# De Branges spaces of exponential type: General theory of growth

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**Abstract**

A de Branges space is a reproducing kernel Hilbert space of entire functions which satisfies additional axioms. We consider such de Branges spaces whose elements possess a certain growth behaviour (e.g. are all of exponential type) and investigate the interplay of Hilbert space structure and growth behaviour. We characterize the presence of certain growth behaviour, prove the existence of spaces with prescribed growth and investigate the structure of subspaces defined by growth restrictions.

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## 1 Introduction

A de Branges Hilbert space (dB-space for short) is a Hilbert space whose elements are *entire functions* and which satisfies certain axioms (for the exact definition we refer to Definition 2.1). Such spaces arise for example by M.G.Krein's method of directing functionals, in which case all elements of the so obtained dB-spaces are of *exponential type*. This method finds various applications, e.g. to interpolation- or extrapolation type problems. Another source is the investigation of canonical systems of differential equations, where the connection to the theory of dB-spaces plays a prominent role. All elements of a dB-space turning up that way are of *exponential type*. Hence, it seems that the class of dB-spaces $\mathcal{H}$ whose elements $F$ satisfy the *growth condition*

$$\limsup_{|z|\to\infty} \frac{\log|F(z)|}{|z|} < \infty,$$

is an object worth being investigated. Indeed, a discussion of the interplay of the dB-space structure on the one hand and growth conditions imposed on the elements on the other hand leads to a fruitful theory.

The aim of the present work is to give some results on dB-spaces $\mathcal{H}$ subject to growth conditions of the type

$$\limsup_{|z|\to\infty} \frac{\log|F(z)|}{\lambda(|z|)} < \infty,\ F\in\mathcal{H}, \tag{1.1}$$

where $\lambda$ is any *growth function*, e.g. one might think of $\lambda(r)=r$, of $\lambda(r)=r^\rho$ for some $\rho>0$, or, even more general, of $\lambda(r)=r^\rho(\log r)^\alpha$ for some $\alpha\in\mathbb{R}$,

$\rho > 0$. The fact that we consider this more general setting is motivated, besides the undoubtedly big gain in generality, by the study of dB-spaces connected to some functions of classical analysis which satisfy the growth condition (1.1) with $\lambda(r) = \sqrt{r}\log r$.

We give an outline the contents of the following sections. In order not to overload this introduction consider the respective sections for the exact definitions of the terms used here to explain the results.

In §2 we recall some basic facts about dB-spaces and give a result on the structure of a given space (Theorem 2.7) which is not connected to any growth conditions but will often be used later on. It says that investigating the structure of a space $\mathcal{H} = \mathcal{H}(E)$ where $E$ is a function of Hermite-Biehler class (cf. Definition 2.1 and the paragraph following it) one can often restrict to the case that

$$\operatorname{mt}\frac{E^\#}{E} = \lim_{y\to+\infty}\frac{1}{y}\log\Big|\frac{E(-iy)}{E(iy)}\Big| = 0.$$

In the consecutive section we define the main objects of our considerations, dB-spaces of finite $\lambda$-type (understanding that (1.1) holds for each element of the space, cf. Definition 3.2), and state three basic results which give conditions for a space to be of finite $\lambda$-type. These are: Theorem 3.4, which says that the growth of any function of the space $\mathcal{H} = \mathcal{H}(E)$ is governed by the growth of $E$. Secondly, Theorem 3.10 asserting that if $\lambda$ grows comparatively fast ($r = O(\lambda(r))$), the existence of a single element of a dB-space $\mathcal{H}$ satisfying the finiteness condition (1.1), implies that also all other elements have this property. Finally Theorem 3.17 which characterizes growth behaviour by means of asymptotic distribution of zeros. This result can be viewed as a link to the spectral theory of the operator of multiplication by the independent variable in the space $\mathcal{H}(E)$. We also give a product representation of functions of Hermite-Biehler class of finite order (a particular form of the Hadamard product, cf. Lemma 3.12) which is not only of great help in many proofs, but also explains that the structure of the space $\mathcal{H}(E)$ must be in an intimate connection with the distribution of zeros of $E$. The section is concluded with a discussion of the indicator function of the elements of $\mathcal{H}(E)$ and with some results on slowly growing functions. In the considerations of this section we encounter the phenomenon that the cases of slowly growing and fast growing $\lambda$ (e.g. think of $\lambda(r) = o(r)$ vs. $r = O(\lambda(r))$) are essentially different from each other and that the case of exponential type ($\lambda(r) = r$) is somewhat special. This fact will also show up in various other places.

The fourth section is mainly concerned with functions $E$ of Hermite-Biehler class which are of completely regular growth. For such functions $E$ the indicator function can be determined explicitly. Conversely, to a prescribed indicator there exist (in the generic case) a function $E$ possessing this indicator function (cf. Theorem 4.3 and Theorem 4.5). These considerations not only give an insight into the growth behaviour of functions of Hermite-Biehler class, but also provides us with a rich variety of examples. Also in this context it is interesting to note how different the behaviour of fast and slow growing $\lambda$ is (cf. case (B) versus all other cases of the above mentioned theorems), and to observe that the case of exponential type (i.e. $\lambda(r) = r$) plays a special role. The reader will recognize this when going through the proof of Theorem 4.5, Case (D).

In §5 we study dB-subspaces of a given dB-space defined by growth conditions. On the first sight there are two natural ways to define such subspaces: One can restrict the growth of $F \in \mathcal{H}(E)$ by either demanding a bound for the growth of $F$ itself or for the growth of $F$ in comparison to $E$. It turns out that the second method is more general. It is a main task of our considerations to find conditions when such growth restrictions give nontrivial (i.e. nonzero) subspaces.

Finally let us make two remarks. Firstly, all considerations of this paper remain valid in the case of dB-Pontryagin spaces (as introduced in [KW2]). This origins in the fact that, as the reader will surely recognize, our arguments employ mainly function theoretic methods and rely on the topology of the space $\mathcal{H}$, but not on the particular inner product. Since a dB-Pontryagin space can be turned into a dB-Hilbert space by a finite dimensional perturbation of the inner product (cf. [KW2, Theorem 3.3]), which clearly does not change the topology of the space, the same arguments will apply. Secondly, dB-spaces are intimitely related to certain operator theoretic concepts like the theory of entire operators or the extension theory of symmetric operators. Growth restrictions on the elements of the space mean restrictions on the asymptotics of the zeros of these functions. Many spectral properties of the related operator model can be expressed in terms of zeros of certain elements of the dB-space, and thus are highly influenced by growth restrictions. As an illustration of this principle let us mention the Q-function of the multiplication operator in the dB-space, in which case this connection is very explicitly given via [KW2, §6]. In the present work we will, however, concentrate on the function theoretic viewpoint. A discussion of connections to these more operator theoretic notions and topics will be subject of forthcoming work.

For the convenience of the reader we tried to recall at least the definitions of the employed notions from either the theory of functions or the theory of dB-spaces. However, it is of course inevitable to refer to lots of results of both of these fields. Our standard references in this respect are [L], [R] and [dB].

# 2 Some preliminary remarks on dB-spaces

Let us give the definition of a dB-space.

**2.1 Definition.** A Hilbert space $(\mathcal{H}, (.,.))$ is called a *dB-(Hilbert) space* if it satisfies the following axioms:

**(dB1)** The elements of $\mathcal{H}$ are entire functions and for each $w \in \mathbb{C}$ the point evaluation $F \mapsto F(w)$ is a continuous linear functional on $\mathcal{H}$.

**(dB2)** If $F \in \mathcal{H}$, also $F^{\#}(z) := \overline{F(\overline{z})}$ belongs to $\mathcal{H}$ and we have

$$(F^{\#}, G^{\#}) = (G, F),\ \ F, G \in \mathcal{H}.$$

**(dB3)** If $w \in \mathbb{C} \setminus \mathbb{R}$ and $F \in \mathcal{H}$, $F(w) = 0$, then $\frac{z-\overline{w}}{z-w}F(z) \in \mathcal{H}$, and

$$\left(\frac{z-\overline{w}}{z-w}F(z), \frac{z-\overline{w}}{z-w}G(z)\right) = \big(F, G\big),\ \ F, G \in \mathcal{H},\ F(w) = G(w) = 0.$$

Recall from [dB] that such spaces can be constructed from certain entire functions. The *Hermite-Biehler class* $\mathcal{H}B$ is defined as the set of all entire functions $E$ which have no zeros in the open upper half plane $\mathbb{C}^+$ and satisfy

$$|E(\overline{z})| \leq |E(z)|, \ z \in \mathbb{C}^+.$$

We denote by $\mathcal{H}B^\times$ the set of all functions in $\mathcal{H}B$ which have no zeros on $\mathbb{R}$. The notation $N(\mathbb{C}^+)$ is used for the ring of all analytic functions on $\mathbb{C}^+$ which can be represented as a quotient of two bounded analytic functions and $\tilde{N}(\mathbb{C}^+)$ will denote its quotient field. We refer to the functions in $N(\mathbb{C}^+)$ as functions of *bounded type*, and to those in $\tilde{N}(\mathbb{C}^+)$ of *bounded characteristic*. If $f \in N(\mathbb{C}^+)$ the number

$$\operatorname{mt} f := \limsup_{y\to+\infty} \frac{1}{y} \log |f(iy)|$$

is finite and will be referred to as the *mean type* of $f$.

Let us recall that the Hardy space $H^2(\mathbb{C}^+)$ is defined to be the set of all functions $f$ analytic in $\mathbb{C}^+$ such that

$$\sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^2 \, dx < \infty.$$

The space $H^2(\mathbb{C}^+)$ is endowed with the norm $\|f\|_{H^2(\mathbb{C}^+)} = \|f(t)\|_{L^2(\mathbb{R})}$ where $f(t)$ denotes the (nontangentially almost everywhere existing) boundary function of $f$. Furthermore, $N^+(\mathbb{C}^+)$ will denote the set of all functions of bounded type such that in the inner-outer factorization the singular inner function in the denominator is not present (cf. e.g. [RR]) and $H^\infty(\mathbb{C}^+)$ those functions analytic on $\mathbb{C}^+$ which are bounded throughout this half plane. We have

$$N(\mathbb{C}^+) \supseteq N^+(\mathbb{C}^+) \supseteq H^2(\mathbb{C}^+), H^\infty(\mathbb{C}^+).$$

In case $f \in N^+(\mathbb{C}^+)$ the mean type of $f$ is nonpositive. If, moreover, the boundary function of $f$ on $\mathbb{R}$ is contained in $L^2(\mathbb{R})$ ($L^\infty(\mathbb{R})$), then $f \in H^2(\mathbb{C}^+)$ ($f \in H^\infty(\mathbb{C}^+)$). This fact can be seen as a kind of maximal principle (cf. e.g. [RR]).

For a function $f$ of bounded type with a continuous continuation to the real axis also the converse is true: If $f$ has a nonpositive mean type, then $f \in N^+(\mathbb{C}^+)$. This can be deduced from the Stieltjes inversion formula (cf. e.g. [RR]).

If $E \in \mathcal{H}B$ is given, the set $\mathcal{H}(E)$ is defined as follows: $F$ belongs to $\mathcal{H}(E)$ if and only if $F$ is an entire function, $\frac{F}{E}$ and $\frac{F^\#}{E}$ are of bounded type and nonpositive mean type in $\mathbb{C}^+$, and

$$\int_{-\infty}^{\infty} \Big|\frac{F(t)}{E(t)}\Big|^2 dt < \infty.$$

Endowed with the inner product

$$(F,G) = \int_{-\infty}^{\infty} F(t)\overline{G(t)} \, \frac{dt}{|E(t)|^2}, \ F,G \in \mathcal{H}(E),$$

$\mathcal{H}(E)$ is a dB-space. Conversely, every dB-space is obtained in this way. Note that, if $t \in \mathbb{R}$ is a zero of $E$ of multiplicity $r$, then $F(t) = 0$ with multiplicity at least $r$ for all $F \in \mathcal{H}(E)$.

The reproducing kernel $K(w,z)$ of the space $\mathcal{H} = \mathcal{H}(E)$ can be expressed in terms of the function $E$ as follows:

$$K(w,z) = \frac{E(z)E^{\#}(\overline{w}) - E(\overline{w})E^{\#}(z)}{2\pi i(\overline{w} - z)}, \ z \neq \bar{w},$$

$$K(\bar{z}, z) = \frac{i}{2\pi}(E'(z)E^{\#}(z) - E(z)E^{\#}(z)').$$

It also can be written using the real and imaginary parts of $E$. Put

$$A := \frac{E + E^{\#}}{2}, \ B := i\frac{E - E^{\#}}{2},$$

so that $E = A - iB$. For $\phi \in \mathbb{R}$ define

$$S_\phi := \sin\phi \cdot A(z) - \cos\phi \cdot B(z),$$

then independently of $\phi \in \mathbb{R}$,

$$\begin{aligned} K(w,z) &= \frac{S_\phi(\overline{w})S_{\phi+\frac{\pi}{2}}(z) - S_\phi(z)S_{\phi+\frac{\pi}{2}}(\overline{w})}{\pi(z - \overline{w})}, \ z \neq \overline{w}, \\ K(\overline{z}, z) &= \frac{1}{\pi}\big(S_\phi(z)S'_{\phi+\frac{\pi}{2}}(z) - S'_\phi(z)S_{\phi+\frac{\pi}{2}}(z)\big). \end{aligned} \tag{2.1}$$

The space $\mathcal{H}(E)$ also can be defined as

$$\mathcal{H}(E) = \Big\{F \text{ entire } : \frac{F}{E}, \frac{F^{\#}}{E} \in H^2(\mathbb{C}^+)\Big\}, \ \|F\|_{\mathcal{H}(E)} = \|\frac{F}{E}\|_{H^2(\mathbb{C}^+)},$$

or as

$$\mathcal{H}(E) = E(H^2(\mathbb{C}^+) \ominus \frac{E^{\#}}{E}H^2(\mathbb{C}^+)).$$

If $\mathcal{H}$ is a dB-space, we will denote by $\operatorname{Assoc}\mathcal{H}$ the set of all entire functions $F$ such that for some $G \in \mathcal{H}$ and $w \in \mathbb{C}$, $G(w) \neq 0$,

$$\frac{F(z)G(w) - F(w)G(z)}{z - w} \in \mathcal{H}.$$

Recall that (cf. [dB, Theorem 25], compare also [LW, Corollary 3.4]) for $E \in \mathcal{H}B$ such that $\mathcal{H} = \mathcal{H}(E)$

$$\operatorname{Assoc}\mathcal{H} = \Big\{F \text{ entire } : \frac{F(z)}{(z+i)E(z)}, \frac{F^{\#}(z)}{(z+i)E(z)} \in H^2(\mathbb{C}^+)\Big\}.$$

It is easily seen from the definitions that $\operatorname{Assoc}\mathcal{H} = \mathcal{H} + z\mathcal{H}$. Moreover let us note that, if $F \in \operatorname{Assoc}\mathcal{H}$ and $F(w) = 0$, then $(z-w)^{-1}F(z) \in \mathcal{H}$, and that $E \in \operatorname{Assoc}\mathcal{H}(E) \setminus \mathcal{H}(E)$.

A subspace $\mathcal{L}$ of a dB-space $\mathcal{H}$ will be called a dB-subspace if it is itself, with the inner product inherited from $\mathcal{H}$, a dB-space. Let us recall that by de Branges' ordering theorem the set of all dB-subspaces $\mathcal{H}(\tilde{E})$ of a given space $\mathcal{H}(E)$ where $\tilde{E}$ has the same real zeros (including multiplicities) as $E$ is totally ordered (with respect to set-theoretic inclusion). Moreover, let us recall that for each such dB-subspace $\mathcal{H}(\tilde{E}) \subseteq \mathcal{H}(E)$ there exists a transfer matrix $M(z)$

which has the property that (write $E = A - iB$ and $\tilde{E} = \tilde{A} - i\tilde{B}$ with $A = A^\#$, $B = B^\#$, etc.)

$$(A, B) = (\tilde{A}, \tilde{B})\, M(z).$$

This transfer matrix measures the difference between $\mathcal{H}(\tilde{E})$ and $\mathcal{H}(E)$.

It is a result of [dB] that if $\mathcal{H}(E_t)$ is an increasing sequence of dB-subspaces of $\mathcal{H}(E)$ and $M_t$ denotes the transfer matrix of $\mathcal{H}(E_t) \subseteq \mathcal{H}(E)$, then

$$\overline{\bigcup_t \mathcal{H}(E_t)} = \mathcal{H}(E)$$

if and only if $\lim M_t = 1$. Dually, if $\mathcal{H}(E_t)$ is a decreasing sequence of dB-subspaces, $\mathcal{H}(\tilde{E}) \subseteq \bigcap_t \mathcal{H}(E_t)$, and $\tilde{M}_t$ denotes the transfer matrix of $\mathcal{H}(\tilde{E}) \subseteq \mathcal{H}(E_t)$, then $\mathcal{H}(\tilde{E}) = \bigcap_t \mathcal{H}(E_t)$ if and only if $\lim \tilde{M}_t = 1$.

According to [L, Kapitel V] we denote by $\mathbb{A}$ the set of all entire functions $F$ whose zeros are close to the real axis in the sense that

$$\sum_{n\in\mathbb{N}} \Big|\operatorname{Im} \frac{1}{z_n}\Big| < \infty, \tag{2.2}$$

where $(z_n)_{n\in\mathbb{N}}$ denotes the sequence of (nonzero) zeros of $F$ counted according to their multiplicities. Note that, if all points $z_n$ are contained in the half plane $\mathbb{C}^+$ (or $\mathbb{C}^-$, respectively), then (2.2) is exactly the condition insuring the convergence of the Blaschke product (see e.g. [RR])

$$B(z) := \prod_{n\in\mathbb{N}} \frac{z - z_n}{z - \overline{z_n}},$$

hence is also frequently called the Blaschke condition. For later reference let us state some elementary properties of complex sequences.

**2.2 Lemma.** *Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of nonzero complex numbers which does not accumulate at* $0$. *If $(z_n)_{n\in\mathbb{N}}$ satisfies* (2.2), *then:*

*(i)* *For all $k \in \mathbb{N}$*

$$\sum_{n\in\mathbb{N}} \Big|\operatorname{Im} \frac{1}{z_n^k}\Big| < \infty. \tag{2.3}$$

*(ii)* *If for some $\delta > 0$ all points $z_n$ lie in the region*

$$\big\{w \in \mathbb{C} : \arg w \in [-\pi + \delta, -\delta] \cup [\delta, \pi - \delta]\big\}, \tag{2.4}$$

*then $\sum_{n\in\mathbb{N}} \frac{1}{|z_n|} < \infty$.*

*(iii)* *For all $k \in \mathbb{N}$*

$$\sum_{|z_n|\le r} \operatorname{Re} \frac{1}{z_n^k} = \sum_{|z_n|\le r} \left(\operatorname{Re} \frac{1}{z_n}\right)^k + O(1).$$

*Proof.*

$\mathtt{ad}\,(i)$: Write $z_n = r_n e^{i\psi_n}$, then

$$\operatorname{Im}\frac{1}{z_n^k} = \frac{\sin(k\psi_n)}{r_n^k} = \frac{\sin\psi_n}{r_n}\cdot\frac{i\sum_{l=0,l\,\text{odd}}^k \binom{k}{l}(-1)^{\frac{l-1}{2}}\sin^{l-1}\psi_n\cos^{k-l}\psi_n}{r_n^{k-1}}.$$

Since the second factor is uniformly bounded with respect to $n$, the convergence of (2.3) follows.

$\mathtt{ad}\,(ii)$: This assertion follows since in the angle (2.4) the estimate $|\operatorname{Re} w| \leq \cot\delta\cdot|\operatorname{Im} w|$ holds.

$\mathtt{ad}\,(iii)$: Similar as in $(i)$

$$\operatorname{Re}\frac{1}{z_n^k} = \frac{\cos(k\psi_n)}{r_n^k} = \frac{(\cos\psi_n)^k}{r_n^k} + \frac{\sin\psi_n}{r_n}\cdot\frac{\sum_{l=2,l\,\text{even}}^k \binom{k}{l}(-1)^{\frac{l}{2}}\sin^{l-1}\psi_n\cos^{k-l}\psi_n}{r_n^{k-1}}.$$

The second factor of the second summand is again bounded uniformly with respect to $n$, and hence by (2.2) the series

$$\sum_{n\in\mathbb{N}}\frac{\sin\psi_n}{r_n}\cdot\frac{\sum_{l=2,l\,\text{even}}^k \binom{k}{l}(-1)^{\frac{l}{2}}\sin^{l-1}\psi_n\cos^{k-l}\psi_n}{r_n^{k-1}}$$

is absolutely convergent.

□

Let us state the following (well known) fact which gives a connection with the theory of dB-spaces.

**2.3 Lemma.** *Let $\mathcal{H}$ be a dB-space and $F \in \operatorname{Assoc}\mathcal{H}$, $F \not\equiv 0$. Then $F$ is of class $\mathbb{A}$. If $(z_n)_{n\in\mathbb{N}}$, $z_n \in \mathbb{C}^+$, is any sequence of zeros of $F$ and $B(z)$ denotes the Blaschke product formed with the sequence $(z_n)_{n\in\mathbb{N}}$, then $\frac{F(z)}{B(z)} \in \operatorname{Assoc}\mathcal{H}$. If $F \in \mathcal{H}$, then $\frac{F(z)}{B(z)} \in \mathcal{H}$. The same conclusions hold if all $z_n$ belong to the lower half plane.*

*Proof.* Choose $E \in \mathcal{H}B$ such that $\mathcal{H} = \mathcal{H}(E)$. Since $\frac{F}{E}$ and $\frac{F^\#}{E}$ are of bounded type in $\mathbb{C}^+$, both satisfy the Blaschke condition. Thus $F \in \mathbb{A}$.

In order to establish the second assertion it suffices to prove that $F \in \mathcal{H}$ implies $\frac{F}{B} \in \mathcal{H}$. Let

$$\frac{F(z)}{E(z)} = B_1(z)S_1(z)U_1(z),\ \frac{F^\#(z)}{E(z)} = B_2(z)S_2(z)U_2(z)$$

be the inner-outer factorization of the elements $\frac{F}{E}, \frac{F^\#}{E} \in H^2(\mathbb{C}^+)$ (cf. [RR]). Put $G(z) := \frac{F(z)}{B(z)}$, then

$$\frac{G(z)}{E(z)} = \frac{B_1(z)}{B(z)}S_1(z)U_1(z),\ \frac{G^\#(z)}{E(z)} = B(z)B_2(z)S_2(z)U_2(z),$$

and hence both functions belong to $H^2(\mathbb{C}^+)$.

□

The following elementary construction is useful.

**2.4 Lemma.** *Let $E \in \mathcal{H}B$, $C$ entire, and put $E_1 := \frac{E}{C}$. Assume that $E_1 \in \mathcal{H}B$ and $\frac{C^\#}{C} \in N^+(\mathbb{C}^+)$. Then $C\mathcal{H}(E_1) \subseteq \mathcal{H}(E)$ isometrically. If $C = C^\#$, then equality holds.*

*Proof.* Since $|\frac{C^\#(t)}{C(t)}| = 1$, $t \in \mathbb{R}$, we conclude that in fact $\frac{C^\#}{C}$ is inner. Let $F \in \mathcal{H}(E_1)$, i.e. $\frac{F}{E_1}, \frac{F^\#}{E_1} \in H^2(\mathbb{C}^+)$. Then $G := CF$ satisfies

$$\frac{G}{E} = \frac{CF}{CE_1} = \frac{F}{E_1} \in H^2(\mathbb{C}^+), \ \frac{G^\#}{E} = \frac{C^\#}{C}\frac{F^\#}{E_1} \in H^2(\mathbb{C}^+). \tag{2.5}$$

Moreover,

$$\|G\|_{\mathcal{H}(E)} = \int_{-\infty}^{\infty} \left|\frac{G(t)}{E(t)}\right|^2 dt = \int_{-\infty}^{\infty} \left|\frac{F(t)}{E_1(t)}\right|^2 dt = \|G\|_{\mathcal{H}(E_1)}.$$

Assume that $C = C^\#$. Since $\frac{E}{C}$ is entire, $C$ can have only real zeros which are also zeros of $E$. Thus, with $G \in \mathcal{H}(E)$ the function $F := \frac{G}{C}$ is entire. Reversing the argument (2.5) yields $F \in \mathcal{H}(E_1)$.

□

The above lemma yields in particular that if $E_1, E_2 \in \mathcal{H}B$, then $E_1\mathcal{H}(E_2) \subseteq \mathcal{H}(E_1E_2)$. The following result provides a more detailed description of the space $\mathcal{H}(E_1E_2)$.

**2.5 Lemma.** *Let $E_1, E_2 \in \mathcal{H}B$. Then*

$$\mathcal{H}(E_1E_2) = E_1\mathcal{H}(E_2) \oplus E_2^\#\mathcal{H}(E_1). \tag{2.6}$$

*Proof.* We calculate for the reproducing kernel of $\mathcal{H}(E_1E_2)$

$$K_{E_1E_2}(w,z) = \frac{E_1(z)E_2(z)E_1^\#(\overline{w})E_2^\#(\overline{w}) - E_1(\overline{w})E_2(\overline{w})E_1^\#(z)E_2^\#(z)}{2\pi i(\overline{w}-z)} =$$

$$E_1(z)K_{E_2}(w,z)\overline{E_1(w)} + E_2^\#(z)K_{E_1}(w,z)\overline{E_2^\#(w)}.$$

By a standard complementation theory argument (For an overview see for example [ADSR], Chapter 1.) the reproducing kernel Hilbert spaces $E_1\mathcal{H}(E_2)$ and $E_2^\#\mathcal{H}(E_1)$ with their respective reproducing kernels $E_1(z)K_{E_2}(w,z)\overline{E_1(w)}$ and $\overline{E_1(w)} + E_2^\#(z)K_{E_1}(w,z)\overline{E_2^\#(w)}$ are contained contractively in $\mathcal{H}(E_1E_2)$. Since by Lemma 2.4 the space $E_1\mathcal{H}(E_2)$ is contained isometrically in $\mathcal{H}(E_1E_2)$, the same is true for $E_2^\#\mathcal{H}(E_1)$ and (2.6) holds (see [ADSR], Chapter 1).

□

The application of Lemma 2.4 will be of special interest in three particular cases: $C(z)$ is real and zerofree; $C(z)$ is the canonical product associated with the real zeros of $E$; $C(z) = e^{-icz}$, $c \geq 0$. The first case tells us that two spaces $\mathcal{H}(E)$ and $\mathcal{H}(E_1)$ such that $\frac{E}{E_1}$ is real and zerofree behave the same with respect to Hilbert space theory. The second shows that in many cases we can restrict our considerations to spaces $\mathcal{H}(E)$ with $E(t) \neq 0$ for all real $t$, i.e. $E \in \mathcal{H}B^\times$. The significance of the third possibility will be apparent from the subsequent discussion. First let us make the following observation:

**2.6 Lemma.** *Let $\mathcal{H}$ be a dB-space and let numbers $\tau_+, \tau_- \le 0$ and $\sigma \in [\tau_-, -\tau_+]$ be given. The mapping $\psi : F(z) \mapsto e^{-i\sigma z} F(z)$ is an isometry of*

$$\mathcal{H}_{(\tau_+,\tau_-)} := \left\{ F \in \mathcal{H} : \operatorname{mt} \frac{F}{E} \le \tau_+, \operatorname{mt} \frac{F^\#}{E} \le \tau_- \right\}$$

*onto*

$$\mathcal{H}_{(\tau_+ + \sigma, \tau_- - \sigma)} = \left\{ F \in \mathcal{H} : \operatorname{mt} \frac{F}{E} \le \tau_+ + \sigma, \operatorname{mt} \frac{F^\#}{E} \le \tau_- - \sigma \right\}.$$

*For all $\tau_+, \tau_- \le 0$ the space $\mathcal{H}_{(\tau_+,\tau_-)}$ is closed.*

*Proof.* Clearly

$$\int_{-\infty}^{\infty} \left| \frac{(\psi F)(t)}{E(t)} \right|^2 dt = \int_{-\infty}^{\infty} \left| \frac{F(t)}{E(t)} \right|^2 dt. \tag{2.7}$$

If $F \in \mathcal{H}$, $\operatorname{mt} \frac{F}{E} \le \tau_+$ and $\operatorname{mt} \frac{F^\#}{E} \le \tau_-$, then $\frac{\psi F}{E}$ is of bounded type and

$$\operatorname{mt} \frac{\psi F}{E} \le \tau_+ + \sigma \le 0.$$

By (2.7) it follows that $\frac{\psi F}{E} \in H^2(\mathbb{C}^+)$. Since $\operatorname{mt} \frac{(\psi F)^\#}{E} \le \tau_- - \sigma \le 0$, similarly $\frac{(\psi F)^\#}{E} \in H^2(\mathbb{C}^+)$. Thus $\psi \mathcal{H}_{(\tau_+,\tau_-)} \subseteq \mathcal{H}_{(\tau_+ + \sigma, \tau_- - \sigma)}$.

Applying this fact once again with $-\sigma$, $\tau_+ + \sigma$ and $\tau_- - \sigma$ in place of $\sigma$, $\tau_+$ and $\tau_-$, yields

$$\psi^{-1} \mathcal{H}_{(\tau_+ + \sigma, \tau_- - \sigma)} \subseteq \mathcal{H}_{(\tau_+,\tau_-)}.$$

To prove the last assertion it suffices to show that for all $\tau_+ \le 0$ the space $\mathcal{H}_{(\tau_+,0)}$ is closed. The mapping (compare [Ka, Lemma 3.2]):

$$\psi : F(z) \mapsto e^{i\tau_+ z} F(z)$$

is an isometry of $\mathcal{H}_{(\tau_+,0)}$ into $\mathcal{H}$, hence can be extended by continuity to an isometry $\psi : \overline{\mathcal{H}_{(\tau_+,0)}} \to \mathcal{H}$. Since point evaluation is continuous, for each $F \in \overline{\mathcal{H}_{(\tau_+,0)}}$ we have $(\psi F)(z) = e^{i\tau_+ z} F(z)$, in particular $e^{i\tau_+ z} F(z) \in \mathcal{H}$. Thus

$$\operatorname{mt} \frac{F(z)}{E(z)} = \operatorname{mt} \left[ e^{-i\tau_+ z} \cdot \frac{e^{i\tau_+ z} F(z)}{E(z)} \right] \le \tau_+,$$

and we obtain $F \in \mathcal{H}_{(\tau_+,0)}$.

$\square$

Standard examples of dB-spaces are the so-called *Paley-Wiener spaces*. A Paley-Wiener space is defined as $\mathcal{H}(e^{-iaz})$ for $a > 0$ . The structure of such spaces is well understood. For example the chain of dB-subspaces of $\mathcal{H}(e^{-iaz})$ is given by $\{\mathcal{H}(e^{-itz}) : 0 \le t \le a\}$.

**2.7 Theorem.** *Let $\mathcal{H} = \mathcal{H}(E)$ be a dB-space. Put $\tau_E := \operatorname{mt} \frac{E^\#}{E}$ and $E_\tau(z) := E(z) e^{-i\frac{\tau}{2} z}$. We have $E_\tau \in \mathcal{H}B$ if and only if $\tau \ge \tau_E$.*

*(i) For each $\tau \in [\tau_E, 0]$ the space $\mathcal{H}(E)$ can be decomposed as*

$$\mathcal{H}(E) = E_\tau \mathcal{H}(e^{i\frac{\tau}{2} z}) \oplus e^{-i\frac{\tau}{2} z} \mathcal{H}(E_\tau).$$

*(ii)* *For each $\tau \in [\tau_E, 0]$ the space $\mathcal{H}(E_\tau)$ is a dB-subspace of $\mathcal{H}(E)$. In fact, $\mathcal{H}(E_\tau) = \mathcal{H}_{(\frac{\tau}{2},\frac{\tau}{2})}$. The interval $[\mathcal{H}(E_{\tau_E}), \mathcal{H}(E)]$ in the chain of all dB-subspaces $\mathcal{H}(\tilde{E})$ of $\mathcal{H}(E)$ where $\tilde{E}$ has the same real zeros as $E$, is given by*

$$[\mathcal{H}(E_{\tau_E}), \mathcal{H}(E)] = \big\{\mathcal{H}(E_\tau) : \tau \in [\tau_E, 0]\big\}.$$

*Proof.* From $\tau \geq \operatorname{mt} \frac{E^\#}{E}$ we conclude that $\frac{E_\tau^\#}{E_\tau} \in N^+(\mathbb{C}^+)$. Since $|\frac{E_\tau^\#(t)}{E_\tau}(t)| = 1$ for $t \in \mathbb{R}$, this function is inner, i.e. $E_\tau$ belongs to $\mathcal{H}B$. Of course for $\tau < \tau_E$ the function $E_\tau$ does not belong to $\mathcal{H}B$ since in this case

$$\operatorname{mt} \frac{E_\tau^\#}{E_\tau} = \operatorname{mt} \frac{E^\#}{E} - \tau > 0.$$

`ad`$(i)$: This assertion is an immediate consequence of Lemma 2.5.

`ad`$(ii)$: Let $F \in \mathcal{H}(E_\tau)$. Then $\frac{F}{E_\tau}, \frac{F^\#}{E_\tau} \in H^2(\mathbb{C}^+)$ and therefore

$$\frac{F(z)}{E(z)} = \frac{F(z)}{E_\tau(z)} \cdot e^{-i\frac{\tau}{2}z} \in H^2(\mathbb{C}^+),\ \frac{F^\#(z)}{E(z)} = \frac{F^\#(z)}{E_\tau(z)} \cdot e^{-i\frac{\tau}{2}z} \in H^2(\mathbb{C}^+),$$

and $\operatorname{mt} \frac{F}{E}, \operatorname{mt} \frac{F^\#}{E} \leq \frac{\tau}{2}$. Since $|E_\tau(t)| = |E(t)|$, $t \in \mathbb{R}$, the $L_2$-conditions are trivially satisfied, i.e. we obtain

$$\mathcal{H}(E_\tau) \subseteq \Big\{F \in \mathcal{H}(E) : \operatorname{mt} \frac{F}{E}, \operatorname{mt} \frac{F^\#}{E} \leq \frac{\tau}{2}\Big\} = \mathcal{H}_{(\frac{\tau}{2},\frac{\tau}{2})}.$$

Thus by Lemma 2.6

$$e^{-i\frac{\tau}{2}z}\mathcal{H}(E_\tau) \subseteq \mathcal{H}_{(\tau,0)} = e^{-i\frac{\tau}{2}z}\mathcal{H}_{(\frac{\tau}{2},\frac{\tau}{2})}. \tag{2.8}$$

If $F$ is contained in $\mathcal{H}_{(\tau,0)}$, then with $\frac{F}{E}, \frac{F^\#}{E}$ also $\frac{e^{i\frac{\tau}{2}z}F}{E_\tau}, \frac{(e^{i\frac{\tau}{2}z}F)^\#}{E_\tau}$ satisfy the $L_2$-condition and are of bounded type. Moreover,

$$\operatorname{mt} \frac{e^{i\frac{\tau}{2}z}F}{E_\tau} = \operatorname{mt} \frac{F}{E} - \tau \leq 0, \quad \operatorname{mt} \frac{(e^{i\frac{\tau}{2}z}F)^\#}{E_\tau} = \frac{F^\#}{E} \leq 0.$$

Thus $e^{i\frac{\tau}{2}z}F \in \mathcal{H}(E_\tau)$, and equality holds in (2.8).

So far we have proved that $\mathcal{H}(E_\tau) = \mathcal{H}_{(\frac{\tau}{2},\frac{\tau}{2})}$ is a dB-subspace of $\mathcal{H}$, i.e.

$$[\mathcal{H}(E_{\tau_E}), \mathcal{H}(E)] \supseteq \big\{\mathcal{H}(E_\tau) : \tau \in [\tau_E, 0]\big\}.$$

To complete the proof it is therefore sufficient to show that for every $s \in [\tau_E, 0]$

$$\bigcap_{t \in (s,0]} \mathcal{H}(E_t) = \mathcal{H}(E_s) = \overline{\bigcup_{t \in [\tau_E, s)} \mathcal{H}(E_s)}.$$

This, however, is an immediate consequence of the relation (write $E_t = A_t - iB_t$ with $A_t^\# = A_t$, $B_t^\# = B_t$)

$$(A_t(z), B_t(z)) = (A_u(z), B_u(z)) \begin{pmatrix} \cos\frac{t-u}{2}z & \sin\frac{t-u}{2}z \\ -\sin\frac{t-u}{2}z & \cos\frac{t-u}{2}z \end{pmatrix},$$

which holds true since ($E = A - iB$ with $A^\# = A$, $B^\# = B$)

$$(A_t(z), B_t(z)) = (A(z), B(z)) \begin{pmatrix} \cos\frac{t}{2}z & \sin\frac{t}{2}z \\ -\sin\frac{t}{2}z & \cos\frac{t}{2}z \end{pmatrix},$$

and

$$\begin{pmatrix} \cos\frac{t}{2}z & \sin\frac{t}{2}z \\ -\sin\frac{t}{2}z & \cos\frac{t}{2}z \end{pmatrix} = \begin{pmatrix} \cos\frac{u}{2}z & \sin\frac{u}{2}z \\ -\sin\frac{u}{2}z & \cos\frac{u}{2}z \end{pmatrix} \cdot \begin{pmatrix} \cos\frac{t-u}{2}z & \sin\frac{t-u}{2}z \\ -\sin\frac{t-u}{2}z & \cos\frac{t-u}{2}z \end{pmatrix}.$$

□

If $\tau_E > 0$ we shall, for obvious reasons, say that the space $\mathcal{H}(E)$ *ends with an interval of Paley-Wiener type.*

# 3 deBranges spaces of finite $\lambda$-type

In this section we give some basic results on the interplay of the dB-space structure of the space under consideration and growth conditions imposed on its elements.

**3.1 Definition.** A function $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ will be called a *growth function* if it satisfies the following axioms:

($i$) The limit $\rho := \lim_{r\to\infty} \frac{\log\lambda(r)}{\log r}$ exists and is a finite nonnegative number.

($ii$) For all sufficiently large values of $r$, the function $\lambda$ is differentiable and $\lim_{r\to\infty} r\frac{\lambda'(r)}{\lambda(r)} = \rho$.

($iii$) $\log r = o(\lambda(r))$.

Hereby the conditions ($i$) and ($ii$) ensure that we have available Valiron's theory of proximate orders (cf. [L, I.12], [LG, I.6]) as well as the theory of value distribution of meromorphic functions (cf. [R]).

Recall that, if $\frac{\log\lambda(r)}{\log r}$ is a proximate order, then

$$\lim_{r\to\infty} \frac{\lambda(Cr)}{\lambda(r)} = C^\rho \tag{3.1}$$

uniformly in $C$ on compact subsets of $\mathbb{R}^+$, and that for sufficiently large values of $r$ the function $\lambda(r)$ strictly increases (cf. [L, I.12 Hilfssatz 5], [LG, Theorem 1.18, Proposition 1.19]). Since the whole importance of a growth function lies in its behaviour at $+\infty$ we can therefore always assume that $\lambda$ is increasing and bounded away from 0.

The condition ($iii$), that $\lambda$ grows sufficiently rapidly, is imposed in order to rule out some more or less trivial cases and to avoid some technical details. It is no significant restriction of generality (compare Lemma 3.3).

Important examples of growth functions are functions of the form

$$\lambda(r) = r^\alpha(\log r)^\beta \tag{3.2}$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha > 0$.

For an entire function $F$ denote by $M(F, r)$ the maximum modulus

$$M(F, r) := \max_{|z|=r} |F(z)|.$$

The function $F$ is said to be of *finite $\lambda$-type* if (all '$O$'- and '$o$'-relations are, unless otherwise specified, understood for $r \to \infty$)

$$\log^+ M(F, r) = O(\lambda(r)).$$

In this case the *$\lambda$-type of $F$* is defined as the number

$$\sigma_F^\lambda := \limsup_{r\to\infty} \frac{\log^+ M(F, r)}{\lambda(r)},$$

and the *indicator function* $h_F^\lambda$ of $F$ with respect to $\lambda$ is

$$h_F^\lambda(\phi) := \limsup_{r\to\infty} \frac{\log |F(re^{i\phi})|}{\lambda(r)}. \tag{3.3}$$

Let us recall (cf. [L, I.Lehrsatz 29]) that, if $F$ is of finite $\lambda$-type,

$$\sigma_F^\lambda = \max_{\phi\in(-\pi,\pi]} h_F^\lambda(\phi).$$

Recall that (cf. [L]) in case $\rho = 1$ the function $h_F^\lambda$ is the support function of a nonempty convex set, the so-called *indicator diagram* of $F$.

The set of all entire functions of finite $\lambda$-type will be denoted by $\Lambda$. If, for example, $\lambda(r) = r^\rho$ for some $\rho > 0$, then $\Lambda$ is the set of all entire functions of growth at most order $\rho$, finite type. If $\lambda$ is any growth function and the number $\rho$ is as in $(i)$ of Definition 3.1, then for any $\varepsilon > 0$ we have $\lambda(r) = O(r^{\rho+\varepsilon})$. Hence $\Lambda$ exclusively contains functions of finite order at most $\rho$.

If, for a set $E \subseteq \mathbb{R}^+$ the limit ($\mu$ denotes the Lebesgue measure)

$$m^*(E) := \lim_{r\to\infty} \frac{\mu\big(E \cap (0, r)\big)}{r}$$

exists, we call $m^*(E)$ the relative measure of $E$ (cf. [L, II.1]).

A function $F$ is said to be of completely regular $\lambda$-growth on a ray $\{re^{i\phi} : r > 0\}$ if, with some exceptional set $E_\phi$ of relative measure 0,

$$h_F^\lambda(\phi) = \lim_{r\to\infty, r\notin E_\phi} \frac{\log |F(re^{i\phi})|}{\lambda(r)}.$$

It is proved in [L, III.1] that, if $F$ is of completely regular $\lambda$-growth on a set of rays which is dense in the plane, then there exists a set $E$ of relative measure zero such that

$$h_F^\lambda(\phi) = \lim_{r\to\infty, r\notin E} \frac{\log |F(re^{i\phi})|}{\lambda(r)}$$

uniformly for all $\phi \in (-\pi, \pi]$. In this case we will call $F$ of completely regular $\lambda$-growth.

Let us recall the notion of Nevanlinna characteristic. For a function $f$, meromorphic in the whole plane, denote by $w_j = \rho_j e^{i\psi_j}$ its nonzero poles (enlisted according to their multiplicities) and let $k^-$ be the order of the (possible) pole at 0, i.e. $k^- = -\min\{k, 0\}$ where $k$ is such that $f(z) \sim z^k$ at 0. Put

$$N(f, r) := \sum_{\rho_j \le r} \log \frac{r}{\rho_j} + k^- \log r,$$

$$m(f, r) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| \, d\theta.$$

Then the *Nevanlinna characteristic* $T(f, r)$ of $f$ is defined to be

$$T(f, r) := m(f, r) + N(f, r).$$

It gives a measure for the total affinity of $f$ to the value $\infty$.

It is elementary to see that

$$T(fg, r) \le T(f, r) + T(g, r).$$

If $F$ is an entire function then (cf. [R, §8]) for any $R > r$

$$T(F, r) \le \log^+ M(F, r) \le \frac{R + r}{R - r} T(F, R). \tag{3.4}$$

If we substitute $R = 2r$ in (3.4) we see from (3.1) that an entire function $F$ is of finite $\lambda$-type if and only if

$$T(F, r) = O(\lambda(r)).$$

The *First Fundamental Theorem* of Nevanlinna (cf. [R, §4]) states that, if $f$ is a meromorphic function in the whole plane and $a \in \mathbb{C}$, then

$$T(\frac{1}{f - a}, r) = T(f, r) + O(1).$$

We denote by $\tilde{\Lambda}$ the set of all meromorphic functions $f$ which satisfy $T(f, r) = O(\lambda(r))$. Obviously $\tilde{\Lambda}$ is a field and contains $\Lambda$. It is a rather deep result (the Miles-Rubel-Taylor Theorem, see [R, §13,14]) that in fact $\tilde{\Lambda}$ is the quotient field of $\Lambda$.

**3.2 Definition.** A dB-space $\mathcal{H}$ is said to be of *finite $\lambda$-type* if every element $F$ of $\mathcal{H}$ has this property, i.e. if $\mathcal{H} \subseteq \Lambda$. In this case the *$\lambda$-type of $\mathcal{H}$* is defined to be the value

$$\sigma_{\mathcal{H}}^{\lambda} := \sup_{F \in \mathcal{H}} \sigma_F^{\lambda}.$$

According to whether $\sigma_{\mathcal{H}}^{\lambda} = 0$ or $0 < \sigma_{\mathcal{H}}^{\lambda} < \infty$, we shall say that $\mathcal{H}$ is of *minimal type* or *normal type*, respectively (we will see in the subsequent Theorem 3.4 that the case $\sigma_{\mathcal{H}}^{\lambda} = \infty$ cannot occur).

In the particular case $\lambda(r) = r$ we shall speak of a dB-space of *exponential type* and will suppress the upper index $\lambda$. A dB-space of *minimal exponential type* is then a space of exponential type with $\sigma_{\mathcal{H}} = 0$. This convention will also apply to all other notations, e.g. $h_F(\theta)$ always means $h_F^{\lambda}(\theta)$ with respect to $\lambda(r) = r$.

First of all let us clarify that the condition $(iii)$ of Definition 3.1 does not essentially reduce the generality of our considerations.

**3.3 Lemma.** *Assume that the function $\lambda$ satisfies (i) and (ii) of Definition 3.1 but not (iii), and let $\mathcal{H}$ be a dB-space, $\mathcal{H} \subseteq \Lambda$. Then $\dim \mathcal{H} < \infty$ and $\mathcal{H}$ contains only polynomials.*

*Proof.* Since $\limsup_{r\to\infty} \frac{\log r}{\lambda(r)} > 0$, the set $\Lambda$ is contained in the set $\mathbb{C}[z]$ of all polynomials (cf. e.g. [R, 10.2]). Hence $\mathcal{H} \subseteq \mathbb{C}[z]$ and thus also $\operatorname{Assoc}\mathcal{H} = \mathcal{H} + z\mathcal{H} \subseteq \mathbb{C}[z]$. As mentioned in Section 2 we have $\mathcal{H} = \mathcal{H}(E)$ for some $E \in \mathcal{H}B$, which is always contained in $\operatorname{Assoc}\mathcal{H} \setminus \mathcal{H}$. In particular, $E \in \mathbb{C}[z]$ and we conclude

$$\mathcal{H}(E) = \big\{p \in \mathbb{C}[z] : \deg p \leq \deg E - 1\big\}.$$

$\square$

We already saw that a dB-space $\mathcal{H} = \mathcal{H}(E)$ is uniquely determined by a single function $E \in \mathcal{H}B$. Also the growth behaviour of $\mathcal{H}$ can therefore be read off $E$.

**3.4 Theorem.** *Let $\mathcal{H}$ be a dB-space, $\mathcal{H} = \mathcal{H}(E)$ for some $E \in \mathcal{H}B$. Then $\mathcal{H}$ is of finite $\lambda$-type if and only if the function $E$ has this property. In this case*

$$\sigma_{\mathcal{H}}^{\lambda} = \max_{F \in \operatorname{Assoc}\mathcal{H}} \sigma_F^{\lambda} = \sigma_E^{\lambda}.$$

For the proof of Theorem 3.4 we need a couple of auxiliary results. Let us first state a consequence of the Phragmen-Lindelöf theorem. It just says that, when knowing in advance that a function $F$ is of finite order, one need only consider its behaviour on a couple of rays in order to get $F \in \Lambda$.

**3.5 Lemma.** *Let $F$ be an entire function of finite order $\hat\rho$ and let $\lambda$ be a growth function, $\rho = \lim_{r\to\infty} \frac{\log \lambda(r)}{\log r}$. Put $\rho_0 := \max\{\rho, \hat\rho\}$. Assume that there exist $\phi_1, \dots, \phi_n \in (-\pi, \pi]$, $\phi_j \leq \phi_{j+1}$, such that*

$$\max\big(\big\{\phi_{j+1} - \phi_j : j = 1, \dots, n-1\big\} \cup \big\{\phi_1 - \phi_n + 2\pi\big\}\big) =: \gamma < \min\Big\{\frac{\pi}{\rho_0}, 2\pi\Big\}.$$

$$h_F^{\lambda}(\phi_j) \leq s < \infty,\ j = 1, \dots, n.$$

*Then $F \in \Lambda$ and $\sigma_F^{\lambda} \leq (\cos\frac{\rho\gamma}{2})^{-1} s$.*

*In particular, if $F$ is any entire function of finite order and there exists a dense set $M \subseteq (-\pi, \pi]$ such that $h_F^{\lambda}(\phi) \leq s$ for all $\phi \in M$, then $F \in \Lambda$ and $\sigma_F^{\lambda} \leq s$.*

*Proof.* Obviously it suffices to prove the following statement: Let $f(z)$ be a function analytic on the angle $A := \{w \in \mathbb{C} : |\arg w| \leq \frac{\gamma}{2}\}$ which satisfies for all $\delta > 0$ an estimation of the form

$$\log |f(z)| \leq M|z|^{\hat\rho + \delta},\ z \in A,$$

where $\gamma < \min\{\frac{\pi}{\rho_0}, 2\pi\}$, and assume that

$$\limsup_{r\to\infty} \frac{\log |f(re^{\pm i\frac{\gamma}{2}})|}{\lambda(r)} \leq s < \infty.$$

Then for all $\epsilon > 0$

$$\limsup_{r\to\infty} \frac{\log|f(re^{i\phi})|}{\lambda(r)} \leq \frac{s}{\cos\frac{\rho\gamma}{2}} + \epsilon,\ \phi \in [-\frac{\gamma}{2}, \frac{\gamma}{2}],$$

uniformly in $\phi$.

To establish this assertion fix $\epsilon > 0$ and choose a function $W(z)$ analytic and zerofree on the whole angle $A$, such that

$$\lim_{r\to\infty} \frac{\log|W(re^{i\phi})|}{\lambda(r)} = -\Big(\frac{s}{\cos\frac{\rho\gamma}{2}} + \epsilon\Big)\cos\rho\phi$$

uniformly for $\phi \in [-\frac{\gamma}{2}, \frac{\gamma}{2}]$. Such a choice is possible by [L, I.17.Hilfssatz 10]. The function $fW$ is analytic on $A$ and bounded on the rays $\arg w = \pm\frac{\gamma}{2}$ (note that $\frac{\rho\gamma}{2} < \frac{\pi}{2}$ and therefore $\cos\frac{\rho\gamma}{2} > 0$). Moreover, it satisfies for all $\delta > 0$ an estimate

$$\log|f(z)W(z)| \leq M|z|^{\rho_0+\delta},\ z \in A.$$

By the principle of Phragmen and Lindelöf it follows that $fW$ is bounded throughout $A$,

$$|f(z)W(z)| \leq C,\ z \in A.$$

In particular,

$$\limsup_{r\to\infty} \frac{\log|f(re^{i\phi})|}{\lambda(r)} \leq -\lim_{r\to\infty} \frac{\log|W(re^{i\phi})|}{\lambda(r)},\ \phi \in [-\frac{\gamma}{2}, \frac{\gamma}{2}].$$

□

**3.6 Lemma.** *Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{C}^+$ and denote by $\rho_1$ its convergence exponent. Assume that $\rho_1 < \infty$ and that $(z_n)_{n\in\mathbb{N}}$ satisfies the Blaschke condition* (2.2). *Denote by $B(z) = \prod_{n\in\mathbb{N}} \frac{z-z_n}{z-\overline{z_n}}$ the associated Blaschke product. Then for any $\epsilon > 0$*

$$T(B, r) = O(r^{\rho_1+\epsilon}).$$

*The same assertion holds if $z_n \in \mathbb{C}^-$, $n \in \mathbb{N}$.*

*Proof.* Denote by $p$ the genus of the sequence $(z_n)_{n\in\mathbb{N}}$. In view of Lemma 2.2, $(i)$, we may write $B$ in the form

$$B(z) = \gamma\frac{D_1(z)}{D_2(z)} \exp\Big[-2i\Big(z\sum_{n\in\mathbb{N}} \operatorname{Im}\frac{1}{z_n} + \ldots + \frac{z^p}{p}\sum_{n\in\mathbb{N}} \operatorname{Im}\frac{1}{z_n^p}\Big)\Big],$$

where $|\gamma| = 1$ and $D_1$ and $D_2$ denote the canonical products

$$D_1(z) := \prod_{n\in\mathbb{N}} \big(1 - \frac{z}{z_n}\big) \exp\Big[\frac{z}{z_n} + \ldots + \frac{z^p}{pz_n^p}\Big],$$

$$D_2(z) := \prod_{n\in\mathbb{N}} \big(1 - \frac{z}{\overline{z_n}}\big) \exp\Big[\frac{z}{\overline{z_n}} + \ldots + \frac{z^p}{p\overline{z_n}^p}\Big].$$

Since the order of $D_1$ and $D_2$ is $\rho_1$ and the genus $p$ does not exceed the convergence exponent, the assertion of the lemma follows by an application of Nevanlinna's First Fundamental Theorem.

□

**3.7 Corollary.** *Let $F$ be an entire function and assume that $(z_n)_{n\in\mathbb{N}}$, $z_n \in \mathbb{C}^+$, is a sequence of zeros of $F$ whose convergence exponent is finite and which satisfies the Blaschke condition. Then the (finite or infinite) orders of $F$ and $\frac{F}{B}$ coincide. The same assertion holds if all $z_n$ belong to $\mathbb{C}^-$.*

*Proof.* Let $\rho_1$ be the convergence exponent of the sequence $(z_n)_{n\in\mathbb{N}}$. Then $\rho_1$ does not exceed the order $\rho$ of $F$. By Lemma 3.6 and the 1.Fundamental Theorem we have ($\epsilon > 0$)

$$T(\frac{F}{B}, r) \leq T(F, r) + T(\frac{1}{B}, r) =$$

$$= T(F, r) + T(B, r) + O(1) = O(r^{\rho+\epsilon}),$$

hence the order of $\frac{F}{B}$ does not exceed that of $F$. Applying the same argument to $\frac{F}{B}$ and the sequence $(\overline{z_n})_{n\in\mathbb{N}}$ in place of $F$ and $(z_n)_{n\in\mathbb{N}}$, we conclude that in fact the orders of $F$ and $\frac{F}{B}$ are equal.

□

Finally let us recall the following perturbation result given in [KW1, Theorem 1.3,Proposition 1.4]: Assume that $F, G \in \mathcal{H}B$ and $\frac{F}{G}$ is of bounded type. Then $F$ is of finite order if and only if $G$ has this property.

*Proof. (of Theorem 3.4)* In the first step assume that $\mathcal{H}$ is of finite $\lambda$-type and let $F \in \operatorname{Assoc}\mathcal{H}$. Choose $G \in \mathcal{H} \setminus \{0\}$ and $w \in \mathbb{C}$ such that $G(w) \neq 0$. Then

$$H(z) := \frac{F(z) - \frac{F(w)}{G(w)} G(z)}{z - w} \in \mathcal{H}.$$

From $F(z) = H(z)(z - w) - \frac{F(w)}{G(w)} G(z)$ we obtain

$$\log^+ |F(z)| \leq \max\Big\{ \log^+ |H(z)(z - w)|, \log^+ \Big|\frac{F(w)}{G(w)} G(z)\Big| \Big\} + \log 2$$

Since

$$\log^+ \Big|\frac{F(w)}{G(w)} G(z)\Big| \leq \log^+ \Big|\frac{F(w)}{G(w)}\Big| + \log^+ |G(z)|,$$

$$\log^+ |H(z)(z - w)| \leq \log^+ |H(z)| + \log^+ |(z - w)|,$$

and the second summand in the last relation is a $o(\lambda(|z|))$, we obtain

$$\sigma_F^\lambda = \limsup_{|z|\to\infty} \frac{\log^+ |F(z)|}{\lambda(|z|)} \leq \max\{\sigma_G^\lambda, \sigma_H^\lambda\} \leq \sigma_{\mathcal{H}}^\lambda.$$

We conclude that every $F \in \operatorname{Assoc}\mathcal{H}$ is of finite $\lambda$-type, in particular $E$ has this property, and that

$$\sup\{\sigma_F^\lambda : F \in \operatorname{Assoc}\mathcal{H}\} = \sigma_{\mathcal{H}}^\lambda.$$

In the second step we shall show that the existence of a function $F \in (\operatorname{Assoc}\mathcal{H}) \setminus \{0\}$ of finite order implies that every function $G \in \operatorname{Assoc}\mathcal{H}$ has finite order.

With a function $F$ also $F_1 = F + F^\#$ ($F_1 = iF$, respectively, in case $F = -F^\#$) is of finite order. By Corollary 3.7 also $F_2 = \frac{F_1}{B}$, where $B$ is the Blaschke product associated with the zeros of $F_1$ located in the upper half plane, has finite order. Note that $F_2 \in \mathcal{H}B$. It readily follows that every function $G_2 \in$

$(\operatorname{Assoc}\mathcal{H})\cap\mathcal{H}B$ is of finite order. If $G_1\in\operatorname{Assoc}\mathcal{H}$ is real, the function $G_2=\frac{G_1}{B}$ ($B$ is now the Blaschke product for the zeros of $G_1$ in $\mathbb{C}^+$) belongs to $\mathcal{H}B$, is thus of finite order, and we conclude from Corollary 3.7 that also $G_1=G_2B$ has finite order. Since every function $G\in\operatorname{Assoc}\mathcal{H}$ can be written as a linear combination of real functions the assertion of the present step follows.

It is the subject of the third step of this proof to show that, if $F\in\mathcal{H}$ and $\phi\in(-\pi,\pi]$, $\phi\neq 0,\pi$, we have $h_F^\lambda(\phi)\leq\sigma_E^\lambda$. An application of the Phragmen-Lindelöf principle (cf. Lemma 3.5) will then yield $F\in\Lambda$, $\sigma_F^\lambda\leq\sigma_E^\lambda$, and henceforth complete the proof of the theorem.

To this end note that the reproducing kernel

$$K(w,z)=\frac{E(z)E^\#(\overline{w})-E^\#(z)E(\overline{w})}{2\pi i(\overline{w}-z)}$$

of the space can easily be estimated off the real axis: If $|\operatorname{Im}z|\geq 1$, we have $|2\pi i(\overline{z}-z)|\geq 4\pi$, and therefore

$$|K(z,z)|\leq\frac{1}{2\pi}\max\big\{|E(z)|^2,|E^\#(z)|^2\big\}.$$

Thus

$$\frac{\log^+|K(z,z)|}{\lambda(|z|)}\leq o(1)+2\max\Big\{\frac{\log^+|E(z)|}{\lambda(|z|)},\frac{\log^+|E^\#(z)|}{\lambda(|z|)}\Big\},\tag{3.5}$$

and it follows from the relation

$$|F(z)|=|(F(.),K(z,.))|\leq\|F\|\sqrt{K(z,z)}$$

that for $\phi\neq 0,\pi$ we must have $h_F^\lambda(\phi)\leq\sigma_E^\lambda$.

□

*3.8 Remark.* Let us explicitly point out the statement of Step 2 of the previous proof: Let $\mathcal{H}$ be a dB-space and assume that there exists a function $F\in(\operatorname{Assoc}\mathcal{H})\setminus\{0\}$ which is of finite order. Then every $G\in\operatorname{Assoc}\mathcal{H}$ is of finite order.

*3.9 Remark.* Since $E$ can be written as a linear combination of $S_\phi$ and $S_{\phi+\frac{\pi}{2}}$, we see that $\mathcal{H}\subseteq\Lambda$ if and only if for one (and hence for all) $\phi\in\mathbb{R}$ both of $S_\phi$ and $S_{\phi+\frac{\pi}{2}}$ are of finite $\lambda$-type. In this case

$$\sigma_{\mathcal{H}}^\lambda=\max\big\{\sigma_{S_\phi}^\lambda,\sigma_{S_{\phi+\frac{\pi}{2}}}^\lambda\big\}.$$

Note that this also can be deduced similarly as the above theorem from the representation (2.1) of $K(w,z)$.

The next theorem can be viewed as the second main result of this section.

**3.10 Theorem.** *Assume that $r=O(\lambda(r))$. If there exists one (not identically vanishing) function $F\in\big(\operatorname{Assoc}\mathcal{H}\big)\cap\Lambda$, then $\mathcal{H}$ is of finite $\lambda$-type.*

*Proof.* The crucial point is to recall (see e.g. [GG, Theorem 4.4] together with [B, 7.2.3]) that for a function $f$ meromorphic in $\mathbb{C}^+$ and of bounded characteristic we have on a dense set of rays $\phi\in(0,\pi)$

$$\lim_{r\to\infty}\frac{\log|f(re^{i\phi})|}{r}=\operatorname{mt}f\cdot\sin\phi.\tag{3.6}$$

It follows that for any two functions $F, G \in \operatorname{Assoc}\mathcal{H}$ (on a dense set of $\phi \in (-\pi, \pi]$)

$$\lim_{r\to\infty} \frac{1}{r} \log\Big|\frac{G(re^{i\phi})}{F(re^{i\phi})}\Big| = \begin{cases} \operatorname{mt} \frac{G}{F} \cdot \sin\phi & , \ \phi > 0 \\ \operatorname{mt} \frac{G^\#}{F^\#} \cdot |\sin\phi| & , \ \phi < 0 \end{cases} .$$

Let us assume that $F \in (\operatorname{Assoc}\mathcal{H}) \cap \Lambda$, $F \not\equiv 0$, and that $G \in \operatorname{Assoc}\mathcal{H}$. We have

$$\frac{1}{\lambda(r)} \log\big|G(re^{i\phi})\big| = \frac{1}{\lambda(r)} \log\big|F(re^{i\phi})\big| + \frac{r}{\lambda(r)} \cdot \frac{1}{r} \log\Big|\frac{G(re^{i\phi})}{F(re^{i\phi})}\Big|$$

The assumption of the theorem concerning $\lambda$ tells us that $\frac{r}{\lambda(r)} \leq C$ and we conclude

$$h_G^\lambda(\phi) \leq h_F^\lambda(\phi) + C \max\Big\{ \operatorname{mt} \frac{G}{F}, \operatorname{mt} \frac{G^\#}{F^\#} \Big\}.$$

It remains to recall that $F$, and hence also $G$ (cf. Remark 3.8), is of finite order and to appeal to the Phragmen-Lindelöf principle Lemma 3.5.

□

*3.11 Remark.* Let us compare the preceeding two theorems. The first one states that if the particular function $E$ belongs to $\Lambda$, then $\mathcal{H} \subseteq \Lambda$. In the above theorem the same conclusion follows on the weaker hypothesis that there exists some function of $\operatorname{Assoc}\mathcal{H}$ which belongs to $\Lambda$ if we know in advance that $r = O(\lambda(r))$. The fact that there cannot be a weaker assumption on $\lambda$ in order to get this result follows already from the example of Paley-Wiener space $\mathcal{H}(e^{-iaz})$, $a > 0$, since $1 \in \operatorname{Assoc}\mathcal{H}(e^{-iaz})$. This counterexample is of course somehow trivial since there are no zeros of $E$ and since the structure of Paley-Wiener space is most easily understood. However, also if we exclude the possibility that the space $\mathcal{H}$ ends with an interval of Paley-Wiener type (cf. Theorem 2.7), the assumption $r = O(\lambda(r))$ cannot be essentially weakened. We will see that for each $\lambda$ with $\lambda(r) = o(r)$ there exist counterexamples (see also Remark 3.26 or Example 5.15).

If $\mathcal{H}(E)$ is a dB-space with $\mathcal{H}(E) \subseteq \Lambda$, then $E$ is necessarily of finite order. Hermite-Biehler functions allows a particular product representation (see [KW1, Proposition 1.4]).

**3.12 Lemma.** *A function $E(z)$ of finite order belongs to $\mathcal{HB}$ if and only if it has no zeros in $\mathbb{C}^+$, belongs to the class $\mathbb{A}$ and can be represented as*

$$E(z) = \gamma C(z) e^{-iaz} \prod_{n\in\mathbb{N}} \big(1 - \frac{z}{z_n}\big) \exp\left[z \operatorname{Re}\frac{1}{z_n} + \ldots + \frac{z^p}{p} \operatorname{Re}\frac{1}{z_n^p}\right] \tag{3.7}$$

*with $a \geq 0$, $|\gamma| = 1$, $z_n \in \mathbb{C}^-$, and a real function $C$ of finite order having only real zeros. Hereby $p$ denotes the genus of the sequence $(z_n)_{n\in\mathbb{N}}$, and the number $a$ is determined by the mean type of $\frac{E^\#}{E}$:*

$$a = -\frac{1}{2} \operatorname{mt} \frac{E^\#(z)}{E(z)}.$$

*Proof.* If $E$ can be represented as in (3.7) it belongs as a locally uniform limit of $\mathcal{HB}$-functions to $\mathcal{HB}$ and is by the Hadamard factorization theorem of finite order.

Conversely, assume that $E \in \mathcal{HB}$ is of finite order and denote by $(z_n)$ its nonreal zeros. Since $E$ belongs to $\mathbb{A}$ (see Lemma 2.3) we can apply Lemma 2.2, $(i)$, and hence write the Hadamard factorization of $E$ in the form

$$E(z) = C(z)e^{ip(z)} \prod_{n\in\mathbb{N}} (1 - \frac{z}{z_n}) \exp \left[ z \operatorname{Re} \frac{1}{z_n} + \ldots + \frac{z^p}{p} \operatorname{Re} \frac{1}{z_n^p} \right]$$

with some real function $C(z)$ of finite order having only real zeros and a real polynomial $p(z)$. We obtain

$$\frac{E^{\#}(z)}{E(z)} = e^{-2ip(z)} B(z), \tag{3.8}$$

where $B$ denotes is the Blaschke product associated with the sequence $(\overline{z_n})_{n\in\mathbb{N}}$. Since the left, and hence also the right hand side of (3.8) belongs to $N^+(\mathbb{C}^+)$, it follows that $e^{-2ip(z)}$ is of bounded type and nonpositive mean type in $\mathbb{C}^+$. Hence $p$ must be a linear polynomial and the coefficient of $z$ must be nonpositive. In fact, by (3.8), it follows that $-2a = \operatorname{mt} \frac{E^{\#}}{E}$.

□

*3.13 Remark.* Together with Lemma 2.4 and Theorem 2.7 this factorization shows us that the structure of a space $\mathcal{H}(E)$ is essentially determined by the distribution of the nonreal zeros of the function $E$. Therefore we may often restrict to the case that $E \in \mathcal{HB}^{\times}$. Moreover, in order to construct "nontrivial" examples we should not think of either real zeros or Payley-Wiener type spaces.

*3.14 Remark.* If $E \in \Lambda \cap \mathcal{HB}$ for some growth function $\lambda$, then the presence of a factor $e^{-iaz}$ in (3.7) implies that $r = O(\lambda(r))$. To see this we consider the cases $\rho = \lim_{r\to\infty} \frac{\log \lambda(r)}{\log r} < 1$ and $\rho = 1$ separately (in the case $\rho > 1$ this relation is trivially satisfied). In the first case the canonical product of the zeros of $E$ is of order at most $\rho < 1$ and therefore $a \neq 0$ would imply that the order of $E$ is equal to 1, a contradiction. Assume next that $\rho = 1$. Then

$$C(z) = e^{\alpha+\beta z} \prod (1 - \frac{z}{w_n}) e^{\frac{z}{w_n}}$$

for some $\alpha, \beta, w_n \in \mathbb{R}$. By [L, I.13.Lehrsatz 18] we know that

$$\frac{r}{\lambda(r)} \left[ \Big( \beta - ia - i \sum_{n\in\mathbb{N}} \operatorname{Im} \frac{1}{z_n} \Big) + \sum_{|z_n|\leq r} \frac{1}{z_n} + \sum_{|w_n|\leq r} \frac{1}{w_n} \right] = O(1).$$

Considering the imaginary part yields

$$\frac{r}{\lambda(r)} \Big( a + \sum_{|z|>r} \operatorname{Im} \frac{1}{z_n} \Big) = O(1).$$

Since all $z_n$ are located in $\mathbb{C}^-$ this is in case $a > 0$ only possible if $\frac{r}{\lambda(r)} = O(1)$.

In the following theorem we will turn the rather vague idea of Remark 3.13, into an exact statement. Before that, however, we have to remind the reader on some notations concerning complex sequences which can for example be found in [R]. Let $Z = (z_n)_{n\in\mathbb{N}}$ be a sequence of (nonzero) complex numbers. Put

$$N(Z, r) := \sum_{|z_n|\leq r} \log \frac{r}{|z_n|}, \ \ S(Z; r_1, r_2; k) := \frac{1}{k} \sum_{r_1<|z_n|\leq r_2} \Big(\frac{1}{z_n}\Big)^k.$$

Following [R, §13] we say that $Z$ has a finite $\lambda$-*density* if (be aware of (3.1) when comparing with the definition in [R])

$$N(Z,r) = O(\lambda(r)),$$

and that $Z$ is $\lambda$-*balanced* if (for $r_1, r_2 \to \infty$)

$$|S(Z;r_1,r_2;k)| = O\Big(\frac{\lambda(r_1)}{r_1^k} + \frac{\lambda(r_2)}{r_2^k}\Big)$$

uniformly in $k$. The sequence $Z$ is called $\lambda$-*admissible* if it is both, of finite $\lambda$-density and $\lambda$-balanced. Note that the subsequence of a sequence of finite $\lambda$-density is also of finite $\lambda$-density, whereas the property being $\lambda$-balanced is in general not inherited by subsequences.

The result we shall employ in the sequel is (cf. [R, §13.5]) that a sequence $Z$ is the precise sequence of zeros (taking into account multiplicities) of a function $F \in \Lambda$ if and only if it is $\lambda$-admissible.

Let us remark that (cf. [R, §13.3]) in the special case $\lambda(r) = r^\rho$ we have that $Z$ is $\lambda$-admissible if and only if

$$\limsup_{r\to\infty} \frac{1}{r^\rho} \sum_{|z_n|\le r} 1 < \infty,$$

and, in the case that $\rho$ is an integer, additionally

$$\left| \sum_{|z_n|\le r} \Big(\frac{1}{z_n}\Big)^\rho \right| = O(1).$$

Hence, the above mentioned result reduces in this particular situation to the classical result of Lindelöf on the distribution of the zeros of an entire function of order $\rho$, finite type.

**3.15 Lemma.** *Assume that $(z_n)_{n\in\mathbb{N}}$ is a $\lambda$-admissible sequence of points $z_n \in \mathbb{C}^+$ such that $\sum_{n\in\mathbb{N}} |\operatorname{Im} \frac{1}{z_n}| < \infty$. If either $\rho \notin \mathbb{Z}$ or $\rho \in \mathbb{Z}$ and $r^\rho = O(\lambda(r))$, then $T(B,r) = O(\lambda(r))$ where $B$ denotes the Blaschke product associated to $(z_n)_{n\in\mathbb{N}}$.*

*Proof.* By [R, 13.5.2] there exists an entire function $H \in \Lambda$ which exactly the zeros $(z_n)_{n\in\mathbb{N}}$. Thus we can write

$$\frac{H(z)}{H^\#(z)B(z)} = e^{P(z)},$$

with some entire function $P(z)$. Since by Lemma 3.6 the left hand side defines a function of finite order $\hat\rho \le \rho$, $P(z)$ is a polynomial of degree at most $\rho$. Under either assumptions on $\lambda$ it follows that $e^{P(z)} \in \Lambda$. Since $\tilde\Lambda$ is a field the assertion of the lemma follows.

□

**3.16 Corollary.** *Let $F \in \Lambda$ and let $(z_n)_{n\in\mathbb{N}}$, $z_n \in \mathbb{C}^+$, be a sequence of zeros of $F$ such that $\sum_{n\in\mathbb{N}} |\operatorname{Im} \frac{1}{z_n}| < \infty$. Denote by $B$ the Blaschke product associated with $(z_n)_{n\in\mathbb{N}}$ and put $G(z) := \frac{F(z)}{B(z)}$. Then the following assertions hold true:*

*(i) If $r = O(\lambda(r))$, then $G \in \Lambda$ and in fact $h_G^\lambda = h_F^\lambda$.*

*(ii) If $\rho < 1$ and $(z_n)_{n\in\mathbb{N}}$ is $\lambda$-balanced, then $G \in \Lambda$.*

*The same results are valid in the case that $z_n \in \mathbb{C}^-$ for all $n \in \mathbb{N}$.*

*Proof.* Assume that $r = O(\lambda(r))$. Since $F \in \Lambda$ the order of $F$ is at most $\rho$. Hence also the convergence exponent of the sequence of all zeros of $F$ is less than or equal to $\rho$. Since $(z_n)_{n\in\mathbb{N}}$ is a subsequence of zeros of $F$, in particular, its convergence exponent does not exceed $\rho$. From Lemma 3.6, the first fundamental theorem and the fact that $\log^+ \max_{|z|=r} |G(z)| \leq \alpha T(G, \beta r) + O(1)$, we conclude that $G$ is of finite order ($\leq 2\rho$). Since $r = O(\lambda(r))$, [B, 7.2.3] implies that

$$\lim_{r\to\infty} \frac{1}{\lambda(r)} \log |B(re^{i\phi})| = 0 \tag{3.9}$$

for a dense set of $\phi \in (-\pi, \pi]$, and we conclude from Lemma 3.5 that $G \in \Lambda$. Together with (3.9) the continuity of $h_G^\lambda$ implies $h_G^\lambda = h_F^\lambda$.

The second case is immediate from Lemma 3.15 and the fact that $(z_n)_{n\in\mathbb{N}}$ has finite $\lambda$-density as it is a subsequence of the sequence of zeros of $F$.

□

We say a space $\mathcal{H}$ is obtained from $\mathcal{H}_1$ by multiplication with $C$, if $F \mapsto CF$ is an isometry of $\mathcal{H}_1$ onto $\mathcal{H}$ (compare Lemma 2.4 and the discussion after it). The following statement is the third main result of this section.

**3.17 Theorem.** *Assume that $r = O(\lambda(r))$. A dB-space $\mathcal{H} = \mathcal{H}(E)$ is obtained by means of multiplication with a real and zerofree function from a dB-space $\mathcal{H}_1 \subseteq \Lambda$ if and only if for one (and hence for all) $\phi \in [0, \pi)$ the sequence $(a_n)_{n\in\mathbb{N}}$ of zeros of $S_\phi$ is $\lambda$-admissible.*

*Proof.* Assume first that $\mathcal{H}$ is obtained from $\mathcal{H}_1 \subseteq \Lambda$ by means of multiplication with $C$. If $\mathcal{H} = \mathcal{H}(E)$, then $\mathcal{H}_1 = \mathcal{H}(E_1)$ with $E_1 = \frac{E}{C}$. Choose $\phi \in [0, \pi)$, then $S_{1,\phi} \in \Lambda$ and hence the sequence $(b_n)_{n\in\mathbb{N}}$ of its zeros is $\lambda$-admissible. However, as $S_\phi = CS_{1,\phi}$, the zeros of $S_\phi$ are exactly those of $S_{1,\phi}$.

We start the proof of the converse with the construction of the multiplicator $C(z)$: If $(a_n)_{n\in\mathbb{N}}$ is $\lambda$-admissible, then by [R, 13.5.2] there exists an entire function $A(z) \in \Lambda$ having $(a_n)_{n\in\mathbb{N}}$ as its precise set of zeros. Tracing back the construction indicated in the proof of [R, 13.5.2], we find that $a_n \in \mathbb{R}$, $n \in \mathbb{N}$, implies $A = A^\#$. Thus

$$C(z) := \frac{S_\phi(z)}{A(z)}$$

is real and zerofree.

Put $E_1(z) := \frac{S_\phi(z)}{C(z)} - i\frac{S_{\phi+\frac{\pi}{2}}(z)}{C(z)}$. The dB-space $\mathcal{H}_1 := \mathcal{H}(E_1) = \mathcal{H}(\frac{E}{C})$ has the property that

$$\frac{E_1(z) + E_1^\#(z)}{2} = A(z) \in \Lambda.$$

By Theorem 3.10 we have $\mathcal{H}_1 \subseteq \Lambda$.

□

We have also proved:

**3.18 Corollary.** *Let $\mathcal{H} = \mathcal{H}(E)$. Assume that $r = O(\lambda(r))$, then $\mathcal{H} \subseteq \Lambda$ if and only if $S_\phi \in \Lambda$ for some $\phi \in [0, \pi)$.*

Let us continue with some remarks on the indicator functions $h_F^\lambda$ of elements of a dB-space of finite $\lambda$-type.

For $E \in \Lambda \cap \mathcal{H}B$, denote by $h_{|E|}^\lambda$ the function

$$h_{|E|}^\lambda(\phi) := h_E^\lambda(|\phi|),\ \ \phi \in (-\pi, \pi].$$

From the relation (3.5) and the argument used in the first part of the proof of Theorem 3.4, we immediately obtain:

**3.19 Corollary.** *Let $\mathcal{H} = \mathcal{H}(E)$ be a dB-space of finite $\lambda$-type. Then*

$$\sup_{F\in\mathcal{H}} h_F^\lambda(\phi) = \sup_{F\in\operatorname{Assoc}\mathcal{H}} h_F^\lambda(\phi) = h_{|E|}^\lambda(\phi).$$

If $\lambda$ grows rapidly this statement can be sharpened in a striking way.

**3.20 Lemma.** *Assume that $r = o(\lambda(r))$ and that $\mathcal{H}$ is a dB-space of finite $\lambda$-type. Then for any two $F, G \in \operatorname{Assoc}\mathcal{H} \setminus \{0\}$ we have $h_F^\lambda(\phi) = h_G^\lambda(\phi)$. In particular,*

$$h_F^\lambda(\phi) = h_{|E|}^\lambda(\phi),\ \ \phi \in (-\pi, \pi],\ F \in \operatorname{Assoc}\mathcal{H}. \tag{3.10}$$

*Proof.* Choose $E \in \mathcal{H}B$ such that $\mathcal{H} = \mathcal{H}(E)$. Since $\frac{F}{E}$ is of bounded type, we conclude from (3.6) that for a dense set of $\phi \in (0, \pi)$

$$\lim_{r\to\infty} \frac{1}{\lambda(r)} \log\Big|\frac{F(re^{i\phi})}{E(re^{i\phi})}\Big| = 0,$$

and therefore

$$h_F^\lambda(\phi) = \limsup_{r\to\infty} \frac{1}{\lambda(r)} \log\Big(|E(re^{i\phi})| \cdot \Big|\frac{F(re^{i\phi})}{E(re^{i\phi})}\Big|\Big) =$$

$$= \limsup_{r\to\infty} \frac{1}{\lambda(r)} \log|E(re^{i\phi})| = h_E^\lambda(\phi).$$

Since the indicator function is continuous it follows that $h_F^\lambda(\phi) = h_E^\lambda(\phi)$ for all $\phi \in [0, \pi]$.

The same argument applied to $\frac{F^\#}{E}$ shows that $h_F^\lambda(\phi) = h_E^\lambda(\phi)$ for all $\phi \in [-\pi, 0]$. In order to see (3.10) we only have to note that

$$h_{|E|}^\lambda(\phi) = \begin{cases} h_E^\lambda(\phi) & ,\ \phi \in [0, \pi] \\ h_{E^\#}^\lambda(\phi) & ,\ \phi \in (-\pi, 0] \end{cases}.$$

$\square$

As it is seen from the example of a Paley-Wiener space $\mathcal{H}(e^{-iaz})$, $a > 0$, and $\lambda(r) = r$ the assumption $r = o(\lambda(r))$ in Lemma 3.20 cannot be dropped. However, also in the case $\lambda(r) = r$ Corollary 3.19 can be sharpened. This requires a more delicate argument and will be proved in Corollary 5.13. In this place we shall only add one remark which is sometimes useful.

**3.21 Lemma.** *Assume that $\limsup_{r\to\infty} \frac{r}{\lambda(r)}$ is finite, and let $\mathcal{H} = \mathcal{H}(E)$ be of finite $\lambda$-type. Then, for all $\phi \in \mathbb{R}$,*

$$h_{S_\phi}^\lambda(\varphi) = h_{|E|}^\lambda(\varphi).$$

*Proof.* Since $\frac{S_\phi}{S_{\phi+\frac{\pi}{2}}} \in \mathcal{N}_0$, we have $\operatorname{mt} \frac{S_\phi}{S_{\phi+\frac{\pi}{2}}} = 0$. Thus

$$0 \leq \operatorname{mt} \frac{E}{S_\phi} = \operatorname{mt} \frac{S_\phi \cos\phi + S_{\phi+\frac{\pi}{2}} \sin\phi}{S_{\phi+\frac{\pi}{2}}} =$$

$$= \operatorname{mt} \Big[ \frac{S_\phi}{S_{\phi+\frac{\pi}{2}}} \cos\phi + \sin\phi \Big] \leq 0.$$

The assertion now follows from (3.6) and since both, $h^\lambda_{S_\phi}$ and $h^\lambda_{|E|}$ are even. $\square$

We conclude this discussion with another corollary of the relation (3.6).

**3.22 Corollary.** *Assume that* $\lim_{r\to\infty} \frac{r}{\lambda(r)} < \infty$ *exists and let* $\mathcal{H}$ *be of finite* $\lambda$*-type. If there exists one function* $F \in \operatorname{Assoc}\mathcal{H}$*,* $F \not\equiv 0$*, which is of completely regular growth, so is any function* $G \in \operatorname{Assoc}\mathcal{H}$*.*

*Assume additionally that* $\rho = \lim_{r\to\infty} \frac{\log\lambda(r)}{\log r} = 1$*. If there exists one function* $F \in \operatorname{Assoc}\mathcal{H}$*,* $F \not\equiv 0$*, whose indicator diagram in* $\mathbb{C}^+$ *is a segment of the imaginary axis, all functions* $G \in \operatorname{Assoc}\mathcal{H}$ *have this property.*

Next we shall study growth functions $\lambda$ which grow comparatively slow, e.g. such $\lambda$ with $\rho = \lim_{r\to\infty} \frac{\log\lambda(r)}{\log r} < 1$. It is apparent from the previous expositions that the cases $\rho < 1$ on the one hand and $r = O(\lambda(r))$ on the other hand essentially differ.

**3.23 Lemma.** *Assume that* $\lambda$ *is a growth function with* $\lambda(r) = O(r)$ *and* $\frac{\lambda(r)}{1+r^2} \in L^1(0,\infty)$*. Then each function* $F \in \Lambda$ *is of bounded type and zero mean type in* $\mathbb{C}^+$ *(as well as in any half other plane).*

*Proof.* Let $F \in \Lambda$. Since $\lambda = O(r)$, the function $F$ is of exponential type. Moreover,

$$\int_{-\infty}^{\infty} \frac{\log^+ |F(t)|}{1+t^2}\, dt \leq \int_{-\infty}^{\infty} \frac{C\lambda(|t|)}{1+t^2}\, dt < \infty.$$

Hence, by Krein's Theorem (see e.g. [RR]), $F$ is of bounded type in $\mathbb{C}^+$. Of course this argument applies to each function $F(e^{i\theta}z + a)$, $\theta \in (-\pi,\pi]$, $a \in \mathbb{C}$. Since $h_F(0) = h_F(\pi) = 0$ (see (3.6)), and since the restrictions of $F$ to both, the right and left half plane, are of bounded type, it follows (cf. (3.6)) that $h_F(\phi) = 0$ for all $\phi$. $\square$

Hence, if $\mathcal{H}(E) \subseteq \Lambda$ where $\lambda$ grows slowly, the conditions $\frac{F}{E}$, $\frac{F^\#}{E}$ being of bounded type and nonpositive mean type in the definition of $\mathcal{H}(E)$ are no restrictions anymore. It seems that, the faster $E$ grows, the more restrictive these bounded type conditions get.

Note that the conditions $\lambda(r) = O(r)$ and $\frac{\lambda(r)}{1+r^2} \in L^1(0,\infty)$ are surely fullfilled when $\rho < 1$.

If $F$ is an entire function of finite $\lambda$-type, put

$$\underline{h}^\lambda_F(\phi) := \liminf_{r\to\infty} \frac{\log|F(re^{i\phi})|}{\lambda(r)}.$$

Let us introduce the numbers

$$\beta_+(F) := \sup\big\{\beta\in\mathbb{R}:\ \int_0^\infty |F(t)|^2 e^{\beta\lambda(t)}\,dt<\infty\big\},$$

$$\beta_-(F) := \sup\big\{\beta\in\mathbb{R}:\ \int_{-\infty}^0 |F(t)|^2 e^{\beta\lambda(|t|)}\,dt<\infty\big\}.$$

For $E\in\mathcal{H}B^\times\cap\Lambda$ denote by $\mathcal{A}(E)$ and $\mathcal{A}_-(E)$ the linear spaces

$$\mathcal{A}(E) := \big\{F\in\Lambda:\ \beta_+(F)\geq -2h_E(0), \beta_-(F)\geq -2h_E(\pi)\big\},$$

$$\mathcal{A}_-(E) := \big\{F\in\Lambda:\ \beta_+(F)> -2\underline{h}_E(0), \beta_-(F)> -2\underline{h}_E(\pi)\big\}.$$

The following relation of $\mathcal{H}(E)$ and the spaces $\mathcal{A}(E)$ and $\mathcal{A}_-(E)$ is sometimes useful.

**3.24 Lemma.** *Assume that* $\lambda(r)=O(r)$, $\frac{\lambda(r)}{1+r^2}\in L^1(0,\infty)$ *and let* $E\in\mathcal{H}B^\times\cap\Lambda$. *Then* $\mathcal{A}_-(E)\subseteq\mathcal{H}(E)\subseteq\mathcal{A}(E)$.

*Proof.* Due to Lemma 3.23 bounded type conditions are immaterial. Hence it suffices to deal with the $L^2$-condition.

Let $\epsilon>0$ be fixed. Then, for sufficiently large values of $t>0$ we have

$$\underline{h}_E(0)-\epsilon\leq\frac{\log|E(t)|}{\lambda(t)}\leq h_E(0)+\epsilon,$$

and therefore

$$e^{(-h_E(0)-\epsilon)\lambda(t)}\leq\frac{1}{|E(t)|}\leq e^{(-\underline{h}_E(0)+\epsilon)\lambda(t)}.$$

Similarly, we obtain for sufficiently large $|t|$, $t<0$,

$$e^{(-h_E(\pi)-\epsilon)\lambda(|t|)}\leq\frac{1}{|E(t)|}\leq e^{(-\underline{h}_E(\pi)+\epsilon)\lambda(|t|)}.$$

Hence $F\in\mathcal{A}_-(E)$ implies that $\int_{-\infty}^\infty |F(t)|^2\frac{dt}{|E(t)|^2}<\infty$. Conversely, $F\in\mathcal{H}(E)$ implies $\int_0^\infty |F(t)|^2 e^{\beta\lambda(t)}\,dt<\infty$ for all $\beta<-2h_E(0)$ and $\int_{-\infty}^0 |F(t)|^2 e^{\beta\lambda(|t|)}\,dt<\infty$ for all $\beta<-2h_E(\pi)$.

□

**3.25 Corollary.** *Assume that* $0<\rho<1$, $E\in\mathcal{H}B^\times\cap\Lambda$ *and* $\underline{h}_E(0),\underline{h}_E(\pi)>0$. *Then* $\mathcal{H}(E)$ *contains all polynomials.*

*3.26 Remark.* The assumption $r=O(\lambda(r))$ in Theorem 3.10 cannot be dropped: Choose a function $E\in\mathcal{H}B^\times$ of order $\frac12$, finite type, of completely regular $\sqrt{r}$-growth with $h_E^{\sqrt{r}}(0),h_E^{\sqrt{r}}(\pi)>0$. Then $1\in\mathcal{H}(E)$ and has zero order. However, clearly $\mathcal{H}(E)$ is not of finite $\lambda$-type e.g. with $\lambda(r)=\sqrt[3]{r}$.

# 4 Hermite-Biehler functions of completely regular growth

We investigate the general form of the indicator of a function $E \in \Lambda \cap \mathcal{H}B^\times$. This gives us, in particular, a rich variety of examples of functions belonging to $\mathcal{H}B^\times$ and having some prescribed growth. The case of functions of completely regular growth can be settled rather completely. Before we do so, let us remark that the occurrence of not completely regular growth seems to be a somewhat involved matter. However, in the particular case of $\lambda(r) = r^\rho$, it is fairly easy to answer this question.

If $(a_n)_{n\in\mathbb{N}}$ is a sequence of nonzero complex numbers (without any finite accumulation point) and $\psi_1 \leq \psi_2$, denote by $n(r;\psi_1,\psi_2;(a_n)_{n\in\mathbb{N}})$ the number

$$n(r;\psi_1,\psi_2;(a_n)_{n\in\mathbb{N}}) := \#\big\{n \in \mathbb{N} : |a_n| \leq r, \arg a_n \in (\psi_1,\psi_2]\big\}.$$

In order to shorten notation put $n(r;(a_n)_{n\in\mathbb{N}}) := n(r;-\pi,\pi;(a_n)_{n\in\mathbb{N}})$. If it is anyway clear from the context which sequence we are dealing with, we shall also drop the argument $(a_n)_{n\in\mathbb{N}}$.

**4.1 Lemma.** *Let $\lambda(r) = r^\rho$. Then there exist functions $E \in \Lambda \cap \mathcal{H}B^\times$ which are not of completely regular $\lambda$-growth if and only if $\rho \notin 2\mathbb{N}$.*

*Proof.* Consider first the case that $\rho \notin 2\mathbb{N}$. Choose a sequence $(r_n)_{n\in\mathbb{N}}$, $0 < r_1 < r_2 < r_3 < \ldots$, such that

$$\liminf_{r\to\infty} \frac{n(r;(r_n)_{n\in\mathbb{N}})}{r^\rho} < \limsup_{r\to\infty} \frac{n(r;(r_n)_{n\in\mathbb{N}})}{r^\rho} < \infty. \tag{4.1}$$

One can take the sequence of integers contained in $\bigcup_{k\in\mathbb{N}}(3^k, 3^k + 3^{k+1})$. Next choose a sequence $(\psi_n)_{n\in\mathbb{N}}$, $-\frac{\pi}{4} < \psi_n < 0$, $\lim_{n\to\infty}\psi_n = 0$, such that

$$\sum_{n\in\mathbb{N}} \Big|\frac{\sin\psi_n}{r_n}\Big| < \infty,$$

For example take $\psi_n$ such that $-\sin\psi_n \leq r_n^{-\rho}$. Finally define a sequence $(z_n)_{n\in\mathbb{N}}$ by

$$z_n := \begin{cases} r_k e^{i\psi_k} & , \ n = 2k-1 \\ r_k e^{i(\pi-\psi_k)} & , \ n = 2k \end{cases}.$$

Clearly, the relation (4.1) holds also for $n(r;(z_n)_{n\in\mathbb{N}})$, and by our choice of $(\psi_n)_{n\in\mathbb{N}}$ the sequence $(z_n)_{n\in\mathbb{N}}$ satisfies (2.2).

Moreover, in case that $\rho$ is an integer, consider the sum

$$\sum_{|z_n|\leq R} \frac{1}{z_n^\rho} = \sum_{|z_n|\leq R} \operatorname{Re}\frac{1}{z_n^\rho} + i\sum_{|z_n|\leq R} \operatorname{Im}\frac{1}{z_n^\rho}.$$

The second summand is bounded with respect to $R$ by Lemma 2.2, $(i)$. If $\rho$ is odd, the first summand vanishes:

$$\sum_{|z_n|\leq R} \operatorname{Re}\frac{1}{z_n^\rho} = \sum_{|r_k|\leq R} \frac{1}{r_k^\rho}\big(\operatorname{Re} e^{i\rho\psi_k} + \operatorname{Re} e^{i\rho(\pi-\psi_k)}\big) =$$

$$= \sum_{|r_k|\leq R} \frac{1}{r_k^\rho}\big(\cos\rho\psi_k + \cos(\rho\pi - \rho\psi_k)\big) = 0.$$

By [R, 13.3.3, 13.5.2] there exists an entire function $F \in \Lambda$ having $(z_n)_{n\in\mathbb{N}}$ as its precise sequence of zeros. Consider the Hadamard factorization of $F$:

$$F(z) = e^{P(z)} \prod_{n\in\mathbb{N}} (1 - \frac{z}{z_n}) e^{\frac{z}{z_n} + \ldots + \frac{1}{p}(\frac{z}{z_n})^p},$$

where $p$ denotes the genus of $(z_n)_{n\in\mathbb{N}}$. We find by virtue of Lemma 2.2 and Lemma 3.12 that

$$E(z) := e^{-P(z)} e^{-iz\sum_{n\in\mathbb{N}} \mathrm{Im}\,\frac{1}{z_n} - \ldots - \frac{i}{p} z^p \sum_{n\in\mathbb{N}} \mathrm{Im}\,\frac{1}{z_n^p}} \cdot F(z) \in \Lambda \cap \mathcal{H}B^\times.$$

By [L, III.3.Lehrsatz 3] this function $E$ cannot be of completely regular growth.

Assume next that $\rho$ is an even integer, and let $E \in \Lambda \cap \mathcal{H}B^\times$. We show that $E$ is of convergence class. For the zeros $(w_n)_{n\in\mathbb{N}}$ of $E$ which are contained in the angle $\{w \in \mathbb{C} : \arg w \in (-\frac{3\pi}{4}, \frac{\pi}{4})\}$ satisfy by Lemma 2.2, $(ii)$, the relation

$$\sum_{n\in\mathbb{N}} \frac{1}{|w_n|} < \infty,$$

and therefore also $\sum_{n\in\mathbb{N}} \frac{1}{|w_n|^\rho} < \infty$. Let $(z_n)_{n\in\mathbb{N}}$ be the sequence of zeros of $E$ not contained in this angle and write $z_n = r_n e^{i\psi_n}$. From $E \in \Lambda$ we obtain that $E$ is of order $\rho$ and not of maximal type. Thus by [L, I.11.Lehrsatz 15] we must have

$$\sum_{|z_n|\leq R} \frac{1}{z_n^\rho} = O(1).$$

Hence, as a consequence of Lemma 2.2, $(iii)$,

$$\sum_{|z_n|\leq R} \Big(\mathrm{Re}\,\frac{1}{z_n}\Big)^\rho = O(1).$$

However, since $\rho$ is even, $\Big(\mathrm{Re}\,\frac{1}{z_n}\Big)^\rho = \frac{\cos^\rho\psi_n}{r_n^\rho} \geq 0$ and $\cos^\rho\psi_n \geq (\frac{1}{2})^{\frac{\rho}{2}}$. It follows that

$$\sum_{n\in\mathbb{N}} \frac{1}{r_n^\rho} = \sum_{n\in\mathbb{N}} \frac{1}{|z_n|^\rho}$$

converges.

Hence $E$ can be written as

$$E(z) = e^{P(z)} \prod_{n\in\mathbb{N}} \big(1 - \frac{z}{w_n}\big) \prod_{n\in\mathbb{N}} \big(1 - \frac{z}{z_n}\big) e^{\frac{z}{z_n} + \ldots + \frac{1}{p}(\frac{z}{z_n})^p},$$

where for the genus $p$ of the sequence $(z_n)_{n\in\mathbb{N}}$ we have $p < \rho$ and $P(z)$ is a polynomial of degree at most $\rho$. Clearly, the factor $e^{P(z)}$ is of completely regular $r^\rho$-growth. The first product is of order at most 1, henceforth of minimal $r^\rho$-type and thus of completely regular $r^\rho$-growth. The same argument applies to the latter product, since it is either of order $< \rho$ (if the convergence exponent of $(z_n)_{n\in\mathbb{N}}$ is less than $\rho$), or of order $\rho$, minimal type (by [L, I.11.Lehrsatz 15]).

□

If in the above proof we use [L, I.13.Lehrsatz 18] instead of [L, I.11.Lehrsatz 15] we obtain the following statement:

**4.2 Corollary.** *If $\lambda$ is a growth function with $\rho \in 2\mathbb{N}$ and $\limsup_{r\to\infty} \frac{r^\rho}{\lambda(r)} > 0$, then any $E \in \Lambda \cap \mathcal{HB}^\times$ is of convergence class (with respect to the order $\rho$).*

In the following we will employ the more refined arguments of [L, Kapitel II,III] in order to discuss the question whether functions of class $\mathcal{HB}$ with prescribed completely regular $\lambda$-growth exist.

The fact that $E \in \mathcal{HB}$ implies $E \in \mathbb{A}$ forces, in case of fast growing $\lambda$, the majority of zeros of $E$ to be close to the real axis (compare also Lemma 4.4). This observation might explain the basic differences between slow and fast growing $\lambda$ (e.g. $\rho < 1$ versus $\rho > 1$) which we shall encounter in the subsequent discussion as well as in various other places.

Let $\lambda$ be a growth function, $\rho = \lim_{r\to\infty} \frac{\log \lambda(r)}{\log r}$, and let $E \in \Lambda \cap \mathcal{HB}^\times$ be of completely regular $\lambda$-growth. Then the indicator function $h_E^\lambda(\theta)$ can be computed explicitly from the distribution of zeros $(z_n)_{n\in\mathbb{N}}$ of $E$ as listed in the subsequent items (A)-(D). We can also prove in most cases that conversely, if $h(\theta)$ is a function of one of the respective forms (A)-(D), then there exists $E \in \Lambda \cap \mathcal{HB}^\times$ of completely regular $\lambda$-growth such that $h = h_E^\lambda$.

Let $(z_n)_{n\in\mathbb{N}}$ be a sequence which has a $\lambda$-density, i.e. is such that for all but at most countably many values of $\psi_1$ and $\psi_2$ the limit

$$\lim_{r\to\infty} \frac{n(r,\psi_1,\psi_2)}{\lambda(r)} =: \Delta\big(\{w \in \mathbb{T} :\, \arg w \in (\psi_1,\psi_2)\}\big) \tag{4.2}$$

exists. Then we shall denote by $\Delta$ the Borel measure on the unit circle $\mathbb{T} = \{w \in \mathbb{C} :\, |w| = 1\}$ determined by (4.2), cf. [L]. If $E$ is a function of completely regular $\lambda$-growth, then the sequence $(z_n)_{n\in\mathbb{N}}$ of its zeros has a $\lambda$-density (cf. [L, III.3.Lehrsatz 3]). Note that, if a function $E$ has no zeros in $\mathbb{C}^+$ (which is surely the case for any $E \in \mathcal{HB}$), then

$$\Delta\big(\{w \in \mathbb{T} :\, \arg w \in (0,\pi)\}\big) = 0. \tag{4.3}$$

We shall always write a function $E$ which belongs to $\mathcal{HB}^\times$ and is of finite order (according to Lemma 3.12) in the form

$$E(z) = e^{P(z)-iaz} e^{-iz\sum_{n\in\mathbb{N}} \operatorname{Im}\frac{1}{z_n} - \ldots - \frac{i}{q} z^q \sum_{n\in\mathbb{N}} \operatorname{Im}\frac{1}{z_n^q}} \times \tag{4.4}$$
$$\times \prod_{n\in\mathbb{N}} \big(1 - \frac{z}{z_n}\big) e^{\frac{z}{z_n} + \ldots + \frac{1}{q}(\frac{z}{z_n})^q}$$

where $q := [\rho]$, $P(z) = c_q z^q + \ldots + c_0$ is some real polynomial and $a \geq 0$. Note that the infinite product in (4.4) is convergent since $q$ is not smaller than the genus of the sequence $(z_n)_{n\in\mathbb{N}}$.

We consider first the (easier) case $\rho \notin \mathbb{N}$.

**4.3 Theorem.** *Assume that $E \in \Lambda \cap \mathcal{HB}^\times$ is of completely regular $\lambda$-growth.*

(A) Case $\rho > 1$, $\rho \notin \mathbb{N}$: *With the numbers*

$$d_+ = \Delta\big(\{1\}\big),\ d_- = \Delta\big(\{-1\}\big),$$

*we have*

$$h_E^\lambda(\theta) = \frac{\pi}{\sin \rho\pi} \cdot \begin{cases} d_+ \cos\rho(\theta-\pi) + d_- \cos\rho\theta & ,\ \theta \in [0,\pi] \\ d_+ \cos\rho(\theta-\pi) + d_- \cos\rho(\theta-2\pi) & ,\ \theta \in [\pi,2\pi] \end{cases} \tag{4.5}$$

(B) Case $0<\rho<1$: *For all $\theta\in[0,2\pi]$ we have*

$$h_E^\lambda(\theta)=\frac{\pi}{\sin\rho\pi}\Bigg(\int\limits_{\{w\in\mathbb{T}:\,\arg w\in(\theta,2\pi]\}}\cos\rho(\theta-\arg w+\pi)\,d\Delta\;+ \tag{4.6}$$
$$+\int\limits_{\{w\in\mathbb{T}:\,\arg w\in[\pi,\theta]\}}\cos\rho(\theta-\arg w-\pi)\,d\Delta\Bigg)$$

*For $\theta\in[0,\pi]$ this formula takes the form*

$$h_E^\lambda(\theta)=\frac{\pi}{\sin\rho\pi}\big(a\cos\rho\theta+b\sin\rho\theta\big),\;\;\theta\in[0,\pi], \tag{4.7}$$

*where*

$$a:=\int\limits_{\mathbb{T}}\cos\rho(\arg w-\pi)\,d\Delta,\;\;b:=\int\limits_{\mathbb{T}}\sin\rho(\arg w-\pi)\,d\Delta.$$

*Conversely, if in case* (A) *two numbers $d_+,d_-\geq 0$ are given, there exists a function $E\in\Lambda\cap\mathcal{H}B^\times$ of completely regular $\lambda$-growth having the function defined by the right hand side of* (4.5) *as its indicator function. If in case* (B) *any finite Borel measure on the unit circle satisfying* (4.3) *is prescribed, there exists a function $E\in\Lambda\cap\mathcal{H}B^\times$ of completely regular $\lambda$-growth having* (4.6) *as its indicator function.*

For the proof we shall use the following

**4.4 Lemma.** *Let $\lambda$ be a growth function and let $(z_n)_{n\in\mathbb{N}}$ be a sequence of nonzero complex numbers (without any finite accumulation point). Then:*

*(i) If $\int_1^\infty\frac{\lambda(r)}{r^2}\,dr<\infty$ and $\limsup_{r\to\infty}\frac{n(r;(z_n)_{n\in\mathbb{N}})}{\lambda(r)}<\infty$, then we have $\sum_{n\in\mathbb{N}}\frac{1}{|z_n|}<\infty$.*

*(ii) If $\int_1^\infty\frac{\lambda(r)}{r^2}\,dr=\infty$ and $\sum_{n\in\mathbb{N}}\frac{1}{|z_n|}<\infty$, then $\liminf_{r\to\infty}\frac{n(r;(z_n)_{n\in\mathbb{N}})}{\lambda(r)}=0$.*

*Proof.* Choose $\epsilon>0$ such that $n(\epsilon)=0$.
`ad`$(i)$: By our assumption (4.2) there exists a constant $M>0$ such that $n(r)\leq M\lambda(r)$, $r\geq\epsilon$. Thus

$$\int_\epsilon^R\frac{n(r)}{r^2}\,dr\leq M\int_\epsilon^R\frac{\lambda(r)}{r^2}\,dr\leq M\int_\epsilon^\infty\frac{\lambda(r)}{r^2}\,dr<\infty.$$

Integration by parts gives

$$\int_\epsilon^R\frac{n(r)}{r^2}\,dr=-\frac{n(R)}{R}+\int_\epsilon^R\frac{dn(r)}{r}=-\frac{n(R)}{R}+\sum_{|z_n|\leq R}\frac{1}{|z_n|}.$$

However, $\frac{n(R)}{R}\leq M\frac{\lambda(R)}{R}$, and since the present condition on $\lambda$ clearly implies $\liminf_{R\to\infty}\frac{\lambda(R)}{R}=0$, we obtain

$$\liminf_{R\to\infty}\sum_{|z_n|\leq R}\frac{1}{|z_n|}<\infty,$$

which means nothing else but $\sum_{n\in\mathbb{N}}\frac{1}{|z_n|}<\infty$.

$\mathtt{ad}(ii)$: Assume on the contrary that for some $m > 0$ we have $n(r) \geq m\lambda(r)$. Then the same computation as in the preceeding paragraph shows that for all $R \geq \epsilon$

$$m \int_\epsilon^R \frac{\lambda(r)}{r^2}\,dr \leq -\frac{n(R)}{R} + \int_\epsilon^R \frac{dn(r)}{r} \leq \sum_{n\in\mathbb{N}} \frac{1}{|z_n|} < \infty,$$

a contradiction.

$\square$

Note that, if $\rho > 1$, surely $\int_1^\infty \frac{\lambda(r)}{r^2}\,dr = \infty$, whereas in case $\rho < 1$ the integral converges.

*Proof. (of Theorem 4.3)*

$\mathtt{ad}$(A), necessity: By Lemma 4.4, $(ii)$, and Lemma 2.2, $(ii)$, we must have

$$\Delta\big(\{w \in \mathbb{T} :\ \arg w \in (\pi, 2\pi)\}\big) = 0,$$

i.e. $\Delta$ is just the sum of two point masses, $d_+ := \Delta(\{1\})$ at 1 and $d_- := \Delta(\{-1\})$ at $-1$. By [L, II.1.Lehrsatz 1] the function $E(z) = e^{Q(z)}V(z)$ where $V$ denotes the canonical product

$$V(z) = \prod_{n\in\mathbb{N}} \big(1 - \frac{z}{z_n}\big) e^{\frac{z}{z_n} + \ldots + \frac{1}{p}(\frac{z}{z_n})^p}, \tag{4.8}$$

the genus $p$ is equal to $[\rho]$ and $Q(z)$ is a polynomial of degree at most $p$, satisfies (with possible exception of a small set, cf. [L]) the asymptotic equation ($0 \leq \theta \leq 2\pi$)

$$\lim_{r\to\infty} \frac{\log|E(re^{i\theta})|}{\lambda(r)} = \frac{\pi}{\sin\rho\pi} \int_{\theta-2\pi}^{\theta} \cos\rho(\theta - \psi - \pi)\,d\Delta(\psi), \tag{4.9}$$

where the integral has to be understood as a Stieltjes integral with respect to the nondecreasing function $\Delta(\psi)$ corresponding to the Borel measure $\Delta$. The formula (4.5) follows on substituting our knowledge that $\Delta$ consists just of two point masses.

$\mathtt{ad}$(B), necessity: Rewriting (4.9) we obtain (4.6). If $\theta \in [0, \pi)$, the second summand in (4.6) vanishes and the first integral is evaluated as (keep in mind that the relation (4.3) is valid)

$$\int\limits_{\{w\in\mathbb{T}:\, \arg w\in(\theta,2\pi]\}} \cos\rho(\theta - \arg w + \pi)\,d\Delta =$$

$$= \cos\rho\theta \quad \cdot \int\limits_{\{w\in\mathbb{T}:\, \arg w\in(\theta,2\pi]\}} \cos\rho(\pi - \arg w)\,d\Delta - \sin\rho\theta \quad \cdot \int\limits_{\{w\in\mathbb{T}:\, \arg w\in(\theta,2\pi]\}} \sin\rho(\pi - \arg w)\,d\Delta =$$

$$= \cos\rho\theta \cdot \int\limits_{\mathbb{T}} \cos\rho(\pi - \arg w)\,d\Delta - \sin\rho\theta \cdot \int\limits_{\mathbb{T}} \sin\rho(\pi - \arg w)\,d\Delta,$$

i.e. (4.7) holds.

ad(A), sufficiency: Once we have constructed a sequence $(z_n)_{n\in\mathbb{N}}$, $z_n \in \mathbb{C}^-$, which satisfies (2.2), has finite $\lambda$-density in every angle and thereby satisfies

$$\begin{gathered}\Delta\big(\{1\}\big) = d_+,\ \Delta\big(\{-1\}\big) = d_-,\\ \Delta\big(\{w \in \mathbb{T} :\ \arg w \in (0,\pi) \cup (\pi, 2\pi)\}\big) = 0,\end{gathered} \tag{4.10}$$

we know (cf. [L, II.4.1,II.1.Lehrsatz 1]) that the function $(p = [\rho])$

$$E(z) := e^{-iz\sum_{n\in\mathbb{N}} \operatorname{Im}\frac{1}{z_n} - \ldots - \frac{i}{p} z^p \sum_{n\in\mathbb{N}} \operatorname{Im}\frac{1}{z_n^p}} \cdot V(z), \tag{4.11}$$

where $V$ is the canonical product (4.8), belongs to $\Lambda$ and its indicator function is given by (4.5). Moreover, by Lemma 3.12, it belongs to $\mathcal{HB}^\times$.

However, it is easy give an example of such a sequence $(z_n)_{n\in\mathbb{N}}$: If $d_+, d_- > 0$, choose $r_n := \lambda^{-1}(\frac{n}{d_+})$ and $t_n := \lambda^{-1}(\frac{n}{d_-})$. If $d_+$ or $d_-$ should be equal to 0 choose $r_n := \lambda^{-1}(n^2)$ or $t_n := \lambda^{-1}(n^2)$, respectively. Next, following the construction of Lemma 4.1, choose $\psi_n \in (-\frac{\pi}{4}, 0)$ and $\phi_n \in (\pi, \frac{3\pi}{4})$ with $-\sin\psi_n \leq r_n^{-\rho}$, $-\sin\phi_n \leq t_n^{-\rho}$, and put

$$z_n := \begin{cases} r_k e^{i\psi_k} & ,\ n = 2k-1 \\ t_k e^{i\phi_k} & ,\ n = 2k \end{cases}. \tag{4.12}$$

Clearly, $\lim_{r\to\infty} \frac{n(r,(r_n)_{n\in\mathbb{N}})}{\lambda(r)} = d_+$ and $\lim_{r\to\infty} \frac{n(r,(t_n)_{n\in\mathbb{N}})}{\lambda(r)} = d_-$. Since for any $\delta > 0$ only finitely many points $z_n$ lie in the angle $\{w \in \mathbb{C} :\ \arg w \in (-\pi + \delta, -\delta)\}$, we conclude that for this sequence $(z_n)_{n\in\mathbb{N}}$ the relations (4.10) hold true.

ad(B), sufficiency: Denote by $\tilde{\Delta}$ the Borel measure given by

$$\tilde{\Delta}(M) := \Delta\big(M \setminus \{1, -1\}\big),$$

and put again $d_+ := \Delta(\{1\})$, $d_- := \Delta(\{-1\})$. By [L, II.4.2] there exists a sequence $(w_n)_{n\in\mathbb{N}}$ with

$$\lim_{r\to\infty} \frac{n(r;\psi_1,\psi_2;(w_n)_{n\in\mathbb{N}})}{\lambda(r)} = \tilde{\Delta}\big(\{w \in \mathbb{T} :\ \arg w \in (\psi_1, \psi_2)\}\big)$$

for all but at most countably many $\psi_1, \psi_2$. Observe that, since $\tilde{\Delta}(\{1\}) = \tilde{\Delta}(\{-1\}) = 0$, the construction of [L, II.4.2] can be made such that $w_n \in \mathbb{C}^-$.

Define a sequence $\tilde{w}_n$ exactly the same as done in the proof of (A) and let $(z_n)_{n\in\mathbb{N}}$ be such that $\{z_n : n \in \mathbb{N}\} = \{w_n : n \in \mathbb{N}\} \cup \{\tilde{w}_n : n \in \mathbb{N}\}$. Then

$$z_n \in \mathbb{C}^-,\ n \in \mathbb{N},$$

and $(z_n)_{n\in\mathbb{N}}$ has a $\lambda$-density which coincides with the prescribed measure $\Delta$. By Lemma 4.4, $(i)$, we have

$$\sum_{n\in\mathbb{N}} \frac{1}{|z_n|} < \infty,$$

and henceforth also (2.2) holds. Defining $E$ by (4.11) concludes the proof of (B).

$\square$

We proceed with the case $\rho \in \mathbb{N}$. Recall that we always factorize $E \in \mathcal{H}B^\times$ as in (4.4) where now $q = \rho$.

Although the given necessary conditions seem likely to be also sufficient we were only able to construct the required functions under certain additional hypotheses on the growth function $\lambda$. However, let us remark that despite this lack in generality a broad class of growth functions is covered by our consideration, for example those $\lambda$ of the form (3.2).

Recall from [L, I.12] the notion of a strong proximate order: We shall call a growth function $\lambda(r)$ a *strong growth function*, if $\frac{\lambda(r)}{r^\rho}$ can be written in the form

$$\frac{\lambda(r)}{r^\rho} = e^{\vartheta_2(\log r) - \vartheta_1(\log r)},$$

with some concave functions $\vartheta_1, \vartheta_2$ which satisfy

$$\lim_{x\to\infty} \vartheta_i(x) = +\infty, \ \lim_{x\to\infty} \frac{\vartheta_i(x)}{x} = 0, \ \lim_{x\to\infty} \frac{\vartheta_i''(x)}{\vartheta_i'(x)} = 0.$$

This means that $\log \lambda(r)$ is a strong proximate order in the sense of [L].

**4.5 Theorem.** *Assume that $E \in \Lambda \cap \mathcal{H}B^\times$ is of completely regular $\lambda$-growth.*

(C) Case $\rho \in \mathbb{N}$, $\rho \neq 1$: *Put $d_+ = \Delta(\{1\})$ and let $\tau, \sigma \in \mathbb{R}$, $\tau \geq 0$ be such that*

$$\tau e^{-i\rho\sigma} = \lim_{r\to\infty} \frac{r^\rho}{\lambda(r)} \Big( c_\rho + \frac{1}{\rho} \sum_{|z_n|\leq r} \operatorname{Re} \frac{1}{z_n^\rho} - \frac{i}{\rho} \sum_{|z_n|>r} \operatorname{Im} \frac{1}{z_n^\rho} \Big). \tag{4.13}$$

*Then $\tau = 0$ or $\rho\sigma \equiv 0 \mod \pi$ and*

$$h_E^\lambda(\theta) = \tau(-1)^{\frac{\rho\sigma}{\pi}} \cos\rho\theta + \pi d_+ \sin\rho|\theta| \tag{4.14}$$

*If $\rho$ is even, then necessarily $d_+ = 0$.*

(D) Case $\rho = 1$: *Let $d_+, \tau$ and $\sigma$ have the same meaning as in* (C)*. Then $\tau = 0$ or $\sigma \in [0, \pi]$ ( $\mathrm{mod}\ 2\pi$). Moreover,*

$$h_E^\lambda(\theta) = \tau\cos\sigma\cos\theta + \tau\sin\sigma\sin\theta + d_+\pi|\sin\theta|, \tag{4.15}$$

*i.e. the indicator diagram of $E$ is a vertical line segment centered at the point $(\tau\cos\sigma, \tau\sin\sigma)$.*

*Conversely, assume that numbers $d_+ \geq 0$, $\tau \geq 0$, $\sigma \in [0, \frac{\pi}{\rho}]$ are given, where $d_+ > 0$ is allowed only if $\rho$ is odd and $\sigma \notin \{0, \frac{\pi}{\rho}\}$ only if $\rho = 1$. Then there exists $E \in \Lambda \cap \mathcal{H}B^\times$ with indicator function given by* (4.14) *or* (4.15)*, respectively, if we additionally assume that either of the following conditions hold:*

(1) *$\tau = 0$ or $r^\rho = O(\lambda(r))$.*

($2_c$) *$\lambda$ is a strong growth function and $\rho > 1$ is odd.*

($2_d$) *$\lambda$ is a strong growth function, $\rho = 1$ and $\sigma \in \{0, \pi\}$.*

(3) $\lambda$ *is a strong growth function with* $\vartheta_2 = 0$.

*Proof. (of Theorem 4.5, necessity)* In the case of integer $\rho$ not only the zeros $(z_n)_{n\in\mathbb{N}}$ must have a density, but also (cf. [L, III.3. p.153 ff]) the limit

$$\lim_{r\to\infty} \frac{r^\rho}{\lambda(r)}\Big(c_\rho - \frac{i}{\rho}\sum_{n\in\mathbb{N}} \operatorname{Im}\frac{1}{z_n^\rho} + \sum_{|z_n|\le r}\frac{1}{z_n^\rho}\Big)$$

must exist. Thus the right hand side of (4.13) is meaningful.

We use [L, II.1.Lehrsatz 2] which states (together with [L, III.3.Lehrsatz 3]), that for $\theta\in[0,2\pi]$

$$\lim_{r\to\infty}\frac{\log|E(re^{i\theta})|}{\lambda(r)} = -\int_{\theta-2\pi}^{\theta}(\psi-\theta)\sin\rho(\psi-\theta)\,d\Delta(\psi) + \tau\cos\rho(\theta-\sigma). \quad (4.16)$$

Moreover, by [L, I.16.Lehrsatz 24] we must have

$$\int_{\mathbb{T}} w^\rho\, d\Delta = 0. \quad (4.17)$$

Assume for the moment that we have already proved that $\Delta$ is the sum of two point masses at 1 and at $-1$. The relation (4.17) then reduces to

$$\Delta\big(\{-1\}\big) = (-1)^{\rho+1}\Delta\big(\{1\}\big).$$

From this it follows that $\Delta(\{-1\}) = \Delta(\{1\}) = 0$ for even $\rho$ and $\Delta(\{-1\}) = \Delta(\{1\})$ for odd $\rho$.

Thus formula (4.16) can be rewritten as

$$\begin{aligned} h_E^\lambda(\theta) &= \tau\cos\rho\sigma\cos\rho\theta + \tau\sin\rho\sigma\sin\rho\theta + \\ &+ \begin{cases} (-1)^{\rho+1}d_+\pi\sin\rho\theta & ,\ \theta\in[0,\pi] \\ -d_+\pi\sin\rho\theta & ,\ \theta\in[\pi,2\pi] \end{cases}. \end{aligned}$$

If $d_+ = \Delta(\{1\}) > 0$, we must have $(-1)^{\rho+1} = 1$, and therefore

$$h_E^\lambda(\theta) = \tau\cos\rho\sigma\cos\rho\theta + \tau\sin\rho\sigma\sin\rho\theta + \pi d_+\sin\rho|\theta|. \quad (4.18)$$

`ad`(C): By Lemma 4.4, $(ii)$, the measure $\Delta$ is concentrated in the points $\pm1$, and hence (4.18) holds. Since $E\in\mathcal{H}B$ we must have $h_E^\lambda(\theta)\ge h_E^\lambda(-\theta)$ for all $\theta\in[0,\pi]$, i.e.

$$2\tau\sin\rho\sigma\sin\rho\theta \ge 0,\ \ \theta\in[0,\pi].$$

Since $\sin\rho\theta$ changes its sign when $\theta$ varies in $[0,\pi]$, this is only possible if $\tau\sin\rho\sigma = 0$ or, equivalently, if $\tau = 0$ or $\rho\sigma\equiv 0 \mod \pi$. The relation (4.14) follows.

`ad`(D): Considering imaginary parts in (4.17) and keeping in mind that $\Delta(\{w\in\mathbb{T}:\ \arg w\in(0,\pi)\}) = 0$, yields

$$0 = \int_{\mathbb{T}}\sin(\arg w)\,d\Delta = \int\limits_{\{w\in\mathbb{T}:\,\arg w\in[\pi,2\pi]\}}\sin(\arg w)\,d\Delta.$$

Hence $\Delta(\{w\in\mathbb{T}:\ \arg w\in(\pi,2\pi)\}) = 0$.

From (4.15) we obtain $\tau\sin\sigma\sin\theta\ge 0$ for all $\theta\in[0,\pi]$. This, however, is only possible if $\sigma\in[0,\pi]$ (or trivially if $\tau = 0$).

$\square$

In the proof of sufficiency we will repeatedly make use of the following elementary observation.

**4.6 Lemma.** *Let $(s_n)_{n\in\mathbb{N}}$ with $1 < s_1 \leq s_2 \leq \ldots$, $s_n \to \infty$, and a strictly decreasing function $l : [1,\infty) \to \mathbb{R}^+$ with $l(s) \to 0$ be given. Then there exists a sequence $(a_n)_{n\in\mathbb{N}}$, $a_n > 0$, such that*

$$\sum_{s_n > s} a_n \leq l(s),\ \ s \geq 1. \tag{4.19}$$

*Proof.* First assume that always $s_n < s_{n+1}$. Choose a continuous bijection $m : [1,\infty) \to [s_1,\infty)$ such that $m(n) = s_n$, $n \in \mathbb{N}$. Then with $l$ also $\hat{l} := l \circ m$ is strictly decreasing and tends to zero. Put

$$a_n := \hat{l}(n) - \hat{l}(n+1) > 0,\ \ n \in \mathbb{N}.$$

Then

$$\sum_{s_n > s} a_n = \sum_{m(n) > s} a_n = \sum_{n > m^{-1}(s)} a_n =$$

$$= \hat{l}\big([m^{-1}(s)] + 1\big) \leq \hat{l}(m^{-1}(s)) = l(s).$$

In the case that $(s_n)_{n\in\mathbb{N}}$ is not strictly increasing, we chose another sequence $(s_n')_{n\in\mathbb{N}}$ such that, if $s_{n-1} < s_n = s_{n+1} = \ldots = s_{n+k} < s_{n+k+1}$, we have

$$s_n = s_n' < s_{n+1}' < \ldots < s_{n+k}' < s_{n+k+1}' = s_{n+k+1}.$$

Since always $s_n' \geq s_n$, we have

$$\{n \in \mathbb{N} :\ s_n > s\} \subseteq \{n \in \mathbb{N} :\ s_n' > s\},$$

and henceforth a sequence $(a_n)_{n\in\mathbb{N}}$ satisfying (4.19) with $s_n'$ in place of $s_n$ a forteriori satisfies this estimate with $s_n$.

$\square$

*Proof. (of Theorem 4.5, sufficiency)* First we construct a function $E_1(z) \in \Lambda \cap \mathcal{H}B^\times$ being of completely $\lambda$-regular growth which has the indicator function $h_E^\lambda(\theta) = \pi d_+ \sin\rho|\theta|$. This is always possible, i.e. the problems in proving a full converse in Theorem 4.5 will lie in the construction of a function $E_2 \in \Lambda \cap \mathcal{H}B^\times$ having a trigonometric indicator.

For $d_+ = 0$ we take $E_1(z) := 1$. So assume that $d_+ > 0$ and $\rho$ is odd. By [L, II.1.Lehrsatz 2] and Lemma 3.12 it is enough to construct a sequence $(z_n)_{n\in\mathbb{N}}$, $z_n \in \mathbb{C}^-$, having the $\lambda$-density

$$\lim_{r\to\infty} \frac{n(r;\psi_1,\psi_2;(z_n)_{n\in\mathbb{N}})}{\lambda(r)} = \Delta\big(\{w \in \mathbb{T} :\ \arg w \in (\psi_1,\psi_2]\}\big)$$

which is the sum of the two point masses at $\pm 1$ with equal weight $d_+$, and satisfies

$$\sum_{|z_n| \leq r} \operatorname{Re} \frac{1}{z_n^\rho} = 0\,, \tag{4.20}$$

$$\lim_{r\to\infty}\frac{r^\rho}{\lambda(r)}\sum_{|z_n|>r}\operatorname{Im}\frac{1}{z_n^\rho}=0,\ \sum_{n\in\mathbb{N}}\operatorname{Im}\frac{1}{z_n}<\infty. \tag{4.21}$$

For if $(z_n)_{n\in\mathbb{N}}$ possesses all these properties, then

$$E_1(z):=e^{-iz\sum_{n\in\mathbb{N}}\operatorname{Im}\frac{1}{z_n}-\ldots-\frac{i}{\rho}z^\rho\sum_{n\in\mathbb{N}}\operatorname{Im}\frac{1}{z_n^\rho}}\prod_{n\in\mathbb{N}}\big(1-\frac{z}{z_n}\big)e^{\frac{z}{z_n}+\cdots\frac{1}{\rho}(\frac{z}{z_n})^\rho}$$

has the desired growth and belongs to $\mathcal{H}B^\times$.

In order to provide an example of such a sequence put $r_n:=\lambda^{-1}(\frac{n}{d_+})$ and

$$z_n:=\begin{cases} r_ke^{i\psi_k} & ,\ n=2k-1\\ r_ke^{i(\pi-\psi_k)} & ,\ n=2k\end{cases},$$

where the $\psi_k\in(-\frac{\pi}{2\rho},0)$ will be chosen as indicated in the following lines. Clearly, by our choice of $(z_n)$ we have already achieved that $(z_n)_{n\in\mathbb{N}}$ has the desired density distribution (as in the proof of Theorem 4.3) and that, by symmetry, (4.20) holds.

Choose a strictly decreasing function $l(s)$, $l(s)\to 0$, such that

$$l(s)\le\frac{1}{s}\frac{\lambda(s)}{r^\rho}.$$

The existence of such an $l$ is obvious. By Lemma 4.6 there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of positive numbers with

$$\sum_{r_n>r}a_n\le l(r),\ r>0.$$

Clearly, it is possible to choose $\psi\in(-\frac{\pi}{2\rho},0)$, $n\in\mathbb{N}$, so small that for all $n\in\mathbb{N}$,

$$\frac{-\sin\rho\psi_n}{r_n^\rho}\le a_n,\ \frac{-\sin\psi_n}{r_n}\le\frac{1}{n^2}.$$

Then $(z_n)_{n\in\mathbb{N}}$ satisfies (4.21).

We turn to the construction of $E_2\in\Lambda\cap\mathcal{H}B^\times$ which is of completely regular $\lambda$-growth and has the trigonometric indicator $h_{E_2}^\lambda(\theta)=\tau\cos\rho(\theta-\sigma)$.

`ad(1)`: Under this hypothesis the assertion is trivial: The function $E_2(z):=e^{\tau e^{-i\rho\sigma}z^\rho}$ belongs to $\Lambda$, and by the assumptions on $\sigma$ in connection with the value of $\rho$ it also is contained in $\mathcal{H}B^\times$.

`ad`$(2_c)$ and $(2_d)$: Assume that $\lambda$ is a strong growth function, that $\rho$ is odd and that $\rho\sigma\in\{0,\pi\}$. By the construction of [L, II.4.4.] we obtain the existence of a sequence $(w_n)_{n\in\mathbb{N}}$ such that the product

$$\prod_{n\in\mathbb{N}}\big(1-\frac{z}{w_n}\big)e^{\frac{z}{w_n}+\cdots\frac{1}{\rho}(\frac{z}{w_n})^\rho}$$

has the desired growth, i.e. has zero $\lambda$-density and satisfies

$$\lim_{r\to\infty}\frac{r^\rho}{\rho\lambda(r)}\sum_{|w_n|\le r}\frac{1}{w_n^\rho}=\tau e^{-i\rho\sigma}. \tag{4.22}$$

In this construction the points $w_n$ are located on the real axis and on the ray $\arg w = \frac{\pi}{\rho}$. However, one sees that those on the second named ray can as well be chosen on the negative real axis (one only needs $w_n^\rho < 0$ for those zeros). Hence we can assume that all $w_n$ are real. Applying Lemma 4.6 (in a similar way as in the last paragraph) separately to the sequence of positive $w_n$ and to the sequence of negative $w_n$, yields the existence of $\psi_n$ contained in $(-\frac{\pi}{2\rho}, 0)$ or $(\pi, \pi + \frac{\pi}{2\rho})$, respectively, such that for the sequence $z_n := w_n e^{i\psi_n}$ (2.2) holds,

$$\lim_{r\to\infty} \frac{r^\rho}{\lambda(r)} \sum_{|z_n|>r} \operatorname{Im} \frac{1}{z_n^\rho} = \lim_{r\to\infty} \frac{r^\rho}{\lambda(r)} \sum_{|w_n|>r} \frac{-\sin\rho\psi_n}{w_n^\rho} = 0,$$

the limit

$$c_\rho := \lim_{r\to\infty} \frac{1}{\rho} \sum_{|z_n|\le r} \Big( \frac{1}{w_n^\rho} - \operatorname{Re} \frac{1}{z_n^\rho} \Big) = \lim_{r\to\infty} \frac{1}{\rho} \sum_{|w_n|\le r} \frac{1}{w_n^\rho} \big(1 - \cos\rho\psi_n\big)$$

exists, and moreover

$$\lim_{r\to\infty} \frac{r^\rho}{\lambda(r)} \Big( c_\rho + \sum_{|z_n|\le r} \operatorname{Re} \frac{1}{z_n^\rho} \Big) =$$

$$= \lim_{r\to\infty} \frac{r^\rho}{\lambda(r)} \left( \sum_{|w_n|\le r} \frac{1}{w_n^\rho} + \sum_{|w_n|>r} \frac{1-\cos\rho\psi_n}{w_n^\rho} \right) = \tau e^{-i\rho\sigma}.$$

Hence the function $E_2$ defined by (4.4) possesses all desired properties.

`ad`$(3), \rho\sigma \in \{0, \pi\}$: In order to establish the assertion under this assumption it suffices to note that in the construction [L, II.4. 4.] the present hypothesis implies that the zeros on the ray $\arg w = \frac{\pi}{\rho}$ are not present. The assertion then follows on shifting the real zeros by Lemma 4.6 in the same way as done in the previous part of the proof.

`ad`$(3), \rho = 1, \sigma \in (0, \pi)$: Again by [L, II.4. Paragraph 4.] we obtain a sequence $(r_n)_{n\in\mathbb{N}}$, $1 < r_1 < r_2 < \ldots$, $r_n \to \infty$, such that it has zero $\lambda$-density and satisfies with an appropriate $\gamma \in \mathbb{R}$,

$$\lim_{r\to\infty} \frac{r}{\lambda(r)} \Big( \gamma + \sum_{r_n \le r} \frac{1}{r_n} \Big) = 1. \tag{4.23}$$

Note that, since $\limsup_{r\to\infty} \frac{r}{\lambda(r)} = \infty$, we must have

$$\sum_{n\in\mathbb{N}} \frac{1}{r_n} = -\gamma, \tag{4.24}$$

and therefore (4.23) just says that

$$\lim_{r\to\infty} \frac{r}{\lambda(r)} \sum_{r_n > r} \frac{1}{r_n} = 1.$$

Consider the sequence $(z_n)_{n\in\mathbb{N}}$ defined as

$$z_n := \frac{r_n}{\tau} e^{-i(\pi-\sigma)}, \; n \in \mathbb{N}.$$

Then $z_n \in \mathbb{C}^-$ and by (4.24) the series $\sum_{n\in\mathbb{N}} \frac{1}{z_n}$ is absolutely convergent. In particular (2.2) is satisfied. Put $c_1 := -\sum_{n\in\mathbb{N}} \operatorname{Re} \frac{1}{z_n}$. Then

$$\lim_{r\to\infty} \frac{r}{\lambda(r)} \Big( c_1 - i \sum_{n\in\mathbb{N}} \operatorname{Im} \frac{1}{z_n} + \sum_{|z_n|\leq r} \frac{1}{z_n} \Big) =$$

$$= - \lim_{r\to\infty} \frac{r}{\lambda(r)} \sum_{r_n > r} \frac{1}{z_n} = - \lim_{r\to\infty} \frac{r}{\lambda(r)} \sum_{r_n > r} \Big( \frac{\tau\cos(\pi-\sigma)}{r_n} + i \frac{\tau \sin(\pi - \sigma)}{r_n} \Big) =$$

$$= \tau\cos\sigma - i\tau\sin\sigma = \tau e^{-i\sigma}.$$

Hence

$$E_2(z) := e^{c_1 z} e^{-iz \sum_{n\in\mathbb{N}} \operatorname{Im} \frac{1}{z_n}} \prod_{n\in\mathbb{N}} \big(1 - \frac{z}{z_n}\big) e^{\frac{z}{z_n}}$$

possesses all desired properties.

$\square$

## 5 Subspaces defined by growth conditions

In this section we investigate dB-subspaces $\tilde{\mathcal{H}}$ of a given dB-space $\mathcal{H}(E)$ which are subject to growth conditions of various kinds. On first sight there seem to be two possibilities of imposing such conditions: Either one might restrict the values of $\operatorname{mt} \frac{F}{E}$ for $F \in \tilde{\mathcal{H}}$, or the growth of the functions $F \in \tilde{\mathcal{H}}$ themselves, e.g. by demanding a bound for $h_F^\lambda(\theta)$.

The overall conclusion of the subsequent considerations is that the first variant is in a way more general than the second, and that the case of spaces of exponential type is somewhat special in comparison with the cases of slow or fast growing $\lambda$, $\lambda(r) = o(r)$ vs. $r = o(\lambda(r))$.

**5.1 Definition.** Let $\mathcal{H}$ be a dB-space.

($i$) Write $\mathcal{H} = \mathcal{H}(E)$ and denote for $\alpha \leq 0$ by $\mathcal{H}_\alpha$ the linear space

$$\mathcal{H}_\alpha := \mathcal{H}_{(\alpha,\alpha)} = \Big\{ F \in \mathcal{H} :\ \operatorname{mt} \frac{F}{E}, \operatorname{mt} \frac{F^\#}{E} \leq \alpha \Big\}.$$

In order to justify this notation note that $\mathcal{H}_\alpha$ does not depend on the particular choice of $E$.

($ii$) For a growth function $\lambda$ and a number $\beta \geq 0$ denote by $\mathcal{H}_{\lambda,\beta}$ the linear space

$$\mathcal{H}_{\lambda,\beta} := \{ F \in \mathcal{H} :\ \sigma_F^\lambda \leq \beta \}.$$

By Lemma 2.6 we have the following

**5.2 Corollary.** *For all $\alpha \leq 0$ the space $\mathcal{H}_\alpha$ is a dB-subspace of $\mathcal{H}$.*

*Proof.* By Lemma 2.6 every space of the form $\mathcal{H}_{(\tau_+,\tau_-)}$, in particular also $\mathcal{H}_\alpha$, is closed.

The fact that $\mathcal{H}_\alpha$ contains with a function $F$ also $F^\#$ is obvious. Finally for $F \in \mathcal{H}_\alpha$ and $F(w) = 0$ one has $\frac{F(z)}{z-w} \in \mathcal{H}_\alpha$ because of $\lim_{y\to\pm\infty} \frac{1}{y} \log|iy-w| = 0$.

$\square$

The space $\mathcal{H}_\alpha$ might consist of the zero element only. It is a matter of interest to obtain conditions for $\mathcal{H}_\alpha \neq \{0\}$.

Let $\mathcal{H} = \mathcal{H}(E)$. In connection with the study of $\mathcal{H}_\alpha$ the spaces

$$\big(\operatorname{Assoc}\mathcal{H}\big)_\alpha := \Big\{F \in \operatorname{Assoc}\mathcal{H} : \operatorname{mt}\frac{F}{E} \leq \alpha, \operatorname{mt}\frac{F^\#}{E} \leq \alpha\Big\}$$

turn out to be useful.

**5.3 Lemma.** *Let $\mathcal{H} = \mathcal{H}(E)$ and $\alpha \leq 0$ be given.*

*(i) If $\mathcal{H}_\alpha \neq \{0\}$ then $\big(\operatorname{Assoc}\mathcal{H}\big)_\alpha = \operatorname{Assoc}\mathcal{H}_\alpha$.*

*(ii) If $\big(\operatorname{Assoc}\mathcal{H}\big)_\alpha \neq \{0\}$ then for all $\beta \in (\alpha, 0]$ we have $\dim\mathcal{H}_\beta = \infty$.*

*(iii) If $\big(\operatorname{Assoc}\mathcal{H}\big)_\alpha \neq \{0\}$ and $\mathcal{H}_\alpha = \{0\}$ then*

$$\big(\operatorname{Assoc}\mathcal{H}\big)_\alpha = \operatorname{span}\{C\}$$

*for some real and zerofree function $C$.*

*Proof.*
`ad`$(i)$: Clearly, $\operatorname{Assoc}\mathcal{H}_\alpha \subseteq \operatorname{Assoc}\mathcal{H}$ and by $\operatorname{Assoc}\mathcal{H}_\alpha = \mathcal{H}_\alpha + z\mathcal{H}_\alpha$ the inclusion $\operatorname{Assoc}\mathcal{H}_\alpha \subseteq \big(\operatorname{Assoc}\mathcal{H}\big)_\alpha$ follows. Assume that $F \in \operatorname{Assoc}\mathcal{H}$ satisfies $\operatorname{mt} E^{-1}F \leq \alpha, \operatorname{mt} E^{-1}F^\# \leq \alpha$. Choose $G \in \mathcal{H}_\alpha \setminus \{0\}$ and $w \in \mathbb{C}$ with $G(w) \neq 0$. Then

$$H(z) := \frac{F(z)G(w) - G(z)F(w)}{z-w}$$

belongs to $\mathcal{H}$ and

$$\operatorname{mt}\frac{H}{E} \leq \alpha,\ \operatorname{mt}\frac{H^\#}{E} \leq \alpha.$$

Hence $H \in \mathcal{H}_\alpha$ and we conclude that $F \in \mathcal{H}_\alpha + z\mathcal{H}_\alpha = \operatorname{Assoc}\mathcal{H}_\alpha$.

`ad`$(ii)$: Choose $F \in \big(\operatorname{Assoc}\mathcal{H}\big)_\alpha \setminus \{0\}$, $\gamma \in (\alpha, \beta]$, and put

$$G_\gamma(z) := F(z)e^{i(\alpha-\gamma)z}.$$

Then $G_\gamma \in \big(\operatorname{Assoc}\mathcal{H}\big)_\gamma \subseteq \big(\operatorname{Assoc}\mathcal{H}\big)_\beta$ and the set of vectors

$$\big\{G_\gamma : \gamma \in [\alpha, \beta]\big\}$$

is linearly independent. It follows that (fix $w \in \mathbb{C}$ with $F(w) \neq 0$)

$$\Big\{\frac{F(z)G_\gamma(w) - G_\gamma(z)F(w)}{z-w} : \gamma \in (\alpha, \beta]\Big\}$$

is contained in $\mathcal{H}_\beta$ and is linearly independent.

`ad`$(iii)$: A similar argument as in the above paragraph shows that under the present hypothesis $\dim\big(\operatorname{Assoc}\mathcal{H}\big)_\alpha = 1$ and that $F \in \big(\operatorname{Assoc}\mathcal{H}\big)_\alpha \setminus \{0\}$ implies that $F$ is zerofree. Since $\big(\operatorname{Assoc}\mathcal{H}\big)_\alpha$ is closed with respect to the involution $F \mapsto F^\#$ we obtain that $\big(\operatorname{Assoc}\mathcal{H}\big)_\alpha = \operatorname{span}\{C\}$ with a real and zerofree function $C$.

$\square$

*5.4 Example.* From the example of a Paley-Wiener space $\mathcal{H} = \mathcal{H}(e^{-iaz})$, $a > 0$, and $\alpha := -a$ we see that the situation described in $(iii)$ actually can occur: We have $1 \in \big(\operatorname{Assoc}\mathcal{H}\big)_{-a}$ and $\mathcal{H}_{-a} = \{0\}$. In fact, the chain of dB-subspaces of $\mathcal{H}$ is given by $\{\mathcal{H}_\alpha : \alpha \in (-a, 0]\}$.

On the other hand consider the function

$$E(z) := \big(z\cos z + \sin z\big) - i\big(z\sin z - \cos z\big)$$

and $\alpha := -1$. Put $\mathcal{H} := \mathcal{H}(E)$. Since $E$ is constructed from the equation $(E = A - iB)$

$$(A, B) = (z, -1)\begin{pmatrix} \cos z & \sin z \\ -\sin z & \cos z \end{pmatrix},$$

we have

$$\mathcal{H}_{-1} = \operatorname{span}\{1\},\ \operatorname{Assoc}\mathcal{H}_{-1} = \operatorname{span}\{1, z\},$$

and (e.g. by Lemma 5.3, $(ii)$) we have $\mathcal{H}_\beta = \{0\}$ for all $\beta < -1$. The chain of dB-subspaces of $\mathcal{H}$ is in this example given by $\{\mathcal{H}_\alpha : \alpha \in [-1, 0]\}$.

**5.5 Lemma.** *If $\mathcal{H}_\alpha \neq \{0\}$ and $\mathcal{H}_\alpha = \mathcal{H}(E_\alpha)$ for some $E_\alpha \in \mathcal{H}B$, then $\operatorname{mt}\frac{E_\alpha}{E} = \alpha$.*

*Proof.* The inequality $\operatorname{mt} E^{-1}E_\alpha \leq \alpha$ follows from Lemma 5.3, $(i)$. Assume that

$$\beta := \operatorname{mt}\frac{E_\alpha}{E} < \alpha.$$

Put $F(z) := E_\alpha(z)e^{-i(\alpha-\beta)z}$, then $E^{-1}F$ and $E^{-1}F^\#$ are of bounded type and

$$\begin{aligned} \operatorname{mt}\frac{F}{E} &= \operatorname{mt}\frac{E_\alpha}{E} + (\alpha - \beta) = \alpha \leq 0, \\ \operatorname{mt}\frac{F^\#}{E} &= \operatorname{mt}\frac{E_\alpha^\#}{E} - (\alpha - \beta) < \alpha \leq 0. \end{aligned} \tag{5.1}$$

Moreover, $E_\alpha \in \operatorname{Assoc}\mathcal{H}_\alpha \subseteq \operatorname{Assoc}\mathcal{H}(E)$. Hence

$$\int_{\mathbb{R}} \Big|\frac{F(t)}{E(t)}\Big|^2 \frac{dt}{1+t^2} < \infty.$$

It follows that $F \in \operatorname{Assoc}\mathcal{H}(E)$ and from (5.1) and Lemma 5.3, $(i)$, we conclude that in fact $F \in \operatorname{Assoc}\mathcal{H}_\alpha$. This contradicts

$$\operatorname{mt}\frac{F}{E_\alpha} = \alpha - \beta > 0.$$

$\square$

**5.6 Lemma.** *Let $\mathcal{H}=\mathcal{H}(E)$ be given.*

(i) *Put $\tau_E = \operatorname{mt} E^{-1}E^{\#}$. If $\alpha\in(\frac{\tau_E}{2},0]$, then $\mathcal{H}_\alpha\neq\{0\}$. Moreover, $\mathcal{H}_{\frac{\tau_E}{2}}\neq\{0\}$ except in the case that $E$ is of the form $C(z)e^{i\frac{\tau_E}{2}z}$ with some real function $C$.*

(ii) *Assume that $\mathcal{H}$ is of exponential type. If $\mathcal{H}_\alpha\neq\{0\}$ then $\alpha\in[-h_E(\frac{\pi}{2}),0]$.*

(iii) *If $\mathcal{H}$ is of minimal exponential type, then $\mathcal{H}_\alpha=\{0\}$ for all $\alpha<0$.*

*Proof.*
`ad`$(i)$: This assertion follows on representing $E$ as in Lemma 3.12 and applying Theorem 2.7.

`ad`$(ii)$: Write $\mathcal{H}_\alpha=\mathcal{H}(E_\alpha)$, then by Lemma 5.5 we have $\operatorname{mt}\frac{E_\alpha}{E}=\alpha$. Thus

$$h_{E_\alpha}(\frac{\pi}{2})=\alpha+h_E(\frac{\pi}{2}).$$

Since $E_\alpha\in\mathcal{H}B$ we have $h_{E_\alpha}(-\frac{\pi}{2})\leq h_{E_\alpha}(\frac{\pi}{2})$ and by [L, I.16.(h)] $0\leq h_{E_\alpha}(-\frac{\pi}{2})+h_{E_\alpha}(\frac{\pi}{2})$. It follows that $h_{E_\alpha}(\frac{\pi}{2})\geq 0$.

`ad`$(iii)$: This follows from assertion $(ii)$.

□

It also can be seen by an elementary consideration that, if $E\in\mathcal{H}B$ is of exponential type, then

$$-h_E(\frac{\pi}{2})\leq\frac{\tau_E}{2}$$

To this end note that we have already seen that for any $E\in\mathcal{H}B$ the inequality $h_E(\frac{\pi}{2})\geq 0$ holds. Hence, by Theorem 2.7,

$$0\leq h_{E(z)e^{-i\frac{\tau_E}{2}z}}(\frac{\pi}{2})=h_E(\frac{\pi}{2})+\frac{\tau_E}{2}.$$

Multiplication of a space $\mathcal{H}$ with a real function (cf. Lemma 2.3) does not change the behaviour of the spaces $\mathcal{H}_\alpha$: Let $E\in\mathcal{H}B$ and $C=C^{\#}$ have only real zeros. Put $E_1:=CE$, then the mapping

$$\psi:\begin{cases}\mathcal{H}(E) & \to & \mathcal{H}(E_1)\\ F & \mapsto & CF\end{cases}$$

has the property that

$$\psi\mathcal{H}(E)_\alpha=\mathcal{H}(E_1)_\alpha. \tag{5.2}$$

In order to establish this fact recall that by Lemma 2.3 the mapping $\psi$ is an isometry from $\mathcal{H}(E)$ onto $\mathcal{H}(E_1)$ satisfying

$$\operatorname{mt}\frac{F}{E}=\operatorname{mt}\frac{\psi F}{E},\ \operatorname{mt}\frac{F^{\#}}{E}=\operatorname{mt}\frac{\psi F^{\#}}{E}.$$

Let us mention a situation where the assertion of Lemma 5.6 can be sharpened.

**5.7 Lemma.** *Assume that there exists a function $C = C^\# \in \operatorname{Assoc}\mathcal{H}$ such that $C^{-1}E$ is entire, i.e. $C$ has no nonreal zeros and its real zeros coincide with those of $E$. Put*

$$\alpha_C := \operatorname{mt}\frac{C}{E},$$

*then*

$$(\alpha_C, 0] \subseteq \big\{\alpha \leq 0 : \mathcal{H}_\alpha \neq \{0\}\big\} \subseteq [\alpha_C, 0].$$

*Proof.* In view of (5.2) we have

$$M := \Big\{\alpha \leq 0 : \mathcal{H}(E)_\alpha \neq \{0\}\Big\} = \Big\{\alpha \leq 0 : \mathcal{H}\big(\frac{E}{C}\big)_\alpha \neq \{0\}\Big\}.$$

However, $1 \in \operatorname{Assoc}\mathcal{H}(\frac{E}{C})$, and hence by Lemma 5.3, $(ii)$, we have $(\alpha_C, 0] \subseteq M$. Since

$$h_{\frac{E}{C}}(\frac{\pi}{2}) = \operatorname{mt}\frac{E}{C} = -\alpha_C,$$

we conclude from Lemma 5.6, $(ii)$, that $M \subseteq [\alpha_C, 0]$.

$\square$

*5.8 Remark.* Note that in the particular case $C = 1 \in \operatorname{Assoc}\mathcal{H}$ in Lemma 5.7 we have

$$\alpha_C = -h_E(\frac{\pi}{2}).$$

*5.9 Example.* Consider the function

$$E(z) := \cos z - i\big(z\cos z + \sin z\big).$$

This function is constructed from the equation ($E = A - iB$)

$$(A, B) = (\cos z, \sin z)\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}.$$

It follows that $E \in \mathcal{H}B^\times$ and that the chain of dB-subspaces of $\mathcal{H} = \mathcal{H}(E)$ is given by

$$\big\{\mathcal{H}(e^{-itz}) : 0 < t \leq 1\big\} \cup \{\mathcal{H}\} \;=\; \big\{\mathcal{H}_\alpha : -1 < \alpha < 0\big\} \cup \{\mathcal{H}(e^{-iz})\} \cup \{\mathcal{H}\}$$

Hereby $\mathcal{H}(e^{-itz}) \subsetneq \mathcal{H}$ with codimension 1.

We have (e.g. by Theorem 2.7) $\tau_E = 0$. Clearly, $h_E(\frac{\pi}{2}) = 1$ and $\sigma_E = 1$. Moreover, $1 \in \operatorname{Assoc}\mathcal{H}(E)$.

*5.10 Example.* One special situation where the conditions of Lemma 5.6, $(i)$ and $(ii)$, coincide is the following: Assume that $E \in \mathcal{H}B^\times$ is of order 1, convergence class. Then $-h_E(\frac{\pi}{2}) = \frac{\tau_E}{2}$. In order to see this represent $E$ as

$$E(z) = \gamma e^{i\frac{\tau_E}{2}z}\prod\big(1 - \frac{z}{z_n}\big).$$

By [L, I.11.Lehrsatz 15] the infinite product is of minimal exponential type. Hence

$$h_E(\frac{\pi}{2}) = -\frac{\tau_E}{2} + h_{\prod(1-\frac{z}{z_n})}(\frac{\pi}{2}) \leq -\frac{\tau_E}{2}.$$

We already saw that the reverse inequality holds in any case.

*5.11 Example.* Choose $E \in \mathcal{HB}$ of minimal exponential type and put $E_1(z) := \sin z \cdot E(z)$. Then

$$h_{E_1}(\frac{\pi}{2}) = 1,\ \tau_{E_1} = 0,$$

and by (5.2) and Lemma 5.6, $(iii)$,

$$\mathcal{H}(E_1)_\alpha = \{0\},\ \alpha < 0.$$

Similarly examples of $E \in \mathcal{HB}$ can be constructed such that for arbitrarily given $\gamma \in [-h_E(\frac{\pi}{2}), 0]$ the set $\{\alpha \leq 0 : \mathcal{H}_\alpha \neq \{0\}\}$ is equal to $[\gamma, 0]$ or to $(\gamma, 0]$. However, note the imperfection that the functions $E$ constructed in this way do not belong to $\mathcal{HB}^\times$.

The fact that the space $\mathcal{H}_{(\tau_+,0)}$ is always closed means just that the mapping

$$\Phi : \left\{ \begin{array}{rcl} \mathcal{H} \setminus \{0\} & \to & \mathbb{R}^- \cup \{0\} \\ F & \mapsto & \operatorname{mt} \frac{F}{E} \end{array} \right.$$

is lower semicontinuous. Let us note that $\Phi$ is, except in trivial cases, nowhere continuous. The proof of this result follows the lines of the proof of [GG, II.Lemma 5.1].

**5.12 Lemma.** *The function $\Phi$ is continuous at a point $F_0 \in \mathcal{H}(E)$ if and only if $\Phi(F_0) = 0$.*

*Proof.* If $\Phi(F_0) = 0$, then for all $F \in \mathcal{H}(E)$ we have $\Phi(F) \leq \Phi(F_0)$. Hence the continuity is a consequence of the lower semicontinuity.

Assume conversely that $\Phi$ is continuous at $F_0$. Then, given $\epsilon > 0$, there exists $\delta > 0$, such that

$$|\Phi(F_0 + F) - \Phi(F_0)| < \epsilon,\ \|F\| < \delta. \tag{5.3}$$

Let $G \in \mathcal{H} \setminus \{0\}$ be given and put $F(z) := \frac{\delta}{2\|G\|} G(z)$. Clearly $\Phi(G) = \Phi(F)$. We obtain, using (5.3), that

$$\Phi(G) = \Phi(F) = \Phi((F - F_0) + F_0) \leq \max\big\{\Phi(F - F_0), \Phi(F_0)\big\} \leq \Phi(F_0) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude

$$\Phi(G) \leq \Phi(F_0),\ G \in \mathcal{H} \setminus \{0\}.$$

Since there exist functions $G$ with $\Phi(G) = 0$, e.g. take $\frac{S_\varphi(z)}{z-w}$ where $w$ is a zero of $S_\varphi$ (cf. the proof of Lemma 3.21) we conclude that $\Phi(F_0) = 0$.

□

For spaces of exponential type we obtain a continuity property of the indicator function:

**5.13 Corollary.** *Let $\mathcal{H} = \mathcal{H}(E)$ be a dB-space of exponential type. For each $\theta$ the function*

$$\chi_\theta : F \mapsto h_F(\theta),\ F \in \mathcal{H},$$

*is lower semicontinuous (uniformly with respect to $\theta$). The function $F$ is a point of continuity of $\chi_\theta$ if and only if $h_F(\theta) = h_{|E|}(\theta)$.*

*Proof.* To deduce this assertion from Lemma 5.12 we only have to note that formula (3.6) implies

$$h_F(\theta) = h_E(|\theta|) + \begin{cases} \Phi(F)\,|\sin\theta| & ,\ \theta \in [0,\pi] \\ \Phi(F^\#)\,|\sin\theta| & ,\ \theta \in [-\pi,0] \end{cases}$$

□

*5.14 Remark.* Note that a similar result for other growth functions $\lambda$ would not be meaningful: If $\lambda$ grows fast, $r = o(\lambda(r))$, then anyway $h_F^\lambda(\theta) = h_E^\lambda(|\theta|)$ does not depend on $F$ (cf. Lemma 3.20), hence the assertion of Corollary 5.13 is trivial. Whereas, if $\lambda$ grows slow, $\lambda(r) = o(r)$, the assertion of Corollary 5.13 is not true in general:

*5.15 Example.* The function

$$E(z) := \frac{\sin\sqrt{iz}}{\sqrt{iz}}$$

is an entire function of order $\frac{1}{2}$ and normal type. In fact

$$h_E^{\sqrt{r}}(\phi) = \sin(\frac{\phi}{2} + \frac{\pi}{4}),\ \phi \in [-\frac{\pi}{2}, \frac{3\pi}{2}],$$

hence $\sigma_E^{\sqrt{r}} = 1$. The zeros of $E$ are all simple and are given by

$$z_n = -in^2,\ n \in \mathbb{N}.$$

Hence we also have

$$E(z) = \prod_{n\in\mathbb{N}} \big(1 + \frac{z}{in^2}\big) \in \mathcal{H}B.$$

The considerations in [KW3] show that $\mathcal{H}(E) = \overline{\mathbb{C}[z]}$. In particular, $1 \in \mathcal{H}(E)$. However, $\mathcal{H}(E)$ is not of finite $\lambda$-type with e.g. $\lambda(r) = r^{\frac{1}{4}}$. Thus the assertion of Theorem 3.10 remains not longer valid if we choose there e.g. $\lambda(r) = r^{\frac{1}{4}}$. Also for each polynomial $p$ surely $h_p^{\sqrt{r}} = 0$, hence also the assertion of Corollary 5.13 cannot hold true for the growth function $\lambda(r) = r^{\frac{1}{2}}$. Also, for all $\beta \in [0,1)$, the space $\mathcal{H}_{\sqrt{r},\beta}$ cannot be closed: Trivially $\mathbb{C}[z] \subseteq \mathcal{H}_{\sqrt{r},\beta}$, hence $\overline{\mathcal{H}_{\sqrt{r},\beta}} = \mathcal{H}(E)$. On the other hand e.g. $F(z) := \frac{E(z)}{z+i}$ belongs to $\mathcal{H}(E)$ but $\sigma_F^{\sqrt{r}} = 1$.

One can construct similar examples for other growth functions $\lambda$ with $\lambda(r) = o(r)$ since then each function $E \in \Lambda$ surely will be of minimal exponential type, and henceforth it is be possible to use the same reasoning.

We turn to the investigation of the spaces $\mathcal{H}_{\lambda,\beta}$.

**5.16 Lemma.** *The linear space $\mathcal{H}_{\lambda,\beta}$ satisfies*

*(i) If $F \in \mathcal{H}_{\lambda,\beta}$, so is $F^\#$.*

*(ii) If $F \in \mathcal{H}_{\lambda,\beta}$ and $F(w) = 0$, then also $\frac{F(z)}{z-w}$ belongs to $\mathcal{H}_{\lambda,\beta}$.*

*Proof.* The first assertion is clear from the definition of $\mathcal{H}_{\lambda,\beta}$. The second follows since we always assume that $\log r = o(\lambda(r))$, for this implies that

$$\lim_{|z|\to\infty} \frac{\log|z-w|}{\lambda(|z|)} = 0.$$

$\square$

Hence $\mathcal{H}_{\lambda,\beta}$ seems to be a candidate for a dB-subspace of $\mathcal{H}$. However, for the reasons mentioned in Remark 5.14, this is trivial if $r = o(\lambda(r))$ as in this case

$$\mathcal{H}_{\lambda,\beta} = \begin{cases} \mathcal{H} & , \ \beta \geq \sigma_E^\lambda \\ \{0\} & , \ \beta < \sigma_E^\lambda \end{cases}$$

and, for $\lambda(r) = o(r)$, in general not true (cf. Example 5.15). In the case $\lambda(r) = r$, indeed, $\mathcal{H}_{\lambda,\beta}$ is a dB-subspace of $\mathcal{H}$, i.e. is always closed, but we already know those subspaces, cf. Lemma 5.18.

Define a mapping $l$ by

$$l : \begin{cases} \mathbb{R}^- \cup \{0\} & \to \quad \mathbb{R} \\ \qquad \mu & \mapsto \quad \max_{\theta\in[0,\pi]}(h_E(\theta) + \mu \sin\theta) \end{cases}$$

and put $\beta_E := \max\{h_E(0), h_E(\pi)\}$. Then $l$ is nondecreasing and continuous. Note that $l(0) = \sigma_E$.

**5.17 Lemma.** *For the function $l$ defined above we have*

$$(\beta_E, \sigma_E] \subseteq \operatorname{ran} l \subseteq [\beta_E, \sigma_E]. \tag{5.4}$$

*Let $B$ be the interval $\overline{l^{-1}((\beta_E, \sigma_E])}$. Then $l|_B$ is a strictly increasing bijection of $B$ onto* $\operatorname{ran} l$.

*Proof.* We always have $l(\mu) \leq l(0) = \sigma_E$ and $l(\mu) \geq h_E(0), h_E(\pi)$. Thus the second $\subseteq$ in (5.4) holds true. A continuity argument will show that $(\beta_E, \sigma_E] \subseteq \operatorname{ran} l$: Trivially $l(0) = \sigma_E$, i.e. $\sigma_E \in \operatorname{ran} l$. Let $\beta \in (\beta_E, \sigma_E)$. Since $l$ is continuous it is enough to show that there exists some $\mu \in \mathbb{R}^- \cup \{0\}$ with $l(\mu) \leq \beta$. To this end choose $\delta > 0$, such that

$$\max_{\phi\in[0,\delta]\cup[\pi-\delta,\pi]} h_E(\phi) \leq \beta.$$

This is possible by the continuity of $h_E$. Since $\sin\phi$ is bounded away from zero on $[\delta, \pi-\delta]$, there exists $\mu \leq 0$ (e.g. take $\mu = -\frac{\sigma_E}{\sin\delta}$) such that

$$\max_{\phi\in[\delta,\pi-\delta]} \big(h_E(\phi) + \mu\sin\phi\big) \leq \beta.$$

Altogether $l(\mu) \leq \beta$.

It remains to prove that $l$ is strictly increasing on $B$. In order to establish this assertion it suffices to show that for every $\mu_0$ with $l(\mu_0) \in (\beta_E, \sigma_E]$ and $\mu \in \mathbb{R}^-$, $\mu < \mu_0$, we have $l(\mu) < l(\mu_0)$. Put

$$C := \big\{\theta \in [0,\pi] : h_E(\theta) + \mu_0 \sin\theta = l(\mu_0)\big\}.$$

Choose $\epsilon > 0$ such that $l(\mu_0) - \epsilon > \beta_E$. Then ($\chi : [0,\pi] \to \mathbb{R}$, $\chi(\theta) := h_E(\theta) + \mu_0 \sin\theta$)

$$D := \chi^{-1}([l(\mu_0) - \epsilon, \infty))$$

is a compact subset of $(0,\pi)$ and contains $C$. Choose $\delta > 0$ such that $D \subseteq [\delta, \pi - \delta]$.

Assume that $\mu < \mu_0$. Then, for $\theta \in [\delta, \pi - \delta]$,

$$h_E(\theta) + \mu \sin\theta = h_E(\theta) + \mu_0 \sin\theta + (\mu - \mu_0)\sin\theta \leq$$

$$\leq h_E(\theta) + \mu_0 \sin\theta + (\mu - \mu_0)\sin\delta \leq l(\mu_0) + (\mu - \mu_0)\sin\delta.$$

For $\theta \in [0,\delta) \cup (\pi - \delta, \pi]$ we have

$$h_E(\theta) + \mu \sin\theta \leq h_E(\theta) + \mu_0 \sin\theta < l(\mu_0) - \epsilon.$$

Altogether it follows that

$$l(\mu) \leq l(\mu_0) - \min\{\epsilon, (\mu_0 - \mu)\sin\delta\} < l(\mu_0).$$

□

**5.18 Lemma.** *Let $\mathcal{H}(E)$ be a dB-space of exponential type. Denote by $\hat{l}$ the function $\hat{l} := \big(l|_B\big)^{-1}$. Then*

$$\mathcal{H}_{r,\beta} = \begin{cases} \mathcal{H}_{\hat{l}(\beta)} & , \ \beta \in (\beta_E, \sigma_E] \\ \{0\} & , \ \beta \in [0, \beta_E) \end{cases} \tag{5.5}$$

*Proof.* In order to establish this relation it is sufficient to note that by (3.6) for all $F \in \mathcal{H}$

$$\sigma_F = \max\big\{l(\Phi(F)), l(\Phi(F^{\#}))\big\}.$$

□

**5.19 Corollary.** *Let $\mathcal{H}(E)$ be a dB-space of exponential type. If for some $\beta < \sigma_E$ we have $\mathcal{H}_{r,\beta} \neq \{0\}$, then there exists $\alpha < 0$ with $\mathcal{H}_\alpha \neq \{0\}$. If $\beta_E < \sigma_E$ also the converse holds.*

*Proof.* Assume that $\beta < \sigma_E$ and $\mathcal{H}_{r,\beta} \neq \{0\}$. Then, by Lemma 5.18, $\beta_E < \sigma_E$. Choose $\beta' \in (\beta_E, \sigma_E)$, then $\mathcal{H}_{\hat{l}(\beta')} = \mathcal{H}_{r,\beta'} \neq \{0\}$. By Lemma 5.17 we have $\hat{l}(\beta') < 0$.

Conversely, assume that $\mathcal{H}_\alpha \neq \{0\}$ for some $\alpha < 0$. Since by our assumption $\beta_E < \sigma_E$, the mapping $l$ is injective locally at 0, there exists $\alpha' \in (\alpha, 0)$ such that $\beta' := l(\alpha') < \sigma_E$. By Lemma 5.18 we have $\mathcal{H}_{r,\beta'} = \mathcal{H}_{\alpha'} \neq \{0\}$.

□

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