PONTRYAGIN SPACES OF ENTIRE FUNCTIONS III

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Abstract

We continue the investigations of the indefinite generalization of L. de Branges theory of Hilbert spaces of entire functions. In this paper we are concerned with the detailed study of degenerated dB-subspaces of a dB-Pontryagin space, and of singularities of maximal chains of matrix functions. These phenomena are typical for the indefinite situation; there are no definite analogues. The main theorem is a continuity result for so-called intermediate Weyl coefficients. As a basic tool we introduce and investigate a certain transformation of maximal chains of matrix functions.

1 Introduction

The present paper is a continuation of our earlier work [KW3] and [KW4]. It is mainly concerned with detailed studies of singularities of maximal chains of matrix functions. These studies are tightly connected with an investigation of the degenerated dB-subspaces of a given dB-Pontryagin space.

In order to provide a more detailed explanation, let us recall a couple of definitions. For a function $F(z)$ denote by $F^{\#}(z)$ the function $F^{\#}(z) := \overline{F(\overline{z})}$. We call F real, if $F^{\#} = F$. A 2×2-matrix function $W(z) = (w_{ij}(z))_{i,j=1}^{2}$ with real and entire entries $w_{ij}(z)$ such that $W(0) = 1$ is said to belong to the class $\mathcal{M}_\kappa$ if $\det W(z) = 1$ and if the kernel

$$H_W(z,w) := \frac{W(z)JW(w)^* - J}{z - \overline{w}}, \ z, w \in \mathbb{C},$$

has κ negative squares. Here J denotes the signature matrix

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Different from [KW2] we added here for notational convenience the condition $W(0) = 1$ to the definition of $\mathcal{M}_\kappa$. For $W \in \mathcal{M}_\kappa$ we denote by $\operatorname{ind}_- W$ the number of negative squares of $H_W(z,w)$, i.e. $\operatorname{ind}_- W = \kappa$.

Definition 1.1. A maximal chain of matrix functions is a family $(W_t)_{t \in \mathcal{I}}$ which satisfies:

(W1) The index set $\mathcal{I}$ equals $(0, M)$, $0 < M \leq \infty$, with the possible exception of finitely many points.

(W2) For each $t \in \mathcal{I}$ the 2×2-matrix function $W_t(z)$ belongs to $\mathcal{M}_{\kappa(t)}$ for some $\kappa(t) \in \mathbb{N} \cup \{0\}$. The function $t \mapsto W_t$ is not constant on any interval contained in $\mathcal{I}$.

(W3) If $s, t \in \mathcal{I}$, $s \leq t$, then $\kappa(s) \leq \kappa(t)$ and $W_{st} := W_s^{-1} W_t \in \mathcal{M}_{\kappa(t)-\kappa(s)}$.

(W4) If $t \in \mathcal{I}$ and for some $W \in \mathcal{M}_\nu$, $\nu \leq \kappa(t)$, we have $W^{-1} W_t \in \mathcal{M}_{\kappa(t)-\nu}$, then there exists a number $s \in \mathcal{I}$ such that $W = W_s$.

(W5) The limit $\lim_{t \nearrow M} W_t$ does not exist, and if $\mathcal{I}$ is not connected, there exist numbers $s < t$, both contained in the last connected component $\mathcal{I}_\infty$ of $\mathcal{I}$ (that is $\sup \mathcal{I}_\infty = M$), such that $W_{st}(z)$ is not a linear polynomial, i.e. $W_{st}'(z)$ is not constant.

The points belonging to $(0, M) \setminus \mathcal{I}$ are called singularities of $(W_t)_{t \in \mathcal{I}}$. The set of all maximal chains of matrix functions is denoted by $\mathfrak{M}$.

A decisive role in the present theory is played by the function ($\operatorname{tr}(M)$ denotes the trace of the matrix M)

$$\mathfrak{t}(W) := \operatorname{tr}(W'(0)J) = w_{12}'(0) - w_{21}'(0).$$

The notation "singularity" for a point $\sigma \in (0, M) \setminus \mathcal{I}$ might be justified by the fact that

$$\lim_{t \nearrow \sigma} \mathfrak{t}(W_t) = +\infty, \ \lim_{t \searrow \sigma} \mathfrak{t}(W_t) = -\infty, \tag{1.1}$$

whereas $\mathfrak{t}(W_t)$ is continuous on $\mathcal{I}$ (cf. [dB]).

The behaviour of a chain $(W_t)_{t \in \mathcal{I}}$ on a single connected component of $\mathcal{I}$ is easy to handle, since it can be reduced immediately to the well studied case of "positive" chains, i.e. chains with $W_t \in \mathcal{M}_0$ for all t. Our aim among others is the investigation of the behaviour of $(W_t)_{t \in \mathcal{I}}$ locally at a singularity. In particular, we are led to a description of all possible types of singularities. One local property of a singularity, which turns out to be of fundamental importance, is the existence of the so called intermediate Weyl coefficient.

Recall in this place from [KW4] that for a chain $(W_t)_{t \in \mathcal{I}} \in \mathfrak{M}$ and a real number τ the limit

$$\lim_{t \nearrow \sup \mathcal{I}} \frac{w_{11}(z)\tau + w_{12}(z)}{w_{21}(z)\tau + w_{22}(z)}, \tag{1.2}$$

exists locally uniformly in $\mathbb{C} \setminus \mathbb{R}$ and does not depend on τ. It is called the Weyl coefficient of the chain. The interest in this notion origins in the fact that the whole chain is uniquely determined by its Weyl coefficient.

The existence of the limit (1.2) follows from the fact that $\lim_{t \nearrow \sup \mathcal{I}} \mathfrak{t}(W_t) = +\infty$. It is an easy consequence of (1.1) that for a singularity σ both limits

$$\lim_{t \nearrow \sigma} \frac{w_{11}(z)\tau + w_{12}(z)}{w_{21}(z)\tau + w_{22}(z)}, \quad \lim_{t \searrow \sigma} \frac{w_{11}(z)\tau + w_{12}(z)}{w_{21}(z)\tau + w_{22}(z)},$$

exist. One of the main results of this note is that these limits coincide. The function represented as such is called the intermediate Weyl coefficient at σ.

As a consequence of the existence of intermediate Weyl coefficients we obtain an algorithm to remove a singularity by repeated application of a certain transformation. In this way we obtain more information about the structure of the singularity. Also this prepares the proper introduction of an indefinite analogue of the so called canonical system of differential equations associated with a "positive" chain. However, in this paper we will not deal with this subject; it is left for forthcoming work.

The inner structure of a singularity becomes much more transparent when the chain of dB-spaces associated with a chain of matrix functions is considered. Recall from [KW3]: An inner product space $\langle\mathfrak{P},[.,.]\rangle$ is called a dB-space, if it satisfies the axioms

(IP1) The isotropic part $\mathfrak{P}^\circ := \mathfrak{P}\cap\mathfrak{P}^\perp$ of $\mathfrak{P}$ is finite dimensional.

(IP2) The factor space $\mathfrak{P}/\mathfrak{P}^\circ$ is a Pontryagin space.

(dB1) The elements of $\mathfrak{P}$ are entire functions, and for each $w\in\mathbb{C}$ the point evaluation $F\mapsto F(w)$ at w is a continuous linear functional.

(dB2) If $F\in\mathfrak{P}$, then also $F^\#\in\mathfrak{P}$, and

$$[F^\#,G^\#]=[G,F],\ F,G\in\mathfrak{P}.$$

(dB3) If $w\in\mathbb{C}\setminus\mathbb{R}$ and $F\in\mathfrak{P}$, $F(w)=0$, then also $\frac{z-\overline{w}}{z-w}F(z)\in\mathfrak{P}$, and

$$[\frac{z-\overline{w}}{z-w}F(z),\frac{z-\overline{w}}{z-w}G(z)]=[F,G],\ F,G\in\mathfrak{P},\ F(w)=G(w)=0.$$

When we speak about continuity in **(dB1)**, the topology under consideration is the Hilbert space topology which is uniquely defined by $\langle\mathfrak{P},[.,.]\rangle$ (see [KW3]).

A subspace $\mathfrak{Q}\subseteq\mathfrak{P}$ provided with the inner product inherited from $\mathfrak{P}$ is called a dB-subspace if it is dB-space itself. The dB-subspaces of any dB-space form a chain $(\mathfrak{P}_s)_{s\in\mathcal{J}}$. A chain of matrix functions $(W_t)_{t\in\mathcal{I}}$ induces a chain of dB-spaces $(\mathfrak{P}_s)_{s\in\mathcal{J}}$ and vice versa. Thereby the singularities of the chain of matrix functions correspond to the degenerated members of the chain of spaces. Hence, the knowledge about dB-subspaces can be used to gain information about singularities.

The paper is divided into several sections. A comprehensive overview is given in the following.

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Section 2 is concerned with the study of the degenerated dB-subspaces of a dB-Pontryagin space, in particular with sequences

$$\mathfrak{P}_1 \subsetneq \mathfrak{P}_2 \subsetneq \ldots \subsetneq \mathfrak{P}_\delta,$$

where each of the spaces $\mathfrak{P}_i$ is degenerated and contained in the subsequent space with codimension one. It turns out that the isotropic parts $\mathfrak{P}_i^\circ$ must be of a very particular form (cf. Theorem 2.3). As a tool we use the switch of inner products from $[.,.]$ to $[.,.]_1$ where

$$[F,G]_1 := [F,G] + mF(0)\overline{G(0)}. \tag{1.3}$$

This transformation already appeared earlier (cf. [KW3]) and turned out to be useful.

In Section 3 we introduce and investigate maximal chains of matrix functions. This notion in fact corresponds to the notion of chains defined in [KW4], also some of the stated results have in essence been proved there. After giving some basic results we turn to the study of singularities. Thereby the chain of dB-spaces associated with a maximal chain of matrix functions plays a vital role. In particular, we prove that singularities correspond to degenerated dB-spaces. A division of singularities into several types is introduced.

The switch of inner products as in (1.3) corresponds to a transformation of matrices. This transformation is investigated in Section 4. Applied to the members of a maximal chain of matrix functions this leads to a transformation of chains (cf. Theorem 4.4).

The existence of intermediate Weyl coefficients is established in Section 5 (cf. Proposition 5.1, Theorem 5.6). The proof of this result involves some operator theoretic arguments concerned with the operator of multiplication by the independent variable in the associated dB-spaces. With the aid of this result we prove a most useful statement giving a certain continuity property of chains of matrix functions (cf. Proposition 5.8).

In Section 6 we investigate in detail the change of singularities when performing the above mentioned transformation of chains. Together with Proposition 5.8 this leads to a deterministic method to remove a singularity by applying certain transformations (cf. Proposition 6.9).

Finally, in Section 7, the question which functions may appear as intermediate Weyl coefficients is answered in terms of the function itself (cf. Theorem 7.4). Basically the given criterion is concerned with the asymptotic distribution of the zeros and poles of the function under consideration (cf. Corollary 7.9).

References to results of [KW2], [KW3], and [KW4] will be given as the following examples indicate: Lemma 0.2.1 refers to Lemma 2.1 of [KW2], (I.2.1) refers to the equation (2.1) of [KW3], and Proposition II.4.1 refers to Proposition 4.1 of [KW4].

2 Degenerated dB-subspaces

In this section we investigate the structure of the degenerated dB-subspaces of a dB-Pontryagin space $\mathfrak{P}$. For simplicity we will assume throughout this whole paper that $\mathfrak{d}(\mathfrak{P}) = 0$ (cf. [KW3]).

Denote by $(\mathfrak{P}_t)_{t\in\mathcal{J}}$ the chain of the dB-subspaces $\mathfrak{P}_t$ of $\mathfrak{P}$ which also satisfy $\mathfrak{d}(\mathfrak{P}_t) = \mathfrak{d}(\mathfrak{P}) = 0$, cf. [KW3], Section 4. Here the index set $\mathcal{J}$ equals the interval $(0, M]$ with the possible exception of certain open, so-called indivisible, intervals. Moreover, $M \in \mathcal{J}$ and $\mathfrak{P}_M = \mathfrak{P}$. First recall from Proposition I.11.4 that the spaces $\mathfrak{P}_t$ depend in a certain sense continuously on t.

Remark 2.1. Let $t_0 \in \mathcal{J}$. Then

$$\dim\left[\mathfrak{P}_{t_0}/\left(\overline{\bigcup_{t\in\mathcal{J},\ t<t_0}\mathfrak{P}_t}\right)\right] = \begin{cases} 0, & \sup\{t\in\mathcal{J} : t<t_0\} = t_0 \\ 1, & \sup\{t\in\mathcal{J} : t<t_0\} < t_0 \end{cases},$$

$$\dim\left[\left(\bigcap_{t\in\mathcal{J},\ t>t_0}\mathfrak{P}_t\right)/\mathfrak{P}_{t_0}\right] = \begin{cases} 0, & \inf\{t\in\mathcal{J} : t>t_0\} = t_0 \\ 1, & \inf\{t\in\mathcal{J} : t>t_0\} > t_0 \end{cases}.$$

We will investigate the behaviour of the isotropic part $\mathfrak{P}_t^\circ$, in particular its dimension in dependence of t. Recall from Theorem I.11.6 that only finitely many spaces $\mathfrak{P}_t$ can be degenerated. It will follow later on (compare the discussion in Section 6) that if $t \in \mathcal{J}$ is not isolated, the dimension of $\mathfrak{P}_t^\circ$ can be arbitrary. However, an immediate dimension argument shows that the isotropic parts of subsequent degenerated spaces $\mathfrak{P}_t$, i.e. degenerated spaces which are corresponding to endpoints of an indivisible interval, are closely related.

Lemma 2.2. *Let $\mathfrak{P}_{\alpha_1}$ and $\mathfrak{P}_{\alpha_2}$ be dB-spaces such that $\mathfrak{P}_{\alpha_1}$ is contained in $\mathfrak{P}_{\alpha_2}$ with codimension one. Then either (i), (ii) or (iii) holds:*

(i) $\mathfrak{P}_{\alpha_1}^\circ \subsetneq \mathfrak{P}_{\alpha_2}^\circ$, $\dim\mathfrak{P}_{\alpha_1}^\circ = \mathfrak{P}_{\alpha_2}^\circ - 1$,

(ii) $\mathfrak{P}_{\alpha_1}^\circ \supsetneq \mathfrak{P}_{\alpha_2}^\circ$, $\dim\mathfrak{P}_{\alpha_1}^\circ = \mathfrak{P}_{\alpha_2}^\circ + 1$,

(iii) $\mathfrak{P}_{\alpha_1}^\circ = \mathfrak{P}_{\alpha_2}^\circ$.

Proof : Clearly, $\mathfrak{P}_{\alpha_2}^\circ \cap \mathfrak{P}_{\alpha_1} \subseteq \mathfrak{P}_{\alpha_1}^\circ$, hence $\dim\mathfrak{P}_{\alpha_2}/\mathfrak{P}_{\alpha_1} = 1$ shows that $\dim\mathfrak{P}_{\alpha_1}^\circ \geq \dim\mathfrak{P}_{\alpha_2}^\circ - 1$. Moreover, for any $H \in \mathfrak{P}_{\alpha_2}\setminus\mathfrak{P}_{\alpha_1}$ we have

$$\mathfrak{P}_{\alpha_2} = \mathfrak{P}_{\alpha_1} \dot{+} \operatorname{span}\{H\}. \tag{2.1}$$

Thus $\mathfrak{P}_{\alpha_1}^\circ \cap \operatorname{span}\{H\}^\perp \subseteq \mathfrak{P}_{\alpha_2}^\circ$ and hence $\dim \mathfrak{P}_{\alpha_2}^\circ \geq \dim \mathfrak{P}_{\alpha_1}^\circ - 1$.

If $\dim \mathfrak{P}_{\alpha_2}^\circ > \dim \mathfrak{P}_{\alpha_1}^\circ$, we must have $\mathfrak{P}_{\alpha_1}^\circ = \mathfrak{P}_{\alpha_2}^\circ \cap \mathfrak{P}_{\alpha_1}$, and $\dim \mathfrak{P}_{\alpha_1}^\circ > \dim \mathfrak{P}_{\alpha_2}^\circ$ causes $\mathfrak{P}_{\alpha_2}^\circ = \mathfrak{P}_{\alpha_1}^\circ \cap \operatorname{span}\{H\}^\perp$.

Consider the case $\dim \mathfrak{P}_{\alpha_1}^\circ = \dim \mathfrak{P}_{\alpha_2}^\circ$ and assume that $\mathfrak{P}_{\alpha_2}^\circ \setminus \mathfrak{P}_{\alpha_1}^\circ \neq \emptyset$. By the above considerations an element $H \in \mathfrak{P}_{\alpha_2}^\circ \setminus \mathfrak{P}_{\alpha_1}^\circ$ cannot belong to $\mathfrak{P}_{\alpha_1}$. It is a consequence of (2.1) that $\mathfrak{P}_{\alpha_1}^\circ \dot{+} \operatorname{span}\{H\} \subseteq \mathfrak{P}_{\alpha_2}^\circ$, in contradiction to the assumed equality of the dimensions of the isotropic parts. We conclude that $\mathfrak{P}_{\alpha_2}^\circ \subseteq \mathfrak{P}_{\alpha_1}^\circ$, and therefore $\mathfrak{P}_{\alpha_2}^\circ = \mathfrak{P}_{\alpha_1}^\circ$.

□

The question arises, how $\mathfrak{P}_t^\circ$ behaves when t runs through $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ for subsequent indivisible intervals $(\alpha_1, \alpha_2), (\alpha_2, \alpha_3), \dots, (\alpha_{n-1}, \alpha_n)$. It is the main purpose of this section to prove the following result.

Theorem 2.3. *Let $\mathfrak{P}$ be a dB-Pontryagin space, and let $(\mathfrak{P}_t)_{t\in\mathcal{J}}$ be the chain of its dB-subspaces. Assume that $(\alpha_1, \alpha_2), \dots, (\alpha_{n-1}, \alpha_n)$ are subsequent indivisible intervals, and that the dB-spaces $\mathfrak{P}_{\alpha_i}$, $i = 1, \dots, n$ are degenerated. Put $d_i := \dim \mathfrak{P}_{\alpha_i}^\circ$. Then there exists a function $F \in \mathfrak{P}$ such that*

$$\mathfrak{P}_{\alpha_i}^\circ = \operatorname{span}\{F(z), \dots, z^{d_i-1}F(z)\}.$$

Moreover, there exists a number $i_{max} \in \{1, \dots, n\}$ such that

$$d_{i+1} = d_i + 1,\ 1 \leq i < i_{max},$$

$$d_{i+1} = d_i - 1,\ i_{max} < i \leq n-1,$$

$$d_{i_{max}+1} = \begin{cases} d_{i_{max}} \\ d_{i_{max}} - 1 \end{cases}.$$

Before we come to the proof of this result we bring some lemmata. First a method to reduce the dimension of the isotropic part is provided. The key construction is the change of inner products according to the formula

$$[F, G]_1 = [F, G] + mF(0)\overline{G(0)}, \tag{2.2}$$

where $m \in \mathbb{R}$.

For a dB-space $\mathfrak{P}$ denote by $\mathfrak{Q}(\mathfrak{P})$ the hyperplane

$$\mathfrak{Q}(\mathfrak{P}) = \{F \in \mathfrak{P} : F(0) = 0\}.$$

Note that for any $F \in \mathfrak{Q}(\mathfrak{P})$ and $G \in \mathfrak{P}$

$$[F, G]_1 = [F, G].$$

There exists an element $K_0 \in \mathfrak{P}$ with the property

$$F(0) = [F, K_0],\ F \in \mathfrak{P},$$

if and only if $\mathfrak{P}^\circ \subseteq \mathfrak{Q}(\mathfrak{P})$. In this case K_0 is uniquely determined modulo $\mathfrak{P}^\circ$. In particular, $K_0(0)$ does not depend on the choice of K_0.

Lemma 2.4. *Let $\langle \mathfrak{P}, [.,.] \rangle$ be a dB-space, and let $\mathfrak{P}_1$ be the set $\mathfrak{P}$ provided with the inner product $[.,.]_1$ defined as in (2.2). Then*

$$\dim \mathfrak{P}_1^\circ = \dim \mathfrak{P}^\circ + \begin{cases} +1, & \mathfrak{P}^\circ \subseteq \mathfrak{Q}(\mathfrak{P}), \quad m = \frac{-1}{K_0(0)} \\ 0, & \mathfrak{P}^\circ \subseteq \mathfrak{Q}(\mathfrak{P}), \quad m \neq \frac{-1}{K_0(0)} \\ -1, & \mathfrak{P}^\circ \not\subseteq \mathfrak{Q}(\mathfrak{P}) \end{cases} .$$

In the respective cases we have

$$\mathfrak{P}_1^\circ = \begin{cases} \mathfrak{P}^\circ \dot{+} \operatorname{span}\{K_0\} \\ \mathfrak{P}^\circ \\ \mathfrak{P}^\circ \cap \mathfrak{Q}(\mathfrak{P}) \end{cases} . \tag{2.3}$$

In the last case every element of $\mathfrak{P}^\circ \setminus \mathfrak{P}_1^\circ$ has nonzero norm.

Proof : Since for $F \in \mathfrak{Q}(\mathfrak{P})$ we have $[F,G] = [F,G]_1$ for all $G \in \mathfrak{P}$, it follows that $\mathfrak{P}^\circ \cap \mathfrak{Q}(\mathfrak{P}) = \mathfrak{P}_1^\circ \cap \mathfrak{Q}(\mathfrak{P})$. In particular, this yields $|\dim \mathfrak{P}^\circ - \dim \mathfrak{P}_1^\circ| \leq 1$.

Assume that $G \in \mathfrak{P}_1^\circ \setminus \mathfrak{P}^\circ$, then for all $F \in \mathfrak{P}$

$$[F,G] = -mF(0)\overline{G(0)}. \tag{2.4}$$

Since $G \notin \mathfrak{P}^\circ$ we have $G(0) \neq 0$, and therefore

$$F(0) = [F, \frac{G}{-mG(0)}], \; F \in \mathfrak{P}. \tag{2.5}$$

In particular, we obtain $\mathfrak{P}^\circ \subseteq \mathfrak{Q}(\mathfrak{P})$ and hence $\mathfrak{P}^\circ = \mathfrak{P}^\circ \cap \mathfrak{Q}(\mathfrak{P}) \subsetneq \mathfrak{P}_1^\circ$. If we take $F = K_0$ in (2.5), we get

$$K_0(0) = [K_0, \frac{G}{-mG(0)}] = -\frac{1}{m}.$$

Moreover, $\frac{G}{-mG(0)} \in K_0 + \mathfrak{P}^\circ$, and from $\mathfrak{P}^\circ \subseteq \mathfrak{P}_1^\circ$ we conclude that $K_0 \in \mathfrak{P}_1^\circ \setminus \mathfrak{P}^\circ$. Hence $\mathfrak{P}_1^\circ = \mathfrak{P}^\circ \dot{+} \operatorname{span}\{K_0\}$.

Conversely, if $\mathfrak{P}^\circ \subseteq \mathfrak{Q}(\mathfrak{P})$ and $m = -\frac{1}{K_0(0)}$, setting $G = K_0$ we obtain (2.4). Thus $K_0 \in \mathfrak{P}_1^\circ$, and hence $\mathfrak{P}_1^\circ = \mathfrak{P}^\circ \dot{+} \operatorname{span}\{K_0\}$.

If $\mathfrak{P}^\circ \subseteq \mathfrak{Q}(\mathfrak{P})$ but $m \neq -\frac{1}{K_0(0)}$, we get $\mathfrak{P}^\circ = \mathfrak{P}^\circ \cap \mathfrak{Q}(\mathfrak{P}) \subseteq \mathfrak{P}_1^\circ$. If we had strict inequality, the second paragraph of the present proof would give $m = -\frac{1}{K_0(0)}$. Hence $\mathfrak{P}^\circ = \mathfrak{P}_1^\circ$.

It remains to consider the case $\mathfrak{P}^\circ \not\subseteq \mathfrak{Q}(\mathfrak{P})$. Again the arguments in the second paragraph show $\mathfrak{P}_1^\circ \subseteq \mathfrak{P}^\circ$. However, for an element $F \in \mathfrak{P}^\circ \setminus \mathfrak{Q}(\mathfrak{P})$ we get $[F,F]_1 = m|F(0)|^2 \neq 0$, and therefore $F \notin \mathfrak{P}_1^\circ$. It follows that $\mathfrak{P}_1^\circ = \mathfrak{P}^\circ \cap \mathfrak{Q}(\mathfrak{P})$.

□

Though in this section we are mainly concerned with the behaviour of the isotropic part, we investigate for later use also the possible change of the negative index when the inner product is switched according to (2.2).

Lemma 2.5. *Let $\langle\mathfrak{P},[.,.]\rangle$ be a dB-space and consider the dB-space $\mathfrak{P}_1 = \langle\mathfrak{P},[.,.]_1\rangle$ where $[.,.]_1$ is defined as in (2.2). If $\mathfrak{P}^\circ \not\subseteq \mathfrak{Q}(\mathfrak{P})$, then*

$$\operatorname{ind}_-\mathfrak{P}_1 = \operatorname{ind}_-\mathfrak{P} + \begin{cases} 0\,, & m > 0 \\ 1\,, & m < 0 \end{cases} . \tag{2.6}$$

If $\mathfrak{P}^\circ \subseteq \mathfrak{Q}(\mathfrak{P})$, then

$$\operatorname{ind}_-\mathfrak{P}_1 = \operatorname{ind}_-\mathfrak{P} + \begin{cases} -1\,, & K_0(0) < 0, m \geq -\frac{1}{K_0(0)} \\ 0\,, & K_0(0) < 0, m < -\frac{1}{K_0(0)} \\ 0\,, & K_0(0) = 0 \\ 0\,, & K_0(0) > 0, m \geq -\frac{1}{K_0(0)} \\ +1\,, & K_0(0) > 0, m < -\frac{1}{K_0(0)} \end{cases} .$$

Proof : Consider first the case that $\mathfrak{P}^\circ \not\subseteq \mathfrak{Q}(\mathfrak{P})$. Then we can write

$$\mathfrak{P} = \mathfrak{Q}(\mathfrak{P})[\dot{+}]\operatorname{span}\{H\},$$

with $H \in \mathfrak{P}^\circ$, hence also

$$\mathfrak{P}_1 = \mathfrak{Q}(\mathfrak{P})[\dot{+}]_1\operatorname{span}\{H\}.$$

Since $[H,H]_1 = [H,H] + m|H(0)|^2 = m|H(0)|^2 \neq 0$ we obtain (2.6).

Next assume that $\mathfrak{P}^\circ \subseteq \mathfrak{Q}(\mathfrak{P})$ and $K_0(0) \neq 0$. Then we have the decomposition

$$\mathfrak{P} = \mathfrak{Q}(\mathfrak{P})[\dot{+}]\operatorname{span}\{K_0\}.$$

The fact that

$$[K_0,K_0]_1 = [K_0,K_0] + m|K_0(0)|^2 = K_0(0)(1 + mK_0(0))$$

yields the assertion in this situation.

It remains to settle the case that $\mathfrak{P}^\circ \subseteq \mathfrak{Q}(\mathfrak{P})$ and $K_0(0) = 0$. Since $K_0 \notin \mathfrak{P}^\circ$ we have a decomposition of $\mathfrak{P}$ of the form

$$\mathfrak{P} = \mathfrak{P}^\circ[\dot{+}]\mathfrak{Q}_1[\dot{+}](\operatorname{span}\{K_0\}\dot{+}\operatorname{span}\{H\}),$$

where $\mathfrak{P}^\circ[\dot{+}]\mathfrak{Q}_1[\dot{+}]\operatorname{span}\{K_0\} = \mathfrak{Q}(\mathfrak{P})$ and H is a neutral element skewly linked with K_0, i.e. $H(0) = [H,K_0] = 1$. It follows that $\mathfrak{P}_1$ can be decomposed orthogonally in the same way

$$\mathfrak{P}_1 = \mathfrak{P}^\circ[\dot{+}]_1\mathfrak{Q}_1[\dot{+}]_1(\operatorname{span}\{K_0\}\dot{+}\operatorname{span}\{H\}).$$

The inner product on the last summand is determined by the Gram matrix

$$\mathcal{G} = \begin{pmatrix} 0 & 1 \\ 1 & m \end{pmatrix},$$

since $[K_0, K_0]_1 = [K_0, K_0] = 0$, $[K_0, H]_1 = [K_0, H] = 1$, $[H, H]_1 = [H, H] + m|H(0)|^2 = m$. By Jacobi's signature rule the inner product $[.,.]_1$ on this two dimensional space has negative index 1.

□

It is not of significant importance that the perturbing term in (2.2) is an evaluation at 0 and not at any other $\alpha \in \mathbb{R}$.

Remark 2.6. Let $\mathfrak{P}$ be a dB-space, and let $\alpha \in \mathbb{R}$. The one-to-one mapping

$$\varphi : F(z) \mapsto F(z + \alpha),$$

takes $\mathfrak{P}$ onto some space $\mathfrak{P}^{(\alpha)}$ of entire functions. If we define an inner product on $\mathfrak{P}^{(\alpha)}$ by declaring φ to be a unitary mapping, we obtain a dB-space. Moreover, a subspace $\tilde{\mathfrak{P}}^{(\alpha)}$ is a dB-subspace of $\mathfrak{P}^{(\alpha)}$ if and only if $\tilde{\mathfrak{P}}$ is a dB-subspace of $\mathfrak{P}$. In this case $\varphi|_{\tilde{\mathfrak{P}}}$ is a unitary mapping from $\tilde{\mathfrak{P}}$ onto $\tilde{\mathfrak{P}}^{(\alpha)}$.

As a corollary of Lemma 2.4 and Remark 2.6 we get:

Corollary 2.7. *For any degenerated dB-space there exists a value $\alpha \in \mathbb{R}$, such that the dimension of the isotropic part actually decreases when we switch to the inner product defined by (2.2) with α in place of 0 and arbitrary $m \neq 0$.*

This will be a basic method of reduction which can be developed further.

Corollary 2.8. *Let $\langle \mathfrak{P}, [.,.] \rangle$ be a dB-space with $\dim \mathfrak{P}^\circ = \Delta$, $\operatorname{ind}_- \mathfrak{P} = \kappa$. Then there exist points $\alpha_1, \dots, \alpha_\Delta$, such that for any choice of $m_i > 0$, $i = 1, \dots, \Delta$, the dB-spaces $\tilde{\mathfrak{P}}_l := \langle \mathfrak{P}, [.,.]_l \rangle$, $l = 1, \dots \Delta$, where*

$$[F, G]_l = [F, G] + m_1 F(\alpha_1)\overline{G(\alpha_1)} + \ldots + m_l F(\alpha_l)\overline{G(\alpha_l)}, \tag{2.7}$$

satisfy $\operatorname{ind}_0 \mathfrak{P}_l = \Delta - l$ *and* $\operatorname{ind}_- \tilde{\mathfrak{P}}_l = \kappa$.

Proof : The existence of $\alpha_1, \dots, \alpha_\Delta$ follows by repeated application of Corollary 2.7. Since $m_1, \dots, m_\Delta > 0$, by (2.6) the negative index does not change.

□

For a dB-Pontryagin space $\mathfrak{P} = \mathfrak{P}(E)$, where $E(z) = A(z) - iB(z)$, recall the notation (cf. Lemma II.5.19)

$$S_\phi(z) = \cos\phi A(z) + \sin\phi B(z), \ \phi \in [0, \pi).$$

From our assumption $\mathfrak{d}(\mathfrak{P}) = 0$ it follows that the functions S_ϕ and S_ψ have no common zeros for distinct ϕ and ψ. Moreover, we will assume that E is normalized by $E(0) = -i$ which means that $S_\phi(0) = \sin\phi$.

Lemma 2.9. *Let $\mathfrak{P} = \mathfrak{P}(E)$ be a dB-Pontryagin subspace of some dB-Pontryagin space $\tilde{\mathfrak{P}}$, such that $\mathfrak{P}$ either contains, or is contained in a degenerated dB-subspace $\mathfrak{P}_1$ of $\tilde{\mathfrak{P}}$ with codimension one. Then $\mathfrak{P}_1^\circ = \operatorname{span}\{S_\phi\}$ for a certain number $\phi \in [0, \pi)$. In the case $\mathfrak{P}_1 \subsetneq \mathfrak{P}$ it is the unique number φ such that S_φ is contained in $\mathfrak{P}$. In the case $\mathfrak{P} \subsetneq \mathfrak{P}_1$ we have $\mathfrak{P}_1 = \mathfrak{P}\dot{+}\operatorname{span}\{S_\phi\}$.*

Proof : If $\mathfrak{P}(E)$ contains the degenerated space, the assertion follows from Corollary I.6.3. In the other case the statement is a consequence of Lemma II.7.6 since we assume the existence of a larger dB-Pontryagin space $\tilde{\mathfrak{P}}$.

□

Lemma 2.10. *Let (α_1, α_2) and (α_2, α_3) be two subsequent indivisible intervals. Consider the triple*

$$(\dim \mathfrak{P}_{\alpha_1}^\circ, \dim \mathfrak{P}_{\alpha_2}^\circ, \dim \mathfrak{P}_{\alpha_3}^\circ). \tag{2.8}$$

this triple cannot equal either of the following triples:

$$(2,1,2),\ (1,1,2),\ (2,1,1),\ (1,1,1). \tag{2.9}$$

Proof : Let $\mathfrak{P}_1 \subsetneq \mathfrak{P}_2 \subsetneq \mathfrak{P}_3$ be degenerated dB-spaces such that $\mathfrak{P}_i$ is contained in $\mathfrak{P}_{i+1}$ with codimension one for $i = 1, 2$. Consider the triple (2.8).

First note that in each case considered in (2.9) the number $\dim \mathfrak{P}_2^\circ$ is the least. By Remark 2.6 we may assume that $\mathfrak{P}_2^\circ \not\subseteq \mathfrak{Q}(\mathfrak{P}_2)$. Then Lemma 2.2 shows that also $\mathfrak{P}_1^\circ \not\subseteq \mathfrak{Q}(\mathfrak{P}_1)$ and $\mathfrak{P}_3^\circ \not\subseteq \mathfrak{Q}(\mathfrak{P}_3)$.

If we had $(\dim \mathfrak{P}_{\alpha_1}^\circ, \dim \mathfrak{P}_{\alpha_2}^\circ, \dim \mathfrak{P}_{\alpha_3}^\circ) = (2,1,2)$, then a change of the inner product according to (2.2) would lead by Lemma 2.4 to spaces $\tilde{\mathfrak{P}}_1 \subsetneq \tilde{\mathfrak{P}}_2 \subsetneq \tilde{\mathfrak{P}}_3$ with $\tilde{\mathfrak{P}}_2^\circ = \{0\}$ and $\dim \tilde{\mathfrak{P}}_1^\circ = \dim \tilde{\mathfrak{P}}_3^\circ = 1$. Lemma 2.9 then showed that

$$\tilde{\mathfrak{P}}_1^\circ = \operatorname{span}\{S_{\phi_1}\},\ \tilde{\mathfrak{P}}_3^\circ = \operatorname{span}\{S_{\phi_3}\},$$

for different functions S_{ϕ_1} and S_{ϕ_3}. However, by (2.3) we had $S_{\phi_1}(0) = S_{\phi_3}(0) = 0$, which would imply $S_{\phi_1} = S_{\phi_3}$.

Now consider the case $(1,1,2)$. The change of the inner product yields $\tilde{\mathfrak{P}}_1^\circ = \tilde{\mathfrak{P}}_2^\circ = \{0\}$, and by Lemma 2.9 we have $\tilde{\mathfrak{P}}_3^\circ = \operatorname{span}\{S_\phi\} \not\subseteq \tilde{\mathfrak{P}}_2,\ S_\phi(0) = 0$, which in turn gives $S_\phi = \tilde{A}_2$ by our usual normalization assumption $E(0) = -i$. However, going one step back in view of Lemma 2.4 we obtain

$$\operatorname{span}\{\tilde{K}_{1,0}(z)\} = \mathfrak{P}_1^\circ = \mathfrak{P}_2^\circ = \operatorname{span}\{\tilde{K}_{2,0}(z)\}, \tag{2.10}$$

i.e. $\tilde{K}_{1,0}(z) = \tilde{K}_{2,0}(z)$. Again by our normalization $E(0) = -i$ we have

$$\frac{\tilde{A}_1(z)}{z} = \tilde{K}_{1,0}(z) = \tilde{K}_{2,0}(z) = \frac{\tilde{A}_2(z)}{z},$$

and therefore $\tilde{A}_1 = \tilde{A}_2$. The transfer matrix from $\tilde{\mathfrak{P}}_1$ to $\tilde{\mathfrak{P}}_2$ (cf.Theorem I.12.2) is linear, and since $\tilde{A}_1 = \tilde{A}_2$, this matrix function is of the form

$$W_{12}(z) = \begin{pmatrix} 1 & lz \\ 0 & 1 \end{pmatrix}, \tag{2.11}$$

for some $l \in \mathbb{R} \setminus \{0\}$. As $\tilde{\mathfrak{P}}_2 = \tilde{\mathfrak{P}}_1 \oplus (\tilde{A}_1, \tilde{B}_1)\mathfrak{K}(W_{12})$, we see that $\tilde{A}_1 = \tilde{A}_2 \in \tilde{\mathfrak{P}}_2$, and we are led to a contradiction.

In the case $(2,1,1)$ we obtain spaces $\tilde{\mathfrak{P}}_1, \tilde{\mathfrak{P}}_2, \tilde{\mathfrak{P}}_3$ such that $\tilde{\mathfrak{P}}_2^\circ = \tilde{\mathfrak{P}}_3^\circ = \{0\}$ and $\tilde{\mathfrak{P}}_1^\circ = \operatorname{span}\{S_\phi\},\ S_\phi = \tilde{A}_2$. As above we get $\tilde{A}_2 = \tilde{A}_3$. Thus the transfer matrix $W_{23}(z)$ has the form (2.11) and $\tilde{\mathfrak{P}}_3 = \tilde{\mathfrak{P}}_2 \oplus (\tilde{A}_2, \tilde{B}_2)\mathfrak{K}(W_{23})$. Hence $\tilde{A}_2 \in \tilde{\mathfrak{P}}_3 \ominus \tilde{\mathfrak{P}}_2$, which contradicts the relation $\tilde{A}_2 \in \tilde{\mathfrak{P}}_1^\circ \subseteq \mathfrak{P}_1$.

It remains to deal with the case $(1,1,1)$. There we obtain three non-degenerated dB-spaces $\tilde{\mathfrak{P}}_1, \tilde{\mathfrak{P}}_2, \tilde{\mathfrak{P}}_3$. Reasoning as in (2.10) we see that $\tilde{A}_1 = \tilde{A}_2 = \tilde{A}_3$, and hence that both transfer matrices from $\tilde{\mathfrak{P}}_1$ to $\tilde{\mathfrak{P}}_2$ and from $\tilde{\mathfrak{P}}_2$ to $\tilde{\mathfrak{P}}_3$ are of the form (2.11). But this contradicts the fact that $\dim \mathfrak{P}_3/\mathfrak{P}_1 = 2$.

□

Corollary 2.11. *Let $(\alpha_1, \alpha_2), \dots, (\alpha_{n-1}, \alpha_n)$ be subsequent indivisible intervals, and assume that the spaces $\mathfrak{P}_{\alpha_i}$ are degenerated. Then there exists a number $i_{max} \in \{1, \dots, n\}$ such that*

$$\mathfrak{P}_{\alpha_1}^\circ \subsetneq \dots \subsetneq \mathfrak{P}_{\alpha_{i_{max}}}^\circ \supseteq \mathfrak{P}_{\alpha_{i_{max}+1}}^\circ \supsetneq \dots \supsetneq \mathfrak{P}_{\alpha_n}^\circ .$$

In other words the function $\dim \mathfrak{P}_{\alpha_i}^\circ$ assumes its maximum on $\{1, \dots, n\}$ at either exactly one point i_{max} or at the two subsequent points i_{max} and $i_{max} + 1$, and on $\{1, \dots, i_{max}\}$ the function $\dim \mathfrak{P}_{\alpha_i}^\circ$ strictly increases, on $\{i_{max} + 1, \dots, n\}$ it strictly decreases.

Proof : It is obvious that the assertion of the theorem can be formulated as follows. No part of the sequence $(\dim \mathfrak{P}_{\alpha_i}^\circ)_{i=1}^n$ has one of the following forms: $(k+1, k, k+1),\ (k, k, k+1),\ (k+1, k, k),\ (k, k, k)$.

Assume the contrary. By Corollary 2.7 we can change the inner product in such a way that the dimension of the isotropic part of the middle space decreases. Then by Lemma 2.2 and Lemma 2.4 also the dimension of the isotropic parts of the left and the right space decreases. Hence we are in the same situation only with $k-1$ replaced by k. Proceeding inductively we end up with one of the forbidden sequences in (2.9).

□

Assume in the following that $\mathfrak{P}_{\alpha_1} \subsetneq \dots \subsetneq \mathfrak{P}_{\alpha_n}$ is a sequence of subsequent and degenerated dB-spaces of maximal length in the chain $(\mathfrak{P}_t)_{t \in \mathcal{J}}$, i.e. either

$$\sup\{t \in \mathcal{J} : t < \alpha_1\} = \alpha_1,$$

or the space corresponding to the left endpoint of the indivisible interval with right endpoint α_1 is non degenerated, and analogously either

$$\inf\{t \in \mathcal{J} : t > \alpha_n\} = \alpha_n,$$

or the space corresponding to the right endpoint of the indivisible interval with left endpoint α_n is non degenerated.

Denote by $\kappa_\pm, \omega, \Delta_\pm$ and μ the numbers

$$\kappa_+ = \min\{\operatorname{ind}_- \mathfrak{P}_t : t \in \mathcal{J}, t > \alpha_n\},\ \kappa_- = \max\{\operatorname{ind}_- \mathfrak{P}_t : t \in \mathcal{J}, t < \alpha_1\},$$

$$\omega = \kappa_+ - \kappa_-,$$

$$\Delta_+ = \dim \mathfrak{P}_{\alpha_n}^\circ,\ \Delta_- = \dim \mathfrak{P}_{\alpha_1}^\circ,\ \mu = \dim \mathfrak{P}_{\alpha_n}/\mathfrak{P}_{\alpha_1} = n-1.$$

From the structure of the isotropic parts $\mathfrak{P}_{\alpha_i}^\circ$ we obtain the following connections between the introduced numbers.

Corollary 2.12. *With the above notation we have*

$$\omega = \left[\frac{\Delta_+ + \Delta_- + \mu}{2}\right] + \begin{cases} 0 \\ 1 \end{cases} . \tag{2.12}$$

If $\Delta_+ - \Delta_- + \mu$ is even, the second summand in (2.12) equals 0. The number i_{max} of Corollary 2.11 is given by

$$i_{max} = \left[\frac{\Delta_+ - \Delta_- + \mu}{2}\right] + 1, \tag{2.13}$$

and $\mathfrak{P}_{\alpha_{i_{max}}}^\circ = \mathfrak{P}_{\alpha_{i_{max}+1}}^\circ$ if and only if $\Delta_+ - \Delta_- + \mu$ is odd.

Proof : Since $\dim \mathfrak{P}_{\alpha_i}^\circ$ is increasing for $i = 1, \ldots, i_{max}$ and decreasing from i_{max} or from $i_{max}+1$ on , we obtain

$$\dim \mathfrak{P}_{\alpha_i}^\circ = \Delta_- + i - 1,\ i \le i_{max},$$

$$\dim \mathfrak{P}_{\alpha_i}^\circ = \Delta_+ + (\mu + 1 - i),\ i \ge i_{max} + 1.$$

From this we immediately get the validity of (2.13).

Since the dimension of $\mathfrak{P}_{\alpha_i}$ increases by one, and the dimension of $\mathfrak{P}_{\alpha_i}^\circ$ increases by one for $i \le i_{max}$ and decreases by one for $i \ge i_{max}+1$, we see that $\operatorname{ind}_- \mathfrak{P}_{\alpha_i}$ is constant for $i \le i_{max}$ and increases by one in each step for $i \ge i_{max}+1$. In the case $\mathfrak{P}_{\alpha_{i_{max}}}^\circ = \mathfrak{P}_{\alpha_{i_{max}+1}}^\circ$ we may have both $\operatorname{ind}_- \mathfrak{P}_{\alpha_{i_{max}+1}} = \operatorname{ind}_- \mathfrak{P}_{\alpha_{i_{max}}}$ or $\operatorname{ind}_- \mathfrak{P}_{\alpha_{i_{max}+1}} = \operatorname{ind}_- \mathfrak{P}_{\alpha_{i_{max}}} + 1$. From these considerations and $\operatorname{ind}_- \mathfrak{P}_{\alpha_1} = \kappa_-$, $\operatorname{ind}_- \mathfrak{P}_{\alpha_n} + \Delta_+ = \kappa_+$ (cf. Proposition I.11.11, Corollary 2.8) the relation (2.12) follows, where the second summand is chosen accordingly whether $\operatorname{ind}_- \mathfrak{P}_{\alpha_{i_{max}}} = \operatorname{ind}_- \mathfrak{P}_{\alpha_{i_{max}+1}}$ or not.

$\square$

Remark 2.13. If $\sup\{t \in \mathcal{J} : t < \alpha_1\} \neq \alpha_1$ ($\inf\{t \in \mathcal{J} : t > \alpha_n\} \neq \alpha_n$), then we must have $\Delta_- = 1$ ($\Delta_+ = 1$). Thus in the respective cases the statement of Corollary 2.12 can be given in a more explicit form.

In the case that $\inf\{t \in \mathcal{J} : t > \alpha_n\} \neq \alpha_n$, the spaces $\mathfrak{P}_{\alpha_i}^\circ$ can be determined, and a formula for Δ_- and for the second summand in (2.12) can be deduced.

So let us assume that α_n is the left endpoint of some indivisible interval, say (α_n, α). Then the space $\mathfrak{P}_\alpha$ is nondegenerated and for a certain number $\phi \in [0, \pi)$ the function S_ϕ belongs to $\mathfrak{P}_\alpha$. We will denote by m the number

$$m := \max\{k \in \mathbb{N} : z^k S_\phi \in \mathfrak{P}_\alpha\},$$

which is finite by Lemma I.7.1. Moreover, put

$$d := \dim \operatorname{span}\{S_\phi, \dots, z^m S_\phi\}^\circ,$$

and in case that $m + d$ is even

$$s := [z^{\frac{m+d}{2}} S_\phi, z^{\frac{m+d}{2}} S_\phi].$$

Proposition 2.14. *In the above described situation we have*

$$\mu = \begin{cases} m-1, & d = 0 \\ m \ , & d \neq 0 \end{cases}, \tag{2.14}$$

$$\Delta_+ = 1, \ \Delta_- = \begin{cases} 1, & d = 0 \\ d, & d \neq 0 \end{cases}. \tag{2.15}$$

The isotropic part $\mathfrak{P}_{\alpha_k}^\circ$ *is given as*

$$\mathfrak{P}_{\alpha_k}^\circ = \operatorname{span}\{S_\phi, \dots, z^{\Delta_- + k - 2} S_\phi\}, \ 1 \leq k \leq i_{max}, \tag{2.16}$$

$$\mathfrak{P}_{\alpha_k}^\circ = \operatorname{span}\{S_\phi, \dots, z^{n-k} S_\phi\}, \ i_{max} + 1 \leq k \leq n. \tag{2.17}$$

For even $\Delta_- + \mu$ $(= m + d)$ *the second summand in (2.12) is given by*

$$\begin{cases} 0, & s > 0 \\ 1, & s < 0 \end{cases}. \tag{2.18}$$

Proof : First observe that the spaces $\mathfrak{P}_{\alpha_i}$ are given in terms of the operator $\mathcal{S}$ in $\mathfrak{P}_\alpha$:

$$\mathfrak{P}_{\alpha_i} = \overline{\operatorname{dom} \mathcal{S}^{n-i+1}}, \ i = 1, \dots, n.$$

From Lemma II.5.19 we obtain that $\mathfrak{P}_{\alpha_1} = \overline{\operatorname{dom} \mathcal{S}^{m+1}}$ or $\mathfrak{P}_{\alpha_1} = \overline{\operatorname{dom} \mathcal{S}^m}$ depending whether $d \neq 0$ or $d = 0$, or equivalently whether $\overline{\bigcup_{t \in \mathcal{J}, t < \alpha_1} \mathfrak{P}_t} = \mathfrak{P}_{\alpha_1}$ or not. In the latter case $\mathfrak{P}_0 := \overline{\operatorname{dom} \mathcal{S}^{m+1}}$ is nondegenerated and is properly contained in $\mathfrak{P}_{\alpha_1}$.

We conclude that $n = m+1$ or $n = m$ depending on which case we are in, but this is just the assertion in (2.14). The fact that $\Delta_+ = 1$ and $\Delta_- = 1$ if $d = 0$ is obvious.

Again by Lemma II.5.19 we have

$$\mathfrak{P}_{\alpha_i} = \operatorname{span}\{S_\phi, \ldots, z^{n-i}S_\phi\}^\perp,\ i = 1, \ldots, n.$$

Therefore

$$\mathfrak{P}^\circ_{\alpha_i} = \operatorname{span}\{S_\phi, \ldots, z^{n-i}S_\phi\}^\circ,\ i = 1, \ldots, n,$$

in particular (2.15) follows. Since the Gram-matrix of the inner product in $\operatorname{span}\{S_\phi, \ldots, z^m S_\phi\}$ is a Hankel matrix with all entries zero above the secondary diagonal, the isotropic part is of the form $\operatorname{span}\{S_\phi, \ldots, z^l S_\phi\}$. We conclude, in fact from Corollary 2.11, that (2.16) and (2.17) are valid.

It remains to consider the increment of negative squares, so assume that $\Delta_- + \mu$ is even. Then

$$n - i_{max} = n - [\frac{1 - \Delta_- + \mu}{2}] - 1 =$$

$$= n - \frac{-\Delta_- + \mu}{2} - 1 = \frac{\Delta_- + \mu}{2} = \frac{m+d}{2}.$$

Since $z^d S_\phi \notin \operatorname{span}\{S_\phi, \ldots, z^m S_\phi\}^\circ$, but $z^{d-1}S_\phi$ is (in case $d \neq 0$), we have

$$0 \neq [z^d S_\phi, z^m S_\phi] = [z^{\frac{m+d}{2}} S_\phi, z^{\frac{m+d}{2}} S_\phi].$$

Clearly

$$0 = [z^d S_\phi, z^{m-1} S_\phi] = [z^{\frac{m+d}{2}} S_\phi, z^{\frac{m+d}{2}-1} S_\phi],$$

and therefore

$$\mathfrak{P}_{\alpha_{i_{max}+1}} = \mathfrak{P}_{\alpha_{i_{max}}} [\dot{+}] \operatorname{span}\{z^{\frac{m+d}{2}} S_\phi\}.$$

The relation (2.18) follows.

□

Now we are in position to prove Theorem 2.3.

Proof (Theorem 2.3): According to Corollary 2.8 there exist points $t_1, \ldots, t_{d_n} \in \mathbb{R}$, and numbers $m_1, \ldots, m_{d_n} > 0$, such that the space $\mathfrak{P}_{\alpha_n}$ endowed with the inner product $[.,.]_{d_n}$ given by (2.7) is nondegenerated. Note that, if the change of inner products is carried out step by step as in the proof of Corollary 2.8, i.e. considering the sequence of inner products

$$[G, H]_k := [G, H] + \sum_{i=1}^{k} m_1 G(t_1)\overline{H(t_i)},\ k = 0, \ldots, d_n,$$

then in each step the dimension of $\mathfrak{P}^\circ_{\alpha_n}$ decreases.

Starting with the dB-Pontryagin space $\langle \mathfrak{P}_{\beta_0}, [.,.]_{d_n} \rangle := \langle \mathfrak{P}_{\alpha_n}, [.,.]_{d_n} \rangle$ we construct larger dB-Pontryagin spaces $\mathfrak{P}_{\beta_j}$, $j = 1, \ldots, d_n+1$, such that $\langle \mathfrak{P}_{\beta_j}, [.,.]_{d_n} \rangle$ is contained isometrically in $\langle \mathfrak{P}_{\beta_{j+1}}, [.,.]_{d_n} \rangle$ with codimension 1. Such a construction is possible by Theorem I.12.2 and its corollary, by successively adding indivisible intervals of appropriately chosen types.

The spaces $\langle \mathfrak{P}_{\beta_j}, [.,.] \rangle$ are degenerated for $j = 1, \ldots, d_n - 1$, since the dimension of the isotropic part of $\mathfrak{P}_{\alpha_n}$ is d_n and the codimension of $\mathfrak{P}_{\alpha_n}$ in $\mathfrak{P}_{\beta_j}$ is j. Moreover, since $[.,.]$ is a d_n-dimensional perturbation of $[.,.]_{d_n}$, the dimension of $\langle \mathfrak{P}_{\beta_j}, [.,.] \rangle^\circ$ does not exceed d_n. Hence, by Corollary 2.11 one of the spaces $\langle \mathfrak{P}_{\beta_{d_n}}, [.,.] \rangle$ and $\langle \mathfrak{P}_{\beta_{d_n+1}}, [.,.] \rangle$ must be nondegenerated. Thus we are, either with $\mathfrak{P}_{\beta_{d_n}}$ or with $\mathfrak{P}_{\beta_{d_n+1}}$ in the situation of Proposition 2.14, which implies the assertion of the theorem.

□

If we write $\mathfrak{P}_{\beta_{d_n}}$ ($\mathfrak{P}_{\beta_{d_n+1}}$, respectively) as $\mathfrak{P}(E)$ with $E = A - iB$, $E(0) = -i$, then the form of the reproducing kernel $K(w, z)$ (cf. Theorem I.5.3) immediately implies the following result:

Corollary 2.15. *Assume that in the situation of Theorem 2.3 for one (and hence for all) $i \in \{1, \ldots, n\}$ we have $\mathfrak{P}_{\alpha_i}^\circ \subseteq \mathfrak{Q}(\mathfrak{P}_{\alpha_i})$, i.e. that $F(0) = 0$ when F denotes the function with the properties stated in Theorem 2.3. Then $K_0(z) := -\frac{F(z)}{z}$ belongs to $\mathfrak{P}_{\alpha_1}$ and satisfies*

$$[G, K_0] = G(0), \ G \in \mathfrak{P}_{\alpha_n}. \tag{2.19}$$

If $\mathfrak{P}_{\alpha_n}$ is the left endpoint of an indivisible interval, then this relation holds for all G belonging to the larger space corresponding to the right endpoint of this interval. If $\mathfrak{P}_{\alpha_1}$ is the right endpoint of an indivisible interval, then K_0 is even contained in the smaller space corresponding to the left endpoint.

Proof : By our assumption $F(0) = 0$, the function $K_0(z)$ equals the reproducing kernel function $K(0, z)$ in the space $\mathfrak{P}(E)$ constructed above. The relation (2.19) follows.

If $\mathfrak{P}_{\alpha_n}$ is left endpoint of an indivisible interval (α_n, α_{n+1}) we can choose $\mathfrak{P}(E)$ such that it isometrically contains $\mathfrak{P}_{\alpha_{n+1}}$. Thus in this case (2.19) also holds for $G \in \mathfrak{P}_{\alpha_{n+1}}$.

If $\mathfrak{P}_{\alpha_1}$ is the right endpoint of an indivisible interval, say (α_0, α_1), then by Lemma II.5.19 we have $K_0 \in \mathfrak{P}_{\alpha_0}$.

□

3 Maximal chains of matrix functions

Now we turn to the investigation of chains of matrix functions. Before we start collecting some easy consequences of the axioms in Definition 1.1, let us note the following facts.

Remark 3.1. If $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$ and $t^\bullet(t)$ is an increasing bijection of $(0,\tilde{M})$, $0<\tilde{M}\leq\infty$, onto $(0,M)$, we may define another chain by

$$\tilde{W}_t(z) := W_{t^\bullet(t)}(z),\ t\in\tilde{\mathcal{I}} := (t^\bullet)^{-1}(\mathcal{I}).$$

We say that the chain $(\tilde{W}_t)_{t\in\tilde{\mathcal{I}}}$ is a reparametrization of $(W_t)_{t\in\mathcal{I}}$. It is apparent that chains which are reparametrizations of each other behave in the same manner.

A source for examples of maximal chains of matrix functions is Theorem II.7.1 and the discussion after its proof. We will see later (cf. Remark 3.10) that this example is in fact universal. Let $W\in\mathcal{M}_\kappa$ and let $\tau\in\mathcal{N}_0$ be such that

$$W\circ\tau\in\mathcal{N}_\kappa. \tag{3.1}$$

Theorem II.7.1 applied to the matrix W shows the existence of a chain $(W_t)_{t\in\mathcal{J}}$ of matrix functions where $\mathcal{J}\subseteq\mathbb{R}^-\cup\{0\}$, $0\in\mathcal{J}$, with $W_0=W$. Linking this chain with the chain $(V_t)_{t\geq 0}$ which has τ as its Weyl coefficient (compare the construction at the end of Section 7 of [KW4]) we obtain another chain of matrix functions. In fact, one can define

$$\tilde{W}_t := \begin{cases} W_t & ,\ t\in\mathcal{J} \\ WV_t & ,\ t\geq 0 \end{cases}.$$

Due to our assumption (3.1) one can check that $(\tilde{W}_t)_{t\in\mathcal{J}\cup\mathbb{R}^+}$ satisfies **(W2)** - **(W5)**. See the end of Section 7 in [KW4]. Note also in this context that the second assertion in Remark II.7.10 is not correct.

The index set $\mathcal{J}\cup\mathbb{R}^+$ is an open interval with finitely many closed intervals contained in $\mathcal{J}\cup\mathbb{R}^+$ taken out. Thus a convenient reparametrization of $(\tilde{W}_t)_{t\in\mathcal{J}\cup\mathbb{R}^+}$ satisfies **(W1)**, and therefore belongs to $\mathfrak{M}$.

Remark 3.2. Before we proceed we have to make some remarks on Theorem II.7.1 and its proof. In the proof first the chain $\mathfrak{P}_t,\ t\in M_{reg}\subseteq(c_-,0]$ of subspaces of $\mathfrak{P}(w_{21}-iw_{22})$ is considered. We can assume that $c_->-\infty$. In order to obtain a complete proof we have to add to this chain of subspaces the trivial space $\mathfrak{P}_{c_-}=\{0\}$. Moreover, if $c_-<t<s$ for all $s\in M_{reg}$ we have to set $t_-(t)=c_-$, and define $W_{c_-}=1$. With this little trick the proof is valid. Moreover, we mention two more properties of the chain constructed in that proof.

(i) On the first component of D^c the function $\mu(t)$ is always zero. If W is not a linear polynomial, all the matrix functions W_t for t in this first component are linear polynomials if and only if $\mathfrak{K}(W)$ contains a constant vector function whose inner product with itself is non-positive.

(ii) Whenever t converges in the first component to c_-, then $W_t(z)$ converges locally uniformly to the 2×2 identity matrix, or equivalently $\mathfrak{t}(W_t)\to 0$.

To see this, consider the chain $\mathfrak{P}_t,\ t \in M_{reg} \subseteq (c_-, 0]$. It is the maximal chain of dB-subspaces of $\mathfrak{P}_0$. The intersection of these spaces is also a dB-subspace of $\mathfrak{P}_0$. If it is not $\{0\}$, then it is one dimensional, since otherwise it would contain a proper dB-subspaces contained in $\mathfrak{P}_0$, which contradicts the maximality of the chain. Hence $\{1\} = \mathfrak{P}_{t_0}$ is the smallest member of $\mathfrak{P}_t,\ t \in M_{reg}$.

If the intersection is zero, then every $\mathfrak{P}_t$ in the chain is of infinite dimension, since otherwise a dimension argument would show that the intersection is non-zero. In particular, 1 belongs to no $\mathfrak{P}_t$, and therefore no $\mathfrak{K}(W_t)$ contains a constant function, where $W_t \in \mathcal{M}_{\mu(t)}$ are the matrix functions constructed in the proof of Theorem II.7.1. Moreover, since we assumed the intersection to be zero, every $\mathfrak{P}_t$ contains infinitely many $\mathfrak{P}_s,\ s \leq t$. By Lemma I.11.10 there is a $t' \in M_{reg}$ such that all $\mathfrak{P}_t,\ t \leq t'$ are Hilbert spaces. Thus the function $\mu(t)$, which shows the number of negative squares of $\mathfrak{K}(W_t) = \mathfrak{K}(W_t)_- \cong \mathfrak{P}_t$, has value zero on the first component of D^c. Since for $t \leq t'$ the space $\mathfrak{K}(W_t) = \mathfrak{K}(W_t)_- \cong \mathfrak{P}_t$ are infinite-dimensional Hilbert spaces, W_t is not a linear polynomial. Hence we have proved *(i)* in the current case.

If *(ii)* were not true, a normal family argument would give rise to non-constant matrix functions $M, M' \in \mathcal{M}_0$ with $W_{t'} = MM'$. But then we would obtain a dB-subspace of $\mathfrak{P}_0$ which is contained in all $\mathfrak{P}_t,\ t \in M_{reg} \subseteq (c_-, 0]$. This contradicts our present assumption.

Now assume that $\{1\}$ is the smallest space $\mathfrak{P}_{t_0}$ in the chain $\mathfrak{P}_t,\ t \in M_{reg}$. In this case the construction in the proof of Theorem II.7.1 shows that $\mu(t) = 0$ on the first component of D^c and that *(ii)* holds.

If $[1, 1] > 0$, then $\mathfrak{P}_{t_0}$ is a Hilbert space, and the space $\mathfrak{K}(W_{t_0}) = \mathfrak{K}(W_{t_0})_- \cong \mathfrak{P}_{t_0}$ exists, and the inner product of the constant function contained in $\mathfrak{K}(W_{t_0})$ with itself coincides with $[1, 1] > 0$. Since $\mathfrak{P}_{t_0} \cong \mathfrak{K}(W_{t_0})$ is contained isometrically in $\mathfrak{P}_0 \cong \mathfrak{K}(W)$, no space $\mathfrak{K}(W_{t_0 t}),\ t > t_0$ contains the same constant function as $\mathfrak{K}(W_{t_0})$ (cf. [KW3], Section 12). Therefore, $\mathfrak{K}(W_t) = \mathfrak{K}(W_{t_0}) \oplus W_{t_0}\mathfrak{K}(W_{t_0 t})$ is of dimension greater than one, and it is a Hilbert space for t sufficiently close at t_0. Thus also in this case *(i)* is true.

For $[1, 1] \leq 0$ the construction in the proof of Theorem II.7.1 shows that *(i)* holds.

In the sequel we collect some simple properties of maximal chains of matrix functions.

Remark 3.3. The function $t \mapsto W_t$ is injective. Moreover, if $s, t \in \mathcal{I}$, $\kappa(s) \leq \kappa(t)$, and $W_s^{-1}W_t \in \mathcal{M}_{\kappa(t)-\kappa(s)}$, then $s \leq t$.

This can be seen as follows: Assume that $W_s = W_t$ for some $s, t \in \mathcal{I},\ s < t$. Then for all $u \in \mathcal{I},\ s < u < t$, we have

$$(W_s^{-1}W_u)(W_u^{-1}W_t) = W_s^{-1}W_t = 1.$$

Since both factors on the left hand side of the above relation belong to $\mathcal{M}_0$ we get $W_u = W_s$ (cf. [dB]), a contradiction to **(W2)**. The second assertion is trivial if

$\kappa(s) < \kappa(t)$, since $\kappa(.)$ is nondecreasing. Let $\kappa(s) = \kappa(t)$, and assume that $s \geq t$. Then, by the axiom **(W3)**, we have $W_t^{-1} W_s \in \mathcal{M}_0$. Since by our assumption also $W_s^{-1} W_t \in \mathcal{M}_0$, we conclude that $W_s = W_t$, i.e. $s = t$ (cf. [dB]).

Recall that the function $\mathfrak{t}$ is defined by $\mathfrak{t}(W) = \operatorname{tr}(W'(0)J)$, $W \in \mathcal{M}_\kappa$. Clearly, $\mathfrak{t}(W)$ depends continuously on W with respect to locally uniform convergence. Recall from [dB], p.121, that for $W \in \mathcal{M}_0$ the relation ($\|.\|$ denotes the Schmidt norm of a matrix)

$$\|W(z) - 1\| \leq e^{\mathfrak{t}(W)|z|} - 1, \tag{3.2}$$

holds. In particular, it follows that the set of all $W \in \mathcal{M}_0$ with $\mathfrak{t}(W) \leq 1$ is compact. This implies the following statement:

Lemma 3.4. *Let $\mathcal{J} \subseteq \mathbb{R}^+$ and let $(W_t)_{t \in \mathcal{J}}$ be a chain of matrix functions which satisfies* **(W2)** *and* **(W3)** *where $\kappa(t)$ is assumed to be constant on $\mathcal{J}$, $\kappa(t) =: \kappa$. If $\mathfrak{t}(W_t)$ is bounded on $\mathcal{J}$, the limit*

$$\lim_{t \nearrow \sup \mathcal{J}, t \in \mathcal{J}} W_t$$

exists locally uniformly and belongs to $\mathcal{M}_{\kappa'}$, $\kappa' \leq \kappa$. If for some $s, t \in \mathcal{J}$ the matrix function W_{st} is not a linear polynomial, then in fact $\kappa' = \kappa$.

Proof : Let $s, t \in \mathcal{J}$, $s \leq t$. Then $W_{st} \in \mathcal{M}_0$, hence $\mathfrak{t}(W_{st}) \geq 0$. As $\mathfrak{t}(W_t) = \mathfrak{t}(W_s) + \mathfrak{t}(W_{st})$, we get $\mathfrak{t}(W_s) \leq \mathfrak{t}(W_t)$. It follows that the limit ($t_+ := \sup \mathcal{J}$)

$$\mathfrak{t}_+ := \lim_{t \nearrow t_+, t \in \mathcal{J}} \mathfrak{t}(W_t),$$

exists in $\mathbb{R}$.

Now fix $s \in \mathcal{J}$. For $t \in \mathcal{J}$, $t \geq s$, we have $\mathfrak{t}(W_{st}) = \mathfrak{t}(W_t) - \mathfrak{t}(W_s) \leq \mathfrak{t}_+ - \mathfrak{t}(W_s)$. Hence there exists a sequence $t_n \in \mathcal{J}$, $t_n \nearrow t_+$, such that the limit $\lim_{n \to \infty} W_{st_n} =: W_{st_+}$ exists and belongs to $\mathcal{M}_0$. Put $W_{t_+} := W_s W_{st_+}$. Clearly, $\mathfrak{t}(W_{t_+}) = \mathfrak{t}_+$ and $\operatorname{ind}_- W_{t_+} = \kappa' \leq \kappa$. Since $W_{st_n} = W_{st} W_{tt_n}$ for $t \in \mathcal{J}$, $s \leq t \leq t_n$, also $\lim_{n \to \infty} W_{tt_n} := W_{tt_+}$ exists and belongs to $\mathcal{M}_0$. Moreover, $W_t W_{tt_+} = W_{t_+}$. Hence,

$$\mathfrak{t}(W_{tt_+}) = \mathfrak{t}(W_{t_+}) - \mathfrak{t}(W_t) = \mathfrak{t}_+ - \mathfrak{t}(W_t) \to 0,$$

for $t \nearrow t_+$. By (3.2) we conclude that $\lim_{t \nearrow t_+} W_{tt_+} = 1$ and thus also $\lim_{t \nearrow t_+} W_t = W_{t_+}$.

As indicated in the proof of Lemma II.8.5 the existence of a nonlinear transfer matrix permits the application of Lemma II.5.17 which yields $\kappa' = \kappa$.

□

The next statement follows basically from Theorem II.7.1 and Remark 3.2.

Lemma 3.5. *Let $(W_t)_{t \in \mathcal{I}} \in \mathfrak{M}$. The function $\kappa(t)$ is constant on each connected component of $\mathcal{I}$ and takes different values on different components. The function $\mathfrak{t}(W_t)$, and hence also W_t, is continuous on $\mathcal{I}$ and strictly increasing on each component. If $\mathcal{I}_0, \ldots, \mathcal{I}_n$ are the components of $\mathcal{I}$ ordered increasingly, then* *(i)* $\kappa(\mathcal{I}_0) = 0$.

(ii) Assume that not the whole chain $(W_t)_{t\in\mathcal{I}}$ *consists of linear polynomials. Then all matrix functions* W_t *for* $t \in \mathcal{I}_0$, *are linear polynomials if and only if for sufficiently large* $t \in \mathcal{I}$ *the space* $\mathfrak{K}(W_t)$ *contains a constant vector function whose inner product with itself is non-positive.*

(iii) For $k = 0, \dots, n$
$$\sup_{t\in\mathcal{I}_k} \mathfrak{t}(W_t) = +\infty.$$

(iv) For $k = 1, \dots, n$
$$\inf_{t\in\mathcal{I}_k} \mathfrak{t}(W_t) = -\infty.$$

(v) On the first component $\mathcal{I}_0$ *we always have*
$$\inf_{t\in\mathcal{I}_0} \mathfrak{t}(W_t) = 0,$$
i.e. $\lim_{t\searrow 0} W_t = 1$.

Proof : Let $t \in \mathcal{I}$ be given. Choose $s \in \mathcal{I}$, $s > t$ sufficiently large, and consider the chain of matrix functions $(\tilde{W}_t)$ going downwards from W_s as constructed in Theorem II.7.1.

Since the chain $(W_t)_{t\in\mathcal{I}}$ is maximal in the sense of **(W4)** each matrix $\tilde{W}_t$ must occur among the W_t's. However, the chain $(\tilde{W}_t)$ possesses the same maximality property by Theorem II.7.1. Hence also each W_t, $t \leq s$, must occur among the $\tilde{W}_t$'s. The assertions of the lemma are now an immediate consequence of Theorem II.7.1 and Remark 3.2.

□

The previous statement allows us to define the negative index of a chain $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$ as the number $\max_{t\in\mathcal{I}} \kappa(t)$. The subset of $\mathfrak{M}$ consisting of all chains with negative index κ will be denoted by $\mathfrak{M}_\kappa$.

For a matrix $W(z)$ and a scalar function $\tau(z)$ recall the notation
$$W(z) \circ \tau(z) := \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_{21}(z)\tau(z) + w_{22}(z)}.$$

It has been proved in Lemma II.8.2 and Lemma II.8.5, that for a maximal chain $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}_\kappa$ the limit
$$q_\infty((W_t)_{t\in\mathcal{I}})(z) := \lim_{t\nearrow\sup\mathcal{I}} (W_t \circ \tau^t)(z) \tag{3.3}$$
exists locally uniformly on $\mathbb{C}\setminus\mathbb{R}$ if $\mathbb{C}\cup\{\infty\}$ is provided with the spherical metric, does not depend on $\tau^t \in \mathcal{N}_0$ and belongs to $\mathcal{N}_\kappa$. The function defined by (3.3) is called the Weyl coefficient of $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$.

The main result of [KW4] implies that a maximal chain $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}_\kappa$ is uniquely determined up to reparametrizations by its Weyl coefficient $q_\infty((W_t)_{t\in\mathcal{I}}) \in \mathcal{N}_\kappa$, and that each function $q \in \mathcal{N}_\kappa$ appears as the Weyl coefficient q_∞ of some maximal chain:

Proposition 3.6. *The map $(W_t)_{t\in\mathcal{I}} \mapsto q_\infty((W_t)_{t\in\mathcal{I}})$ establishes a bijection between $\mathfrak{M}_\kappa$ (up to reparametrizations) and $\mathcal{N}_\kappa$.*

Proof : By Theorem II.8.7 it suffices to show that (with the notation of [KW4], Section 8) a chain $(W_t)_{t>c_-} \in \mathfrak{C}$ can be prolonged uniquely to a chain $(\tilde{W}_t)_{t\in\mathcal{I}} \in \mathfrak{M}$. However, this fact is an immediate consequence of Theorem II.7.1.

□

If $(W_t)_{t\in\mathcal{J}}$ is a chain of matrix functions which is not maximal in the sense of Definition 1.1, it can be completed.

Lemma 3.7. *Let $(W_t)_{t\in\mathcal{J}}$ be a chain of matrix functions which possesses the properties:*

(W1') *The index set $\mathcal{J}$ is contained in $\mathbb{R}^+$.*

(W2') *For each $t \in \mathcal{J}$ the matrix function W_t belongs to $\mathcal{M}_{\kappa(t)}$ for some $\kappa(t) \in \mathbb{N}\cup\{0\}$. The function $\kappa(t)$ is bounded on $\mathcal{J}$, $\max_{t\in\mathcal{J}} \kappa(t) =: \kappa_m$.*

(W3') *If $s,t \in \mathcal{J}$, $s \leq t$, then $\kappa(s) \leq \kappa(t)$ and $W_s^{-1}W_t \in \mathcal{M}_{\kappa(t)-\kappa(s)}$.*

If $\limsup_{t\nearrow\sup\mathcal{J}} \mathfrak{t}(W_t) = +\infty$ assume in addition:

(W4') *If $\kappa_m > 0$, there exist numbers $s,t \in \mathcal{J}$, $\kappa(s) = \kappa(t) = \kappa_m$, such that W_{st} is not a linear polynomial.*

If $\limsup_{t\nearrow\sup\mathcal{J}} \mathfrak{t}(W_t) < +\infty$ assume in addition:

(W5') $\lim_{t\nearrow\sup\mathcal{J},t\in\mathcal{J}} W_t \in \mathcal{M}_{\kappa_m}$.

Then there exists a maximal chain $(\tilde{W}_t)_{t\in\mathcal{I}} \in \mathfrak{M}_{\kappa_m}$ which extends $(W_t)_{t\in\mathcal{J}}$, i.e. for some function $\mathfrak{t}^\bullet : \mathcal{J} \to \mathcal{I}$ we have $\tilde{W}_{\mathfrak{t}^\bullet(t)} = W_t$, $t \in \mathcal{J}$.

If $\limsup_{t\nearrow\sup\mathcal{J}} \mathfrak{t}(W_t) = +\infty$, the extension in $\mathfrak{M}_{\kappa_m}$ is unique, $\sup\mathcal{I} = \sup\mathcal{J}$ *and ($\tau^t \in \mathcal{N}_0$)*

$$q_\infty((\tilde{W}_t)_{t\in\mathcal{I}}) = \lim_{t\in\mathcal{J},t\nearrow\sup\mathcal{J}} (W_t \circ \tau^t).$$

If $\limsup_{t\nearrow\sup\mathcal{J}} \mathfrak{t}(W_t) < +\infty$, a chain $(\tilde{W}_t)_{t\in\mathcal{I}} \in \mathfrak{M}_{\kappa_m}$ extends $(W_t)_{t\in\mathcal{J}}$ if and only if

$$q_\infty((\tilde{W}_t)_{t\in\mathcal{I}}) = \left[\lim_{t\nearrow\sup\mathcal{J},t\in\mathcal{J}} W_t\right] \circ \tau, \tag{3.4}$$

for some $\tau \in \mathcal{N}_0$, which satisfies (3.1) with $W = \lim_{t\nearrow\sup\mathcal{J},t\in\mathcal{J}} W_t$ and $\kappa = \kappa_m$.

Proof : Consider first the case that $\limsup_{t\nearrow\sup\mathcal{J}} \mathfrak{t}(W_t) < +\infty$. Then by Lemma 3.4 the limit

$$W_{\sup\mathcal{J}} := \lim_{t\nearrow\sup\mathcal{J},t\in\mathcal{J}} W_t,$$

exists and belongs to $\mathcal{M}_\kappa$, $\kappa \leq \kappa_m$.

In order to clarify the role of **(W5')** note that if some chain of $\mathfrak{M}$ contains the matrices $(W_t)_{t\in\mathcal{J}}$, Lemma 3.5 implies that $\kappa = \kappa_m$. Also, if there exists a function $\tau \in \mathcal{N}_0$ which satisfies (3.1) we necessarily have $\kappa = \kappa_m$.

Since for each $s, t \in \mathcal{J}$, $t \geq s$, we have $W_t = W_s W_{st}$ with $\operatorname{ind}_- W_{st} = \kappa(t) - \kappa(s)$, it follows that also $W_{\sup\mathcal{J}} = W_s W_{s,\sup\mathcal{J}}$ for some $W_{s,\sup\mathcal{J}} \in \mathcal{M}_{\kappa_m-\kappa(s)}$. Hence each matrix W_s, $s \in \mathcal{J}$, occurs in the chain going downwards from $W_{\sup\mathcal{J}}$ as constructed in Theorem II.7.1. The previously constructed example of a chain in $\mathfrak{M}$ (see Remark 3.1 and the discussion after it) now yields an extension $(\tilde{W}_t)_{t\in\mathcal{I}} \in \mathfrak{M}_{\kappa_m}$ of $(W_t)_{t\in\mathcal{J}}$ for each τ satisfying (3.1).

Conversely, if $(\tilde{W}_t)_{t\in\mathcal{I}} \in \mathfrak{M}_{\kappa_m}$ is an extension of $(W_t)_{t\in\mathcal{J}}$, then by Lemma 3.5 we must have $W_{sup\mathcal{J}} = \tilde{W}_{t_0}$ for some $t_0 \in \mathcal{I}$. Put $\tau := q_\infty((V_t)_{t\geq 0})$ with $V_t := \tilde{W}_{t_0,t_0+t}$. Then, clearly, (3.4) holds. Since $q_\infty((\tilde{W}_t)_{t\in\mathcal{I}}) \in \mathcal{N}_{\kappa_m}$ (c.f. Proposition 3.6), we also see that τ satisfies (3.1).

Now consider the case that $\limsup_{t\nearrow\sup\mathcal{J},t\in\mathcal{J}} \mathfrak{t}(W_t) = +\infty$. Then by Lemma II.8.2 (and its proof) the limit ($\alpha \in \mathbb{R}$)

$$q := \lim_{t\nearrow\sup\mathcal{J},t\in\mathcal{J}} W_t \circ \alpha$$

exists and does not depend on α. By condition **(W4')** it is contained in $\mathcal{N}_{\kappa_m}$ (cf. Lemma II.8.5 and its proof). Consider the (unique) chain $(\tilde{W}_t)_{t\in\mathcal{I}}$ which has q as its Weyl coefficient. For each $s \in \mathcal{J}$ the limit

$$\tau_s := \lim_{t\nearrow\sup\mathcal{J},t\in\mathcal{J}} W_{st} \circ \alpha$$

exists, belongs to $\mathcal{N}_{\kappa'}$, $\kappa' \leq \kappa_m - \kappa(s)$ and satisfies $W_s \circ \tau_s = q$. Hence $\kappa' = \kappa_m - \kappa(s)$, and we conclude that the matrix W_s must occur in the chain $(\tilde{W}_t)_{t\in\mathcal{I}}$ (cf. Corollary II.11.1). Thus we have found an extension of $(W_t)_{t\in\mathcal{J}}$. Clearly, any extension of $(W_t)_{t\in\mathcal{J}}$ must have q as its Weyl coefficient, and thus equals $(\tilde{W}_t)_{t\in\mathcal{I}}$.

□

From the above considerations together with Theorem I.13.1 we obtain the following corollary.

Corollary 3.8. *Let $(W_t)_{t\in\mathcal{J}}$ be as in Lemma 3.7. Assume in addition that for all $\kappa \in \kappa(\mathcal{J})$ we have*

$$\mathfrak{t}(\{W_t : \kappa(t) = \kappa\}) = \begin{cases} \mathbb{R} & , \kappa > 0 \\ (0,\infty) & , \kappa = 0 \end{cases} .$$

Then $(W_t)_{t\in\mathcal{J}}$ *is almost a maximal chain, it satisfies* **(W1)**-**(W3)** *and* **(W5)**. *The maximality axiom* **(W4)** *holds if and only if the existence of a factorization* $W_t = WM$, *where* $W \in \mathcal{M}_\kappa$ *and* $M \in \mathcal{M}_{\kappa(t)-\kappa}$ *for* $0 \leq \kappa \leq \kappa(t)$, *implies* $\kappa \in \kappa(\mathcal{J})$.

Remark 3.9. In the case $\limsup_{t\nearrow\sup\mathcal{J}} \mathfrak{t}(W_t) = +\infty$ there might exist various extensions of $(W_t)_{t\in\mathcal{J}}$ in a set $\mathfrak{M}_\kappa$ with $\kappa > \kappa_m$. A related question will be discussed in Section 7.

Remark 3.10. Let $(\tilde{W}_t)_{t\in\mathcal{I}} \in \mathfrak{M}$ and $t_0 \in \mathcal{I}$. Choosing for $\mathcal{J}$ the one element set $\{t_0\}$, putting $W_{t_0} := \tilde{W}_{t_0}$ and $\tau := q_\infty((V_t))$, $V_t := W_{t_0,t_0+t}$, the previous lemma shows that every chain belonging to $\mathfrak{M}$ can be constructed as in the example discussed after Remark 3.1.

Recall the notion of indivisible intervals. Put (compare (II.7.1))

$$W_{(l,\alpha)}(z) = \begin{pmatrix} 1 - lz\sin\alpha\cos\alpha & lz\cos^2\alpha \\ -lz\sin^2\alpha & 1 + lz\sin\alpha\cos\alpha \end{pmatrix}.$$

If $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$, a nonempty interval $(s,t) \subseteq \mathcal{I}$, $s,t \in \mathcal{I}$, is called indivisible, if W_{st} is a linear polynomial, i.e. $W_{st} = W_{(l,\phi)}$, $l > 0$, $\phi \in [0,\pi)$. The numbers l and ϕ are called the length and the type, respectively, of the indivisible interval. The union of all indivisible intervals will be denoted by $\mathcal{I}^{\text{ind}}$.

Remark 3.11. The chains $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$ which consist only of linear polynomials are exactly those with $q_\infty((W_t)_{t\in\mathcal{I}}) \in \mathbb{R}\cup\{\infty\}$. Such a chain is explicitly given (up to reparametrization) by

$$W_t(z) = W_{(t,\phi)}(z),\ t > 0,$$

where

$$q_\infty((W_t)_{t\in\mathcal{I}}) = \cot\phi.$$

If λ is any number, denote by λ_+ and λ_- the numbers

$$\lambda_+ := \sup\left(\{\lambda\}\cup\{t :\ (\lambda,t) \subseteq \mathcal{I}^{\text{ind}}\}\right), \tag{3.5}$$

$$\lambda_- := \inf\left(\{\lambda\}\cup\{t :\ (t,\lambda) \subseteq \mathcal{I}^{\text{ind}}\}\right). \tag{3.6}$$

One of the main objectives of this paper is to investigate the behaviour of the chain $(W_t)_{t\in\mathcal{I}}$ near a boundary point between two components of $\mathcal{I}$. Those by **(W1)** only finitely many points are called singularities of $(W_t)_{t\in\mathcal{I}}$. We introduce now a subdivision of singularities into several types.

Definition 3.12. Let $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$. A singularity σ of $(W_t)_{t\in\mathcal{I}}$ is said to be

(i) of polynomial type, if $\sigma_- < \sigma < \sigma_+$.

(ii) left (right) dense, if $\sigma_- = \sigma < \sigma_+$ ($\sigma_- < \sigma = \sigma_+$).

(iii) dense, if $\sigma_- = \sigma = \sigma_+$.

The number

$$\omega = \min_{t>\sigma, t\in\mathcal{I}} \operatorname{ind}_- W_t - \max_{t<\sigma, t\in\mathcal{I}} \operatorname{ind}_- W_t,$$

is called the weight of σ.

An important tool in the study of chains of matrix functions is a connection with dB-spaces. Of course this connection had appeared and had been extensively used in our previous work [KW3], [KW4]. Nevertheless, we would like to point it out explicitly.

Let $W = (w_{ij})_{i,j=1}^2 \in \mathcal{M}_\kappa$. Define the function

$$E_W(z) := w_{21}(z) - iw_{22}(z).$$

Then $E_W \in \mathcal{HB}_{\kappa'}$ for some $\kappa' \leq \kappa$ (cf. [KW3]). If

$$\mathfrak{K}_-(W) = \mathfrak{K}(W), \tag{3.7}$$

then in fact $\kappa' = \kappa$. Recall from Theorem II.5.7 that for a chain $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$ the property (3.7) does not depend on t. Moreover, if $\mathfrak{K}_-(W_t) \neq \mathfrak{K}(W_t)$, then an application of the transformation $\mathcal{T}_J$ (cf. Lemma II.10.1) yields a chain $(\tilde{W}_t)_{t\in\tilde{\mathcal{I}}} \in \mathfrak{M}$ which satisfies $\mathfrak{K}_-(\tilde{W}_t) = \mathfrak{K}(\tilde{W}_t)$. Since the chains $(W_t)_{t\in\mathcal{I}}$ and $(\tilde{W}_t)_{t\in\tilde{\mathcal{I}}}$ behave in most respects similar, we frequently will be able to assume without loss of generality that (3.7) holds.

Let a chain of matrix functions $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$ satisfying (3.7) be given. Then we define a chain of dB-Pontryagin spaces $(\mathfrak{P}_t)_{t\in\mathcal{I}}$ by

$$\mathfrak{P}_t := \mathfrak{P}(E_{W_t}),\ t \in \mathcal{I}.$$

This chain of spaces will be of good use when we investigate the structure of singularities of $(W_t)_{t\in\mathcal{I}}$.

Recall from Proposition I.11.4 that the spaces $\mathfrak{P}_t$, $t \in \mathcal{I}$, depend in a way continuously on t.

Remark 3.13. If $s \in \mathcal{I}$ is not right endpoint of an interval contained in $\mathcal{I}^{\text{ind}}$, then

$$\overline{\bigcup_{t\in\mathcal{I}\setminus\mathcal{I}^{\text{ind}}, t<s} \mathfrak{P}_t} = \mathfrak{P}_s.$$

Otherwise,

$$\dim \left[\mathfrak{P}_s \,/\, \overline{\bigcup_{t\in\mathcal{I}\setminus\mathcal{I}^{\text{ind}}, t<s} \mathfrak{P}_t} \right] \geq 1,$$

where strict inequality can occur only if s_- is a singularity. If $s \in \mathcal{I}$ is not left endpoint of an interval contained in $\mathcal{I}^{\rm ind}$, then

$$\bigcap_{t\in\mathcal{I}\setminus\mathcal{I}^{\rm ind}, t>s} \mathfrak{P}_t = \mathfrak{P}_s.$$

Otherwise

$$\dim \left[\bigcap_{t\in\mathcal{I}\setminus\mathcal{I}^{\rm ind}, t>s} \mathfrak{P}_t \,/\, \mathfrak{P}_s \right] \geq 1,$$

where strict inequality can occur only if s_+ is a singularity.

Remark 3.14. For $t \in \mathcal{I}$ we have $t \in \mathcal{I}^{\rm ind}$ or t is the right endpoint of a component of $\mathcal{I}^{\rm ind}$ if and only if $S_\phi \in \mathfrak{P}_t$ where ϕ is the type of the indivisible interval (s, t_+) for one (and hence for all) $s \in [t_-, t_+)$, cf. [KW3], Section 13.

If σ is a singularity denote by $\mathfrak{P}_{\sigma_+}$ ($\mathfrak{P}_{\sigma_-}$) the dB-spaces

$$\mathfrak{P}_{\sigma_+} := \bigcap_{t\in\mathcal{I}\setminus\mathcal{I}^{\rm ind}, t>\sigma} \mathfrak{P}_t, \quad \mathfrak{P}_{\sigma_-} := \overline{\bigcup_{t\in\mathcal{I}\setminus\mathcal{I}^{\rm ind}, t<\sigma} \mathfrak{P}_t}.$$

Note here that this definition agrees with the previous meaning of $\mathfrak{P}_{\sigma_+}$ ($\mathfrak{P}_{\sigma_-}$) in case $\sigma_+ \in \mathcal{I}$ ($\sigma_- \in \mathcal{I}$). Note also that it may happen that $\mathfrak{P}_{\sigma_-}$ is the trivial space $\{0\}$. The number

$$\delta := \dim \left[\mathfrak{P}_{\sigma_+} \,/\, \mathfrak{P}_{\sigma_-} \right]$$

is called the degree of σ. Let us clarify the particular case of a singularity σ of polynomial type with degree one. This means that $\sigma_- < \sigma < \sigma_+$ and that $W_{\sigma_-\sigma_+} = W_{(l,\alpha)}$ for some $l < 0$.

Lemma 3.15. *Let $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}_\kappa$ satisfy (3.7) and consider the chain $(\mathfrak{P}_t)_{t\in\mathcal{I}}$. Then*

(i) $1 \in \operatorname{Ass} \mathfrak{P}_t$, $t \in \mathcal{I}$.

(ii) If $\mathfrak{P}$ ($\mathfrak{d}(\mathfrak{P}) = 0$) is any nondegenerated dB-subspace of $\mathfrak{P}_t$ for some $t \in \mathcal{I}$, then $\mathfrak{P} = \mathfrak{P}_s$ with $s \in \mathcal{I} \setminus \mathcal{I}^{\rm ind}$, $s \leq t$.

(iii) If $s, t \in \mathcal{I}$, $s < t$, then $\mathfrak{P}_s \subseteq \mathfrak{P}_t$ as sets. The equality sign holds if and only if $[s,t) \subseteq \mathcal{I}^{\rm ind}$ or $[s,t) \subseteq (\sigma_-, \sigma_+)$ for some singularity σ of polynomial type with degree 1. The inclusion is isometric if and only if $s \notin \mathcal{I}^{\rm ind}$. Otherwise it is a strict contraction.

Proof : The assertion *(i)* is immediate from Proposition I.10.3. To show *(ii)* consider a nondegenerated dB-subspace $\mathfrak{P}$ of $\mathfrak{P}_t$ where $t \in \mathcal{I}$. By *(i)* and [dB], Problem 72, we have $1 \in \operatorname{Ass} \mathfrak{P}$. Hence there exists a matrix $W \in \mathcal{M}_{\mathrm{ind}_-\mathfrak{P}}$,

$\mathfrak{K}_-(W) = \mathfrak{K}(W)$, $W(0) = 1$, such that $\mathfrak{P} = \mathfrak{P}(E_W)$. Moreover, the same arguments as in the proof of Theorem II.7.1 lead to the conclusion that $W^{-1}W_t \in \mathcal{M}_{\kappa(t)-\text{ind}_-\mathfrak{P}}$ since $\mathfrak{P}$ is contained isometrically in $\mathfrak{P}_t$. Hence $W = W_s$ for some $s \in \mathcal{I}$, $s \leq t$, and thus $\mathfrak{P} = \mathfrak{P}_s$.

It remains to consider *(iii)*. From $(A_{W_t}, B_{W_t}) = (A_{W_s}, B_{W_s})W_{st}$ we conclude by Theorem I.12.2 and Proposition I.13.5 that $\mathfrak{P}_s$ is included contractively in $\mathfrak{P}_t$. Moreover, the inclusion is isometric if and only if $\mathfrak{K}(W_{st})$ contains no constant $\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$, $\phi \in [0, \pi)$, with $S_\phi \in \mathfrak{P}_s$.

If $s \in (s_-, s_+)$, then by Remark 3.14 we have $S_\phi \in \mathfrak{P}_s$. Choose $s_0 \in (s, s_+)$ and write $W_{st} = W_{ss_0}W_{s_0t}$. Since the inclusion of $\mathfrak{P}_s$ in $\mathfrak{P}_{s_0}$ is a strict contraction and the inclusion of $\mathfrak{P}_{s_0}$ in $\mathfrak{P}_t$ is a contraction we obtain that $\mathfrak{P}_s \subseteq \mathfrak{P}_t$ is not isometric.

Now assume that the embedding is not isometric, i.e. for some $\phi \in [0, \pi)$ we have $S_\phi \in \mathfrak{P}_s$ and $\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \in \mathfrak{K}(W_{st})$.

Consider first the case that $[S_\phi, S_\phi]_{\mathfrak{P}_s} > 0$. Then also

$$[\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}]_{\mathfrak{K}(W_{st})} > 0,$$

since the mapping φ defined in the proof of Proposition I.13.5 is a contraction. Hence, by Lemma II.7.5 we may factorize W_{st} as $W_{st} = W_{(l,\phi)}W$ with $l > 0$ and $\text{ind}_- W = \text{ind}_- W_{st}$. It follows that $W_s W_{(l,\phi)} = W_{s'}$ for some $s' \in \mathcal{I}$, and together with Remark 3.14 we conclude that s is contained in some indivisible interval.

Now assume that $[S_\phi, S_\phi]_{\mathfrak{P}_s} \leq 0$. As we have seen in the proof of Proposition I.13.5 this implies that the space $\mathfrak{P}_{s^u}$, which is defined as the set $\mathfrak{P}_s$ endowed with the inner product inherited from $\mathfrak{P}_t$, is nondegenerated. Hence $\mathfrak{P}_{s^u} = \mathfrak{P}(E_{W_{s^u}})$ for some $s^u \in \mathcal{I}$. We have $s^u > s$, because $\mathfrak{P}_s$ is contained strictly contractively in $\mathfrak{P}_{s^u}$. Since $\mathfrak{P}_{s^u} = \mathfrak{P}_s$ as sets, we obtain that $W_{ss^u} = W_{(l,\phi)}$ for some $l \in \mathbb{R} \setminus \{0\}$, and the assertion follows as above.

The stated characterization of the fact that $\mathfrak{P}_s = \mathfrak{P}_t$ as sets follows easily from Remark 3.14, [ADSR] and the considerations in the proof of Proposition I.13.5.

□

The first and last component of $\mathcal{I}$, we denote them by $\mathcal{I}_0$ and $\mathcal{I}_\infty$ ($\inf \mathcal{I}_0 = 0$ and $\sup \mathcal{I}_\infty = \sup \mathcal{I}$, respectively), play a slightly different role than the other components. For $\mathcal{I}_\infty$ this is visible already from the axiom **(W5)**. The peculiarities of $\mathcal{I}_0$ were discussed in Lemma 3.5.

Now we are in position to show that for $t \in \mathcal{I}$, $t \notin \mathcal{I}_0, \mathcal{I}_\infty$, not both of t_- and t_+ can be singularities. The importance of this fact will show up in Proposition 5.1. Note that for the component $\mathcal{I}_\infty$ this fact had to be assumed (cf. **(W5)**).

Proposition 3.16. *Let $\mathcal{I}_n = (\sigma_{n-1}, \sigma_n)$ be a component of $\mathcal{I}$, neither the first nor the last one. Then there exist numbers $s, t \in \mathcal{I}_n$, $s < t$, such that W_{st} is not* *a linear polynomial.*

Proof : Without loss of generality me may assume that $\mathfrak{K}_-(W_t) = \mathfrak{K}(W_t)$. Otherwise we could apply the transformation $\mathcal{T}_J$ (cf. Lemma II.10.1) in order to force this condition to be fulfilled. Since an interval is indivisible in the chain $(W_t)_{t\in\mathcal{I}}$ if and only if it is in the chain $\mathcal{T}_J((W_t)_{t\in\mathcal{I}})$, the general result follows once we have established this particular case.

Assume on the contrary that the component (σ_{n-1}, σ_n) of $\mathcal{I}$ is an indivisible interval, say of type ϕ.

First note that neither σ_{n-1} nor σ_n can be of polynomial type and degree 1. For if e.g. σ_{n-1} were of such kind, some transfer matrix W_{tt_0} for $t_0 \in (\sigma_{n-1}, \sigma_n)$ and $t < \sigma_{n-1}$ would equal $W_{(-l,\phi)}$ for a certain $l > 0$. Since by Lemma 3.5 $\lim_{s\nearrow\sigma_n} \mathfrak{t}(W_s) = +\infty$ by Lemma 3.5, there exists a number $t_1 \in (\sigma_{n-1}, \sigma_n)$ with $W_{t_1} = W_{t_0} W_{(l,\phi)} = W_t$, a contradiction since $\operatorname{ind}_- W_t < \operatorname{ind}_- W_{t_1}$ by Lemma 3.5. For σ_n a similar reasoning applies.

Now choose $t_0 \in (\sigma_{n-1}, \sigma_n)$ and consider the dB-Pontryagin space $\mathfrak{P}_{t_0}$. By the proof of *(iii)* of Lemma 3.15 we are in the situation of Proposition I.13.5; for $\mathfrak{P}(E_c)$ there we can choose any space $\mathfrak{P}_u$, $u \in \mathcal{I}$, $u > \sigma_n$. This is possible, since $\mathcal{I}_n \neq \mathcal{I}_\infty$. From this source we obtain that the closure of the domain of the multiplication operator $\mathfrak{P}_{t_0}^l$ in $\mathfrak{P}_{t_0}$ is a dB-subspace of $\mathfrak{P}_{t_0}$ and is contained isometrically in $\mathfrak{P}_u$. The set $\mathfrak{P}_{t_0}$ endowed with the inner product inherited by $\mathfrak{P}_u$ also is a dB-space and is denoted by $\mathfrak{P}_{t_0}^u$. Moreover, at least one of the spaces $\mathfrak{P}_{t_0}^l$ and $\mathfrak{P}_{t_0}^u$ is nondegenerated.

Consider first the case that the space $\mathfrak{P}_{t_0}^l$ is nondegenerated. Since it is a proper subspace $\mathfrak{P}_{t_0}$ we have $\mathfrak{P}_{t_0}^l = \mathfrak{P}_{t_1}$ for some $t_1 \in \mathcal{I}$, with $t_1 < \sigma_{n-1}$ by *(ii)* of Lemma 3.15. This is a contradiction to the already proved fact that σ_{n-1} is not of polynomial type and degree 1.

It remains to settle the case that $\mathfrak{P}_{t_0}^l$ is degenerated. Then $\mathfrak{P}_{t_0}^u$ is nondegenerated. The space $\mathfrak{P}_{t_0}^u$ is by definition contained isometrically in $\mathfrak{P}_u$, hence we have $\mathfrak{P}_{t_0}^u = \mathfrak{P}_{t_1}$ for some $t_1 \in \mathcal{I}$. Since both, $\mathfrak{P}_{t_0}^u$ and $\mathfrak{P}_{t_0}$, are Pontryagin spaces which contain the degenerated space $\mathfrak{P}_{t_0}^l$ as an isometric subspace with codimension 1, we obtain $\operatorname{ind}_- \mathfrak{P}_{t_0}^u = \operatorname{ind}_- \mathfrak{P}_{t_0}$, thus $t_1 \in (\sigma_{n-1}, \sigma_n)$ by Lemma 3.5. We arrive at a contradiction to *(iii)* of Lemma 3.15.

□

An important observation is that the singularities of the chain of matrix functions correspond to the degenerated dB-subspaces of the $\mathfrak{P}_t$'s.

Corollary 3.17. *Let $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$, let $\mathfrak{P}_t$, $t \in \mathcal{I}$, be the corresponding chain of dB-spaces, and consider a singularity σ of $(W_t)_{t\in\mathcal{I}}$. Then*

(i) If $\mathfrak{P}_{\sigma_-} \subsetneq \mathfrak{P} \subsetneq \mathfrak{P}_{\sigma_+}$ isometrically, then $\mathfrak{P}$ is degenerated.

(ii) The space $\mathfrak{P}_{\sigma_+}$ ($\mathfrak{P}_{\sigma_-}$) is degenerated if and only if σ is dense or right dense (dense or left dense, respectively).

(iii) The degree δ of the singularity σ is finite. There exist (degenerated) dB-subspaces $\mathfrak{P}_1,\dots,\mathfrak{P}_{\delta-1}$ of $\mathfrak{P}_{\sigma_+}$, such that

$$\mathfrak{P}_{\sigma_-} \subsetneq \mathfrak{P}_1 \subsetneq \dots \subsetneq \mathfrak{P}_{\delta-1} \subsetneq \mathfrak{P}_{\sigma_+}. \tag{3.8}$$

The codimension in each step is one.

Proof : To prove *(i)* assume on the contrary that $\mathfrak{P}_{\sigma_-} \subsetneq \mathfrak{P} \subsetneq \mathfrak{P}_{\sigma_+}$ isometrically for some nondegenerated space. By *(ii)* of Lemma 3.15 we have $\mathfrak{P} = \mathfrak{P}_s$ for some $s \in \mathcal{I}$, $s \leq \sigma_+$. Since $(\sigma,\sigma_+] \subseteq \mathcal{I}^{\text{ind}}$ we have $\mathfrak{P}_t = \mathfrak{P}_{\sigma_+}$ for all $t \in (\sigma,\sigma_+]$, which shows that in fact $s < \sigma$. Since the inclusion of $\mathfrak{P}$ in $\mathfrak{P}_{\sigma_+}$ is isometric, *(iii)* of Lemma 3.15 yields that $s \notin (\sigma_-,\sigma)$, i.e. $s \leq \sigma_-$. This contradicts $\mathfrak{P}_{\sigma_-} \subsetneq \mathfrak{P}$.

Now we prove *(ii)*. If $\sigma_+ > \sigma$, then by the considerations in Proposition 3.16 we have $\sigma_+ \in \mathcal{I}$ and the space $\mathfrak{P}_{\sigma_+}$ is nondegenerated. Consider the case $\sigma_+ = \sigma$. If $\mathfrak{P}_{\sigma_+}$ would be nondegenerated, *(ii)* of Lemma 3.15 would imply $\mathfrak{P}_{\sigma_+} = \mathfrak{P}_s$ for some $s \in \mathcal{I}$. In fact, $s < \sigma$ since by its definition $\mathfrak{P}_{\sigma_+} \subseteq \mathfrak{P}_t$ for all $t > \sigma$. But $\mathfrak{P}_t \subseteq \mathfrak{P}_{\sigma_+}$ contractively for all $t < \sigma$. Hence $\sigma \leq s$, a contradiction.

It remains to prove *(iii)*. The fact that $\delta < \infty$ follows from the already proved assertion *(i)* and Theorem I.11.6. Moreover, the corresponding fact for dB-Hilbert spaces implies the existence of spaces $\mathfrak{P}_i$ with the stated properties. Again by *(i)* they must be degenerated.

□

Note in this place that the previous result implies that for any singularity σ which is not of polynomial type with degree 1, there exists a degenerated dB-space $\mathfrak{P}$, such that

$$\mathfrak{P}_{\sigma_-} \subseteq \mathfrak{P} \subseteq \mathfrak{P}_{\sigma_+} \tag{3.9}$$

isometrically. Also the converse is true.

Lemma 3.18. *Assume that $\mathfrak{P}$ is a degenerated dB-space and $\mathfrak{P} \subseteq \mathfrak{P}_s$ isometrically for some $s \in \mathcal{I}$. Then there exists a singularity σ which is not of polynomial type with degree 1, such that (3.9) holds.*

Proof : If $t \in \mathcal{I}$, $t \leq s$, and is not contained in $\mathcal{I}^{\text{ind}}$, then either $\mathfrak{P}_t \subseteq \mathfrak{P}$ or $\mathfrak{P} \subseteq \mathfrak{P}_t$. These inclusions are isometric since $\mathfrak{P}_t$ is contained isometrically in $\mathfrak{P}_s$ by *(iii)* of Lemma 3.15. Hence there exists a number λ, such that

$$\mathfrak{P}_t \subseteq \mathfrak{P},\ t \in \mathcal{I}\setminus\mathcal{I}^{\text{ind}}, t \leq \lambda,$$

$$\mathfrak{P} \subseteq \mathfrak{P}_t,\ t \in \mathcal{I}\setminus\mathcal{I}^{\text{ind}}, t \geq \lambda.$$

Without loss of generality we may assume that $\lambda \notin \mathcal{I}^{\text{ind}}$. If $\lambda \notin \mathcal{I}$ we are done. Assume therefore that $\lambda \in \mathcal{I}$. If λ does not belong to the boundary of $\mathcal{I}^{\text{ind}}$, we can conclude from Remark 3.13 and the above relations that $\mathfrak{P} = \mathfrak{P}_\lambda$, a contradiction. Hence $\lambda = \mu_-$ or $\lambda = \mu_+$ for some $\mu \in \mathcal{I}$. If both μ_- and μ_+ are not singularities, then both $\mathfrak{P}_{\mu_-}$ and $\mathfrak{P}_{\mu_+}$ are contained isometrically in $\mathfrak{P}_s$,

$$\dim \mathfrak{P}_{\mu_+}/\mathfrak{P}_{\mu_-} = 1,$$

and $\mathfrak{P}_{\mu_-} \subseteq \mathfrak{P} \subseteq \mathfrak{P}_{\mu_+}$. Hence $\mathfrak{P} = \mathfrak{P}_{\mu_-}$ or $\mathfrak{P} = \mathfrak{P}_{\mu_+}$, a contradiction. We arrive at the conclusion that μ_+ or μ_- is a singularity, say σ. Obviously $\mathfrak{P}_{\sigma_-} \subseteq \mathfrak{P} \subseteq \mathfrak{P}_{\sigma_+}$, and the same argument as above rules out the case that σ is of polynomial type and degree 1.

□

4 A transformation of chains

In [KW3], Section 3, it is shown that for any dB-space $\mathfrak{P}$ with inner product $[.,.]$ there exist numbers $t_1, \dots, t_n \in \mathbb{R}$ and $C \in \mathbb{R}$, such that the set $\mathfrak{P}$ endowed with the inner product

$$[F,G]_1 := [F,G] + C \sum_{k=1}^{n} F(t_k)\overline{G(t_k)}, \;\; F,G \in \mathfrak{P}, \tag{4.1}$$

is a dB-Hilbert space.

In this section we will investigate some properties of the corresponding transformation of chains. First note that it suffices to consider the transformation corresponding to

$$[F,G]_1 := [F,G] + mF(0)\overline{G(0)},$$

since by repeated application of this and the transformation $\mathcal{T}^\alpha$ of [KW4], Section 10, the transformation corresponding to (4.1) can be generated.

Definition 4.1. Let $W \in \mathcal{M}_\kappa$, $W(0) = 1$, and a number $m \in \mathbb{R} \setminus \{0\}$ be given. Write

$$W(z) = \begin{pmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{pmatrix},$$

and define

$$\alpha(W) := 1 - mw_{21}'(0), \;\; \beta(W) := m\frac{w_{21}''(0)}{2} + mw_{21}'(0)w_{11}'(0) - 2w_{11}'(0). \tag{4.2}$$

If $\alpha(W) \neq 0$, a transformation $\mathcal{T}_m(W)$ is defined by

$$\mathcal{T}_m(W)(z) := \begin{pmatrix} 1 & -\frac{m}{z} \\ 0 & 1 \end{pmatrix} W(z) \begin{pmatrix} \frac{1}{\alpha(W)} & m(\frac{\beta(W)}{\alpha(W)} + \frac{1}{z}) \\ 0 & \alpha(W) \end{pmatrix}. \tag{4.3}$$

If it is clear from the context to which matrix W the transformation is applied, we shall drop the argument W and write α (β) instead of $\alpha(W)$ ($\beta(W)$).

For later use we compute the entries of $\mathcal{T}_m(W)$ explicitly. Put

$$\mathcal{T}_m(W)(z) =: \begin{pmatrix} \tilde{w}_{11}(z) & \tilde{w}_{12}(z) \\ \tilde{w}_{21}(z) & \tilde{w}_{22}(z) \end{pmatrix},$$

then

$$\tilde{w}_{11}(z) = \frac{1}{\alpha}(w_{11}(z) - \frac{m}{z}w_{21}(z)), \tag{4.4}$$

$$\tilde{w}_{21}(z) = \frac{1}{\alpha}w_{21}(z), \tag{4.5}$$

$$\tilde{w}_{12}(z) = (w_{11}(z) - \frac{m}{z}w_{21}(z))m(\frac{\beta}{\alpha} + \frac{1}{z}) + \alpha(w_{12}(z) - \frac{m}{z}w_{22}(z)), \tag{4.6}$$

$$\tilde{w}_{22}(z) = \alpha w_{22}(z) + m(\frac{\beta}{\alpha} + \frac{1}{z})w_{21}(z). \tag{4.7}$$

Some basic properties of $\mathcal{T}_m$ will follow from the next lemma.

Lemma 4.2. *Assume that $M(z) \in \mathcal{M}_\kappa$, $M_{21}'(0) \neq 0$. Let $\chi, \lambda, u, \mu \in \mathbb{R} \setminus \{0\}$, $v, \nu \in \mathbb{R}$, an consider*

$$\tilde{M}(z) := \begin{pmatrix} \chi & -v - \frac{u}{z} \\ 0 & \frac{1}{\chi} \end{pmatrix} M(z) \begin{pmatrix} \frac{1}{\lambda} & \nu + \frac{\mu}{z} \\ 0 & \lambda \end{pmatrix}.$$

The matrix $\tilde{M}(z)$ is entire and satisfies $\tilde{M}(0) = 1$ if and only if

1. $\chi = \lambda + \mu M_{21}'(0)$
2. $u = \mu$
3. $v = \frac{1}{\lambda + \mu M_{21}'(0)} \left[\nu\lambda + \mu\big(2\lambda M_{11}'(0) - \mu\frac{M_{21}''(0)}{2} + \mu M_{11}'(0)M_{21}'(0)\big)\right]$

In this case $\tilde{M}(z) \in \mathcal{M}_{\tilde{\kappa}}$ with

$$\tilde{\kappa} = \kappa + \begin{cases} 1 & , \operatorname{sgn}\lambda \neq \operatorname{sgn}\chi, \operatorname{sgn}\mu = \operatorname{sgn}\lambda \\ 0 & , \operatorname{sgn}\lambda = \operatorname{sgn}\chi \\ -1 & , \operatorname{sgn}\lambda \neq \operatorname{sgn}\chi, \operatorname{sgn}\mu = \operatorname{sgn}\chi \end{cases}. \tag{4.8}$$

Proof : The entries of $\tilde{M}(z)$ are given by

$$\tilde{M}_{11}(z) = \frac{\chi}{\lambda}M_{11}(z) - \frac{v + \frac{u}{z}}{\lambda}M_{21}(z),$$

$$\tilde{M}_{21}(z) = \frac{1}{\lambda\chi}M_{21}(z),$$

$$\tilde{M}_{12}(z) = \chi(\nu+\frac{\mu}{z})M_{11}(z)+\chi\lambda M_{12}(z)-(v+\frac{u}{z})\lambda M_{22}(z)--(v+\frac{u}{z})(\nu+\frac{\mu}{z})M_{21}(z),$$

$$\tilde{M}_{22}(z) = \frac{\nu + \frac{\mu}{z}}{\chi}M_{21}(z) + \frac{\lambda}{\chi}M_{22}(z).$$

Clearly $\tilde{M}_{11}, \tilde{M}_{21}$ and $\tilde{M}_{22}$ are entire. Moreover, $\tilde{M}_{21}(0) = 0$. We have

$$\tilde{M}_{11}(0) = \frac{\chi}{\lambda} - \frac{u}{\lambda}M_{21}'(0),\ \tilde{M}_{22}(0) = \frac{\lambda}{\chi} + \frac{\mu}{\chi}M_{21}'(0).$$

Let $\tilde{M}_{12}(z)=\sum_{k\geq -1} c_k z^k$ be the Laurent expansion of $\tilde{M}_{12}(z)$ at 0. Then

$$c_{-1}=\chi\mu-\lambda u-u\mu M_{21}'(0),$$

$$c_0=\chi\nu-\lambda v+\chi\mu M_{11}'(0)-\lambda u M_{22}'(0)-(v\mu+\nu u)M_{21}'(0)-u\mu\frac{M_{21}''(0)}{2}.$$

The following sets of equations are equivalent:

(I) $\tilde{M}_{11}(0)=1,\tilde{M}_{22}(0)=1,c_{-1}=0,c_0=0.$

(II) $\chi=\lambda+uM_{21}'(0),\chi=\lambda+\mu M_{21}'(0),c_{-1}=0,c_0=0.$

(III) $\chi=\lambda+uM_{21}'(0),u=\mu,c_0=0.$

(IV) $$\begin{array}{c}\chi=\lambda+\mu M_{21}'(0),u=\mu,\\ (\chi\nu-\lambda v)+\mu(\chi+\lambda)M_{11}'(0)-(v+\nu)\mu M_{21}'(0)-\mu^2\frac{M_{21}''(0)}{2}=0\end{array}.$$

The left hand side of the last equation of (IV) equals to

$$\nu(\chi-\mu M_{21}'(0))-v(\lambda+\mu M_{21}'(0))+\mu(2\lambda+\mu M_{21}'(0))M_{11}'(0)-\mu^2\frac{M_{21}''(0)}{2}=$$

$$\nu\lambda-v\chi+\mu\left(2\lambda M_{11}'(0)+\mu M_{21}'(0)M_{11}'(0)-\mu\frac{M_{21}''(0)}{2}\right).$$

Thus the first assertion of the lemma follows. It remains to prove the relation (4.8) between $\tilde{\kappa}$ and κ. Consider the function $\tilde{M}(z)\circ\tau$ with $\tau\in\mathbb{R}\cup\{\infty\}$. Write

$$\begin{pmatrix}\frac{1}{\lambda} & \nu+\frac{\mu}{z}\\ 0 & \lambda\end{pmatrix}\circ\tau=W_{(l,\phi)}\circ\infty$$

with $\cot\phi=\frac{1}{\lambda}(\frac{\tau}{\lambda}+\nu)$ and $l=-\frac{\lambda^2+(\frac{\tau}{\lambda}+\nu)^2}{\lambda\mu}$. Then

$$\tilde{M}(z)\circ\tau=\begin{pmatrix}\chi & -v-\frac{u}{z}\\ 0 & \frac{1}{\chi}\end{pmatrix}M(z)W_{(l,\phi)}(z)\circ\infty=$$

$$=\chi^2(M(z)W_{(l,\phi)}(z))\circ\infty-\chi v-\frac{\chi u}{z}$$

For all τ with one possible exception we have

$$\operatorname{ind}_-\tilde{M}=\operatorname{ind}_-(\tilde{M}(z)\circ\tau). \tag{4.9}$$

Since $\tilde{M}$ is entire and $\tilde{M}(0)=1$ the function $\tilde{M}(z)\circ\tau$ is analytic at 0 if $\tau\neq\infty$, hence

$$\operatorname{ind}_-\chi^2(M(z)W_{(l,\phi)}(z))\circ\infty-\chi v-\frac{\chi u}{z}=$$

$$=\operatorname{ind}_-M(z)W_{(l,\phi)}(z)\circ\infty+\left\{\begin{array}{cl}0 & ,\ (MW_{(l,\phi)})_{21}'(0)\leq 0\\ -1 & ,\ (MW_{(l,\phi)})_{21}'(0)>0\end{array}\right.. \tag{4.10}$$

For all τ with two possible exceptions we have

$$\mathrm{ind}_- M(z)W_{(l,\phi)}(z)\circ\infty = \mathrm{ind}_- M + \mathrm{ind}_- W_{(l,\phi)} =$$

$$= \mathrm{ind}_- M + \begin{cases} 1 \ , \ \lambda\mu > 0 \\ 0 \ , \ \lambda\mu < 0 \end{cases} . \tag{4.11}$$

Finally we compute

$$(MW_{(l,\phi)})'_{21}(0) = M'_{21}(0) + W'_{(l,\phi),21}(0) =$$

$$= M'_{21}(0) + \frac{\lambda}{\mu} = \frac{\mu M'_{21}(0)+\lambda}{\mu} = \frac{\chi}{\mu}. \tag{4.12}$$

Putting together the relations (4.9)-(4.12) we obtain (with an appropriate choice of τ)

$$\mathrm{ind}_- \tilde{M} = \mathrm{ind}_- M + \begin{cases} 1 \ , \ \lambda\mu > 0 \\ 0 \ , \ \lambda\mu < 0 \end{cases} + \begin{cases} 0 \ , \ \frac{\chi}{\mu} < 0 \\ -1 \ , \ \frac{\chi}{\mu} > 0 \end{cases} ,$$

which implies (4.8).

□

Corollary 4.3. *The matrix $\mathcal{T}_m(W)(z)$ is entire and has value* 1 *(identity matrix) at* 0*. It belongs to the class $\mathcal{M}_{\kappa'}$ with*

$$\kappa' := \kappa + \begin{cases} 1 \ , \ [m<0,\ \alpha<0] \\ 0 \ , \ [m<0,\ \alpha>0] \ \text{or} \ [m>0,\ \alpha>0] \\ -1 \ , \ [m>0,\ \alpha<0] \end{cases} .$$

We will apply the transformation $\mathcal{T}_m$ to a given maximal chain of matrix functions $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$. First of all note that the parameter $\alpha(W_t)$ is a continuous and monotone function of t on each component of $\mathcal{I}$. In fact, $\alpha(W_t)$ is increasing if $m>0$ and decreasing otherwise.

Theorem 4.4. *Let $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$ and let $m \in \mathbb{R}\setminus\{0\}$. Assume that $q_\infty((W_t)_{t\in\mathcal{I}})$ is not identically equal to ∞. Consider the chain by $(\mathcal{T}_m(W_t))_{t\in\mathcal{J}}$, $\mathcal{J} := \{t \in \mathcal{I} : W_t \in \operatorname{dom}\mathcal{T}_m\}$ continued to the right by an indivisible interval of type* 0 *and infinite length in case $\lim_{t\nearrow\sup\mathcal{J}} \mathfrak{t}(\mathcal{T}_m(W_t)) < \infty$. This chain can be completed to obtain a maximal chain of matrix functions, denoted by $\mathcal{T}_m((W_t)_{t\in\mathcal{I}})$. We have*

$$q_\infty(\mathcal{T}_m((W_t)_{t\in\mathcal{I}}))(z) = q_\infty((W_t)_{t\in\mathcal{I}})(z) - \frac{m}{z}. \tag{4.13}$$

If $t\in\mathcal{J}$, the spaces $\mathfrak{P}(E_{W_t})$ and $\mathfrak{P}(E_{\mathcal{T}_m(W_t)})$ coincide as sets. Their inner products $[.,.]$ and $[.,.]_1$ are connected by

$$[F,G]_1 = [F,G] + mF(0)\overline{G(0)}. \tag{4.14}$$

The proof of this result will be split into several lemmata.

Lemma 4.5. *Let $t_1, t_2 \in \mathcal{I}$, $t_1 \leq t_2$, and assume that $\alpha(W_{t_1}), \alpha(W_{t_2}) \neq 0$. Then*

$$\mathcal{T}_m(W_{t_1})^{-1}\mathcal{T}_m(W_{t_2}) \in \mathcal{M}_\nu,$$

with $\nu = \operatorname{ind}_-\mathcal{T}_m(W_{t_2}) - \operatorname{ind}_-\mathcal{T}_m(W_{t_1})$. In particular, $\operatorname{ind}_-\mathcal{T}_m(W_{t_2}) \geq \operatorname{ind}_-\mathcal{T}_m(W_{t_1})$.

Proof : We first reduce the problem to a special structure of the considered chain. For $\delta > 0$ sufficiently small, define a perturbed chain $\tilde{W}_t$ by

$$\tilde{W}_t := \begin{cases} W_t^\circ & , 0 < t \leq \delta \\ W_\delta^\circ W_{t-\delta} & , \delta \leq t \leq t_1 - \delta \\ W_\delta^\circ W_{t_1-2\delta} W_{t-t_1+\delta}^\circ & , t_1 - \delta \leq t \leq t_1 \\ W_\delta^\circ W_{t_1-2\delta} W_\delta^\circ W_{t_1 t} & , t_1 \leq t \end{cases},$$

where t is such that the above expressions exist and where W_t° denotes the matrix function

$$W_t^\circ(z) := \begin{pmatrix} \cos(tz) & \sin(tz) \\ -\sin(tz) & \cos(tz) \end{pmatrix}. \tag{4.15}$$

It follows from Lemma II.5.17, Lemma 3.7 and the discussion after it that $(\tilde{W}_t)$ is a maximal chain of matrix functions. Moreover, we clearly have

$$\operatorname{ind}_-\tilde{W}_{t_1} \leq \operatorname{ind}_-W_{t_1},\ \operatorname{ind}_-\tilde{W}_{t_2} \leq \operatorname{ind}_-W_{t_2}.$$

Since

$$\lim_{\delta \searrow 0} \tilde{W}_{t_1} = W_{t_1},\ \lim_{\delta \searrow 0} \tilde{W}_{t_2} = W_{t_2},$$

locally uniformly on $\mathbb{C}$, we conclude that for sufficiently small δ in the above inequalities in fact equality holds. We also conclude that

$$\lim_{\delta \searrow 0} \alpha(\tilde{W}_{t_1}) = \alpha(W_{t_1}),\ \lim_{\delta \searrow 0} \alpha(\tilde{W}_{t_2}) = \alpha(W_{t_2}),$$

and the respective relations for β hold. Hence, again for sufficiently small values of δ, we obtain $\alpha(\tilde{W}_{t_1}), \alpha(\tilde{W}_{t_2}) \neq 0$ and by Corollary 4.3 that

$$\operatorname{ind}_-\mathcal{T}_m(\tilde{W}_{t_1}) = \operatorname{ind}_-\mathcal{T}_m(W_{t_1}),\ \operatorname{ind}_-\mathcal{T}_m(\tilde{W}_{t_2}) = \operatorname{ind}_-\mathcal{T}_m(W_{t_2}).$$

Moreover,

$$\mathcal{T}_m(\tilde{W}_{t_1})^{-1}\mathcal{T}_m(\tilde{W}_{t_2}) = \tag{4.16}$$

$$= \begin{pmatrix} \alpha(\tilde{W}_{t_1}) & -m(\frac{\beta(\tilde{W}_{t_1})}{\alpha(\tilde{W}_{t_1})} + \frac{1}{z}) \\ 0 & \frac{1}{\alpha(\tilde{W}_{t_1})} \end{pmatrix} W_{t_1 t_2} \begin{pmatrix} \frac{1}{\alpha(\tilde{W}_{t_2})} & m(\frac{\beta(\tilde{W}_{t_2})}{\alpha(\tilde{W}_{t_2})} + \frac{1}{z}) \\ 0 & \alpha(\tilde{W}_{t_2}) \end{pmatrix},$$

and

$$\mathcal{T}_m(W_{t_1})^{-1}\mathcal{T}_m(W_{t_2}) = \tag{4.17}$$

$$= \begin{pmatrix} \alpha(W_{t_1}) & -m(\frac{\beta(W_{t_1})}{\alpha(W_{t_1})} + \frac{1}{z}) \\ 0 & \frac{1}{\alpha(W_{t_1})} \end{pmatrix} W_{t_1 t_2} \begin{pmatrix} \frac{1}{\alpha(W_{t_2})} & m(\frac{\beta(W_{t_2})}{\alpha(W_{t_2})} + \frac{1}{z}) \\ 0 & \alpha(W_{t_2}) \end{pmatrix},$$

Since for sufficiently small $\delta > 0$, the signs of $\alpha(W_{t_1})$ and of $\alpha(\tilde{W}_{t_1})$ (of $\alpha(W_{t_2})$ and of $\alpha(\tilde{W}_{t_2})$) coincide, we obtain from Lemma 4.2 that $\mathcal{T}_m(W_{t_1})^{-1}\mathcal{T}_m(W_{t_2})$ and $\mathcal{T}_m(\tilde{W}_{t_1})^{-1}\mathcal{T}_m(\tilde{W}_{t_2})$ belong to the same class $\mathcal{M}_\nu$. Hence it suffices to prove the assertion of the lemma for the chain $(\tilde{W}_t)$.

The chain $(\tilde{W}_t)$ has some pleasant properties:

(i) $\mathfrak{K}_-(\tilde{W}_t) = \mathfrak{K}(\tilde{W}_t)$.

(ii) The domain of the multiplication operator in the space $\mathfrak{P}(E_{\tilde{W}_{t_1}})$ is dense.

The property *(i)* follows immediately, since by Theorem II.5.7 it does not depend on the choice of t and holds for $t = \delta$. To see *(ii)* consider the spaces

$$\mathfrak{P}' := \mathfrak{P}(E_{\tilde{W}_{t_1 - \frac{\delta}{2}}}),\ \mathfrak{P}'' := \mathfrak{P}(E_{\tilde{W}_{t_1}}).$$

Since

$$(0,1)\tilde{W}_{t_1} = (0,1)\tilde{W}_{t_1 - \frac{\delta}{2}} W^\circ_{\frac{\delta}{2}},$$

it is a consequence of Theorem I.12.2 that the space $\mathfrak{P}'$ is contained isometrically in $\mathfrak{P}''$. Now assume on the contrary that the domain of the multiplication operator is not dense. Then Lemma II.7.3 implies that for some number $\phi \in [0, \pi)$

$$W^\circ_{\frac{\delta}{2}} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \in \mathfrak{K}(W^\circ_{\frac{\delta}{2}}).$$

This contradicts the fact that the domain of the multiplication operator is dense in the space $\mathfrak{P}(E_{W^\circ_{\frac{\delta}{2}}})$ (cf. Corollary I.6.3).

The matrices $\mathcal{T}_m(\tilde{W}_t)$ inherit the properties *(i)* and *(ii)*. In fact, Theorem II.5.7 yields *(i)* since $\tilde{W}_t \circ \infty$ and

$$\mathcal{T}(\tilde{W}_t) \circ \infty = \tilde{W}_t \circ \infty - \frac{m}{z}$$

are together regular at ∞ or not. The property *(ii)* follows similar, since by Lemma II.10.4 the fact that the domain of the multiplication operator is dense in $\mathfrak{P}(E_{\tilde{W}_{t_1}})$ is equivalent to the fact that the function $\frac{\tilde{w}_{t_1,22}}{\tilde{w}_{t_1,21}}$ is regular but not finite at ∞. Since

$$\frac{\mathcal{T}_m(\tilde{W}_{t_1})_{22}}{\mathcal{T}_m(\tilde{W}_{t_1})_{21}} = \alpha^2 \frac{\tilde{w}_{t_1,22}}{\tilde{w}_{t_1,21}} + \alpha m(\frac{\beta}{\alpha} + \frac{1}{z}),$$

this is equivalent to the corresponding fact for $\mathfrak{P}(E_{\mathcal{T}_m(\tilde{W}_{t_1})})$.

Since the matrix $\mathcal{T}_m(W_{t_1})^{-1}\mathcal{T}_m(W_{t_2})$ belongs a priori to some class $\mathcal{M}_\nu$ (cf. Lemma II.5.10), the properties *(i)* and *(ii)* allow us to apply Lemma II.5.17, and we conclude that $\nu = \operatorname{ind}_- \mathcal{T}_m(W_{t_2}) - \operatorname{ind}_- \mathcal{T}_m(W_{t_1})$.

□

Remark 4.6. The previous proof is based on the same reasoning as some proofs in [KW4], Section 10. Unfortunately in the proofs of that Section a little mistake occurs at several places. In fact, when for a certain chain a perturbed chain was introduced - in the above proof this is $(\tilde{W}_t)$ - we stated in [KW4] that the transfer matrices of the transformed chain of the original chain and of the transformed chain of the perturbed chain coincide. As can be seen in (4.16) and (4.17) this is not correct. However, using results as in Lemma 4.2 we can conclude that the negative squares of the respective transfer matrices coincide, and that is all what is needed.

For later reference let us record the following simple fact:

Lemma 4.7. *Let $W \in \mathcal{M}_\kappa$, $l \in \mathbb{R}\setminus\{0\}$, $\phi \in [0,\pi)$ and $m \in \mathbb{R}\setminus\{0\}$. Assume that both, W and $W_1 := WW_{(l,\phi)}$, belong to* $\operatorname{dom}\mathcal{T}_m$. *Let $\alpha := \alpha(W)$ and $\beta := \beta(W)$ be as in (4.2). Then*

$$\mathcal{T}_m(W)^{-1}\mathcal{T}_m(W_1) = W_{(\tilde{l},\tilde{\phi})}.$$

For $\phi \neq 0$ we have

$$\cot\tilde{\phi} = \alpha^2 \cot\phi - \beta m,$$

$$\tilde{l} = \frac{l}{\alpha(\alpha + ml\sin^2\phi)}\frac{\sin^2\phi}{\sin^2\tilde{\phi}}.$$

If $\phi = 0$, then also $\tilde{\phi} = 0$, and $\tilde{l} = \alpha^2 l$.

Proof : From the definition of the transformation $\mathcal{T}_m$ we obtain

$$\mathcal{T}_m(W)^{-1}\mathcal{T}_m(W_1) = \begin{pmatrix} \alpha & -m(\frac{\beta}{\alpha} + \frac{1}{z}) \\ 0 & \frac{1}{\alpha} \end{pmatrix} \cdot W^{-1}W_1 \cdot \begin{pmatrix} \frac{1}{\alpha(W_1)} & m(\frac{\beta(W_1)}{\alpha(W_1)} + \frac{1}{z}) \\ 0 & \alpha(W_1) \end{pmatrix}.$$

Since $W^{-1}W_1 = W_{(l,\phi)}$ is a linear polynomial, so is $\mathcal{T}_m(W)^{-1}\mathcal{T}_m(W_1)$. Hence this matrix must be of the form $W_{(\tilde{l},\tilde{\phi})}$ for some $\tilde{l} \in \mathbb{R}\setminus\{0\}$ and $\tilde{\phi} \in [0,\pi)$. The left upper and left lower entries v_{11} and v_{21}, respectively, are computed as follows

$$v_{11}(z) = 1 - z[\frac{\alpha}{\alpha(W_1)} l\sin\phi\cos\phi - \beta(\frac{1}{\alpha} - \frac{1}{\alpha(W_1)})],$$

$$v_{21}(z) = -z\frac{l\sin^2\phi}{\alpha(W_1)\alpha}.$$

Using the relation

$$\alpha(W_1) = \alpha + ml\sin^2\phi,$$

the assertion follows in the case that ϕ, and hence $\tilde{\phi}$, is not equal to 0. Now assume that $\phi = 0$. Then the right upper entry v_{12} computes as

$$v_{12} = m(\beta(W_1) - \beta) + \alpha^2 l z.$$

We conclude that $\beta(W_1) = \beta$ and $\tilde{l} = \alpha^2 l$.

□

Lemma 4.8. *Let $\langle \mathfrak{P}, [.,.] \rangle$ be a dB-Pontryagin space and denote by $K(w,z)$ its reproducing kernel. Consider the new inner product ($t_0 \in \mathbb{R}$, $m \in \mathbb{R} \setminus \{0\}$)*

$$[F,G]_1 := [F,G] + mF(t_0)\overline{G(t_0)}, \ \ F,G \in \mathfrak{P}.$$

Assume that $1 + mK(t_0,t_0) \neq 0$. Then $\langle \mathfrak{P}, [.,.]_1 \rangle$ is nondegenerated and its reproducing kernel $\tilde{K}(w,z)$ is given by

$$\tilde{K}(w,z) = K(w,z) - \frac{m}{1 + mK(t_0,t_0)} K(w,t_0)K(t_0,z). \tag{4.18}$$

If, on the other hand, $1 + mK(t_0,t_0) = 0$, then $\langle \mathfrak{P}, [.,.]_1 \rangle$ is degenerated. In fact

$$\langle \mathfrak{P}, [.,.]_1 \rangle^\circ = \operatorname{span}\{K(t_0,z)\}.$$

Proof : Let $1 + mK(t_0,t_0) \neq 0$. We start with the function $\tilde{K}(w,z)$ defined in (4.18) and show that it has the reproducing kernel property with respect to the inner product $[.,.]_1$. For any $F \in \mathfrak{P}$ a short computation shows that

$$[F(z), \tilde{K}(w,z)]_1 = [F(z), K(w,z)] - \frac{m}{1 + mK(t_0,t_0)} \overline{K(w,t_0)} F(t_0) +$$

$$+ mF(t_0)\overline{K(w,t_0)} - mF(t_0) \frac{m}{1 + mK(t_0,t_0)} \overline{K(w,t_0)} K(t_0,t_0) = F(w).$$

In particular, $\langle \mathfrak{P}, [.,.]_1 \rangle$ is nondegenerated.

Consider the case that $1 + mK(t_0,t_0) = 0$. Then for all $F \in \mathfrak{P}$ we have

$$[F, K(t_0,.)]_1 = [F, K(t_0,.)] + mF(t_0)K(t_0,t_0) =$$

$$= F(t_0)(1 + mK(t_0,t_0)) = 0,$$

i.e. $K(t_0,.) \in \langle \mathfrak{P}, [.,.]_1 \rangle^\circ$. Clearly, the isotropic part of $\langle \mathfrak{P}, [.,.]_1 \rangle$ is at most one-dimensional.

□

Corollary 4.9. *Let* $\mathfrak{P} = \mathfrak{P}(A - iB)$, $A(0) = 0$, $B(0) = 1$, *and* $m \in \mathbb{R} \setminus \{0\}$ *be given such that* $1 - mA'(0) \neq 0$. *If we put*

$$[F,G]_1 := [F,G] + mF(0)\overline{G(0)},\ F,G \in \mathfrak{P},$$

then $\langle \mathfrak{P}, [.,.]_1 \rangle = \mathfrak{P}(\tilde{A} - i\tilde{B})$ *with*

$$\tilde{A}(z) := \frac{1}{(1 - mA'(0))} A(z),\ \tilde{B}(z) := (1 - mA'(0))B(z) + \frac{m}{z} A(z). \tag{4.19}$$

Proof : We apply Lemma 4.8 with $t_0 = 0$. Since $A(0) = 0$ and $B(0) = 1$, we obtain

$$K(0,z) = -\frac{A(z)}{z},\ K(w,0) = -\overline{\left(\frac{A(w)}{w}\right)}.$$

Hence, substituting this into (4.18), we obtain by a short computation

$$\tilde{K}(w,z) = \frac{B(z)\overline{A(w)} - A(z)\overline{B(w)}}{z - \overline{w}} - \frac{m}{1 - mA'(0)} \overline{\left(\frac{A(w)}{w}\right)} \left(\frac{A(z)}{z}\right) =$$

$$\frac{[B(z) + \frac{m}{1-mA'(0)}\frac{A(z)}{z}]\overline{A(w)} - A(z)\overline{[B(w) + \frac{m}{1-mA'(0)}\frac{A(w)}{w}]}}{z - \overline{w}}.$$

Since for some ν we have $\tilde{A} - i\tilde{B} \in \mathcal{HB}_\nu$ (cf. [KW3]) for the functions $\tilde{A}$ and $\tilde{B}$ defined in (4.19), the assertion follows.

□

Proof (of Theorem 4.4): We check the conditions of Lemma 3.7. The validity of **(W1')** for the chain $(\mathcal{T}_m(W_t))_{t\in\mathcal{J}}$ is trivial, the condition **(W2')** follows from Corollary 4.3, **(W3')** from Lemma 4.5.

Consider first the case

$$\limsup_{t\nearrow\sup\mathcal{J}} \mathfrak{t}(\mathcal{T}_m(W_t)) = +\infty. \tag{4.20}$$

By **(W5)** the last component of $\mathcal{I}$ and $\mathcal{J}$ have non-void intersection. If we can find numbers $s < t$ in this intersection such that W_{st} is not a linear polynomial, then Lemma 4.7 implies the validity of **(W4')**, and (4.13) is seen from the definition (4.3) of $\mathcal{T}_m$ by taking $\tau^t = \infty$ in the definition (3.3) of q_∞.

If we cannot find such numbers $s < t$, then the only possibility for the last component $\mathcal{I}_\infty$ of $\mathcal{I}$ to look like, is that $\mathcal{I}_\infty = (t_1, t_2] \cup [t_2, \infty)$, where $W_{t_2 s}$ is a linear polynomial of type zero for $s > t_2$ and W_{st_2} is a linear polynomial of non-zero type for $t_1 < s < t_2$, and where $\alpha(W_{t_2}) = 0$.

By (4.20) the limit $\lim_{t\nearrow t_2} \mathcal{T}_m(W_t) \circ \infty$ exists, and a calculation shows that this limit coincides with $q_\infty((W_t)_{t\in\mathcal{I}}) - \frac{m}{z}$.

Take some fixed $s_0,\ t_1 < s_0 < t_2$. Then $s_0 \in \mathcal{J}$, and by Lemma 4.7 there is a real number τ such that $\mathcal{T}_m(W_{s_0})\circ\tau = q_\infty((W_t)_{t\in\mathcal{I}}) - \frac{m}{z}$. Comparing the respective negative squares we obtain from Corollary 4.3 that $\mathcal{T}_m(W_{s_0}) \in \mathcal{M}_\nu, q_\infty((W_t)_{t\in\mathcal{I}}) - \frac{m}{z} \in \mathcal{N}_\nu$ with $\nu \in \mathbb{N}\cup\{0\}$. By Corollary II.11.1 and by Proposition 3.6 the chain $(\mathcal{T}_m(W_t))_{t\in\mathcal{J}}$ can be completed to a maximal chain.

Now assume that (4.20) is not satisfied. In particular, $\lim_{t\nearrow\sup\mathcal{I}}\alpha(W_t) \neq 0$, hence either $\alpha(W_t) \neq 0$ on all of the last component of $\mathcal{I}$, or if t_0 denotes the largest zero of $\alpha(W_t)$, then there are $s_2 > s_1 > t_0$ such that $W_{s_1 s_2}$ is not a linear polynomial of zero typ. Thus by Lemma 4.7, when adding an indivisible interval of type 0 and infinite length to $(\mathcal{T}_m(W_t))_{t\in\mathcal{J}}$ the conditions **(W3')** and **(W4')** remain valid for the prolonged chain, and we also may apply Lemma 3.7.

The connection (4.14) of the inner products of $\mathfrak{P}(E_{W_t})$ and $\mathfrak{P}(E_{\mathcal{T}_m(W_t)})$ follows from the relations (4.5) and (4.7) together with Corollary 4.9 and Corollary I.6.2.

□

The transformations $\mathcal{T}_m$ and $\mathcal{T}_{-m}$ are inverses of each other.

Lemma 4.10. *Let $m \in \mathbb{R}$. Then $\mathcal{T}_m(\operatorname{dom}\mathcal{T}_m) = \operatorname{dom}\mathcal{T}_{-m}$ and $\mathcal{T}_{-m}(\mathcal{T}_m(W)) = W$, $W \in \operatorname{dom}\mathcal{T}_m$.*

Proof : Let $W \in \operatorname{dom}\mathcal{T}_m$, i.e. $\alpha(W) = 1 - mw_{21}'(0) \neq 0$. By (4.5) we have $(\tilde{W} = \mathcal{T}_m(W))$

$$1 - (-m)\tilde{w}_{21}'(0) = 1 + m\frac{w_{21}'(0)}{1 - mw_{21}'(0)} = \frac{1}{1 - mw_{21}'(0)}, \tag{4.21}$$

which is non-zero. Hence $\mathcal{T}_m(W) \in \operatorname{dom}(\mathcal{T}_{-m})$. We apply $\mathcal{T}_{-m}$ to $\tilde{W} = \mathcal{T}_m(W)$:

$$\mathcal{T}_{-m}(\tilde{W})(z) = \begin{pmatrix} 1 & -\frac{-m}{z} \\ 0 & 1 \end{pmatrix} \tilde{W}(z) \begin{pmatrix} \frac{1}{\alpha(\tilde{W})} & m(\frac{\beta(\tilde{W})}{\alpha(\tilde{W})} + \frac{1}{z}) \\ 0 & \alpha(\tilde{W}) \end{pmatrix} =$$

$$= W(z) \begin{pmatrix} 1 & \frac{1}{\alpha}m(\frac{\beta(\tilde{W})}{\alpha(\tilde{W})} - \frac{\beta(W)}{\alpha(W)}) \\ 0 & 1 \end{pmatrix}.$$

Here we have used that $\alpha(\tilde{W}) = \frac{1}{\alpha(W)}$. Substituting $z = 0$ we conclude that $\frac{\beta(\tilde{W})}{\alpha(\tilde{W})} - \frac{\beta(W)}{\alpha(W)} = 0$, hence $\mathcal{T}_{-m}(\tilde{W}) = W$.

□

Let $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$. By virtue of Theorem 4.4 the chain $(\mathcal{T}_m(W_t))_{t\in\mathcal{J}}$, $\mathcal{J} = \{t \in \mathcal{I} : W_t \in \operatorname{dom}\mathcal{T}_m\}$, can be extended to a maximal chain. In fact it is almost maximal. In order to formulate this fact more accurately we introduce the following notation.

Definition 4.11. Let $(W_t)_{t\in\mathcal{J}}$ be a chain of matrix functions satisfying the conditions of Lemma 3.7 and let $s \in \overline{\mathcal{J}}$. Denote by $(W_t)_{t\in\mathcal{I}}$ a chain belonging to

$\mathfrak{M}$ which extends $(W_t)_{t\in\mathcal{J}}$. We say that $(W_t)_{t\in\mathcal{J}}$ is locally maximal at s if there exists an interval $[s_-,s_+]$, $s_-,s_+\in\mathcal{J}$, which contains s in its interior, such that $[s_-,s_+]\cap\mathcal{J}=[s_-,s_+]\cap\mathcal{I}$.

The following reformulation of the fact that a chain is locally maximal at s is immediate from the axiom **(W4)**.

Remark 4.12. The chain $(W_t)_{t\in\mathcal{J}}$ is locally maximal at s if and only if there exist $s_-,s_+\in\mathcal{J}$, $s_-<s<s_+$, such that:

(W4") If $t\in\mathcal{J}$ and for some $W\in\mathcal{M}_\nu$, $\nu\leq\kappa(t)$, we have $W^{-1}W_t\in\mathcal{M}_{\kappa(t)-\nu}$, then either

(a) $\operatorname{ind}_-W_{s_+}\leq\nu$ and $W_{s_+}^{-1}W\in\mathcal{M}_{\nu-\operatorname{ind}_-W_{s_+}}$, or

(b) $\nu\leq\operatorname{ind}_-W_{s_-}$ and $W^{-1}W_{s_-}\in\mathcal{M}_{\operatorname{ind}_-W_{s_-}-\nu}$, or

(c) there exists a number $t_0\in\mathcal{J}$ such that $W=W_{t_0}$.

Proposition 4.13. *The chain $(\mathcal{T}_m(W_t))_{t\in\mathcal{J}}$ is locally maximal at every point $s\in\mathcal{J}$.*

Proof : Since $\mathcal{J}$ is open, there exist numbers $s_-,s_+\in\mathcal{J}$, such that $[s_-,s_+]\subseteq\mathcal{J}$. Since thus also $[s_-,s_+]\subseteq\mathcal{I}$, we have

$$\operatorname{ind}_-W_{s_-}=\operatorname{ind}_-W_{s_+}.$$

Moreover, $\alpha(W_t)$ has no zeros in $[s_-,s_+]$ and therefore retains its sign on this interval. It follows that also

$$\operatorname{ind}_-\mathcal{T}_m(W_{s_-})=\operatorname{ind}_-\mathcal{T}_m(W_{s_+}),$$

and in turn that $\alpha(\mathcal{T}_m(W_t))$ has the same sign for $t=s_-$ and $t=s_+$.

Denote by $(\tilde{W}_t)_{t\in\tilde{\mathcal{I}}}\in\mathfrak{M}$ an extension of $(\mathcal{T}_m(W_t))_{t\in\mathcal{J}}$, and let $t_-,t_+\in\tilde{\mathcal{I}}$ correspond to s_- and s_+, respectively, i.e. let

$$\mathcal{T}_m(W_{s_-})=\tilde{W}_{t_-},\ \mathcal{T}_m(W_{s_+})=\tilde{W}_{t_+}.$$

Assume that $t_0\in[t_-,t_+]$. Since $\alpha(\tilde{W}_t)$ is a monotone function of t, $\alpha(\tilde{W}_{t_0})\neq 0$, i.e. $\tilde{W}_{t_0}\in\operatorname{dom}\mathcal{T}_{-m}$. From Lemma 4.5 we obtain

$$W_{s_-}^{-1}\mathcal{T}_{-m}(\tilde{W}_{t_0}),\ \mathcal{T}_{-m}(\tilde{W}_{t_0})^{-1}W_{s_+}\in\mathcal{M}_0. \tag{4.22}$$

Since the original chain $(W_t)_{t\in\mathcal{I}}$ is maximal we therefore have $\mathcal{T}_{-m}(\tilde{W}_{t_0})=W_{s_0}$ for some $s_0\in\mathcal{I}$. In fact, by (4.22), $s_0\in[s_-,s_+]$ (compare the corollary of Lemma 3.7). We have proved that $\tilde{W}_{t_0}=\mathcal{T}_m(W_{s_0})$ for convenient $s_0\in\mathcal{J}$, hence $(\mathcal{T}_m(W_t))_{t\in\mathcal{J}}$ is locally maximal at s.

□

Despite the above result the chain $(\mathcal{T}_m(W_t))_{t\in\mathcal{J}}$ need not be maximal. However, those points where there might something to be added necessarily are singularities of either $(W_t)_{t\in\mathcal{I}}$ or $\mathcal{T}_m((W_t)_{t\in\mathcal{I}})$. The behaviour at such points will be discussed in detail in Section 6.

5 Intermediate Weyl coefficients

Let $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$. In the sequel we will denote by $\mathcal{I}_0, \mathcal{I}_1, \dots$ the connected components of $\mathcal{I}$ arranged in increasing order. As already done during the previous sections the last component will also be denoted by $\mathcal{I}_\infty$. The singularity between $\mathcal{I}_n$ and $\mathcal{I}_{n+1}$ will be denoted by σ_n.

With the singularities of a maximal chain of matrix functions certain functions are associated. We call them intermediate Weyl coefficients.

Proposition 5.1. *Let $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$. For each singularity σ_n the limits*

$$\lim_{t\nearrow\sigma_n} (W_t \circ \tau^t)(z) =: q_{\sigma_n -}(z),\ \lim_{t\searrow\sigma_n} (W_t \circ \sigma^t)(z) =: q_{\sigma_n +}(z), \tag{5.1}$$

exist locally uniformly on $\mathbb{C}\setminus\mathbb{R}$, if $\mathbb{C}\cup\{\infty\}$ is provided with the spherical metric, and do not depend on the choice of the functions $\tau^t \in \mathcal{N}_0$ and $-\sigma^t \in \mathcal{N}_0$. We have $q_{\sigma_n -} \in \mathcal{N}_{\kappa(\mathcal{I}_n)}$. For each $t \in \mathcal{I}$ there exist (unique) functions τ_n^t and σ_n^t, such that

$$q_{\sigma_n -} = W_t \circ \tau_n^t,\ q_{\sigma_n +} = W_t \circ \sigma_n^t. \tag{5.2}$$

If $t < \sigma_n$ we have $\tau_n^t \in \mathcal{N}_{\kappa(\mathcal{I}_n)-\kappa(t)}$.

Proof : Observe that, if $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$ and σ is a singularity, then also the chain $(W_t)_{t\in\mathcal{I}, t<\sigma}$ belongs to $\mathfrak{M}$. The validity of **(W5)** hereby follows from Proposition 3.16. The existence of the first limit in (5.1) and the fact that it belongs to $\mathcal{N}_{\kappa(\mathcal{I}_n)}$ thus is immediate from (3.3).

The existence of the second limit in (5.1) is a consequence of [HSW]. In fact choose $s \in \mathcal{I}$, $s \in (\sigma_n, \sigma_{n+1})$, and consider the chain $(W_{st})_{t\in(\sigma_n,\sigma_{n+1})}$. The results of [HSW] are immediately applicable to this chain.

The existence of the factorizations (5.2) also follows from considering the chain $(W_{st})_{t\in\mathcal{I}}$. Since in the case $s < \sigma_n$, the chain $(W_{st})_{t\in\mathcal{I}, t\in(s,\sigma_n)}$ belongs to $\mathfrak{M}$, the relation $\tau_n^t \in \mathcal{N}_{\kappa(\mathcal{I}_n)-\kappa(t)}$ is obvious, since by **(W3)**

$$\max_{t\in(s,\sigma_n)} \operatorname{ind}_- W_{st} = \kappa(\mathcal{I}_n) - \kappa(t).$$

□

If we want to indicate precisely the chain $(W_t)_{t\in\mathcal{I}}$ to which the functions $q_{\sigma_n\pm}(z)$ are associated as its intermediate Weyl coefficients, we shall write $q_{\sigma_n\pm}((W_t)_{t\in\mathcal{I}})(z)$.

The aim of this section is to prove that $q_{\sigma_n -}$ and $q_{\sigma_n +}$ coincide, i.e.

$$q_{\sigma_n -}(z) = q_{\sigma_n +}(z) =: q_{\sigma_n}(z).$$

Once this is shown, we may speak of the intermediate Weyl coefficient q_σ associated with the singularity σ. Essential for the proof of this statement is a representation of $q_{\sigma_n -}$ and $q_{\sigma_n +}$ as certain u-resolvents. In order to establish this representation we need some preliminary information.

Lemma 5.2. *Let $\mathfrak{P}$ be a degenerated dB-space, and denote by $\mathcal{S}$ the operator of multiplication with the independent variable z. The relation $\mathcal{S}/\mathfrak{P}^\circ$ in the Pontryagin space $\mathfrak{P}/\mathfrak{P}^\circ$ is selfadjoint and has discrete spectrum. In fact*

$$\sigma(\mathcal{S}/\mathfrak{P}^\circ) = \{w \in \mathbb{C} : h(w) = 0 \text{ for all } h \in \mathfrak{P}^\circ\}. \tag{5.3}$$

Proof : It is elementary to check, that $\mathcal{S}$ has defect index $(1,1)$ in the sense of [KW1]. Since $\mathfrak{P}^\circ$ consists of entire functions, and since $\mathcal{S}$ is the multiplication operator, we conclude that the conditions (2.3) and (2.4) of [KW1] are satisfied. Thus we obtain from [KW1] that $\mathcal{S}/\mathfrak{P}^\circ$ is selfadjoint.

Clearly, $w \in \rho(\mathcal{S}/\mathfrak{P}^\circ)$ if and only if $\operatorname{ran}(\mathcal{S}/\mathfrak{P}^\circ - w) = \mathfrak{P}/\mathfrak{P}^\circ$, which is the same as condition (2.4) of [KW1] for $z = w$. As $h \in \operatorname{ran}(\mathcal{S} - w)$ if and only if $h(w) = 0$, we obtain (5.3).

□

Lemma 5.3. *Let $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$ be given such that $\mathfrak{K}(W_t) = \mathfrak{K}_-(W_t)$. Consider the transformation $\mathcal{T}^\alpha$ as introduced in Lemma II.10.2. The chain $(\mathcal{T}^\alpha(W_t))_{t\in\mathcal{I}}$ again belongs to $\mathfrak{M}$ and the singularities of these chains are the same. The transformed chain satisfies $\mathfrak{K}(\mathcal{T}^\alpha(W_t)) = \mathfrak{K}_-(\mathcal{T}^\alpha(W_t))$ and the spaces $\mathfrak{P}(E_{W_t})$ are isomorphic to the spaces $\mathfrak{P}(E_{\mathcal{T}^\alpha(W_t)})$ via the unitary mapping $F(z) \mapsto F(z+\alpha)$. Moreover, the dB-subspaces of $\mathfrak{P}(E_{W_t})$ correspond to the dB-subspaces of $\mathfrak{P}(E_{\mathcal{T}^\alpha(W_t)})$ in the same way. In particular, if $\mathfrak{P}$ is a degenerated dB-subspace of $\mathfrak{P}(E_{W_t})$ then the corresponding dB-subspace $\mathfrak{P}_{(\alpha)}$ of $\mathfrak{P}(E_{\mathcal{T}^\alpha(W_t)})$ is also degenerated and the $1/\mathfrak{P}^\circ$ resolvents*

$$r(z) = [(\mathcal{S}/\mathfrak{P}^\circ - z)^{-1}(1/\mathfrak{P}^\circ), (1/\mathfrak{P}^\circ)],$$

and

$$r_{(\alpha)}(z) = [(\mathcal{S}/\mathfrak{P}^\circ_{(\alpha)} - z)^{-1}(1/\mathfrak{P}^\circ_{(\alpha)}), (1/\mathfrak{P}^\circ_{(\alpha)})]_{(\alpha)}$$

are connected by the relation $r_{(\alpha)}(z) = r(z+\alpha)$. The intermediate Weyl coefficients $q_{\sigma\pm}((W_t)_{t\in\mathcal{I}})(z)$ and $q_{\sigma\pm}(\mathcal{T}^\alpha(W_t)_{t\in\mathcal{I}})(z)$ are also connected by

$$q_{\sigma\pm}(\mathcal{T}^\alpha(W_t)_{t\in\mathcal{I}})(z) = q_{\sigma\pm}((W_t)_{t\in\mathcal{I}})(z+\alpha).$$

Proof : Since $\mathcal{T}^\alpha(W_t)$ is everywhere defined on $\mathcal{I}$ and since the inverse transformation of $\mathcal{T}^\alpha$ is $\mathcal{T}^{-\alpha}$, we obtain from Lemma II.10.2 that $(\mathcal{T}^\alpha(W_t))_{t\in\mathcal{I}}$ belongs to $\mathfrak{M}$. Thus the singularities of the transformed chain are the same as those of $(W_t)_{t\in\mathcal{I}}$.

For $t \in \mathcal{I}$ the matrix $W_t(\alpha)^{-1}$ is iJ-unitary. Thus the reproducing kernel Pontryagin spaces $\mathfrak{K}(\mathcal{T}^\alpha(W_t))$ and $\mathfrak{K}(V_t)$, where $V_t(z) = W_t(z+\alpha)$, coincide. A short argument now shows that the mapping $X(z) \mapsto X(z+\alpha)$ is a unitary operator from $\mathfrak{K}(\mathcal{T}^\alpha(W_t))$ onto $\mathfrak{K}(W_t)$. Hence $\mathfrak{K}(\mathcal{T}^\alpha(W_t)) = \mathfrak{K}_-(\mathcal{T}^\alpha(W_t))$ and $\mathfrak{P}(E_{W_t})$ is isomorphic to $\mathfrak{P}(E_{\mathcal{T}^\alpha(W_t)})$ via the unitary mapping $F(z) \mapsto F(z+\alpha)$. Clearly, also the dB-subspaces of $\mathfrak{P}(E_{W_t})$ correspond to the dB-subspaces of $\mathfrak{P}(E_{\mathcal{T}^\alpha(W_t)})$ in the same way.

Let $\mathfrak{P}$ be a degenerated dB-subspace of $\mathfrak{P}(E_{W_t})$ and let $\mathfrak{P}_{(\alpha)}$ be the corresponding dB-subspace of $\mathfrak{P}(E_{\mathcal{T}^\alpha(W_t)})$. Then, for some $h_{(\alpha)} \in \mathfrak{P}_{(\alpha)}^\circ$ we have $(h(z) := h_{(\alpha)}(z-\alpha) \in \mathfrak{P}^\circ)$

$$[(\mathcal{S}/\mathfrak{P}_{(\alpha)}^\circ - w)^{-1}(1/\mathfrak{P}_{(\alpha)}^\circ), (1/\mathfrak{P}_{(\alpha)}^\circ)]_{(\alpha)} = [(\mathcal{S}-w)^{-1}(1-h_{(\alpha)}), 1]_{(\alpha)} =$$

$$= [\frac{1-h_{(\alpha)}(z)}{z-w}, 1]_{(\alpha)} = [\frac{1-h(z)}{z-\alpha-w}, 1] = [(\mathcal{S}-(w+\alpha))^{-1}(1-h), 1] =$$

$$= [(\mathcal{S}/\mathfrak{P}^\circ - (w+\alpha))^{-1}(1/\mathfrak{P}^\circ), (1/\mathfrak{P}^\circ)].$$

The assertion concerning the intermediate Weyl coefficients of $(W_t)_{t\in\mathcal{I}}$ and $\mathcal{T}^\alpha(W_t)_{t\in\mathcal{I}}$ is now obvious.

□

Let $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$ be such that $\mathfrak{K}(W_t) = \mathfrak{K}_-(W_t)$. Moreover, let σ be a singularity of this chain which is not of polynomial type with degree 1.

Denote by $\mathfrak{P}_{\sigma_-,d}$ the smallest degenerated dB-subspace of $\mathfrak{P}_{\sigma_+}$ which contains $\mathfrak{P}_{\sigma_-}$. Note that by Corollary 3.17 the space $\mathfrak{P}_{\sigma_-,d}$ either equals $\mathfrak{P}_{\sigma_-}$ or contains this space as a subspace of codimension 1. The first case occurs if and only if σ is dense or left dense. Note that $\operatorname{ind}_-\mathfrak{P}_{\sigma_-,d} = \operatorname{ind}_-\mathfrak{P}_{\sigma_-}$.

Analogously, let $\mathfrak{P}_{\sigma_+,d}$ be the largest degenerated dB-subspace of $\mathfrak{P}_{\sigma_+}$. Again $\mathfrak{P}_{\sigma_+,d}$ either equals $\mathfrak{P}_{\sigma_+}$ or is contained in this space as a subspace of codimension 1, and the first case occurs if and only if σ is dense or right dense.

Recall from Theorem II.5.7 that the fact $1 \in \mathfrak{P}(E_{W_t})$ does not depend on $t \in \mathcal{I}$. If $\mathfrak{K}_-(W_t) = \mathfrak{K}(W_t)$, then $1 \in \mathfrak{P}(E_{W_t})$ if and only if for some $\alpha \in \mathbb{R}$

$$\begin{pmatrix} \alpha \\ 1 \end{pmatrix} \in \mathfrak{K}(W_t). \tag{5.4}$$

By Corollary II.5.15 α does not depend on $t \in \mathcal{I}$.

Proposition 5.4. *Let $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$ with $\mathfrak{K}_-(W_t) = \mathfrak{K}(W_t)$ and $1 \in \mathfrak{P}(E_{W_t})$ be given, and let α be as in (5.4). Moreover, let σ be a singularity of $(W_t)_{t\in\mathcal{I}}$ which is not of polynomial type with degree 1. Then*

$$q_{\sigma-}(z) = \alpha + [(\mathcal{S}/\mathfrak{P}_{\sigma_-,d}^\circ - z)^{-1}(1/\mathfrak{P}_{\sigma_-,d}^\circ), (1/\mathfrak{P}_{\sigma_-,d}^\circ)], \tag{5.5}$$

and

$$q_{\sigma+}(z) = \alpha + [(\mathcal{S}/\mathfrak{P}_{\sigma_+,d}^\circ - z)^{-1}(1/\mathfrak{P}_{\sigma_+,d}^\circ), (1/\mathfrak{P}_{\sigma_+,d}^\circ)]. \tag{5.6}$$

Proof : The proofs of (5.5) and (5.6) are similar. Therefore we will go into details only in the proof of (5.5).

We will use induction on the number $\dim \mathfrak{P}^\circ_{\sigma_-,d} \in \mathbb{N}$. Basically the idea is to employ the transformation $\mathcal{T}_m$ in order to reduce the degree of degeneracy.

First let us make some preliminary remarks. Recall that by Corollary 2.8 there exists a real number t_0, such that for $m > 0$,

$$\dim \langle \mathfrak{P}_{\sigma_-,d}, [.,.]_0 \rangle^\circ = \dim \langle \mathfrak{P}_{\sigma_-,d}, [.,.] \rangle^\circ - 1,$$

where $[.,.]_0$ is as in (2.7). By Lemma 5.3 we can, in order to prove (5.5), assume without loss of generality that $t_0 = 0$.

If σ is not dense or left dense, it clearly can be achieved by choosing m sufficiently small that the space $\langle \mathfrak{P}_{\sigma_-}, [.,.]_0 \rangle$ is nondegenerated and has the same negative index as $\langle \mathfrak{P}_{\sigma_-}, [.,.] \rangle$. If σ is dense or left dense, by Theorem I.11.6 and the definition of $\mathfrak{P}_{\sigma_-,d}$ the spaces $\langle \mathfrak{P}_t, [.,.]_0 \rangle$, $t < \sigma$, are nondegenerated and have the same negative index as $\langle \mathfrak{P}_{\sigma_-,d}, [.,.]_0 \rangle$ if only t is sufficiently close to σ. In the sequel we choose m fixed with these properties.

It follows from Lemma 4.8 that $W_t \in \operatorname{dom} \mathcal{T}_m$ for $t < \sigma$ sufficiently close, and that in case $\sigma_- < \sigma$ the interval $(\sigma_-, \sigma - \epsilon]$, $\epsilon > 0$, cannot be indivisible of type 0. Note that these conclusions are drawn from the fact that $t_0 = 0$.

Consider the transformed chain $(\tilde{W}_t)_{t \in \tilde{\mathcal{I}}}$ with

$$\tilde{W}_t := \mathcal{T}_m(W_t),\ t \in \mathcal{I} \cap \tilde{\mathcal{I}}.$$

By Theorem 4.4 and Theorem II.5.7 also this chain satisfies $\mathfrak{K}_-(\tilde{W}_t) = \mathfrak{K}(W_t)$ and $1 \in \mathfrak{P}(E_{\tilde{W}_t})$. Hence for some $\tilde{\alpha} \in \mathbb{R}$,

$$\begin{pmatrix} \tilde{\alpha} \\ 1 \end{pmatrix} \in \mathfrak{K}(\tilde{W}_t),\ t \in \tilde{\mathcal{I}}.$$

It follows from Lemma II.5.12 that for $t < \sigma$, and t sufficiently close to σ,

$$W_t \circ \infty \in \mathcal{N}_{\mathrm{ind}_- W_t},\ \tilde{W}_t \circ \infty \in \mathcal{N}_{\mathrm{ind}_- \tilde{W}_t}.$$

By Lemma II.5.16 we therefore have

$$\lim_{y \to +\infty} (W_t \circ \infty)(iy) = \alpha,\ \lim_{y \to +\infty} (\tilde{W}_t \circ \infty)(iy) = \tilde{\alpha}.$$

Note that the formulation of Lemma II.5.16 excludes W_t's which are linear polynomials. However, considering the proof of Lemma II.5.16 in view of our particular situation the assertion of this lemma is still true.

It is elementary to check - in fact, it has already been shown in the proof of Theorem 4.4 - that

$$(W_t \circ \infty)(z) = (\tilde{W}_t \circ \infty)(z) + \frac{m}{z}. \tag{5.7}$$

In particular, we obtain

$$\alpha = \lim_{y\to+\infty}(W_t \circ \infty)(iy) = \lim_{y\to+\infty}\left[(\tilde{W}_t \circ \infty)(iy) + \frac{m}{iy}\right] = \tilde{\alpha}.$$

Now we start our inductive argument.

Assume that $\dim \langle \mathfrak{P}_{\sigma_-,d}, [.,.]\rangle^\circ = 1$. Then $\langle \mathfrak{P}_{\sigma_-,d}, [.,.]_0\rangle$ is nondegenerated. Making use of the fact that there exists some $t \in \mathcal{I}$, $t > \sigma$, with $W_t \in \operatorname{dom}\mathcal{T}_m$ by **(W5)**, we conclude from Lemma 3.15 that $\sigma \in \tilde{\mathcal{I}}$, i.e. the limit

$$\lim_{t\nearrow\sigma} \tilde{W}_t = \tilde{W}_\sigma$$

exists. From (5.7) we obtain

$$q_{\sigma-}(z) = (\tilde{W}_\sigma \circ \infty)(z) + \frac{m}{z}. \tag{5.8}$$

It follows from the fact that regularized 1-resolvents are 1-resolvents plus some real constant if 1 belongs to the considered space (cf. [KW2]), from Lemma II.5.12 and Lemma II.5.16 that

$$(\tilde{W}_\sigma \circ \infty)(z) = \alpha + [(\mathcal{A} - z)^{-1}1, 1]_0, \tag{5.9}$$

where $\mathcal{A}$ is the canonical selfadjoint extension of the multiplication operator $\mathcal{S}$ with the independent variable z in the space $\langle \mathfrak{P}_{\sigma_-,d}, [.,.]_0\rangle$, which is determined by $A_{\tilde{W}_\sigma}(z) = \tilde{w}_{\sigma,21}(z)$ in Lemma I.6.4.

Denote by $K(w,z)$ the reproducing kernel function of $\langle \mathfrak{P}_{\sigma_-,d}, [.,.]_0\rangle$. By Lemma 4.8

$$\langle \mathfrak{P}_{\sigma_-,d}, [.,.]\rangle^\circ = \operatorname{span}\{K(0,.)\}$$

and $K(0,0) = \frac{1}{m}$. Since $A_{\tilde{W}_\sigma}(0) = 0$, we have $K(0,z) = \frac{A_{\tilde{W}_\sigma}(z)}{z}$. Now compute (for w with $K(0,w) \neq 0$)

$$[(\mathcal{S}/\mathfrak{P}^\circ_{\sigma_-,d} - w)^{-1}(1/\mathfrak{P}^\circ_{\sigma_-,d}), (1/\mathfrak{P}^\circ_{\sigma_-,d})] = [(\mathcal{S} - w)^{-1}(1 - \frac{K(0,z)}{K(0,w)}), 1] =$$

$$= [(\mathcal{A} - w)^{-1}(1 - \frac{K(0,z)}{K(0,w)}), 1] =$$

$$= [(\mathcal{A} - w)^{-1}(1 - \frac{K(0,z)}{K(0,w)}), 1]_0 - m\left((A - w)^{-1}(1 - \frac{K(0,z)}{K(0,w)})\right)(0) =$$

$$= [(\mathcal{A} - w)^{-1}1, 1]_0 - \frac{1}{K(0,w)}[K(0,z), (\mathcal{A} - \bar{w})^{-1}1]_0 -$$

$$-m\left(\frac{(1 - \frac{K(0,0)}{K(0,w)}) - \frac{A_{\tilde{W}_\sigma}(0)}{A_{\tilde{W}_\sigma}(w)}(1 - \frac{K(0,w)}{K(0,w)})}{0 - w}\right) =$$

$$= [(\mathcal{A}-w)^{-1}1,1]_0 - \frac{1}{K(0,w)}\frac{1}{-w} + \frac{m}{w} + \frac{1}{K(0,w)}\frac{1}{-w} = [(\mathcal{A}-w)^{-1}1,1]_0 + \frac{m}{w}.$$

Combining this relation with (5.8) and (5.9) yields (5.5).

Now assume that $\dim \langle \mathfrak{P}_{\sigma_-,d}, [.,.]\rangle^\circ > 1$. For notational convenience put

$$\langle \mathfrak{P}_{\sigma_-,d}, [.,.]\rangle =: \mathfrak{P},\ \langle \mathfrak{P}_{\sigma_-,d}, [.,.]_0\rangle =: \mathfrak{P}_0.$$

Clearly, $\{F \in \mathfrak{P}^\circ : F(0) = 0\} \subseteq \mathfrak{P}_0^\circ$. In fact, equality holds since $\dim \mathfrak{P}_0^\circ = \dim \mathfrak{P}^\circ - 1$.

By the inductive hypotheses we have

$$q_{\sigma-}(z) = \lim_{t\nearrow\sigma}(W_t \circ \infty)(z) = \lim_{t\nearrow\sigma}(\tilde{W}_t \circ \infty)(z) + \frac{m}{w} =$$

$$= \alpha + [(\mathcal{S}/\mathfrak{P}_0^\circ - z)^{-1}(1/\mathfrak{P}_0^\circ), (1/\mathfrak{P}_0^\circ)]_0 + \frac{m}{w}. \tag{5.10}$$

We compute for $w \in \rho(\mathcal{S}/\mathfrak{P}_0^\circ)$, where $h \in \mathfrak{P}_0^\circ$ is chosen such that $h(w) \neq 0$ (cf. Lemma 5.2)

$$[(\mathcal{S}/\mathfrak{P}^\circ - w)^{-1}(1/\mathfrak{P}^\circ), (1/\mathfrak{P}^\circ)] = [(\mathcal{S}/\mathfrak{P}_0^\circ - w)^{-1}(1/\mathfrak{P}_0^\circ), (1/\mathfrak{P}_0^\circ)] =$$

$$= [(\mathcal{S}/\mathfrak{P}_0^\circ - w)^{-1}(1/\mathfrak{P}_0^\circ), (1/\mathfrak{P}_0^\circ)]_0 - m((S-w)^{-1}(1 - \frac{h(z)}{h(w)}))(0) =$$

$$= [(\mathcal{S}/\mathfrak{P}_0^\circ - w)^{-1}(1/\mathfrak{P}_0^\circ), (1/\mathfrak{P}_0^\circ)]_0 + \frac{m}{w}.$$

Together with (5.10) this proves (5.5).

The proof of (5.6) is now obtained by substituting in the above arguments σ_- by σ_+ and $t \nearrow \sigma$ by $t \searrow \sigma$.

□

Lemma 5.5. *Let $\mathfrak{P}_1$ and $\mathfrak{P}_2$ be degenerated dB-spaces such that $\mathfrak{P}_1$ is contained in $\mathfrak{P}_2$ with codimension one. Assume that $1 \in \mathfrak{P}_1$. Denote by $\mathcal{S}_1$ and $\mathcal{S}_2$ the operators of multiplication with the independent variable z in the space $\mathfrak{P}_1$ and $\mathfrak{P}_2$, respectively. Then*

$$[(\mathcal{S}_1/\mathfrak{P}_1^\circ - z)^{-1}(1/\mathfrak{P}_1^\circ), (1/\mathfrak{P}_1^\circ)] = [(\mathcal{S}_2/\mathfrak{P}_2^\circ - z)^{-1}(1/\mathfrak{P}_2^\circ), (1/\mathfrak{P}_2^\circ)],$$

for $z \in \sigma(\mathcal{S}_1/\mathfrak{P}_1^\circ) \cap \sigma(\mathcal{S}_2/\mathfrak{P}_2^\circ)$.

Proof : A standard Pontryagin space argument shows that $\mathfrak{P}_1^\circ \subseteq \mathfrak{P}_2^\circ$ or $\mathfrak{P}_2^\circ \subseteq \mathfrak{P}_1^\circ$.

If $\mathfrak{P}_2^\circ \supsetneq \mathfrak{P}_1^\circ$, then clearly $\mathfrak{P}_1/\mathfrak{P}_1^\circ = \mathfrak{P}_2/\mathfrak{P}_2^\circ$, and since $\mathcal{S}_1 \subseteq \mathcal{S}_2$, we have $\mathcal{S}_1/\mathfrak{P}_1^\circ = \mathcal{S}_1/\mathfrak{P}_2^\circ \subseteq \mathcal{S}_2/\mathfrak{P}_2^\circ$. By Lemma 5.2 both are selfadjoint, and hence they coincide.

If $\mathfrak{P}_2^\circ = \mathfrak{P}_1^\circ$, the Pontryagin space $\mathfrak{P}_1/\mathfrak{P}_1^\circ$ is contained in the Pontryagin space $\mathfrak{P}_2/\mathfrak{P}_2^\circ$ with codimension one, and $1/\mathfrak{P}_1^\circ = 1/\mathfrak{P}_2^\circ$ is contained in $\mathfrak{P}_1/\mathfrak{P}_1^\circ$. Now

$\mathcal{S}_1/\mathfrak{P}_1^\circ$ is selfadjoint in $\mathfrak{P}_1/\mathfrak{P}_1^\circ$ and is contained in $\mathcal{S}_2/\mathfrak{P}_2^\circ$, which is also selfadjoint. Therefore $(\mathcal{S}_2/\mathfrak{P}_2^\circ)|_{\mathfrak{P}_1/\mathfrak{P}_1^\circ} = \mathcal{S}_1/\mathfrak{P}_1^\circ$, and the assertion of the lemma follows also in this case.

Finally, if $\mathfrak{P}_2^\circ \subsetneq \mathfrak{P}_1^\circ$, then the image $\mathfrak{P}_1/\mathfrak{P}_2^\circ$ of $\mathfrak{P}_1$ under the factorization by $\mathfrak{P}_2^\circ$ is a degenerated subspace of $\mathfrak{P}_2/\mathfrak{P}_2^\circ$ of codimension one, and $1/\mathfrak{P}_2^\circ \in \mathfrak{P}_1/\mathfrak{P}_2^\circ$. We also have $\mathcal{S}_1/\mathfrak{P}_2^\circ \subseteq \mathcal{S}_2/\mathfrak{P}_2^\circ$ with codimension one, where the latter is selfadjoint. Since $\mathcal{S}_1/\mathfrak{P}_2^\circ$ acts in $\mathfrak{P}_1/\mathfrak{P}_2^\circ$, the operator $(\mathcal{S}_1/\mathfrak{P}_2^\circ - z)^{-1}$ exists and is continuous on $\mathfrak{P}_1/\mathfrak{P}_2^\circ$ for all $z \in \rho(\mathcal{S}_2/\mathfrak{P}_2)$, and

$$(\mathcal{S}_1/\mathfrak{P}_2^\circ - z)^{-1}(1/\mathfrak{P}_2^\circ) = (\mathcal{S}_2/\mathfrak{P}_2^\circ - z)^{-1}(1/\mathfrak{P}_2^\circ). \tag{5.11}$$

Clearly, $(\mathfrak{P}_1/\mathfrak{P}_2^\circ)/(\mathfrak{P}_1/\mathfrak{P}_2^\circ)^\circ$ is isomorphic to $\mathfrak{P}_1/\mathfrak{P}_1^\circ$ and $(\mathcal{S}_1/\mathfrak{P}_2^\circ)/(\mathfrak{P}_1/\mathfrak{P}_2^\circ)^\circ$ is isomorphic to $\mathcal{S}_1/\mathfrak{P}_1^\circ$. Thus

$$[(\mathcal{S}_1/\mathfrak{P}_2^\circ - z)^{-1}(1/\mathfrak{P}_2^\circ), (1/\mathfrak{P}_2^\circ)] =$$
$$= [((\mathcal{S}_1/\mathfrak{P}_2^\circ)/(\mathfrak{P}_1/\mathfrak{P}_2^\circ)^\circ - z)^{-1}(1/\mathfrak{P}_2^\circ)/(\mathfrak{P}_1/\mathfrak{P}_2^\circ)^\circ, (1/\mathfrak{P}_2^\circ)/(\mathfrak{P}_1/\mathfrak{P}_2^\circ)^\circ] =$$
$$= [(\mathcal{S}_1/\mathfrak{P}_1^\circ - z)^{-1}(1/\mathfrak{P}_1^\circ), (1/\mathfrak{P}_1^\circ)].$$

Together with (5.11) this proves the lemma.

□

Now we have collected all necessary ingredients for the proof of the main result of the present section.

Theorem 5.6. *Let $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$ and let σ be a singularity of this chain. Then the functions $q_{\sigma-}$ and $q_{\sigma+}$ coincide.*

Proof : First assume that σ is not of polynomial type with degree one. Moreover, let $\mathfrak{K}(W_t) = \mathfrak{K}_-(W_t)$ and $1 \in \mathfrak{P}(E_{W_t})$. By Proposition 5.4 we know that

$$q_{\sigma-}(z) = \alpha + [(\mathcal{S}_{\sigma_-,d}/\mathfrak{P}^\circ_{\sigma_+,d} - z)^{-1}(1/\mathfrak{P}^\circ_{\sigma_-,d}), (1/\mathfrak{P}^\circ_{\sigma_-,d})], \tag{5.12}$$

and

$$q_{\sigma+}(z) = \alpha + [(\mathcal{S}_{\sigma_+,d}/\mathfrak{P}^\circ_{\sigma_+,d} - w)^{-1}(1/\mathfrak{P}^\circ_{\sigma_+,d}), (1/\mathfrak{P}^\circ_{\sigma_+,d})]. \tag{5.13}$$

Applying Lemma 5.5 step by step to the spaces of the chain (3.8), we obtain that the functions (5.12) and (5.13) coincide.

If it is not true that $\mathfrak{K}(W_t) = \mathfrak{K}_-(W_t)$ and $1 \in \mathfrak{P}(E_{W_t})$, then Lemma I.8.6 and Proposition I.8.3 imply that $\mathfrak{K}(W_t)$ contains no constant function of the form $(\alpha, 1)^T$, $\alpha \in \mathbb{R}$. Let $W_{(1,\frac{\pi}{2})}$ be defined according to (II.7.1). Lemma II.7.2 shows that

$$\mathfrak{K}(W_{(1,\frac{\pi}{2})}) \cap W_{(1,\frac{\pi}{2})}\mathfrak{K}(W_t) = \{0\}.$$

Therefore, $(W_{(1,\frac{\pi}{2})}W_t)_{t\in\mathcal{I}}$ extends to a chain $(V_t)_{t\in\mathcal{J}} \in \mathfrak{M}$ such that $(0,1)^T \in \mathfrak{K}(V_t)$, $t \in \mathcal{J}$. Hence $\mathfrak{K}(V_t) = \mathfrak{K}_-(V_t)$ and $1 \in \mathfrak{P}(E_{V_t})$, cf. Proposition I.8.3. Clearly, the value $\gamma \in \mathcal{J}$ which corresponds to σ is a singularity of $(V_t)_{t\in\mathcal{J}}$, and the corresponding intermediate Weyl coefficients are $W_{(1,\frac{\pi}{2})}q_{\sigma-}$ and $W_{(1,\frac{\pi}{2})}q_{\sigma+}$. By the already settled case we know

$$W_{(1,\frac{\pi}{2})} \circ q_{\sigma-} = W_{(1,\frac{\pi}{2})} \circ q_{\sigma+}.$$

Since $W_{(1,\frac{\pi}{2})}(z)$ is invertible, we obtain $q_{\sigma-} = q_{\sigma+}$.

It remains to consider the case that σ is of polynomial type with degree one, i.e. that $\sigma_-, \sigma_+ \in \mathcal{I}$, $\sigma_- < \sigma < \sigma_+$, and $W_{\sigma_-\sigma_+} = W_{(l,\beta)}$ for some $l < 0$ and $\beta \in [0, \pi)$. Hence for all $t \in [\sigma_-, \sigma) \cup (\sigma, \sigma_+]$ the matrix function $W_{\sigma_- t}$ is of the same form, and hence

$$W_t \circ \cot\beta = W_{\sigma_-} W_{\sigma_-,t} \circ \cot\beta = W_{\sigma_-} \circ \cot\beta.$$

Note that if $\sigma_- = 0$, we have to set $W_0 = 1$ and $W_{0t} = W_t$ above.

Thus also for singularities of polynomial type with degree one we have

$$q_{\sigma-}(z) = (W_{\sigma_-} \circ \cot\beta)(z) = q_{\sigma+}(z).$$

□

The question arises which functions $q \in \mathcal{N}_\kappa$ can be realized as intermediate Weyl coefficient. An immediate consequence of Lemma 5.2, Proposition 5.4 and the proof of Theorem 5.6 is:

Remark 5.7. Assume that q is the intermediate Weyl coefficient at some singularity. Then q is meromorphic in $\mathbb{C}$.

As we will see in Section 7 this necessary condition is far from being sufficient.

From Theorem 5.6 we deduce a continuity property of a maximal chain of matrix functions at a singularity.

Proposition 5.8. *Let $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$ and let σ be a singularity. If one of the values*

$$\limsup_{t\nearrow\sigma, t\in\mathcal{I}} |w'_{21,t}(0)|, \ \limsup_{t\searrow\sigma, t\in\mathcal{I}} |w'_{21,t}(0)|,$$

is finite, then the limits

$$\gamma_- := \lim_{t\nearrow\sigma} w'_{21,t}(0), \ \gamma_+ := \lim_{t\searrow\sigma} w'_{21,t}(0),$$

exist and are equal.

Before we go into the proof of this statement we need a result which follows from [Wi1] (see also [Wi2]).

Lemma 5.9. *Let $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$ and denote by q its Weyl coefficient. Then $w'_{21,t}(0)$ is unbounded for $t \nearrow \sup\mathcal{I}$ if and only if*

$$\lim_{y\searrow 0} yq(iy) = 0. \tag{5.14}$$

Otherwise

$$-i\left[\lim_{y\searrow 0} yq(iy)\right]^{-1} = \lim_{t\nearrow\sup\mathcal{I}} w'_{21,t}(0).$$

Proof : In the case that $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}_0$ the assertion is just Theorem 2.2 from [Wi1]. If $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}_\kappa$ with $\kappa > 0$, choose $t_0 \in \mathcal{I}$ such that $\operatorname{ind}_- W_{t_0} = \kappa$. The result of [Wi1] can be applied to the chain $(\tilde{W}_t)_{t\in\mathcal{I},t>t_0}$ where $\tilde{W}_t := W_{t_0 t}$. Denote its Weyl coefficient by $\tilde{q}$. Since W_{t_0} has the power series expansion

$$W_{t_0} = \begin{pmatrix} 1+O(z) & w'_{12,t_0}(0)z+O(z^2) \\ w'_{21,t_0}(0)z+O(z^2) & 1+O(z) \end{pmatrix}$$

at 0, and a similar expansion holds for $W_{t_0}^{-1}$, we obtain from

$$zq(z) = \frac{w_{11,t_0}(z)z\tilde{q}(z) + zw_{12,t_0}(z)}{w_{21,t_0}(z)\tilde{q}(z) + w_{22,t_0}(z)},$$

that (5.14) is equivalent to $\tilde{w}'_{21,t}(0)$ being unbounded and, since $w'_{21,t}(0) = w'_{21,t_0}(0) + \tilde{w}'_{21,t}(0)$, also to $w'_{21,t}(0)$ being unbounded.

If (5.14) is not valid we obtain

$$i\lim_{y\searrow 0} y\tilde{q}(iy) = \frac{1}{\lim_{t\nearrow\sup\mathcal{J}} \tilde{w}'_{21,t}(0)} =: \beta.$$

Hence

$$i\lim_{y\searrow 0} yq(iy) = \frac{\beta}{w'_{21,t_0}(0)\beta+1} =$$

$$= \left[w'_{21,t_0}(0) + \lim_{t\nearrow\sup\mathcal{J}} \tilde{w}'_{21,t}(0)\right]^{-1} = \left[\lim_{t\nearrow\sup\mathcal{J}} w'_{21,t}(0)\right]^{-1}.$$

□

Proof (of Proposition 5.8): Consider the intermediate Weyl coefficient q_σ at the singularity σ. By the above lemma the number γ_- is finite if and only if $\lim_{y\searrow 0} yq_\sigma(iy) \neq 0$.

For a number $s_1 > \sigma$, such that $\operatorname{ind}_- W_{s_1} = \min_{s\in\mathcal{I},s>\sigma} \operatorname{ind}_- W_s$, consider the chain

$$V_t(z) = W_{s_1,s_1-t}(-z),\ 0 \le t < s_1 - \sigma.$$

It is easy to see that $(V_t)_{t\in(0,s_1-\sigma)} \in \mathfrak{M}_0$. We denote by $\tilde{q}$ the Weyl coefficient of this chain. As

$$w'_{21,s_1-t}(0) = w'_{21,s_1}(0) - v'_{21,t}(0),\ 0 \leq t < s_1 - \sigma,$$

the number γ_+ is finite if and only if $v'_{21,t}(0)$ is bounded for $t \nearrow s_1 - \sigma$. If we set

$$\beta = -i \lim_{y\searrow 0} y\tilde{q}(iy),$$

this is in turn equivalent to $\beta \neq 0$. Note that in any case $\beta \in \mathbb{R}$, and hence,

$$\beta = \overline{\lim_{y\searrow 0} -iy\tilde{q}(iy)} = \lim_{y\searrow 0} iy\tilde{q}(-iy).$$

Since the upper and the lower intermediate Weyl coefficient coincide (cf. Theorem 5.6), we have

$$q_\sigma(z) = W_{s_1}(z) \circ \tilde{q}(-z).$$

Thus

$$\lim_{y\searrow 0} -iyq_\sigma(iy) = \lim_{y\searrow 0} -iy(W_{s_1}(iy) \circ \tilde{q}(-iy)) = \frac{-\beta}{-w'_{21,s_1}(0)\beta + 1}.$$

We see that $\beta \neq 0$ if and only if γ_- is finite. Moreover, in this case

$$\gamma_- = w'_{21,s_1}(0) - \frac{1}{\beta} = \gamma_+.$$

□

6 Evolution of singularities

If we apply the transformation $\mathcal{T}_m$ to the members W_t of a maximal chain $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$ whenever it is possible, we obtain by virtue of Theorem 4.4 a chain of matrix functions which can be extended to a maximal chain $\mathcal{T}_m((W_t)_{t\in\mathcal{I}})$. Of course, during this procedure the structure of the singularities might change, i.e. singularities may get a different type, some may vanish and new ones may appear. The aim of this section is to investigate in detail the evolution of singularities when $\mathcal{T}_m$ is applied. Moreover, we make clear what has to be added to obtain $\mathcal{T}_m((W_t)_{t\in\mathcal{I}})$ from $(\mathcal{T}_m(W_t))_{t\in\mathcal{I},W_t\in\operatorname{dom}\mathcal{T}_m}$. During this whole section we will assume that $\mathfrak{K}_-(W_t) = \mathfrak{K}(W_t)$, in order to have available the related chain of dB-spaces.

First we consider the appearance of new singularities. Recall that the crucial parameter in the transformation $\mathcal{T}_m$ is $\alpha(W) = 1 - mw'_{21}(0)$. Let $\mathcal{I}_n = (\sigma_{n-1}, \sigma_n)$

be a component of $\mathcal{I}$. Singularities may occur if the parameter $\alpha(W_t)$ has a zero in the interior of $\mathcal{I}_n$. Since $\alpha(W_t)$ is monotone, the set of zeros is either empty, consists of a single point or of an interval. Recall in this place the following fact which implies that, if $\alpha(W_t)$ vanishes on an interval, this interval is necessarily indivisible of type 0 (cf. [dB]).

Remark 6.1. Let $W \in \mathcal{M}_0$ and assume that $w_{21}'(0) = 0$. Then $W = W_{(l,0)}$ for some $l \geq 0$.

In particular, Proposition 3.16 and Lemma 3.5 imply that $\alpha(W_t)$ cannot vanish identically on any component of $\mathcal{I}$.

The case that $\alpha(W_t)$ vanishes on an interval which has one endpoint in common with $\mathcal{I}_n$, as well as the case that the zero set is empty but $\lim_{t \nearrow \sigma_n} \alpha(W_t) = 0$ or $\lim_{t \searrow \sigma_{n-1}} \alpha(W_t) = 0$, will be treated later. In these situations the 'zeros' of α do not give rise to a new singularity but contribute to the evolution of σ_n (σ_{n-1}, respectively). Hence we are left with the cases that α has a single zero in $\mathcal{I}_n$ or there exists an indivisible interval of finite length and type 0 on which α vanishes. Considering the definitions of the various types of singularities we obtain the following result.

Proposition 6.2. *The weight of the new singularity σ is* 1.

(i) If α has an isolated zero which is contained in the interior of an indivisible interval, then σ is of polynomial type with degree 1.

(ii) If α has an isolated zero which is left and right (left but not right, right but not left, neither left nor right) endpoint of an indivisible interval, then σ is of polynomial type with degree $\delta = 2$ (left dense with $\delta = 1$, right dense with $\delta = 1$, dense with $\delta = 0$).

(iii) If α vanishes on an interval $[t_-, t_+]$ and t_- is right and t_+ is left (t_- right but t_+ not left, t_+ left but t_- not right, neither t_- right nor t_+ left) endpoint of an indivisible interval, then σ is of polynomial type with $\delta = 3$ (right dense with $\delta = 2$, left dense with $\delta = 2$, dense with $\delta = 1$).

Proof : We prove only the statement *(i)*. The proofs of *(ii)* and *(iii)* are similar.

Assume that α has an isolated zero located in the interior of some indivisible interval. Depending on whether $m < 0$ or $m > 0$ the function $\alpha(W_t)$ is nonincreasing or nondecreasing, respectively. Thus $\alpha(W_t) > 0$ for $t < \sigma$ and $\alpha(W_t) < 0$ for $t > \sigma$ ($\alpha(W_t) < 0$ for $t < \sigma$ and $\alpha(W_t) > 0$ for $t > \sigma$, respectively). By Corollary 4.3,

$$\operatorname{ind}_- \mathcal{T}_m(W_t) = \operatorname{ind}_- W_t + \begin{cases} 0\,, t < \sigma \\ 1\,, t > \sigma \end{cases} \quad \left(\begin{cases} -1\,, t < \sigma \\ \ \ 0\ \,, t > \sigma \end{cases} \right) .$$

Since σ is contained in the interior of an indivisible interval, say (σ_-, σ_+), the transfer matrix $\mathcal{T}_m(W_{\sigma_-})^{-1}\mathcal{T}_m(W_{\sigma_+})$ is by Lemma 4.7 a linear polynomial $W_{(l,\phi)}$.

By Lemma 4.5 we have $l < 0$, hence σ is of polynomial type with degree 1.

□

Now we investigate the circumstances under which a singularity disappears. Since this happens if and only if a new singularity occurs when reversing the process, i.e. when applying the transformation $\mathcal{T}_{-m}$ to the chain $\mathcal{T}_m((W_t)_{t\in\mathcal{I}})$, the above discussion shows us which types of singularities might vanish. It also gives us a hint how the chain $(\mathcal{T}_m(W_t))$ has to be extended in order to obtain maximality. We only have to note that, if the singularity σ is of one of the types mentioned in *(i)* and *(ii)* of Proposition 6.2, then

$$\lim_{t\nearrow\sigma} \mathcal{T}_m(W_t) = \lim_{t\searrow\sigma} \mathcal{T}_m(W_t),$$

whereas in case *(iii)* inequality holds. However, in case of inequality, the transfer matrix V belongs to $\mathcal{M}_0$ and satisfies $v_{21}'(0) = 0$. Thus, in order to extend $(\mathcal{T}_m(W_t))$ locally at the vanishing singularity σ to a maximal chain, in the cases of *(i)* and *(ii)* it suffices to add the matrix $\lim_{t\to\sigma} \mathcal{T}_m(W_t)$, in those of *(iii)* we have to plug in an indivisible interval of type 0 and of appropriate length.

In the remainder of this section we discuss the evolution of an existing singularity σ. As already remarked earlier the change of the type will depend on the behaviour of the parameter $\alpha(W_t)$ at σ. In order to clarify the role of α we need the following result.

Lemma 6.3. *Let $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$ and assume that σ is a singularity which is not of polynomial type with degree 1. Moreover, let $\mathfrak{P}$ be a degenerated dB-space with $\mathfrak{P}_{\sigma_-} \subseteq \mathfrak{P} \subseteq \mathfrak{P}_{\sigma_+}$. Then $\mathfrak{P}^\circ \subseteq \mathfrak{Q}(\mathfrak{P})$ if and only if $w_{21,t}'(0)$ is bounded at σ. In this case*

$$K_0(0) = -\lim_{t\to\sigma} w_{21,t}'(0). \tag{6.1}$$

Proof : First note that by Proposition 5.8 the limit in (6.1) is well defined.

Let us settle the case that σ is neither dense nor left dense, i.e. $\mathfrak{P}_{\sigma_-}$ is nondegenerated. By the (proof of) Theorem 2.3 and Lemma II.7.6 the relation $\mathfrak{P}^\circ \subseteq \mathfrak{Q}(\mathfrak{P})$ is equivalent to the fact that the indivisible interval (σ_-, σ) has type 0. This in turn is equivalent to $w_{21,t}'(0)$ being bounded for $t \nearrow \sigma$. In this case the right hand side of (6.1) equals $-w_{21,\sigma_-}'(0)$ which is $K_{\sigma_-}(0,0)$, when $K_{\sigma_-}(w,z)$ denotes the reproducing kernel of $\mathfrak{P}_{\sigma_-}$, since $K_{\sigma_-}(0,z) = -\frac{w_{21,\sigma_-}(z)}{z}$. It remains to recall from Corollary 2.15 that in fact $K_0 \in \mathfrak{P}_{\sigma_-}$ and hence $K_0 = K_{\sigma_-}(0,.)$.

Consider the case that $\sigma_- = \sigma$, i.e. $\mathfrak{P}_{\sigma_-}$ is degenerated. Again due to Corollary 2.15 we have $K_0 \in \mathfrak{P}_{\sigma_-}$, hence it suffices to prove the statement for $\mathfrak{P} = \mathfrak{P}_{\sigma_-}$. Observe that

$$\overline{\bigcup_{t\in\mathcal{I},t<\sigma} \mathfrak{P}_t} = \mathfrak{P}_{\sigma_-}. \tag{6.2}$$

Moreover, $\mathfrak{P}_t$ can be considered as a subspace of the Pontryagin space $\mathfrak{P}_{\sigma_-}/\mathfrak{P}_{\sigma_-}^\circ$, and the relation (6.2) still holds when $\mathfrak{P}_{\sigma_-}$ is replaced by $\mathfrak{P}_{\sigma_-}/\mathfrak{P}_{\sigma_-}^\circ$.

Assume that $\mathfrak{P}^\circ \subseteq \mathfrak{L}(\mathfrak{P})$. Then point evaluation at 0 is a well defined and continuous functional on $\mathfrak{P}_{\sigma_-}/\mathfrak{P}_{\sigma_-}^\circ$ and is represented as inner product with the element $K_0 + \mathfrak{P}_{\sigma_-}^\circ$. By an elementary Pontryagin space argument we conclude from (6.2) that

$$\lim_{t\nearrow\sigma_-} K_t(0,.) = K_0 + \mathfrak{P}_{\sigma_-}^\circ .$$

In particular (6.1) holds.

Finally assume that $K_t(0,0) = [K_t(0,.), K_t(0,.)]$ is bounded for $t \nearrow \sigma_-$. Again a straightforward argument shows that the family $\{K_t(0,.)\}$ is in fact bounded in a definite norm of $\mathfrak{P}_{\sigma_-}/\mathfrak{P}_{\sigma_-}^\circ$. Hence there exists a weak limit, say $K_0 + \mathfrak{P}_{\sigma_-}^\circ$. It follows that the element $K_0 \in \mathfrak{P}_{\sigma_-}$ represents point evaluation at 0 on the set $\bigcup_{t<\sigma} \mathfrak{P}_t$, and hence, by (6.2) and the fact that point evaluation is continuous on $\mathfrak{P}_{\sigma_-}$, on all of $\mathfrak{P}_{\sigma_-}$. In particular $\mathfrak{P}_{\sigma_-}^\circ \subseteq \mathfrak{L}(\mathfrak{P}_{\sigma_-})$.

□

Given the chain $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$ we consider the chain of transformed matrix functions

$$(\tilde{W}_t)_{t\in\mathcal{I},W_t\in\operatorname{dom}\mathcal{T}_m},\ \tilde{W}_t = \mathcal{T}_m(W_t). \tag{6.3}$$

The dB-spaces associated with this chain will be denoted by $\tilde{\mathfrak{P}}_t$, the singularity σ under consideration as $\tilde{\sigma}$ when thought of as a singularity of $(\tilde{W}_t)_{t\in\mathcal{I},W_t\in\operatorname{dom}\mathcal{T}_m}$, and correspondingly the weight and degree of $\tilde{\sigma}$ by $\tilde{\omega}$ and $\tilde{\delta}$. Moreover, the maximal chain extending $(\tilde{W}_t)_{t\in\mathcal{I},W_t\in\operatorname{dom}\mathcal{T}_m}$ will be denoted by $(\tilde{W}_t)_{t\in\tilde{\mathcal{I}}}$. Let us again state explicitly that we are interested in the evolution of an existing singularity, so we assume throughout the following that σ as well as $\tilde{\sigma}$ actually are singularities of the respective chains.

We will go into details only in the case that σ is, say, right dense. Other types are treated analogously. Listing all possible cases would be an elementary but tedious procedure. Thus let us assume throughout the following that σ is right dense.

To the singularity σ there corresponds a chain

$$\mathfrak{P}_{\sigma_-} \subsetneq \mathfrak{P}_1 \subsetneq \ldots \subsetneq \mathfrak{P}_\delta = \mathfrak{P}_{\sigma_+},$$

of dB-spaces all of which, except $\mathfrak{P}_{\sigma_-}$, are degenerated. In fact, $\Delta_- = \dim \mathfrak{P}_1^\circ = 1$, whereas $\Delta_+ = \dim \mathfrak{P}_\delta^\circ > 0$ can be arbitrary.

Remark 6.4. Notice that the case of a singularity of polynomial type with degree 1 is not covered by arguments similar to those which follow, since then there exist no associated degenerated dB-spaces. However, such a singularity either vanishes or remains of polynomial type with degree 1 as is seen from Lemma 4.7.

Lemma 6.5. *Assume that $\alpha(W_t)$ is bounded at σ and that $\alpha_\sigma := \lim_{t\to\sigma} \alpha(W_t) \neq 0$. Then the chain (6.3) is locally maximal at $\tilde{\sigma}$. The singularity $\tilde{\sigma}$ is again right* *dense. We have*

$$\tilde{\omega} = \omega,\ \tilde{\delta} = \delta.$$

For all $i = 1, \dots, \delta$ *we have*

$$\dim \tilde{\mathfrak{P}}_i^\circ = \dim \mathfrak{P}_i^\circ, \tag{6.4}$$

$$\max_{t<\tilde{\sigma}} \operatorname{ind}_- \tilde{W}_t - \max_{t<\sigma} \operatorname{ind}_- W_t = \min_{t>\tilde{\sigma}} \operatorname{ind}_- \tilde{W}_t - \min_{t>\sigma} \operatorname{ind}_- W_t =$$

$$= \operatorname{ind}_- \tilde{\mathfrak{P}}_i - \operatorname{ind}_- \mathfrak{P}_i = \begin{cases} 1\ , [m<0, \alpha_\sigma<0] \\ 0\ , [m<0, \alpha_\sigma>0] \text{ or } [m>0, \alpha_\sigma>0] \\ -1\, , [m>0, \alpha_\sigma<0] \end{cases}. \tag{6.5}$$

Proof : Since $\alpha_\sigma \neq 0$ we have $W_t \in \operatorname{dom} \mathcal{T}_m$ for t sufficiently close to σ. By Lemma 6.3 we have $\mathfrak{P}_i^\circ \subseteq \mathfrak{Q}(\mathfrak{P}_i)$ and

$$\alpha_\sigma = 1 + mK_0(0) \neq 0.$$

Hence (6.4) follows from Lemma 2.4, the equalities in (6.5) from Lemma 2.5 and Corollary 4.3, respectively. Now it is obvious that $\tilde{\omega} = \omega$ and $\tilde{\delta} = \delta$. Moreover, by Lemma 4.7, $\tilde{\sigma}$ is right dense. The very same argument as in the proof of Proposition 4.13 shows that the chain (6.3) is locally maximal at $\tilde{\sigma}$.

□

The next result follows by a similar argumentation, again with the aid of Lemma 6.3, Lemma 2.4, Lemma 2.5 and Corollary 4.3. For this reason we will omit its proof.

Lemma 6.6. *Assume that* $\alpha(W_t)$ *is bounded at* σ *and that* $\lim_{t\to\sigma} \alpha(W_t) = 0$. *Again the chain (6.3) is locally maximal at* $\tilde{\sigma}$. *The singularity* $\tilde{\sigma}$ *is right dense or dense depending whether* σ_- *is or is not right endpoint of an indivisible interval. We have*

$$\tilde{\omega} = \omega + 1,\ \tilde{\delta} = \delta + \begin{cases} 0\, , \tilde{\sigma} \text{ dense} \\ 1\, , \tilde{\sigma} \text{ right dense} \end{cases}.$$

For all $i = 1, \dots, \delta$ *we have*

$$\dim \tilde{\mathfrak{P}}_i^\circ = \dim \mathfrak{P}_i^\circ + 1,$$

as well as $\dim \tilde{\mathfrak{P}}_{\sigma_-}^\circ = 1$. *Moreover,*

$$\max_{t<\tilde{\sigma}} \operatorname{ind}_- \tilde{W}_t - \max_{t<\sigma} \operatorname{ind}_- W_t = \operatorname{ind}_- \tilde{\mathfrak{P}}_i - \operatorname{ind}_- \mathfrak{P}_i =$$

$$= \operatorname{ind}_- \tilde{\mathfrak{P}}_{\sigma_-} - \operatorname{ind}_- \mathfrak{P}_{\sigma_-} = \begin{cases} -1\, , m>0 \\ 0\ , m<0 \end{cases},$$

and

$$\min_{t>\tilde{\sigma}} \operatorname{ind}_- \tilde{W}_t - \min_{t>\sigma} \operatorname{ind}_- W_t = \begin{cases} 0\, , m>0 \\ 1\, , m<0 \end{cases}.$$

In the case that $\alpha(W_t)$ is unbounded the situation is quite different:

Lemma 6.7. *Assume that $\alpha(W_t)$ is unbounded at σ. In this case the chain (6.3) is not locally maximal. In fact*

$$\lim_{t\nearrow\sigma}\mathcal{T}_m(W_t)=:V_- \tag{6.6}$$

exists. If $\dim\mathfrak{P}^\circ_{\sigma_+}=1$ also $\lim_{t\searrow\sigma}\mathcal{T}_m(W_t)=:V_+$ exists. In order to extend the chain (6.3) to a chain being locally maximal at $\tilde\sigma$, we have to add V_- and plug in an indivisible interval of type zero and infinite length starting with V_-. Moreover, if V_+ exists, we have to add V_+ and plug in another indivisible interval of type zero and infinite length ending with V_+. The singularity $\tilde\sigma$ which remains has polynomial type or is right dense depending whether V_+ exists or not. We have

$$\tilde\omega=\omega-1,\ \tilde\delta=\delta-1, \tag{6.7}$$

$$\dim\tilde{\mathfrak{P}}^\circ_i=\dim\mathfrak{P}^\circ_i-1,\ i=1,\dots,\delta, \tag{6.8}$$

$$\max_{t<\tilde\sigma}\operatorname{ind}_-\tilde W_t-\max_{t<\sigma}\operatorname{ind}_-W_t=$$

$$=\operatorname{ind}_-\tilde{\mathfrak{P}}_i-\operatorname{ind}_-\mathfrak{P}_i=\begin{cases}0\,,\ m>0\\ 1\,,\ m<0\end{cases}, \tag{6.9}$$

$$\min_{t>\tilde\sigma}\operatorname{ind}_-\tilde W_t-\min_{t>\sigma}\operatorname{ind}_-W_t=\begin{cases}-1\,,\ m>0\\ 0\ \ ,\ m<0\end{cases}. \tag{6.10}$$

Proof : The relations (6.7)-(6.10) among negative indices, isotropic parts, weights and degrees follow from the same sources already referred to a couple of times. Hence our task is to show that the extension of (6.3) described in the assertion is in fact locally maximal at $\tilde\sigma$.

First note that, since $\alpha(W_t)$ is unbounded, we have $W_t\in\operatorname{dom}\mathcal{T}_m$ for t sufficiently close to σ.

We show that the limit (6.6) exists. The space $\tilde{\mathfrak{P}}_1$ is nondegenerated, hence there exists a number $t_1\in\tilde{\mathcal{I}}$ such that $\tilde{\mathfrak{P}}_1=\mathfrak{P}(E_{\tilde W_{t_1}})$ (cf. Lemma 3.15). From the relation

$$\max_{t<\sigma}\operatorname{ind}_-W_t=\operatorname{ind}_-W_{\sigma_-}=\operatorname{ind}_-\mathfrak{P}_{\sigma_-}=\operatorname{ind}_-\mathfrak{P}_1,$$

it follows by virtue of (6.9) that

$$\max_{t<\sigma}\operatorname{ind}_-\mathcal{T}_m(W_t)=\operatorname{ind}_-\tilde{\mathfrak{P}}_1=\operatorname{ind}_-\tilde W_{t_1}.$$

Since $\mathfrak{P}(E_{W_t})\subseteq\mathfrak{P}_1$ and $\mathfrak{P}(E_{W_t})=\mathfrak{P}(E_{\mathcal{T}_m(W_t)})$ as sets, we have $\mathcal{T}_m(W_t)=\tilde W_{t^\bullet}$ for $t^\bullet\le t_1$. Thus the limit (6.6) exists,

$$V_-=\lim_{t\nearrow\sigma}\mathcal{T}_m(W_t)=\tilde W_{t_0}$$

for some $t_0 \leq t_1$.

Next we show that in fact $t_0 = t_1$. The space $\tilde{\mathfrak{P}}_{\sigma_-}$ is contained isometrically in $\mathfrak{P}(E_{\tilde{W}_{t\bullet}})$ (for t sufficiently close to σ) as well as in $\mathfrak{P}(E_{\tilde{W}_{t_0}})$ and $\mathfrak{P}(E_{\tilde{W}_{t_1}})$ (cf. Lemma 4.7). Moreover, in any of these spaces it has codimension 1. Let $F \in \mathfrak{P}_1$, span $\{F\} = \mathfrak{P}_1^\circ$. Clearly,

$$[F, F]_{\mathfrak{P}(E_{\tilde{W}_{t_1}})} = [F, F]_{\mathfrak{P}_1} + m|F(0)|^2 = m|F(0)|^2.$$

On the other hand we compute

$$[F, F]_{\mathfrak{P}(E_{\tilde{W}_{t_0}})} = \lim_{t \nearrow \sigma} [F, F]_{\mathfrak{P}(E_{\tilde{W}_{\bullet t}})} = \lim_{t \nearrow \sigma} [F, F]_{\mathfrak{P}(E_{W_t})} + m|F(0)|^2 =$$

$$= [F, F]_{\mathfrak{P}_1} + m|F(0)|^2 = m|F(0)|^2.$$

Hence $\mathfrak{P}(E_{\tilde{W}_{t_0}}) = \mathfrak{P}(E_{\tilde{W}_{t_1}})$ isometrically, i.e. $t_0 = t_1$.

Now assume that $\tilde{W}_s \in \operatorname{dom} \mathcal{T}_{-m}$ for some $s \in \tilde{\mathcal{I}}$, $t_0 \leq s < \tilde{\sigma}$. Using Lemma 4.5 we conclude that

$$W_t^{-1} \mathcal{T}_{-m}(\tilde{W}_s) \in \mathcal{M}_{\operatorname{ind}_- \mathcal{T}_{-m}(\tilde{W}_s) - \operatorname{ind}_- W_t}, \; t < \sigma,$$

as well as

$$\mathcal{T}_{-m}(\tilde{W}_s)^{-1} W_t \in \mathcal{M}_{\operatorname{ind}_- W_t - \operatorname{ind}_- \mathcal{T}_{-m}(\tilde{W}_s)}, \; t > \sigma.$$

This, however, contradicts the maximality of the chain $(W_t)_{t \in \mathcal{I}}$. It follows that $\tilde{W}_s \notin \operatorname{dom} \mathcal{T}_{-m}$ for $t_0 \leq s < \tilde{\sigma}$. Thus $(t_0, \tilde{\sigma})$ is indivisible of type 0.

We have proved that the chain $(\tilde{W}_t)_{t \in \tilde{\mathcal{I}}}$ actually has the described form locally below $\tilde{\sigma}$. A similar argumentation will show the assertion connected with V_+.

□

Remark 6.8. The above discussion enables us to construct all possible types of singularities by starting from an appropriately chosen chain in $\mathfrak{M}_0$ and applying the transformations $\mathcal{T}_{m_i}$ and $\mathcal{T}^{\alpha_i}$ finitely often with appropriate $m_i > 0$ and $\alpha_i \in \mathbb{R}$.

We are led to a method to remove a singularity by performing some transformations. It employs the transformations $\mathcal{T}_m$ and $\mathcal{T}_J$, terminates after ω steps and is deterministic in the sense that it contains no unknown parameters.

Proposition 6.9. *Let $(W_t)_{t \in \mathcal{I}} \in \mathfrak{M}$ and consider a singularity σ. Denote by ω the weight of σ and put $\kappa := \max_{t < \sigma} \operatorname{ind}_- W_t$. Moreover, fix a positive value m. Assume that $w_{21,t}'(0)$ is unbounded at σ. Then the chain $(\mathcal{T}_J \mathcal{T}_m)^\omega((W_t)_{t \in \mathcal{I}})$ has no singularity at σ. In fact, for t sufficiently close to the point $\tilde{\sigma}$ corresponding to the former singularity σ in the chain $(\tilde{W}_t)_{t \in \tilde{\mathcal{I}}} := (\mathcal{T}_J \mathcal{T}_m)^\omega((W_t)_{t \in \mathcal{I}})$ we have*

$$\operatorname{ind}_- \tilde{W}_t = \kappa.$$

Proof : For an accurate proof we would need the above mentioned - but not carried out - discussion of the evolution of singularities of arbitrary type. Hence, here we in fact only prove a particular case.

Assume that σ is right dense. Then we will apply we will apply Lemma 6.7. If $\omega = 1$ we are already done. Otherwise we have to prolong the chain $(\mathcal{T}_m(W_t))_{t<\sigma}$. In fact, we add an indivisible interval of type 0. By virtue of the relation (4.5) the function $\alpha(\mathcal{T}_m((W_t)_{t\in\mathcal{I}}))$ remains bounded if $t \to \tilde{\sigma}$. Since the chain $\mathcal{T}_m((W_t)_{t\in\mathcal{I}})$ has a singularity at $\tilde{\sigma}$, the function

$$\alpha(\mathcal{T}_{\mathcal{J}}\mathcal{T}_m((W_t)_{t\in\mathcal{I}}))$$

must be unbounded at $\tilde{\sigma}$.

If σ is not right dense, we use the analogue of Lemma 6.7 in the respective cases.

Repeating the above procedure we obtain the asserted result.

□

Note that, since $\mathfrak{t}(W_t) \to \infty$ for $t \to \sigma$, the validity of the crucial assumption of Proposition 6.9 that $w_{21,t}'(0)$ is unbounded can always be achieved by a possible application of $\mathcal{T}_J$.

Remark 6.10. The structure of a singularity σ is a local property in the sense that it depends only on the behaviour of the transfer matrices W_{st} where s and t are sufficiently close to σ. Hence, for the study of a particular singularity we may often restrict to the case that $\operatorname{ind}_- W_t = 0$, $t < \sigma$.

7 Identification of intermediate Weyl coefficients

In this section we study the question which functions $q \in \mathcal{N}_\kappa$ can occur as intermediate Weyl coefficients at some singularity. In order to provide an answer to this question we will apply Proposition 6.9. Note that a function q is an intermediate Weyl coefficient if and only if $\frac{-1}{q}$ is. This can be seen by an application of $\mathcal{T}_{\mathcal{J}}$ (cf. Lemma II.10.1). Hence by Lemma 5.9 we may confine our attention to the case that $\lim_{y\searrow 0} yq(iy) = 0$. Moreover, we can assume by Remark 5.7 that q is in fact meromorphic in $\mathbb{C}$.

It is of importance to observe how intermediate Weyl coefficients change when performing the transformation $\mathcal{T}_J\mathcal{T}_m$.

Lemma 7.1. *Let $W \in \operatorname{dom}(\mathcal{T}_J\mathcal{T}_m)$. Then*

$$\mathcal{T}_J\mathcal{T}_m(W) \circ 0 = \frac{-1}{(W \circ \infty) - \frac{m}{z}}.$$

Proof : We have

$$\mathcal{T}_J\mathcal{T}_m(W)\circ 0=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\circ[\mathcal{T}_m(W)\circ\infty]=$$

$$=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & -\frac{m}{z}\\ 0 & 1\end{pmatrix}\circ[W\circ\infty].$$

□

Fix $m>0$. If $q\in\mathcal{N}_\kappa$, we define a sequence $(q_k)_{k\in\mathbb{N}\cup\{0\}}$ by the recursion

$$q_{k+1}(z):=\frac{-1}{q_k(z)-\frac{m}{z}},\ k\in\mathbb{N}\cup\{0\},$$

$$q_0(z):=q(z).$$

The following subclasses of $\mathcal{N}_\kappa$ are relevant for our purposes:

Definition 7.2. Let $\kappa\in\mathbb{N}\cup\{0\}$ and $\Delta\in\mathbb{N}\cup\{0\}$. We denote by $\mathcal{N}_{\kappa/\Delta}$ the set of all functions $q\in\mathcal{N}_\kappa$, such that $q_\Delta\in\mathcal{N}_\kappa$, there exists a matrix $W\in\mathcal{M}_\kappa$, $W(0)=1$, $w_{12}'(0)>0$, which represents q_Δ as

$$q_\Delta=W\circ 0,$$

and no function q_l, $l<\Delta$, admits such a representation.

Obviously a function $q\in\mathcal{N}_\kappa$ can belong to at most one set $\mathcal{N}_{\kappa/\Delta}$. Moreover,

$$\bigcup_{\Delta\in\mathbb{N}\cup\{0\}}\mathcal{N}_{\kappa/\Delta}\subsetneq\mathcal{N}_\kappa.$$

Equality does not hold, since all functions belonging to $\mathcal{N}_{\kappa/\Delta}$ are meromorphic in $\mathbb{C}$.

Lemma 7.3. *Let $q\in\mathcal{N}_{\kappa/\Delta}$, then $q_i\in\mathcal{N}_\kappa$ for all $i\in\{0,\dots\Delta\}$.*
Proof : Since $m>0$ the number $\operatorname{ind}_- q_i$ is nonincreasing.

□

Theorem 7.4. *Let $q\in\mathcal{N}_\kappa$, regular at ∞ and $\lim_{y\searrow 0} yq(iy)=0$. Then q is the intermediate Weyl coefficient of some singularity σ of a certain chain $(W_t)_{t\in\mathcal{I}}\in\mathfrak{M}$ if and only if*

$$q\in\bigcup_{\Delta\in\mathbb{N}}\mathcal{N}_{\kappa/\Delta}. \tag{7.1}$$

For the proof of this statement we need a technical lemma.

Lemma 7.5. *Let $(W_t)_{t\in\mathcal{I}} \in \mathfrak{M}$, σ be a singularity and denote by q_σ the intermediate Weyl coefficient at σ. Assume that $w'_{21,t}(0)$ is unbounded at σ. Then*

$$\operatorname{ind}_-(q_\sigma + \frac{m}{z}) = \operatorname{ind}_- q_\sigma + 1.$$

Proof : Locally at σ we have $W_t \in \operatorname{dom}\mathcal{T}_{-m}$. Put $\tilde{W}_t := \mathcal{T}_{-m}(W_t)$. Since $-m < 0$ we have for $t < \sigma$ sufficiently close to σ (cf. Corollary 4.3),

$$\operatorname{ind}_- \tilde{W}_t = \operatorname{ind}_- W_t + 1.$$

Since (W_t) does not end with an indivisible interval of type 0 at σ, also $(\tilde{W}_t)$ does not. Hence (cf. Lemma II.5.12)

$$\operatorname{ind}_-(q_\sigma + \frac{m}{z}) = \operatorname{ind}_- \lim_{t\nearrow\sigma}(\tilde{W}_t \circ \infty) = \lim_{t\nearrow\sigma} \operatorname{ind}_- \tilde{W}_t =$$

$$= \lim_{t\nearrow\sigma} \operatorname{ind}_- W_t + 1 = \operatorname{ind}_- q_\sigma + 1.$$

□

Proof (of Theorem 7.4): If q is an intermediate Weyl coefficient, then it follows from Proposition 6.9 and Lemma 7.1 that $q \in \mathcal{N}_{\kappa/\Delta}$ for some $\Delta \in \mathbb{N}$.

Conversely, assume that (7.1) holds. Then we have $q_\Delta = W \circ 0$ for some $W \in \mathcal{M}_\kappa$ with the properties cited in Definition 7.2. Thus with $m := \frac{1}{w'_{12}(0)}$ we have $m > 0$. With the help of the discussion after Remark 3.1 we construct a chain $(W_t)_{t\in\mathcal{I}}$ with $W = W_\sigma$, $\sigma \in \mathcal{I}$ and such that $W_{\sigma t}$, $t > \sigma$ is of the form (4.15) with t replaced by $t - \sigma$.

In what follows we will consider the chains $(W_t^{(l)}) := (\mathcal{T}_{-m}\mathcal{T}_{\mathcal{J}})^l((W_t)_{t\in\mathcal{I}})$, $l = 0, \ldots, \Delta$. We will show that q is the intermediate Weyl coefficient of $(W_t^{(\Delta)})$. Note that by Theorem 5.6 and [KW4] the matrix W can be chosen such that it does not end with an indivisible interval of type $\frac{\pi}{2}$.

The case $\Delta = 1$ is easily settled. By our choice of m the chain $(W_t^{(1)})$ has a singularity at σ. Since W does not end with an indivisible interval of type $\frac{\pi}{2}$, we have $W_t \in \operatorname{dom}(\mathcal{T}_{-m}\mathcal{T}_{\mathcal{J}})$ for t sufficiently close to σ, $t < \sigma$. For $t > \sigma$ trivially the same holds, since we linked with a chain that does not start with an indivisible interval. Hence $(W_t^{(1)})$ is given locally at σ as $(\mathcal{T}_{-m}\mathcal{T}_{\mathcal{J}}(W_t))$, and clearly q is the intermediate Weyl coefficient of $(W_t^{(1)})$ at σ.

Let us now treat the case that W ends with an indivisible interval, i.e. $W_{t'\sigma}$ is a linear polynomial of type $\neq \frac{\pi}{2}$ belonging to $\mathcal{M}_0$ for a certain $t' < \sigma$. We show that then necessarily $\Delta = 1$. So assume on the contrary $\Delta \geq 2$. Again by the choice of m the chain $(W_t^{(1)})$ has a singularity at σ. Choose t' so close at σ that $t \in \operatorname{dom}(\mathcal{T}_{-m}\mathcal{T}_{\mathcal{J}})$, $t \in [t', \sigma)$. By Lemma 4.7 $W_{t't}^{(1)}$ is a linear polynomial of non-zero type for $t \in (t', \sigma)$. In fact, the following argument, which involves the chains $(W_t^{(1)})$ and $(W_t^{(2)})$ shows that this type is $\frac{\pi}{2}$. From our choice of m we have for t sufficiently close at σ,

$$\operatorname{ind}_- W_t^{(1)} = \begin{cases} \kappa, \, t < \sigma \\ \kappa + 1, \, t > \sigma \end{cases} .$$

Lemma 7.5 implies that ${w_{12,t}^{(1)}}'(0)$ is bounded at σ, Hence the type is $\frac{\pi}{2}$. Thus ${w_{12,t}^{(1)}}'(0)$ is constant for $t \in (\sigma_-, \sigma)$.

Assume that ${w_{12,t}^{(1)}}'(0) \neq \frac{1}{m}$ on this interval. Then $W_t^{(1)} \in \operatorname{dom}(\mathcal{T}_{-m}\mathcal{T}_{\mathcal{J}})$ for $t \in (\sigma_-, \sigma)$. Consider the chain $(W_t^{(2)})$. Since ${w_{12,t}^{(1)}}'(0)$ is bounded at σ, and since the degree of negativity of $W_t^{(1)}$ increases at σ, Corollary 4.3 shows that also the degree of negativity of $W_t^{(2)}$ increases at σ. Note that the parameter α in that lemma is non-increasing in our situation. Hence σ is a singularity also for $(W_t^{(2)})$. If $\Delta = 2$, the assumption of the theorem implies that ${w_{21,t}^{(2)}}'(0)$ is unbounded at σ (cf. Lemma 5.9). If $\Delta > 2$ another application of Lemma 7.5 yields that ${w_{12,t}^{(2)}}'(0)$ is bounded at σ. Hence also in this case ${w_{21,t}^{(2)}}'(0)$ must be unbounded at σ. By (4.21) we conclude that $\lim_{t\to\sigma} {w_{12,t}^{(1)}}'(0) = \frac{1}{m}$, which is a contradiction to our assumption.

It remains to consider the case that ${w_{12,t}^{(1)}}'(0) = \frac{1}{m}$ for $t \in (\sigma_-, \sigma)$. However, then

$$q_1 = W_{\sigma_-}^{(1)} \circ 0$$

where $W_{\sigma_-}^{(1)} \in \mathcal{M}_\kappa$ and ${w_{12,\sigma_-}^{(1)}}'(0) = \frac{1}{m} > 0$. This contradicts the definition of $\mathcal{N}_{\kappa/\Delta}$ since $\Delta \geq 2$.

In the remaining case we need not bother whether $W_t^{(l)} \in \operatorname{dom}(\mathcal{T}_{-m}\mathcal{T}_{\mathcal{J}})$; this is trivially satisfied for t sufficiently close to σ.

We use induction on l to prove the following statements.

(i) ${w_{12,t}^{(l)}}'(0)$ is bounded at σ, $l = 1, \ldots, \Delta - 1$.

(ii) ${w_{21,t}^{(l)}}'(0)$ is unbounded at σ, $l = 1, \ldots, \Delta$,

(iii) for t sufficiently close to σ, $l = 1, \ldots, \Delta$,

$$\operatorname{ind}_- W_t^{(l)} = \begin{cases} \kappa, \, t < \sigma \\ \kappa + l, \, t > \sigma \end{cases} ,$$

(iv) $q_\sigma((W_t^{(l)})) = q_{\Delta - l}$, $l = 1, \ldots, \Delta$.

This will then finish the proof of the theorem.

Consider the chain $(W_t^{(1)})$. From our choice of m, (4.21) and Corollary 4.3 we conclude that *(ii)* and *(iii)* hold. Keeping in mind that the case $\Delta = 1$ is already settled, i.e. we can assume $\Delta \geq 2$, we obtain from Lemma 7.5 and the fact that $\operatorname{ind}_- q_{\Delta-2} = \kappa$ that *(i)* holds. The relation *(iv)* is obvious.

Let $1 < l < \Delta$ and assume that *(i)*-*(iv)* are already proved for $l' < l$. Since ${w_{12,t}^{(l-1)}}'(0)$ is bounded at σ, and since our parameter, which corresponds to α from Corollary 4.3, is non-increasing, we learn from Corollary 4.3 ($t < \sigma < s$, t, s sufficiently close to σ):

$$\operatorname{ind}_- W_s^{(l)} - \operatorname{ind}_- W_t^{(l)} \geq \operatorname{ind}_- W_s^{(l-1)} - \operatorname{ind}_- W_t^{(l-1)}.$$

It follows that σ is a singularity of the chain $(W_t^{(l)})$. Another application of Lemma 7.5 shows that ${w_{12,t}^{(l)}}'(0)$ is bounded at σ. Hence, necessarily, ${w_{21,t}^{(l)}}'(0)$ must be unbounded at σ, and we conclude ${w_{12,t}^{(l-1)}}'(0) \to \frac{1}{m}$ as $t \to \sigma$ (cf. (4.21)). By virtue of Corollary 4.3 now also *(iii)* follows. The relation *(iv)* is obvious.

It remains to consider the case $l = \Delta$. By the assumption of the theorem ${w_{21,t}^{(l)}}'(0)$ is bounded at σ. Hence *(iii)* and *(iv)* follow as above.

□

In the following we will employ the results of [Wo] in order to give a more internal, however only necessary, condition for $q \in \mathcal{N}_\kappa$ to belong to $\mathcal{N}_{\kappa/\Delta}$.

First note that the recursion defining the sequence $(q_k)_{k\in\mathbb{N}\cup\{0\}}$ can be solved explicitly.

Lemma 7.6. *Let m and q_0 be given, and define a sequence q_k, $k \in \mathbb{N}$, by*

$$q_{k+1} = \frac{-z}{zq_k - m},\ k = 0, 1, 2, \ldots.$$

Then

$$q_k = \frac{1}{r_k}\frac{q_0 - r_k}{q_0 - r_{k+1}},\ k \geq 2, \tag{7.2}$$

$$q_1 = \frac{-1}{q_0 - \frac{m}{z}},$$

where

$$r_k = \frac{\sum_{j=0}^{\left[\frac{k-1}{2}\right]} \lambda_j^{(k-1)} \left(\frac{z}{m}\right)^{2j}}{\sum_{j=0}^{\left[\frac{k}{2}-1\right]} \lambda_j^{(k-2)} \left(\frac{z}{m}\right)^{1+2j}}, \tag{7.3}$$

with some integers $\lambda_j^{(l)}$. The highest powers written in the numerator and the denominator actually occur. In fact, if l is even $\lambda_{[\frac{l}{2}]}^{(l)} = (-1)^{[\frac{l}{2}]}$, and if l is odd $\lambda_{[\frac{l}{2}]}^{(l)} = (-1)^{[\frac{l}{2}]}\frac{l+1}{2}$.

Proof : Put $f(x) = \frac{-z}{zx-m}$. Then $(q_k)_{k\in\mathbb{N}\cup\{0\}}$ is defined by the recurrence $q_{k+1} = f(q_k)$. In order to solve this recursion we assume that $z \neq \pm\frac{m}{2}$. Then $f(x)$ has two distinct fixed points

$$X_{1,2} = \frac{m}{2z} \pm \sqrt{\frac{m^2}{4z^2} - 1}.$$

We use the transformation $p_k = \varphi(q_k)$, where

$$\varphi(x) = \frac{x - X_1}{x - X_2}.$$

The sequence $(p_k)_{k\in\mathbb{N}\cup\{0\}}$ is the solution of the recursion

$$p_{k+1} = (\varphi \circ f \circ \varphi^{-1})(p_k).$$

Since $(\varphi \circ f \circ \varphi^{-1})(x) = \frac{X_1}{X_2}x$, we obtain $p_k = (\frac{X_1}{X_2})^k p_0$ and hence

$$q_k = \varphi^{-1}((\frac{X_1}{X_2})^k\varphi(q_0)).$$

A computation shows that

$$q_k = X_1X_2\frac{X_1^{k-1} - X_2^{k-1}}{X_1^k - X_2^k} \cdot \frac{q_0 - \frac{X_1^k - X_2^k}{X_1^{k-1} - X_2^{k-1}}}{q_0 - \frac{X_1^{k+1} - X_2^{k+1}}{X_1^k - X_2^k}}. \tag{7.4}$$

We use the fact that the coefficients in (7.4) are symmetric functions in X_1, X_2 to rewrite them in terms of $X_1X_2 = 1$ and $X_1 + X_2 = \frac{m}{z}$. The relation (7.3) follows from

$$\frac{X_1^k - X_2^k}{X_1^{k-1} - X_2^{k-1}} = \frac{X_1^{k-1} + X_1^{k-2}X_2 + \ldots + X_1X_2^{k-2} + X_2^{k-1}}{X_1^{k-2} + X_1^{k-3}X_2 + \ldots + X_1X_2^{k-3} + X_2^{k-2}},$$

since $X_1X_2 = 1$ gives

$$X_1^l + X_1^{l-1}X_2 + \ldots + X_1X_2^{l-1} + X_2^l = \sum_{j=0}^{[\frac{l}{2}]} \lambda_j^{(l)}(X_1 + X_2)^{l-2j}. \tag{7.5}$$

It remains to determine the coefficients $\lambda_{[\frac{l}{2}]}^{(l)}$. Assume first that l is even, and let z tend to ∞ in (7.5). We get $\lambda_{[\frac{l}{2}]}^{(l)} = (-1)^{[\frac{l}{2}]}$.

For l odd we can write

$$X_1^l+X_1^{l-1}X_2+\ldots+X_1X_2^{l-1}+X_2^l=(X_1+X_2)(X_1^{l-1}+X_1^{l-3}X_2^2+\ldots+X_1^2X_2^{l-3}+X_2^{l-1}).$$

If z tends to ∞ we obtain

$$\lambda^l_{[\frac{l}{2}]}=(-1)^{[\frac{l}{2}]}\frac{l+1}{2}.$$

□

The asymptotic behaviour of the terms r_k is easy to handle.

Lemma 7.7. *If k is even,*

$$r_k=\frac{mk}{2}\frac{1}{z}+O(\frac{1}{z^3}),$$

and if k is odd,

$$r_k=\frac{2}{m(k-1)}z+O(\frac{1}{z}).$$

For sufficiently large modulus the zeros of $q-r_k$ interlace with the poles of q.

Proof : The asymptotic behaviour of r_k is an immediate consequence of (7.3). Consider the real zeros of $q-r_k$. These are exactly the poles of

$$s(z)=\frac{-1}{q(z)-r_k(z)}.$$

Since r_k is a rational function, s belongs to $\mathcal{N}_\nu$ for some $\nu\in\mathbb{N}\cup\{0\}$. Hence all but finitely many of its real poles must be simple, and therefore the function has negative residuum at those poles.

Consider $t\in\mathbb{R}$ outside a disk which contains all poles of r_k and all multiple poles and poles with positive residuum of s. Clearly, since r_k is continuous outside this disk, the function $q-r_k$ must have at least one zero between two subsequent poles of q. Moreover, any zero must be simple and the derivative of $q-r_k$ at a zero must be positive. Thus there exists exactly one zero of $q-r_k$ between two subsequent poles of q.

□

Let us recall a result from [Wo].

Lemma 7.8. *Let $W\in\mathcal{M}_\kappa$. Denote by $(a_k)_{k\in\mathbb{N}}$ ($(a_k^+)_{k\in\mathbb{N}}$ and $(a_k^-)_{k\in\mathbb{N}}$) the sequences of real simple poles (positive poles and negative poles, respectively) of the function*

$$q(z):=W(z)\circ 0,$$

arranged according to increasing modulus. Then

$$\lim_{r\to\infty}\sum_{0<|a_k|<r}\frac{1}{a_k}=s\in\mathbb{R}, \tag{7.6}$$

$$\lim_{k\to\infty} \frac{k}{a_k^+} = \lim_{k\to\infty} \frac{k}{a_k^-} = \frac{\beta}{2} \in \mathbb{R}. \tag{7.7}$$

This is in fact nothing else but Proposition 2.2 of [Wo]. We only have to note that in the proof given there the assumption $q \in \mathcal{N}_0$ is not used at all.

Now we obtain a necessary condition for $q \in \mathcal{N}_\kappa$ to be an intermediate Weyl coefficient.

Corollary 7.9. *Assume that $q \in \mathcal{N}_\kappa$ is an intermediate Weyl coefficient of some chain. Then (7.6) and (7.7) hold.*

Proof : Consider first the case that $\lim_{y\searrow 0} -iyq(iy) = 0$. Then by Theorem 7.4 we have $q_\Delta = W \circ 0$. Now Lemma 7.8 shows that the poles of q_Δ satisfy (7.6) and (7.7). By Lemma 7.7 they interlace with the poles of q, and hence also the poles of q have these properties.

If $\lim_{y\searrow 0} -iyq(iy) > 0$, consider $-\frac{1}{q}$, which is also a Weyl coefficient of some chain and satisfies the assumption of the first paragraph of this proof. Hence the poles of $-\frac{1}{q}$ satisfy (7.6) and (7.7). Since the poles and the zeros of $-\frac{1}{q}$ interlace, also the poles of q possess these properties.

□

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