A characterization of intermediate Weyl coefficients

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Abstract

In connection with an indefinite analogue of canonical systems of differential equations some subclasses $\mathcal{N}_{0/\Delta}$ ($\Delta \in \mathbb{N}_0$) of the Nevanlinna class $\mathcal{N}_0$ come up. We give a criterion for a function to belong to $\mathcal{N}_{0/\Delta}$ in terms of its poles and residues.

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1 Introduction

In [KW3] some classes $\mathcal{N}_{0/\Delta}$ ($\Delta \in \mathbb{N}_0$) consisting of certain functions meromorphic in the plane and possessing only real poles and zeros were defined (we will recall this definition below, cf. Definition 1.1). It turned out that this notion is most helpful in the study of the indefinite analogue of so-called canonical systems of differential equations (cf. [Wi1], [Wi2], [KL], [KW2]). There one investigates so-called maximal chains of matrix functions.

In order to explain this notion, let us recall that $\mathcal{M}_\kappa$, $\kappa \in \mathbb{N}_0$, denotes the set of all entire 2×2-matrix functions $W(z)$ with $\det W(z) = 1$ that have the property that the matrix kernel

$$H(z,w) := \frac{W(z)JW(w)^* - J}{z - \overline{w}}, \tag{1.1}$$

where

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

has κ negative squares, i. e., the maximum of negative squares of the quadratic forms

$$\sum_{i,j=1}^{n} \big(H(z_i,z_j)x_i, x_j\big)_{\mathbb{C}^2}\, \xi_i \overline{\xi}_j$$

with $n \in \mathbb{N}$, $z_1, \dots, z_n \in \mathbb{C}$, and $x_1, \dots, x_n \in \mathbb{C}^2$, is equal to κ.

A maximal chain of matrix functions is a mapping which assigns to each $t \in \mathcal{I} = (0,M) \setminus \{s_1, \dots, s_n\}$ an entire 2×2-matrix function $W_t(z) \in \mathcal{M}_{\kappa(t)}$ and possesses certain factorization and maximality properties. Here $0 < s_1 < \cdots < s_n < M \leq \infty$ and $\kappa\colon \mathcal{I} \to \mathbb{N}_0$ is non-decreasing and locally constant. The points s_i are called singularities.

In the classical case, i. e., if there exist no singularities and $\kappa(t) = 0$, those maximal chains are exactly the solutions of canonical systems of differential equations. It is a basic result that for any $\tau \in \mathbb{R}$ the limit

$$\lim_{t \nearrow t_0} \frac{W_{t,11}(z)\tau + W_{t,12}(z)}{W_{t,21}(z)\tau + W_{t,22}(z)} =: q_{t_0}(z), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad t_0 \in \{s_1, \dots, s_n, M\},$$

exists, is independent of τ, and defines an analytic function on $\mathbb{C} \setminus \mathbb{R}$. The function q_M is called the Weyl coefficient of the chain and q_{s_i}, $i = 1, \dots, n$, are called *intermediate Weyl coefficients.*

If $(W_t)_{t\in\mathcal{I}}$ is a maximal chain of matrices, then also the family $(W_t)_{t\in(0,s_1)}$ possesses this property and, moreover, has no singularities. A converse question is the following: Given a maximal chain $(W_t)_{t\in(0,m)}$ without singularities and $\kappa(t) = 0$, is it part of a larger chain? It turned out (cf. [KW3]) that this is the case if and only if $q_m \in \mathcal{N}_{0/l}$ for some $l \in \mathbb{N}_0$. In fact, it is part of a longer chain with $\max_{t\in\mathcal{I}} \kappa(t) = \Delta$ if and only if $q_m \in \bigcup_{0\leq l\leq\Delta} \mathcal{N}_{0/l}$.

However, the actual definition of the class $\mathcal{N}_{0/l}$ originates in a recursive application of some transformation of chains and is, as the reader can convince himself below, quite intricate. An internal characterization whether a function q does or does not belong to $\mathcal{N}_{0/l}$ is lacking. It is the aim of this paper to redress this imperfection. We will prove a criterion which involves only the poles and residues (or equivalently the poles and zeros) of the given function q (cf. Theorem 5.1). It turns out that in this respect the distribution of the poles of q as well as the asymptotic behaviour of the residues (or of the canonical product associated with the sequence of zeros of q, respectively) is of significance.

The present results generalize the contents of [Wo] where the particular case $\Delta = 0$ has been settled. Besides the classical theory of growth and distribution of zeros of entire functions (cf. [B], [L]), our proofs rely on a connection with the theory of Hilbert spaces of entire functions by L. de Branges (cf. [dB]).

A function $f\colon \operatorname{dom} f \subseteq \mathbb{C} \to \mathbb{C}$, whose domain is symmetric with respect to the real axis, is called *real* if $f^\#(z) := \overline{f(\overline{z})} = f(z)$. The class $\mathcal{N}_0$ of so-called *Nevanlinna functions* is defined to be the collection of all analytic functions $q : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}$ which are real and map the upper half plane into itself. Note that this notion is different from the notion of functions of bounded type on a certain domain, the entirety of which is sometimes also named Nevanlinna class. Let us recall that a function f analytic on some domain D is called of *bounded type in* D if it can be expressed as a quotient of two functions analytic and bounded in D. In this note the domain D under consideration will always be the upper half plane, so whenever we speak of a function of bounded type, we mean bounded type in $\mathbb{C}^+$. Recall that for a function f of bounded type the limit

$$\operatorname{mt}(f) := \lim_{y\to+\infty} \frac{\log|f(iy)|}{y}$$

exists. This number is called the *mean type* of f.

1.1 Definition. We define the classes $\mathcal{N}_{0/\Delta}$, $\Delta \in \mathbb{N}_0$, separately for $\Delta = 0$ and $\Delta > 0$.

1. A function q is said to belong to the class $\mathcal{N}_{0/0}$ if it belongs to $\mathcal{N}_0$ and can be represented as the quotient $q(z) = \frac{B(z)}{A(z)}$ of two real entire functions which possess the property that there exist real entire functions C and D such that the 2×2-matrix function

$$W(z) := \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

is in $\mathcal{M}_0$ and $W(0) = I$.

2. For a function q we define a sequence $(q_k)_{k\in\mathbb{N}_0}$ recursively by $q_0 := q$ and

$$q_{k+1}(z) := \frac{-1}{q_k(z) - \frac{1}{z}},\ k \in \mathbb{N}_0. \tag{1.2}$$

In the case $\Delta > 0$ we write $q \in \mathcal{N}_{0/\Delta}$ if $q \in \mathcal{N}_0$ and $q_\Delta \in \mathcal{N}_{0/0}$, whereas $q_l \notin \mathcal{N}_{0/0}$ for $l < \Delta$.

Note that a function belonging to $\mathcal{N}_{0/\Delta}$ is necessarily meromorphic in the whole plane and all of its poles and zeros are real.

The criterion for a given function q to belong to $\mathcal{N}_{0/0}$ proved in [Wo] is concerned with the distribution of the poles $(a_k)_{k\in\mathbb{N}}$ of q and with the growth of the canonical product built from the zeros $(b_k)_{k\in\mathbb{N}}$ of q. In order to indicate to the reader where the difficulty in the treatment of the general class $\mathcal{N}_{0/\Delta}$ lies, let us sketch the influence of the recursion (1.2) on the poles and zeros of the respective functions. Obviously q_1 has a zero at t if either q has a pole there or $t = 0$. On the other hand t is a pole of q_1 if it solves the equation $q(z) = \frac{1}{z}$. Hence we might think of the poles of q_1 as perturbed zeros of q. If we proceed to q_2 we find that its zeros are perturbations of the zeros of q plus 0 and its poles are perturbations of the poles of q with a perturbation of 0 added.

Since for a function belonging to $\mathcal{N}_0$ the poles and zeros interlace, the distribution of the sequences of poles (or zeros) of $q_1, q_2, \ldots$ is fairly easy to handle. However, due to the indicated kind of perturbation, the behaviour of the above mentioned canonical product might change tremendously. Moreover, since an accurate solution of the equation $q(z) = \frac{1}{z}$ is out of reach (a standard example for $q(z)$ would be the function $\tan z$), it seems likely that an explicit treatment of these matters is hardly possible. As in [Wo] we will take a detour via de Branges' theory of Hilbert spaces of entire functions. This connection is, however, also of particular interest on its own right.

Let us outline the contents of this article. First, in Section 2, we set up some notation and recall briefly a couple of well-known results. In Section 3 we define the notion of functions l-associated to a de Branges space of entire functions. This is a fairly straightforward generalization of the concept of associated functions as introduced in [dB]. The aim of the fourth section is to provide an approximation method for de Branges spaces containing only functions with a zero of appropriate order at the origin. Our main result (Theorem 5.1) is stated at the beginning of Section 5. This section is mainly devoted to the proof of this statement. Moreover, we draw some conclusions from Theorem 5.1 and give some examples.

2 Preliminaries

In the sequel let us recall a couple of necessary ingredients. For an accurate treatment of the respective subjects we refer to [B], [dB], and [L].

An entire function E is said to belong to the *Hermite-Biehler class* $\mathcal{HB}$ if it satisfies

$$|E(\overline{z})| < |E(z)|,\ z \in \mathbb{C}^+.$$

We will write $E \in \mathcal{HB}^\times$ if E belongs to $\mathcal{HB}$ and has no real zeros. For each function $E \in \mathcal{HB}$ a Hilbert space $\mathcal{H}(E)$ is defined as the collection of all entire functions F such that $\frac{F}{E}$ and $\frac{F^\#}{E}$ are of bounded type, $\operatorname{mt}(\frac{F}{E}), \operatorname{mt}(\frac{F^\#}{E}) \leq 0$, and

$$\|F\|^2_{\mathcal{H}(E)} := \int_{-\infty}^{\infty} |F(t)|^2 \frac{dt}{|E(t)|^2} < \infty.$$

We set $A(z) := \frac{E(z)+E^\#(z)}{2}$ and $B(z) := i\frac{E(z)-E^\#(z)}{2}$, so that $E = A - iB$, and define for $\varphi \in \mathbb{R}$

$$S_\varphi(z) := \sin\varphi \cdot A(z) - \cos\varphi \cdot B(z).$$

In the space $\mathcal{H}(E)$ each point evaluation functional is continuous. The reproducing kernel K of the space, i. e., the family $K(w, \cdot\,)$, $w \in \mathbb{C}$, of functions of $\mathcal{H}(E)$ satisfying

$$F(w) = (F, K(w, \cdot\,)),\ \ F \in \mathcal{H}(E),$$

is given by (independently of $\varphi \in \mathbb{R}$)

$$K(w, z) = \frac{S_\varphi(\overline{w})S_{\varphi+\frac{\pi}{2}}(z) - S_\varphi(z)S_{\varphi+\frac{\pi}{2}}(\overline{w})}{\pi(z - \overline{w})},\ \ z \neq \overline{w}, \tag{2.1}$$

$$K(\overline{z}, z) = \frac{1}{\pi}\big(S_\varphi(z)S'_{\varphi+\frac{\pi}{2}}(z) - S'_\varphi(z)S_{\varphi+\frac{\pi}{2}}(z)\big). \tag{2.2}$$

We will frequently make use of the difference quotient operator associated with a function S ($S(w) \neq 0$)

$$(\mathcal{R}_{S;w}F)(z) := \frac{F(z) - \frac{F(w)}{S(w)}S(z)}{z - w}.$$

Note that for entire functions S and F, the difference quotient $\frac{F(z)S(w)-F(w)S(z)}{z-w}$ is analytic on $\mathbb{C}^2$. For $S = S_\varphi$ the operator $\mathcal{R}_{S;w}$ is the resolvent operator of a self-adjoint extension in $\mathcal{H}(E)$ of the operator of multiplication by the independent variable.

A continuous function $\phi : \mathbb{R} \to \mathbb{R}$ is called a *phase function for E* if, for all $t \in \mathbb{R}$, the relation

$$E(t)e^{i\phi(t)} \in \mathbb{R}$$

holds. Any such function satisfies

$$\phi'(t) = K(t, t)\frac{\pi}{|E(t)|^2},\ \ t \in \mathbb{R}\,,$$

cf. [dB, Problem 48].

A function $E \in \mathcal{H}B$ gives rise to a family of functions $q(\varphi; \cdot\,) \in \mathcal{N}_0$ which are meromorphic in $\mathbb{C}$, namely by ($\varphi \in \mathbb{R}$)

$$q(\varphi; z) = -\frac{S_{\varphi-\frac{\pi}{2}}(z)}{S_\varphi(z)}.$$

Conversely, if $q \in \mathcal{N}_0$ and is meromorphic in $\mathbb{C}$, then there exists a function $E_q \in \mathcal{H}B$, such that q can be obtained in that way. A more complete treatment of this relationship can be found in [dB] and [KW1].

Let us now state accurately the already mentioned conditions on the distribution of a sequence. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence of pairwise distinct real numbers which has no finite limit point. Denote by $(x_k^+)_{k\in\mathbb{N}}$ and $(x_k^-)_{k\in\mathbb{N}}$ the sequences of positive (negative, respectively) members of $(x_k)_{k\in\mathbb{N}}$ arranged according to increasing modulus. The conditions appearing in our context are:

(C1) The limit

$$\lim_{r\to\infty} \sum_{0<|x_k|\le r} \frac{1}{x_k}$$

exists in $\mathbb{R}$.

(C2) The following limits exist in $\mathbb{R}$ and are equal:

$$\lim_{k\to\infty} \frac{k}{x_k^+} = \lim_{k\to\infty} \frac{k}{|x_k^-|}.$$

For a sequence $(x_k)_{k\in\mathbb{N}}$ satisfying those conditions the product

$$f(z) := \lim_{r\to\infty} \prod_{0<|x_k|\le r} \Big(1 - \frac{z}{x_k}\Big)$$

converges locally uniformly on $\mathbb{C}$ and therefore represents an entire function. We define the entire function associated to the sequence $(x_k)_{k\in\mathbb{N}}$ by

$$x(z) := \begin{cases} f(z), & 0 \notin \{x_k : k \in \mathbb{N}\}, \\ zf(z), & 0 \in \{x_k : k \in \mathbb{N}\}. \end{cases} \tag{2.3}$$

The growth of $x(z)$ has been closely investigated, see, e. g., [B] or [L]. The essential facts for our situation have been collected in [Wo, Section 1].

We will also make use of two different representations of a Nevanlinna function q meromorphic in $\mathbb{C}$. First any such function can be written as (cf. [L, VII. Lehrsatz 2])

$$q(z) = a + bz + \sum_{t:q(t)=\infty} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\,\sigma_t, \tag{2.4}$$

where $a \in \mathbb{R}$, $b \ge 0$, and $\sigma_t = -\mathrm{Res}(q;t)$. The point masses σ_t are positive and satisfy

$$\sum_{t:q(t)=\infty} \frac{\sigma_t}{1+t^2} < \infty.$$

Secondly let us recall a multiplicative representation of q (cf. [L, VII. Lehrsatz 1])

$$q(z) = \gamma z^\delta \prod_{k\in\mathbb{N}} \frac{1 - \frac{z}{b_k}}{1 - \frac{z}{a_k}},$$

where $\gamma > 0$, $\delta = 1, 0, -1$ depending whether $q(0) = 0$, $q(0) \in \mathbb{R} \setminus \{0\}$, or $q(0) = \infty$, and $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ denote the sequences of non-zero poles of q and zeros of q, respectively, arranged appropriately. Such a representation is possible since the poles and zeros of q interlace. This fact also implies that the sequence of poles of q meets **(C1)** and **(C2)** if and only if the sequence of zeros of q satisfies **(C1)** and **(C2)**. In fact, for any $\alpha \in \mathbb{R} \cup \{\infty\}$ the sequence $(x_k)_{k\in\mathbb{N}}$ of all real numbers t with $q(t) = \alpha$ behaves in the same way in this respect.

3 Functions l-associated to $\mathcal{H}(E)$

In this section we generalize the notion of associated functions (cf. [dB, Theorem 25]).

3.1 Definition. Let $E \in \mathcal{H}B^\times$. Choose $F_0 \in \mathcal{H}(E)$ and $w_1, w_2, \ldots \in \mathbb{C}\setminus\mathbb{R}$ with $F_0(w_i) \neq 0$, $i = 1, 2, \ldots$. For $l \in \mathbb{N}$ an entire function S is called *l-associated to $\mathcal{H}(E)$* if

$$\mathcal{R}_{F_0;w_1} \cdot \ldots \cdot \mathcal{R}_{F_0;w_l} S \in \mathcal{H}(E).$$

The set of all l-associated functions is denoted by $\operatorname{Assoc}_l \mathcal{H}(E)$. For notational convenience set $\operatorname{Assoc}_0 \mathcal{H}(E) := \mathcal{H}(E)$.

3.2 Remark. Note that for $l = 1$ the notion of an l-associated function coincides with the notion of an associated function in the sense of [dB].

First of all we have to justify our definition of $\operatorname{Assoc}_l \mathcal{H}(E)$ by showing that it does not depend on the actual choice of F_0 and $w_1, w_2, \ldots$. This follows immediately from the next lemma.

3.3 Lemma. *Let $l \in \mathbb{N}_0$. A function S is l-associated to $\mathcal{H}(E)$ if and only if $\frac{S}{E}$ and $\frac{S^\#}{E}$ are of bounded type,* $\operatorname{mt}(\frac{S}{E}), \operatorname{mt}(\frac{S^\#}{E}) \leq 0$, *and*

$$\int_{-\infty}^{\infty} |S(t)|^2 \frac{dt}{(1+t^2)^l |E(t)|^2} < \infty. \tag{3.1}$$

Proof. The case $l = 0$ is trivial by definition. By virtue of Remark 3.2 the case $l = 1$ is nothing else but Theorem 25 of [dB]. We use induction on l.

Assume that the assertion has already been established for some $l \in \mathbb{N}$. Note that by definition $S \in \operatorname{Assoc}_{l+1} \mathcal{H}(E)$ if and only if $\mathcal{R}_{F_0;w_{l+1}} S \in \operatorname{Assoc}_l \mathcal{H}(E)$. The function F_0 satisfies condition (3.1) for all $l \in \mathbb{N}_0$. Hence S satisfies (3.1) for $l+1$ if and only if

$$(\mathcal{R}_{F_0;w_{l+1}} S)(z) = \frac{1}{z - w_{l+1}} \left(S(z) - \frac{S(w_{l+1})}{F_0(w_{l+1})} F_0(z) \right)$$

satisfies (3.1) for l. Moreover, $\frac{F_0}{E}$ and $\frac{F_0^\#}{E}$ are of bounded type and have non-positive mean type. Thus also the conditions that $\frac{S}{E}$ ($\frac{S^\#}{E}$, respectively) is of bounded type and has non-positive mean type and that $\frac{\mathcal{R}_{F_0;w_{l+1}} S}{E}$ ($\frac{(\mathcal{R}_{F_0;w_{l+1}} S)^\#}{E}$, respectively) is of bounded type and has non-positive mean type are equivalent. ☐

3.4 Corollary. *Let $l, k \in \mathbb{N}_0$. Then for any $w \in \mathbb{C}^+$*

$$\operatorname{Assoc}_{l+k} \mathcal{H}(E) = \operatorname{Assoc}_k \mathcal{H}((z+w)^l E(z)).$$

Proof. One only has to note that, for $w \in \mathbb{C}^+$, with E also the function $(z+w)^l E(z)$ belongs to $\mathcal{H}B^\times$. ☐

3.5 Corollary. *Let $l \in \mathbb{N}_0$, $T \in \operatorname{Assoc}_{l+1} \mathcal{H}(E)$ and $w \in \mathbb{C}$ with $T(w) \neq 0$. Then $\mathcal{R}_{T;w}$ maps $\operatorname{Assoc}_{l+1} \mathcal{H}(E)$ onto $\operatorname{Assoc}_l \mathcal{H}(E)$. We have*

$$\ker \mathcal{R}_{T;w} = \operatorname{span}\{T\}.$$

Proof. In the case $l = 0$ the assertion follows from Lemma 4.5 of [KW1] and the remark made before it. Corollary 3.4 implies the general case. ☐

4 Approximating spaces

In the sequel we study the spaces $\mathcal{H}((z+i\varepsilon)^l E(z))$ in dependence of $\varepsilon > 0$. Set

$$E_{l,\varepsilon}(z) := (z+i\varepsilon)^l E(z),\ l \in \mathbb{N}_0, \varepsilon > 0,$$

$$E_l^0(z) := z^l E(z),\ l \in \mathbb{N}_0.$$

Note the essential difference that $E_{l,\varepsilon} \in \mathcal{H}B^\times$, whereas for $l > 0$ we have $E_l^0 \in \mathcal{H}B \setminus \mathcal{H}B^\times$. However, still in some sense the spaces $\mathcal{H}(E_{l,\varepsilon})$ approximate $\mathcal{H}(E_l^0)$. Of course, for $l = 0$ all these spaces coincide since $E_{0,\varepsilon} = E_0^0 = E$.

To shorten notation denote the space $\mathcal{H}(E_{l,\varepsilon})$ by $\mathcal{H}_{l,\varepsilon}$, the space $\mathcal{H}(E_l^0)$ by $\mathcal{H}_l^0$ and let the functions $K_{l,\varepsilon}$, $S_{\varphi;l,\varepsilon}$, K_l^0 and $S_{\varphi;l}^0$ be defined correspondingly, e. g., write $E_{l,\varepsilon} = A_{l,\varepsilon} - iB_{l,\varepsilon}$, let $S_{\varphi;l,\varepsilon} = \sin\varphi \cdot A_{l,\varepsilon} - \cos\varphi \cdot B_{l,\varepsilon}$, and let $K_{l,\varepsilon}$ be as in (2.1) with $S_{\varphi;l,\varepsilon}$ instead of S_φ.

We collect some properties of the introduced spaces. First of all note that $\lim_{\varepsilon\searrow 0} E_{l,\varepsilon} = E_l^0$ locally uniformly on $\mathbb{C}$. Hence also $\lim_{\varepsilon\searrow 0} S_{\varphi;l,\varepsilon} = S_{\varphi;l}^0$ and $\lim_{\varepsilon\searrow 0} K_{l,\varepsilon} = K_l^0$ hold locally uniformly on $\mathbb{C}$ and $\mathbb{C}^2$, respectively.

4.1 Lemma. *Let $l \in \mathbb{N}$ be fixed. With the notation introduced above the following statements are valid:*

(i) The spaces $\mathcal{H}_{l,\varepsilon}$, $\varepsilon > 0$, coincide as sets. For $\varepsilon < \delta$ the identity mapping $\mathrm{id}_\delta^\varepsilon$ of $\mathcal{H}_{l,\varepsilon}$ onto $\mathcal{H}_{l,\delta}$ is a strict contraction, i. e.,

$$\|F\|_{\mathcal{H}_{l,\varepsilon}} > \|F\|_{\mathcal{H}_{l,\delta}}, \quad F \in \mathcal{H}_{l,\varepsilon} = \mathcal{H}_{l,\delta}, F \neq 0. \tag{4.1}$$

(ii) For any $w \in \mathbb{C}$ the function $K_{l,\varepsilon}(w,w)$ is non-increasing with decreasing ε. We have

$$\lim_{\varepsilon\searrow 0} K_{l,\varepsilon}(w,w) = K_l^0(w,w) = |w|^{2l} K(w,w) \tag{4.2}$$

locally uniformly on $\mathbb{C}$.

Proof. Assertion (i) is obvious. The fact that $K_{l,\varepsilon}(w,w)$ is non-increasing with decreasing ε follows by a straightforward argument from (4.1) since

$$K_{l,\varepsilon}(w,w) = \|K_{l,\varepsilon}(w,\cdot)\|^2_{\mathcal{H}_{l,\varepsilon}}$$

and $\|K_{l,\varepsilon}(w,\cdot)\|_{\mathcal{H}_{l,\varepsilon}}$ is the norm of the functional $F \mapsto F(w)$ in the space $\mathcal{H}_{l,\varepsilon}$.

The first equality of (4.2) is immediate from (2.1) and (2.2). It remains to prove

$$K_l^0(w,w) = |w|^{2l} K(w,w),\ w \in \mathbb{C}. \tag{4.3}$$

For $w \in \mathbb{C} \setminus \mathbb{R}$ this equality follows from

$$S_{\varphi;l}^0(z) = z^l S_\varphi(z)$$

and (2.1). Together with the fact that both sides of (4.3) are continuous functions of $w \in \mathbb{C}$, this implies the validity of (4.3) also for real values of w. □

4.2 Lemma. *Let $l \in \mathbb{N}$ and $\varepsilon > 0$ be fixed.*

(*i*) *We have*

$$\mathcal{H}_l^0 = \{\, F \in \mathcal{H}_{l,\varepsilon} \colon\ F(0) = \ldots = F^{(l-1)}(0) = 0 \,\} \tag{4.4}$$

as sets. The identity mapping $\mathrm{id}_\varepsilon^0$ *of* $\mathcal{H}_l^0$ *into* $\mathcal{H}_{l,\varepsilon}$ *is a strict contraction, i. e.,*

$$\|F\|_{\mathcal{H}_l^0} > \|F\|_{\mathcal{H}_{l,\varepsilon}}, \quad F \in \mathcal{H}_l^0, F \neq 0. \tag{4.5}$$

The norm of its inverse can be estimated explicitly:

$$\|(\mathrm{id}_\varepsilon^0)^{-1}\|^2 \leq (1+\varepsilon^2)^l + \int_{-1}^{1} \frac{dt}{|E(t)|^2} \cdot \max_{|z|=1} K_{l,\varepsilon}(z,z). \tag{4.6}$$

This bound is non-increasing with decreasing ε*.*

(*ii*) *If* $F \in \mathcal{H}_{l,\varepsilon} \setminus \mathcal{H}_l^0$*, then*

$$\lim_{\varepsilon \searrow 0} \|F\|_{\mathcal{H}_{l,\varepsilon}} = \infty.$$

Proof. First we deal with (*i*). The inclusion

$$\mathcal{H}_l^0 \subseteq \{F \in \mathcal{H}_{l,\varepsilon}^0 \colon F(0) = \ldots = F^{(l-1)}(0) = 0\}$$

is immediate from the fact that E_l^0 has a zero of order l at the origin and from the inequality

$$\int_{-\infty}^{\infty} |F(t)|^2 \frac{dt}{(\varepsilon^2+t^2)^l |E(t)|^2} < \int_{-\infty}^{\infty} |F(t)|^2 \frac{dt}{t^{2l}|E(t)|^2}.$$

This also implies (4.5).

Let $F \in \mathcal{H}_{l,\varepsilon}$, $F(0) = \ldots = F^{(l-1)} = 0$. We compute

$$\int_{-\infty}^{\infty} |F(t)|^2 \frac{dt}{|E_l^0(t)|^2} = \int_{-1}^{1} \Big|\frac{F(t)}{t^l}\Big|^2 \frac{dt}{|E(t)|^2} + \int_{|t|\geq 1} |F(t)|^2 \frac{1}{t^{2l}} \frac{dt}{|E(t)|^2} \leq$$
$$\leq \|F\|_{\mathcal{H}_{l,\varepsilon}}^2 \cdot \max_{|z|=1} K_{l,\varepsilon}(z,z) \cdot \int_{-1}^{1} \frac{dt}{|E(t)|^2} + (1+\varepsilon^2)^l \|F\|_{\mathcal{H}_{l,\varepsilon}}^2.$$

Here the first integral is estimated with the help of the maximum principle on the unit disk and the fact that

$$|F(z)| \leq \|F\|_{\mathcal{H}_{l,\varepsilon}} \|K_{l,\varepsilon}(z,\cdot)\|_{\mathcal{H}_{l,\varepsilon}}.$$

The estimate of the second integral is obtained from the elementary relation

$$\max_{|t|\geq 1} \frac{\varepsilon^2+t^2}{t^2} = 1+\varepsilon^2.$$

We conclude that equality (4.4) as well as inequality (4.6) hold true.

It remains to prove (*ii*). This, however, is immediate from the monotone convergence theorem. ☐

4.3 Lemma. *Let* ϕ *be a phase function for* E*,* $l \in \mathbb{N}$*,* $\varepsilon > 0$*. Then the function*

$$\phi_{l,\varepsilon}(x) := \begin{cases} \phi(x) - l \operatorname{Arctan} \frac{\varepsilon}{x}, & x > 0, \\ \phi(0) - l\frac{\pi}{2}, & x = 0, \\ \phi(x) - l \operatorname{Arctan} \frac{\varepsilon}{x} - l\pi, & x < 0 \end{cases}$$

is a phase function for $E_{l,\varepsilon}$. Moreover, the following relations hold:

$$\lim_{\varepsilon\searrow 0}\phi_{l,\varepsilon}^{-1}(\varphi)=\begin{cases}\phi^{-1}(\varphi), & \varphi>\phi(0),\\ 0, & \phi(0)-l\pi\le\varphi\le\phi(0),\\ \phi^{-1}(\varphi+l\pi), & \varphi<\phi(0)-l\pi.\end{cases} \tag{4.7}$$

Proof. The function $\phi_{l,\varepsilon}$ is a phase function for $E_{l,\varepsilon}$ since it is continuous and $(x+i\varepsilon)^l E(x)e^{i(\phi(x)-l\operatorname{Arctan}\frac{\varepsilon}{x})}=E(x)e^{i\phi(x)}\Big((x+i\varepsilon)e^{-i\operatorname{Arctan}\frac{\varepsilon}{x}}\Big)^l\in\mathbb{R}$.

To show relation (4.7), consider first the case $\varphi>\phi(0)$. For such a φ the equation $\phi_{l,\varepsilon}(x)=\varphi$ is the same as $\phi(x)-l\operatorname{Arctan}\frac{\varepsilon}{x}=\varphi$ for $\varepsilon>0$. From $\phi_{l,\varepsilon}(\phi^{-1}(\varphi))=\varphi-l\operatorname{Arctan}\frac{\varepsilon}{\phi^{-1}(\varphi)}<\varphi$ it follows

$$\phi_{l,\varepsilon}^{-1}(\varphi)>\phi^{-1}(\varphi). \tag{4.8}$$

This also implies

$$\varphi=\phi(\phi_{l,\varepsilon}^{-1}(\varphi))-l\operatorname{Arctan}\frac{\varepsilon}{\phi_{l,\varepsilon}^{-1}(\varphi)}>\phi(\phi_{l,\varepsilon}^{-1}(\varphi))-l\operatorname{Arctan}\frac{\varepsilon}{\phi^{-1}(\varphi)}$$

and hence $\phi_{l,\varepsilon}^{-1}(\varphi)<\phi^{-1}\big(\varphi+l\operatorname{Arctan}\frac{\varepsilon}{\phi^{-1}(\varphi)}\big)$, which proves, together with (4.8), relation (4.7) in this case.

Next let $\phi(0)-l\frac{\pi}{2}<\varphi\le\phi(0)$. As $\phi_{l,\varepsilon}(0)=\phi(0)-l\frac{\pi}{2}<\varphi$, we have

$$\phi_{l,\varepsilon}^{-1}(\varphi)>0.$$

For every $x>0$ there exists an $\varepsilon_0>0$ such that $\phi_{l,\varepsilon}(x)>\phi(0)\ge\varphi$ for all $\varepsilon\le\varepsilon_0$. Hence

$$\limsup_{\varepsilon\searrow 0}\phi_{l,\varepsilon}^{-1}(\varphi)\le x,$$

which implies $\lim_{\varepsilon\searrow 0}\phi_{l,\varepsilon}^{-1}(\varphi)=0$.

The case $\varphi=\phi(0)-l\frac{\pi}{2}$ is trivial and the cases $\varphi<\phi(0)-l\pi$ and $\phi(0)-l\pi\le\varphi<\phi(0)-l\frac{\pi}{2}$ are handled in a similar way as above. Here the equation $\phi(x)-l\operatorname{Arctan}\frac{\varepsilon}{x}-l\pi=\varphi$ has to be considered. ☐

5 Characterization of $\mathcal{N}_{0/\Delta}$

This section is devoted to the proof of our main result, the characterization of $q\in\mathcal{N}_{0/\Delta}$ in terms of the poles and residues (or poles and zeros) of q on the one hand and in terms of certain spaces $\mathcal{H}(E)$ (with $q=-\frac{S_0}{S_{\frac{\pi}{2}}}$) on the other. The major effort is needed to establish the connection between the asymptotic behaviour of poles and residues and the properties of the spaces $\mathcal{H}(E)$.

Let us first introduce the following notation. Denote by $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ the sequences of the poles and zeros of q, respectively. If $(a_k)_{k\in\mathbb{N}}$ (or equivalently $(b_k)_{k\in\mathbb{N}}$) satisfies **(C1)** and **(C2)**, then the canonical products $a(z)$ and $b(z)$ corresponding to $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$, respectively, can be defined as in (2.3). In this case we have

$$q(z)=\gamma_q\frac{b(z)}{a(z)}$$

with $\gamma_q>0$. Moreover, set $E_q(z):=a(z)-i\gamma_q b(z)$ and $S_{\varphi;q}(z)=\sin\varphi\cdot a(z)-\cos\varphi\cdot\gamma_q b(z)$.

5.1 Theorem. *Let $q \in \mathcal{N}_0$ be given and assume that q is meromorphic in $\mathbb{C}$ and $q \not\equiv 0$. Denote by $(a_k)_{k\in\mathbb{N}}$ and $(\sigma_k)_{k\in\mathbb{N}}$ the sequences of its poles and point masses (cf. (2.4)), respectively. Let $\Delta \in \mathbb{N}_0$, and if $\Delta = 0$, assume in addition that either $q(0) = 0$ or q has a pole at 0. Then the subsequent statements (i), (ii), and (iii) are equivalent.*

(i) *$q \in \bigcup_{l=0}^{\Delta} \mathcal{N}_{0/l}$ in the case that q is analytic at 0, and $q \in \bigcup_{l=0}^{\Delta+1} \mathcal{N}_{0/l}$ if q has a pole at 0.*

(ii) *The sequence $(a_k)_{k\in\mathbb{N}}$ satisfies* **(C1)** *and* **(C2)**. *Moreover,*

(C3)
$$\sum_{k\in\mathbb{N}} \frac{1}{\big(1+a_k^{2(1+\Delta)}\big)a'(a_k)^2\sigma_k} < \infty,$$

where $a(z)$ is defined by (2.3) using the sequence $(a_k)_{k\in\mathbb{N}}$.

(iii) *For one (and hence for all) $E \in \mathcal{H}B^\times$ with $q = -\frac{S_0}{S_{\frac{\pi}{2}}}$ the set $\operatorname{Assoc}_{\Delta+1} \mathcal{H}(E)$ contains a real and zero-free function $S(z)$.*

If $(a_k)_{k\in\mathbb{N}}$ satisfies **(C1)** *and* **(C2)**, *define E_q and $S_{\varphi;q}$ as in the paragraph preceeding the theorem. The condition* **(C3)** *in (ii) can be replaced by any of the following:*

(C3$_\mathbf{I}$) *For one $\varphi \in [0,\pi)$,*

$$\sum_{t:S_{\varphi;q}(t)=0} \frac{1}{(1+t^{2(1+\Delta)})\big|S'_{\varphi;q}(t)S_{\varphi+\frac{\pi}{2};q}(t)\big|} < \infty. \tag{5.1}$$

(C3$_\mathbf{II}$) *Relation (5.1) is valid for all $\varphi \in [0,\pi)$ and the sums in (5.1) are uniformly bounded with respect to φ.*

(C3$_\mathbf{III}$)
$$\int_{-\infty}^{\infty} \frac{1}{(1+t^2)^{1+\Delta}|E_q(t)|^2}\,dt < \infty.$$

5.2 Remark. The condition $q(0) = 0$ is necessary for $q \in \mathcal{N}_{0/0}$.

5.3 Remark. This theorem contains the result of [Wo] as a particular case: Theorem 1.1 of [Wo] states that in the case $\Delta = 0$ and $q(0) = 0$ the assertions (ii) with **(C3)** and (iii) are equivalent.

5.4 Remark. Let us remark that for a zero t of $S_{\varphi;q}$ we have

$$-S'_{\varphi;q}(t)S_{\varphi+\frac{\pi}{2};q}(t) = K_q(t,t),$$

where K_q is the kernel (2.1) corresponding to E_q. Moreover,

$$S_{0;q}(z) = -\gamma_q b(z), \quad S_{\frac{\pi}{2};q}(z) = a(z), \quad \sigma_k = -\gamma_q \frac{b(a_k)}{a'(a_k)}.$$

Hence condition **(C3)** coincides with **(C3$_\mathbf{I}$)** for $\varphi = \frac{\pi}{2}$.

The proof of Theorem 5.1 will be carried out in five steps. In the first step we settle the trivial case that q is rational. The following three steps are concerned with the proof of the equivalence of (ii) and (iii). In the last step we will make use of this fact to establish the equivalence with (i).

Step 1: Assume that q is rational, i. e., it has only finitely many poles and zeros. Then (ii) is trivially satisfied, even for $\Delta = 0$. Let $E \in \mathcal{H}B^\times$ and assume that $q = -\frac{S_0}{S_{\frac{\pi}{2}}}$. Then $S_{\frac{\pi}{2}}$ has only finitely many zeros, namely the poles $a_1, \dots, a_d$ of q. Since $S_{\frac{\pi}{2}} \in \operatorname{Assoc}_1 \mathcal{H}(E)$, the function

$$F(z) := \frac{S_{\frac{\pi}{2}}(z)}{\prod_{i=1}^d (z - a_i)}$$

belongs to $\operatorname{Assoc}_1 \mathcal{H}(E)$. It is real and zero-free, hence (iii) holds. Finally, note that the space $\mathcal{H}(E_q)$ consists of all polynomials with degree at most $d-1$. By [dB, Theorem 27] the function q belongs to $\mathcal{N}_{0/0}$ if $q(0) = 0$, i. e., (i) is valid for $\Delta = 0$. But even if $q(0) \neq 0$ we have $q_1(0) = 0$. So $q_1 \in \mathcal{N}_{0/0}$ and hence $q \in \mathcal{N}_{0/1}$.

Throughout the remainder of this section we will assume that q has infinitely many poles. This implies that for any function $E \in \mathcal{H}B^\times$ satisfying $q = -\frac{S_0}{S_{\frac{\pi}{2}}}$, the space $\mathcal{H}(E)$ is infinite-dimensional.

Step 2: Let us clarify the relationships among the conditions **(C3)**,**(C3$_\mathbf{I}$)**–**(C3$_\mathbf{III}$)**. Assume that **(C1)** and **(C2)** are valid, so that the function E_q is well defined. We shall provide evidence for the following implications:

$$\begin{array}{ccc} & \mathbf{(C3_{II})} & \\ \swarrow & & \searrow \\ \mathbf{(C3)} & & \mathbf{(C3_{III})} \\ \searrow & & \swarrow \\ & \mathbf{(C3_{I})} & \end{array} \tag{5.2}$$

By Remark 5.4 the left branch in the above diagram consists in fact of trivial observations. Next let ϕ be a phase function for E_q and define the functions

$$\phi_n : \begin{cases} [\phi^{-1}(n\pi), \phi^{-1}((n+1)\pi)] \to [0, \pi] \\ \phi_n(x) := \phi(x) - n\pi \end{cases}$$

for $n \in \mathbb{Z}$. With $f(t) := \frac{1}{(1+t^2)^{1+\Delta} K_q(t,t)}$ use Fubini's theorem to compute:

$$\int_0^\pi \sum_{t: S_{\varphi;q}(t)=0} f(t)\, d\varphi = \int_0^\pi \sum_{\phi(t) \equiv \varphi \mod \pi} f(t)\, d\varphi = \int_0^\pi \sum_{n \in \mathbb{Z}} f(\phi_n^{-1}(\varphi))\, d\varphi =$$

$$= \sum_{n \in \mathbb{Z}} \int_0^\pi f(\phi_n^{-1}(\varphi))\, d\varphi = \sum_{n \in \mathbb{Z}} \int_{\phi_n^{-1}(0)}^{\phi_n^{-1}(\pi)} f(u) \phi_n'(u)\, du = \int_{-\infty}^\infty f(u) \phi'(u)\, du =$$

$$= \int_{-\infty}^\infty \frac{\phi'(u)}{(1+u^2)^{1+\Delta} K_q(u,u)}\, du = \int_{-\infty}^\infty \frac{\pi}{(1+u^2)^{1+\Delta} |E_q(u)|^2}\, du.$$

Thus also the implications in the right branch of (5.2) hold true.

Step 3: Next we show that (iii) implies **(C1)** and **(C2)** as well as **(C3$_\mathbf{II}$)**. Let $E \in \mathcal{H}B^\times$, $q = -\frac{S_0}{S_{\frac{\pi}{2}}}$, and assume that $S \in \operatorname{Assoc}_{\Delta+1} \mathcal{H}(E)$ is real and zero-free. Then also $\hat{E} := \frac{E}{S}$ belongs to $\mathcal{H}B^\times$ and by Lemma 3.3 we have $1 \in \operatorname{Assoc}_{\Delta+1} \mathcal{H}(\hat{E})$. Since

$$\hat{S}_0 = \frac{S_0}{S},\ \hat{S}_{\frac{\pi}{2}} = \frac{S_{\frac{\pi}{2}}}{S},$$

we have $q=-\frac{\hat{S}_0}{\hat{S}_{\frac{\pi}{2}}}$. Proposition 2.2 of [Wo] yields that **(C1)** and **(C2)** are valid. Moreover, $\hat{E}$ in fact equals E_q.

Henceforth we may assume without loss of generality that $E=E_q$ and $1\in\operatorname{Assoc}_{\Delta+1}\mathcal{H}(E)$. Consider the space $\mathcal{H}_\Delta^0$ as defined in the beginning of Section 4 and choose $F\in\mathcal{H}_\Delta^0$ with $F(i)=1$, $F'(i)=\ldots=F^{(\Delta)}(i)=0$. Then trivially $F\in\mathcal{H}_{\Delta,\varepsilon}$ and, since $1\in\operatorname{Assoc}_1\mathcal{H}_{\Delta,\varepsilon}$, also

$$G(z):=\frac{F(z)-1}{z-i}\Big(\frac{z}{z-i}\Big)^\Delta\in\mathcal{H}_{\Delta,\varepsilon}.$$

Moreover, $G(0)=\ldots=G^{(\Delta-1)}(0)=0$. By Lemma 4.2, (i), in fact $G\in\mathcal{H}_\Delta^0$ and $\|G\|_{\mathcal{H}_{\Delta,\varepsilon}}\leq\|G\|_{\mathcal{H}_\Delta^0}$. For arbitrary $\varphi\in[0,\pi)$ we have (cf. [dB, Theorem 22])

$$\sum_{t:S_{\varphi;\Delta,\varepsilon}(t)=0}|G(t)|^2\frac{1}{K_{\Delta,\varepsilon}(t,t)}\leq\|G\|^2_{\mathcal{H}_{\Delta,\varepsilon}},\tag{5.3}$$

hence these sums are bounded uniformly with respect to ε and φ. Let ϕ be a phase function for E and $\phi_{\Delta,\varepsilon}$ a phase function for $E_{\Delta,\varepsilon}$ as defined in Lemma 4.3. Keeping in mind that by (4.7) of Lemma 4.3 and (4.2) of Lemma 4.1

$$\lim_{\varepsilon\searrow 0}\frac{|G(\phi_{\Delta,\varepsilon}^{-1}(\varphi+n\pi))|^2}{K_{\Delta,\varepsilon}(\phi_{\Delta,\varepsilon}^{-1}(\varphi+n\pi),\phi_{\Delta,\varepsilon}^{-1}(\varphi+n\pi))}=$$

$$=\begin{cases}\frac{|G(\phi^{-1}(\varphi+n\pi))|^2}{K_\Delta^0(\phi^{-1}(\varphi+n\pi),\phi^{-1}(\varphi+n\pi))}, & \varphi+n\pi\geq\phi(0),\\ \Big|\frac{G^{(\Delta)}(0)}{\Delta!}\Big|^2\frac{1}{K(0,0)}, & \phi(0)-\Delta\pi\leq\varphi+n\pi<\phi(0),\\ \frac{|G(\phi^{-1}(\varphi+(n+\Delta)\pi))|^2}{K_\Delta^0(\phi^{-1}(\varphi+(n+\Delta)\pi),\phi^{-1}(\varphi+(n+\Delta)\pi))}, & \varphi+n\pi<\phi(0)-\Delta\pi,\end{cases}$$

an application of Fatou's lemma yields

$$\begin{aligned}\|G\|^2_{\mathcal{H}_\Delta^0}\geq\|G\|^2_{\mathcal{H}_{\Delta,\varepsilon}}\geq&\sum_{t:S_{\varphi;\Delta,\varepsilon}(t)=0}|G(t)|^2\frac{1}{K_{\Delta,\varepsilon}(t,t)}=\\ =&\sum_{n\in\mathbb{Z}}\frac{|G(\phi_{\Delta,\varepsilon}^{-1}(\varphi+n\pi))|^2}{K_{\Delta,\varepsilon}(\phi_{\Delta,\varepsilon}^{-1}(\varphi+n\pi),\phi_{\Delta,\varepsilon}^{-1}(\varphi+n\pi))}\geq\\ \geq&\sum_{\substack{n\in\mathbb{Z}:\\ \varphi+n\pi\geq\phi(0)}}\frac{|G(\phi^{-1}(\varphi+n\pi))|^2}{K_\Delta^0(\phi^{-1}(\varphi+n\pi),\phi^{-1}(\varphi+n\pi))}+\\ &+\Delta\Big|\frac{G^{(\Delta)}(0)}{\Delta!}\Big|^2\frac{1}{K(0,0)}+\\ &+\sum_{\substack{n\in\mathbb{Z}:\\ \varphi+n\pi+\Delta\pi<\phi(0)}}\frac{|G(\phi^{-1}(\varphi+(n+\Delta)\pi))|^2}{K_\Delta^0(\phi^{-1}(\varphi+(n+\Delta)\pi),\phi^{-1}(\varphi+(n+\Delta)\pi))}\geq\\ \geq&\sum_{t:S_\varphi(t)=0}|G(t)|^2\frac{1}{K_\Delta^0(t,t)}=\\ =&\sum_{t:S_\varphi(t)=0}|G(t)|^2\frac{1}{t^{2\Delta}K(t,t)}.\end{aligned}$$

Since $F \in \mathcal{H}_\Delta^0$, in particular

$$\sum_{t:S_\varphi(t)=0} |F(t)|^2 \frac{1}{(1+t^2)^\Delta K(t,t)} \leq \sum_{t:S_\varphi(t)=0} |F(t)|^2 \frac{1}{t^{2\Delta} K(t,t)} \leq \|F\|_{\mathcal{H}_\Delta^0}.$$

Using the Minkowski inequality we obtain the estimate

$$\Big(\sum_{t:S_\varphi(t)=0} \frac{1}{1+t^2} \cdot \frac{1}{(1+t^2)^\Delta K(t,t)} \Big)^{\frac{1}{2}} \leq$$

$$\leq \Big(\sum_{t:S_\varphi(t)=0} |G(t)|^2 \Big(\frac{1+t^2}{t^2} \Big)^\Delta \frac{1}{(1+t^2)^\Delta K(t,t)} \Big)^{\frac{1}{2}} +$$

$$+ \Big(\sum_{t:S_\varphi(t)=0} \frac{|F(t)|^2}{1+t^2} \cdot \frac{1}{(1+t^2)^\Delta K(t,t)} \Big)^{\frac{1}{2}} \leq$$

$$\leq \|G\|_{\mathcal{H}_\Delta^0} + \|F\|_{\mathcal{H}_\Delta^0}.$$

Since $\frac{1}{1+t^{2(1+\Delta)}} \leq \frac{1}{(1+t^2)^{\Delta+1}}$, condition $(\mathbf{C3_{II}})$ follows.

Step 4: Assume that **(C1)**, **(C2)**, and $(\mathbf{C3_I})$ hold. We shall establish the validity of (iii). For this sake we need a slight generalization of [Wo, Proposition 3.1].

5.5 Lemma. *Let $(x_k)_{k\in\mathbb{N}}$ be a sequence of pairwise distinct real numbers which has no finite limit point. Assume that it satisfies the conditions* **(C1)** *and* **(C2)***, so that the function $x(z)$ in* (2.3) *is well defined. If for some $n \in \mathbb{N}_0$*

$$\sum_{k\in\mathbb{N}} \frac{1}{(1+|x_k|^n)|x'(x_k)|} < \infty, \tag{5.4}$$

then $x(z)$ is of bounded type.

Proof. The case $n = 2$ is nothing else but Proposition 3.1 of [Wo]. For $n = 0$ and $n = 1$ the assertion is therefore a trivial consequence of this. We shall employ a little trick in order to explain the general case by the mentioned result.

Choose n non-zero real numbers $y_1, \dots, y_n$ which do not occur among the x_k's. Consider the extended sequence $(\hat{x}_k)_{k\in\mathbb{N}}$ defined as

$$\hat{x}_k := \begin{cases} y_k, & k = 1, \dots, n \\ x_{k-n}, & k = n+1, n+2, \dots \end{cases}$$

Clearly, with $(x_k)_{k\in\mathbb{N}}$, also the sequence $(\hat{x}_k)_{k\in\mathbb{N}}$ possesses the properties **(C1)** and **(C2)**. Moreover, $\hat{x}(z) = p(z)x(z)$ where

$$p(z) := \prod_{i=1}^n \Big(1 - \frac{z}{y_i}\Big).$$

It follows that

$$|\hat{x}'(x_k)| = |p(x_k)| \cdot |x'(x_k)| \sim \frac{1}{\prod_{i=1}^n |y_i|} (1+|x_k|^n)|x'(x_k)|, \quad k \to \infty.$$

We only have to observe that the sequence $(\hat{x}_k)_{k\in\mathbb{N}}$ satisfies condition (5.4) with 0 in place of n. Then it follows from the known special case that $\hat{x}(z)$, and thus also $x(z)$, is of bounded type. $\square$

We claim that (for some $\lambda \in \mathbb{R} \setminus \{0\}$)

$$S_{\varphi;q}(z) = \lambda x(z),$$

where $x(z)$ denotes the product (2.3) built from the sequence of zeros of $S_{\varphi;q}$. Consider the entire function

$$F(z) := \frac{S_{\varphi;q}(z)}{x(z)} = \frac{\sin\varphi \cdot a(z) - \cos\varphi \cdot \gamma_q b(z)}{x(z)}.$$

By [L, VII. Lehrsatz 1] either the function $\frac{a(z)}{x(z)}$ or its inverse belongs to $\mathcal{N}_0$ and hence is of bounded type. The same applies to $\frac{b(z)}{x(z)}$. Therefore $F(z)$ is of bounded type as well. Since F is real and zero-free, a short argument employing [dB, Problem 34] implies that F is constant and hence establishes our claim.

Recall that, since $-\frac{S_{\varphi;q}}{S_{\varphi+\frac{\pi}{2};q}} \in \mathcal{N}_0$, we have

$$\sum_{t:S_{\varphi;q}(t)=0} \frac{1}{1+t^2} \cdot \frac{S_{\varphi;q}(t)}{S'_{\varphi+\frac{\pi}{2};q}(t)} < \infty.$$

Together with our assumption (5.1) and the elementary inequality $1 \leq u + \frac{1}{u}$, $u > 0$, this implies

$$\sum_{t:S_{\varphi;q}(t)=0} \frac{1}{(1+t^{2(1+\Delta)})|S'_{\varphi;q}(t)|} < \infty.$$

Lemma 5.5 allows us to conclude that $S_{\varphi;q}$, and hence also E_q, is of bounded type.

Since the function $S_{\varphi;q}$ is not a polynomial, there exists a point $w \in \mathbb{C}^+$ such that the value $S_{\varphi;q}(w)$ is non-real and is attained at least Δ times by $S_{\varphi;q}$, say for $z \in \{w_1 = w, w_2, \ldots, w_\Delta\}$. We claim that

$$S(z) := \frac{S_{\varphi;q}(z) - S_{\varphi;q}(w)}{\prod_{i=1}^{\Delta}(z-w_i)} \in \operatorname{Assoc}_1 \mathcal{H}(E_q). \tag{5.5}$$

From this claim it follows at once that $1 \in \operatorname{Assoc}_{\Delta+1} \mathcal{H}(E_q)$ since

$$S(z) = -S_{\varphi;q}(w)\big(\mathcal{R}_{S_{\varphi;q};w_\Delta} \cdots \mathcal{R}_{S_{\varphi;q};w_1} 1\big)(z).$$

It remains to establish relation (5.5).

Since $S_{\varphi;q}$ is of bounded type, by [dB, Problem 34] the function $|S_{\varphi;q}(iy)|$ is non-decreasing for $y > 0$. From the estimate

$$\Big|\frac{S(z)}{S_{\varphi;q}(z)}\Big| = \frac{1}{\prod_{i=1}^{\Delta}|z-w_i|}\Big|1 - \frac{S_{\varphi;q}(w)}{S_{\varphi;q}(z)}\Big| \leq \frac{1}{\prod_{i=1}^{\Delta}|z-w_i|}\Big(1 + \Big|\frac{S_{\varphi;q}(w)}{S_{\varphi;q}(z)}\Big|\Big),$$

we conclude that

$$\limsup_{y\to+\infty} \left|\frac{S(iy)}{S_{\varphi;q}(iy)}\right| < \infty.$$

A similar reasoning will show that

$$\limsup_{y\to+\infty}\left|\frac{S(-iy)}{S_{\varphi;q}(iy)}\right|<\infty.$$

Moreover, by our assumption (5.1) and the fact that all w_i are non-real,

$$\sum_{t:S_{\varphi;q}(t)=0}|S(t)|^2\frac{1}{(1+t^2)K_q(t,t)}=\sum_{t:S_{\varphi;q}(t)=0}\frac{|S_{\varphi;q}(w)|^2}{\prod_{i=1}^{\Delta}|t-w_i|^2}\frac{1}{(1+t^2)K_q(t,t)}<\infty.$$

Assume for the moment that $S_{\varphi;q}\notin\mathcal{H}(E_q)$. Since clearly $\frac{S}{E_q}$ and $\frac{S^\#}{E_q}$ are of bounded type, all assumptions of [dB, Problem 70] are fulfilled and we may conclude that $S\in\operatorname{Assoc}_1\mathcal{H}(E_q)$.

In the case $S_{\varphi;q}\in\mathcal{H}(E_q)$ consider the closure of the domain of the multiplication operator in $\mathcal{H}(E_q)$. It again is a space $\mathcal{H}(E)$, in fact by [dB, Problem 87] we have

$$(A_q,B_q)=(A,B)\begin{pmatrix}1-\beta z & \alpha z\\ -\gamma z & 1+\beta z\end{pmatrix}$$

for certain $\alpha,\beta,\gamma\in\mathbb{R}$. Moreover, $S_\varphi=S_{\varphi;q}$. It follows that $\mathcal{H}(E)$ is contained isometrically in $L^2(\nu)$ where ν is the discrete measure related to $S_{\varphi;q}$ and that $\frac{S_{\varphi;q}}{E}$ is of bounded type. Once again we employ [dB, Problem 70], obtain $S\in\operatorname{Assoc}_1\mathcal{H}(E)$, and hence also $S\in\operatorname{Assoc}_1\mathcal{H}(E_q)$.

We provided evidence to the assertion of (iii) in the case of the particular function E_q. Now let $E\in\mathcal{H}B^\times$ be arbitrary with $q=-\frac{S_0}{S_{\frac{\pi}{2}}}$. Then, by [dB, Theorem 24]

$$E(z)=S(z)E_q(z)$$

for some real and zero-free function S. Clearly S belongs to $\operatorname{Assoc}_{\Delta+1}\mathcal{H}(E)$.

Step 5: We use induction on Δ to establish the equivalence of (i) and (ii). Note that the case $\Delta=0$ is nothing else but Corollary 1.2 of [Wo]. The main ingredients in the proof of the inductive step are on the one hand the already proved fact that under the assumption of **(C1)** and **(C2)** the conditions **(C3$_\mathbf{I}$)** for $\varphi=0$ and $\varphi=\frac{\pi}{2}$, respectively, are equivalent, and on the other hand the lemma below, which is based upon an elementary calculation.

Let us remark that we need not bother about the validity of **(C1)** and **(C2)**. This originates in the fact that the poles of q_{k+1} interlace its zeros which are the poles of q_k plus 0, and therefore possess the same distribution.

5.6 Lemma. *Let $q\in\mathcal{N}_0$ be meromorphic in the plane and let $n\in\mathbb{N}$ be given. Assume that* **(C1)** *and* **(C2)** *are satisfied. If q is analytic at 0, then q satisfies* **(C3$_\mathbf{I}$)** *with $\varphi=\frac{\pi}{2}$ and $\Delta=n$ if and only if q_1 (defined by (1.2)) satisfies* **(C3$_\mathbf{I}$)** *with $\varphi=0$ and $\Delta=n-1$.*

In the case that q has a pole at 0, q satisfies **(C3$_\mathbf{I}$)** *with $\varphi=\frac{\pi}{2}$ and $\Delta=n$ if and only if q_1 satisfies* **(C3$_\mathbf{I}$)** *with $\varphi=0$ and $\Delta=n$.*

Proof. Consider first the case that q is analytic at 0. Denote by $(\hat a_k)_{k\in\mathbb{N}}$ and $(\hat b_k)_{k\in\mathbb{N}_0}$ the sequences of poles and zeros, respectively, of q_1. They can be arranged so that $\hat b_0=0$ and $\hat b_k=a_k$, $k\geq1$. Let the functions $\hat a(z)$ and $\hat b(z)$ be defined correspondingly.

We clearly have $\hat{b}(z) = za(z)$, and therefore

$$\hat{b}'(a_k) = a_k a'(a_k).$$

We compute

$$\operatorname{Res}\Big(-\frac{1}{q_1(z)}; a_k\Big) = -\frac{1}{\gamma_{q_1}}\frac{\hat{a}(a_k)}{\hat{b}'(a_k)} = -\frac{1}{\gamma_{q_1}}\frac{\hat{a}(a_k)}{a_k a'(a_k)},$$

and

$$\operatorname{Res}\Big(q(z) - \frac{1}{z}; a_k\Big) = \gamma_q \frac{b(a_k)}{a'(a_k)}.$$

It follows that ($-\frac{1}{q_1(z)} = q(z) - \frac{1}{z}$)

$$\hat{a}(a_k) = -\gamma_q \gamma_{q_1} a_k b(a_k),$$

and, in view of (2.2), that

$$K_{E_{q_1}}(\hat{b}_k, \hat{b}_k) = \frac{1}{\pi}\hat{a}(\hat{b}_k)\gamma_{q_1}\hat{b}'(\hat{b}_k) = -\frac{1}{\pi}\gamma_q\gamma_{q_1} a_k b(a_k)\cdot\gamma_{q_1}\cdot a_k a'(a_k) =$$
$$= \gamma_{q_1}^2 a_k^2 K_{E_q}(a_k, a_k).$$

Thus for an arbitrary function $g(t)$

$$\frac{1}{\gamma_{q_1}^2}\sum_{t:S_{\frac{\pi}{2};q}(t)=0}\frac{g(t)}{t^2}\frac{1}{K_{E_q}(t,t)} = \sum_{t:S_{0;q_1}(t)=0} g(t)\frac{1}{K_{E_{q_1}}(t,t)} - \pi\frac{g(0)}{\gamma_{q_1}}. \tag{5.6}$$

An application of (5.6) with $g(t) := \frac{1}{1+t^{2n}}$ yields the assertion of the lemma in this case.

If q has a pole at 0, then $\hat{b}_k = a_k$ for all k, provided the sequences are arranged appropriately. Hence $\hat{b}(z) = a(z)$ and

$$K_{E_{q_1}}(\hat{b}_k, \hat{b}_k) = \gamma_{q_1}^2 K_{E_q}(a_k, a_k).$$

The rest of the proof is analogous. □

We will now complete the proof. First note that $q \in \bigcup_{l\leq\Delta_0+1}\mathcal{N}_{0/l}$ if and only if $q_1 \in \bigcup_{l\leq\Delta_0}\mathcal{N}_{0/l}$. This is immediate from the fact that $q \mapsto q_1$ maps $\mathcal{N}_{0/l}$ into $\mathcal{N}_{0/l-1}$ for $l > 0$ by definition, and maps $\mathcal{N}_{0/0}$ into itself by (5.6) and the already established equivalence of (i) and (ii) in case $\Delta = 0$. Furthermore, note that $q_j(0) = 0$ for $j \geq 1$.

We consider first the case that q is analytic at 0, where we use induction on Δ. Assume that the asserted equivalence of (i) and (ii) has already been proved for $\Delta = \Delta_0 - 1$. Let $q \in \bigcup_{l\leq\Delta_0}\mathcal{N}_{0/l}$. Then $q_1 \in \bigcup_{l\leq\Delta_0-1}\mathcal{N}_{0/l}$ and therefore q_1 satisfies **(C1)**, **(C2)**, and **($\mathbf{C3_I}$)** with $\varphi = 0$ and $\Delta = \Delta_0 - 1$. By Lemma 5.6 the function q thus satisfies **($\mathbf{C3_I}$)** with $\varphi = \frac{\pi}{2}$ and $\Delta = \Delta_0$, i.e., (i) implies (ii). Reversing these arguments yields the converse implication.

In the case that q has a pole at 0, then $q \in \bigcup_{l\leq\Delta_0+1}\mathcal{N}_{0/l}$ implies $q_1 \in \bigcup_{l\leq\Delta_0}\mathcal{N}_{0/l}$. Since $q_1(0) = 0$, we can use the already proved case to show that q_1 satisfies **(C1)**, **(C2)**, and **($\mathbf{C3_I}$)** with $\varphi = 0$ and $\Delta = \Delta_0$. Using the second part of Lemma 5.6 we get that q satisfies **($\mathbf{C3_I}$)** with $\varphi = \frac{\pi}{2}$ and $\Delta = \Delta_0$. Again the arguments can be reversed to show the converse implication.

All assertions of Theorem 5.1 are proved.

5.7 Remark. As we have seen in the foregoing arguments, condition (iii) could be replaced by

(iii') $1 \in \operatorname{Assoc}_{\Delta+1} \mathcal{H}(E_q)$.

In the remainder of this section we draw some conclusions and give some examples.

5.8 Corollary. *Let $q \in \mathcal{N}_0$, meromorphic in $\mathbb{C}$, satisfy* **(C1)** *and* **(C2)***, and assume that q is analytic at 0. Define E_q as in Theorem 5.1 and let $\Delta \in \mathbb{N}$. Then $1 \in \operatorname{Assoc}_1 \mathcal{H}((z+i)^\Delta E_q)$ if and only if $1 \in \operatorname{Assoc}_1 \mathcal{H}(E_{q_\Delta})$.*

Proof. Assume that $1 \in \operatorname{Assoc}_1 \mathcal{H}((z+i)^\Delta E_q)$, i. e., $1 \in \operatorname{Assoc}_{1+\Delta} \mathcal{H}(E_q)$. By Theorem 5.1 we have $q \in \bigcup_{l=0}^{\Delta} \mathcal{N}_{0/l}$ and thus $q_\Delta \in \mathcal{N}_{0/0}$. That is by [dB, Theorem 27] nothing else but $1 \in \operatorname{Assoc}_1 \mathcal{H}(E_{q_\Delta})$. Reversing these arguments leads to the converse implication. □

5.9 Corollary. *Let $(a_k)_{k\in\mathbb{Z}}$ and $(b_k)_{k\in\mathbb{Z}}$ be sequences of distinct real numbers with*

$$a_k < b_k < a_{k+1},\ \ k \in \mathbb{Z}.$$

Assume that **(C1)** *and* **(C2)** *are satisfied, and let $a(z)$ and $b(z)$ be defined as in* (2.3)*. Denote by $E(z)$ the function* ($\gamma \in \mathbb{R} \setminus \{0\}$)

$$E(z) := a(z) - i\gamma b(z).$$

If for some $n \in \mathbb{N}$

$$\int_{-\infty}^{\infty} \frac{1}{(1+|t|^n)|E(t)|^2}\, dt < \infty,$$

then also ($\log^+ x := \log\max\{1, x\}$)

$$\int_{-\infty}^{\infty} \frac{\log^+ |E(t)|}{1+t^2}\, dt < \infty. \tag{5.7}$$

Proof. Set $q(z) := \gamma \frac{b(z)}{a(z)}$. Then either q or $-q$ belongs to $\mathcal{N}_0$. Consider first the case $q \in \mathcal{N}_0$. Then $E(z) = E_q(z)$ where E_q is as in Theorem 5.1. Choose $\Delta \in \mathbb{N}$, such that $n \leq 2(1+\Delta)$. Then, by Theorem 5.1, $1 \in \operatorname{Assoc}_{1+\Delta} \mathcal{H}(E)$. Thus E is of bounded type and henceforth satisfies (5.7).

In the case $-q \in \mathcal{N}_0$ consider the function $E^\#$ instead of E. □

As a first example we consider functions $q \in \mathcal{N}_0$ whose poles lie at $\mathbb{Z} \setminus \{0\}$. Then, clearly, **(C1)** and **(C2)** are satisfied and $a(z) = \frac{\sin \pi z}{\pi z}$.

5.10 Corollary. *Let $\sigma_n > 0$, $n \in \mathbb{Z} \setminus \{0\}$, be the point masses at the poles n. Then q belongs to $\bigcup_{l=0}^{\Delta} \mathcal{N}_{0/l}$, $\Delta \in \mathbb{N}$, if and only if*

$$\sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{1}{n^{2\Delta}\sigma_n} < \infty.$$

Proof. For the proof one only has to note that $a'(n) = \frac{(-1)^n}{n}$. Then the assertion follows immediately from Theorem 5.1. □

Next consider functions q with poles at $\mathbb{Z}\setminus\{0,1\}$. Similar as above we obtain:

5.11 Corollary. *Let $\sigma_n > 0$, $n \in \mathbb{Z} \setminus \{0,1\}$, be the point masses at the poles n. Then q belongs to $\bigcup_{l=0}^{\Delta} \mathcal{N}_{0/l}$ if and only if*

$$\sum_{n \in \mathbb{Z} \setminus \{0,1\}} \frac{1}{n^{2(\Delta-1)}\sigma_n} < \infty.$$

A comparison of the last two corollaries shows the sensitivity of the conditions of Theorem 5.1 with respect to the actual location of the poles of q. Hence the condition **(C3)** cannot be thought of as a purely asymptotic condition.

5.12 Remark. In particular the above corollaries show that for all $\Delta \geq 0$ we have $\mathcal{N}_{0/\Delta} \neq \emptyset$, namely for $\Delta > 0$ take, e. g., $\sigma_n = |n|^{2(1-\Delta)}$ and for $\Delta = 0$ take $q(z) = \tan z$. Moreover, there exist functions $q \in \mathcal{N}_0$ which are meromorphic in the whole plane but still do not belong to any class $\mathcal{N}_{0/\Delta}$.

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