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Resolvent Matrices in Degenerated Inner Product Spaces

By Harald Woracek of Vienna

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1. Introduction

Let $\langle \mathfrak{P}, [\,\cdot\,,\,\cdot\,]\rangle$ be a Pontryagin space, S be a densely defined closed symmetric operator in $\mathfrak{P}$ with defect index $(1,1)$ and let u be an element of $\mathfrak{P}$. It has been proved in [KL] that there exists a 2×2 – matrix valued function $W(z) = (w_{ij}(z))_{i,j=1}^2$ which is analytic in a certain open set, such that the formula

$$r_u(z) \;=\; \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_{21}(z)\tau(z) + w_{22}(z)} \tag{1.1}$$

establishes a bijective correspondence between the set of so – called u – resolvents of S

$$r_u(z) \;:=\; \left[(A-z)^{-1}u, u\right],$$

where A runs through the selfadjoint extensions of S acting in some Pontryagin spaces $\widetilde{\mathfrak{P}} \supseteq \mathfrak{P}$, and the set $\bigcup_{\nu=0}^{\infty}\mathcal{N}_\nu$ of parameters $\tau(z)$. Here $\mathcal{N}_\nu$ denotes the set of all functions τ meromorphic in $\mathbb{C}\setminus\mathbb{R}$, $\tau(\overline{z}) = \overline{\tau(z)}$, such that the Nevanlinna kernel

$$N_\tau(z,w) \;:=\; \frac{\tau(z) - \overline{\tau(w)}}{z - \overline{w}}$$

has ν negative squares. For notational convenience we assume that the function $\tau(z) \equiv \infty$ belongs to $\mathcal{N}_0$. A matrix $W(z)$ with the above property is called a u – resolvent matrix of S. The existence of a u – resolvent matrix is a consequence of Krein's formula on the description of generalized resolvents.

In [KW3] the element u was allowed to be a so – called generalized element, which leads to a natural characterization of those matrix functions $W(z)$ which appear as

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resolvent matrices. For a particular subclass of the set of all resolvent matrices, namely for those W where $u \in \mathfrak{P}$, a related characterization can be found in [KL].

Assume now that $\mathfrak{P}$ is an inner product space which satisfies the following two axioms:

(D1) The isotropic part $\mathfrak{P}^\circ$ of $\mathfrak{P}$ is finite dimensional.

(D2) The space $\mathfrak{P}/\mathfrak{P}^\circ$ is a Pontryagin space.

Then an analogue of Krein's formula has been proved in [KW2]. The first aim of this note is to introduce an appropriate notion of generalized elements for a space $\mathfrak{P}$ satisfying **(D1)** and **(D2)** and a closed symmetric relation $S \subseteq \mathfrak{P}^2$ with defect index $(1,1)$, and to derive a formula of the type (1.1) for a generalized element u. Secondly, a characterization of those matrices $W(z)$ shall be given which can be represented as $u-$resolvent matrices in this setting, i. e. with a relation S in a space $\mathfrak{P}$ which is degenerated $\big(\dim \mathfrak{P}^\circ > 0\big)$. Finally, we consider inner product spaces of entire functions which satisfy certain additional axioms (compare [dB], [KW4]) and show that for such spaces the set of generalized elements can be identified with a set of entire functions known as the set of associated functions. This supplements the results of [KW4], Section 10.

In Section 2 we provide the theory of generalized elements and triplet spaces for a closed symmetric relation with defect index (n,n), $n \in \mathbb{N}$, in a Pontryagin space, which is similar to the considerations of [KW3] in the case of defect $(1,1)$. This notion is used to define triplet spaces for a degenerated inner product space $\mathfrak{P}$. Section 3 is concerned with the study of regularized resolvents of $S \subseteq \mathfrak{P}^2$. In particular an appropriate version of Krein's formula is proved (Proposition 3.7). The characterization of resolvent matrices of symmetric relations in degenerated spaces is given in Section 4 (Proposition 4.3). This result is not constructive in the sense that it uses an abstract model for a certain selfadjoint relation (compare [KW3]). However, if the symmetric relation S is minimal, the conditions can be reformulated in terms of the asymptotic behaviour of the entries of W for $z \to i\infty$ (Proposition 4.5). In Section 5 we consider spaces of entire functions and investigate the mentioned interpretation of generalized elements (Proposition 5.1).

Our notation is similar to that of [KW2] and [KW3]. For some elementary facts concerning Pontryagin spaces and linear relations therein we refer to [IKL] and [DS]. In the case that $\mathfrak{P}$ is a Hilbert space different related constructions of spaces of generalized elements can be found e. g. in [B], [GG] or [LT].

2. Triplet spaces

Let $\langle \mathfrak{P}, [\,\cdot\,,\,\cdot\,]\rangle$ be a Pontryagin space and let $S \subseteq \mathfrak{P}^2$ be a symmetric relation with equal and finite defect numbers. Choose a fundamental symmetry $\mathcal{J}$ on $\mathfrak{P}$ and define $(\,\cdot\,,\,\cdot\,) := [\mathcal{J}\,\cdot\,,\,\cdot\,]$. We use the following notation (compare [KW3]):

$$\left(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\right)_+ := (a_1, a_2) + (b_1, b_2)\,, \quad a_i\,, b_i \in \mathfrak{P}, \quad \mathfrak{P}_+ := \langle S^*, (\,\cdot\,,\,\cdot\,)_+\rangle\,,$$

$$\pi\ :\ \begin{cases} S^* & \longrightarrow & \mathfrak{P} \\ \begin{pmatrix} a \\ b \end{pmatrix} & \longmapsto & a \end{cases},\quad \iota\ :\ \begin{cases} \mathfrak{P} & \longrightarrow & \mathfrak{P}/\ker \pi^* \\ a & \longmapsto & \hat{a} \end{cases},$$

$$\left(\hat{a},\hat{b}\right)_- := (\pi^* a, \pi^* b)_+,\ \hat{a},\hat{b},\in \mathfrak{P}/\ker\pi^*,\ \mathfrak{P}_- := \overline{\langle \mathfrak{P}/\ker\pi^*, (\,\cdot\,,\,\cdot\,)_-\rangle} \oplus \langle S^*(0), (\,\cdot\,,\,\cdot\,)\rangle,$$

$$V\ :\ \begin{cases} (\mathfrak{P}/\ker\pi^*) \oplus S^*(0) & \longrightarrow & \mathfrak{P}_+ \\ \hat{a}\oplus b & \longmapsto & \pi^* a + \begin{pmatrix} 0 \\ b \end{pmatrix} \end{cases}.$$

The inner product of the space $\mathfrak{P}_-$ will again be denoted by $(\,\cdot\,,\,\cdot\,)_-$. If $\widetilde{\mathfrak{P}} \supseteq \mathfrak{P}$ is another Pontryagin space, denote

$$\widetilde{\mathfrak{P}}_+ := \mathfrak{P}_+ \oplus \left(\widetilde{\mathfrak{P}}[-]\mathfrak{P}\right)^2,\quad \widetilde{\mathfrak{P}}_- := \mathfrak{P}_- \oplus \left(\widetilde{\mathfrak{P}}[-]\mathfrak{P}\right)^2,$$

$$\widetilde{V} := V \oplus \mathrm{id}_{\widetilde{\mathfrak{P}}[-]\mathfrak{P}}.$$

Moreover, we define a duality

$$\left[\begin{pmatrix} a \\ b \end{pmatrix}, u\right]_\pm := \left(\begin{pmatrix} a \\ b \end{pmatrix}, \widetilde{V}u\right)_+,\quad \begin{pmatrix} a \\ b \end{pmatrix} \in \widetilde{\mathfrak{P}}_+,\quad u \in \left(\mathfrak{P}/\ker\pi^* \dot{+} S^*(0)\right) \oplus \left(\widetilde{\mathfrak{P}}[-]\mathfrak{P}\right).$$

Lemma 2.1. *With the above notation we have:*

$$\ker\pi = S^*_\infty,\quad \operatorname{ran}\pi = \operatorname{dom} S^*,\quad \ker\pi^* = S(0),\quad \overline{\operatorname{ran}\pi^*} = S^{*\,(\perp)_+}_\infty.$$

The mappings V $\left(\widetilde{V}\right)$ are isometric, hence extend by continuity to $\mathfrak{P}_-$ $\left(\widetilde{\mathfrak{P}}_-\right)$. These extensions will again be denoted by V $\left(\widetilde{V}\right)$. Also the duality $[\,\cdot\,,\,\cdot\,]_\pm$ extends to $\mathfrak{P}_+ \times \mathfrak{P}_-$. We have $V\iota = \pi^*$, hence

$$\left[\begin{pmatrix} a \\ b \end{pmatrix}, \iota f\right]_\pm = \left[\pi\begin{pmatrix} a \\ b \end{pmatrix}, f\right],\quad \begin{pmatrix} a \\ b \end{pmatrix} \in \mathfrak{P}_+,\quad f \in \mathfrak{P}.$$

If $f \in \mathfrak{P}$, then $\pi^* f$ is the $(\,\cdot\,,\,\cdot\,)_+$ –orthogonal projection of $\begin{pmatrix} \mathcal{J}f \\ 0 \end{pmatrix}$ onto S^*.

Proof. With exception of the last statement all assertions are proved similar as the corresponding results in [KW3]. To prove the last assertion note that for any $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathfrak{P}_+$

$$\left(\pi^* f, \begin{pmatrix} a \\ b \end{pmatrix}\right)_+ = [f,a] = (\mathcal{J}f, a) = \left(\begin{pmatrix} \mathcal{J}f \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\right)_+. \qquad \square$$

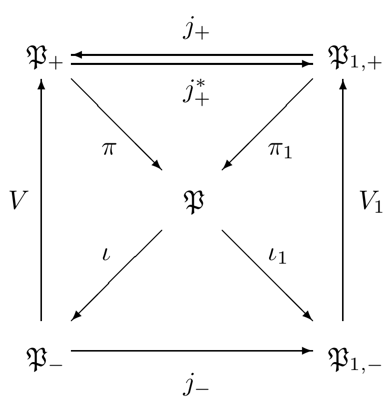
For notational convenience we put

$$\left[u, \begin{pmatrix} a \\ b \end{pmatrix}\right]_\pm := \overline{\left[\begin{pmatrix} a \\ b \end{pmatrix}, u\right]_\pm}.$$

Let S, $S_1 \subseteq \mathfrak{P}^2$ be symmetric relations and assume that $S \subseteq S_1$. Then clearly $S_1^* \subseteq S^*$, hence if $\mathfrak{P}_+$ and $\mathfrak{P}_{1,+}$ denote the spaces constructed with S and S_1, respectively, we have $\mathfrak{P}_{1,+} \subseteq \mathfrak{P}_+$. In the following we investigate how the corresponding spaces $\mathfrak{P}_-$ and $\mathfrak{P}_{1,-}$ are connected. Let π, ι, V (π_1, ι_1, V_1) be constructed as above starting from S (S_1), denote by j_+ the embedding of $\mathfrak{P}_{1,+}$ into $\mathfrak{P}_+$ and let j_+^* be its adjoint with respect to the inner products $(\,\cdot\,,\,\cdot\,)_+$ and $(\,\cdot\,,\,\cdot\,)_{1,+}$. Note that these inner products are in fact the same and that j_+^* is the $(\,\cdot\,,\,\cdot\,)_+$–orthogonal projection of S^* onto S_1^*. Moreover, define a mapping $j_- : \mathfrak{P}_- \to \mathfrak{P}_{1,-}$ by

$$j_- \;:=\; V_1^{-1} j_+^* V\,. \tag{2.1}$$

Then we are in the following situation:



Note the formal similarity with the situation considered in [KW3], Section 7. However, there it is assumed that the relation S has defect $(1,1)$ in a smaller space $\mathfrak{P}' \subseteq \mathfrak{P}$ which need not be the case in the present situation.

Lemma 2.2. *With the above introduced notation the following relations hold:*

$$\pi j_+ \;=\; \pi_1\,, \quad j_-\iota \;=\; \iota_1\,.$$

The mapping j_- is the adjoint of j_+ with respect to the dualities $[\,\cdot\,,\,\cdot\,]_{1,\pm}$ and $[\,\cdot\,,\,\cdot\,]_\pm$.

Proof. The first relation is obvious since j_+ is the embedding of $\mathfrak{P}_{1,+}$ into $\mathfrak{P}_+$. To prove the second relation we compute

$$j_-\iota \;=\; V_1^{-1} j_+^* V\iota \;=\; V_1^{-1} j_+^* \pi^* \;=\; V_1^{-1}\pi_1^* \;=\; \iota_1\,.$$

It follows from the definition (2.1) of j_- that for $u \in \mathfrak{P}_-$ and $\binom{a}{b} \in \mathfrak{P}_{1,+}$ the relation

$$\left[j_- u, \begin{pmatrix} a \\ b \end{pmatrix}\right]_{1,\pm} \;=\; \left[u, j_+ \begin{pmatrix} a \\ b \end{pmatrix}\right]_\pm$$

holds. □

The just introduced general notion of triplet spaces will be used to define spaces $\mathfrak{P}_+$ and $\mathfrak{P}_-$ also if $\mathfrak{P}$ is degenerated. In the remainder of this paper $\mathfrak{P}$ will always be assumed to be an inner product space satisfying the axioms **(D1)** and **(D2)** and which is actually degenerated, i. e. $\Delta := \dim \mathfrak{P}^\circ > 0$. If $\mathfrak{P}_n$ is a nondegerated subspace of $\mathfrak{P}$ with $\mathfrak{P}_n \dot{+} \mathfrak{P}^\circ = \mathfrak{P}$, we define a Pontryagin space

$$\mathfrak{P}_c \; := \; \mathfrak{P}_n [\dot{+}] \big(\mathfrak{P}^\circ \dot{+} \mathfrak{P}'\big)\,, \tag{2.2}$$

where $\mathfrak{P}'$ is an isomorphic copy of $\mathfrak{P}^\circ$ which is skewly linked to $\mathfrak{P}^\circ$ (compare [IKL]).

Let S be a closed symmetric relation in $\mathfrak{P}$ with defect index $(1,1)$; for the notion of defect indices in degenerated spaces compare [KW2]. Then S can be considered as a relation in $\mathfrak{P}_c$ with defect index $(\Delta+1, \Delta+1)$.

If $\{h_1, \ldots, h_\Delta\}$ and $\{h_1', \ldots, h_\Delta'\}$ are skewly linked bases of $\mathfrak{P}^\circ$ and $\mathfrak{P}'$, i. e. if

$$\operatorname{span}\{h_1, \ldots, h_\Delta\} \; = \; \mathfrak{P}^\circ\,, \quad \operatorname{span}\{h_1', \ldots, h_\Delta'\} \; = \; \mathfrak{P}'\,, \quad \big[h_i, h_j'\big] \; = \; \delta_{ij}\,,$$

and if $\mathcal{J}_n$ is a fundamental symmetry of $\mathfrak{P}_n$, then the mapping $\mathcal{J} : \mathfrak{P} \to \mathfrak{P}$ defined by

$$\mathcal{J}\big|_{\mathfrak{P}_n} \; = \; \mathcal{J}_n\,, \quad \mathcal{J}(h_i) \; = \; h_i'\,, \quad \mathcal{J}\big(h_i'\big) \; = \; h_i\,,$$

is a fundamental symmetry of $\mathfrak{P}_c$. Using this fundamental symmetry and the symmetric relation $S \subseteq \mathfrak{P}_c^2$, we construct spaces $\mathfrak{P}_{c,+}$ and $\mathfrak{P}_{c,-}$. Note that $\mathfrak{P}^\circ \times \mathfrak{P}^\circ \subseteq S^* \subseteq \mathfrak{P}_c^2$.

Definition 2.3. Denote in the following

$$\mathfrak{P}_+ \; := \; \mathfrak{P}_{c,+} \cap (\mathfrak{P}_n + \mathfrak{P}')^2 \; = \; \mathfrak{P}_{c,+}(-)_+(\mathfrak{P}^\circ)^2\,, \quad \mathfrak{P}_- \; := \; \overline{\iota\mathfrak{P}} \oplus (S^*(0)(-)_-\mathfrak{P}^\circ)\,,$$

where the closure of $\iota\mathfrak{P}$ has to be understood in the space $\mathfrak{P}_{c,-}$.

Lemma 2.4. *With the above definition we have*

$$V\mathfrak{P}_- \; = \; \mathfrak{P}_+\,. \tag{2.3}$$

Moreover, $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathfrak{P}_+$ *if and only if* $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathfrak{P}_{c,+}$ *and both,* $\mathcal{J}a$ *and* $\mathcal{J}b$*, are contained in* $\mathfrak{P}$*.*

Proof. Since $\mathfrak{P}^\circ \times \{0\} \subseteq S^*$ and $\mathcal{J}\mathfrak{P}' = \mathfrak{P}^\circ$, we conclude from Lemma 2.1 that $\pi^*\mathfrak{P}' = \mathfrak{P}^\circ \times \{0\}$. Since $\mathfrak{P}'$ is finite dimensional, $\iota\mathfrak{P}'$ is closed. Hence it follows that $(\iota\mathfrak{P}' \oplus \mathfrak{P}^\circ)^{(\perp)_-} = \mathfrak{P}_-$. Since V is an isometry of $\mathfrak{P}_{c,-}$ onto $\mathfrak{P}_{c,+}$ and maps $\iota\mathfrak{P}' \oplus \mathfrak{P}^\circ$ onto $(\mathfrak{P}^\circ)^2$, we obtain (2.3). □

3. Regularized resolvents

As in the second part of the previous section let $\mathfrak{P}$ be a fixed inner product space which satisfies the axioms **(D1)** and **(D2)** and assume that $\Delta = \dim \mathfrak{P}^\circ > 0$. Moreover, let $S \subseteq \mathfrak{P}^2$ be a closed symmetric relation with defect index $(1,1)$. Let $\widetilde{\mathfrak{P}}$ be a Pontryagin space which extends $\mathfrak{P}$ and let $\tilde{A} \subseteq \widetilde{\mathfrak{P}}^2$ be a selfadjoint relation with nonempty resolvent set which extends S. A straightforward argument yields:

Lemma 3.1. *The space* $\widetilde{\mathfrak{P}}$ *can be considered as an extension of* $\mathfrak{P}_c$.

Denote by $\widetilde{P}$ the orthogonal projection of $\widetilde{\mathfrak{P}}$ onto $\mathfrak{P}_c$ and by $\widetilde{P}^+$ the orthogonal projection of $\widetilde{\mathfrak{P}}_+$ onto $\mathfrak{P}_+$. As in [KW3] we may define operators $R_z^+ : \widetilde{\mathfrak{P}} \to \widetilde{\mathfrak{P}}_+$ and $R_z^- : \widetilde{\mathfrak{P}}_- \to \widetilde{\mathfrak{P}}$ by

$$R_z^+ f \;:=\; \begin{pmatrix} (A-z)^{-1} f \\ (I + z(A-z)^{-1}) f \end{pmatrix}, \quad f \in \widetilde{\mathfrak{P}}, \quad R_z^- \;:=\; \left(R_{\overline{z}}^+\right)^* \widetilde{V},$$

and $\widetilde{R}_z^+ : \mathfrak{P}_c \to \mathfrak{P}_{c,+}$ and $\widetilde{R}_z^- : \mathfrak{P}_{c,-} \to \mathfrak{P}_c$ by

$$\widetilde{R}_z^+ \;:=\; \widetilde{P}^+ R_z^+|_{\mathfrak{P}_c}, \quad \widetilde{R}_z^- := \widetilde{P} R_z^-|_{\mathfrak{P}_{c,-}}.$$

With similar arguments as in [KW3] we prove that $(z, w \in \rho(A))$

$$\begin{aligned} R_z^+ - R_w^+ &= (z-w) R_z^+ (A-w)^{-1}, \\ R_z^- - R_w^- &= (z-w)(A-z)^{-1} R_w^-, \\ \left[R_z^+ f, u\right]_\pm &= \left[f, R_{\overline{z}}^- u\right], \quad f \in \mathfrak{P}_c, \quad u \in \mathfrak{P}_{c,-}, \end{aligned}$$

$$\begin{aligned} \ker R_z^+ &= \{0\}, \quad \operatorname{ran} R_z^+ \;=\; A, \\ \ker R_z^- &= V^{-1}(\mathfrak{P}_{c,+}(\perp)_+ A), \quad \operatorname{ran} R_z^- \;=\; \mathfrak{P}_c. \end{aligned} \tag{3.1}$$

Lemma 3.2. *Let* $A \subseteq \mathfrak{P}_c^2$, $\rho(A) \neq \emptyset$, *be such that* $(A-z)^{-1}\mathfrak{P} \subseteq \mathfrak{P}$ *for* $z \in \rho(A)$. *Then*

$$\dim(\ker R_w^- \cap \mathfrak{P}_-) \;=\; 1, \quad R_w^- \mathfrak{P}_- \;=\; \mathfrak{P}. \tag{3.2}$$

Proof. First note that the condition $(A-z)^{-1}\mathfrak{P} \subseteq \mathfrak{P}$ implies $R_w^+ \mathfrak{P}^\circ \subseteq \mathfrak{P}^\circ \times \mathfrak{P}^\circ$, hence for $u \in \mathfrak{P}_-$ we have

$$\left[R_w^- u, \mathfrak{P}^\circ\right] \;=\; \left[u, R_{\overline{w}}^+ \mathfrak{P}^\circ\right]_\pm \;=\; 0, \tag{3.3}$$

i. e. $R_w^- u \in \mathfrak{P}$. We also conclude that $\dim(A \cap (\mathfrak{P}^\circ \times \mathfrak{P}^\circ)) \geq \Delta$. The reverse inequality holds anyway as $\rho(A) \neq \emptyset$, since otherwise $A \cap \ker((x; y) \mapsto y - zx) \neq \{0\}$ and we obtain a contradiction if z is chosen in $\rho(A)$. Since $\operatorname{codim}_{\mathfrak{P}_{c,+}} A = \Delta + 1$, this yields the first relation in (3.2).

Denote by P the $(\,\cdot\,,\,\cdot\,)_+$-orthogonal projection of $\mathfrak{P}_{c,+}$ onto $\mathfrak{P}_{c,+}(-)_+ A$, then the above consideration shows that $\dim P(\mathfrak{P}^\circ \times \mathfrak{P}^\circ) = \Delta$. Let $u \in \mathfrak{P}_{c,-}$ be given such that $R_w^- u \in \mathfrak{P}$. By (3.3) we have $u(\perp)_\pm(A \cap (\mathfrak{P}^\circ \times \mathfrak{P}^\circ))$, and by the above proved we can choose $u_1 \in V^{-1}(\mathfrak{P}_{c,+}(-)_+ A)$, such that $u + u_1(\perp)_\pm \mathfrak{P}^\circ \times \mathfrak{P}^\circ$, i. e. $u + u_1 \in \mathfrak{P}_-$. The second relation in (3.2) now follows from (3.1). □

Note that, if $S \subseteq S_1 \subseteq \mathfrak{P}_c^2$, if spaces $\mathfrak{P}_{c,-}$ and $\mathfrak{P}_{c1,-}$ are constructed starting from S and S_1, respectively, and if A is a selfadjoint extension of S_1 and hence also of S, then

$$j_+ R_z^{1,+} \;=\; R_z^+ \,. \tag{3.4}$$

Along the lines of [KW3] we may define a so−called regularized resolvent $\hat{R}_z : \mathfrak{P}_{c,-} \to \mathfrak{P}_{c,+}$ by $\big(z_0 \in \rho(\tilde{A})\big)$

$$\begin{aligned}\hat{R}_z \;&:=\; \begin{pmatrix} \widetilde{R}_z^- - \dfrac{1}{2}\Big(\widetilde{R}_{z_0}^- + \widetilde{R}_{\overline{z_0}}^-\Big) \\ z\widetilde{R}_z^- - \dfrac{1}{2}\Big(z_0\widetilde{R}_{z_0}^- + \overline{z_0}\widetilde{R}_{\overline{z_0}}^-\Big)\end{pmatrix} \\ &=\; (z - \operatorname{Re} z_0)\widetilde{P}^+ R_{\overline{z_0}}^+ R_{z_0}^-\big|_{\mathfrak{P}_{c,-}} \\ &\quad\; + (z - z_0)(z - \overline{z_0})\widetilde{P}^+ R_{\overline{z_0}}^+ \big(\tilde{A} - z\big)^{-1} R_{z_0}^-\big|_{\mathfrak{P}_{c,-}} \,.\end{aligned} \tag{3.5}$$

The function $(u, v \in \mathfrak{P}_-,\ \alpha \in \mathbb{R})$

$$r_{u,v}(z) \;:=\; \alpha + \big[\hat{R}_z u, v\big]_\pm \,, \quad z \in \rho(A)\,,$$

is called a regularized resolvent of $S \subseteq \mathfrak{P}^2$. We shall give a parametrization of the set of all regularized resolvents of $S \subseteq \mathfrak{P}^2$. Note that the relation (3.5) implies

$$\begin{aligned}\big[\hat{R}_z u, v\big]_\pm \;&=\; (z - \operatorname{Re} z_0)\big[R_{z_0}^- u, R_{z_0}^- v\big] \\ &\quad\; + (z - z_0)(z - \overline{z_0})\Big[\big(\tilde{A} - z\big)^{-1} R_{z_0}^- u, R_{z_0}^- v\Big] \,.\end{aligned} \tag{3.6}$$

If we choose another point $z_0 \in \rho\big(\tilde{A}\big)$ for the definition (3.5) of a regularization, the function $r_{u,v}(z)$ changes only by a real additive constant.

In order to make the results of [KW2] applicable we assume in the following that S satisfies the regularity conditions

(R1) For each $h \in \mathfrak{P}^\circ$ we have $S \cap \operatorname{span}\{h\}^2 = \{0\}$.

(R2) There exist numbers $z_+ \in \mathbb{C}^+$ and $z_- \in \mathbb{C}^-$ such that

$$\operatorname{ran}(S - z_\pm) + \mathfrak{P}^\circ \;=\; \mathfrak{P}\,.$$

First we investigate the condition **(R1)** and show that, when studying the set of regularized resolvents, it does not represent an essential restriction. Let us recall from [HSW]:

Lemma 3.3. *Let $A \subseteq \widetilde{\mathfrak{P}}^2$ be a selfadjoint relation in the Pontryagin space $\widetilde{\mathfrak{P}}$, $\rho(A) \neq \emptyset$, and let $\mathfrak{M}$ be a subspace of $\widetilde{\mathfrak{P}}$ which is invariant under each resolvent $(A-z)^{-1}$, $z \in \rho(A)$. Then the relation $A_{\mathfrak{M}} := (A \cap \mathfrak{M}^2)/\mathfrak{M}^\circ \subseteq (\mathfrak{M}/\mathfrak{M}^\circ)^2$ is selfadjoint and $\rho(A_{\mathfrak{M}}) \supseteq \rho(A)$.*

By a straightforward argument using Lemma 4.3 of [KW3] we obtain

Lemma 3.4. *Let $A \subseteq \widetilde{\mathfrak{P}}^2$, $\rho(A) \neq \emptyset$, be a selfadjoint extension of $S \subseteq \mathfrak{P}^2$. Let $\mathcal{L} \subseteq \mathfrak{P}^\circ$ be contained in* $\operatorname{ran}(S - z)$ *for all $z \in \rho(A)$ and assume that $(S - z)^{-1}\mathcal{L} \subseteq \mathcal{L}$ for such z. Put $\mathfrak{M} = \mathcal{L}^\perp$, then $S_1 := S/\mathfrak{L}$ has defect index $(1,1)$ in the space $\mathfrak{P}_1 :=$*

$\mathfrak{P}/\mathfrak{L} \subseteq \mathfrak{P}_{\mathfrak{M}} := \mathfrak{M}/\mathfrak{M}^\circ$. For any element $u_1 \in \mathfrak{P}_{1,-}$ there exists an element $u \in \mathfrak{P}_-$ with $R_z^- u \in \mathfrak{M}$ for one and hence for all $z \in \rho(A)$, such that

$$\big(R_z^- u\big)/\mathfrak{M}^\circ \;=\; R_{\mathfrak{M},z}^- u_1\,, \quad z \in \rho(A)\,,$$

and conversely.

Denote by M_μ the spaces

$$\begin{aligned} M_\mu(S) \;&:=\; \{h \in \mathfrak{P}^\circ \mid (h;\mu h) \in S\}\,, \quad \mu \in \mathbb{C}\,, \\ M_\infty(S) \;&:=\; \mathfrak{P}^\circ \cap S(0)\,, \end{aligned}$$

and let

$$\mathfrak{L}(S) \;:=\; \operatorname{span}\{M_\mu(S) \mid \mu \in \mathbb{C} \cup \{\infty\}\}\,.$$

If S is an operator the spaces M_μ are clearly linearly independent. If S is a proper relation this is not true in general. However, we have the following result:

Lemma 3.5. *Assume that S has an extension $A_0 \subseteq \widetilde{\mathfrak{P}}_0^2$ with nonempty resolvent set in some Pontryagin space $\widetilde{\mathfrak{P}}_0 \supseteq \mathfrak{P}$. Then there exist linearly independent elements $f_1, \ldots, f_m \in \mathfrak{P}^\circ$, $(\lambda_i f_i; \mu_i f_i) \in S$, $i = 1, \ldots m$, such that*

$$M_\mu(S) \;=\; \operatorname{span}\left\{ f_i \;\middle|\; \mu = \frac{\mu_i}{\lambda_i} \right\}.$$

Proof. It suffices to show that there exists no nontrivial linear combination

$$\sum_{i=1}^{n} \gamma_i h_i \in M_\infty(S)\,, \tag{3.7}$$

$\gamma_i \in \mathbb{C} \setminus \{0\}$, $h_i \in M_{\mu_i}(S) \setminus \{0\}$, $\mu_i \in \mathbb{C}$ pairwise different for $i = 1, \ldots, n$. Assume the contrary, and let $n(S)$ be the minimal lenght of a linear combination satisfying (3.7). Since S admits an extension with nonempty resolvent set, we have

$$\operatorname{span}\{h\}^2 \not\subseteq S\,, \quad h \in \mathfrak{P}\,, \tag{3.8}$$

hence $n(S) > 1$. Clearly

$$\left(\sum_{i=1}^{n} \gamma_i h_i ; \sum_{i=1}^{n-1} \gamma_i (\mu_i - \mu_n) h_i \right) \in S - \mu_n\,,$$

hence

$$\sum_{i=1}^{n-1} \gamma_i (\mu_i - \mu_n) h_i \in M_\infty(S/M_\infty(S))\,.$$

Lemma 3.3 applied with $\mathfrak{M} := \widetilde{\mathfrak{P}}_0[-]M_\infty(S)$ shows that the relation $S/M_\infty(S) \subseteq (\mathfrak{P}/M_\infty(S))^2$ admits an extension with nonempty resolvent set. The element h_i is contained in $M_{\mu_i}(S/M_\infty(S))$ and is by (3.8) not zero in the space $\mathfrak{P}/M_\infty(S)$. We conclude that $n(S/M_\infty(S)) < n(S)$. Proceeding inductively we obtain a contradiction. □

Proposition 3.6. *Let $S \subseteq \mathfrak{P}^2$ be a closed symmetric relation and assume that S admits an extension $A_0 \subseteq \widetilde{\mathfrak{P}}_0^2$ with nonempty resolvent set in some Pontryagin space $\widetilde{\mathfrak{P}}_0 \supseteq \mathfrak{P}$. Then the relation $S/\mathfrak{L}(S) \subseteq (\mathfrak{P}/\mathfrak{L}(S))^2$ is a closed symmetric relation with the same defect index as S and satisfies*

$$(S/\mathfrak{L}(S)) \cap \operatorname{span}\{h\}^2 \;=\; \{0\}\,, \quad h \in (\mathfrak{P}/\mathfrak{L}(S))^\circ\,.$$

The relations S and $S/\mathfrak{L}(S)$ have the same family of regularized resolvents.

Proof. First let an extension $A \subseteq \widetilde{\mathfrak{P}}^2$, $\rho(A) \neq \emptyset$, of S be given. If we put $\mathfrak{M} := \mathfrak{L}(S)^\perp$, we clearly have $\mathfrak{M}^\circ = \mathfrak{L}(S)$ and $\mathfrak{P} \subseteq \mathfrak{M}$. Hence the relation $A_{\mathfrak{M}}$ extends $S/\mathfrak{L}(S)$. It follows from $\mathfrak{L}(S) \subseteq \mathfrak{P}^\circ$ that

$$\big[(A-z)^{-1}u, v\big] \;=\; \big[(A/\mathfrak{M}^\circ - z)^{-1}(u/\mathfrak{M}^\circ), (v/\mathfrak{M}^\circ)\big]\,, \quad u\,,\, v \in \mathfrak{P}\,,$$

and (3.6) implies that $A_{\mathfrak{M}}$ induces the same regularized resolvent as A.

Now let an extension $A' \subseteq \left(\widetilde{\mathfrak{P}}'\right)^2$ of $S' := S/\mathfrak{L}(S)$ be given. We may consider $\mathfrak{P}/\mathfrak{L}(S)$ as a subspace of $\mathfrak{P}$, e. g. by

$$\mathfrak{P}/\mathfrak{L}(S) \cong \mathfrak{P}' \;:=\; \mathfrak{P}_n\,[\dot{+}]\,\mathfrak{P}'^\circ\,,$$

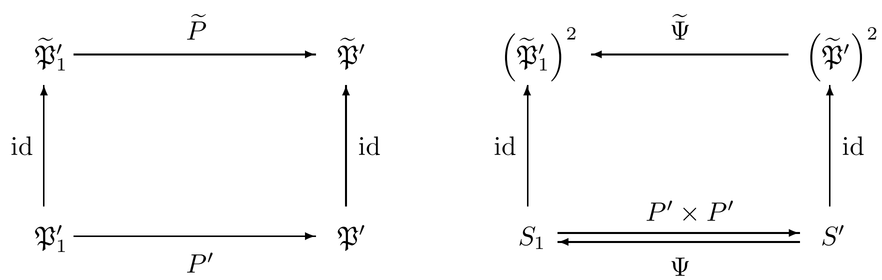
where $\mathfrak{P}'^\circ$ is any complement of $\mathfrak{L}(S)$ in $\mathfrak{P}^\circ$ and where $\mathfrak{P}_n$ is as in (2.2). Choose $h \in \mathfrak{L}(S)$, $(\lambda h; \mu h) \in S$, and define a Pontryagin space

$$\widetilde{\mathfrak{P}}_1' \;:=\; \widetilde{\mathfrak{P}}'\,[\dot{+}]\,\operatorname{span}\{h, h'\}\,,$$

where h and h' are skewly linked, and $\mathfrak{P}_1' := \mathfrak{P}' \dot{+} \operatorname{span}\{h\} \subseteq \widetilde{\mathfrak{P}}_1'$. Note that $\left(\mathfrak{P}_1'\right)^2$ contains the relation $S_1' := S/(\mathfrak{L}(S) \ominus \operatorname{span}\{h\})$. Denote by P' the projection of $\mathfrak{P}$ onto $\mathfrak{P}'$ with kernel $\mathfrak{L}(S)$, and by P_1' the projection of $\mathfrak{P}$ onto $\mathfrak{P}_1'$ with kernel $\mathfrak{L}(S) \ominus \operatorname{span}\{h\}$. Then $S' \cong P' \times P'S$ and $S_1' \cong P_1' \times P_1'S$. Write $S_1' = S_1 \dot{+} \operatorname{span}\{(\lambda h; \mu h)\}$, with a closed subspace S_1 of $\left(\mathfrak{P}_1'\right)^2$. Then $P' \times P'$ maps S_1 bijectively onto S', hence there exists an inverse mapping Ψ. If T is any closed complement of S' in $\left(\widetilde{\mathfrak{P}}'\right)^2$, $S' \dot{+} T = \left(\widetilde{\mathfrak{P}}'\right)^2$, we may extend Ψ to

$$\widetilde{\Psi} \;:=\; \Psi \oplus \operatorname{id}_T \;:\; \left(\widetilde{\mathfrak{P}}'\right)^2 \;\longrightarrow\; \left(\widetilde{\mathfrak{P}}_1'\right)^2\,.$$

We are in the following situation:



Note that $\operatorname{ran}\widetilde{\Psi} \subseteq \operatorname{span}\left\{\widetilde{\mathfrak{P}}', h\right\}^2$, in fact $\operatorname{ran}\left(\widetilde{\Psi} - \mathrm{id}_{\left(\widetilde{\mathfrak{P}}'\right)^2}\right) \subseteq \operatorname{span}\{(\lambda h; \mu h)\}$. It follows that the relation

$$A \;:=\; \operatorname{span}\left\{\widetilde{\Psi}A', (\lambda h; \mu h)\right\} \subseteq \left(\widetilde{\mathfrak{P}}'_1\right)^2$$

is closed, symmetric, extends S and has defect index $(1,1)$.

We show that $\sigma_p(A) \subseteq \sigma_p(A') \cup \left\{\frac{\mu}{\lambda}\right\}$. Assume that $z \in \sigma_p(A) \setminus \sigma_p(A')$, and let $(x; zx) \in A$, $x \neq 0$. By the definition of A we can write for some $(a; b) \in A'$

$$(x; zx) \;=\; \widetilde{\Psi}(a; b) + \sigma(\lambda h; \mu h) \;=\; (a; b) + \sigma'(\lambda h; \mu h)\,. \tag{3.9}$$

Hence $b - za = -\sigma'(\mu - z\lambda)h$, which implies $b - za = 0$ and $-\sigma'(\mu - z\lambda) = 0$. Since $z \notin \sigma_p(A')$ we conclude that $a = b = 0$, and since $x \neq 0$ the relation (3.9) implies that $\sigma' \neq 0$, hence $\mu - z\lambda = 0$.

It follows that there exists a selfadjoint extension $\tilde{A} \subseteq \left(\widetilde{\mathfrak{P}}'_1\right)^2$ of A with nonempty resolvent set. By our construction the relation $\tilde{A}_{\mathfrak{M}}$ as defined in the first part of this proof coincides with A', thus $\tilde{A}$ induces the same regularized resolvent as A'. Proceeding inductively, which is possible by Lemma 3.5, the assertion follows. $\square$

Note that, if S satisfies **(R1)** and **(R2)**, which will be assumed throughout the following, the relation $S/\mathfrak{P}^\circ \subseteq (\mathfrak{P}/\mathfrak{P}^\circ)^2$ is selfadjoint, has nonempty resolvent set and $z \in \mathbb{C} \setminus \mathbb{R}$ is an eigenvalue of $S/\mathfrak{P}^\circ$ if and only if $\operatorname{ran}(S - z) + \mathfrak{P}^\circ \neq \mathfrak{P}$.

We recall some notations and results. Let z_0 be such that $\operatorname{ran}(S - z_0) + \mathfrak{P}^\circ = \mathfrak{P}$. By [KW2] there exists a basis $\{h_1, \ldots, h_\Delta\}$ of $\mathfrak{P}^\circ$ such that

$$S \cap (\mathfrak{P}^\circ)^2 \;=\; \operatorname{span}\{(h_i; h_{i+1}) \mid i = 1, \ldots, \Delta - 1\}$$

and we can write $S = S_1 \dot{+} S \cap (\mathfrak{P}^\circ)^2$ where $\operatorname{ran}(S_1 - z_0)$ is nondegenerated and $\operatorname{ran}(S_1 - z_0) \dot{+} \mathfrak{P}^\circ = \mathfrak{P}$. In the definition (2.2) of $\mathfrak{P}_c$ we choose $\mathfrak{P}_n := \operatorname{ran}(S_1 - z_0)$. Again by [KW2] there exists a selfadjoint extension $\mathring{A} \subseteq \mathfrak{P}_c^2$, $\rho\left(\mathring{A}\right) \neq \emptyset$, of S with

$$h_1 \in \mathring{A}(0)\,. \tag{3.10}$$

Note that $\mathring{A}$ satisfies $\left(\mathring{A} - z\right)^{-1}\mathfrak{P} \subseteq \mathfrak{P}$. If $\left\{h'_1, \ldots, h'_\Delta\right\}$ is a basis of $\mathfrak{P}'$ in (2.2) which is skewly linked to $\{h_1, \ldots, h_\Delta\}$ and

$$\chi(z_0) \;:=\; h'_1 + z_0 h'_2 + \cdots + z_0^{\Delta-1} h'_\Delta\,,$$

then $\chi(z) := \left(I + (z - z_0)\left(\mathring{A} - z\right)^{-1}\right)\chi(z_0)$ defines defect elements of S, i.e. elements satisfying $\chi(z) \perp \operatorname{ran}(S - \overline{z})$, for which additionally

$$[\chi(z), h_i] \;=\; z^{i-1}\,, \quad i \;=\; 1\,,\,\dots\,,\,\Delta\,.$$

Denote by q the (up to additive real constants) unique function with

$$\frac{q(z) - \overline{q(w)}}{z - \overline{w}} \;=\; [\chi(z), \chi(w)]\,.$$

It is shown in [KW2] that the formula

$$(3.11)\qquad \left[(A - z)^{-1}u, v\right] \;=\; \left[\left(\mathring{A} - z\right)^{-1} u, v\right] - [u, \chi(\overline{z})]\,\frac{1}{q(z) + \tau(z)}\,[\chi(z), v]\,, \qquad u\,,\,v \in \mathfrak{P}\,,$$

establishes a correspondence of the set of generalized resolvents of $S \subseteq \mathfrak{P}^2$ and parameters $\tau \in \bigcup_{\nu=0}^{\infty} \mathcal{K}_\nu^\Delta \setminus \{-q\}$. There the set $\mathcal{K}_\nu^\Delta$ is defined as the set of all functions τ meromorphic in $\mathbb{C} \setminus \mathbb{R}$, $\tau(\overline{z}) = \overline{\tau(z)}$, which are such that the maximal number of negative squares of a quadratic form

$$Q(\xi_1, \dots, \xi_m; \eta_1, \dots, \eta_\Delta) \;=\; \sum_{i,j=1}^{m} \frac{\tau(z) - \overline{\tau(w)}}{z - \overline{w}}\,\xi_i \overline{\xi_j} + \sum_{k=1}^{\Delta}\sum_{i=1}^{m} \operatorname{Re}\left(z_i^{k-1}\xi_i \overline{\eta_k}\right)$$

is ν. For an alternative approach to the classes $\mathfrak{K}_\nu^\Delta$ compare [KW1]. These facts imply the following result:

Proposition 3.7. *Let $u, v \in \mathfrak{P}_-$ be given. The formula*

$$\left[\hat{R}_z u, v\right]_\pm \;=\; \left[\hat{\mathring{R}}_z u, v\right]_\pm - \left[u, \begin{pmatrix} \chi(\overline{z}) \\ \overline{z}\chi(\overline{z}) \end{pmatrix}\right]_\pm \frac{1}{q(z) + \tau(z)} \left[\begin{pmatrix} \chi(z) \\ z\chi(z) \end{pmatrix}, v\right]_\pm + \beta(u, v)\,,$$

parametrizes the regularized resolvents of $S \subseteq \mathfrak{P}^2$. Here $\beta(u, v)$ is a constant which depends besides u and v on the choice of z_0 in the definition (3.5). *The meaning of χ, q and τ is as in* (3.11).

Proof. Using (3.11) and the definition of $\widetilde{R}_z^+$ we compute for $\binom{a}{b} \in \mathfrak{P}_+$ and $x \in \mathfrak{P}$:

$$\begin{aligned}\left(\widetilde{R}_z^+ x, \begin{pmatrix} a \\ b \end{pmatrix}\right)_+ \;&=\; \left[(A - z)^{-1}x, \mathcal{J}a\right] + \left[\left(I + z(A - z)^{-1}\right)x, \mathcal{J}b\right] \\ &=\; \left(\mathring{R}_z^+ x, \begin{pmatrix} a \\ b \end{pmatrix}\right)_+ - [x, \chi(\overline{z})]\,\frac{1}{q(z) + \tau(z)} \left(\begin{pmatrix} \chi(z) \\ z\chi(z) \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\right)_+\,,\end{aligned}$$

hence it follows that for $u \in \mathfrak{P}_-$ and $x \in \mathfrak{P}$

$$\left[\tilde{R}_z^- u, x\right] \;=\; \left(Vu, \tilde{R}_{\overline{z}}^+ x\right)_+ \;=\; \left[\mathring{R}_z^- u, x\right] - \left[u, \begin{pmatrix} \chi(\overline{z}) \\ \overline{z}\chi(\overline{z}) \end{pmatrix}\right]_\pm \frac{1}{q(z) + \tau(z)}\,[\chi(z), x]\,.$$

From this formula and the definition of $\hat{R}_z$ we find for $u,\, v \in \mathfrak{P}_-$

$$\left[\hat{R}_z u, v\right]_\pm \;=\; \left[\hat{\hat{R}}_z u, v\right]_\pm - \left[u, \begin{pmatrix} \chi(\overline{z}) \\ \overline{z}\chi(\overline{z}) \end{pmatrix}\right]_\pm \frac{1}{q(z)+\tau(z)} \left[\begin{pmatrix} \chi(z) \\ z\chi(z) \end{pmatrix}, v\right]_\pm$$
$$+ \frac{\left[u, \begin{pmatrix} \chi(\overline{z_0}) \\ \overline{z_0}\chi(\overline{z_0}) \end{pmatrix}\right]_\pm \left[\begin{pmatrix} \chi(z_0) \\ z_0\chi(z_0) \end{pmatrix}, v\right]_\pm}{2(q(z_0)+\tau(z_0))}$$
$$+ \frac{\left[u, \begin{pmatrix} \chi(z_0) \\ z_0\chi(z_0) \end{pmatrix}\right]_\pm \left[\begin{pmatrix} \chi(\overline{z_0}) \\ \overline{z_0}\chi(\overline{z_0}) \end{pmatrix}, v\right]_\pm}{2(q(\overline{z_0})+\tau(\overline{z_0}))}\,. \qquad \square$$

4. Resolvent matrices

Let an element $u \in \mathfrak{P}_-$ be given. The function

$$r(z) \;:=\; \alpha + \left[\hat{R}_z u, u\right]_\pm\,,$$

where $\hat{R}_z$ is the regularized resolvent of some selfadjoint extension of S and $\alpha \in \mathbb{R}$, is called a regularized u–resolvent of S. Note that if $\left[u, \begin{pmatrix} \chi(z) \\ z\chi(z) \end{pmatrix}\right]_\pm = 0$ for all $z \in \mathbb{C}^+$ or all $z \in \mathbb{C}^-$, there exists (up to real additive constants) exactly one regularized u–resolvent. Hence, when investigating the regularized u–resolvents, we may assume that for some $z_+ \in \mathbb{C}^+$ and $z_- \in \mathbb{C}^-$

$$\left[u, \begin{pmatrix} \chi(z_+) \\ z_+\chi(z_+) \end{pmatrix}\right]_\pm \;\neq\; 0\,, \qquad \left[u, \begin{pmatrix} \chi(z_-) \\ z_-\chi(z_-) \end{pmatrix}\right]_\pm \;\neq\; 0\,.$$

Proposition 3.7 has the following corollary:

Corollary 4.1. *Let $u \in \mathfrak{P}_-$ be given. There exists a 2×2 – matrix valued function*

$$W(z) \;=\; \begin{pmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{pmatrix},$$

which is analytic in an open set containing $\mathbb{C}\setminus\mathbb{R}$ with possible exception of a set which has no accumulation point in $\mathbb{C}\setminus\mathbb{R}$, such that for any $\tau \in \bigcup_{\nu=0}^\infty \mathcal{K}_\nu^\Delta \setminus \{-q\}$ the function

$$(W\circ\tau)(z) \;:=\; \frac{w_{11}(z)\tau(z)+w_{12}(z)}{w_{21}(z)\tau(z)+w_{22}(z)}$$

is a regularized u – resolvent of S and, conversely, any regularized u – resolvent $r(z)$ can be written as

$$r(z) \;=\; \alpha + (W\circ\tau)(z)$$

for some choice of $\tau \in \bigcup_{\nu=0}^\infty \mathcal{K}_\nu^\Delta \setminus \{-q\}$ and a certain real constant α.

P r o o f. By (3.11) a matrix $W(z)$ which has the asserted properties is given by

$$
\begin{aligned}
w_{11}(z) &= \frac{\mathring{r}(z)}{\left[u, \begin{pmatrix} \chi(\bar z) \\ \bar z\chi(\bar z) \end{pmatrix}\right]_{\pm}}, \\
w_{21}(z) &= \frac{1}{\left[u, \begin{pmatrix} \chi(\bar z) \\ \bar z\chi(\bar z) \end{pmatrix}\right]_{\pm}}, \\
w_{12}(z) &= \frac{\mathring{r}(z)q(z) - \left[u, \begin{pmatrix} \chi(\bar z) \\ \bar z\chi(\bar z) \end{pmatrix}\right]_{\pm} \left[\begin{pmatrix} \chi(z) \\ z\chi(z) \end{pmatrix}, u\right]_{\pm}}{\left[u, \begin{pmatrix} \chi(\bar z) \\ \bar z\chi(\bar z) \end{pmatrix}\right]_{\pm}}, \\
w_{22}(z) &= \frac{q(z)}{\left[u, \begin{pmatrix} \chi(\bar z) \\ \bar z\chi(\bar z) \end{pmatrix}\right]_{\pm}},
\end{aligned}
\tag{4.1}
$$

where $\mathring{r}(z) = \left[\hat{\hat{R}}_z u, u\right]_{\pm}$. □

The matrix W depends in an obvious way on the choice of $\chi(z)$ and $q(z)$. Note that W depends in general also on the choice of $\mathring{A}$ subject to the condition (3.10).

We will call a 2×2–matrix valued function $W(z)$ a generalized resolvent matrix in a degenerated space, if it equals the matrix

$$
\begin{pmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{pmatrix},
$$

where w_{ij} are given by (4.1) for some choice of $\mathfrak{P}$, $\dim \mathfrak{P}^\circ > 0$, $u \in \mathfrak{P}_-$, $S \subseteq \mathfrak{P}^2$ and $\mathring{A} \supseteq S$, $h_1 \in \mathring{A}(0)$, such that

$$
\operatorname{cls}\left\{\chi(z), R_z^- u \mid z \in \rho\left(\mathring{A}\right)\right\} = \mathfrak{P}_c .
$$

Consider the relation $S_1 \subseteq \mathfrak{P}_c^2$ defined by

$$
S_1 := \left\{(f;g) \in \mathring{A} \mid g - zf \perp \chi(\overline{z}),\, z \in \rho\left(\mathring{A}\right)\right\} . \tag{4.2}
$$

Clearly $S_1 \subseteq \mathfrak{P}_c^2$ is a symmetric relation with defect index $(1,1)$ and $S \subseteq S_1 \subseteq \mathfrak{P}_c^2$. Let $\mathfrak{P}_{c1,-}$ ($\mathfrak{P}_{c,-}$) be constructed starting with S_1 (S) and let $j_- : \mathfrak{P}_{c,-} \to \mathfrak{P}_{c1,-}$ be as introduced in Section 2. Then $\mathfrak{P}_-$ can be identified with the subspace $j_-\mathfrak{P}_-$ of $\mathfrak{P}_{c1,-}$. Now we have from the definition of S_1, Lemma 2.4 and (3.4):

Lemma 4.2. *The matrix* W *defined by* (4.1) *is a generalized* j_-u *–resolvent matrix* (*in the sense of* [KW3]) *of the symmetric relation* $S_1 \subseteq \mathfrak{P}_c^2$.

In particular, if $u_1, u_2 \in \mathfrak{P}_-$ *are such that* $j_-u_1 = j_-u_2$, *then the generalized* u_i *– resolvent matrices* $(i = 1, 2)$ *are equal.*

Denote by $\mathcal{M}_\nu$ the set of all $2 \times 2-$matrix functions, meromorphic in $\mathbb{C} \setminus \mathbb{R}$ which satisfy $W(z)JW(\overline{z}) = J$ and for which the kernel

$$\frac{W(z)JW(w)^* - J}{z - \overline{w}}$$

has ν negative squares. Here J denotes the matrix

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that Lemma 4.2 and the results of [KW3] imply in particular that a generalized resolvent matrix W of $S \subseteq \mathfrak{P}^2$ is contained in $\mathcal{M}_{\kappa+\Delta}$. In the sequel we investigate the question which matrices $W \in \mathcal{M}_\nu$ can be realized as a generalized resolvent matrix of a symmetric relation in some degenerated space.

Recall from [KW3] that for any matrix $W \in \mathcal{M}_\nu$ there exist $(iJ)-$unitary matrices U and V such that $VW(z)U$ is a generalized $u'-$resolvent matrix of a certain symmetric relation S' with defect index $(1,1)$ in a Pontryagin space $\mathfrak{P}'$, $u' \in \mathfrak{P}'_-$. We can assume that $(\mathfrak{P}', S', u')$ is minimal in the sense that $\big(\phi(z) \perp \operatorname{ran}(S' - \overline{z}),\ A' \subseteq \mathfrak{P}'^2,\ A' \supseteq S'\big)$

$$\operatorname{cls}\Big\{\phi(z), R_z'^{\,-} u' \mid z \in \rho(A')\Big\} \;=\; \mathfrak{P}'.$$

If w_{21} does not vanish identically and at least one of $w_{21},\ w_{22},\ \det W$ is not constant, we may choose $U = V = I$, in which case $\mathfrak{P}',\ S'$ and u' are uniquely determined up to unitary equivalence and $W(z)$ is given by the relations (4.1) for some extension A' of S'.

Proposition 4.3. *Let $W \in \mathcal{M}_\nu$ be given and assume that w_{21} does not vanish identically and that at least one of $w_{21},\ w_{22}$ and $\det W$ is not constant. Let $(\mathfrak{P}', S', u')$ be the unique minimal triple such that $W(z)$ is a generalized u' –resolvent matrix of S' and let $\mathring{A}$ be the canonical selfadjoint extension which is used to write W via the formulas* (4.1)*. Denote by $\phi(z)$ defect elements of S' connected with $\mathring{A}$. Then $W(z)$ is a generalized resolvent matrix in a degenerated space if and only if $\mathring{A}(0)$ contains a neutral element h_0 which has the properties*

(i) $[h_0, \phi(z)] \neq 0$ *for one and hence for all* $z \in \rho\big(\mathring{A}\big)$,

(ii) $V'u'(\perp)_+ \begin{pmatrix} 0 \\ h_0 \end{pmatrix}$.

Proof. First assume that the triple $(\mathfrak{P}', S', u')$ has the stated properties. Then define

$$\mathfrak{P} \;:=\; \operatorname{span}\{h_0\}^\perp, \quad S \;:=\; \Big\{(f;g) \in \mathring{A} \mid g - zf \perp \phi(\overline{z}),\, h_0\Big\}.$$

Clearly $\mathfrak{P}^\circ = \operatorname{span}\{h_0\}$ and $S \subseteq \mathfrak{P}^2$, $S \subseteq S'$. The condition (i) shows that for $z \in \rho\big(\mathring{A}\big)$ the relation $h_0 \notin \operatorname{ran}(S - z)$ holds. Since $\rho\big(\mathring{A}\big) \neq \emptyset$, we conclude that S satisfies **(R1)**. Moreover, the relation S has defect index $(2,2)$ in the space $\mathfrak{P}'$, hence the condition **(R2)** follows from the fact that $\phi(z) \notin \mathfrak{P}$ and we conclude that S has defect index $(1,1)$ in the space $\mathfrak{P}$. Clearly $\mathfrak{P}_c \cong \mathfrak{P}'$ and $\mathring{A}$ satisfies (3.10).

Let $\mathfrak{P}'_+ = S_1^* \subseteq \mathfrak{P}'^2$, $\mathfrak{P}_{c,+} = S^* \subseteq \mathfrak{P}_c^2 \cong \mathfrak{P}'^2$, and let j_+, j_- be as in Section 2, then $\mathfrak{P}_+ \subseteq \mathfrak{P}_{c,+}$. We shall construct an element $u \in \mathfrak{P}_-$, such that $j_- u = u'$. Then by Lemma 4.2 the matrix W will be the generalized $u-$resolvent matrix of $S \subseteq \mathfrak{P}^2$.

First note that since $h_0 \in \mathring{A}(0)$, clearly $\begin{pmatrix} 0 \\ h_0 \end{pmatrix} \in S_1^*$. We claim that $\begin{pmatrix} h_0 \\ 0 \end{pmatrix} \notin S_1^*$. Assume the contrary, then $\operatorname{span}\{h_0\}^2 \subseteq S_1^*$, hence $h_0 \in \ker(S_1^* - z)$ for all z. It follows that $\left(\text{for } z \in \rho\left(\mathring{A}\right)\right)$ $\phi(z) = \lambda_z h_0$, hence we obtain

$$[\phi(z), h_0] \;=\; \lambda_z [h_0, h_0] \;=\; 0\,,$$

a contradiction to the condition (i).

Let $\ker j_+^* =: \operatorname{span}\left\{\begin{pmatrix} a_0 \\ b_0 \end{pmatrix}\right\}$, and define $u \in \mathfrak{P}_{c,-}$ by

$$Vu \;:=\; j_+ V' u' - \frac{\left(j_+ V' u', \begin{pmatrix} h_0 \\ 0 \end{pmatrix}\right)_+}{\left(\begin{pmatrix} a_0 \\ b_0 \end{pmatrix}, \begin{pmatrix} h_0 \\ 0 \end{pmatrix}\right)_+} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}.$$

This definition makes sense, since by the above consideration we have

$$\left(\begin{pmatrix} a_0 \\ b_0 \end{pmatrix}, \begin{pmatrix} h_0 \\ 0 \end{pmatrix}\right)_+ \;\neq\; 0\,.$$

It follows that in fact $u \in \mathfrak{P}_-$. Clearly $j_- u = u'$.

Now assume that W is the generalized $u-$resolvent matrix of $S \subseteq \mathfrak{P}^2$. Consider the realization of W as a generalized resolvent matrix given in Lemma 4.2. By the construction of $\mathring{A}$ and the definition of $\mathfrak{P}_-$ the properties (i) and (ii) are satisfied for $(\mathfrak{P}_c, S_1, u)$. □

Corollary 4.4. *Let W be as in Proposition 4.3 and assume that W is a generalized resolvent matrix in a degenerated space $\mathfrak{P}$. Then W can also be represented in a space $\mathfrak{P}_1$ with $\dim \mathfrak{P}_1^\circ = 1$.*

In the case that the relation S' in the representing triple is minimal, i. e. that

$$\mathfrak{P}' \;=\; \operatorname{cls}\left\{\phi(z) \mid z \in \rho\left(\mathring{A}\right)\right\},$$

the conditions given in Proposition 4.3 can be easily read off from the entries of W.

Proposition 4.5. *Let W be as in Proposition 4.3 and assume that S' is minimal. Then W is a generalized resolvent matrix in a degenerated space if and only if*

$$\lim_{y\to+\infty} y \frac{w_{21}(iy)}{w_{22}(iy)} \;=\; 0\,, \tag{4.3}$$

and

$$\lim_{y\to+\infty}\frac{\det W(iy)}{w_{22}(iy)} \;=\; 0\,. \tag{4.4}$$

Proof. Since S' is minimal, we have $S'(0) = \{0\}$ and $\operatorname{codim}\overline{\operatorname{dom} S'} \le 1$, which means $\dim S'^*(0) \le 1$.

First we prove that, if $S'^*(0) = \operatorname{span}\{h_0\}$ and $\mathfrak{P} = \operatorname{span}\{h_0\}^\perp$ for some element $h_0 \neq 0$, then the condition $V'u'(\perp)_+\begin{pmatrix} 0 \\ h_0 \end{pmatrix}$ is equivalent to $u' \in \overline{\iota\mathfrak{P}'}$ which is again equivalent to $\overset{\circ}{R}{}_z^- u' \in \mathfrak{P}$ for one and hence for all $z \in \rho\big(\overset{\circ}{A}\big)$, where $\overset{\circ}{A}$ is the extension of S' with $\overset{\circ}{A}(0) = \operatorname{span}\{h_0\}$. The first equivalence follows since V' is an isometry of $\mathfrak{P}'_-$ onto $\mathfrak{P}'_+$ and $S'^*(0) = \operatorname{span}\{h_0\}$, as by the definition of V'

$$\operatorname{span}\left\{V'^{-1}\begin{pmatrix} 0 \\ h_0 \end{pmatrix}\right\}^{(\perp)_-} \;=\; \overline{\iota\mathfrak{P}'}\,.$$

The second equivalence follows from the fact that $\overset{\circ}{R}{}_z^-$ is a bounded operator, that $\overset{\circ}{R}{}_z^-\,\iota = \big(\overset{\circ}{A} - z\big)^{-1}$ maps $\mathfrak{P}_c$ into $\mathfrak{P}$ and that $\operatorname{ran}\overset{\circ}{R}{}_z^- = \mathfrak{P}_c$.

If $\overline{\operatorname{dom} S'} = \mathfrak{P}'$, then all canonical selfadjoint extensions of S' are operators. Otherwise, if $\operatorname{codim}\overline{\operatorname{dom} S'} = 1$, there exists exactly one proper relational extension of S'. Which of these cases occurs can be seen from the family of Q$-$functions of S': The first case occurs if and only if every Q$-$function q_A of S' and a (canonical) extension A satisfies

$$\lim_{y\to+\infty}\frac{q_A(iy)}{y} \;=\; 0\,.$$

The second case occurs if and only if for some Q$-$function $q_{\overset{\circ}{A}}$

$$\liminf_{y\to+\infty}\left|\frac{q_{\overset{\circ}{A}}(iy)}{y}\right| \;\neq\; 0\,.$$

Then for all other Q$-$functions q_A, $A \neq \overset{\circ}{A}$, the limit

$$i\lim_{y\to+\infty} y(q_A(iy) - \alpha_A) \in \mathbb{R} \tag{4.5}$$

exists for some $\alpha_A \in \mathbb{R}$. The, in this sense exceptional, extension $\overset{\circ}{A}$ is the proper relational extension of S'. This has been proved in [HLS] in the positive definite case, in the Pontryagin space situation a similar argument applies.

Assume now that the conditions (4.3) and (4.4) are satisfied. By the relations (4.1) we have

$$q_{\overset{\circ}{A}}(z) \;=\; \frac{w_{22}(z)}{w_{21}(z)}\,, \tag{4.6}$$

hence by (4.3)

$$\lim_{y\to+\infty}\frac{q_{\overset{\circ}{A}}(iy)}{y} \;=\; \infty\,, \tag{4.7}$$

and we conclude that $\mathring{A}$ is a proper relation. Let $\mathring{A}(0) = \operatorname{span}\{h_0\}$. If A is any (canonical) operator extension of S', defect elements $\phi(z)$ of S' connected to A are given by

$$\phi(z) \;=\; (A-z)^{-1}h_0\,. \tag{4.8}$$

Hence $q_A(z)$ is, up to a real additive constant α_A, equal to $\big[(A-z)^{-1}h_0, h_0\big]$ and we find

$$-i\lim_{y\to+\infty} y(q_A(iy)-\alpha_A) \;=\; [h_0,h_0]\,,$$

in particular h_0 is neutral if and only if the limit (4.5) is zero.

The function q_A is expressed in terms of $q_{\mathring{A}}$ by

$$q_A(z) \;=\; \frac{\big(t+2\operatorname{Re}q_{\mathring{A}}(z_0)\big)q_{\mathring{A}}(z)-\big|q_{\mathring{A}}(z_0)\big|^2}{q_{\mathring{A}}(z)+t}\,, \tag{4.9}$$

when $t\in\mathbb{R}$ is the parameter corresponding to A in Krein's formula. Hence, by (4.7), the limit (4.5) is zero with the choice $\alpha_A = t+2\operatorname{Re}q_{\mathring{A}}(z_0)$.

The defect elements $\phi(z)$ defined by (4.8) satisfy $\phi(z) = \big(I+(z-z_0)(A-z)^{-1}\big)\phi(z_0)$, hence for a certain nonzero constant K

$$K\phi(z) \;=\; \big(I+(z-z_0)(A-z)^{-1}\big)\chi(z_0) \;=\; \frac{q_{\mathring{A}}(z_0)+t}{q_{\mathring{A}}(z)+t}\,\chi(z)\,.$$

We obtain

$$\begin{aligned} h_0 &= -i\lim_{y\to+\infty} y(A-iy)^{-1}h_0 \\ &= -i\lim_{y\to+\infty} y\phi(iy) \\ &= -K\big(t+q_{\mathring{A}}(z_0)\big)\,i\lim_{y\to+\infty}\frac{y}{q_{\mathring{A}}(iy)+t}\,\chi(z)\,, \end{aligned}$$

hence the relation $\mathring{R}_w^- u' \in \mathfrak{P}$, i.e. $\mathring{R}_w^- u' \perp h_0$, is equivalent to

$$\lim_{y\to+\infty}\frac{y}{q_{\mathring{A}}(iy)+t}\Big[\chi(iy),\mathring{R}_w^- u'\Big] \;=\; 0\,.$$

By (4.3) and the fact that $\Big[\chi(z),\mathring{R}_w^- u'\Big]$ is the right upper entry of the Nevanlinna kernel of the Potapov – Ginzburg transform of W (see [KW3]), this limit relation is equivalent to (4.4). We conclude from Proposition 4.3 that W is a generalized resolvent matrix in a degenerated space.

Conversely, if W can be represented as such, then $\mathring{A}$ is a proper relation and the limit (4.5) is zero for all $A\neq\mathring{A}$. Since

$$\liminf_{y\to+\infty}\left|\frac{q_{\mathring{A}}(iy)}{y}\right| \;\neq\; 0\,,$$

it follows from the representation (4.9) that $\alpha_A = t+2\operatorname{Re}q_{\mathring{A}}(z_0)$. Since the point $z_0\in\rho(A)\cap\rho\big(\mathring{A}\big)$ in (4.9) can be chosen such that $q_{\mathring{A}}(z_0)\neq -t$, the condition (4.3) follows. By the previous step of this proof also (4.4) follows. $\square$

5. Associated functions of degenerated dB – spaces

In [dB] L. de Branges developed a theory of Hilbert spaces of entire functions subject to certain additional conditions. Some parts of this theory have been generalized to indefinite inner product spaces $\langle\mathfrak{P}, [\,\cdot\,,\cdot\,]\rangle$, which satisfy besides **(D1)** and **(D2)** the following axioms:

(dB1) The space $\mathfrak{P}$ consists of entire functions. If $(\,\cdot\,,\cdot\,)$ denotes a Hilbert space inner product associated with $[\,\cdot\,,\cdot\,]$, then $\langle\mathfrak{P}, (\,\cdot\,,\cdot\,)\rangle$ is a reproducing kernel space.

(dB2) If $F \in \mathfrak{P}$, then $\overline{F(\overline{z})}$ also belongs to $\mathfrak{P}$ and

$$\left[\overline{F(\overline{z})}, \overline{G(\overline{z})}\right] = [G(z), F(z)], \quad F, G \in \mathfrak{P}.$$

(dB3) If $w \in \mathbb{C} \setminus \mathbb{R}$ and $F \in \mathfrak{P}$, $F(w) = 0$, then $\frac{z-\overline{w}}{z-w} F(z) \in \mathfrak{P}$. If moreover $G \in \mathfrak{P}$, $G(\overline{w}) = 0$, then

$$\left[\frac{z-\overline{w}}{z-w} F(z), G(z)\right] = \left[F(z), \frac{z-w}{z-\overline{w}} G(z)\right].$$

We call such spaces dB – spaces.

An entire function $U(z)$ is said to be an associated function for the dB – space $\mathfrak{P}$, if for one and hence for all $F \in \mathfrak{P}$, $w \in \mathbb{C}$, $F(w) \neq 0$,

$$\frac{U(z)F(w) - F(z)U(w)}{z-w} \in \mathfrak{P}.$$

If $\mathfrak{P}$ is a nondegenerated dB – space, it is shown in [KW4], Section 10, that the space $\mathfrak{P}_-$ can be identified with the set of associated functions for $\mathfrak{P}$. The notion of triplet spaces in the degenerated situation, as introduced in the previous sections, enables us to supplement this result by proving that also if $\mathfrak{P}$ is a degenerated dB – space, one can identify $\mathfrak{P}_-$ with the set of associated functions for $\mathfrak{P}$.

In the following let $\mathfrak{P}$ be a dB – space and assume that $\dim \mathfrak{P}^\circ = \Delta > 0$. For simplicity we assume moreover that for all $w \in \mathbb{C}$ there exists a function $F \in \mathfrak{P}$ with $F(w) \neq 0$. We remark that this may be assumed without loss of generality. The symmetric relation S under consideration is the operator of multiplication by the independent variable

$$(SF)(z) := zF(z),$$

where $\operatorname{dom} S := \{F \in \mathfrak{P} \mid zF(z) \in \mathfrak{P}\}$. Clearly the regularity condition **(R1)** is fullfilled, in fact S has no eigenvalues at all. We have

$$\operatorname{ran}(S-w) = \{F \in \mathfrak{P} \mid F(w) = 0\},$$

hence S has defect index $(1,1)$, satisfies **(R2)** and is minimal, i. e.

$$\bigcap_{w\in\mathbb{C}} \operatorname{ran}(S-w) = \{0\}.$$

Let $h_1 \in \mathfrak{P}$ and $\mathring{A} \subseteq \mathfrak{P}_c^2$ be chosen in accordance with Section 3, (3.10), and let $z_0 \in \rho\big(\mathring{A}\big)$ be such that $h_1(z_0) \neq 0$. The functional $\Phi : \mathfrak{P}_c \to \mathbb{C}$ defined by

$$\Phi F := \begin{cases} F(z_0), & F \in \mathfrak{P}, \\ 0, & F \in \mathfrak{P}', \end{cases}$$

is continuous, hence can be represented as

$$\Phi F = [F, \phi(\overline{z_0})] \,,$$

for some element $\phi(\overline{z_0}) \in \mathfrak{P}_c$. Note that $\phi(\overline{z_0}) \notin \mathfrak{P}$ and $\phi(\overline{z_0}) \perp \operatorname{ran}(S - z_0)$. Define elements $\phi(z)$ by

$$\phi(z) := \left(I + (z - \overline{z_0}) \left(\mathring{A} - z \right)^{-1} \right) \phi(\overline{z_0}) \,, \quad z \in \rho\left(\mathring{A}\right).$$

Then $\begin{pmatrix} \phi(z) \\ z\phi(z) \end{pmatrix} \in S^*$ and since $h_1 \in \mathring{A}(0)$ the value $[h_1, \phi(z)] = h_1(z_0)$ is constant and nonzero.

Now we associate to each element $u \in \mathfrak{P}_-$ a function $\hat{u}(z)$ which is, at the first sight, analytic on $\rho\left(\mathring{A}\right)$:

$$\hat{u}(z) := \frac{h_1(z)}{h_1(z_0)} \left[u, \begin{pmatrix} \phi(\overline{z}) \\ \overline{z}\phi(\overline{z}) \end{pmatrix} \right]_{\pm} .$$

It will turn out in the sequel that $\hat{u}$ is in fact entire. Since

$$\operatorname{cls}\left(\left\{ \phi(z) \mid z \in \rho\left(\mathring{A}\right) \right\} \cup \mathfrak{P}^\circ \right) = \mathfrak{P}_c \,,$$

we conclude similar as in [KW3], Lemma 3.5, to obtain

$$\mathfrak{P}_{c,+} = \operatorname{cls}\left(\left\{ \begin{pmatrix} \phi(z) \\ z\phi(z) \end{pmatrix} \mid z \in \rho\left(\mathring{A}\right) \right\} \cup (\mathfrak{P}^\circ \times \mathfrak{P}^\circ) \right).$$

Hence the correspondence $u \mapsto \hat{u}$ is one-to-one. Note that for $F \in \mathfrak{P}$ we have $\widehat{(\iota F)}(w) = F(w)$. This follows for $w \in \rho\left(\mathring{A}\right)$, $h_1(w) \neq 0$, since then we may write $F(z) = F_1(z) + \frac{F(w)}{h_1(w)} h_1(z)$ for some $F_1 \in \operatorname{ran}(S - w)$, and hence the following relation holds:

$$\begin{aligned} \widehat{(\iota F)}(w) &= \frac{h_1(w)}{h_1(z_0)} \left[\iota F, \begin{pmatrix} \phi(\overline{w}) \\ \overline{w}\phi(\overline{w}) \end{pmatrix} \right]_{\pm,c} \\ &= \frac{h_1(w)}{h_1(z_0)} [F, \phi(\overline{w})] \\ &= \frac{h_1(w)}{h_1(z_0)} \left[\frac{F(w)}{h_1(w)} h_1, \phi(\overline{w}) \right] \\ &= F(w) \,. \end{aligned}$$

Now we come to the mentioned connection of $\mathfrak{P}_-$ with the set of associated functions for $\mathfrak{P}$.

Proposition 5.1. *Let $\mathfrak{P}$ be a degenerated dB-space. An entire function $G(z)$ is an associated function for $\mathfrak{P}$ if and only if $G = \hat{u}$ for some $u \in \mathfrak{P}_-$.*

Proof. Let $u \in \mathfrak{P}_-$, then by Lemma 3.2 we have $\overset{\circ}{R}{}_z^- u \in \mathfrak{P}$. Hence

$$
\begin{aligned}
\left(\overset{\circ}{R}{}_z^- u\right)(w) &= \frac{h_1(w)}{h_1(z_0)}\left[\overset{\circ}{R}{}_z^- u, \phi(\overline{w})\right] \\
&= \frac{h_1(w)}{h_1(z_0)}\left[u, \overset{\circ}{R}{}_{\overline{z}}^+ \phi(\overline{w})\right]_\pm \\
&= \frac{h_1(w)}{h_1(z_0)} \cdot \frac{1}{w-z}\left(\left[u, \begin{pmatrix} \phi(\overline{w}) \\ \overline{w}\phi(\overline{w}) \end{pmatrix}\right]_\pm - \left[u, \begin{pmatrix} \phi(\overline{z}) \\ \overline{z}\phi(\overline{z}) \end{pmatrix}\right]_\pm\right) \\
&= \frac{1}{w-z}\left(\hat{u}(w) - \frac{h_1(w)}{h_1(z)}\,\hat{u}(z)\right),
\end{aligned} \tag{5.1}
$$

and we conclude that $\hat{u}$ is entire and associated for $\mathfrak{P}$.

Let $\ker R_w^- \cap \mathfrak{P}_- = \operatorname{span}\{k\}$ (compare Lemma 3.2) and let $z_1 \in \rho\left(\overset{\circ}{A}\right)$ be such that $h(z_1) \neq 0$, $\hat{k}(z_1) \neq 0$. If $F \in \mathfrak{P}$ is given, there exists an element $u \in \mathfrak{P}_-$ such that $F = \overset{\circ}{R}{}_{z_1}^- u$. By our choice of z_1, we may assume moreover that $\hat{u}(z_1) = 0$. The relation (5.1) shows that $F(w) = \frac{\hat{u}(w)}{w-z}$, i.e. $(w-z)F(w) = \hat{u}(w) \in \mathfrak{P}_-$. Since by [KW4], Lemma 4.5, every associated function G can be written as $(z-z_1)F(z) + F_1(z)$ with appropriate F, $F_1 \in \mathfrak{P}$, we are done. □

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Institut für Analysis
und Technische Mathematik
TU Wien
Wiedner Hauptstr. 8 – 10/114.1
A – 1040 Wien
Austria