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Integral Equations and Operator Theory

Almost Pontryagin spaces

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Abstract. The purpose of this note is to provide an axiomatic treatment of a generalization of the Pontryagin space concept to the case of degenerated inner product spaces.

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1. Introduction

In this note we provide an axiomatic treatment of a generalization of the Pontryagin space concept to the case of degenerated inner product spaces. Pontryagin spaces are inner product spaces which can be written as the direct and orthogonal sum of a Hilbert space and a finite dimensional anti Hilbert space. The subject of our paper are spaces which can be written as the direct and orthogonal sum of a Hilbert space, a finite dimensional anti Hilbert space and a finite dimensional neutral space.

The necessity of a systematic approach to such "almost" Pontryagin spaces became clear in the study of various topics: For example in the investigation of indefinite versions of various classical interpolation problems (e.g. [7]). Related to these questions is the generalization of Krein's formula for generalized resolvents of a symmetric operator (e.g. [8]). Another topic where the occurrence of degeneracy plays a crucial role is the theory of Pontryagin spaces of entire functions which generalizes the theory of Louis de Branges on Hilbert spaces of entire functions.

In Section 2 we generalize the concept of Pontryagin spaces by giving the definition of almost Pontryagin spaces and investigating the basic notion of Gram operator and fundamental decomposition. Moreover, the role played by the topology of an almost Pontryagin space is made clear. In the subsequent Section 3 we investigate some elementary constructions which can be made with almost Pontryagin spaces. We deal with subspaces, product spaces and factor spaces. Related to the last one of these constructions is the notion of morphism between almost Pontryagin spaces. Section 4 deals with the concept of completion. This topic is much more involved than the previous constructions. However, it is clearly of particular importance to be able to construct almost Pontryagin spaces from given linear spaces carrying an inner product. In Section 5 we turn our attention to a particular class of almost Pontryagin spaces, so-called almost reproducing kernel Pontryagin spaces. The intention there is to prove the existence of the correct analogue of a reproducing kernel of a reproducing kernel Pontryagin space. Finally, in Section 6, we explain some circumstances where almost Pontryagin spaces actually occur.

2. Almost Pontryagin spaces

Before we give the definition of almost Pontryagin spaces, recall the definition of Pontryagin spaces. A pair $(\mathfrak{P}, [.,.])$ where $\mathfrak{P}$ is a complex vector space and $[.,.]$ is a hermitian inner product on $\mathfrak{P}$ is called a Pontryagin space if one can decompose $\mathfrak{P}$ as

$$\mathfrak{P} = \mathfrak{P}_-[\dot{+}]\mathfrak{P}_+, \tag{2.1}$$

where $(\mathfrak{P}_-, [.,.])$ is a finite dimensional anti Hilbert space, $(\mathfrak{P}_+, [.,.])$ is a Hilbert space, and $[\dot{+}]$ denotes the direct and $[.,.]$-orthogonal sum. Such decompositions of $\mathfrak{P}$ are called fundamental decompositions. It is worthwhile to note (see [2] or see below) that every Pontryagin space carries a unique Hilbert space topology $\mathcal{O}$ (there exists an inner product $(.,.)$ such that $(\mathfrak{P}, (.,.))$ is a Hilbert space and such that $(.,.)$ induces the topology $\mathcal{O}$, i.e. $\mathcal{O} = \mathcal{O}_{(.,.)}$) such that the inner product $[.,.]$ is continuous with respect to $\mathcal{O}$. This topology is also called the Pontryagin space topology on $\mathfrak{P}$.

With respect to this topology the subspace $\mathfrak{P}_+$ is closed for any fundamental decomposition (2.1). Conversely, the product topology induced on $\mathfrak{P}$ by any fundamental decomposition (2.1) coincides with the unique Hilbert space topology.

It will turn out that for almost Pontryagin spaces the uniqueness assertion about the topology is no longer true. Thus we will include the topology into the definition.

Definition 2.1. Let $\mathfrak{L}$ be a linear space, $[.,.]$ an inner product on $\mathfrak{L}$ and $\mathcal{O}$ a Hilbert space topology on $\mathfrak{L}$. The triplet $(\mathfrak{L}, [.,.], \mathcal{O})$ is called an almost Pontryagin space, if

- **(aPS1)** $[.,.]$ is $\mathcal{O}$-continuous.
- **(aPS2)** There exists a $\mathcal{O}$-closed linear subspace $\mathfrak{M}$ of $\mathfrak{L}$ with finite codimension such that $(\mathfrak{M}, [.,.])$ is a Hilbert space.

Let $(\mathfrak{R}, [.,.])$ be any linear space equipped with a inner product $[.,.]$ and assume that

$$\sup \{ \dim \mathfrak{U} : \mathfrak{U} \text{ negative definite subspace of } \mathfrak{R} \} < \infty.$$

Then (see for example [2, Corollary I.3.4]) the dimensions of all maximal negative definite subspaces of $\mathfrak{R}$ are equal. We denote this number by $\kappa_-(\mathfrak{R}, [.,.])$ and refer to it as the negative index (or the degree of negativity) of $(\mathfrak{R}, [.,.])$. If the above supremum is not finite, we set $\kappa_-(\mathfrak{R}, [.,.]) = \infty$.

The isotropic part of an inner product space $(\mathfrak{R}, [.,.])$ is defined as

$$\mathfrak{R}^{[\circ]} = \{x \in \mathfrak{R} : \ [x, y] = 0, y \in \mathfrak{R}\}.$$

We will denote its dimension by $\Delta(\mathfrak{R}, [.,.]) \in \mathbb{N} \cup \{0, \infty\}$ and call this number the degree of degeneracy of $(\mathfrak{R}, [.,.])$.

Remark 2.2. It immediately follows from the definition that if $(\mathfrak{L}, [.,.], \mathcal{O})$ is an almost Pontryagin space, then $\kappa_-(\mathfrak{L}, [.,.])$ and $\Delta(\mathfrak{L}, [.,.])$ are both finite.

The fact that a given triplet $(\mathfrak{L}, [.,.], \mathcal{O})$ is an almost Pontryagin space can be characterized in several ways. First let us give one characterization via a spectral property of a Gram operator.

Proposition 2.3. *Let $\mathfrak{L}$ be a linear space, $[.,.]$ an inner product on $\mathfrak{L}$ and $\mathcal{O}$ a Hilbert space topology on $\mathfrak{L}$.*

(i) *Assume that $(\mathfrak{L}, [.,.], \mathcal{O})$ is an almost Pontryagin space and let $(.,.)$ be any Hilbert space inner product which induces the topology $\mathcal{O}$. Then there exists a unique $(.,.)$-selfadjoint bounded operator $G_{(.,.)}$ with*

$$[x, y] = (G_{(.,.)}x, y), \ \ x, y \in \mathfrak{L}.$$

There exists $\epsilon > 0$ such that $\sigma(G_{(.,.)}) \cap (-\infty, \epsilon)$ consists of finitely many eigenvalues of finite multiplicity. If we denote by $E(M)$ the spectral measure of $G_{(.,.)}$, this just means that

$$\dim \operatorname{ran} E(-\infty, \epsilon) < \infty. \tag{2.2}$$

Moreover

$$\Delta(\mathfrak{L}, [.,.]) = \dim \ker G_{(.,.)}, \ \ \kappa_-(\mathfrak{L}, [.,.]) = \dim \operatorname{ran} E(-\infty, 0).$$

We will refer to $G_{(.,.)}$ as the Gram operator corresponding to $(.,.)$.

(ii) *Let $(\mathfrak{L}, (.,.))$ be a Hilbert space, and let G be a bounded selfadjoint operator on $(\mathfrak{L}, (.,.))$ which satisfies (2.2) where $E(M)$ denotes the spectral measure of G. Moreover, let $\mathcal{O}$ be the topology induced by $(.,.)$ and define $[.,.] = (G.,.)$. Then $(\mathfrak{L}, [.,.], \mathcal{O})$ is an almost Pontryagin space.*

Proof. `ad` (i): Since $[.,.]$ is continuous with respect to the topology $\mathcal{O}$, the Lax-Milgram theorem ensures the existence and uniqueness of $G_{(.,.)}$. Moreover, since $[.,.]$ is an inner product, $G_{(.,.)}$ is selfadjoint.

Let $\mathfrak{M}$ be a $\mathcal{O}$-closed linear subspace of $\mathfrak{L}$ with finite codimension such that $(\mathfrak{M}, [.,.])$ is a Hilbert space. By the open mapping theorem $[.,.]|_{\mathfrak{M}}$ and $(.,.)|_{\mathfrak{M}}$ are equivalent. Hence $P_{\mathfrak{M}} G_{(.,.)}|_{\mathfrak{M}}$, where $P_{\mathfrak{M}}$ denotes the $(.,.)$-orthogonal projection onto $\mathfrak{M}$, is strictly positive. Choose $\epsilon > 0$ such that $\epsilon I_{\mathfrak{M}} < P_{\mathfrak{M}} G_{(.,.)}|_{\mathfrak{M}}$. Assume that $\dim \operatorname{ran} E(-\infty, \epsilon) > \operatorname{codim}_{\mathfrak{L}} \mathfrak{M}$, then $\operatorname{ran} E(-\infty, \epsilon) \cap \mathfrak{M} \neq \{0\}$. For any $x \in \operatorname{ran} E(-\infty, \epsilon) \cap \mathfrak{M}$, $(x, x) = 1$, we have

$$\epsilon < (P_{\mathfrak{M}} G_{(.,.)}|_{\mathfrak{M}} x, x) = (G_{(.,.)}x, x) \leq \epsilon,$$

a contradiction.

ad(ii): Choose $\mathfrak{M} = \operatorname{ran} E[\epsilon,\infty)$. Then $\mathfrak{M}$ is $(.,.)$-closed, $\operatorname{codim}_{\mathfrak{L}} \mathfrak{M} = \dim \operatorname{ran} E(-\infty,\epsilon) < \infty$ and $[.,.] = (G.,.)$ is a Hilbert space inner product on $\mathfrak{M}$ since $G|_{\mathfrak{M}}$ is strictly positive.

□

Corollary 2.4. *Let an almost Pontryagin space $(\mathfrak{L},[.,.],\mathcal{O})$ be given. If $(.,.)$ and $\langle .,. \rangle$ are two Hilbert space inner products on $\mathfrak{L}$ which both induce the topology $\mathcal{O}$ and T is the $(.,.)$-strictly positive bounded operator on $\mathfrak{L}$ with $\langle .,. \rangle = (T.,.)$, then the Gram operators $G_{(.,.)}$ and $G_{\langle .,. \rangle}$ are connected by*

$$G_{(.,.)} = TG_{\langle .,. \rangle}.$$

There exists a Hilbert space inner product $(.,.)$ on $\mathfrak{L}$ which induces $\mathcal{O}$ such that its Gram operator $G_{(.,.)}$ is a finite dimensional perturbation of the identity.

Proof. The first assertion is clear from

$$(G_{(.,.)}x,y) = [x,y] = \langle G_{\langle .,. \rangle}x,y \rangle = (TG_{\langle .,. \rangle}x,y),\ x,y \in \mathfrak{L}.$$

For the second assertion choose a Hilbert space inner product $\langle .,. \rangle$ on $\mathfrak{L}$ which induces $\mathcal{O}$. Let $G_{\langle .,. \rangle}$ be the corresponding Gram operator, $E(M)$ its spectral measure, and choose $\epsilon > 0$ as in Proposition 2.3, (i). Define

$$(x,y) = \langle (E[\epsilon,\infty)G_{\langle .,. \rangle} + E(-\infty,\epsilon))x,y \rangle,\ x,y \in \mathfrak{L}.$$

Then

$$G_{(.,.)} = (E[\epsilon,\infty)G_{\langle .,. \rangle} + E(-\infty,\epsilon))^{-1}G_{\langle .,. \rangle} = E[\epsilon,\infty) + E(-\infty,\epsilon)G_{\langle .,. \rangle}.$$

□

In the study of Pontryagin spaces so-called fundamental decompositions play an important role. The following is the correct analogue for almost Pontryagin spaces. In particular, it gives us another characterization of this notion.

Proposition 2.5. *The following assertions hold true:*

(i) *Let $(\mathfrak{L},[.,.],\mathcal{O})$ be an almost Pontryagin space. Then there exists a direct and $[.,.]$-orthogonal decomposition*

$$\mathfrak{L} = \mathfrak{L}_+[\dot{+}]\mathfrak{L}_-[\dot{+}]\mathfrak{L}^{[\circ]}, \tag{2.3}$$

where $\mathfrak{L}_+$ is $\mathcal{O}$-closed, $(\mathfrak{L}_+,[.,.])$ is a Hilbert space and $\mathfrak{L}_-$ is negative definite, $\dim \mathfrak{L}_- = \kappa_-(\mathfrak{L},[.,.])$.

(ii) *Let $(\mathfrak{L}_+,(.,.)_+)$ be a Hilbert space, $(\mathfrak{L}_-,(.,.)_-)$ be a finite dimensional Hilbert space, and let $\mathfrak{L}_0$ be a finite dimensional linear space. Define a linear space*

$$\mathfrak{L} = \mathfrak{L}_+\dot{+}\mathfrak{L}_-\dot{+}\mathfrak{L}_0,$$

and inner products

$$[x_+ + x_- + x_0, y_+ + y_- + y_0] = (x_+,y_+) - (x_-,y_-),$$
$$(x_+ + x_- + x_0, y_+ + y_- + y_0) = (x_+,y_+) + (x_-,y_-) + (x_0,y_0)_0,$$

where $(.,.)_0$ *is any Hilbert space inner product on* $\mathfrak{L}_0$*. Moreover, let* $\mathcal{O}$ *be the topology on* $\mathfrak{L}$ *induced by the Hilbert space inner product* $(.,.)$*. Then* $(\mathfrak{L},[.,.],\mathcal{O})$ *is an almost Pontryagin space. Thereby* $\kappa_-(\mathfrak{L},[.,.]) = \dim \mathfrak{L}_-$ *and* $\mathfrak{L}^{[\circ]} = \mathfrak{L}_0$*.*

Proof. Let $(\mathfrak{L},[.,.],\mathcal{O})$ be an almost Pontryagin space. Choose a Hilbert space inner product $(.,.)$ which induces $\mathcal{O}$, let $G_{(.,.)}$ be the corresponding Gram operator, and denote by $E(M)$ the spectral measure of $G_{(.,.)}$. Define

$$\mathfrak{L}_+ = \operatorname{ran} E(0,\infty),\ \ \mathfrak{L}_- = \operatorname{ran} E(-\infty,0).$$

Then $\mathfrak{L}_+$ is $\mathcal{O}$-closed. The inner products $[.,.]$ and $(.,.)$ are equivalent on $\mathfrak{L}_+$ since $G|_{\mathfrak{L}_+}$ is strictly positive. Hence $(\mathfrak{L}_+,[.,.])$ is a Hilbert space. Clearly $(\mathfrak{L}_-,[.,.])$ is negative definite and $\dim \mathfrak{L}_- = \kappa_-(\mathfrak{L},[.,.])$. Since $E(-\infty,0)+E\{0\}+E(0,\infty) = I$, the space $\mathfrak{L}$ is decomposed as in (2.3).

Conversely, let $(\mathfrak{L}_+,(.,.)_+)$, $(\mathfrak{L}_-,(.,.)_-)$ and $\mathfrak{L}_0$ be given. The Gram operator of $[.,.]$ with respect to $(.,.)$ is equal to

$$G = \begin{pmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Obviously, $\operatorname{ran} E(-\infty,\frac{1}{2}) = \mathfrak{L}_- + \mathfrak{L}_0$, $\ker G = \mathfrak{L}_0$ and $\operatorname{ran} E(-\infty,0) = \mathfrak{L}_-$. □

Corollary 2.6. *We have*

- (i) *Let* $(\mathfrak{L}_+,(.,.)_+)$*,* $(\mathfrak{L}_-,(.,.)_-)$ *and* $\mathfrak{L}_0$ *be as in* (ii) *of Proposition 2.5, and let* $(\mathfrak{L},[.,.],\mathcal{O})$ *be the almost Pontryagin space constructed there. Then* $\mathfrak{L} = \mathfrak{L}_+[\dot+]\mathfrak{L}_-[\dot+]\mathfrak{L}_0$ *is a decomposition of the same kind as in (2.3).*
- (ii) *Let* $(\mathfrak{L},[.,.],\mathcal{O})$ *be an almost Pontryagin space, and assume that* $\mathfrak{L}$ *is decomposed as* $\mathfrak{L} = \mathfrak{L}_+[\dot+]\mathfrak{L}_-[\dot+]\mathfrak{L}^{[\circ]}$ *where* $(\mathfrak{L}_+,[.,.])$ *is a Hilbert space and* $(\mathfrak{L}_-,[.,.])$ *is negative definite. Let* $(\mathfrak{L}_1,[.,.]_1,\mathcal{O}_1)$ *be the almost Pontryagin space constructed by means of Proposition 2.5,* (ii)*, from* $(\mathfrak{L}_+,[.,.])$*,* $(\mathfrak{L}_-,-[.,.])$*,* $\mathfrak{L}_0 = \mathfrak{L}^{[\circ]}$*. Then* $\mathfrak{L}_1 = \mathfrak{L}$ *and* $[.,.]_1 = [.,.]$*. We have* $\mathcal{O}_1 = \mathcal{O}$ *if and only if* $\mathfrak{L}_+$ *is* $\mathcal{O}$*-closed.*

Proof. The assertion (i) follows immediately since $\mathfrak{L}_+$ is $(.,.)$-closed. We come to the proof of (ii). The facts that $\mathfrak{L}_1 = \mathfrak{L}$ and $[.,.]_1 = [.,.]$ are obvious.

Assume that $\mathfrak{L}_+$ is $\mathcal{O}$-closed. Note that by the assumption on their dimensions the subspaces $\mathfrak{L}_-$ and $\mathfrak{L}_0$ are closed, too. By the Open Mapping Theorem the linear bijection

$$(x_+;x_-;x_0) \mapsto x_+ + x_- + x_0,$$

is bicontinuous from $\mathfrak{L}_+ \times \mathfrak{L}_- \times \mathfrak{L}_0$ provided with the product topology onto $\mathfrak{L}$ provided with $\mathcal{O}$. On the other hand by the definition of $\mathcal{O}_1$ this mapping is also bicontinuous if we provide $\mathfrak{L}$ with $\mathcal{O}_1$. Thus $\mathcal{O}_1 = \mathcal{O}$.

Finally, assume that $\mathcal{O}_1 = \mathcal{O}$. By the construction of $(.,.)_1$ the space $\mathfrak{L}_+$ is $\mathcal{O}_1$-closed and, therefore, also $\mathcal{O}$-closed. □

From the above results we obtain a statement which shows from another point of view that almost Pontryagin spaces can be viewed as a generalization of Pontryagin spaces.

Corollary 2.7. *Let* $(\mathfrak{P}, [.,.])$ *be a Pontryagin space, and let* $\mathcal{O}$ *be the unique topology on* $\mathfrak{P}$ *such that* $[.,.]$ *is continuous (see* [2]*, compare also Corollary 2.10). Then* $(\mathfrak{P}, [.,.], \mathcal{O})$ *is an almost Pontryagin space. Moreover,* $\Delta(\mathfrak{P}, [.,.]) = 0$*.*

Conversely, if $(\mathfrak{P}, [.,.], \mathcal{O})$ *is an almost Pontryagin space with* $\Delta(\mathfrak{P}, [.,.]) = 0$*, then* $(\mathfrak{P}, [.,.])$ *is a Pontryagin space.*

Proof. Let $(\mathfrak{P}, [.,.])$ be a Pontryagin space. Choose a fundamental decomposition $\mathfrak{P} = \mathfrak{P}_+[\dot{+}]\mathfrak{P}_-$. Then $\mathfrak{P}_+$ is $\mathcal{O}$-closed, $(\mathfrak{P}_+, [.,.])$ is a Hilbert space and $\operatorname{codim}_{\mathfrak{P}} \mathfrak{P}_+ = \dim \mathfrak{P}_- < \infty$.

Let $(\mathfrak{P}, [.,.], \mathcal{O})$ be an almost Pontryagin space with $\Delta(\mathfrak{P}, [.,.]) = 0$. Choose a decomposition $\mathfrak{P} = \mathfrak{P}_+[\dot{+}]\mathfrak{P}_-$ according to (2.3). By Corollary 2.6, *(ii)*, the topology $\mathcal{O}$ coincides with the topology of the Pontryagin space $(\mathfrak{P}_+[\dot{+}]\mathfrak{P}_-, [.,.])$. □

It is a noteworthy fact that in certain cases the topology of an almost Pontryagin space $(\mathfrak{L}, [.,.], \mathcal{O})$ is uniquely determined by the inner product, see the Proposition 2.9 below. However, in general this is not true. This fact goes back to [4],[5].

Lemma 2.8. *For any infinite dimensional almost Pontryagin space* $(\mathfrak{L}, [.,.], \mathcal{O})$ *with* $\Delta(\mathfrak{L}, [.,.]) > 0$ *there exists a topology* $\mathcal{T}$ *different from* $\mathcal{O}$ *such that also* $(\mathfrak{L}, [.,.], \mathcal{T})$ *is an almost Pontryagin space.*

Proof. Choose a Hilbert space inner product $(.,.)$ on $\mathfrak{L}$ inducing $\mathcal{O}$. Let $h \in \mathfrak{L}^{[\circ]}$ and $\mathfrak{K} = h^{(\perp)}$, and let f be a non-continuous linear functional on $\mathfrak{K}$. We define the linear mapping U from $\mathfrak{L} = \mathfrak{K}(\dot{+})\operatorname{span}\{h\}$ onto itself by

$$U(x + \xi h) = x + (\xi + f(x))h, \; x \in \mathfrak{K}, \; \xi \in \mathbb{C}.$$

The mapping U is bijective and non-continuous. In fact, if it were continuous, then also f would be continuous. Nevertheless, U is isometric:

$$[U(x+\xi h), U(y+\eta h)] = [x+(\xi+f(x))h, y+(\eta+f(y))h] = [x, y] = [x+\xi h, y+\eta h].$$

Therefore, with $(\mathfrak{L}, [.,.], \mathcal{O})$ also its isometric copy $(\mathfrak{L}, [.,.], U^{-1}(\mathcal{O}))$ is an almost Pontryagin space. As U is not continuous we have $\mathcal{T} = U^{-1}(\mathcal{O}) \neq \mathcal{O}$. □

The existence of a sufficiently large family of functionals which are required to be continuous guarantees the uniqueness of the topology. Such a family of functionals will show up, in particular, when we deal with spaces consisting of functions such that the point evaluation functionals are continuous.

A family $(f_i)_{i \in I}$ of linear functionals on a linear space $\mathfrak{L}$ is said to be point separating if for each two $x, y \in \mathfrak{L}$, $x \neq y$, there exists $i \in I$ such that $f_i(x) \neq f_i(y)$.

Proposition 2.9. *Let $(\mathfrak{L}, [.,.], \mathcal{O})$ be an almost Pontryagin space and assume that there exists a point separating family of continuous linear functionals $(f_i)_{i\in I}$. Then $\mathcal{O}$ is the unique Banach space topology on $\mathfrak{L}$ such that all functionals f_i, $i \in I$, are continuous.*

Proof. Let $\mathcal{T}$ be a Banach space topology on $\mathfrak{L}$ such that every f_i is continuous. The identity mapping $\text{id} : (\mathfrak{L}, \mathcal{O}) \to (\mathfrak{L}, \mathcal{T})$ has a closed graph. In fact, if $x_n \to x$ with respect to $\mathcal{O}$ and $x_n \to y$ with respect to $\mathcal{T}$, then by assumption

$$f_i(x) = \lim_{n\to\infty} f_i(x_n) = f_i(y), \text{ for all } i \in I,$$

and hence $x = y$. By the Closed Graph Theorem the identity map is bicontinuous, and therefore $\mathcal{T} = \mathcal{O}$. □

As a corollary we obtain the well known result that a Pontryagin space carries a unique Hilbert space topology with respect to which $[.,.]$ is continuous.

Corollary 2.10. *If an almost Pontryagin space $(\mathfrak{P}, [.,.], \mathcal{O})$ is a Pontryagin space, i.e. $\Delta(\mathfrak{P}, [.,.]) = 0$, then $\mathcal{O}$ is the unique Banach space topology $\mathcal{T}$ on $\mathfrak{P}$ such that $[.,.]$ is continuous with respect to $\mathcal{T}$. In particular, it is the unique Hilbert space topology $\mathcal{T}$ on $\mathfrak{P}$ such that $(\mathfrak{P}, [.,.], \mathcal{T})$ is an almost Pontryagin space.*

Proof. The assumption $\Delta(\mathfrak{P}, [.,.]) = 0$ is equivalent to the fact that the family of functionals $f_x = [.,x]$, $x \in \mathfrak{P}$, is point separating. Hence we can apply Lemma 2.9. □

3. Subspaces, products, factors

The next result shows that the class of almost Pontryagin spaces is closed under the formation of subspaces and finite direct products. Note that the first half of this statement is not true for Pontryagin spaces.

Proposition 3.1. *Let $(\mathfrak{L}, [.,.], \mathcal{O})$ be an almost Pontryagin space, $\mathfrak{K}$ a closed linear subspace of $\mathfrak{L}$, and denote by $\mathcal{O} \cap \mathfrak{K}$ the subspace topology induced by $\mathcal{O}$ on $\mathfrak{K}$. Then $(\mathfrak{K}, [.,.], \mathcal{O} \cap \mathfrak{K})$ is an almost Pontryagin space. We have $\kappa_-(\mathfrak{K}, [.,.]) \leq \kappa_-(\mathfrak{L}, [.,.])$.*

Let $(\mathfrak{L}_1, [.,.]_1, \mathcal{O}_1)$ and $(\mathfrak{L}_2, [.,.]_2, \mathcal{O}_2)$ be two almost Pontryagin spaces, and denote by $\mathcal{O}_1 \times \mathcal{O}_2$ the product topology on $\mathfrak{L}_1 \times \mathfrak{L}_2$. Define the inner product

$$[(u;v),(x;y)] = [u,x]_1 + [v,y]_2, \ (u;v),(x;y) \in \mathfrak{L}_1 \times \mathfrak{L}_2.$$

Then $(\mathfrak{L}_1 \times \mathfrak{L}_2, [.,.], \mathcal{O}_1 \times \mathcal{O}_2)$ is an almost Pontryagin space. We have

$$\kappa_-(\mathfrak{L}_1 \times \mathfrak{L}_2, [.,.]) = \kappa_-(\mathfrak{L}_1, [.,.]_1) + \kappa_-(\mathfrak{L}_2, [.,.]_2),$$
$$\Delta(\mathfrak{L}_1 \times \mathfrak{L}_2, [.,.]) = \Delta(\mathfrak{L}_1, [.,.]_1) + \Delta(\mathfrak{L}_2, [.,.]_2).$$

Proof. To establish the first part of the assertion choose an $\mathcal{O}$-closed linear subspace $\mathfrak{M}$ of $\mathfrak{L}$ with finite codimension such that $(\mathfrak{M}, [.,.])$ is a Hilbert space. We already saw that by the closed graph theorem $[.,.]$ induces the topology $\mathcal{O} \cap \mathfrak{M}$ on $\mathfrak{M}$. Thus $\mathfrak{K} \cap \mathfrak{M}$ is at the same time $\mathcal{O} \cap \mathfrak{K}$-closed linear subspace of $\mathfrak{K}$ with finite codimension in $\mathfrak{K}$, and a $\mathcal{O}\cap\mathfrak{M}$-closed (i.e. $[.,.]$-closed) subspace of $\mathfrak{M}$. Hence $(\mathfrak{K}\cap\mathfrak{M},[.,.])$ is a Hilbert space. Thus $(\mathfrak{K},[.,.],\mathcal{O}\cap\mathfrak{K})$ is an almost Pontryagin space. The relation between the negative indices is clear.

To prove the second assertion take for $j=1,2$ a $\mathcal{O}_j$-closed subspace $\mathfrak{M}_j$ with finite codimension in $\mathfrak{L}_j$ such that $(\mathfrak{M}_j,[.,.]_j)$ is a Hilbert space. Then $\mathfrak{M}_1\times\mathfrak{M}_2$ is a $\mathcal{O}_1\times\mathcal{O}_2$-closed subspace of $\mathfrak{L}_1\times\mathfrak{L}_2$ of finite codimension such that $(\mathfrak{M}_1\times\mathfrak{M}_2,[.,.])$ is a Hilbert space. □

We conclude from Corollary 2.7 together with Proposition 3.1 that every closed subspace of a Pontryagin space is an almost Pontryagin space. Also the converse holds true:

Proposition 3.2. *Let $(\mathfrak{L},[.,.],\mathcal{O})$ be an almost Pontryagin space. Then there exists a Pontryagin space $(\mathfrak{P},[.,.])$ such that $\mathfrak{L}$ is a closed subspace of $\mathfrak{P}$ with codimension $\Delta(\mathfrak{L},[.,.])$ and $\mathcal{O}$ is the subspace topology on $\mathfrak{L}$ induced by the Pontryagin space topology on $\mathfrak{P}$. Moreover, $\kappa_-(\mathfrak{P},[.,.])=\kappa_-(\mathfrak{L},[.,.])+\Delta(\mathfrak{L},[.,.])$. Any two Pontryagin spaces with the listed properties are isometrically isomorphic.*

Conversely, let $(\mathfrak{P},[.,.])$ be a Pontryagin space. If $\mathfrak{L}$ is a closed subspace of $\mathfrak{P}$, so that $\mathfrak{L}$ with the inner product and topology inherited from $(\mathfrak{P},[.,.])$ is an almost Pontryagin space, then $\operatorname{codim}_{\mathfrak{P}}\mathfrak{L}\geq\Delta(\mathfrak{L},[.,.])$.

Proof. Fix a decomposition $\mathfrak{L}=\mathfrak{L}_+[\dot+]\mathfrak{L}_-[\dot+]\mathfrak{L}^{[\circ]}$ according to (2.3). Let $\mathfrak{L}'$ be a linear space of dimension $\Delta(\mathfrak{L},[.,.])$ and define

$$\mathfrak{P}=\mathfrak{L}_+\dot+\mathfrak{L}_-\dot+\mathfrak{L}^{[\circ]}\dot+\mathfrak{L}'.$$

We declare an inner product $[.,.]_1$ on $\mathfrak{P}$ by

$$[x,y]_1=[x,y],\ x,y\in\mathfrak{L}_++\mathfrak{L}_-,\ (\mathfrak{L}_++\mathfrak{L}_-)[\perp]_1(\mathfrak{L}^{[\circ]}+\mathfrak{L}'),$$

and the requirement that $\mathfrak{L}^{[\circ]}$ and $\mathfrak{L}'$ are skewly linked neutral subspaces, i.e. for every non-zero $x\in\mathfrak{L}^{[\circ]}$ there exists a $y\in\mathfrak{L}'$ such that $[x,y]\neq 0$ and, conversely, for every non-zero $y\in\mathfrak{L}'$ there exists an $x\in\mathfrak{L}^{[\circ]}$ such that $[x,y]\neq 0$.

Then $(\mathfrak{P},[.,.]_1)$ is a Pontryagin space because it can be seen as the product of the Pontryagin spaces $(\mathfrak{L}_+\dot+\mathfrak{L}_-,[.,.])$ and $(\mathfrak{L}^{[\circ]}\dot+\mathfrak{L}',[.,.]_1)$.

Clearly, $\operatorname{codim}_{\mathfrak{P}}\mathfrak{L}=\Delta(\mathfrak{L},[.,.])$ and $[.,.]_1|_{\mathfrak{L}}=[.,.]$. Since $\mathfrak{L}=(\mathfrak{L}^{[\circ]})^{[\perp]_1}$, the space $\mathfrak{L}$ is a closed subspace of $\mathfrak{P}$. Let $\mathcal{T}$ be the subspace toplogy on $\mathfrak{L}$ induced by the Pontryagin space topology on $\mathfrak{P}$. Then $\mathcal{T}$ coincides with the topology on $\mathfrak{L}$ obtained from the construction of Proposition 2.5, (ii), applied with $(\mathfrak{L}_+,[.,.])$, $(\mathfrak{L}_-,-[.,.])$ and $\mathfrak{L}^{[\circ]}$. Since $\mathfrak{L}_+$ is $\mathcal{O}$-closed, Corollary 2.6, (ii), yields $\mathcal{T}=\mathcal{O}$.

Let $(\mathfrak{P}_2,[.,.]_2)$ be another Pontryagin space which contains $\mathfrak{L}$ with codimension $\Delta(\mathfrak{L},[.,.])$. Then $\mathfrak{P}_2$ can be decomposed as

$$\mathfrak{P}_2=\mathfrak{L}_+[\dot+]\mathfrak{L}_-[\dot+](\mathfrak{L}^{[\circ]}\dot+\mathfrak{L}''),$$

where $\mathfrak{L}''$ is a neutral subspace skewly linked to $\mathfrak{L}^{[\circ]}$. It is now obvious that there exists an isometric isomorphism of $\mathfrak{P}_2$ onto the above constructed space $\mathfrak{P}$.

The second part of the assertion follows from [2, Theorem I.10.9]: Consider the $\Delta(\mathfrak{L},[.,.])$-dimensional subspace $\mathfrak{L}^{[\circ]}$ of $\mathfrak{P}$. Then certainly $\mathfrak{L} \subseteq (\mathfrak{L}^{[\circ]})^{[\perp]}$ and thus

$$\operatorname{codim}_{\mathfrak{P}} \mathfrak{L} \geq \operatorname{codim}_{\mathfrak{P}} (\mathfrak{L}^{[\circ]})^{[\perp]} = \dim \mathfrak{L}^{[\circ]} = \Delta(\mathfrak{L},[.,.]).$$

□

Let us introduce the correct notion of morphism between almost Pontryagin spaces.

Definition 3.3. Let $(\mathfrak{L}_1,[.,.]_1,\mathcal{O}_1)$ and $(\mathfrak{L}_2,[.,.]_2,\mathcal{O}_2)$ be almost Pontryagin spaces. A map $\phi : \mathfrak{L}_1 \to \mathfrak{L}_2$ is called a morphism between $(\mathfrak{L}_1,[.,.]_1,\mathcal{O}_1)$ and $(\mathfrak{L}_2,[.,.]_2,\mathcal{O}_2)$ if ϕ is linear, isometric, continuous and maps $\mathcal{O}_1$-closed subspaces of $\mathfrak{L}_1$ onto $\mathfrak{O}_2$-closed subspaces of $\mathfrak{L}_2$.

A linear mapping ϕ from an almost Pontryagin space $(\mathfrak{L}_1,[.,.]_1,\mathcal{O}_1)$ onto an almost Pontryagin space $(\mathfrak{L}_2,[.,.]_2,\mathcal{O}_2)$ is called an isomorphism if ϕ is bijective, bicontinuous and isometric with respect to $[.,.]_1$ and $[.,.]_2$.

Let us collect a couple of elementary facts.

Lemma 3.4. *The identity map of an almost Pontryagin space onto itself is an isomorphism. Every isomorphism is a morphism. The composition of two (iso)morphisms is a(n) (iso)morphism.*

Let $\phi : (\mathfrak{L}_1,[.,.]_1,\mathcal{O}_1) \to (\mathfrak{L}_2,[.,.]_2,\mathcal{O}_2)$ be a morphism. Then

(*i*) $\ker \phi \subseteq \mathfrak{L}^{[\circ]_1}$
(*ii*) $(\operatorname{ran} \phi,[.,.]_2,\mathcal{O}_2 \cap \operatorname{ran} \phi)$ *is an almost Pontryagin space.*
(*iii*) *If ϕ is surjective, then ϕ is open.*
(*iv*) *If ϕ is bijective, then ϕ is an isomorphism.*

If $\mathfrak{K}$ is a closed subspace of an almost Pontryagin space $(\mathfrak{L},[.,.],\mathcal{O})$, then the inclusion map $\iota : (\mathfrak{K},[.,.],\mathcal{O} \cap \mathfrak{K}) \to (\mathfrak{L},[.,.],\mathcal{O})$ is a morphism.

Proof. The first statement of the lemma is obvious.

`ad`(*i*): Since ϕ is isometric an element $x \in \ker \phi$ must satisfy

$$[x,y]_1 = [\phi x, \phi y]_2 = 0,\ y \in \mathfrak{L},$$

and hence $x \in \mathfrak{L}^{[\circ]_1}$.

`ad`(*ii*): Since $\operatorname{ran}\phi$ is $\mathcal{O}_2$-closed, we may to refer to Proposition 3.1.

`ad`(*iii*): Apply the Open Mapping Theorem.

`ad`(*iv*): This is an immediate consequence of the previous assertion.

The last statement follows since $\mathfrak{K}$ is a closed subspace of $\mathfrak{L}$. □

Morphisms can be constructed in a canonical way from subspaces of $\mathfrak{L}^{[\circ]}$.

Proposition 3.5. *Let $(\mathfrak{L},[.,.],\mathcal{O})$ be an almost Pontryagin space and let $\mathfrak{R}$ be a subspace of $\mathfrak{L}^{[\circ]}$. We consider the factor space $\mathfrak{L}/\mathfrak{R}$ endowed with an inner product $[.,.]_1$ defined by*

$$[x+\mathfrak{R}, y+\mathfrak{R}]_1 = [x,y], \tag{3.1}$$

and with the quotient topology $\mathcal{O}/\mathfrak{R}$. Then $(\mathfrak{L}/\mathfrak{R}, [.,.]_1, \mathcal{O}/\mathfrak{R})$ is an almost Pontryagin space. We have

$$\kappa_-(\mathfrak{L}/\mathfrak{R}, [.,.]_1) = \kappa_-(\mathfrak{L}, [.,.]),$$

$$\Delta(\mathfrak{L}/\mathfrak{R}, [.,.]_1) = \Delta(\mathfrak{L}, [.,.]) - \dim \mathfrak{R}.$$

The quotient map $\pi : \mathfrak{L} \to \mathfrak{L}/\mathfrak{R}$ is a morphism of $(\mathfrak{L}, [.,.], \mathcal{O})$ onto $(\mathfrak{L}/\mathfrak{R}, [.,.]_1, \mathcal{O}/\mathfrak{R})$.

Proof. The inner product on $\mathfrak{L}/\mathfrak{R}$ is well defined by (3.1) because of $\mathfrak{R} \subseteq \mathfrak{L}^{[\circ]}$. Since $\mathcal{O}$ is a Hilbert space topology and $\mathfrak{R}$ is a finite dimensional and, hence, closed subspace of $\mathfrak{L}$, the topology $\mathcal{O}/\mathfrak{R}$ is also a Hilbert space topology.

Denote by $\pi : \mathfrak{L} \to \mathfrak{L}/\mathfrak{R}$ the canonical projection. Since the inner product on $\mathfrak{L}/\mathfrak{R}$ is defined according to

$$\begin{array}{ccc} (\mathfrak{L}/\mathfrak{R})^2 & \overset{[.,.]_1}{\dashrightarrow} & \mathbb{C}\ , \\ {\scriptstyle \pi\times\pi}\uparrow & \nearrow {\scriptstyle [.,.]} & \\ \mathfrak{L}^2 & & \end{array}$$

and the quotient topology is the final topology with respect to π, we obtain that $[.,.]_1$ is $\mathcal{O}/\mathfrak{R}$-continuous.

Choose an $\mathcal{O}$-closed subspace $\mathfrak{M}$ of $\mathfrak{L}$ such that $\operatorname{codim}_{\mathfrak{L}} \mathfrak{M} < \infty$ and such that $(\mathfrak{M}, [.,.])$ is a Hilbert space. Since $\mathfrak{R}$ is finite dimensional $\mathfrak{M} + \mathfrak{R}$ is $\mathcal{O}$-closed. Thus $\pi(\mathfrak{M}) = (\mathfrak{M} + \mathfrak{R})/\mathfrak{R}$ satisfies the requirements of axiom (aPS2).

The formulas for the negative index and the degree of degeneracy are obvious.

The quotient map π is clearly linear, isometric and continuous. If $\mathfrak{U}$ is any $\mathcal{O}$-closed subspace of $\mathfrak{L}$, then also $\mathfrak{U} + \mathfrak{R}$ is $\mathcal{O}$-closed and therefore $\pi(\mathfrak{U}) = (\mathfrak{U} + \mathfrak{R})/\mathfrak{R}$ is $\mathcal{O}/\mathfrak{R}$-closed. This shows that π is a morphism. □

We conclude this section with the 1^{st} homomorphy theorem.

Lemma 3.6. *Let $\phi : (\mathfrak{L}_1, [.,.]_1, \mathcal{O}_1) \to (\mathfrak{L}_2, [.,.]_2, \mathcal{O}_2)$ be a morphism. Then ϕ induces an isomorphism $\hat{\phi}$ between $(\mathfrak{L}_1/\ker\phi, [.,.]_1, \mathcal{O}_1/\ker\phi)$ and $(\operatorname{ran}\phi, [.,.]_2, \mathcal{O}_2 \cap \operatorname{ran}\phi)$ with*

$$\begin{array}{ccc} (\mathfrak{L}_1, [.,.]_1, \mathcal{O}_1) & \xrightarrow{\ \phi\ } & (\mathfrak{L}_2, [.,.]_2, \mathcal{O}_2) \\ {\scriptstyle \pi}\downarrow & & \uparrow{\scriptstyle \iota} \\ (\mathfrak{L}_1/\ker\phi, [.,.]_1, \mathcal{O}_1/\ker\phi) & \underset{\hat{\phi}}{\dashrightarrow} & (\operatorname{ran}\phi, [.,.]_2, \mathcal{O}_2 \cap \operatorname{ran}\phi) \end{array} .$$

Proof. The induced mapping $\hat{\phi}$ is bijective, isometric and continuous. By the Open Mapping Theorem it is also open. Thus it is an isomorphism. □

4. Completions

The generalization of the concept of completion to the almost Pontryagin space setting is a much more delicate topic.

Remark 4.1. Let an inner product space $(\mathfrak{A}, [.,.])$ with $\kappa_-(\mathfrak{A}, [.,.]) = \kappa < \infty$ be given. Then there always exists a Pontryagin space which contains $\mathfrak{A}/\mathfrak{A}^{[\circ]}$ as a dense subspace. We are going to sketch the construction of this so-called completion of $(\mathfrak{A}, [.,.])$ (see e.g. [4]).

Take any subspace $\mathfrak{M}$ of $\mathfrak{A}$ which is maximal with respect to the property that $(\mathfrak{M}, [.,.])$ is an anti Hilbert space. If $e_1, \dots, e_\kappa$ is an orthonormal basis of $(\mathfrak{M}, -[.,.])$, then

$$P_{\mathfrak{M}} = -[., e_1]e_1 \cdots - [., e_\kappa]e_\kappa,$$

is the orthogonal projection of $\mathfrak{A}$ onto $\mathfrak{M}$. By the maximality property of $\mathfrak{M}$ the orthogonal complement $((I-P_{\mathfrak{M}})\mathfrak{A}, [.,.])$ is positive semidefinite. Therefore, setting $J_{\mathfrak{M}} = I - 2P_{\mathfrak{M}}$ we see that $[J_{\mathfrak{M}}., .] = (.,.)_{\mathfrak{M}}$ is a positive semidefinite product on $\mathfrak{A}$. We then have $(J_{\mathfrak{M}}., .) = [.,.]_{\mathfrak{M}}$, and $J_{\mathfrak{M}}$ and $[.,.]$ are continuous with respect to the topology induced by $(.,.)_{\mathfrak{M}}$.

Note that if $\mathfrak{M}'$ is another subspace of $\mathfrak{L}$ which is maximal with respect to the property that $(\mathfrak{M}', [.,.])$ is an anti Hilbert space, then $P_{\mathfrak{M}'}$, and hence $J_{\mathfrak{M}'}$ and $(.,.)_{\mathfrak{M}'}$ are continuous with respect to $(.,.)_{\mathfrak{M}}$. By symmetry we obtain that $(.,.)_{\mathfrak{M}}$ and $(.,.)_{\mathfrak{M}'}$ are equivalent scalar products, i.e. there exist $\alpha, \beta > 0$ such that

$$\alpha(x,x)_{\mathfrak{M}} \le (x,x)_{\mathfrak{M}'} \le \beta(x,x)_{\mathfrak{M}}, \; x \in \mathfrak{A}. \tag{4.1}$$

This in turn means that these two scalar products induce the same topology $\mathcal{T}$ on $\mathfrak{A}$. In particular, $\mathcal{T}$ is determined by $[.,.]$ and not by a particularly chosen $\mathfrak{M}$.

A completion of $(\mathfrak{A}, [.,.])$ is given by $(\mathfrak{P}, [.,.])$, where $\mathfrak{P}$ is the completion of $\mathfrak{A}/\mathfrak{A}^{[\circ]}$ with respect to $(.,.)_{\mathfrak{M}}$. Note that

$$\mathfrak{A}^{(\circ)_{\mathfrak{M}}} = \{x \in \mathfrak{A} : (x,x)_{\mathfrak{M}} = 0\} = \{x \in \mathfrak{A} : [x,y] = 0 \text{ for all } y \in \mathfrak{A}\} = \mathfrak{A}^{[\circ]},$$

and that $\mathfrak{A}^{[\circ]} \cap \mathfrak{M} = \{0\}$.

After factoring out $\mathfrak{A}^{[\circ]}$ by continuity we can extend $P_{\mathfrak{M}}, J_{\mathfrak{M}}, [.,.]$ to $\mathfrak{P}$. Then we have $J_{\mathfrak{M}} = I - 2P_{\mathfrak{M}}$ and $[.,.] = (J_{\mathfrak{M}}., .)_{\mathfrak{M}}$ also on $\mathfrak{P}$. The extension $P_{\mathfrak{M}}$ is the orthogonal projection of $\mathfrak{P}$ onto $\mathfrak{M}/\mathfrak{A}^{[\circ]}$. It is straightforward to check that

$$\mathfrak{P} = P_{\mathfrak{M}}\mathfrak{P}[\dot{+}]((I - P_{\mathfrak{M}})\mathfrak{P}) \tag{4.2}$$

is a fundamental decomposition of $(\mathfrak{P}, [.,.])$. Therefore, $(\mathfrak{P}, [.,.])$ is a Pontryagin space and by (4.1) its construction does not depend on the chosen space $\mathfrak{M}$. Moreover, it is the unique Pontryagin space (up to isomorphisms) which contains $\mathfrak{A}/\mathfrak{A}^{[\circ]}$ such that $[.,.]$ on $\mathfrak{P}$ is a continuation of $[.,.]$ on $\mathfrak{A}/\mathfrak{A}^{[\circ]}$.

To see this let $(\mathfrak{P}', [.,.])$ be another such Pontryagin space, and let $\mathfrak{P}' = \mathfrak{P}_-[\dot{+}]\mathfrak{P}_+$ be a fundamental decomposition of $\mathfrak{P}'$. By a density argument we find a subspace $\mathfrak{M}$ of $\mathfrak{A}$ with the same dimension as $\mathfrak{P}_-$ such that $\mathfrak{M}/\mathfrak{A}^{[\circ]}$ is sufficiently close to $\mathfrak{P}_-$ in order that $(\mathfrak{M}, [.,.])$ is an anti Hilbert space. It follows that $\mathfrak{M}$ is maximal with respect to this property. Let P be the orthogonal projection of $\mathfrak{P}'$ onto $\mathfrak{M}/\mathfrak{A}^{[\circ]}$, and let $(.,.)$ be the Hilbert space inner product $[(I-2P).,.]$ on $\mathfrak{P}$.

If $(\mathfrak{P},[.,.])$ is the completion as constructed above, then the identity ϕ on $\mathfrak{A}/\mathfrak{A}^{[\circ]}$ is a $[.,.]$-isometric linear mapping from a dense subspace of $\mathfrak{P}$ onto a dense subspace of $\mathfrak{P}'$. By construction $\phi P_{\mathfrak{M}} = P\phi$. Hence ϕ is isometric with respect to $(.,.)_{\mathfrak{M}}$ and $(.,.)$. As both induce the topology on the respective spaces $\mathfrak{P}$ and $\mathfrak{P}'$ we see that ϕ can be extended to an isomorphism from $\mathfrak{P}$ onto $\mathfrak{P}'$.

Definition 4.2. Let $(\mathfrak{A},[.,.])$ be an inner product space such that $\kappa_-(\mathfrak{A},[.,.]) = \kappa < \infty$. An almost Pontryagin space with a linear mapping $((\mathfrak{L},[.,.],\mathcal{O}),\iota)$ is called a completion of $\mathfrak{A}$, if ι is an isometric mapping (with respect to $[.,.]$) from $\mathfrak{A}$ onto a dense subspace $\iota(\mathfrak{A})$ of $\mathfrak{L}$.

Two completions $((\mathfrak{L}_1,[.,.]_1,\mathcal{O}_1),\iota_1)$ and $((\mathfrak{L}_2,[.,.]_2,\mathcal{O}_2),\iota_2)$ are called isomorphic if there exists an isomorphism ϕ from $(\mathfrak{L}_1,[.,.]_1,\mathcal{O}_1)$ onto $(\mathfrak{L}_2,[.,.]_2,\mathcal{O}_2)$ such that $\phi\circ\iota_1 = \iota_2$.

Remark 4.3. We saw above that, up to isomorphism, there always exists a unique Pontryagin space which is a completion of $(\mathfrak{A},[.,.])$.

If we allow the almost Pontryagin space of a completion $((\mathfrak{L},[.,.],\mathcal{O}),\iota)$ to be degenerated, i.e. $\Delta(\mathfrak{L},[.,.]) > 0$, then $(\mathfrak{L},[.,.],\mathcal{O})$ is not uniquely determined if we assume $\dim\mathfrak{A}/\mathfrak{A}^{[\circ]} = \infty$. This can be derived immediately from Proposition 2.8.

For $\dim\mathfrak{A}/\mathfrak{A}^{[\circ]} = \infty$ it follows from the subsequent result that for any $\Delta \geq 0$ there exists a completion $((\mathfrak{L},[.,.],\mathcal{O}),\iota)$ of $(\mathfrak{A},[.,.])$ such that $\Delta(\mathfrak{L},[.,.]) = \Delta$. Also for fixed Δ Proposition 2.8 shows that $(\mathfrak{L},[.,.],\mathcal{O})$ is not uniquely determined.

Proposition 4.4. *Let $(\mathfrak{A},[.,.])$ be an inner product space with $\kappa_-(\mathfrak{A},[.,.]) = \kappa < \infty$, and let $\mathcal{T}$ be the topology determined by $[.,.]$ on $\mathfrak{A}$ (see Remark 4.1).*

If $f_1,\dots,f_\Delta$ are complex linear functionals on $\mathfrak{A}$ such that no linear combination of them is continuous with respect to $\mathcal{T}$, then there exists an (up to isomorphic copies) unique completion $((\mathfrak{L},[.,.],\mathcal{O}),\iota)$ with $\Delta(\mathfrak{L},[.,.]) = \Delta$ such that $f_1,\dots,f_\Delta$ are continuous with respect to $\iota^{-1}(\mathcal{O})$.

Proof. The construction made in this proof stems from [6].

Let $(\mathfrak{P},[.,.])$ be the unique Pontryagin space completion of $(\mathfrak{A},[.,.])$, i.e. the completion with respect to $\mathcal{T}$. Let $(.,.)_{\mathfrak{M}}$ be the Hilbert space inner product on $\mathfrak{P}$ from Remark 4.1 constructed with the help of a subspace $\mathfrak{M}$ of $\mathfrak{A}$ being maximal with respect to the property that $(\mathfrak{M},[.,.])$ is an anti Hilbert space. We define

$$\mathfrak{L} = \mathfrak{P}\times\mathbb{C}^\Delta,$$

and provide $\mathfrak{L}$ with the inner product $(.,.)$ such that $(.,.)$ coincides with $(.,.)_{\mathfrak{M}}$ on $\mathfrak{P}$ and with the euclidean product on $\mathbb{C}^\Delta$, and such that $\mathfrak{L} = \mathfrak{P}(\dot{+})\mathbb{C}^\Delta$. Let $[.,.]$ be defined on $\mathfrak{L}$ by

$$[(x;\xi),(y;\eta)] = [x,y].$$

By definition $(\mathfrak{L},[.,.],\mathcal{O}_{(.,.)})$ is an almost Pontryagin space. Hereby $\mathcal{O}_{(.,.)}$ is the topology induced by $(.,.)$ on $\mathfrak{L}$.

Now we embed $\mathfrak{A}$ in $\mathfrak{L}$ via the mapping ι

$$\iota(x) = (x + \mathfrak{A}^{[\circ]}; (f_1(x), \dots, f_\Delta(x))).$$

Then $\iota(\mathfrak{A})$ is dense in $\mathfrak{L}$. In fact, if not, then we could find $(y;\eta) \in \mathfrak{L}$ such that $(y;\eta)(\perp)\iota(\mathfrak{A})$. It would follow that

$$(x + \mathfrak{A}^{[\circ]}, -y) = \sum_{j=1}^{\Delta} \bar{\eta}_j f_j(x),\ x \in \mathfrak{A},$$

and, therefore, the right hand side would be continuous with respect to $\mathcal{T}$. By assumption $\eta = 0$ and further $y(\perp)\mathfrak{A}$ in $\mathfrak{P}$. This is not possible as $\mathfrak{A}$ is dense in $\mathfrak{P}$.

The mapping ι is isometric with respect to $[.,.]$. Thus by defining $\mathcal{O} = \mathcal{O}_{(.,.)}$, $((\mathfrak{L}, [.,.], \mathcal{O}), \iota)$ is a completion of $(\mathfrak{A}, [.,.])$. By the definition of ι the functionals $f_1, \dots, f_\Delta$ are continuous with respect to $\iota^{-1}(\mathcal{O})$.

Assume now that $((\mathfrak{L}', [.,.]', \mathcal{O}'), \iota')$ is another completion of $(\mathfrak{A}, [.,.])$ such that $\Delta(\mathfrak{L}', [.,.]') = \Delta$ and such that $f_1, \dots, f_\Delta$ are continuous with respect to ${\iota'}^{-1}(\mathcal{O}')$. Let $(.,.)'$ be a Hilbert space scalar product on $\mathfrak{L}'$ which induces $\mathcal{O}'$. By elementary considerations from the theory of locally convex vector spaces we can factor $f_1, \dots, f_\Delta$ through the isotropic part $\mathfrak{A}^{(\circ)'}$ of $\mathfrak{A}$ with respect to $(\iota(.), \iota(.))'$. Note that $\mathfrak{A}^{(\circ)'}$ is also the set of all points in $\mathfrak{A}$ which have exactly the same neighbourhoods as 0 with respect to the topology ${\iota'}^{-1}(\mathcal{O}')$.

Clearly, $(\mathfrak{L}', (.,.)')$ is isomorphic to the completion of $\mathfrak{A}/\mathfrak{A}^{(\circ)'}$ with respect to $(\iota(.), \iota(.))'$. Hence by continuation to the completion we obtain continuous linear functionals $g_1, \dots, g_\Delta$ on $(\mathfrak{L}', [.,.]', \mathcal{O}')$ such that $f_1 = g_1 \circ \iota', \dots, f_\Delta = g_\Delta \circ \iota'$.

By Proposition 3.5 $(\mathfrak{L}'/{\mathfrak{L}'}^{[\circ]'}, [.,.]')$ is a Pontryagin space. We denote by π the factorization mapping. As $(\mathfrak{A}/\mathfrak{A}^{[\circ]}, [.,.])$ is isometrically embedded by $\pi \circ \iota'$ in this Pontryagin space Remark 4.1 shows that $(\mathfrak{L}'/{\mathfrak{L}'}^{[\circ]'}, [.,.]')$ is an isomorphic copy of $(\mathfrak{P}, [.,.])$. Let $\phi : \mathfrak{L}'/{\mathfrak{L}'}^{[\circ]'} \to \mathfrak{P}$ be this isomorphism, which satisfies $\phi \circ \pi \circ \iota' = \mathrm{id}_{\mathfrak{A}}$.

Let $0 \neq x \in {\mathfrak{L}'}^{[\circ]'}$ be such that $g_1(x) = \cdots = g_\Delta(x) = 0$. By elementary linear algebra we find a non-trivial linear combination g of the functionals $g_1, \dots, g_\Delta$ which vanishes on ${\mathfrak{L}'}^{[\circ]'}$. Hence, we find a functional f on $\mathfrak{P}$ such that $g = f \circ \phi \circ \pi$. But then $a \mapsto f(a + \mathfrak{A}^{[\circ]})$ is a non-trivial linear combination of $f_1, \dots, f_\Delta$ which is continuous with respect to $\mathcal{T}$. By assumption this is ruled out. Thus the intersection of the kernels of $g_j,\ j = 1, \dots, \Delta$, has no point in common with ${\mathfrak{L}'}^{[\circ]'}$ except of 0. Since the intersection of Δ hyperplanes has codimension at most Δ, we have

$${\mathfrak{L}'}^{[\circ]'} \dot{+} (\ker(g_1) \cap \cdots \cap \ker(g_\Delta)) = \mathfrak{L}', \tag{4.3}$$

and see that the mapping

$$\varphi : \mathfrak{L}' \to \mathfrak{L},\ x \mapsto (\phi \circ \pi(x); (g_1(x), \dots, g_\Delta(x))),$$

is bijective. Moreover, φ is isometrically with respect to $[.,.]$ and satisfies $\varphi \circ \iota' = \iota$. Since in the decomposition (4.3) all subspaces are closed, the Open Mapping Theorem implies that φ is bicontinuous with respect to $\mathcal{O}'$ and $\mathcal{O}$. □

Remark 4.5. With the notation from Proposition 4.4

$$\langle .,. \rangle = (.,.)_{\mathfrak{M}} + \sum_{j=1}^{\Delta} f_j(.)\bar{f}_j(.), \tag{4.4}$$

is a non-negative inner product on $\mathfrak{A}$. It is easy to see that ι induces an isomorphism from the completion of $(\mathfrak{A}/\mathfrak{A}^{\langle\circ\rangle}, \langle .,. \rangle)$ onto $(\mathfrak{L}, (.,.))$. In particular, $\mathcal{O}_{\langle .,. \rangle} = \iota^{-1}(\mathcal{O})$.

The completion constructed in Proposition 4.4 appeared in implicit forms already in various papers. See for example [7].

Definition 4.6. We call the completion of $(\mathfrak{A}, [.,.])$ constructed in Proposition 4.4 the completion of $(\mathfrak{A}, [.,.], (f_i)_{i=1,\dots,\Delta})$.

Corollary 4.7. *Let* $(\mathfrak{A}, [.,.])$ *be an inner product space with* $\kappa_-(\mathfrak{A}, [.,.]) = \kappa < \infty$, *and let* $\mathcal{T}$ *be the topology determined by* $[.,.]$ *on* $\mathfrak{A}$ *(see Remark 4.1).*

Let $(f_i)_{i=1,\dots,\Delta}$ *and* $(f_i')_{i=1,\dots,\Delta}$ *be two sets of complex linear functionals on* $\mathfrak{A}$ *such that no linear combination of* $(f_i)_{i=1,\dots,\Delta}$ *and no linear combination of* $(f_i')_{i=1,\dots,\Delta}$ *is continuous with respect to* $\mathcal{T}$.

The completion of $(\mathfrak{A}, [.,.], (f_i)_{i=1,\dots,\Delta})$ *is isomorphic to the completion of* $(\mathfrak{A}, [.,.], (f_i')_{i=1,\dots,\Delta})$ *if and only if the functionals* $f_1', \dots, f_\Delta'$ *are continuous with respect to the topology induced by* $\langle .,. \rangle$ *defined in (4.4) on* $\mathfrak{A}$.

Proof. We denote by $((\mathfrak{L}, [.,.], \mathcal{O}), \iota)$ and $((\mathfrak{L}', [.,.]', \mathcal{O}'), \iota')$ the completions of the triplets $(\mathfrak{A}, [.,.], (f_i)_{i=1,\dots,\Delta})$ and $(\mathfrak{A}, [.,.], (f_i')_{i=1,\dots,\Delta})$, respectively. Moreover, let $(g_i)_{i=1,\dots,\Delta}$ and $(g_i')_{i=1,\dots,\Delta}$ be the continuous linear functionals on $\mathfrak{L}$ and $\mathfrak{L}'$, respectively, such that $f_i = g_i \circ \iota$ and $f_i' = g_i' \circ \iota'$, respectively.

If the two completions are isomorphic by the isomorphism $\phi : (\mathfrak{L}, [.,.], \mathcal{O}) \to (\mathfrak{L}', [.,.]', \mathcal{O}')$, then $g_i' \circ \phi,\ i = 1, \dots, \Delta$, are continuous functionals on $(\mathfrak{L}, [.,.], \mathcal{O})$. By Remark 4.5 $f_i' = g_i' \circ \iota' = g_i' \circ \phi \circ \iota$ is continuous on $\mathfrak{A}$ with respect to the topology induced by $\langle .,. \rangle$.

Conversely, if $f_1', \dots, f_\Delta'$ are continuous with respect to the topology induced by $\langle .,. \rangle$, then by continuation to the completion we obtain continuous linear functionals $(h_i')_{i=1,\dots,\Delta}$ on $(\mathfrak{L}, [.,.], \mathcal{O})$ such that $f_i' = h_i' \circ \iota$. By the uniqueness assertion in Proposition 4.4 $((\mathfrak{L}, [.,.], \mathcal{O}), \iota)$ is also a completion of $(\mathfrak{A}, [.,.], (f_i')_{i=1,\dots,\Delta})$. □

5. Almost reproducing kernel Pontryagin spaces

Objects of intensive studies are the so-called reproducing kernel Pontryagin spaces. These are Pontryagin spaces $(\mathfrak{P}, [.,.])$ which consist of functions F mapping some set M into $\mathbb{C}$ such that there exist $K(.,t) \in \mathfrak{P},\ t \in M$, with

$$F(t) = [F, K(.,t)],\ \ F \in \mathfrak{P}, t \in M. \tag{5.1}$$

An equivalent definition of reproducing kernel Pontryagin spaces is the assumption that $(\mathfrak{P}, [.,.])$ consists of complex valued functions on a set M such that the point evaluations are continuous at all points of M.

The first approach to reproducing kernel Pontryagin spaces does not have an immediate generalization to almost Pontryagin spaces but the second does.

Definition 5.1. Let $(\mathfrak{L}, [.,.], \mathcal{O})$ be an almost Pontryagin space, and assume that the elements of $\mathfrak{L}$ are complex valued functions on a set M. This space is a called an almost reproducing kernel Pontryagin spaces on M, if for any $t \in M$ the linear functional

$$f_t : F \mapsto F(t),\ F \in \mathfrak{L},$$

is continuous on $\mathfrak{L}$ with respect to $\mathcal{O}$.

Remark 5.2. As the elements of $\mathfrak{L}$ are functions we see that the family $(f_t)_{t\in M}$ of point evaluation functionals is point separating. Hence Proposition 2.9 yields the uniqueness of the topology $\mathcal{O}$ for which the functionals $f_t,\ t \in M$, are continuous. Consequently, we are going to skip the topology and write almost reproducing kernel Pontryagin spaces as pairs $(\mathfrak{L}, [.,.])$.

A major setback to the study of almost reproducing kernel Pontryagin spaces is the fact that in the case $\Delta(\mathfrak{L}, [.,.]) > 0$ we do not find a reproducing kernel $K(s,t)$ which satisfies (5.1). However, we do have the following

Proposition 5.3. *Let $(\mathfrak{L}, [.,.])$ be an almost reproducing kernel Pontryagin space on a set M and put $\Delta = \Delta(\mathfrak{L}, [.,.])$. Moreover, let N be a separating subset of M, i.e. assume that the family $(f_t)_{t\in N}$ is point separating. Then there exist $t_1, \dots, t_\Delta \in N$, $c \in \mathbb{R}$, and $R(.,t) \in \mathfrak{L}$ such that*

$$F(t) = [F, R(.,t)] + c\big(F(t_1)R(t,t_1) + \dots + F(t_\Delta)R(t,t_\Delta)\big),\ F \in \mathfrak{L},\ t \in M.$$

Proof. The number Δ is by definition the dimension of $\mathfrak{L}^{[\circ]} = \ker G$, where $G = G_{(.,.)}$ is the Gram operator with respect to a Hilbert space product $(.,.)$ inducing the topology of $(\mathfrak{L}, [.,.])$. By Corollary 2.4 we may choose $(.,.)$ such that $G = I + L$, where L is a selfadjoint finite rank operator.

Because of the assumption on N by induction one can easily show the existence of points $t_1, \dots, t_\Delta \in N$, such that $h \in \mathfrak{L}^{[\circ]}$ and $h(t_j) = 0,\ j = 1, \dots, \Delta$, implies $h = 0$.

Because of the continuity of point evaluations we find elements $K(.,t) \in \mathfrak{L}$ with

$$F(t) = (F, K(.,t)),\ F \in \mathfrak{L}.$$

We define the following selfadjoint operator H of finite rank on $\mathfrak{L}$

$$H(F) = \sum_{j=1}^{\Delta} F(t_j)K(.,t_j).$$

Let $\mathfrak{K} = \ker(H) \cap \ker(L)$, then $\mathfrak{K}^{(\perp)}$ is finite dimensional since the selfadjoint operators H and L are of finite rank, and $\mathfrak{K}^{(\perp)}$ contains the range of L and H. For $z \in \mathbb{C}$ it follows that the restriction of the operator $I + L + zH$ onto $\mathfrak{K}$ is equal to the identity. Hence $I + L + zH$ is invertible on $\mathfrak{L}$ if and only if it is invertible on $\mathfrak{K}^{(\perp)}$. To show that $I + L + zH$ is invertible for $z = i$, let $(I + L + iH)F = 0$.

Then $(HF, F) = 0 = ((I+L)F, F)$, and the form of H implies that $F(t_j) = 0,\ j = 1, \dots, \Delta$. It follows that $H(F) = 0$, and hence $(I + L)F = 0$, or $F \in \mathfrak{L}^{[\circ]}$ as $I + L$ is the Gram operator. The definition of the points $t_1, \dots, t_\Delta$ implies that $F = 0$, that is, the operator $(I + L + iH)$ is invertible.

Thus $\det((I + L + zH)|_{\mathfrak{K}^{[\perp]}})$ is not identically zero, and therefore has only a discrete zero set. In particular, we find $c \in \mathbb{R}$ such that $(I + L + cH)$ is invertible. Now set

$$R(., t) = (I + L + cH)^{-1}K(., t).$$

Note that because of the selfadjointness of $I + L + cH$,

$$R(s, t) = (R(., t), K(., s)) = (K(., t), R(., s)) = \overline{R(t, s)}.$$

For $F \in \mathfrak{L}$ and $t \in M$ we have

$$[F, R(., t)] = ((I + L + cH)F, R(., t)) - c(HF, R(., t)) = F(t) - c\sum_{j=1}^{\Delta} F(t_j)R(t, t_j).$$

□

6. Examples of almost Pontryagin spaces

As the first topic of this section we are going to sketch the continuation problem for hermitian functions with finitely many negative squares on intervals $[-2a, 2a]$ to the whole real axis. We will meet inner product spaces and completions in the sense of Section 4. Taking into account also a possible degeneracy of this completion one obtains a refinement of classical results on the number of all possible extensions of the given hermitian functions with finitely many negative squares to $\mathbb{R}$. For a complete treatment of this topic see [7].

Definition 6.1. Let $a > 0$ be a real number, and assume that $f : [-2a, 2a] \to \mathbb{C}$ is a continuous function. We say that f is hermitian if it satisfies $f(-t) = \overline{f(t)},\ t \in [-2a, 2a]$, and f is said to be hermitian with $\kappa(\in \mathbb{N} \cup \{0\})$ many negative squares if the kernel $f(t - s),\ s, t \in (-a, a)$, has κ negative squares. The set of all such functions we denote by $\mathcal{P}_{\kappa,a}$.

By $\mathcal{P}_\kappa$ we denote the set of all continuous hermitian functions with κ negative squares on $\mathbb{R}$, i.e. $f(t - s),\ s, t \in \mathbb{R}$, has κ negative squares.

For $\kappa = 0$ the function f is called positve definite.

The continuation problem is to find for given $f \in \mathcal{P}_{\kappa,a}$ and $\tilde{\kappa} \in \mathbb{N} \cup \{0\}$ all possible extensions $\tilde{f}$ of f to the whole real axis such that $\tilde{f} \in \mathcal{P}_{\tilde{\kappa}}$. Trivially, by the definition of the respective classes a necessary condition for the existence of such extensions is $\kappa \le \tilde{\kappa}$. The following classical result can be found for example in [3].

Theorem 6.2. *Let $f \in \mathcal{P}_{\kappa,a}$. Then either f has exactly one extension belonging to $\mathcal{P}_\kappa$, or it has infinitely many extensions in $\mathcal{P}_\kappa$. In the latter case f also has infinitely many extensions in $\mathcal{P}_{\tilde{\kappa}}$ for every $\tilde{\kappa} \ge \kappa$.*

This result originates from the following operator theoretic considerations. First let $(\mathfrak{P}(f), [.,.])$ be the reproducing kernel Pontryagin space on $(-a, a)$ having $k(s,t) = f(s-t)$ as its reproducing kernel. As we assume $f \in \mathcal{P}_{\kappa,a}$ the degree of negativity of $(\mathfrak{P}(f), [.,.])$ is κ. Clearly, $(\mathfrak{P}(f), [.,.])$ is the completion of $(\mathfrak{A}(f), [.,.])$ where $\mathfrak{A}(f)$ is the linear hull of $\{k(.,t) : t \in (-a,a)\}$.

Moreover, a certain differential operator $S(f)$ is constructed on $(\mathfrak{P}(f), [.,.])$. This operator is symmetric and densely defined. Its defect elements are given by

$$\ker (S(f)^* - z) = e^{izs},\ z \in \mathbb{C} \setminus \mathbb{R},$$

as a function of $s \in (-a, a)$ if they belong to $\mathfrak{P}(f)$. Thus $S(f)$ has either defect indices $(1,1)$ or $(0,0)$ depending on whether e^{izs} belongs to this space or not.

A crucial fact in verifying Theorem 6.2 is that all extensions of f belonging to $\mathcal{P}_\kappa$ correspond bijectively to all $\mathfrak{P}(f)$-minimal selfadjoint extensions A of $S(f)$ in a possibly larger Pontryagin space $\tilde{\mathfrak{P}} \supseteq \mathfrak{P}(f)$ with $\kappa_-(\tilde{\mathfrak{P}}, [.,.]) = \kappa$. Hereby $\mathfrak{P}(f)$-minimal means

$$\text{cls}(\mathfrak{P}(f) \cup \{(A-z)^{-1}x : x \in \mathfrak{P}(f),\ z \in \rho(A)\}) = \tilde{\mathfrak{P}}.$$

Hence, in the case that $S(f)$ has defect index $(0,0)$ or, equivalently, that $S(f)$ is selfadjoint there are no $\mathfrak{P}(f)$-minimal selfadjoint extensions of $S(f)$ other than $S(f)$ itself. Therefore, f has exactly one extension in $\mathcal{P}_\kappa$.

If $S(f)$ has defect index $(1,1)$, then there are infinitely many $\mathfrak{P}(f)$-minimal selfadjoint extensions A of $S(f)$ and, hence, infinitely many extensions in $\mathcal{P}_\kappa$. Moreover, in this case the extensions of f in $\mathcal{P}_{\tilde{\kappa}}$ for $\tilde{\kappa} \geq \kappa$ correspond bijectively to all $\mathfrak{P}(f)$-minimal selfadjoint extensions A of $S(f)$ in a Pontryagin space $\tilde{\mathfrak{P}} \supseteq \mathfrak{P}(f)$ with $\kappa_-(\tilde{\mathfrak{P}}, [.,.]) = \tilde{\kappa}$, and there are also infinitely many of them for an arbitrary $\tilde{\kappa} \geq \kappa$.

Theorem 6.2 seems to give a sufficiently satisfactory answer to the continuation problem. But as some examples show it can happen that f has exactly one extension in $\mathcal{P}_\kappa$ but infinitely many extensions in $\mathcal{P}_{\tilde{\kappa}}$ for some $\tilde{\kappa} > \kappa$. How does this fit in with the operator theoretic approach mentioned above?

Here almost Pontryagin spaces come into play. In the case that $S(f)$ has defect index $(1,1)$ the fact that $e^{izs},\ z \in \mathbb{C} \setminus \mathbb{R}$ belongs to $\mathfrak{P}(f)$ can be reformulated by saying that

$$F_z : \sum_j \alpha_j k(.,t_j) \mapsto \sum_j \alpha_j e^{izt_j}$$

are continuous linear functionals on $(\mathfrak{A}, [.,.])$ for all $z \in \mathbb{C} \setminus \mathbb{R}$.

If f has a unique extension $f_0 \in \mathcal{P}_\kappa$, i.e. $S(f)$ has defect $(0,0)$, then these functionals are not continuous. But it can happen that by refining the topology on $(\mathfrak{A}, [.,.])$ by finitely many functionals $F_{z_1}, \ldots, F_{z_\Delta},\ z_j \in \mathbb{C} \setminus \mathbb{R}$ as in Remark 4.5 we obtain a topology $\mathcal{O}$ on $(\mathfrak{A}, [.,.])$ such that all functionals $F_z,\ z \in \mathbb{C} \setminus \mathbb{R}$, are continuous. Hereby let $\Delta \in \mathbb{N}$ always be chosen such that $F_{z_1}, \ldots, F_{z_\Delta}$ is a minimal set of functionals such that all the functionals $F_z,\ z \in \mathbb{C} \setminus \mathbb{R}$, are continuous with respect to $\mathcal{O}$. Then no linear combination of $F_{z_1}, \ldots, F_{z_\Delta}$ is continuous with respect to the topology induced by $[.,.]$ on $\mathfrak{A}$ as in Remark 4.1.

Now let $((\mathfrak{Q}(f),[.,.],\mathcal{O}(f)),\iota)$ be the completion of $(\mathfrak{A},[.,.],(F_{z_j})_{j=1,\dots,\Delta})$. On $(\mathfrak{Q}(f),[.,.],\mathcal{O}(f))$ one can find a symmetric operator $T(f)$ with defect index $(1,1)$. For the concept of symmetric operators on almost Pontryagin spaces see [8]. In that paper almost Pontryagin spaces were always considered as degenerate subspaces of Pontryagin spaces, and they were not yet called almost Pontryagin spaces. Similarly as for $S(f)$ the extensions $\tilde{f}\in\mathcal{P}_{\tilde{\kappa}}$ of f which differ from f_0 correspond bijectively to all $\mathfrak{Q}(f)$-minimal selfadjoint extensions A of $T(f)$ in a Pontryagin space $\tilde{\mathfrak{P}}\supseteq\mathfrak{Q}(f)$ with $\kappa_-(\tilde{\mathfrak{P}},[.,.])=\tilde{\kappa}$.

Since every Pontryagin space $\tilde{\mathfrak{P}}$ which contains $\mathfrak{Q}$ must satisfy $\kappa_-(\tilde{\mathfrak{P}},[.,.])\geq\Delta(\mathfrak{Q},[.,.])+\kappa_-(\tilde{\mathfrak{Q}},[.,.])$, there exist extensions $\tilde{f}\in\mathcal{P}_{\tilde{\kappa}}$, $\tilde{f}\neq f_0$ of f if only if $\tilde{\kappa}\geq\Delta+\kappa$. In fact, for these $\tilde{\kappa}$ there always exist infinitely many extensions in $\mathcal{P}_{\tilde{\kappa}}$. These considerations yield the following refinement of Theorem 6.2.

Theorem 6.3. *Let $f\in\mathcal{P}_{\kappa,a}$. Then there exists $\Delta\in\{0\}\cup\mathbb{N}\cup\{\infty\}$ such that*

- *If $\Delta>0$, then f has a unique extension in $\mathcal{P}_\kappa$.*
- *f has no extensions in $\mathcal{P}_{\tilde{\kappa}}$ for $\kappa<\tilde{\kappa}<\Delta+\kappa$.*
- *f has infinitely many extensions in $\mathcal{P}_{\tilde{\kappa}}$ for $\tilde{\kappa}\geq\Delta+\kappa$.*

As a second topic in the present section we give an example of an interesting class of almost reproducing kernel Pontryagin spaces. In fact, we are going to consider the indefinite generalization of Hilbert space of entire functions introduced by Louis de Branges (see [1], [9],[10],[11]).

Definition 6.4. An inner product space $(\mathfrak{L},[.,.])$ is called a de Branges space (dB-space) if the following three axioms hold true:

(dB1) $(\mathfrak{L},[.,.])$ is an almost reproducing kernel Pontryagin space on $\mathbb{C}$ consisting of entire functions.

(dB2) If $F\in\mathfrak{L}$ then $F^\#\in\mathfrak{L}$, where $F^\#(z)=\overline{F(\bar{z})}$. Moreover,

$$[F^\#,G^\#]=[G,F].$$

(dB3) If $F\in\mathfrak{L}$ and $z_0\in\mathbb{C}\setminus\mathbb{R}$ with $F(z_0)=0$, then

$$\frac{z-\bar{z}_0}{z-z_0}F(z)\in\mathfrak{L},$$

as a function of z. Moreover, if also $G\in\mathfrak{L}$ with $G(z_0)=0$, then

$$\Big[\frac{z-\bar{z}_0}{z-z_0}F(z),\frac{z-\bar{z}_0}{z-z_0}G(z)\Big]=[F,G].$$

In many cases one can assume that a dB-space also satisfies

$$\text{For all } t\in\mathbb{R} \text{ there exists } F\in\mathfrak{L} \text{ such that } F(t)\neq 0. \tag{6.1}$$

One of the main results about dB-spaces is that the set of all admissible dB-subspaces of a given dB-space is totally ordered. To explain this in more detail, let us start with a dB-space satisfying (6.1). We call a subspace $\mathfrak{K}$ of $\mathfrak{L}$ a dB-subspace of $(\mathfrak{L},[.,.])$ if $(\mathfrak{K},[.,.])$ itself is a dB-space. It is called an admissible dB-subspace if $(\mathfrak{K},[.,.])$ also satisfies (6.1). The following result originates from [1] and was generalized to the indefinite situation in [9].

Theorem 6.5. *Let* $(\mathfrak{L}, [.,.])$ *be a dB-space satisfying (6.1). Then the set of all admissible dB-subspaces is totally ordered with respect to inclusion, i.e. if* $\mathfrak{P}$ *and* $\mathfrak{Q}$ *are two admissible dB-subspaces of* $(\mathfrak{L}, [.,.])$*, then* $\mathfrak{P} \subseteq \mathfrak{Q}$ *or* $\mathfrak{Q} \subseteq \mathfrak{P}$*.*

One may think of the degenerate members of the chain of admissible dB-subspaces of a given dB-space as singularities. Thus it is desirable not to have too many of this kind. In [9] the following result was obtained.

Theorem 6.6. *With the same assumptions as in Theorem 6.5 the number of admissible dB-subspaces* $\mathfrak{K}$ *of* $(\mathfrak{L}, [.,.])$ *with* $\Delta(\mathfrak{K}, [.,.]) > 0$ *is finite.*

The presence of singularities is exactly what distinguishes the classical - positive definite- case from the indefinite situation. Thus, to obtain a thorough understanding of the structure of an indefinite dB-space, it is inevitable to deal with degenerated spaces.

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