De Branges spaces of entire functions symmetric about the origin

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Abstract. We define and investigate the class of symmetric and the class of semibounded de Branges spaces of entire functions. A construction is made which assigns to each symmetric de Branges space a semibounded one. By employing operator theoretic tools it is shown that every semibounded de Branges space can be obtained in this way, and which symmetric spaces give rise to the same semibounded space. Those subclasses of Hermite-Biehler functions are determined which correspond to symmetric or semibounded, respectively, nondegenerated de Branges spaces. The above assignment is determined in terms of the respective generating Hermite-Biehler functions.

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1. Introduction and preliminaries

Let us describe the objects of our studies. An almost Pontryagin space is a triple $(\mathfrak{L}, [.,.], \mathcal{O})$, where $\mathfrak{L}$ is a linear space, $[.,.]$ is an inner product on $\mathfrak{L}$, and $\mathcal{O}$ a Hilbert space topology on $\mathfrak{L}$ such that the following two axioms hold:

(aPs1) $[.,.]$ is $\mathcal{O}$-continuous.

(aPs2) There exists a $\mathcal{O}$-closed linear subspace $\mathfrak{M}$ of $\mathfrak{L}$ with finite codimension such that $(\mathfrak{M}, [.,.])$ is a Hilbert space.

This type of inner product space is very close to a Pontryagin space. The fundamental difference is that here the occurence of degeneracy is allowed, i.e. the isotropic part $\mathfrak{L}^\circ := \{f \in \mathfrak{L} : f \perp \mathfrak{L}\}$ might contain nonzero elements. If $\mathfrak{L}^\circ \neq \{0\}$, we call $\mathfrak{L}$ degenerated, otherwise nondegenerated. If $(\mathfrak{L}, [.,.], \mathcal{O})$ is a nondegenerated almost Pontryagin space, then $(\mathfrak{L}, [.,.])$ is a Pontryagin space. In this case the topology $\mathcal{O}$ meeting the above requirements is unique.

In the present context a particular class of almost Pontryagin spaces is of importance: We say that an inner product space $(\mathfrak{L},[.,.])$ is a reproducing kernel almost Pontryagin space on the set Ω, if

(rk1) The elements of $\mathfrak{L}$ are complex valued functions on Ω and the linear operations of $\mathfrak{L}$ are defined pointwise.

(rk2) There exists a Hilbert space topology $\mathcal{O}$ on $\mathfrak{L}$ such that $(\mathfrak{L},[.,.],\mathcal{O})$ is an almost Pontryagin space.

(rk3) For each $w\in\Omega$ the point evaluation functional $\chi_w: f\mapsto f(w)$ is $\mathcal{O}$-continuous.

If $(\mathfrak{L},[.,.])$ is a reproducing kernel almost Pontryagin space, then the topology satisfying the requirements (rk2), (rk3) is unique. If, additionally, $(\mathfrak{L},[.,.])$ is nondegenerated, then there exists a reproducing kernel of the space. This is a mapping $K:\Omega\times\Omega\to\mathbb{C}$ such that for every fixed $w\in\Omega$ the function $z\mapsto K(w,z)$ belongs to $\mathfrak{L}$, and has the property that

$$[f,K(w,.)]=f(w),\ f\in\mathfrak{L}.$$

In the degenerated case it is obvious that there cannot exist a reproducing kernel function. For a more elaborate discussion of almost Pontryagin spaces see [KWW1], a detailed account on reproducing kernel Pontryagin spaces is given in [ADRS].

We can now give the definition of a de Branges space of entire functions, cf. [dB], [KW1]:

1.1. Definition. An inner product space $(\mathfrak{P},[.,.])$ is called a de Branges space (dB-space, for short), if the following axioms hold true:

(dB1) $(\mathfrak{P},[.,.])$ is a reproducing kernel almost Pontryagin space on $\mathbb{C}$ whose elements are entire functions.

(dB2) If $F\in\mathfrak{P}$, then $F^\#\in\mathfrak{P}$, where $F^\#(z):=\overline{F(\overline{z})}$. Moreover,

$$[F^\#,G^\#]=[G,F],\ F,G\in\mathfrak{P}.$$

(dB3) If $F\in\mathfrak{P}$ and $z_0\in\mathbb{C}\setminus\mathbb{R}$ with $F(z_0)=0$, then

$$\frac{z-\overline{z_0}}{z-z_0}F(z)\in\mathfrak{P}.$$

Moreover, if additionally $G\in\mathfrak{P}$ with $G(z_0)=0$, then

$$\Big[\frac{z-\overline{z_0}}{z-z_0}F(z),\frac{z-\overline{z_0}}{z-z_0}G(z)\Big]=[F,G].$$

We will assume throughout this paper that also

(Z) For every $t\in\mathbb{R}$ there exists $F\in\mathfrak{P}$ with $F(t)\neq 0$.

By [KW1] an equivalent formulation of (Z) is: Whenever $F\in\mathfrak{P}$ and $t\in\mathbb{R}$ such that $F(t)=0$, then $(z-t)^{-1}F(z)\in\mathfrak{P}$. Let us note that the assumption of (Z) is no loss of generality. To every space satisfying (dB1)-(dB3) there exists an isometrically isomorphic space which satisfies (dB1)-(dB3) and (Z), cf. [KW1, Corollary 5.5].

If $(\mathfrak{P}, [.,.])$ is a nondegenerated dB-space, then we call it a dB-Pontryagin space. In this case there exists a reproducing kernel $K(w,z): \mathbb{C}\times\mathbb{C}\to\mathbb{C}$ which is analytic in the variables $\overline{w}, z$. It is an important feature that this kernel function is of a very particular form, cf. [KW1]. To explain this in more detail let us recall: The Hermite-Biehler class $\mathcal{HB}_\kappa$ with negative index $\kappa\in\mathbb{N}\cup\{0\}$, is defined as the set of all entire functions E, such that E and $E^\#$ have no common nonreal zeros, $E^{-1}E^\#$ is not constant, and the kernel

$$S_{\frac{E^\#}{E}}(w,z) := i\frac{1-\frac{E^\#(z)}{E(z)}\overline{\frac{E^\#(w)}{E(w)}}}{z-\overline{w}}$$

has κ negative squares on $\mathbb{C}^+$. This condition means that for every choice of $n\in\mathbb{N}$ and $z_1,\dots,z_n\in\mathbb{C}^+$, the quadratic form

$$q(\xi_1,\dots,\xi_n) := \sum_{i,j=1}^n S_{\frac{E^\#}{E}}(z_j,z_i)\xi_i\overline{\xi_j}\,,$$

has at most κ negative squares, and that for some choice of n and $z_1,\dots,z_n$ this upper bound is actually attained. Throughout this paper we shall always assume additionally that E has no real zeros. This condition corresponds to (Z). For notational convenience we set

$$\mathcal{HB}_{\leq\kappa} := \bigcup_{\nu\leq\kappa}\mathcal{HB}_\nu,\ \mathcal{HB}_{<\infty} := \bigcup_{\nu\in\mathbb{N}\cup\{0\}}\mathcal{HB}_\nu\,. \tag{1.1}$$

The Hermite-Biehler class is related to the notion of dB-Pontryagin spaces by the fact that, if $(\mathfrak{P}, [.,.])$ is a dB-Pontryagin space, then its reproducing kernel K is of the form

$$K(w,z) = i\frac{E(z)\overline{E(w)} - E^\#(z)E(\overline{w})}{2\pi(z-\overline{w})}\,, \tag{1.2}$$

for a (not unique) Hermite-Biehler function E. Conversely, every Hermite-Biehler function generates in this way a dB-Pontryagin space.

In the general theory of dB-spaces the operator of multiplication with the independent variable plays a decisive role. Let $(\mathfrak{P}, [.,.])$ be a dB-space. We denote by $\mathcal{S}$ the linear operator in $\mathfrak{P}$ defined by

$$(\mathcal{S}F)(z) := zF(z)$$

with domain

$$\operatorname{dom}\mathcal{S} := \big\{F\in\mathfrak{P} :\ zF(z)\in\mathfrak{P}\big\}\,.$$

The axiom (dB3) means nothing else than that $\mathcal{S}$ is a symmetric operator with defect index $(1,1)$. Note that $\mathcal{S}$ is closed since point evaluation in $\mathfrak{P}$ is continuous.

In the centre of our interests in the present paper are two particular classes of dB-spaces.

1.2. Definition. A dB-space $(\mathfrak{P}, [.,.])$ is called semibounded if the inner product

$$[F,G]_S := [\mathcal{S}F, G],\ \ F,G\in\operatorname{dom}\mathcal{S}\,, \tag{1.3}$$

has a finite number of negative squares on $\operatorname{dom}\mathcal{S}$.

If an operator in some almost Pontryagin space has the property that the corresponding inner product (1.3) has a finite number of negative squares, we shall also say that it is of finite negativity.

Denote by $\mathcal{O}(\mathbb{C})$ the set of all entire functions and let $M : \mathcal{O}(\mathbb{C}) \to \mathcal{O}(\mathbb{C})$ be defined as

$$M : \begin{cases} \mathcal{O}(\mathbb{C}) & \to & \mathcal{O}(\mathbb{C}) \\ F(z) & \mapsto & F(-z) \end{cases}$$

The map M is a linear involution of $\mathcal{O}(\mathbb{C})$.

1.3. Definition. A dB-space $(\mathfrak{P}, [.,.])$ is called symmetric, if M induces an isometric involution on $\mathfrak{P}$, i.e. if $M(\mathfrak{P}) \subseteq \mathfrak{P}$ and

$$[MF, MG] = [F, G], \; F, G \in \mathfrak{P}\,.$$

Semibounded and symmetric dB-spaces on the other hand are most intimately related. It is the aim of the present paper to investigate these classes of dB-spaces and to describe the relationship among them. More concisely formulated: We construct a map from the set of all symmetric dB-spaces to the set of all semibounded dB-spaces, show that it is surjective and determine its kernel, cf. Theorem 3.11. Moreover, in the case of dB-Pontryagin spaces, the action of this map is determined in terms of the generating Hermite-Biehler functions, cf. Theorem 4.5.

Besides the interest on its own right our motivation for the study of symmetry in dB-spaces arises out of different sources.

On the one hand, symmetry appears in dB-spaces defined by certain functions known from classical analysis as a consequence of the presence of functional equations. Gauss's hypergeometric functions, cf. [dB], or the Riemann Ξ-function, cf. [KW4], may serve as examples for this phenomenon. A structural understanding of symmetry in dB-spaces can therefore lead, in turn, to an analysis of functions satisfying a functional equation.

Secondly, symmetry appears in dB-spaces associated with a certain type of differential equations. For instance, we would like to quote here the equation of a vibrating string with inhomogenous mass distribution, cf. [KK], or Regge type boundary problems, cf. [MP]. The solution of inverse spectral problems depends in many cases essentially on the underlying theory of dB-spaces, and hence a thorough understanding of the structure of these spaces is necessary to give inverse results.

The notion of indefiniteness appears in such problems when singularities are allowed, for example in the form of discontinuities of weight functions, for example as in [LW] or [P]. In order to investigate such more general problems, a theory of symmetry in generally indefinite dB-spaces is needed. Particular attention has to be paid to the occurence of degeneracy, since an investigation of degenerated situations generically means an investigation of the singularities themselves.

Forthcoming work will be devoted to the application of the present results in some of the mentioned areas, for example to the investigation of the generalized string, cf. [KWW4].

The contents of this paper is divided into four sections. After this introductory part we deal with symmetric dB-spaces, investigate some of their basic properties and define the desired assignment by constructing a semibounded dB-space out of a given symmetric one, cf. Proposition 2.6. In the next section we elaborate the converse construction, cf. Proposition 3.4, Proposition 3.10. This construction depends on an operator theoretic construction introduced in [KWW3]. We concisely collect our results in Theorem 3.11. In the last section we consider the case of dB-Pontryagin spaces. We characterize those subclasses of Hermite-Biehler functions which give rise to symmetric or semibounded dB-Pontryagin spaces, cf. Proposition 4.3 and Proposition 4.4. Moreover, the correspondence between semibounded and symmetric spaces which was set up in the previous section is characterized in terms of the generating Hermite-Biehler functions, cf. Theorem 4.5.

2. Symmetric de Branges spaces

We start off with some fairly elementary observations. Let $(\mathfrak{P},[.,.])$ be a symmetric dB-space. The presence of the isometric involution M implies that the space $\mathfrak{P}$ splits.

2.1. Lemma. *Let $(\mathfrak{P},[.,.])$ be a symmetric dB-space and put*

$$\mathfrak{P}_e := \ker\big(I - M|_{\mathfrak{P}}\big) = \big\{F\in\mathfrak{P} :\ F \textit{ is even}\big\}\,,$$

$$\mathfrak{P}_o := \ker\big(I + M|_{\mathfrak{P}}\big) = \big\{F\in\mathfrak{P} :\ F \textit{ is odd}\big\}\,.$$

Then $\mathfrak{P}_e$ and $\mathfrak{P}_o$ are closed subspaces of $\mathfrak{P}$, and we have

$$\mathfrak{P} = \mathfrak{P}_e[\dot{+}]\mathfrak{P}_o\,.$$

Proof. The spaces $\mathfrak{P}_e$ and $\mathfrak{P}_o$ are closed subspaces of $\mathfrak{P}$ because point evaluation is continuous. Since $M|_{\mathfrak{P}}$ is an involution, the mappings

$$P_e := \frac{1}{2}\big(I + M|_{\mathfrak{P}}\big),\ P_o := \frac{1}{2}\big(I - M|_{\mathfrak{P}}\big)$$

are projections on $\mathfrak{P}$. We have $\operatorname{ran} P_e = \ker P_o = \mathfrak{P}_e$ and $\operatorname{ran} P_o = \ker P_e = \mathfrak{P}_o$. Since $P_e + P_o = I$, it follows that $\mathfrak{P} = \operatorname{ran} P_e \dot{+} \operatorname{ran} P_o$. Since $M|_{\mathfrak{P}}$ is isometric, we obtain for any $F\in\mathfrak{P}_e, G\in\mathfrak{P}_o$,

$$[F,G] = [MF,MG] = [F,-G] = -[F,G]\,.$$

This shows that $\mathfrak{P}_e \perp \mathfrak{P}_o$. □

Let us note that, by our overall assumption (Z), the space $\mathfrak{P}_e$ contains nonzero elements: For, if $F\in\mathfrak{P}$ is chosen such that $F(0)\neq 0$, then $(P_eF)(0) = F(0)\neq 0$.

Since $\mathfrak{P}_e$ and $\mathfrak{P}_o$ are closed, the projections P_e and P_o are continuous. Moreover, $\mathfrak{P}_e$ and $\mathfrak{P}_o$ are, with the inner product and topology inherited from $\mathfrak{P}$, themselves almost Pontryagin spaces.

Let us recall the notion of functions associated to a dB-space, cf. [dB], [KW1]: An entire function T is said to belong to $\operatorname{Assoc}\mathfrak{P}$, if

$$\frac{F(z)T(w)-F(w)T(z)}{z-w}\in\mathfrak{P} \text{ whenever } F\in\mathfrak{P}, w\in\mathbb{C}\,.$$

It is shown in [KW1] that $T\in\operatorname{Assoc}\mathfrak{P}$ if and only if there exist $F,G\in\mathfrak{P}$ such that

$$T(z)=F(z)+zG(z)\,.$$

In the case of a symmetric dB-space, the linear space $\operatorname{Assoc}\mathfrak{P}$ splits similarly as $\mathfrak{P}$ does. Let us denote

$$\operatorname{Assoc}_e\mathfrak{P}:=\big\{F\in\operatorname{Assoc}\mathfrak{P}:\ F \text{ even}\big\}\,,$$

$$\operatorname{Assoc}_o\mathfrak{P}:=\big\{F\in\operatorname{Assoc}\mathfrak{P}:\ F \text{ odd}\big\}\,.$$

2.2. Lemma. *Let* $(\mathfrak{P},[.,.])$ *be a dB-space and assume that* M *leaves* $\mathfrak{P}$ *invariant. Then* M *leaves* $\operatorname{Assoc}\mathfrak{P}$ *invariant, and we have*

$$\operatorname{Assoc}\mathfrak{P}=\operatorname{Assoc}_e\mathfrak{P}\dot{+}\operatorname{Assoc}_o\mathfrak{P}\,. \tag{2.1}$$

Moreover,

$$\operatorname{Assoc}_e\mathfrak{P}=\mathfrak{P}_e+z\mathfrak{P}_o,\ \operatorname{Assoc}_o\mathfrak{P}=\mathfrak{P}_o+z\mathfrak{P}_e=z\mathfrak{P}_e\,. \tag{2.2}$$

Proof. If $T\in\operatorname{Assoc}\mathfrak{P}$, choose $F,G\in\mathfrak{P}$ such that $T(z)=F(z)+zG(z)$. Then

$$(MT)(z)=(MF)(z)-z(MG)(z)\in\operatorname{Assoc}\mathfrak{P}\,.$$

The relation (2.1) follows since the mappings

$$\frac{1}{2}\big(I+M|_{\operatorname{Assoc}\mathfrak{P}}\big),\ \frac{1}{2}\big(I-M|_{\operatorname{Assoc}\mathfrak{P}}\big)\,,$$

are projections whose sum is the identity.

To prove (2.2) note first that the respective inclusions '$\supseteq$' are obvious. Let $T\in\operatorname{Assoc}_e\mathfrak{P}$ be given. Choose $F\in\mathfrak{P}_e$ with $F(0)=1$. Then

$$G(z):=\frac{F(z)T(0)-T(z)}{z}\in\mathfrak{P}_o$$

and we obtain

$$T(z)=T(0)\cdot F(z)-zG(z)\in\mathfrak{P}_e+z\mathfrak{P}_o\,.$$

If $T\in\operatorname{Assoc}_o\mathfrak{P}$, then $T(0)=0$ and hence

$$G(z):=\frac{T(z)}{z}\in\mathfrak{P}\,.$$

Clearly, G is even, and therefore $T\in z\mathfrak{P}_e$. □

Let us next investigate the multiplication operator $\mathcal{S}$ on a symmetric dB-space. Since $\mathcal{S}$ transforms even functions into odd functions and vice-versa, it splits accordingly.

2.3. Lemma. *Let $(\mathfrak{P},[.,.])$ be a symmetric dB-space and let $\mathcal{S}$ be the multiplication operator in $\mathfrak{P}$. Then*

$$\operatorname{dom}\mathcal{S} = \big(\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_e\big)\,[\dot{+}]\,\big(\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o\big)\,, \tag{2.3}$$

$$\begin{aligned}\operatorname{ran}\mathcal{S} &= \{F\in\mathfrak{P} : F(0)=0\} = \big(\operatorname{ran}\mathcal{S}\cap\mathfrak{P}_e\big)\,[\dot{+}]\,\mathfrak{P}_o\,,\\ \operatorname{ran}\mathcal{S}\cap\mathfrak{P}_e &= \big\{F\in\mathfrak{P}_e :\, F(0)=F'(0)=0\big\}\,.\end{aligned} \tag{2.4}$$

The operator $\mathcal{S}$ admits the following block operator representation:

$$\mathcal{S} = \begin{pmatrix} 0 & S_{oe}\\ S_{eo} & 0\end{pmatrix} : \begin{matrix}\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_e\\ [\dot{+}]\\ \operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o\end{matrix} \longrightarrow \begin{matrix}\operatorname{ran}\mathcal{S}\cap\mathfrak{P}_e\\ [\dot{+}]\\ \mathfrak{P}_o\end{matrix}$$

If $\overline{\operatorname{dom}\mathcal{S}}=\mathfrak{P}$, then $\overline{\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_e}=\mathfrak{P}_e$ and $\overline{\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o}=\mathfrak{P}_o$. Otherwise either

$$\overline{\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_e}=\mathfrak{P}_e,\ \operatorname{codim}_{\mathfrak{P}_o}\overline{\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o}=1\,,$$

or

$$\operatorname{codim}_{\mathfrak{P}_e}\overline{\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_e}=1,\ \overline{\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o}=\mathfrak{P}_o\,.$$

Proof. Since $M(zF(z))=-z(MF)(z)$, $F\in\mathcal{O}(\mathbb{C})$, we have

$$M(\operatorname{dom}\mathcal{S})=\operatorname{dom}\mathcal{S} \text{ and } \mathcal{S}\circ M = -(M\circ\mathcal{S})\,.$$

Hence also

$$\begin{aligned}P_e(\operatorname{dom}\mathcal{S})\subseteq\operatorname{dom}\mathcal{S},\ S\circ P_e = P_o\circ\mathcal{S}\,,\\ P_o(\operatorname{dom}\mathcal{S})\subseteq\operatorname{dom}\mathcal{S},\ S\circ P_o = P_e\circ\mathcal{S}\,.\end{aligned}$$

From this the relation (2.3) is obvious. We have $\mathcal{S}(\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_e)\subseteq\mathfrak{P}_o$. Moreover, $\mathfrak{P}_o\subseteq\operatorname{ran}\mathcal{S}$ since $F(0)=0$ implies that $z^{-1}F(z)\in\mathfrak{P}$, and $\mathcal{S}(\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o)\subseteq\mathfrak{P}_e$. Thus the first line of (2.4) as well as the desired block operator representation follows.

We show the equality stated in the second line of (2.4). If $F\in\operatorname{ran}\mathcal{S}$, then $F(0)=0$. If F is even, then $F'(0)=0$, and the inclusion '$\subseteq$' in the second equality follows. The converse inclusion is clear.

Since the projections P_e and P_o are continuous and map $\operatorname{dom}\mathcal{S}$ into $\operatorname{dom}\mathcal{S}$, we have

$$\overline{\operatorname{dom}\mathcal{S}} = \overline{\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_e}[\dot{+}]\overline{\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o}\,.$$

From this and the fact that the codimension of $\overline{\operatorname{dom}\mathcal{S}}$ in $\mathfrak{P}$ is either 0 or 1, the remaining assertions of the lemma follow. □

Let us moreover note that the isotropic part $\mathfrak{P}^\circ$ of $\mathfrak{P}$ also splits accordingly and that the dimensions of the isotropic parts of $\mathfrak{P}_e$ and $\mathfrak{P}_o$ cannot differ too much.

2.4. Lemma. *Let $(\mathfrak{P},[.,.])$ be a symmetric dB-space. Then M leaves $\mathfrak{P}^\circ$ invariant,*

$$\mathfrak{P}^\circ = (\mathfrak{P}_e\cap\mathfrak{P}^\circ)\,[\dot{+}]\,(\mathfrak{P}_o\cap\mathfrak{P}^\circ) = \mathfrak{P}_e^\circ[\dot{+}]\mathfrak{P}_o^\circ\,,$$

and we have

$$\big|\dim\mathfrak{P}_e^\circ - \dim\mathfrak{P}_o^\circ\big| \leq 1\,. \tag{2.5}$$

Proof. Let $F \in \mathfrak{P}^\circ$, then for all $G \in \mathfrak{P}$,

$$[MF, G] = [F, MG] = 0\,,$$

and, hence, $MF \in \mathfrak{P}^\circ$. From this and the fact that $\mathfrak{P}_e[\perp]\mathfrak{P}_o$ the first assertion follows.

Let $F \in \mathfrak{P}^\circ \cap \operatorname{ran}\mathcal{S}$ and $G \in \operatorname{ran}\mathcal{S}$, then using the fact that $\ker\mathcal{S} = \{0\}$ we get

$$[\mathcal{S}^{-1}F, G] = [\mathcal{S}^{-1}F, \mathcal{S}\mathcal{S}^{-1}G] = [F, \mathcal{S}^{-1}G] = 0\,.$$

Hence $\mathcal{S}^{-1}$ maps $\mathfrak{P}^\circ \cap \operatorname{ran}\mathcal{S}$ into $(\operatorname{ran}\mathcal{S})^{[\perp]}$. Clearly, $\mathcal{S}^{-1}$ is injective. Since $\mathfrak{P}_o \subseteq \operatorname{ran}\mathcal{S}$, we obtain

$$\mathcal{S}^{-1}(\mathfrak{P}_o^\circ) \subseteq (\operatorname{ran}\mathcal{S})^{[\perp]} \cap \mathfrak{P}_e\,,$$

and

$$\mathcal{S}^{-1}(\mathfrak{P}_e^\circ \cap \operatorname{ran}\mathcal{S}) \subseteq (\operatorname{ran}\mathcal{S})^{[\perp]} \cap \mathfrak{P}_o \subseteq \mathfrak{P}_o^\perp \cap \mathfrak{P}_o = \mathfrak{P}_o^\circ\,.$$

Since $\operatorname{ran}\mathcal{S}$ is a hyperplane, we have

$$\dim(\mathfrak{P}_e^\circ \cap \operatorname{ran}\mathcal{S}) \geq \dim\mathfrak{P}_e^\circ - 1\,,$$

and it readily follows that

$$\dim\mathfrak{P}_e^\circ \leq \dim(\mathfrak{P}_e^\circ \cap \operatorname{ran}\mathcal{S}) + 1 \leq \dim\mathfrak{P}_o^\circ + 1\,.$$

On the other hand $(\operatorname{ran}\mathcal{S})^{[\perp]} \cap \mathfrak{P}_e \subseteq (\operatorname{ran}\mathcal{S} \cap \mathfrak{P}_e)^{\perp_{\mathfrak{P}_e}}$ and, since $\operatorname{ran}\mathcal{S} \cap \mathfrak{P}_e$ is a hyperplane in $\mathfrak{P}_e$, we find

$$\dim\left((\operatorname{ran}\mathcal{S} \cap \mathfrak{P}_e)^{\perp_{\mathfrak{P}_e}}\right) \leq \dim\mathfrak{P}_e^\circ + 1\,.$$

Thus

$$\dim\mathfrak{P}_o^\circ \leq \dim\left((\operatorname{ran}\mathcal{S})^\perp \cap \mathfrak{P}_e\right) \leq \dim\mathfrak{P}_e^\circ + 1\,.$$

Alltogether (2.5) follows. ☐

It is a basic observation that the space $\mathfrak{P}_e$ almost carries a dB-space structure. Consider the map

$$\Phi : \begin{cases} \mathcal{O}(\mathbb{C}) & \to \quad \{\, F \in \mathcal{O}(\mathbb{C}) :\, F \text{ even}\} \\ F(z) & \mapsto \quad F(z^2) \end{cases}$$

The map Φ is a linear bijection.

2.5. Definition. Let $(\mathfrak{P}, [.,.])$ be a symmetric dB-space. Define a linear space

$$\mathfrak{P}_+ := \Phi^{-1}(\mathfrak{P}_e)\,,$$

and let $\mathfrak{P}_+$ be endowed with an inner product $[.,.]_+$ and a topology by the requirement that $\Phi|_{\mathfrak{P}_+} : \mathfrak{P}_+ \to \mathfrak{P}_e$ is isometric and homeomorphic.

By its definition $\mathfrak{P}_+$ is an almost Pontryagin space and $\Phi|_{\mathfrak{P}_+}$ is an almost Pontryagin space isomorphism of $\mathfrak{P}_+$ onto $\mathfrak{P}_e$. In the next proposition we show that $\mathfrak{P}_+$ is a semibounded dB-space. Thus we have assigned to each symmetric dB-space a semibounded one.

2.6. Proposition. *Let* $(\mathfrak{P},[.,.])$ *be a symmetric dB-space. Then* $(\mathfrak{P}_+,[.,.]_+)$ *is a semibounded dB-space. If* $\mathcal{S}$ *and* $\mathcal{S}_+$ *denote the respective multiplication operators on* $\mathfrak{P}$ *and* $\mathfrak{P}_+$*, we have*

$$\Phi\circ\mathcal{S}_+\circ\Phi^{-1}=\mathcal{S}^2|_{\mathfrak{P}_e}\,. \tag{2.6}$$

Moreover,

$$\operatorname{Assoc}_e\mathfrak{P}\subseteq\Phi(\operatorname{Assoc}\mathfrak{P}_+)\,.$$

Proof. To see that the point evaluation functionals

$$\chi_w:\begin{cases}\mathcal{O}(\mathbb{C}) & \to & \mathbb{C}\\ F & \mapsto & F(w)\end{cases}$$

are continuous on $\mathfrak{P}_+$ it is enough to note that

$$\chi_{w^2}=\chi_w\circ\Phi,\ w\in\mathbb{C}\,.$$

The axiom (dB2) is immediate since

$$\Phi(F^\#)=(\Phi F)^\#,\ F\in\mathcal{O}(\mathbb{C})\,.$$

Let $F\in\mathfrak{P}_+$ and $z_0\in\mathbb{C}\setminus\mathbb{R}$ with $F(z_0)=0$ be given. Fix $w\in\mathbb{C}^+$ such that $w^2=z_0$. Then $\Phi F\in\mathfrak{P}_e$ and $(\Phi F)(\pm w)=0$. Hence

$$\Phi\Big(\frac{z-\overline{z_0}}{z-z_0}F(z)\Big)=\frac{z^2-\overline{w}^2}{z^2-w^2}(\Phi F)(z)=$$
$$=\frac{z-\overline{w}}{z-w}\frac{z+\overline{w}}{z+w}(\Phi F)(z)\in\mathfrak{P}_e\,.$$

Moreover, if $G\in\mathfrak{P}_+$ also vanishes at z_0, we find with help of this relation

$$\Big[\frac{z-\overline{z_0}}{z-z_0}F(z),\frac{z-\overline{z_0}}{z-z_0}G(z)\Big]_+=\Big[\Phi\Big(\frac{z-\overline{z_0}}{z-z_0}F(z)\Big),\Phi\Big(\frac{z-\overline{z_0}}{z-z_0}G(z)\Big)\Big]=$$
$$=\Big[\frac{z-\overline{w}}{z-w}\frac{z+\overline{w}}{z+w}(\Phi F)(z),\frac{z-\overline{w}}{z-w}\frac{z+\overline{w}}{z+w}(\Phi G)(z)\Big]=[\Phi F,\Phi G]=[F,G]_+\,.$$

We see that $(\mathfrak{P}_+,[.,.]_+)$ is a dB-space. Moreover it satisfies (Z). To see this let $t\in\mathbb{R}$ be given. We have to find $F\in\mathfrak{P}_e$ such that $F(\sqrt{t})\neq 0$. We already saw after Lemma 2.1 that there exists $F\in\mathfrak{P}_e$ with $F(0)\neq 0$, hence the case $t=0$ is settled. Let w be a square root of t. Assume first that $\dim\mathfrak{P}>1$ and hence that $\operatorname{dom}\mathcal{S}\neq\{0\}$. Choose $F\in\operatorname{dom}\mathcal{S}\setminus\{0\}$. Since $\mathfrak{P}$ satisfies (dB3) and (Z) we can assume that $F(w)\neq 0$. If $F(w)+F(-w)\neq 0$, we have $(P_eF)(w)\neq 0$ and are done. Otherwise the function $P_e\mathcal{S}F$ does the job, because $F(w)+F(-w)=0$ implies that

$$(\mathcal{S}F)(w)+(\mathcal{S}F)(-w)=w(F(w)-F(-w))=2wF(w)\neq 0\,.$$

If $\dim\mathfrak{P}=1$, we must have $\mathfrak{P}=\mathfrak{P}_e$, and thus are done.

To prove the semiboundedness of $(\mathfrak{P}_+,[.,.]_+)$, we show that the linear bijection

$${}_z\Phi:\begin{cases}\mathcal{O}(\mathbb{C}) & \to & \{\,G\in\mathcal{O}(\mathbb{C}):\,G\text{ odd}\}\\ F(z) & \mapsto & zF(z^2)\end{cases}$$

induces an isometry of $(\operatorname{dom}\mathcal{S}_+, [.,.]_{\mathcal{S}_+})$ onto $(\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o, [.,.])$. First assume that $F \in \operatorname{dom}\mathcal{S}_+$. Then $zF(z) \in \mathfrak{P}_+$ and hence

$$z({}_z\Phi\, F)(z) = z^2F(z^2) = \Phi(zF(z)) \in \mathfrak{P}\,.$$

It follows that ${}_z\Phi\, F \in \operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o$. Conversely, let $G \in \operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o$. Then $G(0) = 0$ and hence $z^{-1}G(z) \in \mathfrak{P}_e$. Put $F := \Phi^{-1}(z^{-1}G(z))$, then $F \in \mathfrak{P}_+$ and ${}_z\Phi\, F = G$. Moreover,

$$\Phi(zF(z)) = zG(z) \in \mathfrak{P}_e\,,$$

and thus $zF(z) \in \mathfrak{P}_+$. We therefore have $({}_z\Phi)^{-1}G = F \in \operatorname{dom}\mathcal{S}_+$.

In order to check the isometry property of ${}_z\Phi$, let $F, G \in \operatorname{dom}\mathcal{S}_+$ be given and note that

$$(\Phi\mathcal{S}_+F)(z) = z^2(\Phi F)(z) = z({}_z\Phi\, F)(z)\,. \tag{2.7}$$

It follows that

$$[F,G]_{\mathcal{S}_+} = [\mathcal{S}_+F, G]_+ = [\Phi\mathcal{S}_+F, \Phi G] =$$
$$= [z({}_z\Phi\, F)(z), (\Phi G)(z)] = [\,{}_z\Phi\, F,\ {}_z\Phi\, G]\,.$$

The relation (2.6) is an immediate consequence of (2.7). In order to establish the last assertion of the proposition let $T \in \operatorname{Assoc}_e\mathfrak{P}$ be given. By Lemma 2.2 there exist $F \in \mathfrak{P}_e$ and $G \in \mathfrak{P}_o$ such that

$$T(z) = F(z) + zG(z)\,.$$

We have $H := \Phi^{-1}F \in \mathfrak{P}_+$ and, since $z^{-1}G(z) \in \mathfrak{P}_e$, also

$$L(z) := \Phi^{-1}\big(\frac{1}{z}G(z)\big) \in \mathfrak{P}_+\,.$$

Thus $H(z) + zL(z) \in \operatorname{Assoc}\mathfrak{P}_+$. However,

$$\Phi\big(H(z) + zL(z)\big) = (\Phi H)(z) + z^2(\Phi L)(z) = F(z) + zG(z) = T(z)\,.$$

□

The following considerations will point us the way to a converse of this assignment. Let us recall a construction introduced in [KWW3]: To an almost Pontryagin space $(\mathfrak{P}, [.,.], \mathcal{O})$ and a closed symmetric relation S in $\mathfrak{P}$ which is of finite negativity there is assigned a triple $(\mathfrak{P}_S, [.,.]_S, \mathcal{O}_S)$ of a linear space $\mathfrak{P}_S$, an inner product $[.,.]_S$ and a Hilbert space topology $\mathcal{O}_S$. Thereby the space $\mathfrak{P}_S$ can be considered as a linear subspace of $\mathfrak{P}$ and, if this is done, the set theoretic inclusion map is continuous.

If $(\mathfrak{P}, [.,.])$ is a Hilbert space and S is an operator, $\mathfrak{P}_S$ is obtained as the completion of $\operatorname{dom} S$ with respect to the inner product

$$h_m^S[F,G] := m[F,G] + [SF,G],\ \ F,G \in \operatorname{dom} S\,,$$

where m is chosen sufficiently large, so that $S + m$ is strictly positive. The inner product $[.,.]_S$ is given as the extension by continuity of

$$[F,G]_S := [SF,G],\ \ F,G \in \operatorname{dom} S\,.$$

Finally, the topology $\mathcal{O}_S$ is the Hilbert space topology which $\mathfrak{P}_S$ carries as completion of $(\operatorname{dom} S, h_m^S[.,.])$. In the general -indefinite- case, the construction of $(\mathfrak{P}_S, [.,.]_S, \mathcal{O}_S)$ is carried out by reduction to the Hilbert space case.

An intrinsic characterization of the set $\mathfrak{P}_S$ can be given as follows: Let $F \in \mathfrak{P}$, then $F \in \mathfrak{P}_S$ if and only if there exists a sequence $(F_n; G_n) \in \mathcal{S}$ such that $F_n \to F$ with respect to $\mathcal{O}$ and $\lim_{n,m\to\infty}[F_n - F_m, G_n - G_m] = 0$.

It is important to note that in the present setting of symmetric dB-spaces the reduction to the Hilbert space case can always be achieved within the class of symmetric dB-spaces.

2.7. Lemma. *Let $(\mathfrak{P}, [.,.])$ be a symmetric dB-space. Then there exists a finite rank perturbation $(.,.)$ of $[.,.]$ such that $(\mathfrak{P}, (.,.))$ is a symmetric dB-Hilbert space.*

Proof. By [KW1, Theorem 3.3] there exist $t_1, \dots, t_n \in \mathbb{R}$, $c_1, \dots, c_n > 0$ such that the inner product

$$(F,G)_1 := [F,G] + \sum_{i=1}^n c_i F(t_i)\overline{G(t_i)},\ \ F,G \in \mathfrak{P}$$

turns $\mathfrak{P}$ into a dB-Hilbert space. It follows that $\mathfrak{P}$ also becomes a dB-Hilbert space if endowed with the inner product

$$(F,G) := [F,G] + \sum_{i=1}^n c_i\big(F(t_i)\overline{G(t_i)} + F(-t_i)\overline{G(-t_i)}\big),\ \ F,G \in \mathfrak{P}\,.$$

It is straightforward to check that, if $(\mathfrak{P}, [.,.])$ is symmetric, then also $(\mathfrak{P}, (.,.))$ has this property. □

The next result shows us how the space $\mathfrak{P}_o$ can -almost entirely- be recovered from $(\mathfrak{P}_+, [.,.]_+)$.

2.8. Proposition. *Let $(\mathfrak{P}, [.,.])$ be a symmetric dB-space and let us denote by $(\mathfrak{P}_{+,\mathcal{S}_+}, [.,.]_{\mathcal{S}_+}, \mathcal{O}_{\mathcal{S}_+})$ the triple constructed out of the space $(\mathfrak{P}_+, [.,.]_+)$ and the multiplication operator $\mathcal{S}_+$ in $\mathfrak{P}_+$. The map ${}_z\Phi$ induces an isometric and homeomorphic bijection of $(\mathfrak{P}_{+,\mathcal{S}_+}, [.,.]_{\mathcal{S}_+}, \mathcal{O}_{\mathcal{S}_+})$ onto $(\overline{\operatorname{dom}\mathcal{S} \cap \mathfrak{P}_o}, [.,.])$.*

Proof. We saw in the proof of Proposition 2.6 that ${}_z\Phi|_{\operatorname{dom}\mathcal{S}_+}$ is an isometry of $(\operatorname{dom}\mathcal{S}_+, [.,.]_{\mathcal{S}_+})$ onto $(\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o, [.,.])$. Our aim is to show that it is bicontinuous with respect to the topology $\mathcal{O}_{\mathcal{S}_+}|_{\operatorname{dom}\mathcal{S}_+}$ and the topology of $\mathfrak{P}$.

Assume first that $(\mathfrak{P}, [.,.])$ is a dB-Hilbert space. Then the topology on $\operatorname{dom}\mathcal{S}_+$ is induced by the inner product $h_m^{\mathcal{S}_+}[.,.]_+$ where m is sufficiently large, and the one of $\mathfrak{P}$ by $[.,.]$.

Since $\mathcal{S}$ is injective and has a closed range, 0 is a point of regular type for $\mathcal{S}$. Moreover, ${}_z\Phi\, F = \mathcal{S}(\Phi F)$, and by the isometry properties of Φ and ${}_z\Phi$ we find for $F \in \operatorname{dom}\mathcal{S}_+$

$$[F,F]_+ = [\Phi F, \Phi F] \le \|\mathcal{S}^{-1}\|^2[\,{}_z\Phi\, F, {}_z\Phi\, F] = \|\mathcal{S}^{-1}\|^2[F,F]_{\mathcal{S}_+} = \|\mathcal{S}^{-1}\|^2[\mathcal{S}_+F, F]_+\,.$$

Hence, in the Hilbert space case $\mathcal{S}_+$ is strictly positive with a lower bound satisfying

$$m(\mathcal{S}_+) \geq \frac{1}{\|\mathcal{S}^{-1}\|^2} > 0. \tag{2.8}$$

Therefore, we can take $m = 0$ and the topology on $\operatorname{dom}\mathcal{S}_+$ is induced by the inner product $h_0^{S_+}[.,.]_+ = [.,.]_{\mathcal{S}_+}$. The isometry property

$$[\,{}_z\Phi\, F,\, {}_z\Phi\, F] = [F,F]_{\mathcal{S}_+} = h_0^{S_+}[F,F]_+,\ F \in \operatorname{dom}\mathcal{S}_+$$

of ${}_z\Phi$ now shows, in particular, that ${}_z\Phi$ is bicontinuous.

Let $(\mathfrak{P},[.,.])$ be an arbitrary symmetric dB-space. Choose a finite rank perturbation $(.,.)$ of $[.,.]$ such that $(\mathfrak{P},(.,.))$ is a symmetric dB-Hilbert space. Since point evaluation is continuous in $(\mathfrak{P},[.,.])$, the topology of $(\mathfrak{P},(.,.))$ is the same as the one of $(\mathfrak{P},[.,.])$. Moreover, also the inner product $(.,.)_+$ is a finite rank perturbation of $[.,.]_+$. By [KWW3, Corollary 4.11] the topology $\mathcal{O}_{\mathcal{S}_+}$ remains the same, whether constructed from $(\mathfrak{P}_+,[.,.]_+)$ or $(\mathfrak{P}_+,(.,.)_+)$. Hence we may apply what we proved in the previous paragraph and see that also in the general case ${}_z\Phi$ is bicontinuous.

The map ${}_z\Phi\,|_{\operatorname{dom}\mathcal{S}_+}$ can be extended by continuity to $\mathfrak{P}_{\mathcal{S}_+}$ and the map ${}_z\Phi^{-1}\,|_{\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o}$ to $\overline{\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o}$. Since in both spaces point evaluation is continuous (remember that $\mathfrak{P}_{+,\mathcal{S}_+}$ is continuously embedded in $\mathfrak{P}_+$), those extensions must be given by ${}_z\Phi\,|_{\mathfrak{P}_{+,\mathcal{S}_+}}$ and ${}_z\Phi^{-1}\,|_{\overline{\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o}}$, respectively. Since the respective inner products are continuous, also the isometry property is retained. □

We know that $\operatorname{codim}_{\mathfrak{P}_0}\overline{\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o}$ is either 0 or 1. In the following we examine the latter case.

2.9. Lemma. *Assume that* $\operatorname{codim}_{\mathfrak{P}_0}\overline{\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o} = 1$. *Then we have* ${}_z\Phi^{-1}(\mathfrak{P}_o) \subseteq \mathfrak{P}_+$ *and*

$${}_z\Phi^{-1}\big(\overline{\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o}^{\perp_{\mathfrak{P}_o}}\big) = (\operatorname{ran}\mathcal{S}_+)^{\perp}\,.$$

Proof. The first assertion is clear since $\mathfrak{P}_o \subseteq \operatorname{ran}\mathcal{S}$ and ${}_z\Phi^{-1}\,|_{\operatorname{ran}\mathcal{S}} = \Phi^{-1}\circ\mathcal{S}^{-1}$. Let $F \in \mathfrak{P}_o$, $F \perp \overline{\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o}$ be given. If $H \in \operatorname{ran}\mathcal{S}_+$, then $\Phi H \in \operatorname{ran}\mathcal{S}^2\cap\mathfrak{P}_e$, and therefore $\mathcal{S}^{-1}\Phi H \in \operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o$. It follows that

$$[\,{}_z\Phi^{-1}F,H]_+ = [\Phi^{-1}\mathcal{S}^{-1}F,H]_+ = [\mathcal{S}^{-1}F,\Phi H] = [F,\mathcal{S}^{-1}\Phi H] = 0\,.$$

We have shown the inclusion '⊆'. The asserted equality follows since both spaces have codimension 1 in $\mathfrak{P}_+$. □

In order to formulate the next result which shows how to reconstruct $\mathfrak{P}_o$ in case $\overline{\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o} \neq \mathfrak{P}_o$, we need to recall one more notion introduced in [KWW3]: To an almost Pontryagin space $(\mathfrak{P},[.,.],\mathcal{O})$ and a closed symmetric relation $\mathcal{S}$ in $\mathfrak{P}$ of finite negativity and finite defect, there is assigned a linear space $\mathfrak{P}^S$ and a Hilbert space topology $\mathcal{O}^S$ on $\mathfrak{P}^S$. Thereby $\mathfrak{P}^S$ contains $\mathfrak{P}_S$ as a subspace of finite codimension and $\mathcal{O}^S$ is the unique Hilbert space topology on $\mathfrak{P}^S$ such that $\mathcal{O}^S|_{\mathfrak{P}_S} = \mathcal{O}_S$ and $\mathfrak{P}_S$ is $\mathcal{O}^S$-closed.

If $(\mathfrak{P},[.,.])$ is a Hilbert space, every symmetric relation of finite negativity is semibounded from below. Denote by $m(S)$ its lower bound. In this case we have for all $m>-m(S)$

$$\mathfrak{P}^S=\mathfrak{P}_S\dot{+}\ker(S^*+m)\,. \tag{2.9}$$

The general indefinite case is again treated by reduction to the Hilbert space case. An intrinsic characterization of $\mathfrak{P}^S$ is:

$$\mathfrak{P}^S=\mathfrak{P}_S+L$$

where L is the set of all $a\in\mathfrak{P}$ such that

$$\big\{(x;[Sx,a]):\,x\in\operatorname{dom}\mathcal{S}\big\}$$

is (the graph of) an $\mathcal{O}$-continuous linear functional on $\operatorname{dom}\mathcal{S}$.

2.10. Proposition. *Let $(\mathfrak{P},[.,.])$ be a symmetric dB-space and assume that we have $\overline{\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o}\neq\mathfrak{P}_o$. Then the map ${}_z\Phi$ induces a linear homeomorphism of $(\mathfrak{P}_+^{S_+},\mathcal{O}^{S_+})$ onto $(\mathfrak{P}_o,\mathcal{O}|_{\mathfrak{P}_o})$. If $\mathfrak{P}_+^{S_+}$ is endowed with an inner product $[.,.]^{S_+}$ by the requirement that ${}_z\Phi$ is an isometry of $(\mathfrak{P}_+^{S_+},[.,.]^{S_+})$ onto $(\mathfrak{P}_o,[.,.])$, then $[.,.]^{S_+}$ satisfies*

$$\begin{gathered}[.,.]^{S_+}|_{\mathfrak{P}_{+,S_+}\times\mathfrak{P}_{+,S_+}}=[.,.]_{S_+}\,,\\ [F,G]^{S_+}=[F(z),zG(z)]_+,\ F\in\mathfrak{P}^{S_+},G\in\operatorname{dom}\mathcal{S}_+\,.\end{gathered} \tag{2.10}$$

Proof. In order to prove the first assertion we can by Lemma 2.7 and [KWW3, Proposition 4.14] assume without loss of generality that $(\mathfrak{P},[.,.])$ is a Hilbert space. In this case we saw in (2.8) that the operator $\mathcal{S}_+$ is strictly positive and that we may chose $m=0$ for the topology defining inner product $h_m^{S_+}[.,.]_+$ on $\mathfrak{P}_{+,S_+}$. Thus, we may also chose $m=0$ in (2.9) and obtain

$$\mathfrak{P}_+^{S_+}=\mathfrak{P}_{+,S_+}\dot{+}(\operatorname{ran}\mathcal{S}_+)^\perp\,.$$

Since $\dim(\operatorname{ran}\mathcal{S}_+)^\perp=1$, we conclude from Lemma 2.9 that ${}_z\Phi^{-1}(\mathfrak{P}_o)=\mathfrak{P}_+^{S_+}$. Moreover, since $\overline{\operatorname{dom}\mathcal{S}\cap\mathfrak{P}_o}$ is a closed subspace of $\mathfrak{P}_o$, $\mathfrak{P}_{+,S_+}$ is a closed subspace of $\mathfrak{P}_+^{S_+}$, and since ${}_z\Phi$ restricted to these spaces is a homeomorphism, it follows that ${}_z\Phi$ is also homeomorphic with respect $\mathcal{O}^{S_+}$ and $\mathcal{O}|_{\mathfrak{P}_o}$.

Let $\mathfrak{P}_+^{S_+}$ be endowed with the inner product $[.,.]^{S_+}$ so that ${}_z\Phi$ is an isometry onto $(\mathfrak{P}_o,[.,.])$. The first of the two assertions concerning $[.,.]^{S_+}$ is immediate from Proposition 2.8. In order to see the second one, let $F\in\mathfrak{P}_+^{S_+}$ and, $G\in\operatorname{dom}\mathcal{S}_+$ be given. Then $\Phi G\in\operatorname{dom}\mathcal{S}^2$ and hence, ${}_z\Phi\,G\in\operatorname{dom}\mathcal{S}$. We compute

$$[F,G]^{S_+}=[\,{}_z\Phi\,F,\,{}_z\Phi\,G]=[\Phi F,z^2\cdot\Phi G]=[\Phi F,\Phi(zG(z))]=[F,zG(z)]_+\,.$$

$\square$

3. Semibounded dB-spaces. The converse construction

Let a semibounded dB-space $(\mathfrak{P},[.,.])$ be given, then the multiplication operator $\mathcal{S}$ is a closed symmetric operator with finite negativity. Applying the construction of [KWW3] with $(\mathfrak{P},[.,.])$ and $\mathcal{S}$, we obtain the triple $(\mathfrak{P}_S,[.,.]_S,\mathcal{O}_S)$.

3.1. Lemma. *Let $(\mathfrak{P},[.,.])$ be a semibounded dB-space. Then $(\mathfrak{P}_S,[.,.]_S,\mathcal{O}_S)$ satisfies (dB1) and (dB2).*

Proof. First of all note that, since $\mathfrak{P}_S\subseteq\mathfrak{P}$, its elements are entire functions. In order to show that $(\mathfrak{P}_S,[.,.]_S,\mathcal{O}_S)$ is an almost Pontryagin space consider first the case that $(\mathfrak{P},[.,.])$ is a Hilbert space. Choose a canonical selfadjoint extension A of $\mathcal{S}$. From [KW1, Proposition 6.1] we know that $\sigma(A)$ is discrete. Since $\mathcal{S}$ is of finite negativity and has finite defect indices, also A is of finite negativity. This means nothing else but $\#(\sigma(A)\cap(-\infty,0))<\infty$. It follows from the discreteness of the spectrum that also for all $\epsilon>0$,

$$\#\big(\sigma(A-\epsilon I)\cap(-\infty,0)\big)<\infty\,.$$

In particular, the operator $S-\epsilon I$ is of finite negativity. By [KWW3, Proposition 4.7] this implies that $(\mathfrak{P}_S,[.,.]_S,\mathcal{O}_S)$ is an almost Pontryagin space.

Let $(\mathfrak{P},[.,.])$ be an arbitrary semibounded dB-space. By [KW1, Theorem 3.3] there exists a finite rank perturbation $(.,.)$ of $[.,.]$ such that $(\mathfrak{P},(.,.))$ is a dB-Hilbert space. By [KWW3, Corollary 4.11] and what we just proved it follows that $(\mathfrak{P}_S,[.,.]_S,\mathcal{O}_S)$ is an almost Pontryagin space.

Since the set theoretic inclusion of $\mathfrak{P}_S$ into $\mathfrak{P}$ is continuous and point evaluation is continuous on $\mathfrak{P}$, it is also continuous on $\mathfrak{P}_S$. Thus $(\mathfrak{P}_S,[.,.]_S)$ is a reproducing kernel almost Pontryagin space.

The map $F\mapsto F^\#$ is continuous in the topology of $(\mathfrak{P},[.,.])$, leaves $\operatorname{dom}\mathcal{S}$ invariant, and satisfies

$$\mathcal{S}(F^\#)=(\mathcal{S}F)^\#,\ F\in\operatorname{dom}\mathcal{S}\,.$$

Hence it induces an anti-linear isometry of $(\operatorname{dom}\mathcal{S},[.,.]_S)$ onto itself. The same reasoning as above, using [KW1, Theorem 3.3] and [KWW3, Corollary 4.11], shows that it is continuous with respect to $\mathcal{O}_S$. Hence it extends by continuity to $\mathfrak{P}_S$. Since point evaluation is continuous, this extension must actually be given by the map $F\mapsto F^\#$. $\square$

From [KWW1, Proposition 3.1] we obtain:

3.2. Corollary. *The linear space $\mathfrak{P}\times\mathfrak{P}_S$ turns into an almost Pontryagin space if it is endowed with the sum inner product $[.,.]+[.,.]_S$ and the product of the topologies of $(\mathfrak{P},[.,.])$ and $\mathcal{O}_S$.*

Consider the map defined as

$$\Psi:\left\{\begin{array}{lcl}\mathcal{O}(\mathbb{C})\times\mathcal{O}(\mathbb{C}) & \to & \mathcal{O}(\mathbb{C})\\ \big(F(z);G(z)\big) & \mapsto & (\Phi F)(z)+(\,_z\Phi\,G)(z)\end{array}\right.$$

The map Ψ is a linear bijection. Note that

$$\Psi(F,0)=\Phi(F) \text{ and } \Psi(0,F)={}_z\Phi(F),\ F\in\mathcal{O}(\mathbb{C})\,. \tag{3.1}$$

3.3. Definition. Let $(\mathfrak{P},[.,.])$ be a semibounded dB-space. Define a linear space

$$\mathfrak{Q}(\mathfrak{P}):=\Psi(\mathfrak{P}\times\mathfrak{P}_S)\,.$$

Let $\mathfrak{Q}(\mathfrak{P})$ be endowed with an inner product $[.,.]_{\mathfrak{Q}(\mathfrak{P})}$ and a topology defined by the requirement that $\Psi|_{\mathfrak{P}\times\mathfrak{P}_S}$ is isometric and homeomorphic.

By its definition $\mathfrak{Q}(\mathfrak{P})$ is an almost Pontryagin space and $(\Psi|_{\mathfrak{P}\times\mathfrak{P}_S})^{-1}$ is an almost Pontryagin space isomorphism of $\mathfrak{Q}(\mathfrak{P})$ onto $\mathfrak{P}\times\mathfrak{P}_S$.

3.4. Proposition. *Let $(\mathfrak{P},[.,.])$ be a semibounded dB-space. Then $\mathfrak{Q}(\mathfrak{P})$ is a symmetric dB-space. We have*

$$\big(\mathfrak{Q}(\mathfrak{P}),[.,.]_{\mathfrak{Q}(\mathfrak{P})}\big)_+=(\mathfrak{P},[.,.])\,. \tag{3.2}$$

Proof. First of all note that we have

$$\begin{array}{ccc} \mathfrak{P}\times\mathfrak{P}_S & \xrightarrow{\ \Psi\ } & \mathfrak{Q}(\mathfrak{P}) \\ {\scriptstyle \chi_{w^2}\times w\chi_{w^2}}\downarrow & & \downarrow{\scriptstyle \chi_w} \\ \mathbb{C}\times\mathbb{C} & \xrightarrow[+]{} & \mathbb{C} \end{array} \tag{3.3}$$

Hence, the point evaluation functional χ_w is continuous on $\mathfrak{Q}(\mathfrak{P})$.

The validity of the axiom (dB2) is immediate from the fact that it holds for $(\mathfrak{P},[.,.])$ as well as for $(\mathfrak{P}_S,[.,.]_S)$, and from the relation

$$\Psi(F^\#,G^\#)=\Psi(F,G)^\#\,.$$

In order to establish that $\mathfrak{Q}(\mathfrak{P})$ is a dB-space, we shall show that the multiplication operator $\mathcal{S}_{\mathfrak{Q}(\mathfrak{P})}$ in $\mathfrak{Q}(\mathfrak{P})$ is a symmetric operator with defect index $(1,1)$.

Let $H\in\mathfrak{Q}(\mathfrak{P})$ and write $H=\Psi(F,G)$ with $F\in\mathfrak{P}$ and $G\in\mathfrak{P}_S$. Since

$$zH(z)=z\big(F(z^2)+zG(z^2)\big)=z^2G(z^2)+zF(z^2)=\Psi(zG(z),F(z))\,, \tag{3.4}$$

we have $zH(z)\in\mathfrak{Q}(\mathfrak{P})$ if and only if $zG(z)\in\mathfrak{P}$ and $F\in\mathfrak{P}_S$. It follows that

$$\Psi^{-1}\big(\operatorname{dom}\mathcal{S}_{\mathfrak{Q}(\mathfrak{P})}\big)=\Psi^{-1}\big(\{H\in\mathfrak{Q}(\mathfrak{P}):\,zH(z)\in\mathfrak{Q}(\mathfrak{P})\}\big)=$$

$$=\big\{(F,G)\in\mathfrak{P}\times\mathfrak{P}_S:\,zG(z)\in\mathfrak{P},F\in\mathfrak{P}_S\big\}=\mathfrak{P}_S\times\operatorname{dom}\mathcal{S}\,.$$

Moreover, relation (3.4) yields

$$\big(\Psi^{-1}\circ\mathcal{S}_{\mathfrak{Q}(\mathfrak{P})}\circ\Psi\big)(F,G)=\big(zG(z),F(z)\big),\ (F,G)\in\mathfrak{P}_S\times\operatorname{dom}\mathcal{S}\,.$$

To see that $\mathcal{S}_{\mathfrak{Q}(\mathfrak{P})}$ is symmetric note first that, since $\mathfrak{P}_S$ is continuously embedded in $\mathfrak{P}$ and $\operatorname{dom}\mathcal{S}$ is dense in $\mathfrak{P}_S$,

$$[F,G]_S=[F(z),zG(z)],\ F\in\mathfrak{P}_S,G\in\operatorname{dom}\mathcal{S}\,. \tag{3.5}$$

Let $H=\Psi(F,G), L=\Psi(C,D)\in\operatorname{dom}\mathcal{S}_{\mathfrak{Q}(\mathfrak{P})}$ be given. By the isometric nature of Ψ and the relations (3.4) and (3.5) we obtain

$$[\mathcal{S}_{\mathfrak{Q}(\mathfrak{P})}H,L]_{\mathfrak{Q}(\mathfrak{P})}=[zG(z),C(z)]+[F,D]_S= \\ =[G,C]_S+[F(z),zD(z)]=[H,\mathcal{S}_{\mathfrak{Q}(\mathfrak{P})}L]_{\mathfrak{Q}(\mathfrak{P})}\,. \tag{3.6}$$

Our next objective is to show that

$$\operatorname{ran}\mathcal{S}_{\mathfrak{Q}(\mathfrak{P})}=\big\{H\in\mathfrak{Q}(\mathfrak{P}):\ H(0)=0\big\}\,. \tag{3.7}$$

The inclusion '$\subseteq$' is clear. Conversely, assume that $H\in\mathfrak{Q}(\mathfrak{P})$, $H(0)=0$, and write $H=\Psi(F,G)$. It follows that $F(0)=0$ and thus (remember that $\mathfrak{P}$ satisfies (Z)) that $z^{-1}F(z)\in\operatorname{dom}\mathcal{S}$. Hence

$$\Psi\big(G(z),\frac{1}{z}F(z)\big)\in\operatorname{dom}\mathcal{S}_{\mathfrak{Q}(\mathfrak{P})}\,,$$

and, by (3.4),

$$\mathcal{S}_{\mathfrak{Q}(\mathfrak{P})}\Psi\big(G(z),\frac{1}{z}F(z)\big)=\Psi(F,G)=H\,.$$

Thus in (3.7) equality holds.

By (3.7), the range of $\mathcal{S}_{\mathfrak{Q}(\mathfrak{P})}$ is closed. Since $\mathcal{S}_{\mathfrak{Q}(\mathfrak{P})}$ is injective, this implies that 0 is a point of regular type for $\mathcal{S}_{\mathfrak{Q}(\mathfrak{P})}$. Moreover, again by (3.7), the codimension of $\operatorname{ran}\mathcal{S}_{\mathfrak{Q}(\mathfrak{P})}$ in $\mathfrak{Q}(\mathfrak{P})$ is 1. We conclude that $\mathcal{S}_{\mathfrak{Q}(\mathfrak{P})}$ has defect index $(1,1)$. Alltogether it follows that $\mathfrak{Q}(\mathfrak{P})$ is a dB-space.

The fact that $\mathfrak{Q}(\mathfrak{P})$ is actually a symmetric dB-space is immediate from the relation

$$(M\circ\Psi)(F,G)=\Psi(F,-G),\ \ F,G\in\mathcal{O}(\mathbb{C})\,. \tag{3.8}$$

Finally, the assertion (3.2) follows from (3.1), the relation

$$\mathfrak{Q}(\mathfrak{P})_e=\Psi(\mathfrak{P}\times\{0\}),$$

and the fact that by the definition of the inner product

$$[\Psi(F,0),\Psi(G,0)]_{\mathfrak{Q}(\mathfrak{P})}=[F,G]\,.$$

Finally we note that $\mathfrak{Q}(\mathfrak{P})$ satisfies (Z). For if $t\in\mathbb{R}$, then there exists $F\in\mathfrak{P}$ with $F(t^2)\neq 0$. □

3.5. Corollary. *Let $(\mathfrak{P},[.,.])$ be a semibounded dB-space and let $\mathfrak{Q}(\mathfrak{P})$ be the symmetric dB-space as in the above proposition. Then*

$$\overline{\operatorname{dom}\mathcal{S}_{\mathfrak{Q}(\mathfrak{P})}\cap\mathfrak{Q}(\mathfrak{P})_o}=\mathfrak{Q}(\mathfrak{P})_o\,.$$

Proof. We have

$$\Psi^{-1}\big(\operatorname{dom}\mathcal{S}_{\mathfrak{Q}(\mathfrak{P})}\cap\mathfrak{Q}(\mathfrak{P})_o\big)=\{0\}\times\operatorname{dom}\mathcal{S}\,,$$

and

$$\Psi^{-1}\big(\mathfrak{Q}(\mathfrak{P})_o\big)=\{0\}\times\mathfrak{P}_S\,.$$

Since $\operatorname{dom}\mathcal{S}$ is dense in $\mathfrak{P}_S$, the assertion follows. □

3.6. *Remark.* The splitting in Lemma 2.4 of the isotropic part of $\mathfrak{Q}(\mathfrak{P})$ corresponds to the following facts which hold true in general: Let $(\mathfrak{L},[.,.],\mathcal{O})$ be an almost Pontryagin space, and let $\mathcal{S}$ be an injective and closed symmetric operator in $\mathfrak{L}$ with finite negativity and finite defect. Then

$$\mathcal{S}^{-1}\big(\mathfrak{L}^\circ\cap\operatorname{ran}\mathcal{S}\big)\subseteq\mathfrak{L}_S^\circ \text{ and } \mathfrak{L}_S^\circ\subseteq(\operatorname{ran}\mathcal{S})^\perp .$$

To see the first relation let $F\in\mathfrak{L}^\circ\cap\operatorname{ran}\mathcal{S}$ and $G\in\operatorname{dom}\mathcal{S}$ be given. Then $\mathcal{S}^{-1}F\in\operatorname{dom}\mathcal{S}$ and

$$[\mathcal{S}^{-1}F,G]_S=[\mathcal{S}\mathcal{S}^{-1}F,G]=[F,G]=0 .$$

Since $\operatorname{dom}\mathcal{S}$ is dense in $\mathfrak{L}_S$, it follows that $F\in\mathfrak{L}_S^\circ$. The second relation is established in a similar manner: Let $F\in\mathfrak{L}_S^\circ$ and $G\in\operatorname{ran}\mathcal{S}$, then

$$[F,G]=[F,\mathcal{S}\mathcal{S}^{-1}G]=[F,\mathcal{S}^{-1}G]_S=0 .$$

This computation is justified, since by the density of $\operatorname{dom}\mathcal{S}$ in $\mathfrak{L}_S$,

$$[H,K]_S=[H,\mathcal{S}K],\ H\in\mathfrak{L}_S,K\in\operatorname{dom}\mathcal{S} .$$

A similar construction is now made with $\mathfrak{P}^S$ in place of $\mathfrak{P}_S$ in order to obtain also those symmetric dB-spaces $\mathfrak{Q}$ with $\mathfrak{Q}_+=\mathfrak{P}$ and $\overline{\operatorname{dom}\mathcal{S}_\mathfrak{Q}\cap\mathfrak{Q}_o}\neq\mathfrak{Q}_o$. We call an inner product $[.,.]^S$ on $\mathfrak{P}^S$ admissible, if it satisfies the relations (2.10).

3.7. Lemma. *Let $(\mathfrak{P},[.,.])$ be a semibounded dB-space and let $[.,.]^S$ be an admissible inner product on $\mathfrak{P}^S$. Then $(\mathfrak{P}^S,[.,.]^S,\mathcal{O}^S)$ satisfies (dB1) and (dB2).*

Proof. Since $[.,,.]^S$ extends $[.,.]_S$ and $\mathfrak{P}_S$ is a closed subspace of $\mathfrak{P}^S$ with codimension 1, the validity of (aPs1) follows. For the subspace $\mathfrak{M}$ required in (aPs2) choose the same one which exists for $\mathfrak{P}_S$. The same reasoning yields that point evaluation is continuous on $\mathfrak{P}^S$.

To see that $\mathfrak{P}^S$ is closed with respect to $.^\#$, we can by [KWW3, Proposition 4.14] assume without loss of generality that $(\mathfrak{P},[.,.])$ is a Hilbert space. Then $\mathfrak{P}^S=\mathfrak{P}_S\dot{+}(\operatorname{ran}\mathcal{S})^\perp$. Since $\mathfrak{P}_S$ is invariant under $.^\#$, $\mathcal{S}$ is real with respect to $.^\#$ and $.^\#$ is an anti-linear isometry on $(\mathfrak{P},[.,.])$, also $(\operatorname{ran}\mathcal{S})^\perp$ is invariant under $.^\#$.

Let $F\in\mathfrak{P}^S$ and $G\in\operatorname{dom}\mathcal{S}$ be given. Then, by the second relation in (2.10),

$$[F^\#,G^\#]^S=[F^\#(z),zG^\#(z)]=[F^\#(z),(zG(z))^\#]=$$
$$=[zG(z),F(z)]=[G,F]^S .$$

Since $.^\#$ is continuous on $\mathfrak{P}_S$, it is also continuous on $\mathfrak{P}^S$. Thus this relation extends to

$$[F^\#,G^\#]^S=[G,F]^S,\ \ F\in\mathfrak{P}^S,G\in\mathfrak{P}_S . \tag{3.9}$$

Since $\mathfrak{P}^S$ is invariant under $.^\#$, the real and the imaginary part of an element in $\mathfrak{P}^S\setminus\mathfrak{P}_S$ belongs to $\mathfrak{P}^S$, and at least one of them does not belongs to $\mathfrak{P}_S$. Thus, we can choose an $H\in\mathfrak{P}^S\setminus\mathfrak{P}_S$ with $H=H^\#$. Let $F,G\in\mathfrak{P}^S$ and write $F=F_1+\lambda H$, $G=G_1+\mu H$, with $F_1,G_1\in\mathfrak{P}_S$. Then $F^\#=F_1^\#+\overline{\lambda}H^\#$, $G^\#=G_1^\#+\overline{\mu}H^\#$ and we compute

$$[F^\#,G^\#]^S=[F_1^\#,G_1^\#]^S+\mu[F_1^\#,H^\#]^S+\overline{\lambda}[H^\#,G_1^\#]^S+\overline{\lambda}\mu[H^\#,H^\#]^S=$$

$$= [G_1, F_1]^S + \mu[H, F_1]^S + \overline{\lambda}[G_1, H]^S + \overline{\lambda}\mu[H^\#, H^\#]^S .$$

Since we chose H to be real, we have $[H^\#, H^\#]^S = [H, H]^S$, and thus the last expression is just $[G, F]^S$. □

3.8. *Remark.* The set of all admissible inner products $[.,.]^S$ on $\mathfrak{P}^S$ can be parameterized by one real parameter. To see this it suffices to note that the action of $[.,.]^S$ is determined by (2.10) on $\mathfrak{P}^S \times \mathfrak{P}_S$. Hence with the above notation, only the choice of $[H, H]$ is free.

As a corollary we again obtain that, if $\mathfrak{P}^S$ is endowed with an admissible inner product, $\mathfrak{P} \times \mathfrak{P}^S$ becomes an almost Pontryagin space if endowed with the sum inner product $[.,.] + [.,.]^S$ and the product topology $\mathcal{O} \times \mathcal{O}^S$.

3.9. Definition. Let $(\mathfrak{P}, [.,.])$ be a semibounded dB-space and let $[.,.]^S$ be an admissible inner product on $\mathfrak{P}^S$. Define a linear space

$$\mathfrak{Q}_{[.,.]^S}(\mathfrak{P}) := \Psi(\mathfrak{P} \times \mathfrak{P}^S) .$$

Let $\mathfrak{Q}_{[.,.]^S}(\mathfrak{P})$ be endowed with an inner product $[.,.]_{\mathfrak{Q}_{[.,.]^S}(\mathfrak{P})}$ and a topology defined by the requirement that $\Psi|_{\mathfrak{P}\times\mathfrak{P}^S}$ is isometric and homeomorphic.

The space $\mathfrak{Q}_{[.,.]^S}(\mathfrak{P})$ is an almost Pontryagin space and $(\Psi|_{\mathfrak{P}\times\mathfrak{P}^S})^{-1}$ is an almost Pontryagin space isomorphism of $\mathfrak{Q}_{[.,.]^S}(\mathfrak{P})$ onto $\mathfrak{P} \times \mathfrak{P}^S$.

3.10. Proposition. *Let $(\mathfrak{P}, [.,.])$ be a semibounded dB-space and let $[.,.]^S$ be an admissible inner product on $\mathfrak{P}^S$. Then $\mathfrak{Q}_{[.,.]^S}(\mathfrak{P})$ is a symmetric dB-space. We have*

$$\big(\mathfrak{Q}_{[.,.]^S}(\mathfrak{P}), [.,.]_{\mathfrak{Q}_{[.,.]^S}(\mathfrak{P})}\big)_+ = (\mathfrak{P}, [.,.]) .$$

Proof. The fact that $(\mathfrak{Q}_{[.,.]^S}(\mathfrak{P}), [.,.]_{\mathfrak{Q}_{[.,.]^S}(\mathfrak{P})}, \mathcal{O}^S)$ satisfies (dB1) and (dB2) follows with the help of Lemma 3.7 in the same way as in the proof of Proposition 3.4, since the diagram (3.3) is valid for arbitrary entire functions.

To show (dB3) we also proceed along the same lines as in the proof of Proposition 3.4. In the present case we have, due to (3.4),

$$\Psi^{-1}\big(\operatorname{dom} \mathcal{S}_{\mathfrak{Q}_{[.,.]^S}(\mathfrak{P})}\big) = \mathfrak{P}^S \times \operatorname{dom} \mathcal{S} .$$

The relation (3.5) holds for all $F \in \mathfrak{P}^S$ and $G \in \operatorname{dom} \mathcal{S}$, since $[.,.]^S$ is admissible. The computation (3.6) remains valid and hence $\mathcal{S}_{\mathfrak{Q}_{[.,.]^S}(\mathfrak{P})}$ is symmetric. The fact that

$$\operatorname{ran} \mathcal{S}_{\mathfrak{Q}_{[.,.]^S}(\mathfrak{P})} = \big\{H \in \mathfrak{P}^S : H(0) = 0\big\}$$

follows by repeating the argument used in the proof of Proposition 3.4 word by word.

Also in this case (3.8) applies and proves that $\mathfrak{Q}_{[.,.]^S}(\mathfrak{P})$ is a symmetric dB-space. The validity of (Z) follows since it already holds for the subspace $\mathfrak{Q}(\mathfrak{P})$.

Finally $\big(\mathfrak{Q}_{[.,.]^S}(\mathfrak{P}), [.,.]_{\mathfrak{Q}_{[.,.]^S}(\mathfrak{P})}\big)_+ = (\mathfrak{P}, [.,.])$ is also proved in the same way as in the end of the proof of Proposition 3.4. □

Let us now collect our results and concisely formulate the connection between symmetric and semibounded dB-spaces.

3.11. Theorem. *The assigment $\Upsilon : (\mathfrak{P}, [.,.]) \mapsto (\mathfrak{P}_+, [.,.]_+)$ maps the set of all symmetric dB-spaces surjectively onto the set of all semibounded dB-spaces. Whenever $(\mathfrak{P}_+, [.,.])$ is a semibounded dB-space, then*

$$\Upsilon^{-1}\big((\mathfrak{P}_+, [.,.])\big) = \big\{\mathfrak{Q}(\mathfrak{P}_+)\big\} \cup \big\{\mathfrak{Q}_{[.,.]^{S_+}}(\mathfrak{P}_+) : [.,.]^{S_+} \text{ admissible}\big\} \tag{3.10}$$

No two of the spaces $\mathfrak{Q}$ listed on the right hand side of (3.10) are equal as almost Pontryagin spaces. As topological vector spaces we have $\mathfrak{Q}_{[.,.]^{S_+}}(\mathfrak{P}_+) = \mathfrak{Q}_{[.,.]_1^{S_+}}(\mathfrak{P}_+)$ for any two admissible inner products $[.,.]^{S_+}$ and $[.,.]_1^{S_+}$. Moreover, $\mathfrak{Q}(\mathfrak{P}_+)$ is a closed subspace of $\mathfrak{Q}_{[.,.]^{S_+}}(\mathfrak{P}_+)$ of codimension 1.

Proof. The fact that Υ maps symmetric dB-spaces to semibounded dB-spaces was proved in Proposition 2.6. That every semibounded dB-space can be obtained in this way and that in fact the inclusion '$\supseteq$' in (3.10) holds, is the content of Proposition 3.4 and Proposition 3.10. The inclusion '$\subseteq$' is a consequence of Proposition 2.8 and Proposition 2.10.

Since the inner product $[.,.]^{S_+}$ on $\mathfrak{P}^{S_+}$ can be recovered from the inner product on $\mathfrak{Q}_{[.,.]^{S_+}}(\mathfrak{P}_+)$ by means of the isometry property of ${}_z\Phi$, it follows that different admissible inner products give rise to different almost Pontryagin spaces $\mathfrak{Q}_{[.,.]^{S_+}}(\mathfrak{P}_+)$. However, as topological vector spaces we have

$$\mathfrak{Q}_{[.,.]^{S_+}}(\mathfrak{P}_+) = \Psi\big(\mathfrak{P}_+ \times \mathfrak{P}_+^{S_+}\big),$$

and

$$\mathfrak{Q}(\mathfrak{P}_+) = \Psi\big(\mathfrak{P}_+ \times \mathfrak{P}_{+,S_+}\big).$$

Thus any two of the spaces $\mathfrak{Q}_{[.,.]^{S_+}}(\mathfrak{P}_+)$ are equal as topological vector spaces and contain $\mathfrak{Q}(\mathfrak{P})$ as a closed subspace of codimension 1. □

4. Symmetric and semibounded Hermite-Biehler functions

In this section we consider in more detail the nondegenerated case. We will thereby obtain generalizations of the results of [dB]. As already noted, any dB-Pontryagin space $\mathfrak{P}$ is generated by a function $E \in \mathcal{H}B_{<\infty}$ by means of (1.2). In this case we write $\mathfrak{P} = \mathfrak{P}(E)$.

Hermite-Biehler functions are related to generalized Nevanlinna functions. In order to describe this connection denote for $E \in \mathcal{H}B_{<\infty}$ by A and B its 'real' and 'imaginary' parts

$$A(z) := \frac{E(z) + E^{\#}(z)}{2},\ B(z) := i\frac{E(z) - E^{\#}(z)}{2}. \tag{4.1}$$

Note that, since E and $E^{\#}$ do not have common zeros, also A and B cannot have common zeros.

By the formula

$$S_{\frac{E^\#}{E}}(w,z) = 2\frac{1}{E(z)A(z)}\frac{\frac{B(z)}{A(z)} - \overline{\left(\frac{B(w)}{A(w)}\right)}}{z-\overline{w}}\overline{\left(\frac{1}{E(z)A(z)}\right)}$$

we have $E \in \mathcal{HB}_\kappa$ if and only if the function $q := A^{-1}B$ belongs to the generalized Nevanlinna class $\mathcal{N}_\kappa$, i.e. the kernel

$$L_q(w,z) := \frac{q(z) - \overline{q(w)}}{z - \overline{w}}$$

has κ negative squares, see e.g. [KL]. Conversely, if A and B are real entire functions which have no common zeros and are such that $A^{-1}B \in \mathcal{N}_\kappa$, then $E := A - iB \in \mathcal{HB}_\kappa$.

Let $(\mathfrak{P}, [.,.])$ be a dB-Pontryagin space with negative index κ and let $K(w,z)$ be its reproducing kernel. A computation shows that (1.2) can be rewritten to

$$K(w,z) = \frac{B(z)A(\overline{w}) - A(z)B(\overline{w})}{\pi(z-\overline{w})}\,. \tag{4.2}$$

Let $(\mathfrak{P}, [.,.])$ be a dB-Pontryagin space. We already noted in the very beginning that the choice of $E \in \mathcal{HB}_{<\infty}$ with $(\mathfrak{P}, [.,.]) = \mathfrak{P}(E)$ is not unique. Recall from [KW1, Corollary 6.2] that two spaces $\mathfrak{P}(E)$ and $\mathfrak{P}(\hat{E})$ are equal isometrically, i.e. coincide as sets of entire functions and carry the same inner product, if and only if there exist real numbers u_1, u_2, v_1, v_2 with $u_1v_2 - u_2v_1 = 1$, such that

$$(\hat{A}, \hat{B}) = (A, B)\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}. \tag{4.3}$$

4.1. Definition. We define $\mathcal{HB}_\kappa^{sym}$ to be the subset of $\mathcal{HB}_\kappa$ consisting of all functions E which have the property that $E^\#(z) = E(-z)$. Moreover, we denote by $\mathcal{HB}_\kappa^{sb}$ the set of all functions $E = A - iB \in \mathcal{HB}_\kappa$ such that B has only finitely many zeros in $\mathbb{C} \setminus [0,\infty)$.

The notations $\mathcal{HB}_{\leq\kappa}^{sym}$ and $\mathcal{HB}_{<\infty}^{sym}$ as well as $\mathcal{HB}_{\leq\kappa}^{sb}$ and $\mathcal{HB}_{<\infty}^{sb}$ are defined correspondingly, cf. (1.1).

4.2. *Remark.*

- (i) Note that the symmetry property $E^\#(z) = E(-z)$ is equivalent to A being even and B being odd.
- (ii) Since for a function $E \in \mathcal{HB}_\kappa$ the function $A(z)^{-1}B(z)$ belongs to $\mathcal{N}_\kappa$, at most finitely many zeros of B are located in $\mathbb{C} \setminus \mathbb{R}$. Hence, $E \in \mathcal{HB}_{<\infty}^{sb}$ if and only if B has only finitely many zeros on $\mathbb{R}^-$.
- (iii) Let $E \in \mathcal{HB}_{<\infty}$ and put $q := A^{-1}B$. Then $E \in \mathcal{HB}_{<\infty}^{sb}$ if and only if q has only finitely many poles in $\mathbb{C} \setminus [0,\infty)$.
- (iv) If $E \in \mathcal{HB}_{<\infty}^{sym}$, then q is odd.

The items (iii) and (iv) of Remark 4.2 enable us to employ the theory of symmetric and essentially positive generalized Nevanlinna functions developed in

[KWW2], in particular, the powerful tool [KWW2, Theorem 4.1]. For the convenience of the reader we shall recall this result: If $q \in \mathcal{N}_{<\infty}$ and has only finitely many poles on $\mathbb{R}^-$, then the function $\hat{q}(z) := zq(z^2)$ is odd and belongs to $\mathcal{N}_{<\infty}$. Conversely, if $\hat{q} \in \mathcal{N}_\kappa$ is odd, then the functions q defined by the relation $\hat{q}(z) = zq(z^2)$ belongs to $\mathcal{N}_{\leq\kappa}$ and has only finitely many poles on $\mathbb{R}^-$.

The following result generalizes [dB, Theorem 47].

4.3. Proposition. *Let $(\mathfrak{P}, [.,.])$ be a dB-Pontryagin space. Then $(\mathfrak{P}, [.,.])$ is symmetric if and only if there exists a function $E \in \mathcal{HB}^{sym}_{<\infty}$ such that $(\mathfrak{P}, [.,.]) = \mathfrak{P}(E)$. If, in this case, we additionally demand that $E(0) = 1$, then the function E is unique.*

Proof. Assume that $(\mathfrak{P}, [.,.]) = \mathfrak{P}(E)$ for some $E \in \mathcal{HB}^{sym}_{<\infty}$. Then, by (4.2), the reproducing kernel function $K(w,z)$ has the symmetry property $K(w,-z) = K(-w,z)$. Consider the linear space

$$\mathcal{L} := \operatorname{span}\big\{K(w,z) : w \in \mathbb{C}\big\}$$

and the linear operator on $\mathcal{L}$ defined by

$$L : K(w,z) \mapsto K(w,-z)\,.$$

By the symmetry of K this operator maps $\mathcal{L}$ onto itself and is isometric:

$$[LK(w_1,z), LK(w_2,z)] = [K(w_1,-z), K(w_2,-z)] = [K(-w_1,z), K(-w_2,z)] = \\ = K(-w_1,-w_2) = K(w_1,w_2) = [K(w_1,z), K(w_2,z)]\,.$$

Hence L has a continuation to an isometry of $\mathfrak{P}(E)$ onto itself, cf. [ADRS]. Since point evaluation in $\mathfrak{P}(E)$ is continuous, we have $(LF)(z) = F(-z)$ for all $F \in \mathfrak{P}(E)$, i.e. $L = M$. This shows that $\mathfrak{P}(E)$ is symmetric.

Conversely, assume that $(\mathfrak{P}, [.,.])$ is symmetric, so that the map $M|_{\mathfrak{P}}$ is an isometry of $\mathfrak{P}$ onto itself. We obtain

$$[F(z), MK(w,z)] = [MF(z), K(w,z)] = F(-w) = [F, K(-w,z)],\ F \in \mathfrak{P}\,,$$

and hence that $K(w,-z) = MK(w,z) = K(-w,z)$.

Write $(\mathfrak{P}, [.,.]) = \mathfrak{P}(E)$ with some $E \in \mathcal{HB}_{<\infty}$. We have to show that the choice of E can be made such that $E \in \mathcal{HB}^{sym}_{<\infty}$. If $B(0) \neq 0$ and $A(0) = 0$, consider $E_1(z) := iE(z)$. If $B(0) \neq 0$ and $A(0) \neq 0$, consider $E_1(z) := A(z) - i(B(z) - A(0)^{-1}B(0)A(z))$. In the case $B(0) = 0$, put $E_1(z) := E(z)$. In any case $\mathfrak{P}(E_1) = \mathfrak{P}(E)$, $B_1(0) = 0$ and $A_1(0) \neq 0$. It follows that

$$\frac{B_1(z)}{z}A_1(0) = K(0,z) = K(0,-z) = \frac{B_1(-z)}{-z}A_1(0)\,,$$

and, hence, $B_1(-z) = -B_1(z)$. Fix $w_0 \in \mathbb{R}$ such that $B_1(w_0) \neq 0$, and put

$$E_2(z) := \big(A_1(z) + \frac{A_1(-w_0) - A_1(w_0)}{2B_1(w_0)}B_1(z)\big) - iB_1(z)\,.$$

Then $\mathfrak{P}(E_2) = \mathfrak{P}(E)$, B_2 is odd, $B_2(w_0) \neq 0$ and $A_2(-w_0) = A_2(w_0)$. We have

$$-B_2(z)A_2(w_0) + A_2(-z)B_2(w_0) = B_2(-z)A_2(-w_0) - A_2(-z)B_2(-w_0) = \\ = (-z + w_0)K(-\overline{w}_0, -z) = (-z + w_0)K(\overline{w}_0, z) = -B_2(z)A_2(w_0) + A_2(z)B_2(w_0)\,,$$

and conclude that $A_2(z) = A_2(-z)$. This yields $E_2 \in \mathcal{HB}^{sym}_{<\infty}$. Finally put

$$E_3(z) := \frac{1}{A_2(0)} A_2(z) - iA_2(0)B_2(z)\,.$$

Then we still have $E_3 \in \mathcal{HB}^{sym}_{<\infty}$, $\mathfrak{P}(E_3) = \mathfrak{P}(E)$, and in addition ensured that $E_3(0) = 1$.

To prove the uniqueness part of the present proposition let $E, \hat{E} \in \mathcal{HB}^{sym}_{<\infty}$, $E(0) = \hat{E}(0) = 1$, and assume that $\mathfrak{P}(E) = \mathfrak{P}(\hat{E})$ isometrically. Let $u_1, u_2, v_1, v_2 \in \mathbb{R}$ with $u_1 v_2 - u_2 v_1 = 1$ be such that (4.3) holds. Since $B(0) = \hat{B}(0) = 0$ and $A(0) = \hat{A}(0) = 1$ we conclude that $u_1 = 1$ and $u_2 = 0$. Since both A and $\hat{A}$ are even and B is odd, it follows that $v_1 = 0$, and hence that $v_2 = 1$. This means $E = \hat{E}$. □

Also the fact that $\mathfrak{P}(E)$ is a semibounded dB-Pontryagin space can be read off the function E.

4.4. Proposition. *Let $E \in \mathcal{HB}_{<\infty}$. Then $\mathfrak{P}(E)$ is a semibounded dB-Pontryagin space if and only if $E \in \mathcal{HB}^{sb}_{<\infty}$. In this case for all $\phi \in [0, \pi)$ the function*

$$S_\phi(z) := \sin\phi\, A(z) - \cos\phi\, B(z)$$

has only finitely many zeros in $\mathbb{C} \setminus [0, \infty)$.

Proof. Since a selfadjoint extension of $\mathcal{S}$ is a rank-one extension of $\mathcal{S}$, the operator $\mathcal{S}$ is of finite negativity if and only if one (and hence all) of its selfadjoint extensions has this property. Recall from [KW1, Proposition 6.1] that the selfadjoint extensions $\mathcal{A}_\phi$ of $\mathcal{S}$ are related to the functions S_ϕ by

$$\big(\mathcal{A}_\phi - w\big)^{-1} F(z) = \frac{F(z) - \frac{S_\phi(z)}{S_\phi(w)} F(w)}{z - w}$$

and

$$\sigma(\mathcal{A}_\phi) = \big\{ w \in \mathbb{C} :\, S_\phi(w) = 0 \big\}\,.$$

It follows from the spectral theory in Pontryagin spaces that the relation $\mathcal{A}_\phi$ has finite negativity if and only if $\#(\sigma(\mathcal{A}_\phi) \cap \mathbb{R}^-) < \infty$ and, in turn, if and only if $\#(|\sigma(\mathcal{A}_\phi) \cap \mathbb{C} \setminus [0, \infty)) < \infty$. □

Finally let us deduce how the generating Hermite-Biehler functions transform when we proceed from $\mathfrak{P}(E)$ to $\mathfrak{P}(E)_+$ and vice versa, respectively.

4.5. Theorem. *The following assertions hold:*

(i_a) *Let $E \in \mathcal{HB}^{sym}_\kappa$, $E = A - iB$, and $\gamma \in \mathbb{R}$ be given. Define an entire function $E_{+,\gamma} := A_{+,\gamma} - iB_{+,\gamma}$ by*

$$B_{+,\gamma}(z^2) := zB(z),\ A_{+,\gamma}(z^2) := A(z) + \gamma z B(z)\,. \tag{4.4}$$

Then $E_{+,\gamma} \in \mathcal{HB}^{sb}_{\leq\kappa}$, $B_+(0) = 0$, and we have $\mathfrak{P}(E_{+,\gamma}) = \mathfrak{P}(E)_+$. If we additionally assume that $E(0) = 1$, then also $E_{+,\gamma}(0) = 1$.

(i_b) *Let* $E \in \mathcal{HB}^{sym}_{<\infty}$, $E(0) = 1$, *and* $E_+ \in \mathcal{HB}^{sb}_{<\infty}$, $E_+(0) = 1$, *be such that* $\mathfrak{P}(E_+) = \mathfrak{P}(E)_+$. *Then there exists a number* $\gamma \in \mathbb{R}$ *such that* $E_+ = E_{+,\gamma}$.

(ii_a) *Let* $E_+ \in \mathcal{HB}^{sb}_{<\infty}$, $E_+ = A_+ - iB_+$, $B_+(0) = 0$, *and let* $\gamma \in \mathbb{R}$. *Define an entire function* $E_\gamma := A_\gamma - iB_\gamma$ *by*

$$B_\gamma(z) := \frac{1}{z} B_+(z^2),\ A_\gamma(z) := A_+(z^2) - \gamma B_+(z^2)\,. \tag{4.5}$$

Then $E_\gamma \in \mathcal{HB}^{sym}_{<\infty}$ *and we have* $\mathfrak{P}(E_+) = \mathfrak{P}(E_\gamma)_+$. *If* $\gamma \neq \gamma'$, *then the spaces* $\mathfrak{P}(E_\gamma)$ *and* $\mathfrak{P}(E_{\gamma'})$ *are not equal isometrically. If we additionally assume that* $E_+(0) = 1$, *then also* $E_\gamma(0) = 1$.

(ii_b) *Let* $E_+ \in \mathcal{HB}^{sb}_{<\infty}$, $E_+(0) = 1$, *and* $E \in \mathcal{HB}^{sym}_{<\infty}$, $E(0) = 1$, *be such that* $\mathfrak{P}(E)_+ = \mathfrak{P}(E_+)$. *Then there exists a number* $\gamma \in \mathbb{R}$ *such that* $E = E_\gamma$.

Proof. *ad* (i_a): The reproducing kernel $K_e(w,z)$ of the space $\mathfrak{P}_e$ is equal to

$$K_e(w,z) = \frac{I+M}{2} K(w,z) = \frac{1}{2}\big(K(w,z) + K(w,-z)\big)\,.$$

We compute

$$K(w,z) + K(w,-z) = \frac{B(z)A(\overline{w}) - A(z)B(\overline{w})}{\pi(z-\overline{w})} +$$

$$+ \frac{B(-z)A(\overline{w}) - A(-z)B(\overline{w})}{\pi(-z-\overline{w})} =$$

$$= \frac{1}{\pi(z^2-\overline{w}^2)}\big[(z+\overline{w})\big(B(z)A(\overline{w}) - A(z)B(\overline{w})\big) + (z-\overline{w})\big(B(z)A(\overline{w}) + A(z)B(\overline{w})\big)\big] =$$

$$= \frac{2}{\pi(z^2-\overline{w}^2)}\big[zB(z)A(\overline{w}) - A(z)\overline{w}B(\overline{w})\big]\,.$$

Since $(\mathfrak{P}, [.,.])_+$ is isometrically isomorphic to $\mathfrak{P}_e$ via the mapping $\Phi : F(z) \mapsto F(z^2)$, the reproducing kernel $K_+(w,z)$ of $(\mathfrak{P}, [.,.])_+$ must satisfy $K_+(w^2,z^2) = K_e(w,z)$. The functions $A_{+,\gamma}$ and $B_{+,\gamma}$ defined by (4.4) are real and entire, and satisfy by the above computation

$$K_+(w,z) = \frac{B_{+,\gamma}(z)A_{+,\gamma}(\overline{w}) - A_{+,\gamma}(z)B_{+,\gamma}(\overline{w})}{\pi(z-\overline{w})}\,.$$

Hence $E_{+,\gamma} \in \mathcal{HB}_{\leq\kappa}$ and $\mathfrak{P}(E_{+,\gamma}) = \mathfrak{P}(E)_+$. A zero of $B_{+,\gamma}$ located on $\mathbb{R}^-$ corresponds to a pair of nonreal zeros of B lying on the imaginary axis. Therefore, $B_{+,\gamma}$ can have only finitely many zeros on $\mathbb{R}^-$, i.e. $E_{+,\gamma} \in \mathcal{HB}^{sb}_{\leq\kappa}$. The remaining assertions of (i_a) are obvious.

ad (i_b): Consider the function $E_{+,0}$. Then, by (i_a), we have $\mathfrak{P}(E_+) = \mathfrak{P}(E)_+ = \mathfrak{P}(E_{+,0})$. Since $E_+(0) = E_{+,0}(0) = 1$, it follows that for some $\gamma \in \mathbb{R}$

$$(A_+, B_+) = (A_{+,0}, B_{+,0}) \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} = (A_{+,\gamma}, B_{+,\gamma})\,.$$

ad (ii_a): The functions A_γ and B_γ are real and entire. Moreover, B_γ is odd and A_γ is even. We have

$$-\frac{A_\gamma(z)}{B_\gamma(z)} = z\Big(-\frac{A_+(z^2)}{B_+(z^2)}\Big) + \gamma z\,.$$

Theorem 4.1 of [KWW2] implies that $-B_\gamma(z)^{-1}A_\gamma(z)$ is a generalized Nevanlinna function, and thus $E_\gamma \in \mathcal{HB}^{sym}_{<\infty}$. With the notation of (4.4) we have $(E_\gamma)_{+,\gamma} = E_+$, and hence the already proved assertion (i_a) implies $\mathfrak{P}(E_\gamma)_+ = \mathfrak{P}(E_+)$.

Let $\gamma < \gamma'$ be given. Then

$$(A_{\gamma'}, B_{\gamma'}) = (A_\gamma, B_\gamma)\begin{pmatrix} 1 & 0 \\ (\gamma-\gamma')z & 1\end{pmatrix}.$$

In case $B_\gamma \notin \mathfrak{P}(E_\gamma)$ we conclude that $\mathfrak{P}(E_{\gamma'}) = \mathfrak{P}(E_\gamma)\dot{+}\operatorname{span}\{B_\gamma\}$. Otherwise $\mathfrak{P}(E_{\gamma'}) = \mathfrak{P}(E_\gamma)$ as sets but not isometrically. The last assertion of (ii_a) is obvious.

ad (ii_b): By the already proved assertion (i_b) there exists a number $\gamma \in \mathbb{R}$ such that $E_+ = E_{+,\gamma}$. Since by definition $E_+ = (E_\gamma)_{+,\gamma}$, it follows that $E = E_\gamma$. ☐

4.6. *Remark.* If $(\mathfrak{P}, [.,.])$ is a dB-Pontryagin space, there exist functions $E \in \mathcal{HB}_{<\infty}$ which generate this space, i.e. $(\mathfrak{P}, [.,.]) = \mathfrak{P}(E)$, and which additionally satisfy $E(0) = 1$. Hence we may restrict all discussions to $\mathcal{HB}^0 := \{E \in \mathcal{HB}_{<\infty} : E(0) = 1\}$.

By Proposition 4.3 there is a bijective correspondence between $\mathcal{HB}^{sym}_{<\infty} \cap \mathcal{HB}^0$ and the set of all symmetric dB-Pontryagin spaces. By Proposition 4.4 there is a (surjective but not injective) correspondence between $\mathcal{HB}^{sb}_{<\infty} \cap \mathcal{HB}^0$ and the set of all semibounded dB-Pontryagin spaces.

If $(\mathfrak{P}, [.,.]) = \mathfrak{P}(E)$ is a symmetric dB-Pontryagin space, then $(\mathfrak{P}, [.,.])_+$ is a semibounded dB-Pontryagin space. The totality of all functions in $\mathcal{HB}^{sb}_{<\infty} \cap \mathcal{HB}^0$ which generate $(\mathfrak{P}, [.,.])_+$ is given by $\{E_{+,\gamma} : \gamma \in \mathbb{R}\}$, cf. (4.4).

Conversely, let $(\mathfrak{P}_+, [.,.]_+)$ be a semibounded dB-Pontryagin space, and write $(\mathfrak{P}_+, [.,.]_+) = \mathfrak{P}(E_+)$ for some $E_+ \in \mathcal{HB}^{sb}_{<\infty} \cap \mathcal{HB}^0$. Then we have a bijective correspondence between all symmetric dB-Pontryagin spaces $(\mathfrak{P}, [.,.])$ with $(\mathfrak{P}, [.,.])_+ = (\mathfrak{P}_+, [.,.]_+)$ and $\{E_\gamma : \gamma \in \mathbb{R}\}$, cf. (4.5). If $E_+, \hat{E}_+ \in \mathcal{HB}^{sb}_{<\infty} \cap \mathcal{HB}^0$ both generate $(\mathfrak{P}_+, [.,.]_+)$, then for some $\beta \in \mathbb{R}$

$$(\hat{A}_+, \hat{B}_+) = (A_+, B_+)\begin{pmatrix} 1 & 0 \\ \beta & 1\end{pmatrix}.$$

It follows that $(\hat{E})_\gamma = E_{\gamma-\beta}$.

Let $\mathfrak{P}_+$ be a semibounded dB-space. We use Theorem 4.5 to investigate the structure of $\Upsilon^{-1}(\mathfrak{P}_+)$ more closely.

4.7. Proposition. *Let $\mathfrak{P}_+$ be a semibounded dB-Pontryagin space. Then there exists exactly one space $\mathfrak{Q}_0 \in \Upsilon^{-1}(\mathfrak{P}_+)$ which is degenerated. Choose $E_+ = A_+ - iB_+ \in$*

$\mathcal{H}B^{sb}_{<\infty}$, $E_+(0)=1$, such that $\mathfrak{P}_+=\mathfrak{P}(E_+)$ and define

$$\lambda := \lim_{t\to-\infty} \frac{A_+(t)}{B_+(t)} \in \mathbb{R}\cup\{\pm\infty\}\,.$$

Then $\mathfrak{Q}_0=\mathfrak{Q}(\mathfrak{P}_+)$ if and only if $\lambda\in\{\pm\infty\}$. If $\lambda\in\mathbb{R}$, we have $\mathfrak{Q}(\mathfrak{P}_+)=\mathfrak{P}(E_\lambda)$.

Proof. Let $[.,.]^{S_+}$ be an admissible inner product. The space

$$\mathfrak{Q}_{[.,.]^{S_+}}(\mathfrak{P}_+)\,[-]_{\mathfrak{Q}_{[.,.]^{S_+}}(\mathfrak{P}_+)}\,\mathfrak{Q}(\mathfrak{P}_+)$$

is one-dimensional and does, by (2.10), not depend on the choice of $[.,.]^{S_+}$. Fix $H\in\mathfrak{Q}_{[.,.]^{S_+}}(\mathfrak{P}_+)$ so that it is equal to $\operatorname{span}\{H\}$.

Assume first that $H\notin\mathfrak{Q}(\mathfrak{P}_+)$. Then $\mathfrak{Q}(\mathfrak{P}_+)$ is nondegenerated and

$$\mathfrak{Q}_{[.,.]^{S_+}}(\mathfrak{P}_+) = \mathfrak{Q}(\mathfrak{P}_+)\,[+]_{\mathfrak{Q}_{[.,.]^{S_+}}(\mathfrak{P}_+)}\,\operatorname{span}\{H\}\,.$$

By Remark 3.8 the set of all admissible inner products $[.,.]^{S_+}$ is in a bijective correspondence with $\alpha\in\mathbb{R}$ via $[H,H]^{S_+}=\alpha$. We see that $\mathfrak{Q}_0=\mathfrak{Q}_{[.,.]^{S_+}}(\mathfrak{P}_+)$ if and only if $\alpha=0$.

Next consider the case that $H\in\mathfrak{Q}(\mathfrak{P}_+)$. Then $\mathfrak{Q}(\mathfrak{P}_+)$ is degenerated, in fact $\mathfrak{Q}(\mathfrak{P}_+)^\circ=\operatorname{span}\{H\}$. Assume that for some admissible inner product $[.,.]^{S_+}$ the space $\mathfrak{Q}_{[.,.]^{S_+}}(\mathfrak{P}_+)$ is degenerated. By [KW3, Lemma 2.2] it follows that $H\in\mathfrak{Q}_{[.,.]^{S_+}}(\mathfrak{P}_+)^\circ$. Since $H\in\mathfrak{Q}(\mathfrak{P}_+)$ the condition $H\perp\mathfrak{Q}_{[.,.]^{S_+}}(\mathfrak{P}_+)$ does not depend on the choice of the admissible inner product $[.,.]^{S_+}$. Hence all spaces $\mathfrak{Q}_{[.,.]^{S_+}}(\mathfrak{P}_+)$ are degenerated. A contradiction, since we know from Theorem 4.5 that there exist nondegenerated spaces $\mathfrak{P}(E)$ with $\mathfrak{P}(E)_+=\mathfrak{P}_+$.

In order to prove the remaining assertions we identify the element H. Choose $\gamma\in\mathbb{R}$ such that $\mathfrak{P}(E_\gamma)\neq\mathfrak{Q}(\mathfrak{P}_+)$. Then

$$\overline{\operatorname{dom}\mathcal{S}_{\mathfrak{P}(E_\gamma)}} = \mathfrak{Q}(\mathfrak{P}_+)$$

and hence $\mathfrak{Q}(\mathfrak{P}_+)^\perp$ is of the form $\operatorname{span}\{uA_\gamma+vB_\gamma\}$. Since $\mathfrak{P}(E_\gamma)_e=\mathfrak{Q}(\mathfrak{P}_+)_e$ is nondegenerated and contained in $\overline{\operatorname{dom}\mathcal{S}_{\mathfrak{P}(E_\gamma)}}$, it follows that $\mathfrak{Q}(\mathfrak{P}_+)^\perp\subseteq\mathfrak{P}(E_\gamma)_o$, i.e. $uA_\gamma+vB_\gamma$ is odd. This implies that $u=0$. Thus we can choose $H=B_\gamma=z^{-1}B_+(z^2)$. Note that by its definition B_γ does not depend on γ.

If $\mathfrak{Q}(\mathfrak{P}_+)$ is nondegenerated, we can choose $\gamma\in\mathbb{R}$ such that $\mathfrak{P}(E_\gamma)=\mathfrak{Q}(\mathfrak{P}_+)$. Since $B_\gamma=H\notin\mathfrak{Q}(\mathfrak{P}_+)$, the selfadjoint extension of $\mathcal{S}_{\mathfrak{Q}(\mathfrak{P}_+)}$ which is induced by B_γ is an operator and hence, cf. [KW1, Lemma 6.4], [KW2, Lemma 5.2],

$$\lim_{y\to+\infty}\frac{1}{iy}\frac{A_\gamma(iy)}{B_\gamma(iy)}=0\,.$$

However, we have

$$\lim_{y\to+\infty}\frac{1}{iy}\frac{A_\gamma(iy)}{B_\gamma(iy)} = \lim_{y\to+\infty}\frac{1}{iy}\frac{A_+(-y^2)-\gamma B_+(-y^2)}{(iy)^{-1}B_+(-y^2)} = \lambda-\gamma\,.$$

It follows that $\lambda=\gamma\in\mathbb{R}$.

Conversely, if λ belongs to $\mathbb{R}$, it follows that $H = B_\lambda \notin \mathfrak{P}(E_\lambda)$. Hence $\mathfrak{P}(E_\lambda) = \mathfrak{Q}(\mathfrak{P}_+)$, in particular $\mathfrak{Q}(\mathfrak{P}_+)$ is nondegenerated. $\square$

4.8. *Remark.* The previous results could be used for 'ad hoc'-definitions of semi-bounded dB-Pontryagin spaces, the notion of $\mathfrak{P}_+$ and the relation between $\mathfrak{P}$ and $\mathfrak{P}_+$. However, when trying to incorporate the degenerated case, one would run by use of this approach into serious difficulties.

Let $E \in \mathcal{HB}_\kappa^{sym}$. By the above results the space $\mathfrak{P}(E)_e$ is characterized as the isomorphic image under the mapping Φ of $\mathfrak{P}(E_+)$ where e.g. $E_+ = E_{+,0}$. The space $\mathfrak{P}(E)_o$ of all odd functions in $\mathfrak{P}(E)$ can be obtained in a similar way.

4.9. Proposition. *Let $E = A - iB \in \mathcal{HB}_\kappa^{sym}$. Define entire functions A_-, B_- by*

$$A_-(z^2) := A(z),\ B_-(z^2) := \frac{B(z)}{z},$$

and put $E_- := A_- - iB_-$. Then $E_- \in \mathcal{HB}_{\leq\kappa}$ and the mapping ${}_z\Phi : F(z) \mapsto zF(z^2)$ is an isometry of $\mathfrak{P}(E_-)$ onto $\mathfrak{P}(E)_o$.

Proof. Denote by $K(w,z), K_o(w,z)$ and $K_-(w,z)$ the reproducing kernels of the spaces $\mathfrak{P}(E), \mathfrak{P}(E)_o$ and $\mathfrak{P}(E_-)$, respectively.

First note that

$$z\frac{B_-(z^2)}{A_-(z^2)} = \frac{B(z)}{A(z)}$$

is an odd generalized Nevanlinna function. Theorem 4.1 of [KWW2] implies that also $A_-(z)^{-1}B_-(z)$ is a generalized Nevanlinna function, and this means $E_- \in \mathcal{HB}_{\leq\kappa}$.

We have $(w, z \neq 0)$

$$2K_o(w,z) = K(w,z) - K(w,-z) = \frac{B(z)A(\overline{w}) - A(z)B(\overline{w})}{\pi(z-\overline{w})} -$$

$$-\frac{B(-z)A(\overline{w}) - A(-z)B(\overline{w})}{\pi(-z-\overline{w})} = 2\frac{\overline{w}B(z)A(\overline{w}) - zA(z)B(\overline{w})}{\pi(z^2-\overline{w}^2)} =$$

$$= 2\overline{w}z\frac{\frac{B(z)}{z}A(\overline{w}) - A(z)\frac{B(\overline{w})}{\overline{w}}}{\pi(z^2-\overline{w}^2)} = 2\overline{w}zK_-(w^2,z^2).$$

It follows that ${}_z\Phi$ maps $K_-(w^2,z)$ into $\mathfrak{P}(E)_o$. Moreover,

$$\big[{}_z\Phi\, K_-(w^2,z),\, {}_z\Phi\, K_-(v^2,z)\big]_{\mathfrak{P}(E)_o} = \big[\frac{1}{\overline{w}}K_o(w,z), \frac{1}{\overline{v}}K_o(v,z)\big]_{\mathfrak{P}(E)_o} =$$

$$= \frac{1}{\overline{w}v}K_o(w,v) = K_-(w^2,v^2) = \big[K_-(w^2,z), K_-(v^2,z)\big]_{\mathfrak{P}(E_-)}.$$

Since

$$\operatorname{cls}\big\{K_o(w,z) :\ w \in \mathbb{C}\setminus\{0\}\big\} = \mathfrak{P}(E)_o,\ \operatorname{cls}\big\{K_-(w^2,z) :\ w \in \mathbb{C}\setminus\{0\}\big\} = \mathfrak{P}(E_-),$$

and point evaluation is continuous in both spaces $\mathfrak{P}(E_-)$ and $\mathfrak{P}(E)_o$ it follows that ${}_z\Phi$ is in fact an isometry of $\mathfrak{P}(E_-)$ onto $\mathfrak{P}(E)_o$. $\square$

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