Алгебра и анализ
Том 18 (2006), №5

SUBSPACES OF DE BRANGES SPACES WITH PRESCRIBED GROWTH

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The growth properties of de Branges spaces and their subspaces are studied. It is shown that, for each given pair of growth functions $\lambda(r) = O(r)$ and $\lambda_1 = o(\lambda)$, there exist de Branges spaces of growth λ that have a de Branges subspace of growth λ_1. This phenomenon cannot occur for a class of de Branges spaces that, in a certain sense, behave regularly along the real axis.

§1. Introduction

A *de Branges space* is a Hilbert space $\langle \mathcal{H}, (.,.)\rangle$ with the following properties:

(dB1) The elements of $\mathcal{H}$ are entire functions, and for each $w \in \mathbb{C}$ the point evaluation $F \mapsto F(w)$ is a continuous linear functional on $\mathcal{H}$.

(dB2) If $F \in \mathcal{H}$, then $F^{\#}(z) := \overline{F(\bar{z})}$ also belongs to $\mathcal{H}$, and $\|F^{\#}\| = \|F\|$.

(dB3) If $w \in \mathbb{C} \setminus \mathbb{R}$ and $F \in \mathcal{H}$, $F(w) = 0$, then

$$\frac{z-\bar{w}}{z-w}F(z) \in \mathcal{H} \quad \text{and} \quad \Big\|\frac{z-\bar{w}}{z-w}F(z)\Big\| = \|F\|.$$

The theory of such Hilbert spaces of entire functions was founded by L. de Branges (cf. [dB]). It was further developed by many authors (see, e.g., [B1, B2, GM, KWW2, KW1, KW3, RR]), and found applications in various contexts (see, e.g., [BP1, DK1, DK2, Li, OS, Re, PW]).

Throughout this paper it is assumed that a de Branges space is additionally subject to the condition

(Z) For every $t \in \mathbb{R}$ there exists $F \in \mathcal{H}$ with $F(t) \neq 0$.

In most respects, this assumption causes no loss of generality (cf. [dB, Problem 44; KW3, Lemma 2.4]).

Key words: de Branges space, growth function, de Branges subspace.

Prominent examples of de Branges spaces are the so-called *Paley–Wiener spaces*. For a real positive number a, let $\mathcal{P}W_a$ be the set of all entire functions of exponential type at most a whose restriction to $\mathbb{R}$ belongs to $L^2(\mathbb{R})$. Endowed with the norm $\|F\| := \|F|_{\mathbb{R}}\|_{L^2(\mathbb{R})}$, the space $\mathcal{P}W_a$ becomes a de Branges space.

In complex analysis, in particular in the theory of entire functions, the notion of growth plays a central role. A function $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *growth function* if it satisfies the following axioms:

- **(gf1)** The limit $\rho := \lim_{r\to\infty} \frac{\log \lambda(r)}{\log r}$ exists and is a finite nonnegative number.
- **(gf2)** For all sufficiently large values of r, the function λ is differentiable, and $\lim_{r\to\infty} r\frac{\lambda'(r)}{\lambda(r)} = \rho$.
- **(gf3)** $\log r = o(\lambda(r))$.

Conditions (gf1) and (gf2) ensure that Valiron's theory of proximate orders is available, cf. [L, I.12] or [LG, I.6], as well as the theory of value distribution of meromorphic functions, cf. [Ru]. Condition (gf3), saying that λ grows sufficiently rapidly, is imposed to exclude trivial cases and is not an essential restriction. Unless specified, "O"- and "o"-relations are always understood for $r \to \infty$.

Classical examples of growth functions are presented by functions of the form

$$\lambda(r) = r^{\alpha}(\log r)^{\beta},$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha > 0$.

Since the elements of a de Branges space are entire functions, it is a natural task to bring together the concepts of a de Branges space and a growth function, i.e., to study de Branges spaces from the viewpoint of growth properties of their elements. For general growth functions, a systematic study was initiated in [KW3]. For the particular case of exponential growth, i.e., $\lambda(r) = r$, such investigations go back to the very beginning of the theory of de Branges spaces (see, e.g., [dB]).

Let us briefly describe the main theme of our present work. If F is an entire function and λ is a growth function, the *λ-type of F* is defined as the number

$$\sigma_F^{\lambda} := \limsup_{|z|\to\infty} \frac{\log^+ |F(z)|}{\lambda(|z|)} \in [0, \infty].$$

If $\mathcal{H}$ is a de Branges space and λ is a growth function, the *λ-type of $\mathcal{H}$* is the number

$$\sigma_{\mathcal{H}}^{\lambda} := \sup_{F\in\mathcal{H}} \sigma_F^{\lambda} \in [0, \infty].$$

A closed linear subspace $\mathcal{L}$ of a de Branges space $\mathcal{H}$ is called a *dB-subspace* if $\mathcal{L}$ itself, with the norm inherited from $\mathcal{H}$, is a de Branges space. The set of all dB-subspaces of a given space $\mathcal{H}$ will be denoted by $\operatorname{Sub}\mathcal{H}$. One of the most important results in the theory of de Branges spaces, the so-called Ordering Theorem, states that $\operatorname{Sub}\mathcal{H}$ is totally ordered with respect to inclusion; see [dB, Theorem 35], where an even stronger version is proved.

We address our paper to the following question: Let λ be a growth function and let $\mathcal{H}$ be a de Branges space of finite and positive λ-type. Do there exist subspaces $\mathcal{L} \in \operatorname{Sub}\mathcal{H}$ of strictly smaller growth?

Our answer is twofold. First, we show that, given a pair (λ, λ_1) of growth functions with

$$\lambda_1(r) = o(\lambda(r)) \quad \text{and} \quad \lambda(r) = O(r),$$

there exists a de Branges space $\mathcal{H}$ of finite and positive λ-type, such that one of its dB-subspaces is of finite and positive λ_1-type, cf. Theorem 3.6. It should be emphasized that, by [KW3, Theorem 3.10], the second condition is in a sense necessary for the existence of subspaces with smaller growth, see Remark 3.7. Second, for a de Branges space $\mathcal{H}$ with finite and positive λ-type we give a condition ensuring that no infinite dimensional dB-subspace can be of finite λ_1-type for any growth function λ_1 with $\lambda_1(r) = o(\lambda(r))$, see Theorem 4.1.

For the proof of these results we apply methods of various kinds. On the one hand, we use hard analysis, e.g., growth estimates; on the other hand, we employ more functional analytic tools, e.g., the theory of symmetric and semi-bounded de Branges space or the notion of transfer matrices for dB-subspaces. A cornerstone for proving the existence of examples of dB-subspaces with prescribed smaller growth is to establish the existence of de Branges spaces $\mathcal{H}$ of finite and positive λ-type (where $\lambda(r) = O(r)$) that have the property that the constant function 1 belongs to $\mathcal{H} + z\mathcal{H}$ (cf. Theorem 3.1), a result which is of interest on its own right. For the case where $\lambda(r) = r^\rho$, a result similar to Theorem 3.1 was proved in [BP2] by different methods.

§2. Preliminaries

In this section we recall some basic facts concerning de Branges spaces and set up some notation. Moreover, we provide a couple of results which supplement [KW3], and which will be used later on.

a. The Hermite–Biehler class. It is a basic fact that a given de Branges space $\mathcal{H}$ is completely determined by a single entire function. We say that an entire function E belongs to the *Hermite–Biehler class* $\mathcal{HB}$ if

$$|E^{\#}(z)| < |E(z)|, \quad z \in \mathbb{C}^+, \tag{2.1}$$

and E has no real zeros. For $E \in \mathcal{H}B$, define

$$\mathcal{H}(E) := \Big\{ F \text{ entire} : \frac{F}{E}, \frac{F^{\#}}{E} \in H^2(\mathbb{C}^+) \Big\},$$

where $H^2(\mathbb{C}^+)$ denotes the Hardy class in the upper half-plane, see, e.g., [RosR]. Moreover, we define

$$\|F\|_E^2 := \int_{\mathbb{R}} \left| \frac{F(t)}{E(t)} \right|^2 dt, \quad F \in \mathcal{H}(E).$$

Then $\mathcal{H}(E)$ is a de Branges space. Conversely, every nonzero de Branges space can be obtained in this way. For example, the Paley–Wiener space $\mathcal{P}W_a$ is obtained as $\mathcal{H}(e^{-iaz})$.

By the axiom (dB1), a de Branges space $\mathcal{H}$ is a reproducing kernel Hilbert space. If $\mathcal{H}$ is a de Branges space and $E \in \mathcal{H}B$ is such that $\mathcal{H} = \mathcal{H}(E)$, then the reproducing kernel $K(w, \cdot)$ of $\mathcal{H}$ can be expressed in terms of E:

$$K(w, z) = \frac{E(z)E^{\#}(\bar{w}) - E(\bar{w})E^{\#}(z)}{2\pi i(\bar{w} - z)}. \tag{2.2}$$

2.1. Remark. (i) In the literature, the condition that E has no real zeros is often dropped from the definition of $\mathcal{H}B$. This corresponds to dropping the condition (Z) in our definition of de Branges space. As we have already remarked, dropping this condition is no essential gain in generality.

(ii) Condition (2.1) in the definition of $\mathcal{H}B$ means that the kernel $K(w, z)$ is positive semidefinite, i.e., that each of the quadratic forms ($z_1, \ldots, z_n \in \mathbb{C}$)

$$\sum_{i,j=1}^{n} K(z_j, z_i)\zeta_i\overline{\zeta_j}$$

is positive semidefinite. This is a classical result of analysis, see [P].

(iii) With an entire function E we associate a pair of entire functions A and B:

$$A := \frac{E + E^{\#}}{2}, \quad B := i\frac{E - E^{\#}}{2}. \tag{2.3}$$

Then A and B are real entire functions (we say that entire function F is real if it is real on the real axis or, equivalently, if $F = F^{\#}$), and we have $E = A - iB$. If $E \in \mathcal{H}B$, then A and B have only real zeros, and these zeros are simple and interlace, that is, between every two zeros of A there is a zero of B and *vice versa*.

(iv) The *Nevanlinna class* $\mathcal{N}_0$ is defined as the set of all functions q that are analytic in $\mathbb{C} \setminus \mathbb{R}$, satisfy $q = q^{\#}$, and have nonnegative imaginary part in $\mathbb{C}^+$. This class of functions is closely related to $\mathcal{H}B$. More precisely, let E be an entire function and write $E = A - iB$ with A and B as in

(2.3). Then $E \in \mathcal{H}B$ if and only if E has no real zeros, is not constant, and $\frac{B}{A} \in \mathcal{N}_0$.

b. The growth of $\mathcal{H}(E)$.

2.2. Remark. In the context of growth properties of de Branges spaces, it is a basic result that the growth of any function $F \in \mathcal{H}(E)$ is governed by the growth of E. In fact, we always have $\sigma_{\mathcal{H}}^{\lambda} = \sigma_{E}^{\lambda}$, see [KW3, Theorem 3.4].

We come to the announced generalizations of two results of [KW3]. They will be deduced from the next proposition, which also provides a more general viewpoint on some results concerning the exponential growth of the so-called Nevanlinna matrices, see, e.g., [BP1, Theorem 4.2; BP2, Theorem 4.8; K1].

2.3. Proposition. *Let A, B be entire functions such that $\frac{B}{A} \in \mathcal{N}_0$, and let λ be a growth function. Then $\sigma_A^{\lambda} = \sigma_B^{\lambda}$.*

Proof. If both σ_A^{λ} and σ_B^{λ} are equal to ∞, we are done. We show that $\sigma_A^{\lambda} < \infty$ implies $\sigma_B^{\lambda} \leqslant \sigma_A^{\lambda}$. Since with $\frac{B}{A}$ also $-\frac{A}{B}$ belongs to $\mathcal{N}_0$, this will yield the desired conclusion.

Since the function $\frac{B}{A}$ belongs to $\mathcal{N}_0$, it has, by the Herglotz theorem, an integral representation of the form

$$\frac{B(z)}{A(z)} = az + b + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{t^2+1} \Big) \, d\mu(t), \quad z \in \mathbb{C}^+,$$

where $a \geqslant 0$, $b \in \mathbb{R}$, and μ is a Borel measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} (t^2 + 1)^{-1} d\mu(t) < \infty$. In the present case, in fact, μ is a discrete measure with point masses at the zeros of A. Hence,

$$\begin{aligned} \Big| \frac{B(z)}{A(z)} \Big| &= \Big| az + b + \int_{\mathbb{R}} \frac{tz+1}{(t-z)(t^2+1)} \, d\mu(t) \Big| \\ &\leqslant \Big| az + b + z \int_{\mathbb{R}} \frac{d\mu(t)}{t^2+1} \Big| + \Big| (z^2+1) \int_{\mathbb{R}} \frac{d\mu(t)}{(t-z)(t^2+1)} \Big|, \quad z \in \mathbb{C}^+. \end{aligned}$$

Therefore,

$$\log |B(z)| \leqslant \log |A(z)| + C_1 \log(|z| + 2) + \log^+ \frac{1}{|\operatorname{Im} z|}$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$. In particular,

$$\log |B(z)| \leqslant \log |A(z)| + C_1 \log(|z| + 2), \quad |\operatorname{Im} z| \geqslant 1. \tag{2.4}$$

Now, let $|\operatorname{Im} z| < 1$. Then, by the subharmonicity of $\log|B|$,

$$\begin{aligned}\log|B(z)| &\leqslant \frac{1}{2\pi}\int_0^{2\pi}\log|B(z+e^{i\phi})|\,d\phi\\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi}\log|A(z+e^{i\phi})|d\phi + C_2\log(|z|+2)\\ &\quad+\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|\operatorname{Im}(z+e^{i\phi})|}\,d\phi.\end{aligned}$$

Clearly,

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|\operatorname{Im}(z+e^{i\phi})|}\,d\phi \leqslant C_3$$

for all z with $|\operatorname{Im} z| < 1$. Consequently,

$$\log|B(z)| \leqslant \frac{1}{2\pi}\int_0^{2\pi}\log^+|A(z+e^{i\phi})|\,d\phi + C_2\log(|z|+2) + C_3, \quad |\operatorname{Im} z| < 1. \tag{2.5}$$

Since $\sigma_A^\lambda < \infty$ and $\log r = o(\lambda(r))$, from (2.4) and (2.5) it follows that $\sigma_B^\lambda < \infty$ and $\sigma_B^\lambda \leqslant \sigma_A^\lambda$. •

For a function $E \in \mathcal{H}B$, we shall always use the notation $E = A - iB$ with the real entire functions A, B as in (2.3). We denote

$$S_\phi := A\sin\phi - B\cos\phi, \quad \phi \in \mathbb{R}.$$

The family of functions S_ϕ contains significant information about the space $\mathcal{H}(E)$. In fact, the reproducing kernel $K(w, z)$ can be written in terms of the functions S_ϕ. Also, the functions S_ϕ describe the selfadjoint extensions of the operator of multiplication by the independent variable in $\mathcal{H}(E)$. Clearly, $A = S_{\frac{\pi}{2}}$ and $B = S_\pi$. We note that

$$S_{\phi+\frac{\pi}{2}} = \frac{1}{2}\big(e^{i\phi}E + e^{-i\phi}E^\#\big).$$

2.4. Corollary. *Let $E \in \mathcal{H}B$. Then E is of finite λ-type if and only if there exists one value $\phi_0 \in \mathbb{R}$ such that S_{ϕ_0} is of finite λ-type. In this case every function S_ϕ, $\phi \in \mathbb{R}$, is of finite λ-type, and*

$$\sigma_E^\lambda = \sigma_{S_\phi}^\lambda, \quad \phi \in \mathbb{R}.$$

Proof. It suffices to show that $\sigma^\lambda_{S_\phi} = \sigma^\lambda_E$ for any $\phi \in \mathbb{R}$. Since the functions $E(z)$ and $e^{i(\phi-\frac{\pi}{2})}E(z)$ have the same λ-type, we can restrict the explicit proof to the case where $\phi = \frac{\pi}{2}$, i.e., $S_\phi = A$.

Assume that $\sigma^\lambda_A < \infty$. Then, by the definition of A and B and by Proposition 2.3,

$$\sigma^\lambda_E \leqslant \max\{\sigma^\lambda_A, \sigma^\lambda_B\} = \sigma^\lambda_A \leqslant \sigma^\lambda_E.$$

It follows that $\sigma^\lambda_E < \infty$ and, in fact, σ^λ_E is equal to σ^λ_A. Conversely, if $\sigma^\lambda_E < \infty$, then, clearly, $\sigma^\lambda_A \leqslant \sigma^\lambda_E$ and, thus, $\sigma^\lambda_A < \infty$. •

We obtain a generalization of [KW3, Corollary 3.18], where the following statement was proved under the additional assumption $r = O(\lambda(r))$.

2.5. Corollary. *Let $\mathcal{H}$ be a de Branges space, and let λ be a growth function. Then $\mathcal{H}$ is of finite λ-type if and only if for some $\phi \in [0,\pi)$ the function S_ϕ is of finite λ-type. In this case $\sigma^\lambda_{\mathcal{H}} = \sigma^\lambda_{S_\phi}$.*

Proof. Combine Corollary 2.4 with Remark 2.2. •

We also obtain a similar generalization of [KW3, Theorem 3.17], i.e., we can drop the assumption $r = O(\lambda(r))$ made there. In order to explain this result, we need to recall the notion of a λ-admissible sequence of complex numbers, cf. [Ru].

We say that a sequence $\{z_n\}$ has a *finite λ-density* if

$$\sum_{|z_n|\leqslant r} \log \frac{r}{|z_n|} = O(\lambda(r)).$$

When comparing with the definition in [Ru] one should be aware of the property $\lambda(r) \asymp \lambda(cr)$ for any $c > 1$, which follows from the definition of a growth function (we write $f \asymp g$ if there exist constants $c_1, c_2 > 0$ such that $c_1 f(x) \leqslant g(x) \leqslant c_2 f(x)$ for all admissible values of x). The sequence $\{z_n\}$ is said to be *λ-balanced* if, uniformly in $k \in \mathbb{N}$, we have

$$\Big|\frac{1}{k} \sum_{r_1<|z_n|\leqslant r_2} \Big(\frac{1}{z_n}\Big)^k\Big| = O\Big(\frac{\lambda(r_1)}{r_1^k} + \frac{\lambda(r_2)}{r_2^k}\Big), \quad r_1, r_2 \to \infty.$$

Finally, the sequence $\{z_n\}$ is *λ-admissible* if it is both of finite λ-density and λ-balanced.

2.6. Corollary. *A de Branges space $\mathcal{H}(E)$ is obtained from a de Branges space $\mathcal{H}(E_1)$ of finite λ-type by multiplication by a zero free entire function real on $\mathbb{R}$, if and only if for one* (*and, hence, for all*) *$\phi \in [0,\pi)$ the sequence of zeros of S_ϕ is λ-admissible.*

Proof. We repeat briefly the proof in [KW3]. If the sequence $\{a_n\}$ of zeros of S_ϕ is λ-admissible, then, by [Ru, Theorem 13.5.2], there exists an entire function A_1 of finite λ-type, $A_1 = A_1^\#$, having $\{a_n\}$ as its precise set of zeros. Hence, $C = \frac{S_\phi}{A_1}$ is a zero free entire function real on $\mathbb{R}$. Put

$$E_1(z) = \frac{S_\phi(z) - iS_{\phi+\frac{\pi}{2}}(z)}{C(z)} = -\frac{ie^{i\phi}E(z)}{C(z)}.$$

Then $E_1 \in \mathcal{H}B$ and $\mathcal{H}(E) = \mathcal{H}(CE_1) = C\mathcal{H}(E_1)$. Note also that $\frac{E_1+E_1^\#}{2} = A_1$ is of finite λ-type. By Corollary 2.4, E_1 is of finite λ-type and, thus, $\mathcal{H}(E_1)$ is of finite λ-type.

The converse implication is almost immediate, see [KW3] for the details. •

c. Functions associated with a de Branges space. An entire function S is said to be associated with the space $\mathcal{H}(E)$ if for each $w \in \mathbb{C}$ the difference quotient operator

$$F(z) \mapsto \frac{F(z)S(w) - S(z)F(w)}{z - w}$$

maps the space $\mathcal{H}(E)$ into itself. The set of all functions associated with a space $\mathcal{H}(E)$ is denoted by $\operatorname{Assoc}\mathcal{H}(E)$.

The functions of class $\operatorname{Assoc}\mathcal{H}(E)$ can be characterized in various ways. For instance, it is straightforward to verify that

$$\operatorname{Assoc}\mathcal{H}(E) = \mathcal{H}(E) + z\mathcal{H}(E).$$

Another characterization of $\operatorname{Assoc}\mathcal{H}(E)$ employs a certain class of entire matrix functions. This is a deep result, see [dB, Theorem 27].

The spaces $\mathcal{H}(E)$ with the property that $1 \in \operatorname{Assoc}\mathcal{H}(E)$ are of particular interest. Let us formulate explicitly the characterization [dB, Theorem 27] (mentioned above) for this case. For this, we need to recall the notion of matrix functions of class $\mathcal{M}_0$, a notion closely related to the so-called Nevanlinna matrices. A (2×2)-matrix $W = (w_{ij})_{i,j=1,2}$ is said to belong to the class $\mathcal{M}_0$ if its entries w_{ij} are real entire functions, $W(0) = I$, $\det W \equiv 1$, and the kernel

$$H_W(w, z) := \frac{W(z)JW^*(w) - J}{z - \bar{w}},$$

where

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.6}$$

is positive semidefinite. We note that if

$$W = \begin{pmatrix} A_W & B_W \\ C_W & D_W \end{pmatrix} \in \mathcal{M}_0,$$

then each of the functions $A_W - iB_W$, $D_W + iC_W$, $D_W - iB_W$, and $A_W + iC_W$ belongs to $\mathcal{H}B$.

The relationship between $\mathcal{M}_0$ and the spaces $\mathcal{H}(E)$ with the property that $1 \in \operatorname{Assoc}\mathcal{H}(E)$ is now the following.

2.7. Remark. Let $W \in \mathcal{M}_0$ and define

$$E_W := A_W - iB_W, \quad (A_W, B_W) := (1,0)W.$$

Then $E_W \in \mathcal{H}B$ and $1 \in \operatorname{Assoc}\mathcal{H}(E_W)$. Conversely, if $\mathcal{H}$ is a de Branges space with $1 \in \operatorname{Assoc}\mathcal{H}$, and we write $\mathcal{H} = \mathcal{H}(E)$, $E = A - iB$, with $E(0) = 1$, then there exist entire functions C, D such that

$$W := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{M}_0. \tag{2.7}$$

In this case the functions D, C can be chosen so that

$$\lim_{y\to+\infty} \frac{1}{y} \operatorname{Im} \frac{D(iy)}{B(iy)} = 0. \tag{2.8}$$

We shall also employ the following result, which was proved in [W, Theorem 1.1], and from which a characterization of $1 \in \operatorname{Assoc}\mathcal{H}(E)$ can be deduced.

2.8. Remark. Let $E = A - iB \in \mathcal{H}B$ and $E(0) = 1$. Then the set $\operatorname{Assoc}\mathcal{H}(E)$ contains a real and zero free function if and only if the following conditions are satisfied:

(C1) Let $(x_k)_{k\in\mathbb{N}}$ be the sequence of zeros of A. Then the limit $\lim_{r\to\infty} \sum_{|a_k|\leqslant r} \frac{1}{x_k}$ exists in $\mathbb{R}$.

(C2) Let $(x_k^+)_{k\in\mathbb{N}}$ and $(x_k^-)_{k\in\mathbb{N}}$ be the sequence of positive and negative, respectively, zeros of A arranged according to increasing modulus. Then the limits $\lim_{k\to\infty} \frac{k}{x_k^+}$ and $\lim_{k\to\infty} \frac{k}{x_k^-}$ exist in $\mathbb{R}$ and are equal.

(C3) Let $X(z) := \lim_{r\to\infty} \prod_{|x_k|\leqslant r}(1 - \frac{z}{x_k})$ and $Y(z) := z \lim_{r\to\infty} \prod_{|y_k|\leqslant r}(1 - \frac{z}{y_k})$, where $(y_k)_{k\in\mathbb{N}}$ denotes the sequence of nonzero zeros of B. Then

$$\sum_{k\in\mathbb{N}} \left| \frac{1}{x_k^2 X'(x_k) Y(x_k)} \right| < \infty. \tag{2.9}$$

In this case we have $\frac{A(z)}{X(z)} \in \operatorname{Assoc}\mathcal{H}(E)$.

The class $\mathcal{M}_0$ plays also in other respects an important role in the theory of de Branges spaces, cf. [dB, Theorems 33,34].

2.9. Remark. If W and W_1 belong to $\mathcal{M}_0$, then $WW_1 \in \mathcal{M}_0$. We have $\mathcal{H}(E_W) \subseteq \mathcal{H}(E_{WW_1})$, and the set-theoretic inclusion map is a contraction.

If the set $\mathcal{H}(E_W)$ is endowed with the norm $\|\cdot\|_{E_{WW_1}}$, then it becomes a dB-subspace of $\mathcal{H}(E_{WW_1})$.

d. Transformation of matrices of class $\mathcal{M}_0$. . We shall use some transformation of matrices.

2.10. Definition. Let W be a (2×2)-matrix function whose elements are entire functions.

(i) We put

$$V := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and define

$$\mathfrak{v}(W) := VW^{-1}V.$$

(ii) For $a \in \mathbb{R}$, we define

$$\mathfrak{w}_a(W)(z) := W(a)^{-1}W(z+a).$$

Then, as a computation of reproducing kernels shows, for $a \in \mathbb{R}$ the maps $\mathfrak{v}$ and $\mathfrak{w}_a$ take $\mathcal{M}_0$ into itself; see, e.g., [KWW1, Lemma 2.3; Wi].

If λ is a growth function and $W = (w_{ij})_{i,j=1,2} \in \mathcal{M}_0$, then, by Proposition 2.3, all the entries w_{ij} of W are of the same λ-type. This allows us to put

$$\sigma_W^\lambda := \sigma_{w_{ij}}^\lambda, \quad i,j = 1,2.$$

2.11. Lemma. *Let $W \in \mathcal{M}_0$. Then $\sigma^\lambda_{\mathcal{H}(E_W)} = \sigma_W^\lambda$. We have*

$$\sigma^\lambda_{\mathfrak{v}(W)} = \sigma^\lambda_{\mathfrak{w}_a(W)} = \sigma^\lambda_W, \quad a \in \mathbb{R}.$$

If, moreover, $W_1 \in \mathcal{M}_0$, then

$$\sigma^\lambda_W \leqslant \sigma^\lambda_{WW_1} \leqslant \sigma^\lambda_W + \sigma^\lambda_{W_1}. \tag{2.10}$$

Proof. The first assertion follows from Corollary 2.4. The relation $\sigma^\lambda_{\mathfrak{v}(W)} = \sigma^\lambda_W$ is immediate from the explicit computation

$$\mathfrak{v}(W) = \begin{pmatrix} w_{11} & -w_{21} \\ -w_{12} & w_{22} \end{pmatrix}.$$

The fact that $\sigma^\lambda_{\mathfrak{w}_a(W)} = \sigma^\lambda_W$ follows because $W(a)$ is invertible and because for any entire function f we have $\sigma^\lambda_{f(z)} = \sigma^\lambda_{f(z+a)}$.

The first inequality in (2.10) is a consequence of the fact that, as a set, $\mathcal{H}(E_W)$ is contained in $\mathcal{H}(E_{WW_1})$. The second inequality in (2.10) is trivial. •

e. The square root transformation. The following transformation of entire functions is a useful tool. Let E be an entire function and assume that E satisfies the functional equation

$$E^{\#}(z) = E(-z). \tag{2.11}$$

If E is written as $E = A - iB$ with A and B as in (2.3), we define $E_+ := A_+ - iB_+$, where A_+ and B_+ are defined by the relations

$$A_+(z^2) = A(z), \quad B_+(z^2) = zB(z). \tag{2.12}$$

Note that the validity of (2.11) is equivalent to A being even and B being odd, so that A_+ and B_+ are well-defined entire functions. Clearly, $B_+(0) = 0$.

The assignment $E \mapsto E_+$ maps the set of all entire functions satisfying (2.11) bijectively onto the set of all entire functions that take a real value at 0. Its inverse is given by

$$E_+ = A_+ - iB_+ \mapsto E := A_+(z^2) - \frac{i}{z}B_+(z^2).$$

In connection with this transformation, the so-called Stieltjes class $\mathcal{S}$ is of importance. Recall that a function q is said to belong to $\mathcal{S}$ if $q \in \mathcal{N}_0$ and is analytic and nonnegative on $(-\infty, 0)$. The fact that $q \in \mathcal{S}$ can be characterized in several ways, see [KaK1, KaK2]. We recall that

$$q \in \mathcal{S} \iff zq(z^2) \in \mathcal{N}_0. \tag{2.13}$$

Moreover, for any function $q \in \mathcal{S}$ we have $\lim_{y\to+\infty} \frac{1}{y} \operatorname{Im} q(iy) = 0$ and $\lim_{x\to-\infty} q(x) \in \mathbb{R}$.

2.12. Remark. Let $A - iB \in \mathcal{H}B$, and assume that A and B do not vanish on $(-\infty, 0)$ while $B(0) = 0$. Then $q = -\frac{A}{B} \in \mathcal{S}$. Indeed, $q \in \mathcal{N}_0$, and therefore, by the Herglotz theorem,

$$q(z) = -\frac{\sigma_0}{z} + \sum_{n=1}^{\infty} \sigma_n \Big(\frac{1}{x_n - z} - \frac{1}{x_n} \Big) + a + bz$$

for some $a \in \mathbb{R}$, $b \geqslant 0$, and $\sigma_n > 0$, $n = 0, 1, \ldots$ Here the x_n are the nonzero zeros of B. Clearly, $\lim_{x\to-0} q(x) = +\infty$. Since q does not change the sign on $(-\infty, 0)$, it follows that $q > 0$ on $(-\infty, 0)$.

2.13. Remark. Let $E = A - iB$ and $E_+ = A_+ - iB_+$ be entire functions related to each other as in (2.12). Then:

(i) $E \in \mathcal{H}B$ if and only if $E_+ \in \mathcal{H}B$ and $A_+(x)B_+(x) < 0$, $x < 0$, which is equivalent to $-\frac{A_+}{B_+} \in \mathcal{S}$. This follows by combining Remark 2.1(iv), with (2.13) and the fact that $q \in \mathcal{N}_0$ if and only if $-\frac{1}{q} \in \mathcal{N}_0$.

(ii) If $E \in \mathcal{H}B$ and $1 \in \operatorname{Assoc}\mathcal{H}(E)$, then also $1 \in \operatorname{Assoc}\mathcal{H}(E_+)$. This follows from some geometric arguments concerning the generated de Branges spaces. A proof can be found in [KWW2, Proposition 2.6], where even a more general setting was considered.

(iii) Assume that $E \in \mathcal{H}B$ and $1 \in \operatorname{Assoc}\mathcal{H}(E_+)$. Let C_+, D_+ be entire functions as in Remark 2.7 and satisfying (2.8). Then $1 \in \operatorname{Assoc}\mathcal{H}(E)$ if and only if

$$\lim_{x\to-\infty} \frac{D_+(x)}{B_+(x)} \in \mathbb{R}.$$

On the first sight, this result may look surprising. It follows from a deeper discussion concerning the reproducing kernel space generated by the matrix W in (2.7) (see [KWW1, Proposition 3.14]), which involves the theory of de Branges Pontryagin spaces, cf. [KW1]. No direct proof is known to the authors.

§3. Existence of subspaces with prescribed growth

This section is devoted to the proof of the existence of de Branges spaces that have subspaces of smaller growth, see Theorem 3.6. Our method is based on the following result, which is also of interest on its own right.

3.1. Theorem. *Let λ be a growth function with $\lambda(r) = O(r)$. Then there exists a de Branges space $\mathcal{H}$ of finite and positive λ-type with $1 \in \operatorname{Assoc}\mathcal{H}$.*

Before proceeding to the proof of this theorem, we discuss the statement in more detail.

3.2. Remark. (i) By [KW3, Theorem 3.10], every de Branges space $\mathcal{H}$ with $1 \in \operatorname{Assoc}\mathcal{H}$ must be of finite exponential type. Hence, the condition $\lambda(r) = O(r)$ is natural.

(ii) A different approach to this theorem could proceed via [BP2, Theorems 3.6, 5.1]; the method employed there is function theoretic in its nature and goes back to a result of M.G.Krein, see [K2]. By means of this approach the case where $\lambda(r) = r^\rho$ was treated in [BP2, Theorems 5.6]. In the present paper, however, we prefer a more Hilbert space theoretic point of view.

(iii) The hard part of the theorem is to deal with the case where $\lambda(r)$ grows slower, but almost as fast, as r. If the growth function λ, $\lambda(r) = O(r)$, satisfies a very mild additional condition, a space $\mathcal{H}$ with finite and positive λ-type and with $1 \in \operatorname{Assoc}\mathcal{H}$ can be constructed much more explicitly. Indeed, denoting by μ the inverse function of λ, let us assume

that
$$\int_0^\infty \frac{dt}{\mu(t)} < \infty. \tag{3.1}$$
Put $y_n := \mu(n)$, $n \in \mathbb{N}$. Then, by (3.1), we have $\sum_{n=1}^\infty \frac{1}{y_n} < \infty$. Hence, we may define an entire function by
$$E(z) := \prod_{n\in\mathbb{N}} \Big(1 + \frac{z}{iy_n}\Big).$$
Clearly, $E \in \mathcal{HB}$. Moreover, e.g., by [B2, Theorem 1], the space $\mathcal{H}(E)$ contains the set of all polynomials as a dense linear subspace. In particular, $1 \in \operatorname{Assoc}\mathcal{H}(E)$. Since
$$\Delta_\lambda((y_n)_{n\in\mathbb{N}}) = 1, \quad \limsup_{r\to\infty} \Big| \sum_{|iy_n|\leqslant r} \frac{1}{iy_n} \Big| = \sum_{n=1}^\infty \frac{1}{y_n} < \infty,$$
the results of [L, I. Lehrsatz 17,18] imply that E is of finite and positive λ-type.

Note that if (3.1) fails, a similar construction is not possible. This follows from the fact that, due to the presence of the Blaschke condition, a function of finite and positive λ-type must, in the case where (3.1) fails, have most of its zeros close to the real axis in the sense that there exists no angle $\{z \in \mathbb{C} : \arg z \in (-\pi+\delta, -\delta)\}$, $\delta > 0$, containing all zeros of E.

Let us come to the proof of Theorem 3.1. It is based on a perturbation argument. Let $E = A - iB \in \mathcal{HB}$, $E(0) = 1$, and let $(\gamma_k)_{k\in\mathbb{N}}$ be a monotone nondecreasing sequence of positive real numbers. Assume the following:

(a) All zeros of A lie in $(0,\infty)$.

(b) Denote by $(a_k)_{k\in\mathbb{N}}$ the sequence of zeros of A arranged increasingly. Then
$$\sum_{k=1}^\infty \frac{1}{a_k} < \infty.$$

Note that $B(0) = 0$ because $E(0) = 1$. Taking into account (a) and the fact that the zeros of A and B interchange, we see that the zeros of B lie in $[0,\infty)$.

We construct a function $\tilde{E}(z)$ by proceeding as follows.

(1) The function $\frac{B(z)}{A(z)}$ belongs to the class $\mathcal{N}_0$ and is meromorphic in $\mathbb{C}$. Hence, if we put $\sigma_n := -\frac{B(a_n)}{A'(a_n)}$, then $\sigma_n > 0$, $\sum_{k=1}^\infty \sigma_k a_k^{-2} < \infty$, and we have
$$\frac{B(z)}{A(z)} = az + \sum_{k=1}^\infty \Big(\frac{1}{a_k - z} - \frac{1}{a_k}\Big)\sigma_k,$$
where $a \geqslant 0$.

(2) Define a sequence $(\tilde{a}_k)_{k\in\mathbb{N}}$ as $\tilde{a}_k := a_k\gamma_k$, $k\in\mathbb{N}$. Then
$$0<\tilde{a}_1<\tilde{a}_2<\dots \quad\text{and}\quad \tilde{a}_k\geqslant \gamma_1 a_k,\quad k\in\mathbb{N}.$$
(3) We have
$$\sum_{k=1}^{\infty}\frac{\sigma_k}{1+\tilde{a}_k^2}\leqslant\sum_{k=1}^{\infty}\frac{\sigma_k}{1+\gamma_1^2a_k^2}<\infty.$$
Consequently, the series
$$\tilde{q}(z):=\sum_{k=1}^{\infty}\Big(\frac{1}{\tilde{a}_k-z}-\frac{1}{\tilde{a}_k}\Big)\sigma_k$$
determines a function $\tilde{q}\in\mathcal{N}_0$ meromorphic in $\mathbb{C}$ and such that its poles are the $\tilde{a}_k$, and its residue at $\tilde{a}_k$ is $-\sigma_k\neq 0$.

(4) We have
$$\sum_{k=1}^{\infty}\frac{1}{\tilde{a}_k}\leqslant\sum_{k=1}^{\infty}\frac{1}{\gamma_1 a_k}<\infty,$$
so that the product
$$\tilde{A}(z):=\prod_{k=1}^{\infty}\Big(1-\frac{z}{\tilde{a}_k}\Big)$$
determines an entire function.

(5) Put $\tilde{B}(z):=\tilde{q}(z)\cdot\tilde{A}(z)$; then $\tilde{B}$ is a real entire function and $\tilde{B}(0)=0$. If we now define
$$\tilde{E}:=\tilde{A}-i\tilde{B},$$
then $\tilde{E}\in\mathcal{H}B$ (see Remark 2.1(iv)).

3.3. Lemma. *Assume that $E\in\mathcal{H}B$ is subject to the above conditions* (a) *and* (b)*, and let $\tilde{E}$ be constructed as in* (1)–(5). *If $1\in\operatorname{Assoc}\mathcal{H}(E)$, then also $1\in\operatorname{Assoc}\mathcal{H}(\tilde{E})$.*

Proof. Assume that $1\in\operatorname{Assoc}\mathcal{H}(E)$. In order to prove the lemma, we shall verify the conditions (C1)–(C3) of Remark 2.8 for the function $\tilde{E}=\tilde{A}-i\tilde{B}$.

Because of the positivity of the numbers $\tilde{a}_k$, condition (C1) is equivalent to the convergence of the series $\sum_{k\in\mathbb{N}}\frac{1}{\tilde{a}_k}$, and this was already seen in the item (4) above. Condition (C2) is fulfilled because $(\gamma_k)_{k\in\mathbb{N}}$ is monotone increasing (thus, $\lim_{k\to\infty}\frac{1}{\gamma_k}$ exists), and because, by the assumption of the lemma, the sequence $(a_k)_{k\in\mathbb{N}}$ satisfies (C2). Indeed, since all zeros are positive, $\lim_{k\to\infty}\frac{k}{a_k}=0$ and hence also $\lim_{k\to\infty}\frac{k}{\tilde{a}_k}=0$.

To establish (C3), we need to estimate $\tilde{A}'(\tilde{a}_k)$. Since $1\in\operatorname{Assoc}\mathcal{H}(E)$ and all zeros of A are positive, [KWW1, Proposition 3.12] implies that E, and hence also A and B, are of order at most $\frac{1}{2}$. Thus, $A(z)=\prod_{n=1}^{\infty}(1-\frac{z}{a_n})$ and

$B(z) = \alpha z \prod_{n=1}^{\infty}(1-\frac{z}{b_n})$, where $(b_n)_{n\in\mathbb{N}}$ denotes the sequence of nonzero zeros of B, and $\alpha > 0$. Note that, by locally uniform convergence of the respective products and analyticity,

$$\tilde{A}'(\tilde{a}_k) = -\frac{1}{\tilde{a}_k}\prod_{\substack{n\in\mathbb{N}\\ n\neq k}}\Big(1-\frac{\tilde{a}_k}{\tilde{a}_n}\Big), \quad A'(a_k) = -\frac{1}{a_k}\prod_{\substack{n\in\mathbb{N}\\ n\neq k}}\Big(1-\frac{a_k}{a_n}\Big).$$

We have $\frac{\tilde{a}_k}{\tilde{a}_n} = \frac{a_k}{a_n}\frac{\gamma_k}{\gamma_n}$, whence

$$\frac{\tilde{a}_k}{\tilde{a}_n}\begin{cases}\leqslant \frac{a_k}{a_n} & \text{if } k<n,\\ \geqslant \frac{a_k}{a_n} & \text{if } k>n.\end{cases}$$

Moreover,

$$\Big|1-\frac{\tilde{a}_k}{\tilde{a}_n}\Big| = \left\{\begin{array}{ll} 1-\frac{\tilde{a}_k}{\tilde{a}_n} & \text{if } k<n\\ \frac{\tilde{a}_k}{\tilde{a}_n}-1 & \text{if } k>n\end{array}\right\} \geqslant \left\{\begin{array}{ll} 1-\frac{a_k}{a_n} & \text{if } k<n\\ \frac{a_k}{a_n}-1 & \text{if } k>n\end{array}\right\} = \Big|1-\frac{a_k}{a_n}\Big|.$$

It follows that

$$|\tilde{A}'(\tilde{a}_k)| \geqslant \frac{1}{\gamma_k}|A'(a_k)|.$$

The series (2.9) for E can be written as

$$\sum_{k\in\mathbb{N}}\Big|\frac{1}{a_k^2 A'(a_k)B(a_k)}\Big| = \sum_{k\in\mathbb{N}}\frac{1}{a_k^2 A'(a_k)^2\sigma_k},$$

and, since by our construction the numbers σ_k are the same for E and $\tilde{E}$, the series (2.9) for $\tilde{E}$ is nothing else but

$$\sum_{k\in\mathbb{N}}\frac{1}{\tilde{a}_k^2\tilde{A}'(\tilde{a}_k)^2\sigma_k}.$$

It follows that

$$\sum_{k\in\mathbb{N}}\frac{1}{\tilde{a}_k^2\tilde{A}'(\tilde{a}_k)^2\sigma_k} \leqslant \sum_{k\in\mathbb{N}}\frac{1}{(\gamma_k a_k)^2(\frac{1}{\gamma_k}A'(a_k))^2\sigma_k} = \sum_{k\in\mathbb{N}}\frac{1}{a_k^2 A'(a_k)^2\sigma_k} < \infty,$$

i.e., condition (C3) is satisfied. •

The next lemma is completely elementary. It simply ensures that the numbers γ_n can be chosen appropriately. Recall that for a growth function μ the upper μ-density of a sequence $(w_k)_{k\in\mathbb{N}}$ of complex numbers is defined as

$$\Delta_\mu((w_k)_{k\in\mathbb{N}}) := \limsup_{r\to\infty}\frac{\#\{k\in\mathbb{N}:\ |w_k|\leqslant r\}}{\mu(r)}.$$

3.4. Lemma. *Let λ and λ_1 be growth functions with $\lambda_1(r) = O(\lambda(r))$, and let $(a_k)_{k\in\mathbb{N}}$ be a monotone increasing sequence of positive real numbers that has a finite and positive upper λ-density. Then there exists a monotone nondecreasing sequence $(\gamma_k)_{k\in\mathbb{N}}$ of positive real numbers such that the sequence $(\gamma_k a_k)_{k\in\mathbb{N}}$ has a finite and positive upper λ_1-density.*

Proof. Without loss of generality we can assume that λ_1 is continuous and monotone increasing, and that $\lim_{r\searrow 0}\lambda_1(r) \in (0,1)$.

First, observe that, since $\lambda_1(r) \leqslant c\lambda(r)$ for an appropriate real constant c, we certainly have

$$\Delta_{\lambda_1}((a_k)_{k\in\mathbb{N}}) \geqslant \frac{1}{c}\Delta_\lambda((a_k)_{k\in\mathbb{N}}) > 0.$$

If $\Delta_{\lambda_1}((a_k)_{k\in\mathbb{N}}) < \infty$, then we set $\gamma_n := 1$, $n\in\mathbb{N}$, and are done. Hence, for the rest of the proof we may assume that $\Delta_{\lambda_1}((a_k)_{k\in\mathbb{N}}) = \infty$.

Inductively, we construct numbers $n_k \in \mathbb{N}$ and real numbers $\gamma_1,\dots,\gamma_{n_k}$ such that

(i) $n_k < n_{k+1}, \quad k\in\mathbb{N}$;
(ii) $0 < \gamma_1 \leqslant \dots \leqslant \gamma_{n_k}$;
(iii) for all $0 < r \leqslant \gamma_{n_k} a_{n_k}$ we have

$$\frac{\#\{n : \gamma_n a_n \leqslant r\}}{\lambda_1(r)} \leqslant 1.$$

Moreover, the construction will be carried out in such a way that

(iv) for all $k > 1$ there exists $r_k > 0$ such that $r_k \to \infty$ and

$$\lim_{k\to\infty} \frac{\#\{\gamma_n a_n \leqslant r_k\}}{\lambda_1(r_k)} = 1.$$

For $k = 1$ we put $n_1 := 1$, and let γ_1 be such that $\lambda_1(\gamma_1 a_1) = 1$. Then (i) and (ii) are trivial, and (iii) is fulfilled because $\{\gamma_n a_n \leqslant r\} = \varnothing$ for $r < \gamma_1 a_1$.

We come to the inductive step. Assume that numbers n_k and γ_l satisfying (i)–(iii) are given for all $k \leqslant K$ and $l = 1,\dots,n_K$. Consider the sequence

$$\tilde{a}_n := \begin{cases} \gamma_n a_n & \text{if } n \leqslant n_K, \\ \gamma_{n_K} a_n & \text{if } n > n_K. \end{cases}$$

Consider the function $g(r) := \frac{\#\{\tilde{a}_n \leqslant r\}}{\lambda_1(r)}$. It is continuous from the right, has positive jumps at the points $\tilde{a}_n$, and is monotone decreasing between two successive jumps. By (iii) of the inductive hypothesis,

$$g(\tilde{a}_{n_K}) = \frac{\#\{n : \gamma_n a_n \leqslant \gamma_{n_K} a_{n_K}\}}{\lambda_1(\gamma_{n_K} a_{n_K})} \leqslant 1.$$

Moreover, since $\lambda_1(\gamma_{n_K} r) \asymp \lambda_1(r)$ for any fixed K, we have $\Delta_{\lambda_1}((\tilde{a}_n)_{n\in\mathbb{N}}) = \infty$, i.e., $\limsup_{r\to\infty} g(r) = \infty$. Hence, there exists a smallest number r that satisfies

$$r > \tilde{a}_{n_K} \quad \text{and} \quad g(r) \geqslant 1.$$

Clearly, we must have $r = \tilde{a}_n$ for some $n \in \mathbb{N}$. We define n_{K+1} as this number n. Then $n_{K+1} > n_K$, and (i) is satisfied. Consider the function

$$f(\alpha) := \frac{n_K + \#\{n > n_K : \alpha\tilde{a}_n \leqslant \tilde{a}_{n_{K+1}}\}}{\lambda_1(\tilde{a}_{n_{K+1}})}.$$

This is a monotone nonincreasing step function with jumps of height $\lambda_1(\tilde{a}_{n_{K+1}})^{-1}$. We have $f(1) = g(\tilde{a}_{n_{K+1}}) \geqslant 1$, and

$$f(\alpha) = \frac{n_K}{\lambda_1(\tilde{a}_{n_{K+1}})} \leqslant \frac{n_K}{\lambda_1(\tilde{a}_{n_K})} \leqslant 1$$

whenever $\alpha > \tilde{a}_{n_{K+1}}\tilde{a}^{-1}_{n_K+1}$. Thus, there exists a value $\alpha_0 \in [1, \infty)$ such that $1 \geqslant f(\alpha_0) \geqslant 1 - \lambda_1(\tilde{a}_{n_{K+1}})^{-1}$. We define

$$\gamma_l := \alpha_0\gamma_{n_K}, \quad n_K < l \leqslant n_{K+1}.$$

The requirement (iii) for the values $r \leqslant \gamma_{n_K}$ is fulfilled by the inductive hypothesis. For $\gamma_{n_K} < r \leqslant \gamma_{n_{K+1}}$, this requirement follows from our choice of n_{K+1} and the fact that $\alpha_0 \geqslant 1$.

Finally, put $r_k = \lambda_1(\tilde{a}_{n_{K+1}}) = \lambda_1(\gamma_{n_K} a_{n_{K+1}})$. Then, by our choice of α_0,

$$\frac{\#\{\gamma_n a_n \leqslant r_k\}}{\lambda_1(r_k)} = f(\alpha_0) \geqslant 1 - \lambda_1(\tilde{a}_{n_{K+1}})^{-1},$$

and condition (iv) is satisfied. •

Proof of Theorem 3.1. Consider the Paley–Wiener space $\mathcal{P}W_1 = \mathcal{H}(E)$, where $E(z) := e^{-iz} = \cos z - i \sin z$. This is a de Branges space of finite and positive exponential type, and with $1 \in \operatorname{Assoc} \mathcal{P}W_1$.

We define a function $E_+ = A_+ - iB_+$ by the relations

$$A_+(z^2) = \cos z, \quad B_+(z^2) = z \sin z.$$

Then $E_+ \in \mathcal{H}B$ and $1 \in \operatorname{Assoc} \mathcal{H}(E_+)$, see Remark 2.13(ii). Moreover, clearly, $\mathcal{H}(E_+)$ is of finite and positive $\sqrt{r}$-type, and the zeros of A_+ are precisely the numbers $a_k := \pi^2(k - \frac{1}{2})^2$, $k \in \mathbb{N}$.

The sequence $(a_k)_{k\in\mathbb{N}}$ has $\sqrt{r}$-density $\frac{1}{\pi^2}$. Since we assume that $\lambda(r) = O(r)$, we have $\sqrt{\lambda(r)} = O(\sqrt{r})$. By Lemma 3.4, we can choose a monotone nondecreasing sequence $(\gamma_k)_{k\in\mathbb{N}}$ of positive real numbers so that the sequence $(\gamma_k a_k)_{k\in\mathbb{N}}$ has finite and positive upper $\sqrt{\lambda(r)}$-density.

Clearly, E_+ satisfies the assumptions (a) and (b). Therefore, we may apply Lemma 3.3 to obtain a function $\tilde{E}_+ = \tilde{A}_+ - i\tilde{B}_+ \in \mathcal{H}B$ with $1 \in \operatorname{Assoc}\mathcal{H}(\tilde{E}_+)$, $\tilde{B}_+(0) = 0$, and $\tilde{A}_+(z) = \prod_{k=1}^{\infty}(1 - \frac{z}{\gamma_k a_k})$. By [L, I.Lehrsatz 17], the function $\tilde{A}_+$ has finite and positive $\sqrt{\lambda(r)}$-type.

We choose $\tilde{C}_+, \tilde{D}_+$ as in Remark 2.7 (and satisfying (2.8)), and put

$$\tilde{W}_+ := \begin{pmatrix} \tilde{A}_+ & \tilde{B}_+ \\ \tilde{C}_+ & \tilde{D}_+ \end{pmatrix}.$$

By the properties of the class $\mathcal{M}_0$, we have $\tilde{A}_+ + i\tilde{C}_+ \in \mathcal{H}B$ and $\tilde{C}_+(0) = 0$. Since the zeros of $\tilde{A}_+$ and $\tilde{C}_+$ should interlace and all zeros of $\tilde{A}_+$ are in $(0,\infty)$, it follows that all zeros of $\tilde{C}_+$ are in $[0,\infty)$. We also have $\tilde{D}_+ + i\tilde{C}_+ \in \mathcal{H}B$, and thus the zeros of $\tilde{D}_+$ and $\tilde{C}_+$ interlace. Consequently, there are two possibilities:

(i) all zeros of $\tilde{D}_+$ are in $(0,\infty)$;
(ii) there exists a unique $a < 0$ such that $\tilde{D}_+(a) = 0$.

In the case (i) the function $q = -\frac{\tilde{D}_+}{\tilde{B}_+}$ is in the class $\mathcal{N}_0$, does not vanish on $(-\infty, 0)$, and has a pole at the zero. Hence, $q \in \mathcal{S}$ by Remark 2.12. We conclude that

$$\lim_{x\to-\infty} \frac{\tilde{D}_+(x)}{\tilde{B}_+(x)} \in \mathbb{R}.$$

By Remark 2.13, (iii), this implies that $1 \in \operatorname{Assoc}\mathcal{H}(\tilde{E})$ where $\tilde{E}$ is defined by

$$\tilde{E}(z) := \tilde{A}_+(z^2) - \frac{i}{z}\tilde{B}_+(z^2).$$

In the case (ii) we make use of the transformation $\mathfrak{w}_a$ introduced in Subsection 2, d. Put

$$\hat{W}_+ = \begin{pmatrix} \hat{A}_+ & \hat{B}_+ \\ \hat{C}_+ & \hat{D}_+ \end{pmatrix} := \mathfrak{w}_a(\tilde{W}_+).$$

A computation shows that

$$\hat{W}_+(z) = \begin{pmatrix} -\tilde{B}_+(a)\tilde{C}_+(z+a) & -\tilde{B}_+(a)\tilde{D}_+(z+a) \\ -\tilde{C}_+(a)\tilde{A}_+(z+a)+\tilde{A}_+(a)\tilde{C}_+(z+a) & -\tilde{C}_+(a)\tilde{B}_+(z+a)+\tilde{A}_+(a)\tilde{D}_+(z+a) \end{pmatrix}.$$

Hence, we have

$$\frac{\hat{D}_+(z)}{\hat{B}_+(z)} = \frac{\tilde{C}_+(a)}{\tilde{B}_+(a)}\frac{\tilde{B}_+(z+a)}{\tilde{D}_+(z+a)} - \frac{\tilde{A}_+(a)}{\tilde{B}_+(a)}$$

and

$$-\frac{\hat{A}_+(z)}{\hat{B}_+(z)} = -\frac{\tilde{C}_+(z+a)}{\tilde{D}_+(z+a)}.$$

By Remark 2.12, the functions $-\frac{\hat{A}_+}{\hat{B}_+}$ and $\frac{\tilde{B}_+(z+a)}{\tilde{D}_+(z+a)}$ belong to the Stieltjes class. Thus,

$$\lim_{y\to+\infty} \frac{1}{y} \operatorname{Im} \frac{\hat{D}_+(iy)}{\hat{B}_+(iy)} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{\hat{D}_+(x)}{\hat{B}_+(x)} \in \mathbb{R}.$$

By Remark 2.13, (iii), we have $1 \in \operatorname{Assoc} \mathcal{H}(\hat{E})$, where

$$\hat{E}(z) := \hat{A}_+(z^2) - \frac{i}{z} \hat{B}_+(z^2).$$

Clearly, $\hat{E}$ is of finite and positive λ-type. •

3.5. Remark. By using the construction of Remark 3.2(iii), for λ growing sufficiently slowly, it can be shown that the space $\mathcal{H}$ in Theorem 3.1 can be chosen in such a way that the domain of the multiplication operator in $\mathcal{H}$ is dense in $\mathcal{H}$.

Now, from Theorem 3.1 we can deduce the existence of de Branges spaces with subspaces of smaller growth.

3.6. Theorem. *Let λ, λ_1 be growth functions with*

$$\lambda_1(r) = o(\lambda(r)) \quad \text{and} \quad \lambda(r) = O(r).$$

Then there exists a de Branges space $\mathcal{H}$ of finite and positive λ-type that has a subspace $\mathcal{L} \in \operatorname{Sub} \mathcal{H}$ of finite and positive λ_1-type.

Proof. By Theorem 3.1, we can choose $W, W_1 \in \mathcal{M}_0$ such that $\sigma_{W_1}^{\lambda_1}, \sigma_W^{\lambda} \in (0,\infty)$. Then $\sigma_{W_1}^{\lambda} = 0$ and

$$\sigma_W^{\lambda} \leqslant \sigma_{WW_1}^{\lambda} \leqslant \sigma_W^{\lambda} + \sigma_{W_1}^{\lambda} = \sigma_W^{\lambda},$$

whence

$$\sigma_{\mathfrak{v}(WW_1)}^{\lambda} = \sigma_{WW_1}^{\lambda} = \sigma_W^{\lambda} \in (0,\infty).$$

Consider the de Branges space $\mathcal{H} := \mathcal{H}(E_{\mathfrak{v}(WW_1)})$. Then, as we have just established, $\mathcal{H}$ is of finite and positive λ-type. Since $\mathfrak{v}(WW_1) = \mathfrak{v}(W_1)\mathfrak{v}(W)$, there exists $\mathcal{L} \in \operatorname{Sub} \mathcal{H}$ such that $\mathcal{L} = \mathcal{H}(E_{\mathfrak{v}(W_1)})$ as sets. However,

$$\sigma_{\mathcal{L}}^{\lambda_1} = \sigma_{\mathfrak{v}(W_1)}^{\lambda_1} = \sigma_{W_1}^{\lambda_1} \in (0,\infty). \quad \bullet$$

3.7. Remark. The following statement shows that the condition $\lambda(r) = O(r)$ in Theorem 3.6 is natural. Let λ, λ_1 be growth functions, and assume that

$$\lambda_1(r) = o(\lambda(r)) \quad \text{and} \quad r = o(\lambda(r)).$$

Moreover, let $\mathcal{H}$ be a de Branges space of finite and positive λ-type. Then no nonzero subspace $\mathcal{L} \in \operatorname{Sub} \mathcal{H}$ can be of finite λ_1-type.

To see this, we recall that, by [KW3, Theorem 3.10], if $r = O(\mu(r))$ and a de Branges space contains one nonzero function of finite μ-type, then it is itself of finite μ-type. Applying this with

$$\mu(r) := \max\{\lambda_1(r), r\},$$

yields the desired conclusion.

§4. A condition for nonexistence of subspaces with small growth

In this section we show that the existence of subspaces of a given de Branges space $\mathcal{H}$ that have smaller growth than $\mathcal{H}$ is by no means the generic situation. To be more precise, we show that if $\mathcal{H}$ is written as $\mathcal{H}(E)$ and the function E has maximal admissible growth along the real axis, then the space $\mathcal{H}$ cannot contain infinite-dimensional subspaces of smaller growth.

4.1. Theorem. *Let λ and λ_1 be growth functions with $\lambda_1(r) = o(\lambda(r))$, and let $\mathcal{H}$ be a de Branges space of finite and positive λ-type. Assume that for one (and hence for all) functions $E \in \mathcal{H}B$ with $\mathcal{H} = \mathcal{H}(E)$ we have $\log|E(x)| \asymp \lambda(|x|)$, $x \in \mathbb{R}$. Then no infinite-dimensional subspace $\mathcal{L} \in \operatorname{Sub}\mathcal{H}$ is of finite λ_1-type.*

Proof. Suppose that an infinite-dimensional de Branges subspace $\mathcal{H}(E_1)$ of $\mathcal{H}(E)$ has finite λ_1-type. Then

$$\lim_{|x|\to\infty} |E_1(x)/E(x)| = 0.$$

On the other hand, we have

$$\|F\|_{E_1} = \|F\|_E, \quad F \in \mathcal{H}(E_1). \tag{4.1}$$

Recall that the reproducing kernel $K_1(\zeta, \cdot)$ of the space $\mathcal{H}(E_1)$ corresponding to the point $\zeta \in \mathbb{C}$ is of the form

$$K_1(\zeta, z) = \frac{E(z)E^\#(\overline{\zeta}) - E(\overline{\zeta})E^\#(z)}{2\pi i(\overline{\zeta} - z)}.$$

We shall show that there exists a sequence $x_n \in \mathbb{R}$, $|x_n| \to \infty$, such that

$$\|K_1(x_n, \cdot)\|_E = o(\|K_1(x_n, \cdot)\|_{E_1}), \quad n \to \infty,$$

which contradicts (4.1).

Let $R > 0$ and $x \in \mathbb{R}$. We have

$$\|K_1(x, \cdot)\|_E^2 = \int\limits_{|t|\leqslant R} \Big|\frac{K_1(x,t)}{E(t)}\Big|^2 dt + \int\limits_{|t|>R} \Big|\frac{K_1(x,t)}{E_1(t)}\Big|^2 \Big|\frac{E_1(t)}{E(t)}\Big|^2 dt.$$

Choosing R sufficiently large, we can make the last summand smaller than $\varepsilon\|K_1(x,\cdot)\|^2_{E_1}$ for any given $\varepsilon > 0$ and for all $x \in \mathbb{R}$. Thus, it suffices to show that for a fixed R there exists a sequence $\{x_n\}$ such that

$$\|K_1(x_n,\cdot)\|^{-2}_{E_1} \int\limits_{|t|\leqslant R} \Big|\frac{K_1(x_n,t)}{E(t)}\Big|^2 dt \to 0, \quad n \to \infty.$$

Observe that $|K_1(x,t)| \leqslant \pi^{-1}|t-x|^{-1}|E(x)E(t)|$, and so

$$\int\limits_{|t|\leqslant R} \Big|\frac{K_1(x,t)}{E(t)}\Big|^2 dt \leqslant C(R)|E_1(x)|^2 x^{-2}, \quad |x| > R+1.$$

We also recall that

$$\|K_1(x,\cdot)\|^2_{E_1} = \frac{|E_1(x)|^2\varphi_1'(x)}{2\pi},$$

where φ_1 is the so-called phase function for E_1. Moreover,

$$\varphi_1'(x) = a + \sum_n \frac{|\operatorname{Im} z_n|}{|x-z_n|^2},$$

where the $z_n \in \mathbb{C}^-$ are the zeros of E_1 and $a \geqslant 0$ is the parameter in the factor e^{-iaz} of the standard factorization of the Hermite–Biehler functions ([L]; see also [B1, KW2]).

To complete the proof, we need to show that

$$\liminf_{|x|\to\infty} x^{-2}(\varphi_1'(x))^{-1} = 0. \tag{4.2}$$

If $a > 0$, then (4.2) is trivial. Otherwise, E_1 has infinitely many zeros, because we assumed the subspace $\mathcal{H}(E_1)$ to be infinite dimensional. Put $x_n = \operatorname{Re} z_n + \operatorname{sign}(\operatorname{Re} z_n)|\operatorname{Im} z_n|$. Then $|x_n| \to \infty$ and

$$\varphi_1'(x_n) \geqslant |2\operatorname{Im} z_n|^{-1} \geqslant |2x_n|^{-1}.$$

Thus, we have $x_n^2\varphi'(x_n) \to \infty$, which proves (4.2). •

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Поступило 22 апреля 2006 г.

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