DE BRANGES SPACES OF ENTIRE FUNCTIONS CLOSED UNDER FORMING DIFFERENCE QUOTIENTS

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With a de Branges space $\mathcal{H}(E)$ of entire functions a function q, analytic in $\mathbb{C}^+$ and satisfying there $\operatorname{Im} q(z) \geq 0$, is associated. In this note we give necessary and sufficient conditions for $\mathcal{H}(E)$ to be closed under forming certain difference quotients in terms of the poles and zeros of q. Moreover, we obtain a criterion whether a function q possessing the above mentioned properties can be written as the quotient of the right upper and right lower entry of an entire matrix function $W(z)$ satisfying a certain kernel condition.

1 Introduction and results

Let $\mathcal{H}(E)$ be a de Branges Hilbert space (cf. [dB]), i.e. let $E(z)$ be an entire function satisfying the inequality

$$|E(\overline{z})| < |E(z)|, \ z \in \mathbb{C}^+,$$

and let $\mathcal{H}(E)$ be the set of all entire functions $F(z)$, such that ($F^\#(z) := \overline{F(\overline{z})}$)

$$\frac{F(z)}{E(z)}, \frac{F^\#(z)}{E(z)}$$

are of bounded type and nonpositive mean type in $\mathbb{C}^+$ and

$$\int_{-\infty}^{\infty} \left| \frac{F(t)}{E(t)} \right|^2 dt < \infty.$$

A question which arises naturally in the discussion of de Branges Hilbert spaces and their generaliztion to the indefinite setting (cf. [dB], [KW1], [KW2]) is whether the space $\mathcal{H}(E)$ is closed under forming difference quotients, i.e. whether ($z_0 \in \mathbb{C}$)

$$\frac{F(z) - F(z_0)}{z - z_0} \in \mathcal{H}(E),$$

whenever $F \in \mathcal{H}(E)$. In the notation of [dB] this means that the function 1 is an associated function; $1 \in \operatorname{Assoc} \mathcal{H}(E)$.

We write $E(z) = A(z) - iB(z)$ where the functions A and B are defined as

$$A(z) := \frac{E(z) + E^{\#}(z)}{2}, B(z) := i\frac{E(z) - E^{\#}(z)}{2}.$$

Then the function

$$q(z) := \frac{B(z)}{A(z)} \tag{1.1}$$

contains essential informations about the space $\mathcal{H}(E)$. In fact, if we assume that E has no real zeros, which can be done without loss of generality (cf. [KW1]), then q determines E up to real zero-free factors (cf. [dB], Theorem 24).

In [dB] there can be found criteria for 1 to be an associated function in terms of the function E (cf. [dB], Theroems 25 and 27). The aim of this note is to decide whether $1 \in \operatorname{Assoc} \mathcal{H}(E)$ in terms of the function q, in fact in terms of its poles and zeros.

Of course, since q remains unchanged when multiplying E with some real (meaning $S^{\#} = S$) zero-free entire function S, we can only expect an answer to the question whether $\operatorname{Assoc} \mathcal{H}(E)$ contains some real zero-free function (compare the discussion at the beginning of Section 2), or - formulated differently - whether $\mathcal{H}(E)$ is equivalent to a space $\mathcal{H}(E_1)$ with $1 \in \operatorname{Assoc} \mathcal{H}(E_1)$. However, if the answer is positive, this function is explicitly determined. Our characterization is of different nature than those found in [dB]. Recall e.g. that Theorem 25 of [dB] states that $1 \in \operatorname{Assoc} \mathcal{H}(E)$ if and only if $\frac{1}{E(z)}$ is of bounded type and nonpositive mean type in $\mathbb{C}^+$ and

$$\int_{-\infty}^{\infty} \frac{1}{|E(t)|^2} \frac{dt}{1+t^2} < \infty.$$

So this criterion deals with growth conditions on the function E itself. In contrast we give asymptotic conditions on the sequences of poles and zeros of q, i.e. the sequences of points where $E(t) \equiv \frac{\pi}{2} \bmod \pi$ and $E(t) \equiv 0 \bmod \pi$, respectively (compare the below stated Theorem 1.1).

The following conditions turn out to be essential. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence of distinct real numbers and denote by $(x_k^+)_{k\in\mathbb{N}}$ and $(x_k^-)_{k\in\mathbb{N}}$ the sequence of positive (negative, respectively) $x_k's$ arranged according to increasing modulus. If $0 \in \{x_k | k \in \mathbb{N}\}$ we agree to rearrange the sequence $(x_k)_{k\in\mathbb{N}}$ as $(x_k)_{k\in\mathbb{N}\cup\{0\}}$ with $x_0 = 0$. We will make use of the conditions

(C1) $$\lim_{r\to\infty} \sum_{0<|x_k|\le r} \frac{1}{x_k} = s \in \mathbb{R},$$

and

(C2) $$\lim_{k\to\infty} \frac{k}{x_k^+} = \lim_{k\to\infty} \frac{k}{x_k^-} = \frac{\beta}{2} < \infty.$$

Note that **(C2)** implies that for all $\rho > 1$

$$\sum_{k\in\mathbb{N}} \frac{1}{|x_k|^\rho} < \infty.$$

Hence, if a sequence $(x_k)_{k\in\mathbb{N}}$ satisfies **(C2)**, the canonical product $\prod_{k\in\mathbb{N}}(1-\frac{z}{x_k})e^{\frac{z}{x_k}}$ converges locally uniformly and represents an entire function of order at most 1. If additionally **(C1)** is satisfied, we define an entire function by

$$x(z) := e^{-sz}\prod_{k\in\mathbb{N}}(1-\frac{z}{x_k})e^{\frac{z}{x_k}} = \lim_{r\to\infty}\prod_{|x_k|\le r}(1-\frac{z}{x_k}), \tag{1.2}$$

if $0\notin\{x_k|k\in\mathbb{N}\}$ and

$$x(z) := ze^{-sz}\prod_{k\in\mathbb{N}}(1-\frac{z}{x_k})e^{\frac{z}{x_k}} = z\lim_{r\to\infty}\prod_{|x_k|\le r}(1-\frac{z}{x_k}), \tag{1.3}$$

otherwise. The growth of the function $x(z)$ is well understood. In fact $x(z)$ is of exponential type and

$$\lim_{r\to\infty}\frac{\ln|x(re^{i\phi})|}{r} = \pi\beta|\sin\phi|,\ \ \phi\in[0,2\pi),\phi\neq 0,\pi. \tag{1.4}$$

The fact that $x(z)$ is of exponential type is a consequence of Lindelöfs Theorem (cf. [B], 2.10.3). An application of [B], 8.3.1, yields (1.4).

With a sequence $(x_k)_{k\in\mathbb{N}}$ satisfying **(C1)** and **(C2)** we associate the sequence $(x_k')_{k\in\mathbb{N}}$ defined by

$$x_k' := x'(x_k),\ \ k\in\mathbb{N}. \tag{1.5}$$

By virtue of the locally uniform convergence of the products (1.2) and (1.3), respectively, we have $x_k' = -\frac{1}{x_k}\lim_{r\to\infty}\prod_{|x_i|\le r, i\neq k}\left(1-\frac{x_k}{x_i}\right)$ in the case that $0\notin\{x_k|k\in\mathbb{N}\}$. Otherwise $x_k' = -\lim_{r\to\infty}\prod_{|x_i|\le r, i\neq k}\left(1-\frac{x_k}{x_i}\right)$.

We may always assume without loss of generality that $E(0)=1$, since multiplication of E with a constant does not change the set $\mathcal{H}(E)$ and the inner product is multiplied by a constant factor. Note that the set $\{x_k|k\in\mathbb{N}\}$ is always infinite. The treated questions are of course also meaningful if q has only finitely many poles and zeros. However, then the answers are trivial.

Theorem 1.1. *Let $\mathcal{H}(E)$ be given, $E(t)\neq 0$ for $t\in\mathbb{R}$, $E(0)=1$, and write $E=A-iB$. Then* $\operatorname{Assoc}\mathcal{H}(E)$ *contains a real zero-free function if and only if*

(i) The sequence $(a_k)_{k\in\mathbb{N}}$ of zeros of A, or equivalently the sequence $(b_k)_{k\in\mathbb{N}}$ of zeros of B, satisfies the conditions **(C1)** *and* **(C2)***.*

(ii) If $a(z)$ and $b(z)$ are defined as in (1.2) or (1.3) using the sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$, respectively, we have

(C3) $$\sum_{k\in\mathbb{N}}\left|\frac{1}{a_k^2 a_k' b(a_k)}\right| < \infty.$$

In the case these conditions are fullfilled $S(z) := \frac{A(z)}{a(z)}\in\operatorname{Assoc}\mathcal{H}(E)$.

Recall that a function q which is analytic in $\mathbb{C}^+\cup\mathbb{C}^-$ is said to belong to the class $\mathcal{N}_0$ if it is real, i.e. $q^\# = q$, and $\operatorname{Im} q(z)\geq 0$, $z\in\mathbb{C}^+$. In order to avoid unnecessary misunderstandings let us point out that sometimes, in the context of analytic functions, the symbol $\mathcal{N}_0$ is used in a quite different manner. A 2×2-matrix function $W(z)$ with real and entire entries is said to belong to the class $\mathcal{M}_0^1$ if $\det W(z) = 1$ and the kernel

$$\frac{W(z)JW(w)^* - J}{z - \overline{w}}, \; z, w \in \mathbb{C},$$

where

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

is nonnegative. If $W(z)$ is any 2×2-matrix valued function

$$W(z) =: \begin{pmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{pmatrix},$$

and $\tau(z)$ is a scalar function, then we define

$$W(z) \circ \tau(z) := \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_{21}(z)\tau(z) + w_{22}(z)}.$$

As a corollary we obtain a result on the representability of a function $q \in \mathcal{N}_0$.

Corollary 1.2. *Let $q \in \mathcal{N}_0$ be given. In order that q can be represented as*

$$q(z) = W(z) \circ 0 \tag{1.6}$$

for some matrix $W \in \mathcal{M}_0^1$, $W(0) = 1$, it is necessary and sufficient that $q(0) = 0$ and that the conditions **(C1)** - **(C3)** *are satisfied for the poles $(a_k)_{k\in\mathbb{N}}$ and zeros $(b_k)_{k\in\mathbb{N}}$ of q.*

In Section 2 we reduce the problem to the question whether a certain entire function is of bounded type (we always mean bounded type in $\mathbb{C}^+$) and derive the necessity of the conditions given in Theorem 1.1. Thereby we make use of another criterion for 1 to be an associated function (Lemma 2.1) which, however, follows easily from [dB]. Section 3 is concerned with the proof of sufficiency. The essential ingredients is a condition which ensures that the function (1.2) is of bounded type (Proposition 3.1). This result is also of interest on its own right. Finally we give a proof of Corollary 1.2.

The methods employed are in their nature function theoretic. Besides [dB] we will frequently refer to the textbooks [B] and [L].

2 Proof of necessity

Let $\mathcal{H}(E)$ be given, $E = A - iB$. In order that $\operatorname{Assoc}\mathcal{H}(E)$ contains a zero-free function it is necessary that E has no real zeros (cf. [dB], Theorem 25). Consequently our overall assumption $E(t) \neq 0$, $t \in \mathbb{R}$, is no loss of generality.

If $\operatorname{Assoc}\mathcal{H}(E)$ contains a real zero-free function S, then we may consider the space $\mathcal{H}(E_1)$ with

$$E_1(z) := \frac{E(z)}{S(z)}.$$

The mapping $F \mapsto \frac{F}{S}$ is an isometry of $\mathcal{H}(E)$ onto $\mathcal{H}(E_1)$ and also a bijection of $\operatorname{Assoc}\mathcal{H}(E)$ onto $\operatorname{Assoc}\mathcal{H}(E_1)$. Thus $1 \in \operatorname{Assoc}\mathcal{H}(E_1)$. Since the conditions *(i)* and *(ii)* of Theorem 1.1 are not changed when multiplying E with some real zero-free function, we may assume for the proof of necessity that $1 \in \operatorname{Assoc}\mathcal{H}(E)$.

We start with a lemma which basically follows from [dB]. Denote by $(a_k)_{k\in\mathbb{N}}$ the sequence of zeros of A. Note here that A has only real and simple zeros and that $A(0) = 1$.

Lemma 2.1. *We have* $1 \in \operatorname{Assoc}\mathcal{H}(E)$ *if and only if* A *is of bounded type in* $\mathbb{C}^+$ *and*

$$\sum_{k\in\mathbb{N}} \frac{1}{a_k^2 A'(a_k)B(a_k)} < \infty. \tag{2.1}$$

Proof : Denote by $d\nu$ the discrete measure with point masses of weight $\frac{-1}{A'(a_k)B(a_k)}$ at a_k, $k \in \mathbb{N}$. Note here that $A'(a_k)B(a_k) < 0$ since $\frac{B(z)}{A(z)} \in \mathcal{N}_0$.

Consider first the case that $A \notin \mathcal{H}(E)$. Then $\mathcal{H}(E)$ is equal isometrically to $L^2(d\nu)$ (cf. [dB], Theorem 22 together with Problem 69). Assume that $1 \in \operatorname{Assoc}\mathcal{H}(E)$. By [dB], Theorem 25, the function $\frac{1}{E(z)}$ and thus also $E(z)$ is of bounded type in $\mathbb{C}^+$. From the fact that $|E^\#(z)| < |E(z)|$, $z \in \mathbb{C}^+$, we conclude that $E^\#(z)$ and thus also $A(z)$ possesses the same property. Since for any $F \in \mathcal{H}(E)$ with $F(i) = 1$, we have

$$\frac{F(t) - 1}{t - i} \in \mathcal{H}(E),$$

(2.1) holds. Conversely let A be of bounded type and assume that the condition (2.1) is satisfied. Then [dB], Problem 71, shows that $1 \in \operatorname{Assoc}\mathcal{H}(E)$.

To complete the proof consider the case that $A \in \mathcal{H}(E)$. Denote by $\mathcal{H}(E_1)$ the closure of the domain of multiplication by z in $\mathcal{H}(E)$. By [dB], Problem 72, we have $1 \in \operatorname{Assoc}\mathcal{H}(E)$ if and only if $1 \in \operatorname{Assoc}\mathcal{H}(E_1)$. Since (cf. [dB], Problem 87)

$$(A, B) = (A_1, B_1) \begin{pmatrix} 1 & lz \\ 0 & 1 \end{pmatrix},$$

for some $l > 0$, the conditions (2.1) for E and E_1 coincide. As $A_1 = A \notin \mathcal{H}(E_1)$, the already proved statement may be applied and shows that also in the now considered case the assertion of the lemma holds.

□

Recall that any function $q \in \mathcal{N}_0$ is of bounded type (cf. [GG]). Hence A is of bounded type if and only if B is. The necessity of the conditions of Theorem 1.1 will follow together with the subsequent proposition, since $1 \in \operatorname{Assoc}\mathcal{H}(E)$ implies by Lemma 2.1 that the function q of (1.1) is represented as a quotient of two entire functions of bounded type.

Proposition 2.2. *Let* $q \in \mathcal{N}_0$ *be meromorphic in the plane,* $q(0) = 0$. *Assume that* q *admits the representation*

$$q(z) = \frac{b(z)}{a(z)}$$

with real entire functions a and b which are of bounded type, have no common zeros and satisfy $a(0) = 1$, $b(0) = 0$. Then the sequence $(a_k)_{k\in\mathbb{N}}$ of zeros of a satisfies the conditions **(C1)** *and* **(C2)**. *The constant in* **(C2)** *is the mean type of A. Moreover,*

$$a(z) = \lim_{r\to\infty} \prod_{|a_k|\le r} (1 - \frac{z}{a_k}). \tag{2.2}$$

Similarly

$$b(z) = \gamma z \lim_{r\to\infty} \prod_{|a_k|\le r} (1 - \frac{z}{b_k}), \tag{2.3}$$

when $(b_k)_{k\in\mathbb{N}}$ denotes the zeros of b and $\gamma = q'(0)$. The sequence $(b_k)_{k\in\mathbb{N}}$ satisfies **(C1)** *and* **(C2)** *with the same constant β as $(a_k)_{k\in\mathbb{N}}$.*

Proof : Let β be the mean type of a in $\mathbb{C}^+$. By Krein's theorem (cf. [RR], Theorem 6.17) the functions a and b are of exponential type β and satisfy

$$\int_{-\infty}^{\infty} \frac{\log_+ |a(t)|}{1+t^2}\, dt < \infty, \quad \int_{-\infty}^{\infty} \frac{\log_+ |b(t)|}{1+t^2}\, dt < \infty.$$

Hence we may apply [L], V.Lehrsatz 11, p.249, to obtain **(C1)** and, with the aid of [B], Lemma 1.5.1, **(C2)** with the constant β for the sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$. Moreover, a and b can be written in the form (2.2) and (2.3), respectively.

□

Putting together the statements of Lemma 2.1 and Proposition 2.2 we obtain one half of Theorem 1.1.

Proof (of Theorem 1.1, necessity): Assume that $1 \in \operatorname{Assoc} \mathcal{H}(E)$. Then by Lemma 2.1 the function A is of bounded type. Therefore Proposition 2.2 is applicable to the function q defined by (1.1). We obtain that $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ satisfy **(C1)** and **(C2)**. Moreover, A and B coincide up to the constant γ with the functions defined from their zeros by means of (1.2) and (1.3), respectively. Hence (2.1) is the same as **(C3)**.

□

3 A condition for bounded type

In order to establish the sufficiency of the conditions of Theorem 1.1 we prove the following result.

Proposition 3.1. *Let $(x_k)_{k\in\mathbb{N}}$ be a sequence of real nonzero numbers. Assume that $(x_k)_{k\in\mathbb{N}}$ satisfies* **(C1)**, **(C2)** *and*

(C4) $$\sum_{k\in\mathbb{N}} \frac{1}{x_k^2 |x_k'|} < \infty.$$

Then the function $x(z)$ given by (1.2) is of bounded type.

For the proof of Proposition 3.1 we need two lemmata which give lower estimates for certain analytic functions.

Let $0 < \eta < 1$. A set $\mathcal{E} \subseteq \mathbb{R}^+$ is called an η-set, if ($\mu := \frac{2}{\eta}$) for all $n \in \mathbb{Z}$ the set $\mathcal{E} \cap [\mu^{n-1}, \mu^n]$ has measure at least $(1-\eta)\mu^n = (1 - \frac{\eta}{2-\eta})(\mu^n - \mu^{n-1})$.

Lemma 3.2. *Let f, $f(0) \neq 0$, be an entire function which satisfies ($C, \alpha, \beta > 0$)*

$$|f(z)| \leq |f(0)| C e^{\alpha |z|^\beta}, \ z \in \mathbb{C},$$

and let $0 < \eta < 1$. Then there exists an η-set $\mathcal{E}$ such that f satisfies

$$|f(z)| \geq \tilde{C} e^{-\tilde{\alpha}|z|^\beta}, \ |z| \in \mathcal{E},$$

with ($H(\eta) := 2 + \ln \frac{24e}{\eta}$) $\tilde{C} := (|f(0)|C)^{-H(\eta)}$, $\tilde{\alpha} := H(\eta)\alpha\eta^{-\beta}2^{2\beta}e^\beta$. The set $\mathcal{E}$ depends only on η.

Proof : We apply [L], I.Lehrsatz 11, p.20, to the function $g(z) := \frac{f(z)}{f(0)}$ with $R = \mu^n$ and a change of scale in η by a factor of 16. From this source we obtain that

$$\ln |g(z)| \geq -H(\eta) \ln \sup_{|z| = 2e\mu^n} |g(z)|,$$

for z with $|z| \leq \mu^n$ which lie outside of certain exceptional disks surrounding the zeros of f, the total sum of radii of which does not exceed $\frac{\eta}{4}\mu^n$. Hence the measure of the set $\mathcal{E}_n$ of all radii $r \in [\mu^{n-1}, \mu^n]$ such that the circle $\{|z| = r\}$ does not intersect any of the exceptional discs has measure at least

$$\mu^n - \mu^{n-1} - \frac{\eta}{2}\mu^n = \mu^n(1 - \frac{1}{\mu} - \frac{\eta}{2}) = \mu^n(1-\eta).$$

For $r \in \mathcal{E}_n$ we have

$$\ln|g(z)| \geq -H(\eta)(\ln(|f(0)|C) + \alpha(2e\mu^n)^\beta) = -H(\eta)\ln(|f(0)|C) - H(\eta)\alpha\mu^\beta(2e)^\beta(\mu^{n-1})^\beta \geq$$

$$\geq -H(\eta)\ln(|f(0)|C) - H(\eta)\alpha\eta^{-\beta}2^{2\beta}e^\beta r^\beta.$$

Put $\mathcal{E} := \bigcup_{n \in \mathbb{Z}} \mathcal{E}_n$. The asserted estimate follows.

□

Lemma 3.3. *Let $q \in \mathcal{N}_0$ be meromorphic in $\mathbb{C}$. Assume that the convergence exponent ρ of the sequence $(a_k)_{k \in \mathbb{N}}$ of poles of q is finite. Moreover, let $0 < \eta < 1$. Then there exists an η-set $\mathcal{E}$, such that for any $\epsilon > 0$ an estimate*

$$|q(z)| \leq D_1 e^{D_2 |z|^{\rho+\epsilon}}, \ |z| \in \mathcal{E},$$

with conveniently chosen $D_1, D_2 > 0$ holds.

Proof : According to [L], VII.Lehrsatz 1, p.308, we can write ($c > 0$)

$$q(z) = c\frac{z - b_1}{z - a_1} \prod_{k \geq 2} \frac{1 - \frac{z}{b_k}}{1 - \frac{z}{a_k}}. \tag{3.1}$$

Since the zeros and poles of q interlace, the convergence exponent of the sequence $(b_k)_{k\in\mathbb{N}}$ also equals ρ. Let p denote the genus of $(a_k)_{k\in\mathbb{N}}$ (which equals the genus of $(b_k)_{k\in\mathbb{N}}$). Then we may represent q in the form

$$q(z) = \frac{s(z)}{r(z)},$$

with the entire functions

$$s(z) := c(z-b_1)\prod_{k\geq 2}(1-\frac{z}{b_k})e^{\frac{z}{b_k}+\ldots+\frac{1}{p}(\frac{z}{b_k})^p},$$

$$r(z) := (z-a_1)\prod_{k\geq 2}(1-\frac{z}{a_k})e^{\frac{z}{b_k}+\ldots+\frac{1}{p}(\frac{z}{b_k})^p}.$$

The convergence of the product $r(z)$ follows from the convergence of the products $s(z)$ and (3.1). By [L], I.Lehrsatz 7, p.15, the functions s and r have order ρ. Hence ($\epsilon > 0$)

$$|s(z)| \leq C_1 e^{\alpha_1 |z|^{\rho+\epsilon}},\ z\in\mathbb{C},$$

and by Lemma 3.2

$$\left|\frac{1}{r(z)}\right| \leq C_2 e^{\alpha_2 |z|^{\rho+\epsilon}},\ |z|\in\mathcal{E},$$

for some η-set $\mathcal{E}$. The asserted estimate follows.

$\square$

Now we are in position to prove Proposition 3.1.

Proof (of Proposition 3.1): The theorem of Mittag-Leffler (cf. [BS], II.7, p.243f., shows that the series

$$H(z) := \sum_{k\in\mathbb{N}} \frac{1}{x_k'}\left(\frac{1}{z-x_k}+\frac{1}{x_k}\right)$$

converges locally uniformly on $\mathbb{C}\setminus\{x_k|k\in\mathbb{N}\}$. Moreover, $H(z)$ has a simple pole at x_k with residuum $\frac{1}{x_k'}$.

The function $\frac{1}{x(z)}$ is analytic in $\mathbb{C}\setminus\{x_k|k\in\mathbb{N}\}$ and also has simple poles at x_k with residuum $\frac{1}{x_k'}$. Hence the function

$$F(z) := \frac{1}{x(z)} - H(z)$$

is entire. Write

$$H_+ := -\sum_{k\in\mathbb{N},x_k'>0} \frac{1}{x_k'}\left(\frac{1}{z-x_k}+\frac{1}{x_k}\right),\ H_- := \sum_{k\in\mathbb{N},x_k'<0} \frac{1}{x_k'}\left(\frac{1}{z-x_k}+\frac{1}{x_k}\right).$$

Then $F(z) = \frac{1}{x(z)} + H_+(z) - H_-(z)$. The functions H_+ and H_- are contained in the class $\mathcal{N}_0$.

We assert that F is of zero exponential type. Let $0 < \eta < 1$. The order of $x(z)$ is at most 1. Hence, by Lemma 3.2 there exists an η-set $\mathcal{E}$ such that ($\epsilon > 0$)

$$\left|\frac{1}{x(z)}\right| \le C_1 e^{C_2|z|^{1+\epsilon}}, \ |z| \in \mathcal{E}.$$

Similar estimates hold for H_+ and H_- by Lemma 3.3 for $|z| \in \mathcal{E}_+$ ($\in \mathcal{E}_-$, respectively) with certain η-sets $\mathcal{E}_+$ and $\mathcal{E}_-$. If η is chosen sufficiently small, the sets $\mathcal{E}, \mathcal{E}_+, \mathcal{E}_-$ have an intersection $\tilde{\mathcal{E}}$ which is not bounded. In particular then there exists for arbitrary $\gamma > 0$ a constant C such that

$$|F(z)| \le C e^{\gamma|z|^2}, \ |z| \in \tilde{\mathcal{E}}.$$

Consider the growth of F along the rays $\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$. On such a ray the functions H_+ and H_- satisfy (cf. [GG], I.Theorems 4.2,4.4, [dB], Problem 30)

$$\lim_{r\to\infty} \frac{\ln|H_+(re^{i\theta})|}{r} = 0, \ \lim_{r\to\infty} \frac{\ln|H_-(re^{i\theta})|}{r} = 0.$$

Together with (1.4) we obtain

$$\lim_{r\to\infty} \frac{\ln|F(re^{i\theta})|}{r} = -\frac{\pi\beta}{\sqrt{2}} \le 0,$$

thus for any positive δ the function $F(z)e^{-\delta z}$ ($F(z)e^{i\delta z}$, $F(z)e^{\delta z}$, $F(z)e^{-i\delta z}$) is bounded along the rays $\theta = -\frac{\pi}{4}, \frac{\pi}{4}$ ($\theta = \frac{\pi}{4}, \frac{3\pi}{4}$, $\theta = \frac{3\pi}{4}, -\frac{3\pi}{4}$, $\theta = -\frac{3\pi}{4}, -\frac{\pi}{4}$). By the version [dB], Theorem 1, of the Phragmen-Lindelöf principle applied for each of the mentioned angles (instead of the upper half plane as in [dB]) we obtain

$$|F(z)| \le C e^{\delta|z|}$$

also inside the above angles. Hence this estimate is valid in the whole plane, which shows that F is of zero exponential type.

With F also the function $\tilde{F}$ defined by

$$\tilde{F}(z) := \frac{F(z) - F(0)}{z}$$

is of zero exponential type. Consider $\tilde{F}$ along the positive imaginary axis. Since $|x(iy)|$ is increasing with y, $|\frac{1}{x(z)}|$ is decreasing and hence bounded. Moreover,

$$\left|\frac{1}{iyx_k'}\left(\frac{1}{iy - x_k} + \frac{1}{x_k}\right)\right| = \left|\frac{1}{iyx_k'}\frac{iy}{(iy - x_k)x_k}\right| \le \frac{1}{x_k^2 x_k'},$$

and condition **(C4)** shows that $\frac{H(z)}{z}$ is bounded along $i\,\mathbb{R}^+$. Similar arguments apply to the negative imaginary axis. Altogether we obtain that $\tilde{F}$ is bounded on $i\,\mathbb{R}$. An application of the Phragmen-Lindelöf principle shows that $\tilde{F}$ is bounded in the whole plane and hence a constant.

From the foregoing paragraphs we obtain that

$$\frac{1}{x(z)} = c_1 z + c_2 - H(z)_+ + H_-(z).$$

Thus $x(z)$ is of bounded type.

□

The proof of the remaining half of Theorem 1.1 now follows.
Proof (of Theorem 1.1, sufficiency): We show that the conditions of the theorem imply that for

$$E_1(z) := a(z) - i\gamma b(z),$$

where a and b are defined by (1.2) and (1.3), respectively, and $\gamma > 0$, we have $1 \in \operatorname{Assoc}\mathcal{H}(E_1)$. Since, by [dB], Theorem 24, the distribution of the zeros and poles of $\frac{B(z)}{A(z)}$ determines E up to real zero-free factors (note that $E(t) \neq 0$, $t \in \mathbb{R}$, and $E(0) = 1$ for all considered E's), the statement of the theorem will follow.

Condition **(C3)** is the same as (2.1) of Lemma 2.1, hence we are done if we can show that $a(z)$ is of bounded type. Since

$$\frac{b(z)}{a(z)} \in \mathcal{N}_0,$$

and has poles at a_k, $k \in \mathbb{N}$, with residues $\frac{b(a_k)}{a'(a_k)}$, we conclude that (cf. [L], VII.Lehrsatz 2, p.310)

$$\sum_{k\in\mathbb{N}} \frac{1}{a_k^2} \left(\frac{-b(a_k)}{a'(a_k)} \right) < \infty. \tag{3.2}$$

From $|x| + \frac{1}{|x|} \geq 1$ we see that **(C3)** and (3.2) imply **(C4)**:

$$\sum_{k\in\mathbb{N}} \frac{1}{a_k^2 |a_k'|} \leq \sum_{k\in\mathbb{N}} \frac{1}{a_k^2 |a_k'|} \left(|b(a_k)| + \frac{1}{|b(a_k)|} \right) =$$

$$= \sum_{k\in\mathbb{N}} \frac{1}{a_k^2} \left(\frac{-b(a_k)}{a'(a_k)} \right) + \sum_{k\in\mathbb{N}} \frac{1}{a_k^2} \left(\frac{-1}{a'(a_k) b(a_k)} \right) < \infty.$$

Proposition 3.1 can be applied and yields the assertion of the theorem.

□

It remains to deduce Corollary 1.2.
Proof (of Corollary 1.2): Write

$$W(z) = \begin{pmatrix} D(z) & B(z) \\ C(z) & A(z) \end{pmatrix},$$

and put

$$\tilde{W}(z) = K W(z)^{-1} K, \; K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

As is seen from the formula

$$\frac{W(z)JW(w)^* - J}{z - \overline{w}} = W(z)K\frac{\tilde{W}(z)J\tilde{W}(w)^* - J}{z - \overline{w}}KW(w)^*,$$

the matrix W belongs to $\mathcal{M}_0^1$ if and only if $\tilde{W}$ does. The matrix $\tilde{W}$ computes as

$$\tilde{W}(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}.$$

Now assume that $q \in \mathcal{N}_0$ is represented as in (1.6). Then [dB], Theorem 27, employing the matrix $\tilde{W}$ shows that $1 \in \operatorname{Assoc}\mathcal{H}(A - iB)$. Thus **(C1) - (C3)** are satisfied by Theorem 1.1.

Assume conversely that **(C1) - (C3)** hold. Then by the proof of sufficiency of Theorem 1.1 we have $1 \in \operatorname{Assoc}\mathcal{H}(a - i\gamma b)$ where a and b are defined by (1.2) and (1.3), respectively, from the poles and zeros of q. It follows from [L], VII.Lehrsatz 1, p.308, that the function q admits the representation

$$q(z) = \frac{\gamma b(z)}{a(z)},$$

with $\gamma = q'(0)$. Now the assertion follows from [dB], Theorem 27, and the above mentioned transformation $W \mapsto \tilde{W}$.

□

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