pmatrix}0\\1\end{pmatrix},$$

$$h^* Jf|_{s_-}^{s_+} = -\mathfrak{w}_\Delta(s_-)^* J \begin{pmatrix}1\\0\end{pmatrix} = -\mathfrak{w}_\Delta(s_-)^* \begin{pmatrix}0\\1\end{pmatrix}.$$

Case $(f;g) = (\chi_+\binom{1}{0};\delta_0)$: Treated similar as the case $(f;g) = (\chi_-\binom{1}{0};-\delta_0)$.

Case $(f;g) = (\delta_{j-1};\delta_j), j = \Delta,\dots,\Delta+ö-1$:

$\cdot)(h;k) = (\delta_{l-1};\delta_l), l = \Delta,\dots,\Delta+ö-1$: By the definition of the inner product on X^δ we have

$$[f,k] - [g,h] = [\delta_{j-1},\delta_l] - [\delta_j,\delta_{l-1}] = 0\,,$$

and by the definition of Γ surely $h^*Jf|_{s_-}^{s_+} = 0$.

$\cdot)(h;k) = (p_0;0)$: Both sides of (5.1) are equal to 0.

$\cdot)(h;k) = (p_l;p_{l-1} + [p_{l-1},p_0]\delta_0)$, $l = 1,\dots,\Delta-1$: We have $[f,k] = [g,h] = 0$, and again both sides of (5.1) are equal to 0.

$\cdot)(h;k) = (\mathfrak{w}_\Delta + \mathfrak{b};p_{\Delta-1} + [p_{\Delta-1},p_0]\delta_0)$: The right hand side of (5.1) is clearly equal to 0. We compute

$$[f,k] = [\delta_{j-1},p_{\Delta-1}] = \begin{cases} 0 & ,\ j\neq\Delta \\ -1 & ,\ j=\Delta \end{cases}\quad,$$

and from the definition of $\mathfrak{b}$

$$[g,h] = [\delta_j,\mathfrak{b}] = \sum_{i=1}^{ö+1} b_i[\delta_j,\delta_{\Delta+ö-i}] = \sum_{i=1}^{ö+1} b_i c_{-\Delta-i+j+2} =$$

$$= \sum_{i=1}^{j+1-\Delta} b_i c_{j+2-\Delta-i} = \begin{cases} 0 & ,\ j<\Delta \\ -1 & ,j=\Delta \\ 0 & ,\ j>\Delta \end{cases}\quad.$$

Case $(f;g) = (p_0;0)$:

$\cdot)(h;k) = (p_0;0)$: The left hand side of (5.1) is clearly equal to 0, the right hand side is equal to 0 since $\mathfrak{w}_0(s_\pm)\in\operatorname{span}\{(0\ 1)^T\}$.

$\cdot)(h;k) = (p_j;p_{j-1} + [p_{j-1},p_0]\delta_0)$, $j = 1,\dots,\Delta-1$:

$$[f,k] = [g,h] = [p_0,p_{j-1} + [p_{j-1},p_0]\delta_0] = 0\,,$$

$$h^*Jf|_{s_-}^{s_+} = \mathfrak{w}_j^*J\mathfrak{w}_0|_{s_-}^{s_+} = 0\,,$$

since $\mathfrak{w}_j(s_\pm),\mathfrak{w}_0(s_\pm)\in\operatorname{span}\{(0\ 1)^T\}$.

$\cdot)(h;k) = (\mathfrak{w}_\Delta + \mathfrak{b};p_{\Delta-1} + [p_{\Delta-1},p_0]\delta_0)$: The same argument as in the previous case applies.

Case $(f;g) = (p_j;p_{j-1} + [p_{j-1},p_0]\delta_0), j = 1,\dots,\Delta-1$:

$\cdot)(h;k) = (p_l;p_{l-1} + [p_{l-1},p_0]\delta_0)$, $l = 1,\dots,\Delta-1$:

$$[f,k] = [p_j,p_{l-1} + [p_{l-1},p_0]\delta_0] = d_{j+l-1} =$$

$$= [p_{j-1} + [p_{j-1},p_0]\delta_0,p_l] = [g,h]\,,$$

$$h^*Jf|_{s_-}^{s_+} = \mathfrak{w}_l^*J\mathfrak{w}_j|_{s_-}^{s_+} = 0\,.$$

$\cdot)(h;k) = (\mathfrak{w}_\Delta + \mathfrak{b}; p_{\Delta-1} + [p_{\Delta-1}, p_0]\delta_0)$: As $[p_{j-1}, \mathfrak{b}] = 0$,

$$[f,k] = [p_j, p_{\Delta-1} + [p_{\Delta-1}, p_0]\delta_0] = d_{j+\Delta-1} =$$
$$= [p_{j-1} + [p_{j-1}, p_0]\delta_0, \mathfrak{w}_\Delta + \mathfrak{b}] = [g,h]\,,$$
$$h^* J f|_{s_-}^{s_+} = \mathfrak{w}_\Delta^* J \mathfrak{w}_j|_{s_-}^{s_+} = 0\,.$$

Case $(f;g) = (\mathfrak{w}_\Delta + \mathfrak{b}; p_{\Delta-1} + [p_{\Delta-1}, p_0]\delta_0)$:
$\cdot)(h;k) = (\mathfrak{w}_\Delta + \mathfrak{b}; p_{\Delta-1} + [p_{\Delta-1}, p_0]\delta_0)$: Since $[p_{\Delta-1}, \mathfrak{b}] = 0$,

$$[f,k] = [\mathfrak{w}_\Delta + \mathfrak{b}, p_{\Delta-1} + [p_{\Delta-1}, p_0]\delta_0] = d_{2\Delta-1} - b_{\ddot{o}+1} =$$
$$= [p_{\Delta-1} + [p_{\Delta-1}, p_0]\delta_0, \mathfrak{w}_\Delta + \mathfrak{b}] = [g,h]\,,$$
$$h^* J f|_{s_-}^{s_+} = \mathfrak{w}_\Delta^* J \mathfrak{w}_\Delta|_{s_-}^{s_+} = 0\,.$$

In all cases the relation (5.1) holds.

❑

5.3 Corollary. *If $\alpha_1^-(H_-) < s$ and $\alpha_1^+(H_+) > s$, then we have*

$$T(\mathfrak{h}) \cap (\mathcal{C}^\circ + X^\delta)^2 = \operatorname{span}\big\{(\delta_{k-1};\delta_k) :\, k = 1,\dots,\Delta + \ddot{o} - 1\big\}\,. \tag{5.2}$$

If $\alpha_1^+(H_+) = s$ or $\alpha_1^-(H_-) = s$, then

$$\begin{gathered} T(\mathfrak{h}) \cap (\mathcal{C}^\circ + X^\delta)^2 = \\ \operatorname{span}\big\{(\delta_{k-1};\delta_k) :\, k = 1,\dots,\Delta + \ddot{o} - 1\big\} \dot{+} \operatorname{span}\{(0;\delta_0)\}\,. \end{gathered} \tag{5.3}$$

In any case the kernel of Ψ^{ac} coincides with the space on the right hand side of (5.2).

Proof. The inclusions '$\supseteq$' in (5.2) and (5.3) follow from Proposition 4.17 and the fact that, if e.g. $\alpha_1^+(H_+) = s$ we have $\chi_+(1,0)^T =_H 0$ and thus $(0;\delta_0) \in T(\mathfrak{h})$. Conversely, let $(f;g) \in T(\mathfrak{h}) \cap (\mathcal{C}^\circ + X^\delta)^2$ be given. In order to prove that $(f;g)$ belongs to the right side of (5.2) or (5.3), respectively, we can by the already established inclusion '$\supseteq$' assume without loss of generality that

$$(f;g) = \sum_{k=0}^{\Delta+\ddot{o}-1} \alpha_k(\delta_k;0) + \alpha(0;\delta_0)\,,$$

where $\alpha = 0$ if $\alpha_1^+(H_+) = s$ or $\alpha_1^-(H_-) = s$. Of course, it is sufficient to show that any element of this form which belongs to T must be equal to 0.

We will use Proposition 5.2. To this end we need to know $\Gamma(f;g)$. If $\alpha_1^-(H_-) < s$ and $\alpha_1^+(H_+) > s$, Lemma 4.18 implies $\Gamma(f;g) = 0$. Assume that e.g. $\alpha_1^+(H_+) = s$. By Lemma 4.18 we can write

$$(f;g) = (f_1;g_1) + \beta(\chi_+\begin{pmatrix}1\\0\end{pmatrix};\delta_0)$$

with some $(f_1; g_1) \in \ker \Psi^{ac}$ and $\beta \in \mathbb{C}$. We apply Proposition 5.2 with $(f; g)$ and $(p_0; 0)$. Since $g = 0$, the left side of (5.1) is zero. However, $\Gamma(f_1; g_1) = 0$ and thus $\Gamma(f; g) = \beta(0; (1, 0)^T)$. Hence the right side of (5.1) equals $-\beta$. We conclude that $\beta = 0$ and hence $\Gamma(f; g) = 0$.

In case that $\alpha_1^+(H_+) = s$ or $\alpha_1^-(H_-) = s$, we have $\alpha = 0$. If $\alpha_1^-(H_-) < s$ and $\alpha_1^+(H_+) > s$ this follows from an application of (5.1) with $(f; g)$ and $(p_0; 0)$. Using $(p_k; p_{k-1} + [p_{k-1}, p_0]\delta_0)$, $k = 1, \ldots, \Delta - 1$, we obtain that $\alpha_k = 0$ for $k = 0, \ldots, \Delta - 2$. With $(\mathfrak{w}_\Delta + \mathfrak{b}; p_{\Delta-1} + [p_{\Delta-1}, p_0]\delta_0)$ it follows that $\alpha_{\Delta-1} = 0$. Finally, using $(\delta_{k-1}; \delta_k)$, $k = \Delta, \ldots, \Delta + \ddot{o} - 1$ and the fact that $c_1 \neq 0$, we obtain that also $\alpha_k = 0$, $k = \Delta, \ldots, \Delta + \ddot{o} - 1$.

The assertion that

$$\ker \Psi^{ac} = \operatorname{span}\big\{(\delta_{k-1}; \delta_k) :\ k = 1, \ldots, \Delta + \ddot{o} - 1\big\}$$

is a consequence of (5.2) and (5.3), since we know that $\ker \Psi^{ac} \subseteq \ker \Psi = (\mathcal{C}^\circ + X^\delta)^2$.

❑

5.2 The adjoint of T

The second powerful condition in the definition of a boundary triplet besides the Green's identity is that $\ker \Gamma = T^*$. The first step in our study of $S(\mathfrak{h}) = T(\mathfrak{h})^*$ is to establish this condition.

5.4 Proposition. *We have* $\ker \Gamma(\mathfrak{h}) = T(\mathfrak{h})^*$.

Proof. Assume first that $\mathfrak{h}$ is of kind (C). Then $T = \mathcal{P}(\mathfrak{h}) \times \mathcal{P}(\mathfrak{h})$, and thus $T^* = \{0\}$. Assume that $(\lambda p_0; \mu p_0) \in \ker \Gamma(\mathfrak{h})$. Then, by the definition of $\Gamma(\mathfrak{h})$, we have $\Lambda(f; g)_1 = \Lambda(f; g)_2 = 0$, where Λ is defined as in (4.3). It follows that $\lambda = 0$. Since $[p_0, p_0] = d_0 \neq 0$, also $\mu = 0$.

The case that $\mathfrak{h}$ is of kind (B) is also treated explicitly. First of all note that, by Proposition 5.2 and the definition of $\Gamma(\mathfrak{h})$,

$$\begin{aligned} \operatorname{span}\big\{(\delta_{k-1}; \delta_k) :\ k = 1, \ldots, \ddot{o}\big\} + \operatorname{span}\big\{(\mathfrak{b}; p_0 + [p_0, p_0]\delta_0)\big\} \subseteq \\ \ker \Gamma(\mathfrak{h}) \subseteq T(\mathfrak{h})^* \,. \end{aligned} \tag{5.4}$$

Let $(f; g) \in T(\mathfrak{h})^*$. Our aim is to show that $(f; g)$ belongs to the left linear span in (5.4). Write $f = \alpha p_0 + \sum_{j=0}^{\ddot{o}} \beta_j \delta_j$, $g = \lambda p_0 + \sum_{j=0}^{\ddot{o}} \mu_j \delta_j$. In order to reach our aim, we can without loss of generality assume that $\lambda = \mu_1 = \ldots = \mu_{\ddot{o}} = 0$. Applying the Green's identity with $(f; g)$ and $(p_0; 0)$, we obtain $\mu_0 = 0$, i.e. $g = 0$. An application with $(f; g)$ and $(0; \delta_0)$ yields $\alpha = 0$. Since we already have shown that $g = 0$, an application with $(f; g)$ and $(\delta_{k-1}; \delta_k)$ yields

$$[f, \delta_k] = 0,\ k = 1, \ldots, \ddot{o}\,.$$

Thus $\beta_1 = \ldots = \beta_{\ddot{o}} = 0$. Finally, an application with $(f; g)$ and $(\mathfrak{b}; p_0 + [p_0, p_0]\delta_0)$ yields $\beta_0 = 0$. It follows that in both inclusions in (5.4) the equality sign must hold.

We come to the case that $\mathfrak{h}$ is of kind (A). Let S be the linear relation $\ker \Gamma(\mathfrak{h})$. By the continuity of Γ the relation S is closed. Hence it is sufficient to show that $S^* = T$. Since Γ is a boundary relation, certainly $T \subseteq S^*$. In particular, S is symmetric. Moreover, $\Psi(S \cap \mathcal{C}^2) = T_{min}$. Finally, let us remark that S is real with respect to the involution $\bar{\cdot}$.

In the first step we show that $S^* \cap (\mathcal{C}^\circ)^2 \subseteq T$. Let $(f;g) \in S^* \cap (\mathcal{C}^\circ)^2$, and write $f = \sum_{k=0}^{\Delta-1} \alpha_k \delta_k$, $g = \sum_{k=0}^{\Delta-1} \beta_k \delta_k$. For $l = 1, \dots, \Delta$, choose $h_l \in X_L$ such that

$$\Gamma_{L^2(H)}(Bh_l; h_l) = \begin{cases} \Gamma(p_l; p_{l-1} + [p_{l-1}, p_0]\delta_0) & , \ l = 1, \dots, \Delta - 1 \\ \Gamma(\mathfrak{w}_\Delta + \mathfrak{b}; p_{\Delta-1} + [p_{\Delta-1}, p_0]\delta_0) & , l = \Delta \end{cases} .$$

Note that this is always possible, because, if $\alpha_1^\pm(H_\pm) \neq s$, then $\Gamma_{L^2(H)}$ is surjective. If $\alpha_1^\pm(H_\pm) = s$, we also have no difficulties, since then $\mathfrak{w}_j(s_\pm) = 0$ for $j = 1, \dots, \Delta$.

Since, by this choice, $(p_l - Bh_l; p_{l-1} - h_l + [p_{l-1} - h_l, p_0]\delta_0) \in S$, $l = 1, \dots, \Delta - 1$, we obtain

$$0 = [f, p_{l-1} - h_l + [p_{l-1} - h_l, p_0]\delta_0] - [g, p_l - Bh_l] = -\alpha_{l-1} + \beta_l \, .$$

Similarly, $(\mathfrak{w}_\Delta + \mathfrak{b} - Bh_\Delta; p_{\Delta-1} - h_\Delta + [p_{\Delta-1} - h_\Delta, p_0]\delta_0) \in S$ yields

$$0 = [f, p_{\Delta-1} - h_\Delta + [p_{\Delta-1} - h_\Delta, p_0]\delta_0] - [g, \mathfrak{w}_\Delta + \mathfrak{b} - Bh_\Delta] = -\alpha_{\Delta-1} \, .$$

It follows that $(f;g) = \sum_{k=0}^{\Delta-2} \alpha_k(\delta_k; \delta_{k+1}) + \beta_0(0; \delta_0)$. If $\alpha_1^\pm(H_\pm) = s$, we are done since then $(f;g) \in T$.

Otherwise let $h_0 \in X_L$ be such that $\Gamma_{L^2(H)}(Bh_0; h_0) = \Gamma(p_0; 0)$. From $(p_0 - Bh_0; -h_0 - [h_0, p_0]\delta_0) \in S$ we conclude

$$0 = [f, -h_0 - [h_0, p_0]\delta_0] - [g, p_0 - Bh_0] = -[g, p_0] = \beta_0 \, ,$$

and, hence, $(f;g) \in T$.

In the next step we show that $S^* \cap \mathcal{C}^2 \subseteq T$. Let $(f;g) \in S^* \cap \mathcal{C}^2$. Then

$$\Psi(f;g) \in \Psi(S \cap \mathcal{C}^2)^* = T_{min}^* = T_{max} \, .$$

Hence, there exists $(\tilde{f}; \tilde{g}) \in T \cap \mathcal{C}^2$ with $\Psi(\tilde{f}; \tilde{g}) = \Psi(f;g)$. Since $T \subseteq S^*$ we have $(\tilde{f} - f; \tilde{g} - g) \in S^*$. Moreover, since $\ker \Psi = (\mathcal{C}^\circ)^2$, this pair belongs to $(\mathcal{C}^\circ)^2$. By the first step of this proof it, therefore, belongs to T. We conclude that also $(f;g) \in T$.

In the last step let $(f;g) \in S^*$ be given. Write

$$\begin{aligned} f &= f_0 + \sum_{j=\Delta}^{\Delta+\ddot{o}-1} \alpha_j \delta_j + \sum_{i=0}^{\Delta-1} \beta_i p_i \, , \\ g &= g_0 + \sum_{j=\Delta}^{\Delta+\ddot{o}-1} \gamma_j \delta_j + \sum_{i=0}^{\Delta-1} \epsilon_i p_i \, , \end{aligned} \tag{5.5}$$

with $f_0, g_0 \in \mathcal{C}$. Since $(\delta_{k-1}; \delta_k) \in S$, $k = 1, \dots, \Delta - 1$, we see that

$$0 = [f, \delta_k] - [g, \delta_{k-1}] = -\beta_k + \epsilon_{k-1}, \ k = 1, \dots, \Delta - 1 \, .$$

The element

$$(f_1; g_1) := \beta_0(p_0; 0) + \sum_{k=1}^{\Delta-1} \beta_k(p_k; p_{k-1} + [p_{k-1}, p_0]\delta_0) +$$

$$+\epsilon_{\Delta-1}(\mathfrak{w}_\Delta + \mathfrak{b}; p_{\Delta-1} + [p_{\Delta-1}, p_0]\delta_0)$$

belongs to T. Hence it is enough to show that $(f - f_1; g - g_1) \in T$. This element, however, belongs to S^* and has the property that in its decomposition (5.5) all β_i and ϵ_i vanish. Therefore we can assume without loss of generality that $(f; g) \in S^*$ is given such that in (5.5) all β_i and ϵ_i vanish. Moreover, since $\sum_{j=\Delta}^{\Delta+\ddot{o}-1} \gamma_j(\delta_{j-1}; \delta_j) \in T$, we can also assume without loss of generality that $\gamma_j = 0$, $j = \Delta, \dots, \Delta + \ddot{o} - 1$.

Since $(\delta_{k-1}; \delta_k) \in S$, $k = \Delta, \dots, \Delta + \ddot{o} - 1$, we obtain

$$0 = [f, \delta_k] - [g, \delta_{k-1}] = \sum_{j=\Delta}^{\Delta+\ddot{o}-1} \alpha_j [\delta_j, \delta_k], \;\; k = \Delta, \dots, \Delta + \ddot{o} - 1\,,$$

i.e.

$$\big([\delta_j, \delta_k]\big)_{j,k=\Delta}^{\Delta+\ddot{o}-1} \begin{pmatrix} \alpha_\Delta \\ \vdots \\ \alpha_{\Delta+\ddot{o}-1} \end{pmatrix} = 0.$$

Since the matrix on the left hand side of this equation is invertible, we must have $\alpha_j = 0$, $j = \Delta, \dots, \Delta + \ddot{o} - 1$. We see that $(f; g) \in \mathcal{C}^2$, and hence have reduced the problem to what was already proved in the previous step.

❑

The above proposition has a couple of corollaries:

5.5 Corollary. *The relation S has defect index $(2,2)$ or $(1,1)$ dependig whether $\mathfrak{h}$ is of kind (A) or (B),(C).*

Proof. In case that $\mathfrak{h}$ is of kind (B) or (C), it follows from our explict computation of $\ker \Gamma(\mathfrak{h})$ in the proof of Proposition 5.4 that $\dim T/T^* = 2$.

Assume that $\mathfrak{h}$ is of kind (A). Since in this case $\Gamma(\mathfrak{h})$ is a map and maps T onto $\mathbb{C}^2 \times \mathbb{C}^2$, we have $\dim T / \ker \Gamma(\mathfrak{h}) = 4$.

❑

5.6 Corollary. *For every pair $(\phi_-; \phi_+) \in [0, \pi)^2$ the relation*

$$A(\phi_-; \phi_+) := \big\{(f; g) \in T : \, (\xi_{\phi_-}^T, \xi_{\phi_+}^T)\Gamma(f; g) = 0\big\}$$

is a selfadjoint extension of $S(\mathfrak{h})$, cf. Remark 2.10. We have

$$\Psi\big(A(\phi_-; \phi_+) \cap \mathcal{C}^2\big) = A_-(\phi_-) \oplus A_+(\phi_+)\,. \tag{5.6}$$

Proof. That the left hand side of (5.6) is contained in the right hand side follows from $\psi(\mathcal{C}) = L^2(H)$ and from (4.18). The converse direction is a consequence of Proposition 4.17.

❑

5.7 Corollary. *We have* $\operatorname{mul} S(\mathfrak{h}) = \{0\}$.

Proof. Since $\operatorname{mul} S(\mathfrak{h}) = \operatorname{dom} T(\mathfrak{h})^\perp$, this follows from Proposition 4.17.

❑

5.8 Corollary. *We have*

$$S(\mathfrak{h}) \cap \big(\mathcal{C}^\circ + X^\delta\big)^2 = \operatorname{span}\big\{(\delta_{k-1};\delta_k):\ k = 1,\dots,\Delta+\ddot{o}-1\big\}\,.$$

Proof. This follows from Corollary 5.3.

❑

Our second aim in the study of $S(\mathfrak{h})$ is to show the validity of (CR).

5.9 Proposition. *Put $C := A(0,\frac{\pi}{2})^{-1}$. Then C is a compact operator defined on all of $\mathcal{P}$.*

Proof. If $\mathfrak{h}$ is of kind (B) or (C), this is trivial. Hence assume that $\mathfrak{h}$ is of kind (A).

In the first step we show that $\operatorname{ran} A(0,\frac{\pi}{2})$ contains the set $X_L + \operatorname{span}\{p_0,\dots,p_{\Delta-1}\} + X^\delta$. Note that $\operatorname{ran} A(0,\frac{\pi}{2}) \supseteq \mathcal{C}^\circ$, since

$$(\chi_+\begin{pmatrix}1\\0\end{pmatrix};\delta_0), (\delta_{k-1};\delta_k) \in A(0,\frac{\pi}{2}),\ k = 1,\dots,\Delta-1\,.$$

Next, let $h \in X_L$ be given. Then, for some $\lambda_1,\lambda_2 \in \mathbb{C}$ we have

$$\Gamma(Bh; h + [h,p_0]\delta_0) = (0,\lambda_1,0,\lambda_2)^T\,.$$

It follows that $(Bh-\lambda_2 p_0; h+[h,p_0]\delta_0) \in A(0,\frac{\pi}{2})$. Hence $h+[h,p_0]\delta_0$, and with it also h, belongs to $\operatorname{ran} A(0,\frac{\pi}{2})$. Since

$$\Gamma(p_k; p_{k-1} + [p_{k-1},p_0]\delta_0) \in \operatorname{span}\Bigg\{\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\Bigg\},\ k = 1,\dots,\Delta-1\,,$$

$$\Gamma(\mathfrak{w}_\Delta + \mathfrak{b}; p_{\Delta-1} + [p_{\Delta-1},p_0]\delta_0) \in \operatorname{span}\Bigg\{\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\Bigg\}\,,$$

the same argument will show that $\operatorname{span}\{p_0,\dots,p_{\Delta-1}\} \subseteq \operatorname{ran} A(0,\frac{\pi}{2})$. Since $(\delta_{k-1};\delta_k) \in A(0,\frac{\pi}{2})$, $k = \Delta,\dots,\Delta+\ddot{o}-1$, we also have $X^\delta \subseteq A(0,\frac{\pi}{2})$.

Our next objective is to show that $\operatorname{ran} A(0,\frac{\pi}{2})$ is closed. Since $\mathcal{C}^2$ has finite codimension in $\mathcal{P}(\mathfrak{h})$, also $A(0,\frac{\pi}{2}) \cap \mathcal{C}^2$ has this property in $A(0,\frac{\pi}{2})$. Thus also $\operatorname{ran}(A(0,\frac{\pi}{2}) \cap \mathcal{C}^2)$ has finite codimension in $\operatorname{ran} A(0,\frac{\pi}{2})$. As we saw in the proof of the first step we have

$$\operatorname{ran}\big(A(0,\frac{\pi}{2}) \cap \mathcal{C}^2\big) = \operatorname{ran}\big(A(0,\frac{\pi}{2}) \cap \mathcal{C}^2\big) + \mathcal{C}^\circ = \psi^{-1}\psi\big(\operatorname{ran}(A(0,\frac{\pi}{2}) \cap \mathcal{C}^2)\big) =$$

$$= \psi^{-1}\big(\operatorname{ran}\big(\Psi(A(0,\frac{\pi}{2}) \cap \mathcal{C}^2)\big)\big) = \psi^{-1}\big(\operatorname{ran}\big(A_-(0) \oplus A_+(\frac{\pi}{2})\big)\big)\,.$$

Since $A(0) \oplus A(\frac{\pi}{2})$ is selfadjoint and has compact resolvents, its range is closed. Hence $\operatorname{ran}(A(0,\frac{\pi}{2}) \cap \mathcal{C}^2)$ and with it also $\operatorname{ran} A(0,\frac{\pi}{2})$ is closed.

We have proved in the previous paragraphs that $\operatorname{ran} A(0,\frac{\pi}{2}) = \mathcal{P}$, and, therefore, that $0 \in \rho(A(0,\frac{\pi}{2}))$. Hence $C := A(0,\frac{\pi}{2})^{-1}$ is an everywhere defined bounded operator. We come to the proof of compactness. Consider the set

$$D := \left\{ h \in X_L : \phi_0(h) = \ldots = \phi_{\Delta-1}(h) = (h, \mathfrak{w}_\Delta) = 0 \right\}.$$

Then D is a linear subspace of finite codimension in X_L. Since B is an operator, also $\{(Bh;h) : h \in D\}$ is a subspace of finite codimension in $\{(Bh; h + [h,p_0]\delta_0 : h \in X_L\}$. Hence $T_1 := \overline{\{(Bh;h) : h \in D\}}$ has finite codimension in (4.13) and thus also in T. Hence there exists a continuous projection P of T, regarded as a Banach space with the graph norm, onto T_1.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the unit ball of $\mathcal{P}$. Then we have

$$\|(I-P)(Cx_n;x_n)\| \leq \|I-P\|(\|C\|+1),\ n \in \mathbb{N}.$$

Since $\operatorname{ran}(I-P)$ is finite dimensional, we may extract a convergent subsequence $(I-P)(Cx_n;x_n)$. For notational convenience we will again denote this subsequence by $(x_n)_{n\in\mathbb{N}}$. We have

$$\|P(Cx_n;x_n)\| \leq \|P\|(\|C\|+1),\ n \in \mathbb{N}.$$

Choose $h_n \in D$, such that

$$\|P(Cx_n;x_n) - (Bh_n;h_n)\| \leq \frac{1}{n},\ n \in \mathbb{N}.$$

Certainly,

$$\|h_n\| \leq \|P\|(\|C\|+1)+1,\ n \in \mathbb{N}.$$

By the compactness of B there exists a subsequence $(h_{n_k})_{k\in\mathbb{N}}$ such that $(Bh_{n_k})_{k\in\mathbb{N}}$ is a Cauchy-sequence in $L^2(H)$. However, since all h_{n_k} belong to D, we obtain from Lemma 4.16

$$\|Bh_{n_k} - Bh_{n_l}\| = \|Bh_{n_k} - Bh_{n_l}\|_{L^2(H)},\ k,l \in \mathbb{N}.$$

Thus the Bh_{n_k}, and with them also the first components of $P(Cx_n;x_n)$, form a Cauchy-sequence in $\mathcal{P}$. Alltogether we see that $(Cx_{n_k})_{k\in\mathbb{N}}$ is convergent in $\mathcal{P}$. Thus C is compact.

❑

Let us determine the set of regular points of $S(\mathfrak{h})$.

5.10 Lemma. *For all $z \in \mathbb{C}$ the subspace* $\operatorname{ran}(S(\mathfrak{h}) - z)$ *is closed and we have* $\ker(S(\mathfrak{h}) - z) = \{0\}$.

Proof. Assume that $(f;zf) \in S$. Consider the element $(\tilde{f};\tilde{g}) := \Psi^{ac}(f;zf)$. Then, by (4.17),

$$\tilde{g} =_H z\tilde{f}$$

and hence, by (4.18), the function $\tilde{f}$ is a solution of the differential equation

$$\tilde{f}' = zJH\tilde{f},\ t \in I,$$

with the initial conditions $\tilde{f}(s_-) = 0, \tilde{f}(s_+) = 0$. By the uniqueness of the solution of this initial value problem we must have $\tilde{f} = 0$. We see that $(f;zf) \in$

$\ker \Psi^{ac}$ and, with the help of Corollary 5.3 we conclude that $(f; zf)$ can be written as

$$(f; zf) = \sum_{k=1}^{\Delta+\ddot{o}-1} \alpha_k (\delta_{k-1}; \delta_k) \,.$$

This gives the equations

$$f = \sum_{k=1}^{\Delta+\ddot{o}-1} \alpha_k \delta_{k-1}, \; zf = \sum_{k=1}^{\Delta+\ddot{o}-1} \alpha_k \delta_k \,.$$

Comparing coefficients yields

$$\alpha_k = z\alpha_{k+1}, \; k = 1, \ldots, \Delta + \ddot{o} - 2, \quad \alpha_{\Delta+\ddot{o}-1} = 0 \,.$$

Hence all α_k must vanish, and we have $f = 0$. Thus $\ker(S(\mathfrak{h}) - z) = \{0\}$.

The fact that $0 \in \rho(A(0, \frac{\pi}{2}))$ implies that $\operatorname{ran} S$ is closed. Again let C denote the compact and selfadjoint extension $A(0; \frac{\pi}{2})^{-1}$ of S^{-1}. We have

$$(I - zC) \cdot S = S - z \,.$$

Hence $\operatorname{ran}(S-z) = (I-zC) \operatorname{ran} S$. Since C is compact it follows that $\operatorname{ran}(S-z)$ is closed.

❑

The fact that the relation S is completely nonselfadjoint will follow from the next general observation.

5.11 Lemma. *Let S be a closed symmetric operator in a Pontryagin space $\mathcal{P}$ which satisfies (CR) and has no eigenvalues. Then S is completely nonselfadjoint.*

Proof. Choose a selfadjoint extension A of S and put $C := A^{-1}$. Let $X := \bigcap_{z\in\mathbb{C}} \operatorname{ran}(S - z)$, then

$$X^\perp = \operatorname{cls}\{\operatorname{ran}(S - z)^\perp : z^{-1} \in \rho(C)\} \,.$$

Since $I + (z - w)(A - z)^{-1}$ maps $\operatorname{ran}(S - \bar{w})^\perp$ bijectively onto $\operatorname{ran}(S - \bar{z})^\perp$ the space X is invariant under C.

Assume that $X \neq \{0\}$. If we had $X^\circ \neq \{0\}$, then it would be a finite dimensional invariant subspace of C and, hence, would contain an eigenvector of C. If X were indefinite and nondegenerated, there would exist a maximal nonpositive subspace of X which is invariant under C, cf. [IKL]. Thus, also in this case we would find an eigenvector of C in X. Finally, if X were a Hilbert space, then $C|_X$ would be a compact and selfadjoint operator in the Hilbert space X, and hence would have an eigenvector.

Since $C \cap X^2 \subseteq S^{-1}$, we have reached a contradiction.

❑

5.12 Corollary. *The relation $S(\mathfrak{h})$ is completely nonselfadjoint.*

Proof. By virtue of Corollary 5.7, Proposition 5.9 and Lemma 5.10 we may apply Lemma 5.11 to $S(\mathfrak{h})$.

❑

Finally, we shall establish the condition (E). For this we need another lemma.

5.13 Lemma. *Let $(f;zf) \in T(\mathfrak{h})$ be a nonzero defect element, and let $(\tilde{f};\tilde{h}) = \Psi^{ac}(f;zf)$. Moreover, write f as*

$$f = f_0 + \sum_{j=\Delta}^{\Delta+\ddot{o}-1} \alpha_j \delta_j + \sum_{i=0}^{\Delta-1} \beta_i p_i, \tag{5.7}$$

where $f_0 \in \mathcal{C}$. Then, $\tilde{f}|_{[s_-,s)} \in L^2(H_-)$ if and only if $\tilde{f}|_{(s,s_+]} \in L^2(H_+)$.

In this case $\beta_0 = \cdots = \beta_{\Delta-1} = 0$. Moreover, $\alpha_j = 0,\ j = \Delta, \ldots, \Delta + \ddot{o} - 1$.

Proof. Clearly,

$$\tilde{f} =_H \psi(f_0) + \sum_{i=0}^{\Delta-1} \beta_i \mathfrak{w}_i.$$

Assume that $\tilde{f}|_{[s_-,s)} \in L^2(H_-)$. Then by (4.18) we can apply Lemma 3.14 or Remark 3.16 to $[s_-, s)$, and we obtain from the fact that $\mathfrak{w}_0, \ldots, \mathfrak{w}_{\Delta_- -1}$ are linearly independent modulo $L^2(H_-)$ that $\beta_0 = \cdots = \beta_{\Delta_- -1} = 0$. Again by Lemma 3.14 (or Remark 3.16), but this time applied to $(s, s_+]$, we obtain from $\beta_0 = 0$ that $\tilde{f}|_{(s,s_+]} \in L^2(H_+)$, and hence $\beta_0 = \cdots = \beta_{\Delta-1} = 0$. The converse is proved in the same way.

By (5.1) we have

$$[f, \delta_k] = z[f, \delta_{k-1}],\ k = \Delta, \ldots, \Delta + \ddot{o} - 1.$$

Since $\beta_i = 0,\ i = 0, \ldots, \Delta-1$, for $k = \Delta$ the previous relation yields $c_1 \alpha_{\Delta+\ddot{o}-1} = 0$. Hence $\alpha_{\Delta+\ddot{o}-1} = 0$. For $k = \Delta + 1$ we now obtain $c_1 \alpha_{\Delta+\ddot{o}-2} = 0$, and so on. Finally, for $k = \Delta + \ddot{o} - 1$ we get $c_1 \alpha_\Delta = 0$.

❑

5.14 Proposition. *The condition (E) holds for $T(\mathfrak{h})$.*

Proof. Let $z \in \mathbb{C}$, $(f;zf) \in T(\mathfrak{h})$, $f \neq 0$, be given and assume that that $((f;zf);(a;0)) \in \Gamma(\mathfrak{h})$. We shall derive a contradiction. If $a = 0$, we would have $(f;zf) \in S(\mathfrak{h})$, which contradicts Lemma 5.10. Thus assume that $a \neq 0$.

The cases that $\mathfrak{h}$ is of kind (B) or (C) are treated by explicit inspection. Consider now the case that $\mathfrak{h}$ is of kind (A). Put $(\tilde{f};\tilde{g}) := \Psi^{ac}(f;zf)$, then $f_+ := \tilde{f}|_{(s,s_+]}$ is a solution of

$$y'(t) = zJH(t)y(t),\ y(s_+) = 0,\ t \in (s, s_+]\,,$$

and thus $f_+ = 0$. By Lemma 5.13 it follows that $f \in \mathcal{C}$.

Since the function $f_- := \tilde{f}|_{[s_-,s)}$ is a solution of

$$y'(t) = zJH(t)y(t),\ y(s_-) = a,\ t \in [s_-, s)\,,$$

we have ($a =: (a_1, a_2)$)

$$f_- = a_1 \chi_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2 \chi_- \begin{pmatrix} 0 \\ 1 \end{pmatrix} + zBf_- \,.$$

Since $f_- \in L^2(H)$ we conclude that $a_2 = 0$.

By the definition of $T(\mathfrak{h})$, we have

$$\big(f_-; zf_- + z[f_-, p_0]\delta_0 - a_1\delta_0\big) \in T(\mathfrak{h}) \,.$$

Hence

$$(h;k) := (f; zf) - \big(f_-; zf_- + (z[f_-, p_0] - a_1)\delta_0\big) \in \ker \Psi^{ac} \,,$$

and Corollary 5.3 implies that

$$(h;k) = \sum_{l=1}^{\Delta+\ddot{o}-1} \alpha_l (\delta_{l-1}; \delta_l) \,.$$

Since both, f and f_-, belong to $\mathcal{C}$, also $(h;k) \in \mathcal{C}^2$. Thus $\alpha_\Delta = \ldots = \alpha_{\Delta+\ddot{o}-1} = 0$, and we obtain

$$(f; zf) = \big(f_-; zf_- + (z[f_-, p_0] - a_1)\delta_0\big) + \sum_{l=1}^{\Delta-1} \alpha_l (\delta_{l-1}; \delta_l) \,.$$

Hence

$$z\big(f_- + \sum_{l=1}^{\Delta-1} \alpha_l \delta_{l-1}\big) = zf_- + (z[f_-, p_0] - a_1)\delta_0 + \sum_{l=1}^{\Delta-1} \alpha_l \delta_l \,,$$

and thus

$$\sum_{l=0}^{\Delta-2} \alpha_{l+1}\delta_l = (z[f_-, p_0] - a_1)\delta_0 + \sum_{l=1}^{\Delta-1} \alpha_l \delta_l \,.$$

We conclude inductively that $\alpha_{\Delta-1} = 0, \ldots, \alpha_1 = 0$, $z[f_-, p_0] = a_1$. Since $a_1 = (1,0)f_-(s_-)$, the last relation contradicts Lemma 3.17

❑

As the reader has certainly recognized, we have by the time collected proofs of all the assertions of Theorem 5.1. For completeness let us state this explicitly.
Proof. (of Theorem 5.1) The relation $T(\mathfrak{h})$ is closed by Proposition 4.17. The relation $\Gamma(\mathfrak{h})$ is a boundary relation by Proposition 5.2. By Lemma 4.19 it is closed, defined on all of $T(\mathfrak{h})$ and compatible with $\bar{\cdot}$. Proposition 5.4 states that $\ker \Gamma(\mathfrak{h}) = T(\mathfrak{h})^*$.

If $\mathfrak{h}$ is of kind (A), by Lemma 4.19 $\Gamma(\mathfrak{h})$ is an operator and by Corollary 5.5 we have $\dim T(\mathfrak{h})/T(\mathfrak{h})^* = 4$. If $\mathfrak{h}$ is of kind (B) or (C) the same sources show that $\operatorname{mul}\Gamma(\mathfrak{h}) = \operatorname{span}\{(m;m)\}$ with $m = (1,0)^T$ and $\dim T(\mathfrak{h})/T(\mathfrak{h})^* = 2$. We see that $(\mathcal{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h}))$ is a boundary triplet of defect 2.

The relation $S(\mathfrak{h})$ is a completely nonselfajoint operator by Corollary 5.12 and Corollary 5.7. It satisfies (CR) and (E) by Proposition 5.9 and Proposition 5.14. Finally, an application of the closed graph theorem yields in view of

Lemma 5.10 that $r(S(\mathfrak{h})) = \mathbb{C}$.

✌

We close this subsection with determining the multivalued part of $T(\mathfrak{h})$ explicitly. This follows from a result which deals with the embeddings of spaces $L^2(H|_J)$, cf. Proposition 4.14.

5.15 Lemma. *Let $\mathfrak{h}$ be an elementary indefinite Hamiltonian of kind (A). Let $s^- \in (s_-, s)$ and assume that s^- is not inner point of an H-indivisible interval. Put $J := (s_-, s^-)$, and let ι_J be the embedding of $L^2(H|_J)$ into $\mathcal{P}(\mathfrak{h})$ as in Proposition 4.14.*

Let $(f; g) \in \mathrm{AC}(J) \times \mathcal{M}(J)$ with $f' = JHg$ be given, so that $(f; g) \in T_{max}(H|_J)$. Assume that $f(s^-) = 0$, then $(\iota_J f; \iota_J g) \in T(\mathfrak{h})$. Moreover, $\Gamma(\mathfrak{h})(\iota_J f; \iota_J g) = (f(s_-); 0)$. More abstractly expressed this says that

$$(\iota_J \times \iota_J)(\ker \Gamma(H|_J)_2) \subseteq \ker \Gamma(\mathfrak{h})_2 \,.$$

Proof. Define functions f_1, g_1 by

$$f_1(t) := \begin{cases} f(t) & , \ t \in J \\ 0 & , \ t \in I \setminus J \end{cases} \ , \ g_1(t) := \begin{cases} g(t) & , \ t \in J \\ 0 & , \ t \in I \setminus J \end{cases}$$

Note that, since $H|_J$ is regular at s_- and s^-, the function f is in fact absolutely continuous on $[s_-, s^-]$. Since we assume that $f(s^-) = 0$, we have $f_1 \in \mathrm{AC}(I)$. Moreover,

$$f_1'(t) = JHg_1(t), \ t \in I \text{ a.e.}$$

and, clearly, $f_1, g_1 \in L^2(H)$. Thus $(f_1; g_1) \in T_{max}(H)$, and we conclude from Proposition 4.17 that there exists a pair $(h; k) \in T(\mathfrak{h}) \cap \mathcal{C}^2$ with $\psi h = f_1, \psi k = g_1$. Since $\psi \circ \iota_J$ is the identity, it follows that

$$\psi(\iota_J f - h) = \psi(\iota_J g - k) = 0 \,,$$

and hence

$$\iota_J f - h, \iota_J g - k \in \operatorname{span}\{\delta_0, \dots, \delta_{\Delta-1}\} \,.$$

By adding an appropriate linear combination of $(\delta_0; \delta_1), \dots, (\delta_{\Delta-2}; \delta_{\Delta-1})$ to $(h; k)$, we can assume without loss of generality that

$$(\iota_J f - h; \iota_J - g) = \sum_{j=0}^{\Delta-1} \alpha_j (\delta_j; 0) + \alpha(0; \delta_0) \,.$$

We obtain

$$-\alpha = [\iota_j g - k, p_0] = [\iota_J g, p_0] - [k, p_0] \,.$$

Since $\Psi^{ac}(h; k) = (f_1; g_1)$, we have

$$\Gamma(\mathfrak{h})(h; k) = \big(f(s_-); 0\big) \,,$$

and hence by Proposition 5.2

$$[k, p_0] = [k, p_0] - [h, 0] = \mathfrak{w}_0(s_-)^* J f(s_-) \,.$$

On the other hand

$$[\iota_J g, p_0] = \int_J \mathfrak{w}_0^* H g = (g, \mathfrak{w}_0)_{L^2(H|_J)} .$$

Since $(\mathfrak{w}_0|_J; 0) \in T_{max}(H|_J)$ it follows from the Green's identity and our assumption $f(s^-) = 0$ that

$$(g, \mathfrak{w}_0)_{L^2(H|_J)} = (g, \mathfrak{w}_0)_{L^2(H|_J)} - (h, 0)_{L^2(H|_J)} = \mathfrak{w}_0(s_-)^* J f(s_-) .$$

We conclude that $\alpha = 0$.

Let $j \in \{1, \ldots, \Delta - 1\}$. Then

$$\alpha_j = [\iota_J g - k, p_j] - [\iota_J f - h, p_{j-1} + [p_{j-1}, p_0]\delta_0] =$$

$$= \big([\iota_J g, p_j] - [\iota_J f, p_{j-1}]\big) - \big([k, p_j] - [h, p_{j-1} + [p_{j-1}, p_0]\delta_0]\big)$$

The respective Green's identities yield $((\mathfrak{w}_j|_J; \mathfrak{w}_{j-1}|_J) \in T_{max}(H|_J))$

$$[k, p_j] - [h, p_{j-1} + [p_{j-1}, p_0]\delta_0] = \mathfrak{w}_j(s_-)^* J f(s_-)$$

$$[\iota_J g, p_j] - [\iota_J f, p_{j-1}] = (g, \mathfrak{w}_j)_{L^2(H|_J)} - (h, \mathfrak{w}_{j-1})_{L^2(H|_J)} = \mathfrak{w}_j(s_-)^* J f(s_-)$$

and it follows that $\alpha_{j-1} = 0$. The same argument applied with the element $(\mathfrak{w}_\Delta + \mathfrak{b}; p_{\Delta-1} + [p_{\Delta-1}, p_0]\delta_0) \in T(\mathfrak{h})$ yields $\alpha_{\Delta-1} = 0$. It follows that

$$(\iota_J f; \iota_J g) = (h; k) \in T(\mathfrak{h}) .$$

❑

5.16 Proposition. *Let $\mathfrak{h}$ be an elementary indefinite Hamiltonian. Then*

Kind (A): We have

$$\dim \operatorname{mul} T(\mathfrak{h}) = \begin{cases} 0 & , \ \alpha_1^-(H_-) = s_-, \alpha_1^+(H_+) = s_+ \\ 1 & , \ {\scriptstyle \alpha_1^-(H_-) > s_-, \alpha_1^+(H_+) = s_+ \atop or\ \alpha_1^-(H_-) = s_-, \alpha_1^+(H_+) < s_+} \\ 2 & , \ \alpha_1^-(H_-) > s_-, \alpha_1^+(H_+) < s_+ \end{cases} \tag{5.8}$$

$$\begin{aligned} \Gamma(0;g)_1 &\begin{cases} = 0 & , \ \alpha_1^-(H_-) = s_- \\ \in \operatorname{span}\{J\xi_\phi\} & , \ {\scriptstyle \alpha_1^-(H_-) > s_- \atop \phi\ type\ of\ (s_-, \alpha_1^-(H_-))} \end{cases} \\ \Gamma(0;g)_2 &\begin{cases} = 0 & , \ \alpha_1^+(H_+) = s_+ \\ \in \operatorname{span}\{J\xi_\phi\} & , \ {\scriptstyle \alpha_1^+(H_+) < s_+ \atop \phi\ type\ of\ (\alpha_1^+(H_+), s_+)} \end{cases} \end{aligned} \tag{5.9}$$

Kind (B): We have $\operatorname{mul} T(\mathfrak{h}) = \operatorname{span}\{\delta_0\}$ *and*

$$\Gamma(\mathfrak{h})(0;\delta_0) = \Big\{ \Big(\begin{pmatrix} \lambda \\ 0 \end{pmatrix}; \begin{pmatrix} \lambda + 1 \\ 0 \end{pmatrix} \Big) : \lambda \in \mathbb{C} \Big\}$$

Kind (C): We have $\operatorname{mul} T(\mathfrak{h}) = \operatorname{span}\{p_0\}$ *and*

$$\Gamma(\mathfrak{h})(0;p_0) = \Big\{ \Big(\begin{pmatrix} d_0 + \lambda \\ 0 \end{pmatrix}; \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \Big) : \lambda \in \mathbb{C} \Big\}$$

Proof. If $\mathfrak{h}$ is of kind (C) this is trivial. Let $\mathfrak{h}$ be of kind (B). Assume that $(0;g) \in T(\mathfrak{h})$. Write $(0;g)$ as a sum according to the definition of $T(\mathfrak{h})$:

$$(0;g) = \alpha(0;\delta_0) + \lambda(p_0;0) + \sum_{k=1}^{\ddot{o}} \mu_k(\delta_{k-1},\delta_k) + \nu(\mathfrak{b};p_0 + [p_0,p_0]\delta_0)\,.$$

We see that

$$0 = \lambda p_0 + \sum_{k=1}^{\ddot{o}} \mu_k \delta_{k-1} + \nu(\mathfrak{b})\,. \tag{5.10}$$

It follows that $\lambda = 0$. Since $b_1 \neq 0$ the relation (5.10) implies that $\nu = 0$ and, in turn, that also $\mu_1 = \ldots = \mu_{\ddot{o}} = 0$.

Assume throughout the following that $\mathfrak{h}$ is of kind (A). First of all note that, since $\operatorname{mul} S(\mathfrak{h}) = 0$ by Corollary 5.7, the map

$$\Gamma(\mathfrak{h})|_{\operatorname{mul} T(\mathfrak{h})} : \operatorname{mul} T(\mathfrak{h}) \to \mathbb{C}^2 \times \mathbb{C}^2$$

is injective.

Let $(0;k) \in T(\mathfrak{h})$ be given and put $(f;g) := \Psi^{ac}(0;k)$. Then, by (4.17), $f =_H 0$ and by (4.18) we have $f' = JHg$.

Assume that $s_- < \alpha_1^-(H_-) < s$. Then Lemma 2.23 implies that $f(\alpha_1^-(H_-)) = 0$. Thus

$$f(t) = \int_{\alpha_1^-(H_-)}^{t} h(x)J\xi_\phi\xi_\phi^T g(x),\ t \in (s_-, \alpha_1^-(H_-))$$

and we obtain $f(s_-) \in \operatorname{span}\{J\xi_\phi\}$. If $s_- = \alpha_1^-(H_-) < s$, then Lemma 2.23 implies that $f(t) = 0$, $t \in (s_-, \alpha_1^+(H_-)) \neq \emptyset$, and thus $f(s_-) = 0$.

Assume that $\alpha_1^-(H_-) = s$. Due to the condition (I) the interval (s_-, s) is H-indivisible of type $\frac{\pi}{2}$. Write $(0;k)$ as a sum according to Definition 4.11. Inspecting the first component we see that from the summands (4.15) only a multiple of $(\mathfrak{w}_\Delta + \mathfrak{b}; p_{\Delta-1} + [p_{\Delta-1}, p_0]\delta_0)$ may occur. By Remark 3.8 we have $(0,1)\mathfrak{w}_\Delta = 0$. For the summand $(x;y) \in$(4.13) we have $\psi x \in L^2(H)$ and thus $(0,1)\Psi^{ac}(x;y)_1 = 0$. For the summand (4.14) the same relation holds by definition, and $\Psi^{ac}((4.16)) = 0$. We conclude that $(0,1)\Psi^{ac}(0;k)_1 = 0$, in particular $\Psi^{ac}(0;k)_1 \in \operatorname{span}\{(1,0)^T\}$.

We have established the first relation in (5.9). The second relation is seen in the same manner. It readily follows from the injectivity of $\Gamma(\mathfrak{h})|_{\operatorname{mul} T(\mathfrak{h})}$ that in (5.8) the inequality '$\leq$' holds.

Assume that $\alpha_1^-(H_-) > s_-$. Then, by Proposition 2.24 we know that $\operatorname{mul} T(H|_{(s_-,\alpha_1^-(H_-))}) \neq \{0\}$. By Lemma 5.15 and the explicit construction of $\operatorname{mul} T(H|_{(s_-,\alpha_1^-(H_-))})$ in the proof of Proposition 2.24 it follows that

$$\iota_{(s_-,\alpha_1^-(H_-))} \operatorname{mul} T_{max}(H|_{(s_-,\alpha_1^-(H_-))}) \subseteq \operatorname{mul} T(\mathfrak{h})\,,$$

and we see that $\dim \operatorname{mul} T(\mathfrak{h}) \geq 1$ in this case. The same argument works if $\alpha_1^+(H_+) < s_+$. If H starts with indivisible intervals on both ends, it is enough to note that the correspondingly constructed elements of $\operatorname{mul} T(\mathfrak{h})$ cannot be linearly dependent, since they have disjoint support (if ψ is applied to them). Hence, in this case, the dimension of $\operatorname{mul} T(\mathfrak{h})$ is at least 2.

❑

5.3 Reparameterization

Similarly as in the positive definite case, also in the indefinite setting the concept of reparameterization of Hamiltonians is of importance. In the following we shall discuss this topic for elementary indefinite Hamiltonians.

5.17 Proposition. *Let $\mathfrak{h} = (H_\pm, \ddot{o}, b_j, d_j)$ be an elementary indefinite Hamiltonian given on $I = (s_-, s) \cup (s, s_+)$, let $\tilde{s}_-, \tilde{s}_+ \in \mathbb{R}$, $\tilde{s}_- < \tilde{s}_+$, and let φ be an absolutely continuous and increasing bijection of $[\tilde{s}_-, \tilde{s}_+]$ onto $[s_-, s_+]$ such that also φ^{-1} is absolutely continuous. Put $\tilde{s} := \varphi^{-1}(s)$.*

If $\tilde{\mathfrak{h}} = (\tilde{H}_\pm, \tilde{\ddot{o}}, \tilde{b}_j, \tilde{d}_j)$ is an elementary indefinite Hamiltonian on $\tilde{I} := (\tilde{s}_-, \tilde{s}) \cup (\tilde{s}, \tilde{s}_+)$ such that

$$\tilde{H}_\pm = (H_\pm \circ \varphi) \cdot \varphi'; \ \tilde{\ddot{o}} = \ddot{o}; \tilde{b}_1 = b_1, \dots, \tilde{b}_{\ddot{o}} = b_{\ddot{o}}; \tag{5.11}$$

$$\tilde{d}_0 = d_0, \dots, \tilde{d}_{2\Delta-2} = d_{2\Delta-2}; \tilde{d}_{2\Delta-1} - \tilde{b}_{\ddot{o}+1} = d_{2\Delta-1} - b_{\ddot{o}+1} \, .$$

then there exists an isometric isomorphism $\varpi : \mathcal{P}(\mathfrak{h}) \to \mathcal{P}(\tilde{\mathfrak{h}})$ such that

- (i) *(ϖ, id) is an isomorphism of the boundary triplets $(\mathcal{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h}))$ and $(\mathcal{P}(\tilde{\mathfrak{h}}), T(\tilde{\mathfrak{h}}), \Gamma(\tilde{\mathfrak{h}}))$.*
- (ii) *For all $x \in \mathcal{P}(\mathfrak{h})$ we have*

$$\psi(\tilde{\mathfrak{h}})(\varpi(x)) = \psi(\mathfrak{h})(x) \circ \varphi \, .$$

Proof. Clearly, $\tilde{\mathfrak{h}}$ is an elementary indefinite Hamiltonian of the same kind as $\mathfrak{h}$. Let us first deal with the case that $\mathfrak{h}$ is of kind (A). We define a map $\varpi : \mathcal{P}(\mathfrak{h}) \to \mathcal{P}(\tilde{\mathfrak{h}})$ in terms of the respective realizations (4.10). To this end note that, by the definition of $\tilde{H}_\pm$ and Lemma 2.4, the map

$$C_\varphi : f \mapsto f \circ \varphi$$

induces an isometric isomorphism of $L^2(H)$ onto $L^2(\tilde{H})$. Moreover, by Remark 3.19, we have $\Delta(H) = \Delta(\tilde{H}) =: \Delta$ and $\mathfrak{w}_j = \tilde{\mathfrak{w}}_j \circ \varphi$, $j \in \mathbb{N} \cup \{0\}$. Now we define $\varpi : \mathcal{P}(\mathfrak{h}) \to \mathcal{P}(\tilde{\mathfrak{h}})$ by

$$\delta_j \mapsto \tilde{\delta}_j, \ j = \Delta, \dots, \Delta + \ddot{o} - 1, \quad f \mapsto C_\varphi(f), \ f \in X_1 + X_2,$$

$$\mathfrak{w}_{\Delta-1+j} + b_{\ddot{o}+1}\delta_{\Delta-j} \mapsto \tilde{\mathfrak{w}}_{\Delta-1+j} + \tilde{b}_{\ddot{o}+1}\tilde{\delta}_{\Delta-j}, \ j = 1, \dots, \Delta;$$
$$p_j \mapsto \tilde{p}_j, \ j = 0, \dots, \Delta - 1.$$

It is elementary to check, that by our assumption this assignment operates isometrically and that it has dense domain and dense range.

Hence it can be continued to the unitary mapping $\varpi : \mathcal{P}(\mathfrak{h}) \to \mathcal{P}(\tilde{\mathfrak{h}})$ such that

$$\iota \circ \varpi \circ \iota^{-1} : \left\{ \begin{array}{ccc} L^2(H)[\dot{+}](\mathbb{C}^\Delta \dot{+} \mathbb{C}^\Delta)[\dot{+}]X^\delta & \to & L^2(\tilde{H})[\dot{+}](\mathbb{C}^\Delta \dot{+} \mathbb{C}^\Delta)[\dot{+}]\tilde{X}^\delta \\ (x, \xi, \eta, \sum_{j=\Delta}^{\Delta+\ddot{o}-1} \alpha_j \delta_j) & \mapsto & (C_\varphi x, \xi, \eta, \sum_{j=\Delta}^{\Delta+\ddot{o}-1} \alpha_j \tilde{\delta}_j) \end{array} \right.$$

Clearly it is compatible with the respective conjugations. Moreover, by (4.11), (4.12) and Remark 3.19, condition (ii) is satisfied.

Regarding to the summands of $T(\mathfrak{h})$ and $T(\tilde{\mathfrak{h}})$, respectively, in (4.13) and (4.14)-(4.16) $\varpi \times \varpi$ acts as ($j = 1, \ldots, \Delta - 1$)

$$(\mathfrak{w}_{\Delta+j}; \mathfrak{w}_{\Delta-1+j} + [\mathfrak{w}_{\Delta-1+j}, p_0]\delta_0) + b_{\ddot{o}+1}(\delta_{\Delta-j-1}; \delta_{\Delta-j}) \mapsto$$

$$(\tilde{\mathfrak{w}}_{\Delta+j}; \tilde{\mathfrak{w}}_{\Delta-1+j} + [\tilde{\mathfrak{w}}_{\Delta-1+j}, \tilde{p}_0]\tilde{\delta}_0) + \tilde{b}_{\ddot{o}+1}(\tilde{\delta}_{\Delta-j-1}; \tilde{\delta}_{\Delta-j}),$$

$$(\mathfrak{w}_{2\Delta}; \mathfrak{w}_{2\Delta-1} + [\mathfrak{w}_{2\Delta-1}, p_0]\delta_0) = (\mathfrak{w}_{2\Delta}; \mathfrak{w}_{2\Delta-1} + b_{\ddot{o}+1}\delta_0 + (d_{2\Delta-1} - b_{\ddot{o}+1})\delta_0) \mapsto$$

$$(\tilde{\mathfrak{w}}_{2\Delta}; \tilde{\mathfrak{w}}_{2\Delta-1} + \tilde{b}_{\ddot{o}+1}\tilde{\delta}_0 + (\tilde{d}_{2\Delta-1} - \tilde{b}_{\ddot{o}+1})\tilde{\delta}_0) = (\tilde{\mathfrak{w}}_{2\Delta}; \tilde{\mathfrak{w}}_{2\Delta-1} + [\tilde{\mathfrak{w}}_{2\Delta-1}, \tilde{p}_0]\tilde{\delta}_0).$$

and

$$(\mathfrak{w}_\Delta + \mathfrak{b}; p_{\Delta-1} + [p_{\Delta-1}, p_0]\delta_0) =$$

$$(\mathfrak{w}_\Delta + b_{\ddot{o}+1}\delta_{\Delta-1} + \sum_{l=1}^{\ddot{o}} b_l \delta_{\Delta+\ddot{o}-l}; p_{\Delta-1} + [p_{\Delta-1}, p_0]\delta_0) \mapsto$$

$$(\tilde{\mathfrak{w}}_\Delta + \tilde{b}_{\ddot{o}+1}\tilde{\delta}_{\Delta-1} + \sum_{l=1}^{\ddot{o}} \tilde{b}_l \tilde{\delta}_{\Delta+\ddot{o}-l}; \tilde{p}_{\Delta-1} + [\tilde{p}_{\Delta-1}, \tilde{p}_0]\tilde{\delta}_0) =$$

$$(\tilde{\mathfrak{w}}_\Delta + \tilde{\mathfrak{b}}; \tilde{p}_{\Delta-1} + [\tilde{p}_{\Delta-1}, \tilde{p}_0]\tilde{\delta}_0).$$

All other summands in the definition of $T(\mathfrak{h})$ are mapped to the corresponding elements in the definition of $T(\tilde{\mathfrak{h}})$. Thus,

$$(\varpi \times \varpi)T(\mathfrak{h}) = T(\tilde{\mathfrak{h}}).$$

It remains to show that $\varpi \times \varpi$ is compatible with boundary values. This, however, is immediate since

$$\widetilde{\Psi^{ac}} \circ (\varpi \times \varpi)|_{T(\mathfrak{h})} = (C_\varphi \times C_\varphi) \circ \Psi^{ac}.$$

Next let us deal with the case that $\mathfrak{h}$ is of kind (B), where $d_{2\Delta-1} = 0 = \tilde{d}_{2\Delta-1}$. Then the last condition in (5.11) implies $b_{\ddot{o}+1} = \tilde{b}_{\ddot{o}+1}$. We define $\varpi : \mathcal{P}(\mathfrak{h}) \to \mathcal{P}(\tilde{\mathfrak{h}})$ by linearity and

$$\varpi(\delta_j) := \tilde{\delta}_j,\ j = 0, \ldots, \ddot{o},\ \varpi(p_0) := \tilde{p}_0.$$

Since $\tilde{d}_0 = d_0$ and $\tilde{c}_j = c_j$, $j = 1, \ldots, \ddot{o}$, this defines an isometric isomorphism. Here the c_j and $\tilde{c}_j$ are to be obtained from the data b_j and $\tilde{b}_j$ according to (4.2). Clearly it is compatible with the respective conjugations. The condition (ii) follows from Remark 3.19. The fact that $(\varpi \times \varpi)T(\mathfrak{h}) = T(\tilde{\mathfrak{h}})$ follows from the definition (4.4) of $T(\mathfrak{h})$, since $\tilde{b}_j = b_j$, $j = 1, \ldots, \ddot{o} + 1$.

Compatibility with boundary values follows since, by (4.5),

$$\Lambda(\mathfrak{h}) \circ (\varpi \times \varpi)|_{T(\mathfrak{h})} = \Lambda(\tilde{\mathfrak{h}}).$$

Thus also the condition (i) holds.

Finally let us settle the case that $\mathfrak{h}$ is of kind (C). We define $\varpi : \mathcal{P}(\mathfrak{h}) \to \mathcal{P}(\tilde{\mathfrak{h}})$ by linearity and $\varpi(p_0) := \tilde{p}_0$. In the same manner as above the desired properties of ϖ are verified.

❑

It is interesting to note that a converse of Proposition 5.17 holds.

5.18 Proposition. *Let* $\mathfrak{h} = (H_\pm, \ddot{o}, b_j, d_j)$ *and* $\tilde{\mathfrak{h}} = (\tilde{H}_\pm, \tilde{\ddot{o}}, \tilde{b}_j, \tilde{d}_j)$ *be elementary indefinite Hamiltonians given on* $I = (s_-, s) \cup (s, s_+)$ *and* $\tilde{I} := (\tilde{s}_-, \tilde{s}) \cup (\tilde{s}, \tilde{s}_+)$, *respectively. Assume that there exists an isometric isomorphism* $\varpi : \mathcal{P}(\mathfrak{h}) \to \mathcal{P}(\tilde{\mathfrak{h}})$ *and an absolutely continuous and increasing bijection* φ *of* $[\tilde{s}_-, \tilde{s}_+]$ *onto* $[s_-, s_+]$ *with the property that also* φ^{-1} *is absolutely continuous, and such that* (i) *and* (ii) *of Proposition 5.17 hold. Then the relations* (5.11) *hold.*

Proof. Let $\hat{\mathfrak{h}}$ be the elementary indefinite Hamiltonian given by the data

$$\hat{H}_\pm := \big(\tilde{H}_\pm \circ (\varphi^{-1})\big) \cdot (\varphi^{-1})',\ \hat{\ddot{o}} := \tilde{\ddot{o}}, \hat{b}_j := \tilde{b}_j, \hat{d}_j := \tilde{d}_j \,.$$

By Proposition 5.17 there exists an isomorphism ϖ' of $\mathcal{P}(\tilde{\mathfrak{h}})$ onto $\mathcal{P}(\hat{\mathfrak{h}})$ which satisfies (i) and (ii) of Proposition 5.17. In order to establish the present assertion, we may therefore assume without loss of generality that $\varphi = \mathrm{id}$.

The relation (ii) implies that $\varpi(\ker \psi(\mathfrak{h})) = \ker \psi(\tilde{\mathfrak{h}})$. It follows that $\Delta + \ddot{o} = \tilde{\Delta} + \tilde{\ddot{o}}$. Passing to the orthogonal complements, we conclude that $\varpi(\mathcal{C}(\mathfrak{h})) = \mathcal{C}(\tilde{\mathfrak{h}})$. Since $\psi(\mathfrak{h})$ maps $\mathcal{C}(\mathfrak{h})$ isometrically onto $L^2(\mathfrak{h})$ and $\psi(\tilde{\mathfrak{h}})$ has the respective property, it follows that id maps $L^2(H)$ isometrically onto $L^2(\tilde{H})$. Thus $H = \tilde{H}$. This implies $\Delta = \tilde{\Delta}$ and, hence, $\ddot{o} = \tilde{\ddot{o}}$.

The condition (i) implies that $(\varpi \times \varpi)S(\mathfrak{h}) = S(\tilde{\mathfrak{h}})$ and thus that

$$(\varpi \times \varpi)\big(S(\mathfrak{h}) \cap (\mathcal{C}(\mathfrak{h})^\circ \dot{+} X^\delta)^2\big) = \big(S(\tilde{\mathfrak{h}}) \cap (\mathcal{C}(\tilde{\mathfrak{h}})^\circ \dot{+} \tilde{X^\delta})^2\big) \,. \tag{5.12}$$

By Corollary 5.8 the relation $S(\mathfrak{h}) \cap (\mathcal{C}(\mathfrak{h})^\circ \dot{+} X^\delta)^2$ is just the shift operator given by

$$\delta_0 \mapsto \delta_1, \dots, \delta_{\Delta+\ddot{o}-2} \mapsto \delta_{\Delta+\ddot{o}-1} \,. \tag{5.13}$$

The same holds for $S(\tilde{\mathfrak{h}}) \cap (\mathcal{C}(\tilde{\mathfrak{h}})^\circ \dot{+} \tilde{X^\delta})^2$. Since

$$\operatorname{dom} \big(S(\mathfrak{h}) \cap (\mathcal{C}(\mathfrak{h})^\circ \dot{+} X^\delta)^2\big)^{\Delta+\ddot{o}-1} = \operatorname{span}\{\delta_0\} \,,$$

we see that $\varpi\delta_0 = \lambda\tilde{\delta}_0$ for some $\lambda \in \mathbb{C} \setminus \{0\}$.

We have $(\chi_+\binom{1}{0}; \delta_0) \in T(\mathfrak{h})$ and thus $(\varpi\chi_+\binom{1}{0}; \lambda\tilde{\delta}_0) \in T(\tilde{\mathfrak{h}})$. By (ii)

$$\varpi\chi_+\begin{pmatrix}1\\0\end{pmatrix} = \tilde{\chi}_+\begin{pmatrix}1\\0\end{pmatrix} + \tilde{x} \,,$$

for some $\tilde{x} \in \ker \psi(\tilde{\mathfrak{h}})$. It follows that

$$(\tilde{x}; (\lambda - 1)\tilde{\delta}_0) = (\varpi\chi_+\begin{pmatrix}1\\0\end{pmatrix}; \lambda\tilde{\delta}_0) - (\tilde{\chi}_+\begin{pmatrix}1\\0\end{pmatrix}; \tilde{\delta}_0) \in T(\tilde{\mathfrak{h}}) \,.$$

By (i)

$$(\begin{pmatrix}1\\0\end{pmatrix}; 0) \in \Gamma(\tilde{\mathfrak{h}})(\varpi\chi_+\begin{pmatrix}1\\0\end{pmatrix}; \varpi\delta_0) \cap \Gamma(\tilde{\mathfrak{h}})(\tilde{\chi}_+\begin{pmatrix}1\\0\end{pmatrix}; \tilde{\delta}_0) \,,$$

and thus

$$\big(\tilde{x}; (\lambda - 1)\tilde{\delta}_0\big) \in S(\tilde{\mathfrak{h}}) \,.$$

If $\lambda \neq 1$, this would contradict the fact that

$$\operatorname{ran} \big(S(\tilde{\mathfrak{h}}) \cap (\mathcal{C}(\tilde{\mathfrak{h}})^\circ \dot{+} \tilde{X^\delta})^2\big) = \operatorname{span}\{\delta_1, \dots, \delta_{\Delta+\ddot{o}-1}\} \,.$$

Thus $\lambda = 1$, i.e. $\varpi\delta_0 = \tilde{\delta}_0$. From (5.12) and (5.13) it now follows that also $\varpi\delta_j = \tilde{\delta}_j$, $j = 1, \ldots, \Delta + \ddot{o} - 1$. We conclude that $\tilde{c}_j = c_j$ and hence $\tilde{b}_j = b_j$, $j = 1, \ldots, \ddot{o}$.

By (i) we have $\varpi(\ker T(\mathfrak{h})) = \ker T(\tilde{\mathfrak{h}})$. Thus $\varpi p_0 \in \ker T(\tilde{\mathfrak{h}})$ ($=$ span$\{p_0, \binom{1}{0}\}$ in case (A) and $=$ span$\{p_0\}$ in case (B) and (C)). Using once more (i) we get $((\varpi p_0 - \tilde{p}_0; 0); \binom{0}{0}) \in \Gamma(\tilde{\mathfrak{h}})$. As in any case $(\ker T(\tilde{\mathfrak{h}}) \times \{0\}) \cap \ker \Gamma(\tilde{\mathfrak{h}}) = \{(0,0)\}$ it follows that $\varpi p_0 = \tilde{p}_0$, and in turn $d_0 = \tilde{d}_0$. We proceed inductively. Assume that $k \in \{0, \ldots, \Delta - 2\}$ and that

$$\varpi p_j = \tilde{p}_j, j = 0, \ldots, k,\ d_i = \tilde{d}_i, i = 0, \ldots, 2k\,.$$

We have

$$\varpi(p_k + d_k\delta_0) = \tilde{p}_k + \tilde{d}_k\tilde{\delta}_0\,,$$

and thus

$$(\varpi p_{k+1} - \tilde{p}_{k+1}; 0) = (\varpi \times \varpi)(p_{k+1}; p_k + d_k\delta_0) - (\tilde{p}_{k+1}; \tilde{p}_k + \tilde{d}_k\tilde{\delta}_0) \in T(\tilde{\mathfrak{h}})\,.$$

Since $H = \tilde{H}$, also $\mathfrak{w}_j = \tilde{\mathfrak{w}}_j$, $j \in \mathbb{N} \cup \{0\}$. Therefore, and by (i),

$$((\varpi p_{k+1} - \tilde{p}_{k+1}; 0); \begin{pmatrix}0\\0\end{pmatrix}) \in \Gamma(\tilde{\mathfrak{h}})$$

As above it follows that $\varpi p_{k+1} = \tilde{p}_{k+1}$. This implies

$$\tilde{d}_{2k+1} = [\tilde{p}_{k+1}, \tilde{p}_k] = [\varpi p_{k+1}, \varpi p_k] = d_{2k+1}$$

and, similarly, $\tilde{d}_{2k+2} = d_{2k+2}$.

Thus $\varpi p_j = \tilde{p}_j$, $j = 0, \ldots, \Delta - 1$, and $\tilde{d}_j = d_j$, $j = 0, \ldots, 2\Delta - 2$.

Since we already know that $\varpi\delta_j = \tilde{\delta}_j$ and $\tilde{b}_j = b_j$, $j = 1, \ldots, \ddot{o}$, we also have (recall $\mathfrak{b} := \sum_{l=1}^{\ddot{o}+1} b_l\delta_{1+\ddot{o}-l}$)

$$(\varpi(\mathfrak{w}_\Delta + b_{\ddot{o}+1}\delta_{\Delta-1}) - \tilde{\mathfrak{w}}_\Delta + \tilde{b}_{\ddot{o}+1}\tilde{\delta}_{\Delta-1}; 0) = (\varpi \times \varpi)(\mathfrak{w}_\Delta + \mathfrak{b}; p_{\Delta-1} + d_{\Delta-1}\delta_{\Delta-1}) -$$

$$-(\tilde{\mathfrak{w}}_\Delta + \tilde{\mathfrak{b}}; \tilde{p}_{\Delta-1} + \tilde{d}_{\Delta-1}\tilde{\delta}_{\Delta-1}) \in T(\tilde{\mathfrak{h}})\,,$$

and the same argument as above allows us to conclude that $\varpi(\mathfrak{w}_\Delta + b_{\ddot{o}+1}\delta_{\Delta-1}) = \tilde{\mathfrak{w}}_\Delta + \tilde{b}_{\ddot{o}+1}\tilde{\delta}_{\Delta-1}$. Scalar multiplication with $p_{\Delta-1}$ ($\tilde{p}_{\Delta-1}$) yields $\tilde{d}_{2\Delta-1} - \tilde{b}_{\ddot{o}+1} = d_{2\Delta-1} - b_{\ddot{o}+1}$.

Finally let us remark that repeating the above arguments we also see that $(j = 1, \ldots, \Delta)$ $\varpi(\mathfrak{w}_{\Delta-1+j} + b_{\ddot{o}+1}\delta_{\Delta-j}) = \tilde{\mathfrak{w}}_\Delta + \tilde{b}_{\ddot{o}+1}\tilde{\delta}_{\Delta-1}$.

❑

Similar as in the definite situation also order-reversing reparameterizations can be studied.

5.19 Lemma. *Let $\mathfrak{h} = (H_\pm, \ddot{o}, b_j, d_j)$ be an elementary indefinite Hamiltonian given on $I = (s_-, s) \cup (s, s_+)$, let $\tilde{s}_-, \tilde{s}_+ \in \mathbb{R}$, $\tilde{s}_- < \tilde{s}_+$, and let φ be an absolutely continuous and decreasing bijection of $[\tilde{s}_-, \tilde{s}_+]$ onto $[s_-, s_+]$ such that also φ^{-1} is absolutely continuous. Put $\tilde{s} := \varphi^{-1}(s)$.*

If $\tilde{\mathfrak{h}} = (\tilde{H}_\pm, \tilde{\ddot{o}}, \tilde{b}_j, \tilde{d}_j)$ is an elementary indefinite Hamiltonian on $\tilde{I} := (\tilde{s}_-, \tilde{s}) \cup (\tilde{s}, \tilde{s}_+)$ such that

$$\tilde{H}_\pm = -(H_\pm \circ \varphi) \cdot \varphi',\ \tilde{\ddot{o}} = \ddot{o}; \tilde{c}_j = (-1)^{j+\ddot{o}} c_j,\ j = 1 \ldots, \ddot{o};$$

$$\tilde{d}_j = (-1)^j d_j,\ j = 0, \dots, 2\Delta - 2; -\tilde{d}_{2\Delta-1} - \tilde{b}_{\ddot{o}+1} = d_{2\Delta-1} - b_{\ddot{o}+1}\,,$$

then there exists an isometric isomorphism $\varpi : \mathcal{P}(\mathfrak{h}) \to \mathcal{P}(\tilde{\mathfrak{h}})$ which is compatible with the respective involutions, such that

(i) *We have*

$$(\varpi \times \varpi) T(\mathfrak{h}) = -T(\tilde{\mathfrak{h}})\,,$$

and

$$\Gamma(\tilde{\mathfrak{h}}) \circ ((-\varpi) \times \varpi)|_{T(\mathfrak{h})} = \psi \circ \Gamma(\mathfrak{h})\,,$$

where

$$\psi : \begin{cases} \mathbb{C}^2 \times \mathbb{C}^2 & \to & \mathbb{C}^2 \times \mathbb{C}^2 \\ (a;b) & \mapsto & (b;a) \end{cases}$$

(ii) *For all $x \in \mathcal{P}(\mathfrak{h})$ we have*

$$\psi(\tilde{\mathfrak{h}})(\varpi(x)) = \psi(\mathfrak{h})(x) \circ \varphi\,.$$

Proof. The proof is in essence the same as that of Proposition 5.17. The necessary changes, cf. the appearance of different signs in the correspondence of $\tilde{\mathfrak{h}}$ and $\mathfrak{h}$, origins in the fact that in the present situation

$$\tilde{\mathfrak{w}}_j = (-1)^j \mathfrak{w}_j,\ \text{and } C_\varphi \circ B = -\tilde{B} \circ C_\varphi\,.$$

We leave the details to the reader.

❏

6 Pasting of boundary value spaces

The idea of gluing together boundary value problems by means of "continuous boundary values" appeared in various places, see e.g. [HSW]. We will use this idea to glue together elementary indefinite Hamiltonians. For this reason we need a formal definition of this proceedure in sufficient generality and we have to establish some properties of it. We do not state results in their most general form; we content ourselves with what is needed in the sequel.

6.1 Definition. Let $(\mathcal{P}_1, T_1, \Gamma_1)$ be a boundary triplet of defect 2 and $(\mathcal{P}_2, T_2, \Gamma_2)$ a boundary triplet of defect 2 or 1. Consider the Pontryagin space $\mathcal{P}_1 \oplus \mathcal{P}_2$, whose elements will be written as $f_1 + f_2$, $f_1 \in \mathcal{P}_1$, $f_2 \in \mathcal{P}_2$, and define

$$\begin{aligned} T_1 \uplus T_2 := \big\{ (f_1 + f_2; g_1 + g_2) \in (\mathcal{P}_1 \oplus \mathcal{P}_2)^2 :\ & (f_1; g_1) \in T_1, (f_2; g_2) \in T_2, \\ \exists\, a, b, c \in \mathbb{C}^2 : \big((f_1; g_1); (a; b)\big) \in \Gamma_1, & \big((f_2; g_2); (b; c)\big) \in \Gamma_2 \big\} \end{aligned} \tag{6.1}$$

$$\begin{aligned} \Gamma_1 \uplus \Gamma_2 := \big\{ \big((f_1 + f_2; g_1 + g_2); (a; c)\big) \in (T_1 \uplus T_2) \times (\mathbb{C}^2 \times \mathbb{C}^2) : & \\ \exists\, b \in \mathbb{C}^2 : \big((f_1; g_1); (a; b)\big) \in \Gamma_1, \big((f_2; g_2); (b; c)\big) \in \Gamma_2 & \big\} \end{aligned} \tag{6.2}$$

We will use the notation

$$\big(\mathcal{P}_1, T_1, \Gamma_1\big) \uplus \big(\mathcal{P}_2, T_2, \Gamma_2\big) := \big(\mathcal{P}_1 \oplus \mathcal{P}_2, T_1 \uplus T_2, \Gamma_1 \uplus \Gamma_2\big)$$

Note that, by definition, $\operatorname{dom}(\Gamma_1 \uplus \Gamma_2) = T_1 \uplus T_2$. This choice of notation is justified by the following result.

6.2 Proposition. *Let $(\mathcal{P}_1, T_1, \Gamma_1)$ be a boundary triplet of defect 2 and $(\mathcal{P}_2, T_2, \Gamma_2)$ a boundary triplet of defect 2 (or of defect 1). Write* $\operatorname{mul}\Gamma_1 = \operatorname{span}\{(m_1; m_1)\}$ *and* $\operatorname{mul}\Gamma_2 = \operatorname{span}\{(m_2; m_2)\}$ *(or* $\operatorname{mul}\Gamma_2 = \operatorname{span}\{(m_2; 0)\}$*, respectively) with some (possibly vanishing) real elements m_1 and m_2. Assume that:*

(LI) *If* $\operatorname{mul}\Gamma_1 \neq \{0\}$ *and* $\operatorname{mul}\Gamma_2 \neq \{0\}$*, then m_1 and m_2 are linearly independent.*

Then $\big(\mathcal{P}_1, T_1, \Gamma_1\big) \uplus \big(\mathcal{P}_2, T_2, \Gamma_2\big)$ is a boundary triplet of defect 2 (or 1, respectively). Moreover, $\operatorname{mul}(\Gamma_1 \uplus \Gamma_2) = \{0\}$.

The proof of Proposition 6.2 will be carried out in several steps.

Step 1: The fact that $\operatorname{dom}(\Gamma_1 \uplus \Gamma_2) = T_1 \uplus T_2$ holds by definition. The compatibility of $T_1 \uplus T_2$ and $\Gamma_1 \uplus \Gamma_2$ with the (componentwise defined) involution $\bar{\cdot}$ is also immediate from the definition. We show that $\Gamma_1 \uplus \Gamma_2$ satisfies the Green's identity (2.6). Let

$$\big((f_1 + f_2; g_1 + g_2); (x_1; x_2)\big), \big((h_1 + h_2; k_1 + k_2); (y_1; y_2)\big) \in \Gamma_1 \uplus \Gamma_2$$

be given, and let $b, c \in \mathbb{C}^2$ be such that

$$\big((f_1; g_1); (x_1; b)\big) \in \Gamma_1, \big((f_2; g_2); (b; x_2)\big) \in \Gamma_2 \,,$$

$$\big((h_1; k_1); (y_1; c)\big) \in \Gamma_1, \big((h_2; k_2); (c; y_2)\big) \in \Gamma_2 \,.$$

Then

$$[g_1 + g_2, h_1 + h_2]_{\mathcal{P}_1 \oplus \mathcal{P}_2} - [f_1 + f_2, k_1 + k_2]_{\mathcal{P}_1 \oplus \mathcal{P}_2} =$$
$$\Big([g_1, h_1]_{\mathcal{P}_1} - [f_1, k_1]_{\mathcal{P}_1}\Big) + \Big([g_2, h_2]_{\mathcal{P}_2} - [f_2, k_2]_{\mathcal{P}_2}\Big) =$$
$$= \big(y_1^* J x_1 - c^* J b\big) + \big(c^* J b - y_2^* J x_2\big) = y_1^* J x_1 - y_2^* J x_2 \,.$$

Step 2: We show that $\Gamma_1 \uplus \Gamma_2$ is closed. Consider the map

$$Q : \left\{ \begin{array}{ccc} \big(\mathcal{P}_1^2 \times (\mathbb{C}^2 \times \mathbb{C}^2)\big) \times \big(\mathcal{P}_2^2 \times (\mathbb{C}^2 \times \mathbb{C}^2)\big) & \to & (\mathcal{P}_1 \times \mathcal{P}_2)^2 \times (\mathbb{C}^2 \times \mathbb{C}^2) \\ \big((x_1; y_1), (a; b), (x_2; y_2), (c; d)\big) & \mapsto & \big(((x_1; x_2), (y_1; y_2)), (a; d)\big) \end{array} \right.$$

Then Q is continuous, surjective and $\ker Q$ is finite-dimensional. Hence Q maps closed subspaces to closed subspaces. However, we have

$$\Gamma_1 \uplus \Gamma_2 = Q\Big(\big(\Gamma_1 \oplus \Gamma_2\big) \cap \big\{((x_1; y_1), (a; b), (x_2; y_2), (c; d)) :\, b = c\big\}\Big) \,.$$

In order to see that $T_1 \uplus T_2$ is closed we consider the map

$$R : \left\{ \begin{array}{ccc} (\mathcal{P}_1 \times \mathcal{P}_2)^2 \times (\mathbb{C}^2 \times \mathbb{C}^2) & \to & (\mathcal{P}_1 \times \mathcal{P}_2)^2 \\ \big(((x_1; x_2), (y_1; y_2)), (a; d)\big) & \mapsto & \big((x_1; x_2), (y_1; y_2)\big) \end{array} \right.$$

Again this map is continuous, surjective, has a finite-dimensional kernel, and thus maps closed subspaces onto closed subspaces. However,

$$T_1 \uplus T_2 = R(\Gamma_1 \uplus \Gamma_2) \,.$$

Step 3: Next we show that $\Gamma_1 \uplus \Gamma_2$ is an operator. Assume that $((0;0);(a;c)) \in \Gamma_1 \uplus \Gamma_2$ and let $b \in \mathbb{C}^2$ such that

$$((0;0);(a;b)) \in \Gamma_1,\ ((0;0);(b;c)) \in \Gamma_2\,.$$

It follows that $(a;b) = \lambda(m_1;m_1)$ and $(b;c) = \mu(m_2;m_2)$ (or $(b;c) = \mu(m_2;0)$ in case of defect 1, respectively). If $m_1 = 0$ then $a = b = 0$ and thus $m_2 = 0$ or $\mu = 0$. We obtain $c = 0$. If $m_2 = 0$ the same argument applies. If m_1 and m_2 are both nonzero, and thus linearly independent, we conclude from $\lambda m_1 = \mu m_2$ that $\lambda = \mu = 0$. Again it follows that $a = b = c = 0$.

To complete the proof of Proposition 6.2 it remains to establish two assertions: Firstly, that $\operatorname{ran}(\Gamma_1 \uplus \Gamma_2) = \mathbb{C}^2 \times \mathbb{C}^2$ or, in case of defect 1, $\operatorname{ran}(\Gamma_1 \uplus \Gamma_2) = \mathbb{C}^2 \times \{0\}$, respectively. Secondly, that $(T_1 \uplus T_2)^* = \ker(\Gamma_1 \uplus \Gamma_2)$. In order to show these assertions, we need the following two statements which deal, in essence, with the case that $\operatorname{mul}\Gamma \neq \{0\}$.

6.3 Lemma. *Let $(\mathcal{P}, T, \Gamma)$ be a boundary triplet of defect 2. Then*

- (i) *For all $x \in \mathbb{C}^2$ there exists $y \in \mathbb{C}^2$ such that $(x;y) \in \operatorname{ran}\Gamma$.*
- (ii) *For all $y \in \mathbb{C}^2$ there exists $x \in \mathbb{C}^2$ such that $(x;y) \in \operatorname{ran}\Gamma$.*
- (iii) *Write $\operatorname{mul}\Gamma = \operatorname{span}\{(m;m)\}$ with a real element m. Then we have*

$$\operatorname{span}\{m\} \times \operatorname{span}\{m\} \subseteq \operatorname{ran}\Gamma\,. \tag{6.3}$$

Let $(\mathcal{P}, T, \Gamma)$ be a boundary triplet of defect 1 and assume that $\operatorname{mul}\Gamma = \operatorname{span}\{(m;0)\} \neq \{0\}$. Then $\operatorname{ran}\Gamma = \operatorname{span}\{(m;0)\}$.

Proof. Assume first that $(\mathcal{P}, T, \Gamma)$ is of defect 2. If $\operatorname{mul}\Gamma = \{0\}$ the assertions (i)-(iii) immediately follow from Remark 2.9, (ii). Hence assume moreover, that $\operatorname{mul}\Gamma \neq \{0\}$, and write $\operatorname{mul}\Gamma = \operatorname{span}\{(m;m)\}$ with a real and nonzero element m.

Certainly, for $x = m$ there exists y, namely $y = m$. Assume that $\operatorname{ran}\Gamma \subseteq \operatorname{span}\{m\} \times \mathbb{C}^2$. Since $(m;m) \in \operatorname{mul}\Gamma$, we obtain from (2.6) that for all $(x;y) \in \operatorname{ran}\Gamma$

$$0 = m^* J x - m^* J y = -m^* J y\,,$$

and thus that $y \in \operatorname{span}\{m\}$. Therefore we would have $\operatorname{ran}\Gamma \subseteq \operatorname{span}\{m\} \times \operatorname{span}\{m\}$. This, however, implies by (2.6) that $T \subseteq T^*$, a contradiction. The second assertion follows in the same way. To see (6.3) choose $(f_1;g_1), (f_2;g_2) \in T$ which are linearly independent modulo T^*, and let $(a;b), (c;d) \in \mathbb{C}^2 \times \mathbb{C}^2$ be such that

$$\big((f_1;g_1);(a;b)\big), \big((f_2;g_2);(c;d)\big) \in \Gamma\,.$$

We show that $(a;b), (c;d)$ and $(m;m)$ are linearly independent: If $\lambda, \mu, \nu \in \mathbb{C}$ are such that

$$\lambda(a;b) + \mu(c;d) + \nu(m;m) = 0\,,$$

then

$$\big(\lambda(f_1;g_1) + \mu(f_2;g_2); \lambda(a;b) + \mu(c;d) + \nu(m;m)\big) \in \Gamma$$

and thus $\lambda(f_1;g_1) + \mu(f_2;g_2) \in \ker\Gamma = T^*$. This implies that $\lambda = \mu = 0$ and thus also that $\nu = 0$. We can therefore choose a nonvanishing linear combination of these three elements which is of the form $(x;0)$. By (2.6) it follows that $0 = m^*Jx - m^*J0 = m^*Jx$ and hence that $x \in \operatorname{span}\{m\}$. Since certainly $(m;m) \in \operatorname{ran}\Gamma$, the relation (6.3) follows.

Assume now that $(\mathcal{P},T,\Gamma)$ is of defect 1. Certainly

$$\{0\} \neq \operatorname{span}\{(m;0)\} \subseteq \operatorname{ran}\Gamma \subseteq \mathbb{C}^2 \times \{0\}\,.$$

Since in the present case $T = T^*$, $\operatorname{ran}\Gamma$ must be neutral (with respect to the J-inner product on $\mathbb{C}^2 \times \{0\}$, cf. the second half of Remark 2.10). Therefore, it is not possible that $\operatorname{ran}\Gamma = \mathbb{C}^2 \times \{0\}$. We conclude that $\operatorname{ran}\Gamma = \operatorname{mul}\Gamma = \operatorname{span}\{(m;0)\}$.

❑

6.4 Corollary. *Let $(\mathcal{P},T,\Gamma)$ be a boundary triplet of defect 2. Write $\operatorname{mul}\Gamma = \operatorname{span}\{(m;m)\}$ with a (possibly vanishing) real element m. Let $y,a \in \mathbb{C}^2$ and $\alpha \in \mathbb{C}$ be given and assume that $a \notin \operatorname{span}\{m\}$. Then there exist $x,z \in \mathbb{C}^2$ such that $(x;y) \in \operatorname{ran}\Gamma$, $(y;z) \in \operatorname{ran}\Gamma$, and*

$$x^*Ja = \alpha,\ z^*Ja = \alpha\,.$$

Proof. If $\operatorname{mul}\Gamma = \{0\}$, choose any $x \in \mathbb{C}^2$ with $x^*Ja = \alpha$. Then, by Remark 2.9, (ii), we have $(x;y) \in \operatorname{ran}\Gamma$ and the assertion is proved. Consider the case that $\operatorname{mul}\Gamma \neq \{0\}$. Choose $x \in \mathbb{C}^2$ such that $(x;y) \in \operatorname{ran}\Gamma$. Since by our assumption $m^*Ja \neq 0$, we can choose $\lambda \in \mathbb{C}$ such that $(x+\lambda m)^*Ja = \alpha$. By (6.3) we have $(x+\lambda m;y) \in \operatorname{ran}\Gamma$. The existence of z is established in the same way.

❑

Step 4: We come to the proof that, in case that $(\mathcal{P}_2,T_2,\Gamma_2)$ is of defect 2, the relation $\Gamma_1 \uplus \Gamma_2$ is surjective. The case that $(\mathcal{P}_2,T_2,\Gamma_2)$ is of defect 1 is treated similar and we will omit the details.

Let $a,c \in \mathbb{C}^2$ be given. Assume first that $\operatorname{mul}\Gamma_1 = \{0\}$. Choose $b \in \mathbb{C}^2$ such that $(b;c) \in \operatorname{ran}\Gamma_2$, cf. Lemma 6.3. By Remark 2.9, (ii), the element $(a;b)$ belongs to $\operatorname{ran}\Gamma_1$. It follows that $(a;c) \in \operatorname{ran}(\Gamma_1 \uplus \Gamma_2)$. If $\operatorname{mul}\Gamma_2 = \{0\}$, we can argue in the same way.

Assume now that

$$\operatorname{mul}\Gamma_1 = \operatorname{span}\{(m_1;m_1)\} \neq \{0\},\ \operatorname{mul}\Gamma_2 = \operatorname{span}\{(m_2;m_2)\} \neq \{0\}\,.$$

Choose $x,y \in \mathbb{C}^2$ such that $(a;x) \in \operatorname{ran}\Gamma_1$ and $(y;c) \in \operatorname{ran}\Gamma_2$. Then, by our assumption (LI), there exist $\lambda,\mu \in \mathbb{C}$ such that $y - x = \lambda m_1 - \mu m_2$. It follows from (6.3) that

$$(a;x+\lambda m_1) \in \operatorname{ran}\Gamma_1,\ (y+\mu m_2;c) \in \operatorname{ran}\Gamma_2\,.$$

Since $x + \lambda m_1 = y + \mu m_2$, we conclude that $(a;c) \in \operatorname{ran}(\Gamma_1 \uplus \Gamma_2)$.

Step 5: Our final task is to prove that $(T_1 \uplus T_2)^* = \ker(\Gamma_1 \uplus \Gamma_2)$. By the Green's identity (2.6) clearly

$$\ker(\Gamma_1 \uplus \Gamma_2) \subseteq (T_1 \uplus T_2)^*\,.$$

Conversely, let $(f_1 + f_2; g_1 + g_2) \in (T_1 \uplus T_2)^*$ be given. For all $(h_1; k_1) \in T_1^* = \ker \Gamma_1$ we have $(h_1 + 0; k_1 + 0) \in T_1 \uplus T_2$, and hence

$$0 = [g_1 + g_2, k_1 + 0] - [f_1 + f_2, h_1 + 0] = [g_1, k_1] - [f_1, h_1] \,.$$

It follows that $(f_1; g_1) \in T_1^{**} = T_1$. The same argument will show that $(f_2; g_2) \in T_2$, and we conclude that $(f_1 + f_2; g_1 + g_2) \in T_1 \oplus T_2$.

Next we show that in fact $(f_1 + f_2; g_1 + g_2) \in T_1 \uplus T_2$. Again, we will restrict explicit proof to the case that $(\mathcal{P}_2, T_2, \Gamma_2)$ is of defect 2. Choose $(a; b), (b'; c) \in \mathbb{C}^2 \times \mathbb{C}^2$ such that

$$\big((f_1; g_1); (a; b)\big) \in \Gamma_1, \ \big((f_2; g_2); (b'; c)\big) \in \Gamma_2 \,.$$

Since $\operatorname{mul} \Gamma_i = \operatorname{span}\{(m_i; m_i)\}$, we can assume that $a \notin \operatorname{span}\{m_1\}$ or $a = 0$ and similarly that $c \notin \operatorname{span}\{m_2\}$ or $c = 0$.

Let $y \in \mathbb{C}^2$ be given. If $a = c = 0$, choose any $x, z \in \mathbb{C}^2$ with $(x; y) \in \operatorname{ran} \Gamma_1$ and $(y; z) \in \operatorname{ran} \Gamma_2$, cf. Lemma 6.3, and let $(h_1; k_1) \in T_1$, $(h_2; k_2) \in T_2$ be such that

$$\big((h_1; k_1); (x; y)\big) \in \Gamma_1, \ \big((h_2; k_2); (y; z)\big) \in \Gamma_2 \,.$$

Then we compute

$$\begin{aligned} 0 &= [g_1 + g_2, h_1 + h_2] - [f_1 + f_2, k_1 + k_2] = \\ &= \big([g_1, h_1] - [f_1, k_1]\big) + \big([g_2, h_2] - [f_2, k_2]\big) = \\ &= \big(x^* J a - y^* J b\big) + \big(y^* J b' - z^* J c\big) = y^* J (b' - b) \,. \end{aligned} \tag{6.4}$$

If one of a and c is not zero, say, $a \neq 0$, choose $z \in \mathbb{C}^2$ with $(y; z) \in \operatorname{ran} \Gamma_2$. By Corollary 6.4 there exists $x \in \mathbb{C}^2$, with $(x; y) \in \operatorname{ran} \Gamma_1$ and $x^* J a = z^* J c$. Again the computation (6.4) can be carried out. Since y was arbitrary, it follows that $b' = b$, and thus that

$$(f_1 + f_2; g_1 + g_2) \in T_1 \uplus T_2 \,.$$

Since $\Gamma_1 \uplus \Gamma_2$ is surjective (or, in case of defect 1 satisfies $\operatorname{ran} \Gamma = \mathbb{C}^2 \times \{0\}$), we conclude from the (by the previous paragraph applicable) Green's identity (2.6) that in fact $(f_1 + f_2; g_1 + g_2) \in \ker(\Gamma_1 \uplus \Gamma_2)$.

As an immediate consequence we obtain that

$$\dim (T_1 \uplus T_2)/(T_1 \uplus T_2)^* = \dim (T_1 \uplus T_2)/\ker(\Gamma_1 \uplus \Gamma_2) = 4$$

or, in case of defect 1, that $\dim (T_1 \uplus T_2)/(T_1 \uplus T_2)^* = \dim (T_1 \uplus T_2)/\ker(\Gamma_1 \uplus \Gamma_2) = 2$. This finishes the proof of Proposition 6.2.

❑

6.5 Remark. Let us note explicitly that $(T_1 \uplus T_2)^*$ is a finite-dimensional extension of $T_1^* \oplus T_2^*$:

$$T_1^* \oplus T_2^* \subseteq (T_1 \uplus T_2)^* \subseteq T_1 \uplus T_2 \subseteq T_1 \oplus T_2 \,. \tag{6.5}$$

6.6 Remark. The operation of pasting boundary triplets is associative: Let $(\mathcal{P}_1, T_1, \Gamma_1)$, $(\mathcal{P}_2, T_2, \Gamma_2)$ be boundary triplets of defect 2 and let $(\mathcal{P}_3, T_3, \Gamma_3)$ be of either defect 2 or 1. Then

$$\big(\mathcal{P}_1 \oplus \mathcal{P}_2, T_1 \uplus T_2, \Gamma_1 \uplus \Gamma_2\big) \uplus (\mathcal{P}_3, T_3, \Gamma_3) = (\mathcal{P}_1, T_1, \Gamma_1) \uplus \big(\mathcal{P}_2 \oplus \mathcal{P}_3, T_2 \uplus T_3, \Gamma_2 \uplus \Gamma_3\big) \,.$$

In fact

$$T_1 \uplus \ldots \uplus T_n = \Big\{ \ (\sum_{j=1}^{n} f_j; \sum_{j=1}^{n} g_j) \in \big(\mathcal{P}_1 \oplus \ldots \oplus \mathcal{P}_n\big)^2 : \\ \exists\, a_0, \ldots, a_n \in \mathbb{C}^2 : \ \big((f_j; g_j); (a_{j-1}; a_j)\big) \in \Gamma_j, \ j = 1, \ldots, n \ \Big\}$$

and

$$\Gamma_1 \uplus \ldots \uplus \Gamma_n = \Big\{ \ (\Big(\sum_{j=1}^{n} f_j; \sum_{j=1}^{n} g_j\Big); (a_0; a_n)) \in \big(\mathcal{P}_1 \oplus \ldots \oplus \mathcal{P}_n\big)^2 : \\ \exists\, a_1, \ldots, a_{n-1} \in \mathbb{C}^2 : \ \big((f_j; g_j); (a_{j-1}; a_j)\big) \in \Gamma_j, \ j = 1, \ldots, n \ \Big\}.$$

Note here that for $n > 1$, $\Gamma_1 \uplus \ldots \uplus \Gamma_n$ is an operator and by Step 3 of the proof of Proposition 6.2 it follows that, in fact, $(\sum_{j=1}^n f_j; \sum_{j=1}^n g_j) \mapsto (a_0; \ldots; a_n)$ is a proper linear mapping.

Next we show that some spectral properties of boundary triplets are inherited by pastings.

6.7 Lemma. *Let $(\mathcal{P}_1, T_1, \Gamma_1)$ be a boundary triplet of defect 2 and $(\mathcal{P}_2, T_2, \Gamma_2)$ be of either defect 2 or 1.*

- *(i) If $z \in \mathbb{C}$ and both $\operatorname{ran}(T_1^* - z)$ and $\operatorname{ran}(T_2^* - z)$ are closed, so is $\operatorname{ran}((T_1 \uplus T_2)^* - z)$.*
- *(ii) If both T_1^* and T_2^* have the property (CR), so has $(T_1 \uplus T_2)^*$.*
- *(iii) If both $(\mathcal{P}_1, T_1, \Gamma_1)$ and $(\mathcal{P}_2, T_2, \Gamma_2)$ satisfy (E), so does $(\mathcal{P}_1 \oplus \mathcal{P}_2, T_1 \uplus T_2, \Gamma_1 \uplus \Gamma_2)$.*

Proof. The assertions (i) and (ii) are immediate since, by (6.5), we deal with finite dimensional extensions and perturbations, respectively.

We come to the proof of (iii). Let $((f_1 + f_2; z(f_1 + f_2)); (a; c)) \in \Gamma_1 \uplus \Gamma_2$. Then there exists $b \in \mathbb{C}^2$ such that

$$\big((f_1; zf_1); (a, b)\big) \in \Gamma_1, \ \big((f_2; zf_2); (b; c)\big) \in \Gamma_2 \,.$$

Assume that $a = 0$. Since (E) holds for Γ_1, we conclude that $f_1 = 0$ and hence that $(a; b) \in \operatorname{mul}\Gamma_1 = \operatorname{span}\{(m_1; m_1)\}$. Since $a = 0$, it follows that also $b = 0$. The condition (E) for Γ_2 now implies that $f_2 = 0$.

❑

6.8 Remark. Let $(\mathcal{P}_j, T_j, \Gamma_j)$ and $(\mathcal{P}_j', T_j', \Gamma_j')$, $j = 1, 2$, be boundary triplets, and assume that (ϖ_j, ϕ_j), $j = 1, 2$, is an isomorphism of $(\mathcal{P}_j, T_j, \Gamma_j)$ and $(\mathcal{P}_j', T_j', \Gamma_j')$, cf. Definition 2.12. Assume that ϕ_1 and ϕ_2 are of the particular form

$$\phi_1 = \hat{\phi}_1 \times \hat{\phi}, \ \phi_2 = \hat{\phi} \times \hat{\phi}_2 \,.$$

Then the pair (ϖ, ϕ), where

$$\varpi := \varpi_1 \oplus \varpi_2, \ \phi := \hat{\phi}_1 \times \hat{\phi}_2 \,,$$

is an isomorphism of

$$(\mathcal{P}_1, T_1, \Gamma_1) \uplus (\mathcal{P}_2, T_2, \Gamma_2)$$

and

$$(\mathcal{P}_1', T_1', \Gamma_1') \uplus (\mathcal{P}_2', T_2', \Gamma_2') .$$

We shall write $(\varpi, \phi) =: (\varpi_1, \phi_1) \uplus (\varpi_2, \phi_2)$. Clearly, this construction can iteratively be applied to any finite number of summands, and is associative.

b. An example

An instructive example of the introduced notion of pasting is found in the theory of positive definite canonical systems.

6.9 Lemma. *Let H be a Hamiltonian on $I = (s_-, s_+)$ which is regular at s_-, and let $s_0 \in (s_-, s_+)$. Assume that s_0 is not inner point of an H-indivisible interval. Put $J_1 := (s_-, s_0)$, $J_2 := (s_0, s_+)$, and $H_1 := H|_{J_1}$, $H_2 := H|_{J_2}$. Then*

$$\big(L^2(H), T_{max}(H), \Gamma(H)\big) =$$

$$= \big(L^2(H_1), T_{max}(H_1), \Gamma(H_1)\big) \uplus \big(L^2(H_2), T_{max}(H_2), \Gamma(H_2)\big) .$$

Proof. We consider $\mathcal{M}(J_1)$ and $\mathcal{M}(J_2)$ as subsets of $\mathcal{M}(I)$. Since s_0 is not contained in an indivisible interval, we have

$$\mathcal{M}(I) = \mathcal{M}(J_1) \dot{+} \mathcal{M}(J_2) \tag{6.6}$$

Clearly, $L^2(H) = L^2(H_1) \oplus L^2(H_2)$.

Let $f \in \mathrm{AC}(I), g \in \mathcal{M}(I)$ be such that $(f; g) \in T_{max}(H)$. Write $f = f_1 + f_2, g = g_1 + g_2$ according to the decomposition (6.6). Then $(f_1; g_1) \in T_{max}(H_1), (f_2; g_2) \in T_{max}(H_2)$, and, since $f_1(s_0) = f(s_0) = f_2(s_0)$, we have

$$(f; g) \in T_{max}(H_1) \uplus T_{max}(H_2) .$$

Moreover, if H is regular at s_+,

$$\Gamma(H)(f; g) = (f(s_-); f(s_+)) \in \big(\Gamma(H_1) \uplus \Gamma(H_2)\big)(f; g) .$$

If H is singular at s_+, the pair $\Gamma(H)(f; g) = (f(s_-); 0)$ has the corresponding property.

Conversely, if $(f_1 + f_2; g_1 + g_2) \in T_{max}(H_1) \uplus T_{max}(H_2)$, then there exist representants $\hat{f}_1, \hat{f}_2$ of f_1, f_2 with $\hat{f}_1' = JHg_1, \hat{f}_2' = JHg_2$ and $\hat{f}_1(s_0) = \hat{f}_2(s_0)$. Thus $\hat{f}_1 + \hat{f}_2 \in \mathrm{AC}(I)$ and we conclude that $(f_1 + f_2; g_1 + g_2) \in T_{max}(H)$. Assume that H is regular at s_+. If $(a; b) \in \Gamma(H_1)(f_1; g_1)$ and $(b; c) \in \Gamma(H_2)(f_2; g_2)$, the choice of $\hat{f}_1, \hat{f}_2$ can be made such that $\hat{f}_1(s_-) = a$, $\hat{f}_1(s_0) = b = \hat{f}_2(s_0)$, $\hat{f}_2(s_+) = c$. It follows that $(a; c) \in \Gamma(H)(f_1 + f_2; g_1 + g_2)$. The case that H is singular at s_+ is treated similarly.

❑

6.10 Remark. This example for the appearance of pasting of boundary triplets also explains the need to give the unusual definition of 'defect 2' and 'defect 1', when the defect of the boundary triplet $(\mathcal{P}, T, \Gamma)$ does not coincide with the defect of the relation T^*. Consider e.g. the case that J_1 is indivisible.

Moreover, it shows that the conditions of Proposition 6.2 are natural. For if $\operatorname{mul}\Gamma(H_1)\neq\{0\}$ and $\operatorname{mul}\Gamma(H_2)\neq\{0\}$, then J_1 and J_2 must end (start) with indivisible intervals, however, their types must be different since we assumed that s_0 is not inner point of an indivisible interval. Thus the assumption of Proposition 6.2 is satisfied. The assumption that s_0 is not contained in an indivisible interval is however necessary. In fact, if s_0 is contained in an indivisible interval, then $L^2(H)$ is not equal to $L^2(H_1)\oplus L^2(H_2)$.

6.11 Corollary. *Assume that H is singular at s_+, that $s_0=\alpha_1^+(H)$, and let ϕ be the type of $(\alpha_1^+(H),s_+)$. Then, with the notation of Lemma 6.9, $L^2(H)=L^2(H_1)$ and*

$$T_{max}(H)=\big\{(f;g)\in T_{max}(H_1):\ \exists(a;b)\in\Gamma(H_1)(f;g):\xi_\phi^T b=0\big\}\tag{6.7}$$

If $(f;g)\in T_{max}(H)$, then the element a appearing on the right side of this formula is unique and $\Gamma(H)(f;g)=(a;0)$.

Proof. Since in the present case $L^2(H_2)=\{0\}$, we have $L^2(H)=L^2(H_1)$. The description (6.7) of $T_{max}(H)$ follows since $T_{max}(H_2)=\{(0;0)\}$ and $\Gamma(H_2)=\operatorname{span}\{(J\xi_\phi;0)\}$. For every element a appearing on the right side of (6.7), we have $(a;0)\in\Gamma(H)(f;g)$. Since $\alpha_1^+(H)=s_0\in(s_-,s_+)$, $\operatorname{mul}\Gamma(H)=\{0\}$. Thus a is unique.

❑

7 Splitting of the model space

Let $\mathfrak{h}$ be an elementary indefinite Hamiltonian of kind (A), let $s^-\in(s_-,s)$ be not inner point of an H-indivisible interval, put $J:=(s_-,s^-)$, and let ι_J be the embedding of the Hilbert space $L^2(H|_J)$ into $\mathcal{P}(\mathfrak{h})$ as in Proposition 4.14. Denote by P_J the orthogonal projection of $\mathcal{P}(\mathfrak{h})$ onto $\iota_J(L^2(H|_J))$, and set $\hat{P}:=I-P_J$. Then, cf. Proposition 4.14, the space $\mathcal{P}(\mathfrak{h})$ splits as

$$\mathcal{P}(\mathfrak{h})=\iota_J(L^2(H|_J))\oplus\hat{P}\mathcal{P}(\mathfrak{h})\,.$$

The present section is devoted to a detailed investigation of this splitting. In fact, our aim is to give an appropriate analogue of Lemma 6.9.

First of all let us determine the action of P_J and $\hat{P}$ in the realization (4.10) of $\mathcal{P}(\mathfrak{h})$.

7.1 Lemma. *Consider $\mathcal{P}(\mathfrak{h})$ as $L^2(H)\oplus(\mathbb{C}^\Delta\times\mathbb{C}^\Delta)\oplus X^\delta$, cf. (4.10), and let $f=(x;\xi;\eta;\alpha)\in\mathcal{P}(\mathfrak{h})$, $\xi=(\xi_i)_{i=0}^{\Delta-1},\eta=(\eta_i)_{i=0}^{\Delta-1},\alpha=\sum_{j=\Delta}^{\Delta+\ddot{o}-1}\alpha_j\delta_j$, be given. Then*

$$P_Jf=\big(x|_J+\sum_{i=0}^{\Delta-1}\eta_i\mathfrak{w}_i|_J;\big(\int_J\mathfrak{w}_j^*H(x+\sum_{i=0}^{\Delta-1}\eta_i\mathfrak{w}_i)\big)_{j=0}^{\Delta-1};0;0\big)\,,$$

$$\hat{P}f=\big(x|_{I\setminus J}-\sum_{i=0}^{\Delta-1}\eta_i\mathfrak{w}_i|_J;\big(\xi_j-\int_J\mathfrak{w}_j^*H(x+\sum_{i=0}^{\Delta-1}\eta_i\mathfrak{w}_i)\big)_{j=0}^{\Delta-1};\eta;\alpha\big)\,.$$

Proof. For $f=(x;\xi;\eta;\alpha)\in\mathcal{P}(\mathfrak{h})$ denote the respective expression on the right side of our asserted equalities by f_0 and $\hat{f}$.

First we show that for each two elements $f=(x;\xi;\eta;\alpha), g=(y;\sigma;\tau;\beta)\in\mathcal{P}(\mathfrak{h})$ we have $[f_0,\hat{g}]=0$. This, however, is immediate from the definition of the inner product:

$$[f_0,\hat{g}]=\big(x|_J+\sum_{i=0}^{\Delta-1}\eta_i\mathfrak{w}_i|_J, y|_{I\setminus J}-\sum_{i=0}^{\Delta-1}\tau_i\mathfrak{w}_i|_J\big)_{L^2(H)}+$$
$$+\sum_{j=0}^{\Delta-1}\overline{\tau_j}\int_J\mathfrak{w}_j^*H(x+\sum_{i=0}^{\Delta-1}\eta_i\mathfrak{w}_i)=0\,.$$

Next note that, clearly, $f_0+\hat{f}=f$. Moreover,

$$f_0=\iota_J\big(x|_J+\sum_{i=0}^{\Delta-1}\eta_i\mathfrak{w}_i|_J\big)\in\iota_J(L^2(H|_J))=P_J\mathcal{P}(\mathfrak{h})\,.$$

Finally, by the explicit form of ι_J obtained in the proof of Proposition 4.14, we see that $f_0=f$ whenever $f=\iota_J x$ for some $x\in L^2(H|_J)$

❑

We obtain a formula for the action of ψP_J and $\psi\hat{P}$.

7.2 Corollary. *Denote by χ_J and $\chi_{I\setminus J}$ the characteristic functions of J and $I\setminus J$, respectively. Then*

$$\psi(P_Jf)=\chi_J\psi f,\ \psi(\hat{P}f)=\chi_{I\setminus J}\psi f,\ f\in\mathcal{P}(\mathfrak{h})\,.$$

Proof. Let the notation be as in Lemma 7.1. Then

$$\psi f=x+\sum_{j=0}^{\Delta-1}\eta_j\mathfrak{w}_j\,.$$

By the above formulas for $P_J,\hat{P}$ we have

$$\psi(P_Jf)=x|_J+\sum_{i=0}^{\Delta-1}\eta_i\mathfrak{w}_i|_J=\chi_J\psi f\,,$$
$$\psi(\hat{P}f)=x|_{I\setminus J}-\sum_{i=0}^{\Delta-1}\eta_i\mathfrak{w}_i|_J+\sum_{i=0}^{\Delta-1}\eta_i\mathfrak{w}_i=\chi_{I\setminus J}\psi f\,.$$

❑

Our next task is to show that the relation $T(\mathfrak{h})$ also splits according to the decomposition $\mathcal{P}(\mathfrak{h})=P_J\mathcal{P}(\mathfrak{h})[\dot{+}]\hat{P}\mathcal{P}(\mathfrak{h})$. To this end define

$$\mathcal{P}_0:=P_J\mathcal{P}(\mathfrak{h}),\ T_0:=(P_J\times P_J)T(\mathfrak{h})\,,$$
$$\Gamma_0:=\big\{\big((f_0;g_0);(x_1;x_2)\big)\in T_0\times(\mathbb{C}^2\times\mathbb{C}^2):\ \exists(f;g)\in T(\mathfrak{h}):$$
$$(P_J\times P_J)(f;g)=(f_0;g_0),x_1=\Psi^{ac}(f;g)_1(s_-),x_2=\Psi^{ac}(f;g)_1(s^-)\big\}$$

and

$$\hat{\mathcal{P}} := \hat{P}\mathcal{P}(\mathfrak{h}),\ \hat{T} := (\hat{P}\times\hat{P})T(\mathfrak{h})\,,$$

$$\hat{\Gamma} := \big\{\big((\hat{f};\hat{g});(x_1;x_2)\big)\in\hat{T}\times(\mathbb{C}^2\times\mathbb{C}^2):\ \exists(f;g)\in T(\mathfrak{h}):$$
$$(\hat{P}\times\hat{P})(f;g)=(\hat{f};\hat{g}), x_1=\Psi^{ac}(f;g)_1(s^-), x_2=\Psi^{ac}(f;g)_1(s_+)\big\}$$

We do not yet know that Γ_0 and $\hat{\Gamma}$ are boundary maps. However, we have the following

7.3 Proposition. *Let $\mathfrak{h}$ be an elementary indefinite Hamiltonian of kind (A), let $s^-\in(s_-,s)$ be not inner point of an H-indivisible interval, and let $P_J,\hat{P}$ etc. be defined as above. Then $T(\mathfrak{h})=T_0\uplus\hat{T}$ and $\Gamma(\mathfrak{h})=\Gamma_0\uplus\hat{\Gamma}$ in the sense of the formulas* (6.1) *and* (6.2).

Proof. Let $(f;g)\in T(\mathfrak{h})$, and put $(f_0;g_0):=(P_J\times P_J)(f;g),(\hat{f};\hat{g}):=(\hat{P}\times\hat{P})(f;g)$. Then $(f_0;g_0)\in T_0,(\hat{f};\hat{g})\in\hat{T}$ and $(f_0;g_0)+(\hat{f};\hat{g})=(f;g)$. Since

$$\big((f_0;g_0);(\Psi^{ac}(f;g)_1(s_-);\Psi^{ac}(f;g)_1(s^-))\big)\in\Gamma_0\,,$$

$$\big((\hat{f};\hat{g});(\Psi^{ac}(f;g)_1(s^-);\Psi^{ac}(f;g)_1(s_+))\big)\in\hat{\Gamma}\,,$$

we obtain

$$(f;g)\in T_0\uplus\hat{T},\ \big((f;g);(\Psi^{ac}(f;g)_1(s_-);\Psi^{ac}(f;g)_1(s_+)\big)\in\Gamma_0\uplus\hat{\Gamma}\,.$$

Thus $T(\mathfrak{h})\subseteq T_0\uplus\hat{T}$ and, since $\operatorname{mul}\Gamma(\mathfrak{h})=\{0\}$, also $\Gamma(\mathfrak{h})\subseteq\Gamma_0\uplus\hat{\Gamma}$.

Conversely, let $((f_0;g_0);(a;b))\in\Gamma_0$, $((\hat{f};\hat{g});(b;c))\in\hat{\Gamma}$ be given. Then there exist $(f_1;g_1),(f_2;g_2)\in T(\mathfrak{h})$ with

$$(P_J\times P_J)(f_1;g_1)=(f_0;g_0),\ (\hat{P}\times\hat{P})(f_2;g_2)=(\hat{f};\hat{g})$$

and

$$\Psi^{ac}(f_1;g_1)(s_-)=a,\ \Psi^{ac}(f_1;g_1)(s^-)=b\,,$$
$$\Psi^{ac}(f_2;g_2)(s^-)=b,\ \Psi^{ac}(f_2;g_2)(s_+)=c\,.$$

Let functions h,k be defined by

$$(h;k):=\Psi^{ac}\big((f_1;g_1)-(f_2;g_2)\big)\,,$$

then $h'=JHk$ on I. It follows that $(h|_J;k|_J)\in T_{max}(H|_J)$. Moreover, we have $h(s^-)=0$. Thus, by Lemma 5.15, $(\iota_J h|_J;\iota_J k|_J)\in T(\mathfrak{h})$. Put

$$(f;g):=(\iota_J h|_J;\iota_J k|_J)+(f_2;g_2)\,,$$

then $(f;g)\in T(\mathfrak{h})$ and, clearly, $(\hat{P}\times\hat{P})(f;g)=(\hat{P}\times\hat{P})(f_2;g_2)=(\hat{f};\hat{g})$. We have

$$\psi(f-f_1)=\psi\big(\iota_J h|_J+f_2-f_1\big)=h|_J+\psi(f_2-f_1)=\begin{cases}0 & ,\ \text{on } J\\ \psi(f_2-f_1) & ,\ \text{on } I\setminus J\end{cases}$$

and thus, by Corollary 7.2, $\psi(P_J(f-f_1))=0$. By Corollary 4.15 the map ψ acts injectively on $\iota_J(L^2(H|_J))$ and we conclude that $P_Jf=P_Jf_1=f_0$. The same argument applies to the second component and yields that also $P_J g = g_0$. We see that $(f_0; g_0) + (\hat{f}; \hat{g}) = (f; g) \in T(\mathfrak{h})$. We have shown $T(\mathfrak{h}) = T_0 \uplus \hat{T}$.

Let us compute $\Gamma(\mathfrak{h})(f; g)$. By Lemma 5.15 we have

$$\Gamma(\mathfrak{h})(f;g)_1 = \Gamma(\mathfrak{h})(\iota_J h|_J; \iota_J k|_J)_1 + \Gamma(\mathfrak{h})(f_2; g_2)_1 = h(s_-) + \Gamma(\mathfrak{h})(f_2; g_2)_1 =$$

$$= \Psi^{ac}\big((f_1; g_1) - (f_2; g_2)\big)_1(s_-) + \Psi^{ac}(f_2; g_2)_1(s_-) = \Psi^{ac}(f_1; g_1)_1(s_-) = a\,,$$

and

$$\Gamma(\mathfrak{h})(f;g)_2 = \Gamma(\mathfrak{h})(\iota_J h|_J; \iota_J k|_J)_2 + \Gamma(\mathfrak{h})(f_2; g_2)_2 = \Gamma(\mathfrak{h})(f_2; g_2)_2 = c\,.$$

It follows that $((f; g); (a; c)) \in \Gamma(\mathfrak{h})$.

❑

It is no surprise that $(\mathcal{P}_0, T_0, \Gamma_0)$ can be identified with $(L^2(H|_J), T_{max}(H|_J), \Gamma(H|_J))$.

7.4 Proposition. *The triple* $(\mathcal{P}_0, T_0, \Gamma_0)$ *is a boundary triplet of defect* 2. *The pair* $(\iota_J; \mathrm{id})$ *is an isomorphism of the boundary triplets* $(L^2(H|_J), T_{max}(H|_J), \Gamma(H|_J))$ *and* $(\mathcal{P}_0, T_0, \Gamma_0)$. *The isomorphism* ι_J *is compatible with the map* ψ *in the sense that*

$$\big(\chi_J \cdot \psi(\mathfrak{h})\big) \circ \iota_J = \mathrm{id}_{L^2(H|_J)}\,. \tag{7.1}$$

Proof. First of all note that, once we have proved that (ι_J, id) has the properties (i), (ii), (iii) in the definition of an isomorphism of boundary triplets, it follows that $(\mathcal{P}_0, T_0, \Gamma_0)$ is a boundary triplet of defect 2, cf. Remark 2.14.

The fact that ι_J is an isometric isomorphism compatible with $\bar{\cdot}$ was shown in Proposition 4.14. The relation (7.1) is nothing else but Corollary 4.15; we restated this property in this place just to point out the structural similarity with the situation in the below Proposition 7.8.

We prove that $(\iota_J \times \iota_J)T_{max}(H|_J) = T_0$. Assume first that $(f_0; g_0) \in T_0$, and define

$$(h; k) := (\psi \times \psi)(f_0; g_0)\,.$$

By Corollary 4.15 it follows that $(h; k) \in L^2(H|_J)$ and $(\iota_J \times \iota_J)(h; k) = (f_0; g_0)$. Let $(f; g) \in T(\mathfrak{h})$ be such that $(P_J \times P_J)(f; g) = (f_0; g_0)$, and write

$$(f; g) = (f_1; g_1) + (f_2; g_2) + (f_3; g_3)$$

with $(f_1; g_1) \in T(\mathfrak{h}) \cap \mathcal{C}^2$, $(f_2; g_2) \in$(4.15) and $(f_3; g_3) \in$(4.16). Then $(\psi f_1; \psi g_1) \in T_{max}(H)$ and hence $(\rho_J \psi f_1; \rho_J \psi g_1) \in T_{max}(H|_J)$. By the definition of ψ and the fact that $\rho_J \psi$ maps $\mathcal{P}(\mathfrak{h})$ onto $L^2(H|_J)$ we find that also $(\rho_J \psi f_2; \rho_J \psi g_2) \in T_{max}(H|_J)$. Finally, $f_3, g_3 \in \ker \rho_J \psi$. Alltogether we see that $(\rho_J \psi f; \rho_J \psi g) \in T_{max}(H|_J)$. However, by Corollary 7.2, $\rho_J \psi f = \psi P_J f = \psi f_0 = h$ and $\rho_J \psi g = \psi P_J g = \psi g_0 = k$.

Conversely, let $(h; k) \in T_{max}(H|_J)$ be given and choose $(\alpha; \beta)^T \in \Gamma(H|_J)(h; k)_2$. Put

$$(h_1; k_1) := (h; k) - \alpha\Big(\binom{1}{0}\Big|_J; 0\Big) - \beta\Big(\binom{0}{1}\Big|_J; 0\Big)\,.$$

Then $(h_1;k_1) \in T_{max}(H|_J)$ and $0 \in \Gamma(H|_J)(h_1;k_1)_2$. It follows from Lemma 5.15 that $(\iota_J \times \iota_J)(h_1;k_1) \in T(\mathfrak{h})$. Since $\iota_J h_1, \iota_J k_1 \in \mathcal{P}_0$, it follows that $(\iota_J \times \iota_J)(h_1;k_1) \in T_0$. Since

$$\Big(\iota_J \begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big|_J; 0\Big) = (P_J \times P_J)\Big(\begin{pmatrix} 1 \\ 0 \end{pmatrix}; 0\Big), \ \Big(\iota_J \begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big|_J; 0\Big) = (P_J \times P_J)\big(p_0; 0\big),$$

we see that also $(\iota_J \times \iota_J)(h;k) \in T_0$.

Finally we have to show that $\Gamma_0 \circ (\iota_J \times \iota_J) = \Gamma(H|_J)$. Let $(h;k) \in T_{max}(L^2(H|_J)$ be given. We use the same notation as in the previous paragraph. Choose $\tilde{h}_1 \in \mathrm{AC}(J)$ with $\tilde{h}_1' = JHk_1$ and $\tilde{h}_1(s^-) = 0$. By the proof of Lemma 5.15, we have $\Psi^{ac}(\iota_J h_1; \iota_J k_1) = (\chi_J \tilde{h}_1; \chi_J k_1)$. Hence $((\iota_J h_1; \iota_J k_1); (\tilde{h}_1(s_-); 0)) \in \Gamma_0$. Since $\Psi^{ac}((1;0)^T;0) = (1;0)^T$ and $\Psi^{ac}(p_0;0) = (0;1)^T$, the assertion follows.

❑

It is a more subtle topic to establish that the triplet $(\hat{\mathcal{P}}, \hat{T}, \hat{\Gamma})$ is isomorphic to a triplet $(\mathcal{P}(\tilde{\mathfrak{h}}), T(\tilde{\mathfrak{h}}), \Gamma(\tilde{\mathfrak{h}}))$ for a certain elementary indefinite Hamiltonian $\tilde{\mathfrak{h}}$ of kind (A).

Assume that $\mathfrak{h}$ consists of the data $H, (d_l)_{l=0}^{2\Delta-1}, \ddot{o}, (b_i)_{i=1}^{\ddot{o}+1}$. Let $\tilde{\mathfrak{w}}_l$ be defined relative to the Hamiltonian $H|_{I\setminus J}$, and let λ_k be the unique real numbers such that

$$\tilde{\mathfrak{w}}_k = \Big(\mathfrak{w}_k - \sum_{j=0}^{k-1} \lambda_{k-j} B^k \chi_- \begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big)\Big|_{I\setminus J}, \ k \in \mathbb{N} \cup \{0\},$$

cf. Lemma 3.12. Here χ_- denotes the indicator function of the interval (s_-, s). Set

$$\mathfrak{d}_k := \sum_{j=0}^{k-1} \lambda_{k-j} B^j \chi_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in X_L, \ k \in \mathbb{N} \cup \{0\}.$$

Note that $(1,0)\mathfrak{d}_k(s_-) = \lambda_k$ and that $(1,0)\mathfrak{d}_k(s^-) = (1,0)\mathfrak{w}_k(s^-)$. Define numbers $\tilde{d}_k$ for $k \in \mathbb{N} \cup \{0\}$ by

$$\tilde{d}_k := d_k - [\mathfrak{d}_k, p_0] + \lambda_{k+1}.$$

Then the data $H|_{I\setminus J}, (\tilde{d}_k)_{k=0}^{2\Delta-1}, \ddot{o}, (b_i)_{i=1}^{\ddot{o}+1}$ represents an elementary indefinite Hamiltonian $\tilde{\mathfrak{h}}$ of kind (A). Define a linear map $\varpi : \mathcal{P}(\tilde{\mathfrak{h}}) \to \hat{\mathcal{P}}$ by

$$\begin{aligned} \varpi\big((x;\xi;0;\alpha)\big) &:= \big(x; \xi + \big((x, \mathfrak{d}_j)_{L^2(H|_{I\setminus J})}\big)_{j=0}^{\Delta-1}; 0; \alpha\big), \\ \varpi(\tilde{p}_k) &:= \hat{P}(p_k - \mathfrak{d}_k), \ k = 0, \dots, \Delta - 1. \end{aligned} \tag{7.2}$$

The first formula has to be understood with respect to the respective realizations (4.10) of $\mathcal{P}(\tilde{\mathfrak{h}})$ and $\mathcal{P}(\mathfrak{h})$. Moreover, we consider $\mathcal{M}(I \setminus J)$ naturally as a subset of $\mathcal{M}(I)$.

7.5 Remark. It is useful to know that in the realizations (4.10) the following set of correspondences hold. Hereby δ_{ij} denotes the Kronecker-Delta.

$$\tilde{p}_k = \big(0; (\tfrac{1}{2}\tilde{d}_{k+j})_{j=0}^{\Delta-1}; (\delta_{kj})_{j=0}^{\Delta-1}; 0\big) \in \mathcal{P}(\tilde{\mathfrak{h}}),$$

$$p_k = \big(0; (\tfrac{1}{2}d_{k+j})_{j=0}^{\Delta-1}; (\delta_{kj})_{j=0}^{\Delta-1}; 0\big) \in \mathcal{P}(\mathfrak{h}),$$

$$\mathfrak{d}_k = \big(\mathfrak{d}_k; ([\mathfrak{d}_k, p_j])_{j=0}^{\Delta-1}; 0; 0\big) \in \mathcal{P}(\mathfrak{h}),$$

$$\hat{P}p_k = \big(-\mathfrak{w}_k|_J; \big(\frac{1}{2}d_{k+j} - (\mathfrak{w}_k, \mathfrak{w}_j)_{L^2(H|_J)}\big)_{j=0}^{\Delta-1}; (\delta_{kj})_{j=0}^{\Delta-1}; 0\big),$$

$$\hat{P}\mathfrak{d}_k = \big(\mathfrak{d}_k|_{I\setminus J}; \big([\mathfrak{d}_k, p_j] - (\mathfrak{d}_k, \mathfrak{w}_j)_{L^2(H|_J)}\big)_{j=0}^{\Delta-1}; 0; 0\big),$$

$$\mathfrak{w}_k = \big(\mathfrak{w}_k; (d_{k+j})_{j=0}^{\Delta-1}; 0; 0\big),\ \tilde{\mathfrak{w}}_k = \big(\tilde{\mathfrak{w}}_k; (\tilde{d}_{k+j})_{j=0}^{\Delta-1}; 0; 0\big),$$

$$\hat{P}\mathfrak{w}_k = \big(\mathfrak{w}_k|_{I\setminus J}; \big(d_{k+j} - (\mathfrak{w}_k, \mathfrak{w}_j)_{L^2(H|_J)}\big)_{j=0}^{\Delta-1}; 0; 0\big),$$

$$\varpi(\tilde{\mathfrak{w}}_k) = \big(\tilde{\mathfrak{w}}_k; \big(\tilde{d}_{k+j} + (\tilde{\mathfrak{w}}_k, \mathfrak{d}_j)_{L^2(H|_{I\setminus J})}\big)_{j=0}^{\Delta-1}; 0; 0\big).$$

Moreover, note that $\varpi\delta_j = \delta_j$, $j = 0, \ldots, \Delta + \ddot{o} - 1$.

We need the following technical lemmata, whose proofs are carried out by persistent application of the Green's identity in various spaces.

7.6 Lemma. *Set $p_k := \mathfrak{w}_k \in X_L$ for $k \geq \Delta$, so that p_k is a well-defined element of $\mathcal{P}(\mathfrak{h})$ for all $k \in \mathbb{N} \cup \{0\}$. Then*

$$\big[\hat{P}(p_k - \mathfrak{d}_k), \hat{P}(p_l - \mathfrak{d}_l)\big] = \tilde{d}_{k+l},\ k, l \in \mathbb{N} \cup \{0\}. \tag{7.3}$$

Proof. Consider first the case that $k \in \mathbb{N} \cup \{0\}$ and $l = 0$. Then the asserted equality has the form

$$\big[\hat{P}(p_k - \mathfrak{d}_k), \hat{P}p_0\big] = \tilde{d}_k.$$

We compute

$$\big[\hat{P}(p_k - \mathfrak{d}_k), \hat{P}p_0\big] = [p_k - \mathfrak{d}_k, p_0] - \big[P_J(p_k - \mathfrak{d}_k), P_J p_0\big] =$$

$$= d_k - [\mathfrak{d}_k, p_0] - (\mathfrak{w}_k - \mathfrak{d}_k, \mathfrak{w}_0)_{L^2(H|_J)}.$$

The last summand can be computed with the help of Green's identity in the space $L^2(H|_J)$. Applied with the pairs $(\mathfrak{w}_{k+1}; \mathfrak{w}_k), (\mathfrak{w}_0; 0) \in T(H|_J)$ we get

$$(\mathfrak{w}_k, \mathfrak{w}_0)_{L^2(H|_J)} = \mathfrak{w}_0(s_-)^* J \mathfrak{w}_{k+1}(s_-) - \mathfrak{w}_0(s^-)^* J \mathfrak{w}_{k+1}(s^-) =$$

$$= -(1, 0)\mathfrak{w}_{k+1}(s^-).$$

Applied with the pairs $(\mathfrak{d}_{k+1}; \mathfrak{w}_k), (\mathfrak{w}_0; 0) \in T(H|_J)$ we get

$$(\mathfrak{d}_k, \mathfrak{w}_0)_{L^2(H|_J)} = \mathfrak{w}_0(s_-)^* J \mathfrak{d}_{k+1}(s_-) - \mathfrak{w}_0(s^-)^* J \mathfrak{d}_{k+1}(s^-) =$$

$$(1, 0)\mathfrak{d}_{k+1}(s_-) - (1, 0)\mathfrak{d}_{k+1}(s^-) = \lambda_{k+1} - (1, 0)\mathfrak{w}_{k+1}(s^-).$$

Hence

$$\big[\hat{P}(p_k - \mathfrak{d}_k), \hat{P}p_0\big] = d_k - [\mathfrak{d}_k, p_0] + (1, 0)\mathfrak{w}_{k+1}(s^-) + \lambda_{k+1} - (1, 0)\mathfrak{w}_{k+1}(s^-) = \tilde{d}_k.$$

Next we show that for all $k \in \mathbb{N} \cup \{0\}$ and $l \geq 1$,

$$\big[\hat{P}(p_k - \mathfrak{d}_k), \hat{P}(p_l - \mathfrak{d}_l)\big] = \big[\hat{P}(p_{k+1} - \mathfrak{d}_{k+1}), \hat{P}(p_{l-1} - \mathfrak{d}_{l-1})\big]. \tag{7.4}$$

The relation (7.3) then follows from this relation. Assume that $k \leq \Delta - 2$. We apply the Green's identity in $\mathcal{P}(\mathfrak{h})$. To this end note that

$$\mathfrak{d}_{k+1} = B(\mathfrak{d}_k) + \lambda_{k+1}\chi_- \binom{1}{0},$$

and hence, remember $\mathfrak{d}_k \in X_L$, that

$$\big(\mathfrak{d}_{k+1};\mathfrak{d}_k + [\mathfrak{d}_k,p_0]\delta_0 - \lambda_{k+1}\delta_0\big) \in T(\mathfrak{h})\,.$$

We now use the pairs

$$\big((p_{k+1}-\mathfrak{d}_{k+1};p_k-\mathfrak{d}_k+\delta_0(d_k-[\mathfrak{d}_k,p_0]+\lambda_{k+1})\big) \in T(\mathfrak{h})\,,$$

$$\big((p_l-\mathfrak{d}_l;p_{l-1}-\mathfrak{d}_{l-1}+\delta_0(d_{l-1}-[\mathfrak{d}_{l-1},p_0]+\lambda_l)\big) \in T(\mathfrak{h})\,,$$

and obtain, remember $l \geq 1$, that

$$[p_k-\mathfrak{d}_k,p_l-\mathfrak{d}_l]-[p_{k+1}-\mathfrak{d}_{k+1},p_{l-1}-\mathfrak{d}_{l-1}]=$$

$$=(\mathfrak{w}_l-\mathfrak{d}_l)(s_-)^*J(\mathfrak{w}_{k+1}-\mathfrak{d}_{k+1})(s_-)-\underbrace{(\mathfrak{w}_l-\mathfrak{d}_l)(s_+)^*J(\mathfrak{w}_{k+1}-\mathfrak{d}_{k+1})(s_+)}_{=0}\,.$$

An application with the pairs

$$(\mathfrak{w}_{k+1}-\mathfrak{d}_{k+1};\mathfrak{w}_k-\mathfrak{d}_k),(\mathfrak{w}_l-\mathfrak{d}_l;\mathfrak{w}_{l-1}-\mathfrak{d}_{l-1}) \in T(H|_J)$$

gives

$$(\mathfrak{w}_k-\mathfrak{d}_k,\mathfrak{w}_l-\mathfrak{d}_l)-(\mathfrak{w}_{k+1}-\mathfrak{d}_{k+1},\mathfrak{w}_{l-1}-\mathfrak{d}_{l-1})=$$

$$=(\mathfrak{w}_l-\mathfrak{d}_l)(s_-)^*J(\mathfrak{w}_{k+1}-\mathfrak{d}_{k+1})(s_-)-(\mathfrak{w}_l-\mathfrak{d}_l)(s^-)^*J(\mathfrak{w}_{k+1}-\mathfrak{d}_{k+1})(s^-)\,.$$

Since $(\mathfrak{w}_l-\mathfrak{d}_l)(s^-),(\mathfrak{w}_{k+1}-\mathfrak{d}_{k+1})(s^-) \in \operatorname{span}\{(0,1)^T\}$, the second summand vanishes. We obtain that

$$\big[\hat{P}(p_k-\mathfrak{d}_k),\hat{P}(p_l-\mathfrak{d}_l)\big]-\big[\hat{P}(p_{k+1}-\mathfrak{d}_{k+1}),\hat{P}(p_{l-1}-\mathfrak{d}_{l-1})\big]=$$

$$=\Big([p_k-\mathfrak{d}_k,p_l-\mathfrak{d}_l]-[p_{k+1}-\mathfrak{d}_{k+1},p_{l-1}-\mathfrak{d}_{l-1}]\Big)-$$

$$-\Big(\big[P_J(p_k-\mathfrak{d}_k),P_J(p_l-\mathfrak{d}_l)\big]-\big[P_J(p_{k+1}-\mathfrak{d}_{k+1}),P_J(p_{l-1}-\mathfrak{d}_{l-1})\big]\Big)=0\,.$$

The relation (7.4) follows. If $k=\Delta-1$, we use the pair $(\mathfrak{w}_\Delta+\mathfrak{b};p_{\Delta-1}+d_{\Delta-1}\delta_0) \in T(\mathfrak{h})$, if $k \geq \Delta+1$, we use $(\mathfrak{w}_{k+1};\mathfrak{w}_k) \in T(\mathfrak{h})$ in a similar computation.

❑

7.7 Lemma. *Put $\tilde{p}_k := \tilde{\mathfrak{w}}_k \in \tilde{X}_L$ for $k \geq \Delta$, so that $\tilde{p}_k$ is a well-defined element of $\mathcal{P}(\tilde{\mathfrak{h}})$ for all $k \in \mathbb{N}\cup\{0\}$. Analogously, let again $p_k = \mathfrak{w}_k$, $k \geq \Delta$. Then*

$$\varpi(\tilde{p}_k)=\hat{P}(p_k-\mathfrak{d}_k),\ k \in \mathbb{N}\cup\{0\}\,.$$

Proof. If $k<\Delta$ this holds by definition. Thus let $k \geq \Delta$. We see from Remark 7.5 that in the realization (4.10)

$$\varpi(\tilde{p}_k)-\hat{P}(p_k-\mathfrak{d}_k)=\big(0;\big(\tilde{d}_{k+j}+(\tilde{\mathfrak{w}}_k,\mathfrak{d}_j)_{L^2(H|_{I\setminus J})}-d_{k+j}+(\mathfrak{w}_k,\mathfrak{w}_j)_{L^2(H|_J)}+$$

$$+[\mathfrak{d}_k,p_j]-(\mathfrak{d}_k,\mathfrak{w}_j)_{L^2(H|_J)}\big)_{j=0}^{\Delta-1};0;0\big)\,.$$

Inductive applications of the Green's identity yields

$$[\mathfrak{d}_k,p_j]=[\mathfrak{d}_{k+j},p_0]+\sum_{l=1}^{j}\mathfrak{w}_l(s_-)^*J\mathfrak{d}_{k+j+1-l}(s_-)\,,$$

$$(\mathfrak{d}_k,\mathfrak{w}_j)_{L^2(H|_J)}=\sum_{l=0}^{j}\mathfrak{w}_l(s_-)^*J\mathfrak{d}_{k+j+1-l}(s_-)-\sum_{l=0}^{j}\mathfrak{w}_l(s^-)^*J\mathfrak{d}_{k+j+1-l}(s^-)\,,$$

$$(\mathfrak{w}_k,\mathfrak{w}_j)_{L^2(H|_J)}=\underbrace{\sum_{l=0}^{j}\mathfrak{w}_l(s_-)^*J\mathfrak{w}_{k+j+1-l}(s_-)}_{=0}-\sum_{l=0}^{j}\mathfrak{w}_l(s^-)^*J\mathfrak{w}_{k+j+1-l}(s^-)\,,$$

$$(\tilde{\mathfrak{w}}_k,\mathfrak{d}_j)_{L^2(H|_{I\setminus J})}=\sum_{l=1}^{j}\mathfrak{d}_l(s^-)^*J\tilde{\mathfrak{w}}_{k+j+1-l}(s^-)\,.$$

The very last expression can be further rewritten. If we keep in mind that $\tilde{\mathfrak{w}}_{k+j+1-l}(s^-)\in\operatorname{span}\{(0,1)^T\}$ and $(1,0)\mathfrak{d}_l(s^-)=(1,0)\mathfrak{w}_l(s^-)$, it follows that

$$\sum_{l=1}^{j}\mathfrak{d}_l(s^-)^*J\tilde{\mathfrak{w}}_{k+j+1-l}(s^-)=\sum_{l=1}^{j}\mathfrak{w}_l(s^-)^*J\tilde{\mathfrak{w}}_{k+j+1-l}(s^-)=$$

$$=\sum_{l=0}^{j}\mathfrak{w}_l(s^-)^*J\tilde{\mathfrak{w}}_{k+j+1-l}(s^-)=\sum_{l=0}^{j}\mathfrak{w}_l(s^-)^*J\big(\mathfrak{w}_{k+j+1-l}(s^-)-\mathfrak{d}_{k+j+1-l}(s^-)\big)\,.$$

If we keep in mind that $\lambda_{k+j+1}=(1,0)\mathfrak{d}_{k+j+1}(s_-)=\mathfrak{w}_0(s_-)^*J\mathfrak{d}_{k+j+1}(s_-)$, and plug all these expressions together, it follows that $\varpi(\tilde{p}_k)-\hat{P}(p_k-\mathfrak{d}_k)=0$.

❑

7.8 Proposition. *The triple $(\hat{\mathcal{P}},\hat{T},\hat{\Gamma})$ is a boundary triplet of defect 2. The pair $(\varpi;\mathrm{id})$, where ϖ is given by (7.2), is an isomorphism of the boundary triplets $(\mathcal{P}(\tilde{\mathfrak{h}}),T(\tilde{\mathfrak{h}}),\Gamma(\tilde{\mathfrak{h}}))$ and $(\hat{\mathcal{P}},\hat{T},\hat{\Gamma})$. It is compatible with the respective maps ψ in the sense that*

$$\big(\chi_{I\setminus J}\cdot\psi(\mathfrak{h})\big)\circ\varpi=\psi(\tilde{\mathfrak{h}})\,.$$

Moreover, we have

$$\varpi|_{X^\delta}=\mathrm{id}_{X^\delta},\;\;\varpi(\mathcal{C}(\tilde{\mathfrak{h}}))=\hat{\mathcal{P}}\cap\mathcal{C}(\mathfrak{h})\,.$$

Proof.
Step 1: Our first task is to show that ϖ is an isometry of $\mathcal{P}(\tilde{\mathfrak{h}})$ onto $\hat{\mathcal{P}}$ with the desired compatibilities. Let $f,g\in\mathcal{P}(\tilde{h})$ be given. If f,g both belong to $\tilde{\mathcal{C}}+X^\delta$, so that the first formula of our definition of ϖ has to be applied, the validity of $[\varpi f,\varpi g]=[f,g]_\sim$ is clear. If $f=(x;\xi;0;\alpha)$ and $g=\tilde{p}_k$, we have

$$\big[\varpi(x;\xi;0;\alpha),\varpi\tilde{p}_k\big]=\Big[\big(x;\xi+\big((x,\mathfrak{d}_j)_{L^2(H|_{I\setminus J})}\big)_{j=0}^{\Delta-1};0;\alpha\big),\big(-\mathfrak{w}_k|_J-\mathfrak{d}_k|_{I\setminus J};$$

$$\big(\frac{1}{2}d_{k+j}-(\mathfrak{w}_k,\mathfrak{w}_j)_{L^2(H|_J)}-[\mathfrak{d}_k,p_j]+(\mathfrak{d}_k,\mathfrak{w}_j)_{L^2(H|_J)}\big)_{j=0}^{\Delta-1};(\delta_{kj})_{j=0}^{\Delta-1};0\big)\Big]=$$

$$=-\underbrace{(x,\mathfrak{w}_k|_J)_{L^2(H)}}_{=0}-(x,\mathfrak{d}_k)_{L^2(H|_{I\setminus J})}+\xi_k+(x,\mathfrak{d}_k)_{L^2(H|_{I\setminus J})}=\xi_k=[f,\tilde{p}_k]_\sim\,.$$

Finally assume that $f=\tilde{p}_k$ and $g=\tilde{p}_l$. By virtue of Lemma 7.6 and the definition of ϖ we have

$$[\tilde{p}_k,\tilde{p}_l]_\sim=\tilde{d}_{k+l}=[\varpi\tilde{p}_k,\varpi\tilde{p}_l]\,.$$

To show that ϖ is surjective, let $f = (x;\xi;\eta;\alpha) \in \mathcal{P}(\mathfrak{h})$ be given. Then

$$\hat{P}f - \varpi\big((x|_{I\setminus J} + \sum_{j=0}^{\Delta-1} \eta_j \mathfrak{d}_j|_{I\setminus J}; 0; 0; \alpha) + \sum_{i=0}^{\Delta-1} \eta_i \tilde{p}_i\big) \in \operatorname{span}\{\delta_0, \dots, \delta_{\Delta-1}\}\,.$$

Since $\varpi\delta_j = \delta_j$, the assertion follows. The fact that ϖ is compatible with $\bar{.}$ follows since all numbers λ_j are real, and thus $\overline{\mathfrak{d}_k} = \mathfrak{d}_k$. Let $f = (x;\xi;0;\alpha)$, then $\psi(\tilde{\mathfrak{h}})f = x$. On the other hand,

$$\psi(\mathfrak{h})(\varpi f) = \psi(\mathfrak{h})\big(x; \xi + \big((x, \mathfrak{d}_j)_{L^2(H|_{I\setminus J})}\big)_{j=0}^{\Delta-1}; 0; \alpha\big) = x\,.$$

If $f = \tilde{p}_k$, then $\psi(\tilde{\mathfrak{h}})f = \tilde{w}_k$ and, by our choice of $\mathfrak{d}_k$, also

$$\psi(\mathfrak{h})(\varpi f) = \psi(\mathfrak{h})(\hat{P}(p_k - \mathfrak{d}_k)) = \chi_{I\setminus J}(\mathfrak{w}_k - \mathfrak{d}_k) = \tilde{w}_k\,.$$

Since, by its definition, $\varpi(\delta_j) = \delta_j$ for all j, we certainly have $\varpi|_{X^\delta} = \mathrm{id}_{X^\delta}$. The inclusion $\varpi(\tilde{\mathcal{C}}) \subseteq \mathcal{C}$ is also immediate from the definition. Assume that $\varpi(\sum_{k=0}^{\Delta-1} \eta_k \tilde{p}_k) \in \mathcal{C}$. Since ϖ is compatible with ψ, this implies

$$\psi\big(\hat{P}\sum_{k=0}^{\Delta-1} \eta_k p_k - \hat{P}\sum_{k=0}^{\Delta-1} \eta_k \mathfrak{d}_k\big) = \chi_{I\setminus J}\sum_{k=0}^{\Delta-1} \eta_k \mathfrak{w}_k - \chi_{I\setminus J}\sum_{k=0}^{\Delta-1} \eta_k \mathfrak{d}_k \in L^2(H|_{I\setminus J})\,.$$

It follows that $\eta_k = 0$ for all k.

Step 2: We show that $(\varpi\times\varpi)(T(\tilde{\mathfrak{h}}) \cap \mathcal{C}(\tilde{\mathfrak{h}})^2) = \hat{T} \cap \mathcal{C}(\mathfrak{h})^2$. Let $(\tilde{f};\tilde{g}) \in T(\tilde{\mathfrak{h}}) \cap \mathcal{C}(\tilde{\mathfrak{h}})^2$ be given, and put $(\tilde{h};\tilde{g}) := \Psi^{ac}(\tilde{f};\tilde{g})$. Then $\tilde{h}, \tilde{g} \in L^2(H|_{I\setminus J})$ and $\tilde{h}' = JH\tilde{g}$. Define

$$h(t) := \begin{cases} \tilde{h}(t) & , \ t \in I\setminus J \\ \tilde{h}(s^-) & , \ t \in J \end{cases} \quad , \quad k(t) := \begin{cases} \tilde{k}(t) & , \ t \in I\setminus J \\ 0 & , \ t \in J \end{cases}$$

Then $h \in \mathrm{AC}(I)$, $k \in \mathcal{M}(I)/_{=_H}$, $h, k \in L^2(H)$, and $h' = JHk$, i.e. $(h;k) \in T_{max}(H)$. Thus there exists $(f;g) \in T(\mathfrak{h}) \cap \mathcal{C}(\mathfrak{h})^2$ such that $\psi f = h, \psi g = k$ and, in fact $\Psi^{ac}(f;g) = (h;k)$. We obtain

$$\psi\hat{P}f = \chi_{I\setminus J}\psi f = \tilde{h} = \tilde{\psi}\tilde{f} = \psi\varpi\tilde{f}\,,$$

$$\psi\hat{P}g = \chi_{I\setminus J}\psi g = \tilde{k} = \tilde{\psi}\tilde{g} = \psi\varpi\tilde{g}\,,$$

and hence

$$\varpi\tilde{f} - \hat{P}f, \varpi\tilde{g} - \hat{P}g \in \operatorname{span}\{\delta_0, \dots, \delta_{\Delta-1}\}\,.$$

Since $\hat{P}\delta_j = \delta_j$, we can assume without loss of generality that

$$(\varpi\tilde{f} - \hat{P}f; \varpi\tilde{g} - \hat{P}g) = \sum_{k=0}^{\Delta-1} \alpha_k(\delta_k; 0) + \alpha(0;\delta_0)\,.$$

Our first task is to compute $\alpha = -[\varpi\tilde{g} - \hat{P}g, p_0]$. Note that, since $g \in \mathcal{C}$ and $\psi g|_J = 0$, we have $\hat{P}g = g$. Moreover,

$$[g, p_0] = \mathfrak{w}_0(s_-)^* Jh(s_-) - \mathfrak{w}_0(s_+)^* Jh(s_+) = (1,0)h(s_-) - (1,0)h(s_+) =$$

$$= (1,0)h(s^-) - (1,0)h(s_+) = [\tilde{g}, \tilde{p}_0]_\sim = [\varpi\tilde{g}, \varpi\tilde{p}_0] \,.$$

If we write $\tilde{g} = (\tilde{k}; \xi; 0; 0)$, we get

$$\varpi\tilde{g} = \big(\tilde{k}; \xi + \big((\tilde{k}, \mathfrak{d}_j)_{L^2(H|_{I\setminus J})}\big)_{j=0}^{\Delta-1}; 0; 0\big) \,.$$

Since

$$\varpi\tilde{p}_0 = \big(-\mathfrak{w}_0|_J; \big(\frac{1}{2}d_j - (\mathfrak{w}_0, \mathfrak{w}_j)_{L^2(H|_J)}\big)_{j=0}^{\Delta-1}; (\delta_{0j})_{j=0}^{\Delta-1}; 0\big)$$

we see that

$$[\varpi\tilde{g}, \varpi\tilde{p}_0] = \xi_0 + (\tilde{k}, \mathfrak{d}_0)_{L^2(H|_{I\setminus J})} = [\varpi\tilde{g}, p_0] \,.$$

Hence $\alpha = 0$, and thus $\varpi\tilde{g} = \hat{P}g$.

Let $k \in \{0, \ldots, \Delta - 2\}$. We have $\alpha_k = -[\varpi\tilde{f} - \hat{P}f, p_k] = [\hat{P}f, p_k] - [\varpi\tilde{f}, p_k]$. Write $\tilde{f} = (\tilde{h}; \gamma; 0; 0)$, then

$$\varpi\tilde{f} = \big(\tilde{h}; \gamma + \big((\tilde{h}, \mathfrak{d}_j)_{L^2(H|_{I\setminus J})}\big)_{j=0}^{\Delta-1}; 0; 0\big) \,,$$

and hence

$$[\varpi\tilde{f}, p_k] = \gamma_k + (h, \mathfrak{d}_k)_{L^2(H|_{I\setminus J})} \,.$$

On the other hand

$$[\hat{P}f, p_k] = [f, p_k] - (h, \mathfrak{w}_k)_{L^2(H|_J)} \,.$$

As $g = \hat{P}g = \varpi\tilde{g}$,

$$[g, p_{k+1}] = [\varpi\tilde{g}, p_{k+1}] = \xi_{k+1} + (\tilde{k}, \mathfrak{d}_{k+1})_{L^2(H|_{I\setminus J})} =$$

$$= \xi_{k+1} + (\tilde{h}, \mathfrak{d}_k)_{L^2(H|_{I\setminus J})} + \mathfrak{d}_{k+1}(s^-)^* J h(s^-) \,.$$

Since $h|_J$ is constant, we have

$$-(h, \mathfrak{w}_k)_{L^2(H|_J)} = \mathfrak{w}_{k+1}(s_-)^* J h(s_-) - \mathfrak{w}_{k+1}(s^-)^* J h(s^-) \,.$$

Moreover,

$$[g, p_{k+1}] - [f, p_k] = \mathfrak{w}_{k+1}(s_-)^* J h(s_-) - \mathfrak{w}_{k+1}(s_+)^* J h(s_+) =$$

$$= \mathfrak{w}_{k+1}(s^-)^* J h(s^-) - (h, \mathfrak{w}_k)_{L^2(H|_J)} - \mathfrak{w}_{k+1}(s_+)^* J h(s_+) =$$

$$= \tilde{\mathfrak{w}}_{k+1}(s^-)^* J h(s^-) + \mathfrak{d}_{k+1}(s^-)^* J h(s^-) - (h, \mathfrak{w}_k)_{L^2(H|_J)} - \tilde{\mathfrak{w}}_{k+1}(s_+)^* J h(s_+) =$$

$$= [\tilde{g}, \tilde{p}_{k+1}]_\sim - [\tilde{f}, \tilde{p}_k]_\sim + \mathfrak{d}_{k+1}(s^-)^* J h(s^-) - (h, \mathfrak{w}_k)_{L^2(H|_J)} =$$

$$= \xi_{k+1} - \gamma_k + \mathfrak{d}_{k+1}(s^-)^* J h(s^-) - (h, \mathfrak{w}_k)_{L^2(H|_J)} \,.$$

We obtain

$$[f, p_k] = \gamma_k + (h, \mathfrak{d}_k)_{L^2(H|_{I\setminus J})} + (h, \mathfrak{w}_k)_{L^2(H|_J)} \,,$$

and thus

$$[\hat{P}f, p_k] = \gamma_k + (h, \mathfrak{d}_k)_{L^2(H|_{I\setminus J})} \,.$$

It follows that $\alpha_k = 0$. Finally, let $k = \Delta - 1$. If we keep in mind that $g, \mathfrak{w}_\Delta \in \mathcal{C}$ and $\tilde{g}, \tilde{\mathfrak{w}}_\Delta \in \tilde{\mathcal{C}}$, it follows that

$$[g, \mathfrak{w}_\Delta] = (k, \mathfrak{w}_\Delta)_{L^2(H)} = (\tilde{k}, \mathfrak{w}_\Delta)_{L^2(H|_{I\setminus J})}$$

and

$$[\tilde{g},\tilde{\mathfrak{w}}_\Delta]_\sim = (\tilde{k},\tilde{\mathfrak{w}}_\Delta)_{L^2(H|_{I\setminus J})} = (\tilde{k},\mathfrak{w}_\Delta)_{L^2(H|_{I\setminus J})} - (\tilde{k},\mathfrak{d}_\Delta)_{L^2(H|_{I\setminus J})} =$$

$$= (\tilde{k},\mathfrak{w}_\Delta)_{L^2(H|_{I\setminus J})} - (\tilde{h},\mathfrak{d}_{\Delta-1})_{L^2(H|_{I\setminus J})} - \mathfrak{d}_\Delta(s^-)^*Jh(s^-)\,.$$

Hence the same computation as above using the pair $(\mathfrak{w}_\Delta+\mathfrak{b};p_{\Delta-1}+d_{\Delta-1}\delta_0)$ yields $\alpha_{\Delta-1}=0$. Alltogether it follows that

$$(\varpi\tilde{f};\varpi\tilde{g}) = (\hat{P}f;\hat{P}g)\in\hat{T}\,.$$

We have shown the inclusion $(\varpi\times\varpi)(T(\tilde{\mathfrak{h}})\cap\tilde{\mathcal{C}}^2)\subseteq\hat{T}\cap\mathcal{C}^2$. Conversely, let $(\tilde{f};\tilde{g})\in\hat{T}\cap\mathcal{C}^2$ and choose $(f;g)\in T(\mathfrak{h})$ with $\hat{f}=\hat{P}f,\hat{g}=\hat{P}g$. Since $\ker\hat{P}\subseteq\mathcal{C}$, it follows that $(f;g)\in T(\mathfrak{h})\cap\mathcal{C}^2$. Thus

$$(\psi\hat{f};\psi\hat{g}) = (\chi_{I\setminus J}\psi f;\chi_{I\setminus J}\psi g)\in T_{max}(H|_{I\setminus J})\,,$$

and hence there exists $(\tilde{f};\tilde{g})\in T(\tilde{\mathfrak{h}})\cap\tilde{\mathcal{C}}^2$ with $\psi\tilde{f}=\psi\hat{f},\psi\tilde{g}=\psi\hat{g}$. We have therefore

$$(\hat{f};\hat{g})-(\varpi\tilde{f};\varpi\tilde{g})\in\hat{T}\cap\ker(\psi\times\psi)\,.$$

Since $\hat{P}\delta_j=\delta_j$ and $\varpi\delta_j=\delta_j$, it follows that $(\hat{f};\hat{g})\in(\varpi\times\varpi)(T(\tilde{\mathfrak{h}})\cap\tilde{\mathcal{C}}^2)$.

Step 3: We deduce that $(\varpi\times\varpi)T(\tilde{\mathfrak{h}})=\hat{T}$. Certainly

$$(\varpi\times\varpi)(T(\tilde{\mathfrak{h}})\cap(X^\delta)^2) = \hat{T}\cap(X^\delta)^2\,. \tag{7.5}$$

Since $\varpi\tilde{p}_0=\hat{P}p_0$, we have $(\varpi\times\varpi)(\tilde{p}_0;0)\in\hat{T}$. Let $k\in\{1,\dots,\Delta-1\}$, then

$$(\varpi\times\varpi)(\tilde{p}_k;\tilde{p}_{k-1}+\tilde{d}_{k-1}\delta_0)-(\hat{P}\times\hat{P})(p_k;p_{k-1}+d_{k-1}\delta_0)+$$
$$+(\hat{P}\times\hat{P})(\mathfrak{d}_k;\mathfrak{d}_{k-1}+[\mathfrak{d}_{k-1},p_0]\delta_0-\lambda_k\delta_0) = \big(0;\delta_0(\tilde{d}_{k-1}-d_{k-1}+$$
$$+[\mathfrak{d}_{k-1},p_0]-\lambda_k)\big)=0\,.$$

Consider the case $k=\Delta$. Note that, since the numbers b_i are the same for $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$, also the respective vectors $\mathfrak{b}$ coincide. We obtain

$$(\varpi\times\varpi)(\tilde{\mathfrak{w}}_\Delta+\mathfrak{b};\tilde{p}_{\Delta-1}+\tilde{d}_{\Delta-1}\delta_0)-(\hat{P}\times\hat{P})(\mathfrak{w}_\Delta+\mathfrak{b};p_{\Delta-1}+d_{\Delta-1}\delta_0)+$$
$$+(\hat{P}\times\hat{P})(\mathfrak{d}_\Delta;\mathfrak{d}_{\Delta-1}+[\mathfrak{d}_{\Delta-1},p_0]\delta_0-\lambda_\Delta\delta_0) =$$
$$=\big(\varpi\tilde{\mathfrak{w}}_\Delta-\hat{P}\mathfrak{w}_\Delta+\hat{P}\mathfrak{d}_\Delta;\delta_0(\tilde{d}_{k-1}-d_{k-1}+[\mathfrak{d}_{k-1},p_0]-\lambda_k\big)=0\,.$$

We have proved the inclusion '$\subseteq$'. If conversely, $(\hat{f};\hat{g})\in\hat{T}$ is given, choose $(f;g)\in T(\mathfrak{h})$ with $\hat{P}f=\hat{f},\hat{P}g=\hat{g}$, and write $(f;g)=(f_1;g_1)+(f_2;g_2)$ where $(f_1;g_1)$ is the component in (4.15). By the above computations, we can realize $(f_1;g_1)$ as $(\varpi\times\varpi)(\tilde{f};\tilde{g})$, $(\tilde{f};\tilde{g})\in T(\tilde{\mathfrak{h}})$, up to a summand in $\hat{T}\cap\mathcal{C}^2$. By Step 2 and (7.5), thus also $(\hat{f};\hat{g})\in(\varpi\times\varpi)T(\tilde{\mathfrak{h}})$.

Step 4: It remains to show that ϖ is compatible with boundary values. This, however, is done with similar arguments as used in the previous discussion, and we will not carry out the details.

❑

7.9 Remark. Let $\mathfrak{h}$ be an elementary indefinite Hamiltonian of kind (A), and let $s^+\in(s,s_+)$ be not inner point of an H-indivisible interval. Then the results analogous to Proposition 7.3, Proposition 7.4 and Proposition 7.8 hold true. This can be seen for example by applying an order reversing reparameterization, cf. Lemma 5.19.

8 Indefinite canonical systems

Let H be a Hamiltonian defined on the interval (s_-, s_+) which is singular at both endpoints. Fix $s_0 \in (s_-, s_+)$. We say that H satisfies the condition (HS$_+$) or (HS$_-$), if $H|_{(s_0,s_+)}$ or $H|_{(s_-,s_0)}(-x)$, respectively, satisfies (HS). Moreover, we define (for the definition of $\phi(.)$ see Theorem 2.27)

$$\Delta_+(H) := \Delta(H|_{(s_0,s_+)}),\ \Delta_-(H) := \Delta(H|_{(s_-,s_0)}(-x)),$$

$$\phi_+(H) := \phi(H|_{(s_0,s_+)}),\ \phi_-(H) := \phi(H|_{(s_-,s_0)}(-x)).$$

By Section 2.3.c. and Lemma 3.12 these notions are well-defined, i.e. do not depend on the choice of $s_0 \in (s_-, s_+)$.

8.1 Definition. A general Hamiltonian $\mathfrak{h}$ is a collection of data of the following kind:

(i) $n \in \mathbb{N} \cup \{0\}$, $\sigma_0, \ldots, \sigma_{n+1} \in \mathbb{R} \cup \{\pm\infty\}$ with $\sigma_0 < \sigma_1 < \ldots < \sigma_{n+1}$.

(ii) Hamiltonians H_i, $i = 0, \ldots, n$, defined on the respective intervals (σ_i, σ_{i+1}),

(iii) numbers $\ddot{o}_1, \ldots, \ddot{o}_n \in \mathbb{N} \cup \{0\}$ and $b_{i,1}, \ldots, b_{i,\ddot{o}_i+1} \in \mathbb{R}$, $i = 1, \ldots, n$, with $b_{i,1} \neq 0$ in case $\ddot{o}_i \geq 1$,

(iv) numbers $d_{i,0}, \ldots, d_{i,2\Delta_i-1} \in \mathbb{R}$, $i = 1, \ldots, n$, where $\Delta_i := \max\{\Delta_+(H_{i-1}), \Delta_-(H_i)\}$,

(v) a finite subset E of $\{\sigma_0, \sigma_{n+1}\} \cup \bigcup_{i=0}^{n} (\sigma_i, \sigma_{i+1})$,

which is assumed to be subject to the following conditions

(H1) H_0 is regular at σ_0 and, if $n \geq 1$, singular at σ_1. H_i is singular at both endpoints σ_i and σ_{i+1}, $i = 1, \ldots, n-1$. If $n \geq 1$, then H_n is singular at σ_n.

(H2) We have $\alpha_1^+(H_i) > \sigma_i$ for $i = 1, \ldots, n-1$. If H_n is singular at σ_{n+1}, then also $\alpha_1^+(H_n) > \sigma_n$.

(H3) We have $\Delta_i < \infty$, $i = 1, \ldots, n$. Moreover, H_0 satisfies (HS$_+$), H_i satisfies (HS$_-$) and (HS$_+$), $i = 1, \ldots, n-1$, and H_n satisfies (HS$_-$).

(H4) We have $\phi_+(H_{i-1}) = \phi_-(H_i)$, $i = 1, \ldots, n$.

(H5) Let $i \in \{1, \ldots, n\}$. If $\alpha_1^+(H_{i-1}) < \sigma_i$ and $\alpha_1^-(H_i) > \sigma_i$, then $d_1 = 0$. If additionally $b_{i,1} = 0$, then also $d_0 < 0$.

(E1) $\sigma_0, \sigma_{n+1} \in E$, and $E \cap (\sigma_i, \sigma_{i+1}) \neq \emptyset$ for $i = 1, \ldots, n-1$. If H_n is singular at σ_{n+1}, then also $E \cap (\sigma_n, \sigma_{n+1}) \neq \emptyset$. Moreover, E contains all endpoints of indivisible intervals of infinite length.

(E2) No point of E is an inner point of an indivisible interval.

The general Hamiltonian $\mathfrak{h}$ is called definite if $n = 0$, and indefinite otherwise. It is called regular or singular, if H_n is regular or singular, respectively, at σ_{n+1}. The common value of $\phi_+(H_{i-1})$ and $\phi_-(H_i)$ will be denoted by ϕ_i. The subset E is called an admissible partition.

In order to shorten notation we shall write a Hamiltonian $\mathfrak{h}$ which is given by the data n, $\sigma_0, \dots, \sigma_{n+1}$, $H_1, \dots, H_n$, $\ddot{o}_1, \dots, \ddot{o}_n$, $b_{i,j}$, $d_{i,j}$, E as

$$\mathfrak{h} = (H, \mathfrak{c}, \mathfrak{d})$$

where

- H represents the Hamiltonians H_i, including their number n and their domains of definition (σ_i, σ_{i+1}),
- $\mathfrak{c}$ represents the numbers $\ddot{o}_i$ and $b_{i,j}$,
- $\mathfrak{d}$ represents the numbers $d_{i,j}$ and the subset E.

Moreover, we set $I := \bigcup_{i=0}^{n} (\sigma_i, \sigma_{i+1})$.

8.2 Remark. If just the data H and $\mathfrak{c}$ are given, then it is elementary to check, that one always can construct a finite subset E and numbers $d_{i,j}$, such that we obtain an in general indefinite Hamiltonian.

8.3 Remark. Let us give an intuitive explanation of the definition of this notion. Its purpose is to model an indefinite canonical system. So we deal with the differential equation $f' = JHf$ given on an interval (σ_0, σ_{n+1}) which involves some kind of singularities. These singularities are the points σ_i, $i = 1, \dots, n$. The condition (H1) says that we deal with an initial value problem at σ_0 and that $\sigma_1, \dots, \sigma_n$ actually are singularities. Moreover, and this is the condition (H2), no two singularities σ_i and σ_{i+1} may lump together. The meaning of (H3) is that the growth of H_i towards a singularity is, if measured appropriately, not too fast. Moreover, (H4) is an interface condition at σ_i.

The numbers $\ddot{o}_i \in \mathbb{N} \cup \{0\}$ and $b_{i,1}, \dots, b_{i,\ddot{o}_i+1}$ model the part of the singularity σ_i which is concentrated at σ_i, whereas the numbers $d_{i,0}, \dots, d_{i,2\Delta_i-1}$ model the part of this singularity which is in interaction with the local behaviour around σ_i. The elements of E in the vicinity of σ_i determine what local here means. The freedom of this interaction is, by (H5), restricted if $\alpha_1^+(H_{i-1}) < \sigma_i$ and $\alpha_1^-(H_i) > \sigma_i$.

Finally, the possibility that $\alpha_1^+(H_{i-1}) < \sigma_i$, $\alpha_1^-(H_i) > \sigma_i$ and $b_{i,1} = 0$ occurs only in the case of 'indivisible intervals of negative length', i.e. elementary indefinite Hamiltonians of kind (C).

In the following we will associate to a general Hamiltonian $\mathfrak{h}$ a model which consists of a space $\mathcal{P}(\mathfrak{h})$ together with a conjugate linear involution, a linear relation $T(\mathfrak{h}) \subseteq \mathcal{P}(\mathfrak{h}) \times \mathcal{P}(\mathfrak{h})$ together with a boundary relation $\Gamma(\mathfrak{h})$, and a map $\psi(\mathfrak{h})$ of $\mathcal{P}(\mathfrak{h})$ onto spaces of functions on I. To do so, we shall split up $\mathfrak{h}$ into simpler pieces by cutting I at the points of E.

The next statement follows immediately on inspecting the definition of a general Hamiltonian and of an elementary indefinite Hamiltonian. Therefore we will not formulate its proof.

8.4 Lemma. *Let $\mathfrak{h}$ be a general Hamiltonian defined on $I = \bigcup_{l=0}^{n} (\sigma_l, \sigma_{l+1})$. We write the corresponding admissible partition of I as $E = \{s_0, \dots, s_{N+1}\}$, $s_0 < s_1 < \dots < s_{N+1}$. Then, for every $l \in \{0, \dots, N\}$, exactly one of the following cases takes place:*

(def) *There exists $i(l) \in \{0, \dots, n\}$ such that $(s_l, s_{l+1}) \subseteq (\sigma_{i(l)}, \sigma_{i(l)+1})$,*

(indef) *There exists $i(l) \in \{1, \ldots, n\}$ such that $\sigma_{i(l)} \in (s_l, s_{l+1})$ and $\sigma_i \notin (s_l, s_{l+1})$, $i \neq i(l)$.*

If for some $l \in \{0, \ldots, N\}$ we are in the case (def), then

$$\mathfrak{h}^l := H_{i(l)}|_{(s_l, s_{l+1})}$$

is a Hamiltonian which is regular at s_l. It is also regular at s_{l+1}, unless $\mathfrak{h}$ is singular and $l = N$.

If $\mathfrak{h}$ is singular and $l = N$, then we are in the case (def) and $\mathfrak{h}^N$ is singular at s_{N+1}.

If we are in the case (indef), then the data (for the definition of N_α see (2.21))

$$N_{\phi_{i(l)}} H_{i(l)-1}|_{(s_l, \sigma_{i(l)})} N_{\phi_{i(l)}}^T, N_{\phi_{i(l)}} H_{i(l)}|_{(\sigma_{i(l)}, s_{l+1})} N_{\phi_{i(l)}}^T,$$
$$\ddot{o}_{i(l)}, b_{i(l),1}, \ldots, b_{i(l), \ddot{o}_{i(l)}+1}, \; d_{i(l),0}, \ldots, d_{i(l), 2\Delta_{i(l)}-1},$$

constitutes an elementary indefinite Hamiltonian $\mathfrak{h}^l$. Thereby $\mathfrak{h}^l$ is of kind (B) or (C) if and only if $s_l = \alpha_1^+(H_{i-1})$, $s_{l+1} = \alpha_1^-(H_i)$. If this happens we are in case (C) if and only if $b_1 = 0$.

Otherwise it is of kind (A).

8.5 Definition. Let $\mathfrak{h}$ be a general Hamiltonian defined on $I = \bigcup_{l=0}^n (\sigma_l, \sigma_{l+1})$. Let the (definite or indefinite) Hamiltonians $\mathfrak{h}^l$ be defined with as in Lemma 8.4.

If $l \in \{0, \ldots, N\}$ is in the case (indef), we define a boundary triplet $(\mathcal{P}^l, T^l, \Gamma^l)$ as (for the definition of N_α see (2.21))

$$\mathcal{P}^l := \mathcal{P}(\mathfrak{h}^l), T^l := T(\mathfrak{h}^l), \Gamma^l := \begin{pmatrix} N_{-\phi_{i(l)}} & 0 \\ 0 & N_{-\phi_{i(l)}} \end{pmatrix} \Gamma(\mathfrak{h}^l).$$

If $l \in \{0, \ldots, N\}$ is in the case (def) put

$$(\mathcal{P}^l, T^l, \Gamma^l) := (L^2(\mathfrak{h}^l), T_{max}(\mathfrak{h}^l), \Gamma(\mathfrak{h}^l)).$$

The boundary triplet associated to $\mathfrak{h}$ is defined as

$$\big(\mathcal{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h})\big) := \biguplus_{l=0}^{N} \big(\mathcal{P}^l, T^l, \Gamma^l\big). \tag{8.1}$$

Moreover, on $\mathcal{P}(\mathfrak{h}) = \mathcal{P}^0 \oplus \ldots \oplus \mathcal{P}^N$ we define a conjugate linear involution $\bar{.}$ by componentwise application of the respective involutions $\bar{.}^l$ of $\mathcal{P}^l$. Also, a linear map $\psi(\mathfrak{h}) : \mathcal{P}(\mathfrak{h}) \to \mathcal{M}(I)/_{=_H}$ can be defined, if we canonically identify $\mathcal{M}(I)/_{=_H}$ with $\prod_{l=0}^N \mathcal{M}(I \cap (s_l, s_{l+1}))/_{=_l}$:

$$f \simeq \big(f|_{I \cap (s_0, s_1)}, \ldots, f|_{I \cap (s_N, s_{N+1})}\big),$$

by componentwise application of $\psi_l := \mathrm{id} : \mathcal{P}^l \to \mathcal{M}(I \cap (s_l, s_{l+1}))/_{=_l}$, for l in the case (def), and

$$N_{-\phi_{i(l)}} \psi_l : \mathcal{P}^l \to \mathcal{M}(I \cap (s_l, s_{l+1}))/_{=_l}$$

for l in the case (indef).

We are now in the position to formulate the main results of this paper, which show the complete analogy of the operator theory of the model of a general Hamiltonian in comparison to the classical (positive definite) case. They are easily obtained by putting together what we have established so far. Let us first deal with the regular case.

8.6 Theorem. *Let $\mathfrak{h}$ be a regular general Hamiltonian. Then $\mathfrak{P}(\mathfrak{h})$ is a Pontryagin space with negative index*

$$\operatorname{ind}_- \mathcal{P}(\mathfrak{h}) = \sum_{i=1}^{n} \big(\Delta_i + [\frac{\ddot{o}}{2}]\big) + \big|\big\{1 \leq i \leq n : \ddot{o}_i \text{ odd}, c_{i,1} < 0\big\}\big|$$

and $\bar{.}$ is a conjugate linear isometric involution of $\mathcal{P}(\mathfrak{h})$. The map $\psi(\mathfrak{h})$ is real with respect to $\bar{.}$ and maps $\mathcal{P}(\mathfrak{h})$ onto a subspace of $\mathcal{M}(I)/_{=_H}$ which contains $L^2(H)$ with codimension $\sum_{i=1}^n \Delta_i$.

The triple $(\mathcal{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h}))$ is a boundary triplet of defect 2 which satisfies the condition (E). The adjoint

$$S(\mathfrak{h}) := T(\mathfrak{h})^*$$

is a completely nonselfadjoint symmetric operator, satisfies the condition (CR) and has the property that $r(S(\mathfrak{h})) = \mathbb{C}$. Its defect index is $(2,2)$ and $\operatorname{mul}(\Gamma(\mathfrak{h})) = \{0\}$ unless $\mathfrak{h}$ consists of just one elementary indefinite Hamiltonian of kind (B) or (C) or is positive definite and consists of just one indivisible interval. In these cases the defect index of $S(\mathfrak{h})$ is $(1,1)$ and $\operatorname{mul}(\Gamma(\mathfrak{h})) \neq \{0\}$.

Proof. The asserted formula for the negative index of $\mathcal{P}(\mathfrak{h})$ is immediate from Proposition 4.13. The fact that $\bar{.}$ is a conjugate linear isometric involution and is compatible with boundary values is obvious from the definition and the fact that N_ϕ is real. The assertion concerning $\psi(\mathfrak{h})$ follows from Proposition 4.13.

We show that $(\mathcal{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h}))$ is a boundary triplet of defect 2. By Theorem 2.18 and Theorem 5.1 each summand $(\mathcal{P}^l, T^l, \Gamma^l)$ has this property. If $N = 0$, we are done. Hence assume that $N \geq 1$. In order to apply Proposition 6.2, we have to show that $\operatorname{mul}\Gamma_0$ and $\operatorname{mul}\Gamma_1$ do not coincide unless both equal $\{0\}$. However, by Lemma 4.19, we have

$$\operatorname{mul}\Gamma_j = \operatorname{span}\big\{(J\xi_{\phi_j}, J\xi_{\phi_j})\big\},\ j = 0, 1\,,$$

where ϕ_0 is the type of the indivisible interval ending at s_1 and ϕ_1 is the type of the indivisible interval beginning at s_1. Since, by our choice, s_1 is not contained in an indivisible interval, we must have $\phi_0 \neq \phi_1 \mod \pi$. It now follows from Proposition 6.2 that

$$(\mathcal{P}^0, T^0, \Gamma^0) \uplus (\mathcal{P}^1, T^1, \Gamma^1)$$

is a boundary triplet of defect 2 with $\operatorname{mul}(\Gamma^0 \uplus \Gamma^1) = \{0\}$. Thus we may step by step add the remaining summands $(\mathcal{P}^l, T^l, \Gamma^l)$, and, finally, obtain a boundary triplet of defect 2.

Note that we have $\operatorname{mul}\Gamma(\mathfrak{h}) = \{0\}$ unless $N = 0$ and $\operatorname{mul}\Gamma^0 \neq \{0\}$. This is the case if and only if $\mathfrak{h} = \mathfrak{h}^0$ is either elementary indefinite of kind (B), (C), or is positive definite and consists of just one indivisible interval. Our assertion on the defect index of $S(\mathfrak{h})$ follows.

Our next task is to show that $\operatorname{mul} S(\mathfrak{h}) = \{0\}$. If $N = 0$, this is granted by Corollary 5.7. Assume that $N \geq 1$. Let $(0; g) \in S(\mathfrak{h})$ and write $g = g_0 + \ldots + g_N$ according to $\mathcal{P}(\mathfrak{h}) = \mathcal{P}_0 \oplus \ldots \oplus \mathcal{P}_N$. Then $(0; g_l) \in T(\mathfrak{h}^l)$ and there exist $a_1, \ldots, a_N \in \mathbb{C}^2$ with

$$((0; g_0); (0; a_1)) \in \Gamma^0, ((0; g_1); (a_1; a_2)) \in \Gamma^1, \ldots, ((0; g_N); (a_N; 0)) \in \Gamma^N .$$

It follows from Corollary 2.25, Proposition 5.16 and our definition of Γ^l that

$$a_1 \in \begin{cases} \operatorname{span}\{J\xi_{\phi_0^+}\} & , \ {\scriptstyle \alpha_1^+(H|_{(s_0,s_1)})<s_1,\ \phi_0^+ \text{ type of } (\alpha_1^+(H|_{(s_0,s_1)}),s_1)} \\ \{0\} & , \ \alpha_1^+(H|_{(s_0,s_1)}) = s_1 \end{cases}$$

$$a_1 \in \begin{cases} \operatorname{span}\{J\xi_{\phi_1^-}\} & , \ {\scriptstyle \alpha_1^-(H|_{(s_1,s_2)})>s_1,\ \phi_1^- \text{ type of } (s_1,\alpha_1^-(H|_{(s_1,s_2)}))} \\ \{0\} & , \ \alpha_1^-(H|_{(s_1,s_2)}) = s_1 \end{cases}$$

Since s_1 is not contained in an indivisible interval, we have $\phi_0^+ \neq \phi_1^- \mod \pi$, and thus $a_1 = 0$. We see that $(0; g_0) \in S(\mathfrak{h}_0)$ and thus that $g_0 = 0$ (see Theorem 5.1). We now proceed inductively, to obtain $g_1 = \ldots = g_N = 0$.

By Theorem 2.18 and Theorem 5.1 each summand in (8.1) satisfies (CR) and (E). By Lemma 6.7 these properties transfer to their sum. The same sources imply that $\operatorname{ran}(S(\mathfrak{h}) - z)$ is closed for all $z \in \mathbb{C}$. Due to the property (E) we have $\ker(S(\mathfrak{h}) - z) = \{0\}$, $z \in \mathbb{C}$, and thus conclude that $r(S(\mathfrak{h})) = \mathbb{C}$. An application of Lemma 5.11 shows that $S(\mathfrak{h})$ is completely nonselfadjoint.

❑

In the case that $\mathfrak{h}$ is singular we obtain:

8.7 Theorem. *Let $\mathfrak{h}$ be a singular general Hamiltonian. Then $\mathfrak{P}(\mathfrak{h})$ is a Pontryagin space with negative index*

$$\operatorname{ind}_- \mathcal{P}(\mathfrak{h}) = \sum_{i=1}^{n} \big(\Delta_i + [\frac{\ddot{o}_i}{2}]\big) + \big|\big\{1 \leq i \leq n : \ddot{o}_i \text{ odd}, c_{i,1} < 0\big\}\big|$$

and $\bar{.}$ is a conjugate linear isometric involution of $\mathcal{P}(\mathfrak{h})$. The map $\psi(\mathfrak{h})$ is real with respect to $\bar{.}$ and maps $\mathcal{P}(\mathfrak{h})$ onto a subspace of $\mathcal{M}(I)/_{=_H}$ which contains $L^2(H)$ with codimension $\sum_{i=1}^n \Delta_i$.

The triple $(\mathcal{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h}))$ is a boundary triplet of defect 1 which satisfies (E). The adjoint $S(\mathfrak{h}) := T(\mathfrak{h})^*$ is a symmetric operator with defect index $(1,1)$ unless $\mathfrak{h}$ is positive and consists of just one indivisible interval, in which case $S(\mathfrak{h})$ is selfadjoint.

Proof. This result is established similar as Theorem 8.6, by putting together our previous results. Only minor changes are necessary in the proof that the pasting is well-defined and that $\operatorname{mul} S(\mathfrak{h}) = \{0\}$, however, we will not carry this out in detail.

❑

8.8 Remark. The operator $S(\mathfrak{h})$ is also completely nonselfadjoint. This, however, will be seen only later.

8.9 Remark. In the case that $\mathfrak{h}$ does not consist of just one indivisible intervall - by this we mean either a definite indivisible intervall or an elementary indefinite Hamiltonian of kind (B) or (C) - we are going to define a mapping $\Psi^{ac} : T(\mathfrak{h}) \to AC(I) \times \mathcal{M}(I)/_=$ on all of I. We will also write $\Psi^{ac}(\mathfrak{h})$ if we want to emphasize the underlaying Hamiltonian.

To do so, let $(f; g) = ((f_0 + \cdots + f_N); (g_0 + \cdots + g_N)) \in T(\mathfrak{h})$. By our assumption $\Gamma(\mathfrak{h})$ is a function. So let

$a_0 := \Gamma(\mathfrak{h})((f;g))_1 (\in \mathbb{C}^2)$, and let $a_1, \dots, a_{N+1} \in \mathbb{C}^2$ be such that $((f_l; g_l); (a_l; a_{l+1})) \in \Gamma^l$, $l = 0, \dots, N$. We define Ψ^{ac} on (s_l, s_{l+1}) inductively for $l = 0, \dots, N$. Note that by Remark 6.6 the numbers $a_0, \dots, a_{N+1}$ are uniquely determined by $(f; g)$.

If $\mathfrak{h}^l$ is positive and no indivisible interval, then according to Remark 2.2 there is a unique $\hat{f}_l \in AC(s_l, s_{l+1})$ such that $\hat{f}_l' = JH_{i(l)} g_l$ a.e. on (s_l, s_{l+1}). We define $\Psi_l^{ac}((f;g)) := (\hat{f}_l; g_l)$. By the definition of boundary values for the definite situation we have $a_{l+1} = \Psi_l^{ac}((f;g))_1(s_{l+1})$ in case of a regular $\mathfrak{h}^l$.

If $\mathfrak{h}^l$ is indefinite of kind (A), then we set $\Psi_l^{ac}((f;g)) := ((N_{-\phi_{i(l)}} \cdot) \times (N_{-\phi_{i(l)}} \cdot)) \circ \Psi^{ac}(\mathfrak{h}^l)((f_l; g_l))$, where $\Psi^{ac}(\mathfrak{h}^l)$ is the function definied as in (4.17) for the elementary indefinite Hamiltonian $\mathfrak{h}^l$ and where $N_{-\phi_{i(l)}} \cdot$ maps a two-vector function f to $N_{-\phi_{i(l)}} f$. Again by definition $a_{l+1} = \Psi_l^{ac}((f;g))_1(s_{l+1})$.

For a positive $\mathfrak{h}^l$ being an indivisible interval of type ϕ, i.e. $H_{i(l)}(t) = h(t)\xi_\phi \xi_\phi^T$, $t \in (s_l, s_{l+1})$, we set

$$\Psi_l^{ac}((f;g))(x) := ((\xi_\phi^T f_l) \cdot \xi_\phi + (\xi_\phi^T g_l)(\int_{s_l}^x h) \cdot J\xi_\phi + \gamma J \xi_\phi; g_l(x)),$$

where $\gamma \in \mathbb{C}$ is chosen so that $a_l = \Psi_l^{ac}((f;g))_1(s_l)$. By (2.2) this is possible and, in case of a regular $\mathfrak{h}^l$, we have $a_{l+1} = \Psi_l^{ac}((f;g))_1(s_{l+1})$. Moreover, by (2.1) $\Psi_l^{ac}((f;g))_1' = JH_{i(l)} \Psi_l^{ac}((f;g))_2$.

If $\mathfrak{h}^l$ is indefinite of kind (B) or (C), we have

$$N_{\phi_{i(l)}} H_{i(l)-1}(t) N_{\phi_{i(l)}}^T = \begin{pmatrix} 0 & 0 \\ 0 & h(t) \end{pmatrix},\ t \in (s_l, \sigma_{i(l)})$$

and

$$N_{\phi_{i(l)}} H_{i(l)}(t) N_{\phi_{i(l)}}^T = \begin{pmatrix} 0 & 0 \\ 0 & h(t) \end{pmatrix},\ t \in (\sigma_{i(l)}, s_{l+1}).$$

Moreover, using the notation from Definition 4.3 and Definition 4.5 we write f_l as $\lambda p_0 + r$ and g_l as $\mu p_0 + s$, where $\lambda, \mu \in \mathbb{C}$ and r, s are zero (case (C)) or are sums of certain δ_j (case (B)). Note that $(N_{\phi_{i(l)}} a_l)_2 = (N_{\phi_{i(l)}} a_{l+1})_2 = \lambda$.

For $t \in (s_l, \sigma_{i(l)})$ we set

$$\Psi_l^{ac}((f;g))(t) := ((N_{-\phi_{i(l)}} \cdot) \times (N_{-\phi_{i(l)}} \cdot)) \circ (\begin{pmatrix} (N_{\phi_{i(l)}} a_l)_1 - \mu(\int_{s_l}^t h) \\ \lambda \end{pmatrix}; \begin{pmatrix} 0 \\ \mu \end{pmatrix}),$$

and for $t \in (\sigma_{i(l)}, s_{l+1})$

$$\Psi_l^{ac}((f;g))(t) := ((N_{-\phi_{i(l)}} \cdot) \times (N_{-\phi_{i(l)}} \cdot)) \circ (\begin{pmatrix} (N_{\phi_{i(l)}} a_{l+1})_1 - \mu(\int_{s_{l+1}}^t h) \\ \lambda \end{pmatrix}; \begin{pmatrix} 0 \\ \mu \end{pmatrix}).$$

One easily sees that $\Psi_l^{ac}((f;g))_1' = JH_{i(l)-1}\Psi_l^{ac}((f;g))_2$ on $(s_l, \sigma_{i(l)})$ and $\Psi_l^{ac}((f;g))_1' = JH_{i(l)}\Psi_l^{ac}((f;g))_2$ on $(\sigma_{i(l)}, s_{l+1})$ and that $a_{l+1} = \Psi_l^{ac}((f;g))_1(s_{l+1})$.

Setting $\Psi^{ac}(f;g)(t) := \Psi_l^{ac}(f;g)(t)$ for $t \in I \cap (s_l, s_{l+1})$ we have defined $\Psi^{ac} : T(\mathfrak{h}) \to AC(I) \times \mathcal{M}(I)/_{=}$ such that $\Psi^{ac}((f;g))_1' = JH\Psi^{ac}((f;g))_2$.

Moreover, $a_l = \Psi^{ac}((f;g))_1(s_l)$ for $l = 0, \dots, N$ and, in the regular case, also for $l = N+1$. In particular, $\Psi^{ac}((f;g))_1(s_0) = \Gamma(\mathfrak{h})(f;g)_1$ and, in case of a regular general Hamiltonian, $\Psi^{ac}((f;g))_1(s_{N+1}) = \Gamma(\mathfrak{h})(f;g)_2$.

Finally, it is easy to verify that Ψ^{ac} is linear and commutes with conjugation.

The above construction of $(\mathcal{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h}))$ one first sight essentially depends on the points E, which, in particular, determine the number of pieces, into which one cuts the general Hamiltonian $\mathfrak{h}$. But if one choses different points E and one changes the numbers $d_{i,j}$ accordingly, we will see that we end up with an isomorphic copy of the originally boundary triplet. Moreover, it is possible to get an isomorphic copy of the originally boundary triplet if we reparametrize the orginal Hamiltonian. Therefore, we define

8.10 Definition. Two general Hamiltonians $\mathfrak{h} = (H, \mathfrak{c}, \mathfrak{d})$ and $\tilde{\mathfrak{h}} = (\tilde{H}, \tilde{\mathfrak{c}}, \tilde{\mathfrak{d}})$ having the same number of singularities are called equivalent, if there exists an absolutely continuous and increasing bijection φ from $[\tilde{\sigma}_0, \tilde{\sigma}_{n+1}]$ onto $[\sigma_0, \sigma_{n+1}]$ such that φ^{-1} is absolutely continuous and $\varphi(\tilde{\sigma}_i) = \sigma_i$, and if there exists an isometric isomorphism $\varpi : \mathcal{P}(\mathfrak{h}) \to \mathcal{P}(\tilde{\mathfrak{h}})$, such that

(*i*) (ϖ, id) is an isomorphism of the boundary triplets $(\mathcal{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h}))$ and $(\mathcal{P}(\tilde{\mathfrak{h}}), T(\tilde{\mathfrak{h}}), \Gamma(\tilde{\mathfrak{h}}))$.

(*ii*) For all $x \in \mathcal{P}(\mathfrak{h})$ we have

$$\psi(\tilde{\mathfrak{h}})(\varpi(x)) = \psi(\mathfrak{h})(x) \circ \varphi \,. \tag{8.2}$$

Clearly, this defines an equivalence relation on the set of all general Hamiltonians.

8.11 Proposition. *Let $\mathfrak{h} = (H, \mathfrak{c}, \mathfrak{d})$ be a general Hamiltonian with the admissible partitions E of I. Moreover, let $\tilde{E}$ be another admissible partition of I. Then there exist numbers $\tilde{d}_{i,j}$ such that*

$$\tilde{\mathfrak{h}} = (H, \mathfrak{c}, \tilde{\mathfrak{d}})\,,$$

is equivalent to $\mathfrak{h}$, where $\tilde{\mathfrak{d}}$ stands for the data $\tilde{d}_{i,j}$ and $\tilde{E}$. The corresponding mapping φ (see Definition 8.10), hereby, is the identity on $[\sigma_0, \sigma_{n+1}]$, and the isomorphism ϖ satisfies ($(f;g) \in T(\mathfrak{h})$)

$$\Psi^{ac}(\mathfrak{h})(f;g) = \Psi^{ac}(\tilde{\mathfrak{h}})(\varpi(f);\varpi(g)). \tag{8.3}$$

Proof. With E and $\tilde{E}$ also $E \cup \tilde{E}$ is an admissible partition of I. Since being equivalent is an equivalence relation, it is enough to prove the present statement for the case that $E \subseteq \tilde{E}$. Since an admissible partition is a finite set, it is for this task enough to consider the case that $\tilde{E} = E \cup \{s\}$, $s \notin E$.

Write $E = \{s_0, \dots, s_{N+1}\}$, $s_0 < s_1 < \dots < s_{N+1}$, and assume that $\tilde{E} = E \cup \{s\}$ and that $l \in \{0, \dots, N\}$ with $s_l < s < s_{l+1}$.

Case 1, (def): Assume that l is in the case (def). Then the obvious identification ϖ_l of $\mathcal{P}^l = L^2(H_l|_{(s_l,s_{l+1})}$ with $L^2(H_l|_{(s_l,s)}) \oplus L^2(H_l|_{(s,s_{l+1})})$ is an isomorphism which satisfies (i) and (ii), cf. Lemma 6.9. Let

$$\big(\mathcal{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h})\big) = \biguplus_{l=0}^{N} (\mathcal{P}^l, T^l, \Gamma^l)\,.$$

If we put $\tilde{d}_{i,j} := d_{i,j}$, we have

$$\big(\mathcal{P}(\tilde{\mathfrak{h}}), T(\tilde{\mathfrak{h}}), \Gamma(\tilde{\mathfrak{h}})\big) = (\mathcal{P}^0, T^0, \Gamma^0) \uplus \ldots \uplus (\mathcal{P}^{l-1}, T^{l-1}, \Gamma^{l-1}) \uplus$$

$$\uplus \Big[(L^2(H_l|_{(s_l,s)}), T_{max}(H_l|_{(s_l,s)}), \Gamma(H_l|_{(s_l,s)})) \uplus$$

$$\uplus (L^2(H_l|_{(s,s_{l+1})}), T_{max}(H_l|_{(s,s_{l+1})}), \Gamma(H_l|_{(s,s_{l+1})}))\Big] \uplus$$

$$\uplus (\mathcal{P}^{l+1}, T^{l+1}, \Gamma^{l+1}) \uplus \ldots \uplus (\mathcal{P}^{l-1}, T^{l-1}, \Gamma^{l-1}) \uplus \ldots \uplus (\mathcal{P}^N, T^N, \Gamma^N)$$

It follows that (cf. Remark 6.8)

$$(\varpi, \mathrm{id}) := (\mathrm{id}_{\mathcal{P}^0}, \mathrm{id}) \uplus \ldots \uplus (\mathrm{id}_{\mathcal{P}^{l-1}}, \mathrm{id}) \uplus (\varpi_l, \mathrm{id}) \uplus (\mathrm{id}_{\mathcal{P}^{l+1}}, \mathrm{id}) \uplus \ldots \uplus (\mathrm{id}_{\mathcal{P}^N}, \mathrm{id})$$

is an isomorphism of $(\mathcal{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h}))$ and $(\mathcal{P}(\tilde{\mathfrak{h}}), T(\tilde{\mathfrak{h}}), \Gamma(\tilde{\mathfrak{h}}))$ which satisfies the requirements in Definition 8.10.

Finally, relation (8.3) follows from the corresponding property of the mapping ϖ_l which, in turn, is verified in an obvious way.

Case 2, (indef): Assume that l is in the case (indef). Note that, since s is not inner point of an indivisible interval, the elementary indefinite Hamiltonian $\mathfrak{h}^l$ must be of kind (A). Let us first consider the case that $s < \sigma(i(l))$. By Proposition 7.4 and Proposition 7.8 there exists an elementary indefinite Hamiltonian $\tilde{\mathfrak{h}^l}$ which consists of data of the form

$$H_l|_{(s,\sigma_{i(l)})}, H_{l+1}|_{(\sigma_{i(l)},s_{l+1})},\ \ddot{o}_{i(l)}, b_{i(l),1}, \ldots, b_{i(l),\ddot{o}_{i(l)}+1},$$

$$\tilde{d}_{i(l),0}, \ldots, \tilde{d}_{i(l),2\Delta_{i(l)}-1}\,,$$

and an isomorphism ϖ_l of $\mathcal{P}^l$ onto $L^2(H_l|_{(s_l,s)}) \oplus \mathcal{P}(\tilde{\mathfrak{h}^l})$ which satisfies the requirements in Definition 8.10. Moreover, in view of the definition of Ψ^{ac} (see Remark 8.9) (8.3) follows in a straight forward way from Proposition 7.4 and Proposition 7.8.

If $s \in (\sigma_{i(l)}, s_{l+1})$, we refer to Remark 7.9 to obtain the same conclusion. The proof is finished similar as in Case 1.

❑

Inspecting the previous proof, we obtain without extra work the following corollary.

8.12 Corollary. *Let $\mathfrak{h} = (H, \mathfrak{c}, \mathfrak{d})$ be a general Hamiltonian. Let $s \in I$ be not an inner point of an indivisible interval, and let i be such that $\sigma_i < s < \sigma_{i+1}$.*

If we set

$$\sigma_k^1 := \sigma_k,\ k=0,\dots,i,\ \sigma_k^2 := \sigma_{k+i},\ k=1,\dots,n-i+1,\ \sigma_{i+1}^1 := s =: \sigma_0^2;$$

$$H_k^1 := H_k,\ k=0,\dots,i-1, H_i^1 := H_i|_{(\sigma_i,s)},$$

$$H_k^2 := H_{k+i},\ k=1,\dots,n-i,\ H_0^2 := H_i|_{(s,\sigma_{i+1})};$$

$$\ddot{o}_k^1 := \ddot{o}_k,\ k=1,\dots,i,\ \ddot{o}_k^2 := \ddot{o}_{k+i},\ k=1,\dots,n-i;$$

$$b_{k,j}^1 := b_{k,j},\ k=1,\dots,i,\ b_{k,j}^2 := b_{k+i,j},\ k=1,\dots,n-i;$$

$$E^1 := (E\cap[\sigma_0,s))\cup\{s\}, E^2 := (E\cap(s,\sigma_{n+1}])\cup\{s\}$$

$$d_{k,j}^1 := d_{k,j},\ k=1,\dots,i-1,\ d_{k,j}^2 := d_{k+i,j},\ k=2,\dots,n-i;$$

then there exist numbers $d_{i,j}^1$ and $d_{1,j}^2$, such that building with these data the general Hamiltonians $\mathfrak{h}^1$ and $\mathfrak{h}^2$ on $[\sigma_0,s]$ and $[s,\sigma_{n+1}]$, respectively, there exists an isomorphism (ϖ,id) of the boundary triplet $(\mathcal{P}(\mathfrak{h}),T(\mathfrak{h}),\Gamma(\mathfrak{h}))$ onto $(\mathcal{P}(\mathfrak{h}^1),T(\mathfrak{h}^1),\Gamma(\mathfrak{h}^1))\uplus(\mathcal{P}(\mathfrak{h}^2),T(\mathfrak{h}^2),\Gamma(\mathfrak{h}^2))$ such that for all $x\in\mathcal{P}(\mathfrak{h})$ we have

$$(\psi(\tilde{\mathfrak{h}^1})\times\psi(\tilde{\mathfrak{h}^2}))(\varpi(x)) = \psi(\mathfrak{h})(x)\,,$$

and such that $((f;g)\in T(\mathfrak{h}))$

$$\Psi^{ac}(\mathfrak{h})(f;g) = (\Psi^{ac}(\mathfrak{h}^1)\times\Psi^{ac}(\mathfrak{h}^2))(\varpi(f);\varpi(g)).$$

At the end of this section we consider another situation when two general Hamiltonians are equivalent.

8.13 Proposition. *Let $\mathfrak{h}=(H,\mathfrak{c},\mathfrak{d})$ and $\tilde{\mathfrak{h}}=(\tilde{H},\tilde{\mathfrak{c}},\tilde{\mathfrak{d}})$ be two general Hamiltonians having the same number of singularities.*

If there is an absolutely continuous and increasing bijection φ of $[\tilde\sigma_0,\tilde\sigma_{n+1}]$ onto $[\sigma_0,\sigma_{n+1}]$ such that also φ^{-1} is absolutely continuous, $\varphi(\tilde\sigma_i)=\sigma_i$ and

$$\tilde{H}_i = (H_i\circ\varphi)\cdot\varphi',\ \tilde{\ddot{o}}_i = \ddot{o}_i; \tilde{b}_{i,1} = b_{i,1},\dots,\tilde{b}_{i,\ddot{o}_i} = b_{i,\ddot{o}_i};$$

$\varphi(\tilde{E}) = E$ and

$$\tilde{d}_{i,0} = d_{i,0},\dots,\tilde{d}_{i,2\Delta_i-2} = d_{i,2\Delta_i-2}; \tilde{d}_{i,2\Delta_i-1} - \tilde{b}_{i,\ddot{o}_i+1} = d_{i,2\Delta_i-1} - b_{i,\ddot{o}_i+1}\,,$$

then $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ are equivalent. Moreover, if these Hamiltonians do not consist of only one indivisible intervall, then $((f;g)\in T(\mathfrak{h}))$

$$\Psi^{ac}(\mathfrak{h})(f;g)\circ(\varphi\times\varphi) = \Psi^{ac}(\tilde{\mathfrak{h}})(\varpi(f);\varpi(g)). \tag{8.4}$$

Proof. Assume that there is a bijection φ of $[\tilde\sigma_0,\tilde\sigma_{n+1}]$ onto $[\sigma_0,\sigma_{n+1}]$ such that $\varphi(\tilde\sigma_i)=\sigma_i$ with the indicated properties.

Both boundary triplets, $(\mathcal{P}(\mathfrak{h}),T(\mathfrak{h}),\Gamma(\mathfrak{h}))$ and $(\mathcal{P}(\tilde{\mathfrak{h}}),T(\tilde{\mathfrak{h}}),\Gamma(\tilde{\mathfrak{h}}))$, are obtained as pasting of elementary indefinite Hamiltonians. As we have $\varphi(\tilde{E})=E$ these building blocks have their counterpart. By our assumptions we can apply Proposition 5.17 to each of the building blocks, and obtain for each of them an isomorphism which has the desired properties. Putting together these isomorphisms by means of Remark 6.8 yields the equivalence of $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$.

The relation (8.4) is also shown by the corresponding property for the building blocks, which is elementary but a bit lengthy. We will not work out the details.

❑

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M. Kaltenbäck, H. Woracek
Institut für Analysis und Scientific Computing
Technische Universität Wien
Wiedner Hauptstr. 8–10/101
A–1040 Wien
AUSTRIA
email: michael.kaltenbaeck@tuwien.ac.at, harald.woracek@tuwien.ac.at