PONTRYAGIN SPACES OF ENTIRE FUNCTIONS II

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We continue the study of a generalization of L. de Branges's theory of Hilbert spaces of entire functions to the Pontryagin space setting. In this - second - part we investigate isometric embeddings of spaces of entire functions into spaces $L^2(\mu)$ understood in a distributional sense and consider Weyl coefficients of matrix chains. The main task is to give a proof of an indefinite version of the inverse spectral theorem for Nevanlinna functions. Our methods use the theory developed by L. de Branges and the theory of extensions of symmetric operators of M.G.Krein.

1 Introduction

In [KW3] we have studied inner product spaces $\langle \mathfrak{P}, [.,.]\rangle$ which satisfy certain additional axioms:

(i) The isotropic part $\mathfrak{P}^\circ$ of $\mathfrak{P}$ is finite dimensional.

(ii) The factor space $\mathfrak{P}/\mathfrak{P}^\circ$ is a Pontryagin space.

(iii) The space $\mathfrak{P}$ consists of entire functions. If $(.,.)$ denotes a Hilbert space inner product associated with $[.,.]$, then $\langle \mathfrak{P}, (.,.)\rangle$ is a reproducing kernel space.

(iv) If $F \in \mathfrak{P}$, then $F^\#(z) := \overline{F(\overline{z})} \in \mathfrak{P}$ and

$$[F^\#, G^\#] = [G, F].$$

(v) If $w \in \mathbb{C} \setminus \mathbb{R}$ and $F \in \mathfrak{P}$, $F(w) = 0$, then $\frac{z-\overline{w}}{z-w}F(z) \in \mathfrak{P}$. If moreover $G \in \mathfrak{P}$, $G(\overline{w}) = 0$, then

$$[\frac{z-\overline{w}}{z-w}F(z), G(z)] = [F(z), \frac{z-w}{z-\overline{w}}G(z)].$$

We call such spaces dB-spaces. In the case that $\langle \mathfrak{P}, [.,.]\rangle$ already is a Hilbert space, the axioms *(i)* and *(ii)* are trivially satisfied and *(iii)*-*(v)* are exactly those axioms used by L. de Branges in [dB7] to characterize certain Hilbert spaces of entire functions. Thus the notion of a dB-space is an immediate generalization of a Hilbert space of entire functions in the sense of L. de Branges.

A motivation for choosing this notion may be found e.g. when studying so-called hermitian functions on $\mathbb{R}$ with a finite number of negative squares (see e.g. [GG], [GL], [KW1], [KL3] or [KL4]), or so-called generalized strings (see e.g. [LW], [W1]). To make the connections with these subjects explicit and to give some applications of the present results will be the subject of a following note.

Recall that a real entire 2×2-matrix function $W(z)$ is said to belong to the class $\mathcal{M}_\kappa^1$ if $\det W(z) = 1$ and the kernel

$$\frac{W(z)JW(w)^* - J}{z - \overline{w}}, \ z, w \in \mathbb{C}, \tag{1.1}$$

where

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

has κ negative squares. It has been proved in [W1], [W3] that for each Nevanlinna function $\tau \in \mathcal{N}_0$ there exists a chain $(W_t)_{t\geq 0}$ of matrices $W_t \in \mathcal{M}_0^1$, $W_0(z) = 1$, such that τ is the Weyl coefficient of $(W_t)_{t\geq 0}$, i.e. such that for any function $\theta \in \mathcal{N}_0$

$$\lim_{t\to\infty} (W_t \circ \theta)(z) = \tau(z). \tag{1.2}$$

For our purposes we allow in in the sequel $\infty \in \mathcal{N}_0$. Here $W_t \circ \tau$ is the function defined by

$$(W_t \circ \tau)(z) := \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_{21}(z)\tau(z) + w_{22}(z)},$$

if $W(z) = (w_{ij}(z))_{i,j=1}^2$. The constructed chain satisfies the following additional conditions:

(i) $W_t(0) = 1$.

(ii) If $0 \leq s < t$, then there exists a matrix $W_{st} \in \mathcal{M}_0^1$ such that $W_t = W_s W_{st}$.

(iii) The trace of $W_t'(0)J$ is equal to t.

Conversely, for every chain $(W_t)_{t\geq 0}$ which satisfies *(i)*-*(iii)*, the limit (1.2) exists and does not depend on $\theta \in \mathcal{N}_0$. Moreover, the chain $(W_t)_{t\geq 0}$ satisfying *(i)*-*(iii)* and having a given $\tau \in \mathcal{N}_0$ as its Weyl coefficient is unique, i.e. there is a bijective correspondence of chains and functions of the class $\mathcal{N}_0$.

The main purpose of this paper is to give a generalization of this result to functions $\tau \in \mathcal{N}_\kappa$ and chains $(W_t)_{t>c_-}$ of matrices which belong to $\mathcal{M}_\kappa^1$ and satisfy *(i)*-*(iii)*. It turns out (Theorem 8.7) that the correspondence established by (1.2) is again bijective.

Beside of this result we prove some statements concerning isometric embeddings of dB-spaces into spaces $L^2(\phi)$ understood in a certain distributional sense (Proposition 4.6), concerning regularized resovents of the operator of multiplication by z in a dB-Pontryagin space (Theorem 5.7), and concerning chains $(W_t)_{t\leq 0}$ of matrices $W_t \in \mathcal{M}_{\kappa(t)}^1$. For example the fact that in each dB-space there exists a unique chain of dB-subspaces will show that each single matrix $W_0 \in \mathcal{M}_\kappa^1$ invents a chain $(W_t)_{t\leq 0}$ of matrices $W_t \in \mathcal{M}_{\kappa(t)}^1$ (Theorem 7.1).

The structure of this chain which goes downwards from W_0 reflects the structure of the chain of subspaces of a certain dB-Pontryagin space related to W_0.

The proof of the main Theorem 8.7 will make use of some transformations of chains which generalize the results of [W2]. In [W2], for the case $\kappa = 0$, some transformations have also been studied in connection with so-called canonical systems of differential equations. It will be the subject of a forthcoming note to develop these connections further.

Table of contents

The contents of this paper splits into two parts.

The first part consists of the Sections 2-6. In Sections 2 and 3 we investigate distributions ϕ on $\mathbb{R} \cup \{\infty\}$ and Pontryagin spaces $\Pi(\phi)$ which are connected to such distributions and generalize the concept of L^2-spaces. This continues some investigations of [JLT] and yields model spaces for selfadjoint relations in Pontryagin spaces. The Sections 4 and 6 are concerned with isometric embeddings of dB-spaces into such spaces $\Pi(\phi)$. We use the integral representation of a resolvent similar as given in [KL2] (compare also [GG]). If ϕ is a bounded measure in some neighbourghhood of ∞, then we obtain generalizations of the results of [dB7] concering isometric embeddings into spaces $L^2(\mu)$. In Section 5 we investigate functions of the form $(W \circ \tau)(z)$ where $W \in \mathcal{M}_\kappa^1$ and $\tau \in \mathcal{N}_\nu$. Theorem 5.7 relates growth conditions on $(W \circ \tau)$ to properties of the reproducing kernel space generated by the kernel (1.1). In particular we obtain conditions in order to ensure that $W \circ \tau \in \mathcal{N}_{\kappa+\nu}$. We also give some results supplementing [KW3], Sections 6 and 7.

In the second part, consisting of Sections 7-11, we study chains of matrices (W_t). Section 7 is first concerned with the chain going downwards from some given matrix W_0 (Theorem 7.1). Some results supplementing [KW3], Sections 12 and 13, are needed. We also show how chains going downwards and going upwards from some matrix W_0 can be linked together. In Section 8 we introduce a certain class of chains going upwards and investigate their Weyl coefficient. Also our main result Theorem 8.7 is formulated. The Sections 9 and 10 provide some essential tools used in the proof of Theorem 8.7. In Section 9 we use the integral representation of a function q belonging to $\mathcal{N}_\kappa$ as given in [KL1], in order to investigate a certain rational transform of q (Proposition 9.1). It is the aim of Section 10 to study some transformations of matrix chains, which correspond to the transformation of the function $q \in \mathcal{N}_\kappa$ studied in Section 9. Some particular cases of such transformation rules have been introduced in [DK], [W1] or [W2]. Finally, in Section 11 we complete the proof of Theorem 8.7 and give some corollaries.

For the use of notation we refer to [KW3] and the literature cited there, in particular [ADSR], [dB7], [IKL], [DS1]. References to results of [KW2] or [KW3] will be given as the following examples indicate: Lemma 0.2.1 refers to Lemma 2.1 of [KW2] and (I.2.1) refers to the equation (2.1) of [KW3].

2 Distributions on $\overline{\mathbb{R}}$

In the sequel denote by $\overline{\mathbb{R}}$ the closed real line $\mathbb{R} \cup \{\infty\}$, i.e. the one-point compactification of $\mathbb{R}$, and let $\mathbb{T}$ be the unit circle in the complex plane. Clearly, $\overline{\mathbb{R}}$ is homoeomorphic to $\mathbb{T}$, e.g. via the mapping

$$\gamma : \begin{cases} \mathbb{T} \to \overline{\mathbb{R}} \\ z \mapsto \tan(\frac{\arg z}{2}) \end{cases},$$

where we choose $-\pi < \arg z \leq \pi$. We shall work with $\overline{\mathbb{R}}$ and $\mathbb{T}$ interchangeably and will always identify them via γ.

Denote by $C^\infty(\overline{\mathbb{R}})$ the set of functions

$$C^\infty(\overline{\mathbb{R}}) := \{f \circ \gamma^{(-1)} | f \in C^\infty(\mathbb{T})\},$$

and provide $C^\infty(\overline{\mathbb{R}})$ with the topology induced by the topology of test functions on $C^\infty(\mathbb{T})$ via the mapping γ. If $C_0^\infty(\mathbb{R})$ denotes the set of all infinitely differentiable functions on $\mathbb{R}$ which vanish at infinity, we clearly have $C_0^\infty(\mathbb{R}) \subseteq C^\infty(\overline{\mathbb{R}})$ as sets.

If φ' is a distribution on $\mathbb{T}$, i.e. $\varphi' \in \mathcal{D}(\mathbb{T})$, we define a distribution φ on $\overline{\mathbb{R}}$ by

$$\varphi \cdot f := \varphi' \cdot (f \circ \gamma),\ f \in C^\infty(\overline{\mathbb{R}}).$$

The set of all functionals φ obtained in this way is denoted by $\mathcal{D}(\overline{\mathbb{R}})$. An element $\varphi \in \mathcal{D}(\overline{\mathbb{R}})$ is called real if $\varphi \cdot f \in \mathbb{R}$ for all real valued functions $f \in C^\infty(\overline{\mathbb{R}})$. Note that $\mathcal{D}(\overline{\mathbb{R}})$ is the dual space of $C^\infty(\overline{\mathbb{R}})$, when $C^\infty(\overline{\mathbb{R}})$ is provided with the above introduced topology.

With the conventions

$$b \cdot \infty = \infty,\ \infty - a = \infty,\ \frac{1}{0} = \infty,\ \frac{1}{\infty} = 0,\ a \in \mathbb{R}, b \in \mathbb{R} \setminus \{0\},$$

it is elementary to check that the mappings

$$M_b : f(t) \mapsto f(bt),\ T_a : f(t) \mapsto f(t-a),\ \mathrm{Inv} : f(t) \mapsto f(\frac{1}{t}),$$

are automorphisms of $C^\infty(\overline{\mathbb{R}})$. Hence the adjoints of these mappings are automorphisms of $\mathcal{D}(\overline{\mathbb{R}})$. In particular, if the distribution φ has support $E \subseteq \overline{\mathbb{R}}$, i.e. $E = \sigma(\varphi)$, then $M_b^*(\varphi)$ ($T_a^*(\varphi)$, $\mathrm{Inv}^*(\varphi)$) has support $b \cdot E$ ($E - a$, $\frac{1}{E}$). Similar as in [JLT] we give the following

Definition 2.1. Let $\varphi \in \mathcal{D}(\overline{\mathbb{R}})$. We write $\varphi \in \mathcal{F}(\overline{\mathbb{R}})$ if φ is real and if there exists a finite set $s(\varphi) \subseteq \overline{\mathbb{R}}$, such that φ restricted to $\overline{\mathbb{R}} \setminus s(\varphi)$ is a positive (possibly unbounded) measure.

In [JLT] those distrubutions $\varphi \in \mathcal{F}(\overline{\mathbb{R}})$ which have compact support in $\mathbb{R}$ are considered, we study the whole of $\mathcal{F}(\overline{\mathbb{R}})$.

In the sequel, $s(\varphi)$ will always denote the smallest set such that φ restricted to $\overline{\mathbb{R}} \setminus s(\varphi)$ is a positive measure. For $F \subseteq \overline{\mathbb{R}}$ we denote by χ_F the characteristic function of F. If $F \cap s(\varphi) = \emptyset$, then $\chi_F \varphi$ is a positive measure.

Denote by $\mathcal{B}_2(\varphi)$ the linear space of all functions f on $\overline{\mathbb{R}}$ such that for all compact subsets F of $\overline{\mathbb{R}}$ with $s(\varphi) \cap F = \emptyset$ the restriction $f|_F$ belongs to $L^2(\chi_F\varphi)$, and such that for some open set U, $s(\varphi) \subseteq U \subseteq \overline{\mathbb{R}}$, the restriction $f|_U$ coincides on U with some function in $C^\infty(\overline{\mathbb{R}})$.

For a finite set $B \subseteq \mathbb{C} \setminus \mathbb{R}$ such that $B = \bar{B}$, i.e. B is symmetric with respect to $\mathbb{R}$, we introduce the set $\mathcal{F}(\mathbb{C} \setminus \mathbb{R}, B)$ of functionals exactly in the same way as in [JLT, p. 257]. For a functional $\psi \in \mathcal{F}(\mathbb{C} \setminus \mathbb{R}, B)$ let $\sigma(\psi)$ be the minimal set such that ψ still belongs to $\mathcal{F}(\mathbb{C} \setminus \mathbb{R}, \sigma(\psi))$.

Definition 2.2. Let $\mathcal{F}$ be the set of functionals

$$\mathcal{F} := \cup_B(\mathcal{F}(\overline{\mathbb{R}}) + \mathcal{F}(\mathbb{C} \setminus \mathbb{R}, B)),$$

where B runs through all finite $\mathbb{R}$-symmetric subsets of $\mathbb{C} \setminus \mathbb{R}$.

If $\phi \in \mathcal{F}$, $\phi = \varphi + \psi$, we set $s(\phi) = s(\varphi)$ and define the support $\sigma(\phi)$ of ϕ as $\sigma(\varphi) \cup \sigma(\psi)$. We call φ the real part of ϕ and ψ the complex part of ϕ. Denote by $\mathcal{B}_2(\phi)$ the set of all functions, which are defined and holomorphic in a neighbourhood of $\sigma(\psi)$ and whose restrictions to $\overline{\mathbb{R}}$ belong to $\mathcal{B}_2(\varphi)$.

From these definitions we immediately have the following

Lemma 2.3. *Let $\phi \in \mathcal{F}$, and let Δ be a finite intervall such that its endpoints do not belong to $s(\phi)$. Then we can write the distribution ϕ as*

$$\phi = \varphi_1 + \varphi_2 + \psi,$$

where $\varphi_1 = \chi_\Delta \varphi$, $\varphi_2 = \chi_{\overline{\mathbb{R}} \setminus \Delta}\varphi$, $\psi \in \mathcal{F}(\mathbb{C} \setminus \mathbb{R}, B)$ and $s(\phi) = s(\varphi_1) \cup s(\varphi_2)$, $\sigma(\phi) = \sigma(\varphi_1) \cup \sigma(\varphi_2) \cup \sigma(\psi)$. The distribution φ_1 has compact support in $\mathbb{R}$, and is of the same kind of distributions considered in [JLT]. If 0 belongs to the interior of Δ the same statement holds for $\mathrm{Inv}^*(\varphi_2)$.

3 Pontryagin spaces associated with distributions

In this section we associate to each distribution in $\phi \in \mathcal{F}$ a Pontryagin space and a selfadjoint relation. This construction is universal in the sense that any selfadjoint relation in a Pontryagin space which is minimal in some sense can be identified with a model constructed from some distribution. In [JLT] this construction was made for functionals ϕ, whose real part has compact support in $\mathbb{R}$.

If $\phi \in \mathcal{F}$ define an inner product on $\mathcal{B}_2(\phi)$ by

$$[f,g]_\phi := \phi \cdot (f\bar{g}),\ \ f,g \in \mathcal{B}_2(\phi).$$

Consider the relation

$$A_\phi = \{(f;g) \in \mathcal{B}_2^2(\phi) | g(t) = tf(t), t \in \mathbb{R} \cup B\}.$$

Proposition 3.1. *Let $\phi \in \mathcal{F}$ be given. The inner product space $(\mathcal{B}_2(\phi), [.,.]_\phi)$ has finite negative index. Hence, it yields by factorization by its isotropic part $\mathcal{B}_2(\phi)^\circ$ and completion a Pontryagin space $\Pi(\phi)$. The relation A_ϕ gives rise by factorization by $(\mathcal{B}_2(\phi)^\circ)^2$ and closure to a selfadjoint relation in $\Pi(\phi)$ (again denoted by A_ϕ) with nonempty resolvent set. In fact we have $\sigma(A_\phi) = \sigma(\phi)$.*

Moreover, $s(\phi)$ coincides with $s(A_\phi)$, which the union of the set of all critical points of A_ϕ and the set of points of negative type of A_ϕ, and

$$\sigma_p(A_\phi) = s(\phi) \cup \{x \in \overline{\mathbb{R}} \setminus s(\phi) | (\chi_{\overline{\mathbb{R}} \setminus s(\phi)} \phi)\{x\} > 0\}.$$

Finally, we have for intervalls $\Delta \subseteq \overline{\mathbb{R}}$, whose endpoints do not belong to $s(\phi)$, and symmetric (with respect to $\mathbb{R}$) neighbourhoods $V \subseteq \mathbb{C} \setminus \mathbb{R}$ of subsets of $\sigma(\phi) \setminus \overline{\mathbb{R}}$

$$E(\Delta)f = \chi_\Delta f,\ \ E_V f = \chi_V f,\ \ f \in \mathcal{B}_2(\phi).$$

Here $E(.)$ denote the spectral projections of A_ϕ and E_V denotes the Riesz projector corresponding to $V \cap \sigma(A_\phi)$.

Proof : We will use the respective properties of the distributions considered in [JLT]. First we decompose ϕ according to Lemma 2.3 as $\phi = \varphi_1 + \varphi_2 + \psi$, where $\Delta = (-N,N)$ so that $-N, N \notin s(\phi)$.

From [JLT] we know that the inner product spaces $(\mathcal{B}_2(\varphi_1), [.,.]_{\varphi_1})$ and $(\mathcal{B}_2(\psi), [.,.]_\psi)$ have finitely many negative squares. Since Inv is an automorphism on $C^\infty(\overline{\mathbb{R}})$, the space $(\mathcal{B}_2(\varphi_2), [.,.]_{\varphi_2})$ is isomorphic to $(\mathcal{B}_2(\mathrm{Inv}^*\varphi_2), [.,.]_{\mathrm{Inv}^*\varphi_2})$ via Inv. As $\mathrm{Inv}^*\varphi_2$ has compact support in $\mathbb{R}$ it follows again from [JLT] that $(\mathcal{B}_2(\varphi_2), [.,.]_{\varphi_2})$ has finitely many negative squares. Clearly, Inv can be extended to an isomorphism (also denoted by Inv) from the completion $\Pi(\varphi_2)$ of $(\mathcal{B}_2(\varphi_2)/\mathcal{B}_2(\varphi_2)^\circ, [.,.]_{\varphi_2})$ to the completion $\Pi(\mathrm{Inv}^*\varphi_2)$ of

$$(\mathcal{B}_2(\mathrm{Inv}^*\varphi_2)/\mathcal{B}_2(\mathrm{Inv}^*\varphi_2)^\circ, [.,.]_{\mathrm{Inv}^*\varphi_2}).$$

Let $\Pi(\varphi_2)$ be the completion of $(\mathcal{B}_2(\varphi_1)/\mathcal{B}_2(\varphi_1)^\circ, [.,.]_{\varphi_1})$ and $\Pi(\psi)$ be the completion of $(\mathcal{B}_2(\psi)/\mathcal{B}_2(\psi)^\circ, [.,.]_\psi)$. Since $(\mathcal{B}_2(\psi), [.,.]_\psi)$ is finite dimensional and, due to the minimality of $\sigma(\psi)$, nondegenerated we have in fact $\Pi(\psi) = \mathcal{B}_2(\psi)$. Note that the spaces $\Pi(\varphi_1)$, $\Pi(\mathrm{Inv}^*\varphi_2)$, $\Pi(\psi)$ are of the same kind as the spaces considered in [JLT].

Now we provide the space $\mathcal{B}_2(\varphi_1) \times \mathcal{B}_2(\varphi_2) \times \mathcal{B}_2(\psi)$ with the inner product

$$[(f_1;f_2;f_3),(g_1;g_2;g_3)]_\phi = [f_1,g_1]_{\varphi_1} + [f_2,g_2]_{\varphi_2} + [f_3,g_3]_\psi,$$

and see that $\mathcal{B}_2(\varphi_1) \times \mathcal{B}_2(\varphi_2) \times \mathcal{B}_2(\psi)$ has finitely many negative squares. Define a mapping Ψ by

$$\Psi : \left\{ \begin{array}{l} \mathcal{B}_2(\phi) \to \mathcal{B}_2(\varphi_1) \times \mathcal{B}_2(\varphi_2) \times \mathcal{B}_2(\psi) \\ f \mapsto (f|_{\overline{\mathbb{R}}}; f|_{\overline{\mathbb{R}}}; f|_B) \end{array} \right. ,\ \ f \in \mathcal{B}_2(\phi).$$

Then Ψ is injective and isometric with respect to $[.,.]_\phi$ and has dense range in $\Pi(\varphi_1)\oplus\Pi(\varphi_2)\oplus\Pi(\psi)$. Hence $(\mathcal{B}_2(\phi),[.,.]_\phi)$ has finitely many negative squares, and the completion $\Pi(\phi)$ of $\mathcal{B}_2(\phi)/\mathcal{B}_2(\phi)^\circ$ is isomorphic to $\Pi(\varphi_1)\oplus\Pi(\varphi_2)\oplus\Pi(\psi)$. We will denote this isomorphism again by Ψ.

Now consider the relation A_ϕ. Since ϕ is real, A_ϕ is symmetric. Moreover, for $z\notin(\mathbb{R}\cup B)$, consider the relation

$$(A_\phi - z)^{-1} = \{(f;g)\in\mathcal{B}_2(\phi) | g(t) = \frac{f(t)}{t-z},\ t\in\mathbb{R}\cup B\}.$$

It is easy to check that $(A_\phi - z)^{-1}$ has domain $\mathcal{B}_2(\phi)$. This shows that the closure of A_ϕ (again denoted by A_ϕ) is a symmetric relation on $\Pi(\phi)$ such that $\operatorname{ran}(A_\phi - z) = \Pi(\phi)$. By [DS1] we conclude that A_ϕ is selfadjoint with $\sigma(A_\phi)\subseteq\overline{\mathbb{R}}\cup B$. Since A_{φ_1}, A_{φ_2} and A_ψ are also selfadjoint in $\Pi(\varphi_1)$, $\Pi(\varphi_2)$, $\Pi(\psi)$, respectively, with nonempty resolvent set, we easily see that $(\Psi^2)(A_\phi) = A_{\varphi_1}\oplus A_{\varphi_2}\oplus A_\psi$. An elementary consideration also shows that $(\mathrm{Inv}^2)A_{\varphi_2} = A^{-1}_{\mathrm{Inv}^*\varphi_2}$.

Note that A_{φ_1}, $A_{\mathrm{Inv}^*\varphi_2}$, A_ψ are the same relations as those introduced in [JLT]. They are in fact bounded selfadjoint operators. In the mentioned paper it is shown that the asserted properties for ϕ and A_ϕ hold for φ_1 and A_{φ_1}, $\mathrm{Inv}^*\varphi_2$ and $A_{\mathrm{Inv}^*\varphi_2}$, and for ψ and A_ψ. Since $(\mathrm{Inv}^2)A_{\varphi_2} = A^{-1}_{\mathrm{Inv}^*\varphi_2}$ and

$$\sigma(A^{-1}_{\mathrm{Inv}^*\varphi_2}) = \frac{1}{\sigma(A_{\mathrm{Inv}^*\varphi_2})},\ \sigma_p(A^{-1}_{\mathrm{Inv}^*\varphi_2}) = \frac{1}{\sigma_p(A_{\mathrm{Inv}^*\varphi_2})},$$

$$s(A^{-1}_{\mathrm{Inv}^*\varphi_2}) = \frac{1}{s(A_{\mathrm{Inv}^*\varphi_2})},$$

and since $\tilde{E}_2(\Delta) = E_2(\frac{1}{\Delta})$ for intervalls Δ with endpoints not in $s(A^{-1}_{\mathrm{Inv}^*\varphi_2})$, where $\tilde{E}_2(.)$ and $E_2(.)$ are the spectral projections of $A^{-1}_{\mathrm{Inv}^*\varphi_2}$ and $A_{\mathrm{Inv}^*\varphi_2}$, respectively, we see that the asserted properties also hold for φ_2 and A_{φ_2}. We are almost done. In fact, since Ψ^2 maps A_ϕ onto $A_{\varphi_1}\oplus A_{\varphi_2}\oplus A_\psi$, the asserted properties also hold for ϕ and A_ϕ.

□

We would like to point out again that the construction of the space $\Pi(\phi)$ in the previous proof shows that for a distribution $\phi\in\mathcal{F}$, whose real part has compact support in $\mathbb{R}$, this spaces coincides with the space constructed in [JLT].

With the same notation as in the previous proof, it is an immediate consequence of Proposition 3.1 and its proof that

$$\Psi((E_V + E((-N,N))\Pi(\phi)) = \Pi(\varphi_1+\psi),\ \Psi(E(\overline{\mathbb{R}}\setminus(-N,N))\Pi(\phi)) = \Pi(\varphi_2). \tag{3.1}$$

Here $V\subseteq\mathbb{C}\setminus\mathbb{R}$ is a symmetric neighbourhood of $\sigma(\psi)$, and E_V denotes the spectral projection onto the spectral subspace corresponding to $\sigma(\psi)$.

Lemma 3.2. *Let A be a selfadjoint relation in a Pontryagin space $\mathfrak{P}$, and denote by $E(\Delta)$ the spectral projections associated with A and by E_B the orthogonal projection onto the span* *of the generalized eigenspaces of all nonreal eigenvalues B. Let $N \in \mathbb{R}$, $N > 0$, be such that $-N, N \notin s(A)$.*

Assume that families $\gamma_1(z) \in (E(-N,N) + E_B)\mathfrak{P}$ and $\gamma_2(z) \in (I - E(-N,N) - E_B)\mathfrak{P}$, $z \in \rho(A)$, are given such that

$$\gamma_j(z) = (I + (z-w)(A-z)^{-1})\gamma_j(w),\ \ z, w \in \rho(A),\ \ j = 1, 2,$$

holds. If $\operatorname{cls}\{\gamma_1(z) | z \in \rho(A)\} = (E(-N,N) + E_B)\mathfrak{P}$ *and* $\operatorname{cls}\{\gamma_2 | z \in \rho(A)\} = (I - E(-N,N) - E_B)\mathfrak{P}$*, then*

$$\operatorname{cls}\{\gamma_1(z) + \gamma_2(z) | z \in \rho(A)\} = \mathfrak{P}.$$

Proof : Assume that $x \in \mathfrak{P}$ such that

$$[x, \gamma_1(z) + \gamma_2(z)] = 0, z \in \rho(A). \tag{3.2}$$

If $z_0 \in \rho(A)$ is fixed, we find

$$[(A-z)^{-1}\gamma_1(z_0), x] = -[(A-z)^{-1}\gamma_2(z_0), x],\ \ z \in \rho(A).$$

Since the right hand side of this relation is analytic on $\mathbb{C} \setminus ((-\infty, -N] \cup [N, \infty))$, so is the left hand side. For a closed interval $\Delta \subseteq (-N, N)$, it follows from the definition of $E(\Delta)\gamma_1(z_0)$ and $E_B\gamma_1(z_0)$ by means of integrals that

$$[(E(\Delta) + E_B)\gamma_1(z_0), x] = 0.$$

We obtain by a limiting argument that

$$[(E(-N,N) + E_B)\gamma_1(z_0), x] = 0.$$

Since z_0 was arbitrarily chosen, this implies $(E(-N,N) + E_B)x = 0$, therefore $x \in (I - E(-N,N) - E_B)\mathfrak{P}$, and $[x, \gamma_1(z)] = 0$ for all $z \in \rho(A)$. Since by (3.2) also $[x, \gamma_2(z)] = 0$, $z \in \rho(A)$, we conclude $x = 0$.

□

Proposition 3.3. *Let $\phi \in \mathcal{F}$ and let $\Pi(\phi)$ and A_ϕ be as above. Choose $z_0 \in \rho(A_\phi)$ and consider the functions*

$$\gamma(z) := \frac{t - z_0}{t - z},\ \ z \in \rho(A_\phi).$$

Then $\gamma(z) \in \mathcal{B}_2(\phi)$,

$$\gamma(z) = (I + (z-w)(A_\phi - z)^{-1})\gamma(w),\ \ z, w \in \rho(A_\phi), \tag{3.3}$$

and

$$\Pi(\phi) = \operatorname{cls}\{\gamma(z) | z \in \rho(A_\phi)\}. \tag{3.4}$$

Proof : Clearly, $\frac{t-z_0}{t-z} \in \mathcal{B}_2(\phi)$ and satisfies (3.3). We decompose ϕ according to Lemma 2.3 as $\phi = (\varphi_1 + \varphi_2) + \psi \in \mathcal{F}$ with $\Delta = (-N, N), \ -N, N \notin s(\phi)$. Since by the proof of Proposition 3.1 φ_1 and $\mathrm{Inv}^*\varphi_2$ have compact support in $\mathbb{R}$, the considerations of [JLT] imply that 1 is a generating element of the spaces $\Pi(\varphi_1 + \psi)$ and $\Pi(\mathrm{Inv}^*\varphi_2)$. Since by the Riesz-Dunford functional calculus

$$1, t, t^2, t^3, \ldots \in \mathrm{cls}\,\{1, \frac{1}{t-z} | z \in \mathbb{C} \setminus (\mathbb{R} \cup B)\},$$

we see that this closed span coincides with $\Pi(\varphi_1 + \psi)$ or $\Pi(\mathrm{Inv}^*\varphi_2)$, respectively. Now we know from Proposition 3.1 and its proof that Inv is an isomorphism form $\Pi(\varphi_2)$ onto $\Pi(\mathrm{Inv}^*\varphi_2)$ and $(\mathrm{Inv}^2)A_{\varphi_2} = A^{-1}_{\mathrm{Inv}^*\varphi_2}$. From $\mathrm{Inv}(1) = 1$ and $\mathrm{Inv}(\frac{1}{t-z}) = -\frac{1}{z} - \frac{1}{z^2}\frac{1}{t-\frac{1}{z}}$ we then obtain that also the closed span

$$\mathrm{cls}\,\{1, \frac{1}{t-z} | z \in \mathbb{C} \setminus (\mathbb{R} \cup B)\} \subseteq \Pi(\varphi_2)$$

is equal to $\Pi(\varphi_2)$. By (3.1) and Lemma 3.2 we see that (3.4) holds.

□

We associate by Proposition 3.1 and Proposition 3.3 to each $\phi \in \mathcal{F}$ a triple $(\Pi(\phi), A_\phi, \gamma(z))$ consisting of a Pontryagin space, a selfadjoint relation therein with nonempty resolvent set and elements satisfying (3.3) and (3.4). The following proposition gives a converse result. By the relation (3.3) the family $\gamma(z)$ is determined by each of its members $\gamma(z_0)$. Hence we will also use the notation of a triple $(\Pi(\phi), A_\phi, \gamma(z_0))$.

Proposition 3.4. *Let $\mathfrak{P}$ be a Pontryagin space, $A \subseteq \mathfrak{P}^2$ be a selfadjoint relation, $\rho(A) \neq \emptyset$, and let $\gamma(z) \in \mathfrak{P}$ be elements satisfying (3.3) and (3.4). If a number $z_0 \in \rho(A)$ is chosen, there exists a unique $\phi \in \mathcal{F}$ such that the following triples are isomorphic:*

$$(\mathfrak{P}, A, \gamma(z)) \cong (\Pi(\phi), A_\phi, \frac{t-z_0}{t-z}).$$

Proof : Let $s(A)$ be again the union of the set of critical points and of the set of points of negative type of A and let B be the set of nonreal eigenvalues of A. Denote again by $E(\Delta)$ the spectral projections associated with A, and let E_B be the Riesz projector corresponding to B. Let $N \in \mathbb{R}$, $N \geq 2$, be chosen such that $s(A) \cap \mathbb{R} \subseteq (-N+1, N-1)$.

The space $\mathfrak{P}$ can be decomposed as $\mathfrak{P} = \mathfrak{P}_1 \oplus \mathfrak{P}_2$, where $\mathfrak{P}_1 = (E(-N,N) + E_B)\mathfrak{P}$ and $\mathfrak{P}_2 = (I - E(-N,N) - E_B)\mathfrak{P}$, and the relation A can be written accordingly as $A = A_1 \oplus A_2$, where A_j is a selfadjoint relation in $\mathfrak{P}_j$ for $j = 1, 2$. Moreover, A_1 and A_2^{-1} are bounded selfadjoint operators on $\mathfrak{P}_1$ and $\mathfrak{P}_2$, respectively, and it follows from (3.4) that A_1 and A_2^{-1} are cyclic operators with the generating elements $(A_1 - z_0)\gamma(z_0)$ and $(A_2^{-1} - z_0)\gamma(\frac{1}{z_0})$ (see [K]).

By [JLT] there exist unique functionals

$$\phi_1' = \varphi_1' + \psi' \in \mathcal{F}(\mathbb{R}) + \mathcal{F}(\mathbb{C} \setminus \mathbb{R}, B), \ \varphi_2' \in \mathcal{F}(\mathbb{R}),$$

such that $(\mathfrak{P}_1, A_1, (A_1 - z_0)\gamma(z_0)) \cong (\Pi(\phi_1'), A_{\phi_1'}, 1)$, and $(\mathfrak{P}_2, A_2^{-1}, (A_2^{-1} - z_0)\gamma(\frac{1}{z_0})) \cong (\Pi(\varphi_2'), A_{\varphi_2'}, 1)$.

We set $\varphi_1 = \varsigma\varphi_1'$, $\psi = \varsigma\psi'$ and $\phi_1 = \varphi_1 + \psi$, where $\varsigma(t) = \frac{1}{|t-z_0|^2}$, $t \in \mathbb{R} \cup B$. Moreover, let $\varphi_2 = \vartheta\varphi_2'$, where $\vartheta(t) = \frac{1}{|z_0^2 t - z_0|^2}$, $t \in \mathbb{R} \cup B$. It follows that $(\mathfrak{P}_1, A_1, \gamma(z_0))$ is isomorphic to $(\Pi(\phi_1), A_{\phi_1}, 1)$, and $(\mathfrak{P}_2, A_2^{-1}, \gamma(\frac{1}{z_0}))$ is isomorphic to $(\Pi(\varphi_2), A_{\varphi_2}, \frac{tz_0^2 - z_0}{t - z_0})$, and by the above considerations (see proof of Proposition 3.1) to $(\Pi(\mathrm{Inv}^*\varphi_2), A^{-1}_{\mathrm{Inv}^*\varphi_2}, \frac{t-z_0}{t-\frac{1}{z_0}})$. Hence, $(\mathfrak{P}_2, A_2, \gamma(z_0))$ is isomorphic to $(\Pi(\mathrm{Inv}^*\varphi_2), A_{\mathrm{Inv}^*\varphi_2}, 1)$.

We see from the considerations in [JLT] that the support of φ_1 (φ_2) is contained in $[-N, N]$ ($[-\frac{1}{N}, \frac{1}{N}]$). Hence the support of $\mathrm{Inv}^*\varphi_2$ is contained in $\overline{\mathbb{R}} \setminus (-N, N)$. Moreover, by [JLT] the restriction of φ_1 to a sufficiently small neighbourhood of $\{-N, N\}$ is a positive measure, which has no point mass at N and $-N$.

Now set $\phi = (\varphi_1 + \mathrm{Inv}^*\varphi_2) + \psi$. We find that $\phi \in \mathcal{F}$, and $\varphi_1 = \chi_{(-N,N)}(\varphi_1 + \mathrm{Inv}^*\varphi_2)$, $\mathrm{Inv}^*\varphi_2 = \chi_{\overline{\mathbb{R}}\setminus(-N,N)}(\varphi_1 + \mathrm{Inv}^*\varphi_2)$. Since by (3.1) the triple $(\Pi(\phi), A_\phi, 1)$ is isomorphic to $(\Pi(\phi_1) \oplus \Pi(\mathrm{Inv}^*\varphi_2), A_{\phi_1} \oplus A_{\mathrm{Inv}^*\varphi_2}, (1; 1))$, we obtain that $(\Pi(\phi), A_\phi, 1)$ is isomorphic to $(\mathfrak{P}, A, \gamma(z_0))$. Hence $(\Pi(\phi), A_\phi, \frac{t-z_0}{t-z})$ is isomorphic to $(\mathfrak{P}, A, \gamma(z))$.

Assume that $\tilde{\phi} = \tilde{\varphi} + \tilde{\psi}$, $\tilde{\varphi} \in \mathcal{F}(\overline{\mathbb{R}})$, $\tilde{\psi} \in \mathcal{F}(\mathbb{C} \setminus \mathbb{R})$ is another element from $\mathcal{F}$, such that $(\Pi(\tilde{\phi}), A_{\tilde{\phi}}, 1)$ is isomorphic to $(\mathfrak{P}, A, \gamma(z_0))$. Then by (3.1) we see that $(\Pi(\chi_{(-N,N)}\tilde{\varphi} + \tilde{\psi}), A_{\chi_{(-N,N)}\tilde{\varphi}+\tilde{\psi}}, 1)$ is isomorphic to $(\Pi(\varphi_1 + \psi), A_{\varphi_1+\psi}, 1)$ and $(\Pi(\chi_{\overline{\mathbb{R}}\setminus(-N,N)}\tilde{\varphi}), A_{\chi_{\overline{\mathbb{R}}\setminus(-N,N)}\tilde{\varphi}}, 1)$ is isomorphic to $(\Pi(\mathrm{Inv}^*\varphi_2), A_{\mathrm{Inv}^*\varphi_2}, 1)$. Reversing the some steps from above and keeping in mind that by [JLT] the uniqueness statement for selfadjoint relations wich are bounded operators hold, we find that in fact $\chi_{(-N,N)}\tilde{\varphi} + \tilde{\psi} = \varphi_1 + \psi$ and $\chi_{\overline{\mathbb{R}}\setminus(-N,N)}\tilde{\varphi} = \mathrm{Inv}^*\varphi_2$. Hence $\tilde{\phi} = \phi$.

□

Corollary 3.5. *Let $q(z)$ belong to $\mathcal{N}_\kappa$ and let $z_0 \in \mathbb{C} \setminus \mathbb{R}$ belong to the domain of holomorphy $\rho(q)$ of $q(z)$. Then there exists a unique distribution $\phi \in \mathcal{F}$ with $\sigma(q) := \mathbb{C} \setminus \rho(q) = \sigma(\phi)$ and a number $\alpha \in \mathbb{R}$ such that*

$$q(z) = \alpha + \phi \cdot \left(\left(\frac{1}{t-z} - \frac{t - \operatorname{Re} z_0}{|t-z_0|^2}\right)|t - z_0|^2 \right). \tag{3.5}$$

Conversely, for any $\phi \in \mathcal{F}$ and $\alpha \in \mathbb{R}$ the function $q(z)$ defined by (3.5) belongs to $\mathcal{N}_\kappa$ for some $\kappa \in \mathbb{N}_0$. The number κ is the index of negativity of $\Pi(\phi)$.

Proof : It follows for example from [HSW] (see also [KL1]) that there exists a (up to isomorphisms) unique triple $(\Pi(q), A_q, \gamma_q(z))$, where $\Pi(q)$ is a Pontryagin space, A_q is a selfadjoint relation with $\sigma(A_q) = \sigma(q)$, and where $\gamma_q(z) \in \Pi(q)$, $z \in \rho(A_q)$, are elements which satisfy (3.3) and (3.4) such that

$$\frac{q(z) - \overline{q(w)}}{z - \bar{w}} = [\gamma_q(z), \gamma_q(w)],\ z, w \in \mathbb{C} \setminus \sigma(q). \tag{3.6}$$

By Proposition 3.4 there exists a unique $\phi \in \mathcal{F}$ such that

$$(\Pi(q), A_q, \gamma_q(z)) \cong (\Pi(\phi), A_\phi, \frac{t - z_0}{t - z}).$$

We obtain

$$\frac{q(z)-\overline{q(w)}}{z-\bar{w}} = \phi\cdot\Big(\frac{|t-z_0|^2}{(t-z)(t-\bar{w})}\Big),$$

and further with a little calculation

$$q(z)-\operatorname{Re}(q(z_0)) = \phi\cdot\Big(\Big(\frac{1}{t-z}-\frac{t-\operatorname{Re} z_0}{|t-z_0|^2}\Big)|t-z_0|^2\Big).$$

To prove the converse, note first that the function defined in (3.5) satisfies (3.6), if $\gamma_q(z) = \frac{t-z_0}{t-z} \in \Pi(\phi)$. Hence the kernel on the left hand side has the same number of negative squares κ as $\Pi(\phi)$. This gives $q \in \mathcal{N}_\kappa$.

□

4 Embeddings of dB-spaces

Let a dB-Pontryagin space $\mathfrak{P} = \mathfrak{P}(E)$ be given and assume for simplicity that $\mathfrak{d}(\mathfrak{P}) = 0$. Recall that $\mathfrak{d}(\mathfrak{P})(w)$ is the minimal order of a zero at w of $\in \mathfrak{P}$ (see (I.4.1)). If $\tilde{A} \subseteq \tilde{\mathfrak{P}}^2$, $\rho(\tilde{A}) \neq \emptyset$, is a selfadjoint extension of the operator $\mathcal{S}$ of multiplication by z in $\mathfrak{P}$ acting in some Pontryagin space $\tilde{\mathfrak{P}} \supseteq \mathfrak{P}$ and $\Delta \subseteq \mathbb{R}$ is an interval whose endpoints are not critical points of $\tilde{A}$, we will denote by $\tilde{E}(\Delta)$ the spectral projection associated with $\tilde{A}$ and Δ. The spaces $\mathfrak{P}_+, \mathfrak{P}_-, \tilde{\mathfrak{P}}_+, \tilde{\mathfrak{P}}_-$ and the operators $R_z^+ : \tilde{\mathfrak{P}} \to \tilde{\mathfrak{P}}_+$, $R_z^- : \tilde{\mathfrak{P}}_- \to \tilde{\mathfrak{P}}$ are defined as in [KW2].

Proposition 4.1. *Let $\tilde{A} \subseteq \tilde{\mathfrak{P}}^2$, $\rho(\tilde{A}) \neq \emptyset$, be a selfadjoint extension of $\mathcal{S}$ which is $\mathfrak{P}$-minimal, and let $u \in \mathfrak{P}_- = \operatorname{Ass}\mathfrak{P}$ (compare Proposition I.10.2). Moreover, let $\Delta \subseteq \mathbb{R}$ be a closed interval whose endpoints are not critical points for $\tilde{A}$, and assume that $u(t) \neq 0$ for $t \in \Delta$. If Δ contains exactly one critical point α of $\tilde{A}$, then*

$$\tilde{E}(\Delta)f = \int_\Delta \left(\frac{f(t)}{u(t)}(t-z_0) - \sum_{l=0}^{2\nu-1}\frac{1}{l!}\frac{d^l}{dz^l}\Big(\frac{f(z)}{u(z)}(z-z_0)\Big)\Big|_{z=\alpha}(t-\alpha)^l\right) d\tilde{E}_t R_{z_0}^- u -$$

$$-\sum_{l=0}^{2\nu}\frac{1}{l!}\frac{d^l}{dz^l}\Big(\frac{f(z)}{u(z)}(z-z_0)\Big)\Big|_{z=\alpha} u_l,\ f \in \mathfrak{P}. \tag{4.1}$$

Here z_0 is any fixed point $z_0 \in \rho(\tilde{A}) \setminus \Delta$, and u_l are certain elements of $\tilde{\mathfrak{P}}$ (depending on z_0). If $w \in \rho(\tilde{A}) \setminus \Delta$, then

$$\tilde{E}(\Delta)R_w f = \int_\Delta \left(\frac{f(t)}{u(t)}\frac{t-z_0}{t-w} - \sum_{l=0}^{2\nu-1}\frac{1}{l!}\frac{d^l}{dz^l}\Big(\frac{f(z)}{u(z)}\frac{z-z_0}{z-w}\Big)\Big|_{z=\alpha}(t-\alpha)^l\right) d\tilde{E}_t R_{z_0}^- u -$$

$$-\sum_{l=0}^{2\nu}\frac{1}{l!}\frac{d^l}{dz^l}\Big(\frac{f(z)}{u(z)}\frac{z-z_0}{z-w}\Big)\Big|_{z=\alpha} u_l,\ f \in \mathfrak{P}. \tag{4.2}$$

Proof : Choose an open interval $\Delta_0 \supseteq \Delta$ which contains no critical point besides α, such that $u(t) \neq 0$, $t \in \Delta_0$. We decompose the space $\tilde{\mathfrak{P}}$ as

$$\tilde{\mathfrak{P}} = \tilde{E}(\Delta_0)\tilde{\mathfrak{P}}[\dot{+}](I - \tilde{E}(\Delta_0)\tilde{\mathfrak{P}}),$$

and write the resolvent $R_z = (\tilde{A} - z)^{-1}$ correspondingly as $R_z = R_z^{(0)} + R_z^{(1)}$. By [L] (compare also [DS2]) we have

$$R_z^{(0)} = \int_{\Delta_0} \left(\frac{1}{t-z} + \sum_{l=1}^{2\nu} \frac{(t-\alpha)^{l-1}}{(z-\alpha)^l} \right) d\tilde{E}_t + \sum_{l=1}^{2\nu+1} B_l \frac{1}{(z-\alpha)^l}. \tag{4.3}$$

Here $\nu \in \mathbb{N}_0$ and B_l are certain bounded operators on $\tilde{E}(\Delta_0)\tilde{\mathfrak{P}}$.

Choose a point $z_1 \in \rho(\tilde{A}) \setminus \mathbb{R}$ and an element $\varphi^{z_1}(z_1) \in \mathfrak{P}$, $\varphi^{z_1}(z_1) \perp \operatorname{ran}(\mathcal{S} - \overline{z}_1)$. Let elements $\varphi^{z_1}(z) \in \tilde{\mathfrak{P}}$, $z \in \rho(\tilde{A})$ be defined as

$$\varphi^{z_1}(z) = (I + (z - z_1)R_z)\varphi^{z_1}(z_1),$$

then clearly $\varphi^{z_1}(z) \perp \operatorname{ran}(\mathcal{S} - \overline{z})$ and the family $\varphi^{z_1}(z)$ satisfies

$$\varphi^{z_1}(z) = (I + (z - w)R_z)\varphi^{z_1}(w), \; z, w \in \rho(\tilde{A}). \tag{4.4}$$

The orthogonal projection of $\tilde{\mathfrak{P}}$ onto $\mathfrak{P}$ maps $\varphi^{z_1}(\overline{z})$ onto a certain multiple of $K(z, .)$ and vanishes by analyticity only on a set which has no accumulation point in $\rho(\tilde{A})$. Hence, for all z with $u(z) \neq 0$ we find

$$[f, \varphi^{z_1}(\overline{z})] = \frac{f(z)}{u(z)}[u, \begin{pmatrix} \varphi^{z_1}(\overline{z}) \\ \overline{z}\varphi^{z_1}(\overline{z}) \end{pmatrix}]_\pm.$$

We compute, using (4.4) and (0.5.7),

$$(z - z_0)[R_{z_0}^- u, \varphi^{z_1}(\overline{z})] = [u, (\overline{z} - \overline{z_0})R_{\overline{z_0}}^+ \varphi^{z_1}(\overline{z})]_\pm =$$

$$= [u, \begin{pmatrix} \varphi^{z_1}(\overline{z}) \\ \overline{z}\varphi^{z_1}(\overline{z}) \end{pmatrix}]_\pm - [u, \begin{pmatrix} \varphi^{z_1}(\overline{z_0}) \\ \overline{z_0}\varphi^{z_1}(\overline{z_0}) \end{pmatrix}]_\pm.$$

It follows that

$$[f, \varphi^{z_1}(\overline{z})] = \frac{f(z)}{u(z)}(z - z_0)[R_{z_0}^- u, \varphi^{z_1}(\overline{z})] + \frac{f(z)}{u(z)}[u, \begin{pmatrix} \varphi^{z_1}(\overline{z_0}) \\ \overline{z_0}\varphi^{z_1}(\overline{z_0}) \end{pmatrix}]_\pm. \tag{4.5}$$

Choose $z_2 \in \rho(\tilde{A}) \setminus \mathbb{R}$, then (4.4) and (4.5) imply

$$[\frac{1}{z - \overline{z_2}} f + R_z f, \varphi^{z_1}(z_2)] =$$

$$= \frac{f(z)}{u(z)}(z - z_0)[\frac{1}{z - \overline{z_2}} R_{z_0}^- u + R_z^{(0)} R_{z_0}^- u + R_z^{(1)} R_{z_0}^- u, \varphi^{z_1}(z_2)] + \tag{4.6}$$

$$+ \frac{f(z)}{u(z)}[u, \begin{pmatrix} \varphi^{z_1}(\overline{z_0}) \\ \overline{z_0}\varphi^{z_1}(\overline{z_0}) \end{pmatrix}]_\pm.$$

Now we integrate (4.6) with respect to z along the path consisting of the line segments $\Delta+i\varepsilon$ and $\Delta-i\varepsilon$, $\varepsilon>0$, and let ε tend to 0. The first summand on the left hand side of (4.6) as well as the first, third and last summand on the right hand side of (4.6) are (for sufficiently small ε) analytic in the domain $\Delta\times[-\varepsilon,\varepsilon]$, hence do not contribute to the integral. By the definition of the spectral projection $\tilde{E}(\Delta)$, relation (4.3) and the Stieltjes-Livsic inversion formula we obtain

$$[\tilde{E}(\Delta)f,\varphi^{z_1}(z_2)]=$$

$$=\int_\Delta\left(\frac{f(t)}{u(t)}(t-z_0)-\sum_{l=0}^{2\nu-1}\frac{1}{l!}\frac{d^l}{dz^l}(\frac{f(z)}{u(z)}(z-z_0))|_{z=\alpha}(t-\alpha)^l\right)\,d[\tilde{E}_tR_{z_0}^-u,\varphi^{z_1}(z_2)]-$$

$$-\sum_{l=0}^{2\nu}\frac{1}{l!}\frac{d^l}{dz^l}(\frac{f(z)}{u(z)}(z-z_0))|_{z=\alpha}[B_lR_{z_0}^-u,\varphi^{z_1}(z_2)].$$

Since z_1 and z_2 were arbitrary, $\mathcal{S}$ is simple, and $\tilde{A}$ is $\mathfrak{P}$-minimal, the span of $\{\varphi^{z_1}(z_2)|z_1,z_2\in\rho(\tilde{A})\setminus\mathbb{R}\}$ is dense in $\tilde{\mathfrak{P}}$. Hence (4.1) holds.

To prove the relation (4.2), note that

$$[R_wf,\varphi^{z_1}(\overline{z})]=[f,R_{\overline{w}}\varphi^{z_1}(\overline{z})]=\frac{1}{z-w}[f,\varphi^{z_1}(\overline{z})]-\frac{1}{z-w}[f,\varphi^{z_1}(\overline{w})],$$

hence by (4.5) we have

$$[R_wf,\varphi^{z_1}(\overline{z})]=$$

$$=\frac{f(z)}{u(z)}\frac{z-z_0}{z-w}[R_{z_0}^-u,\varphi^{z_1}(\overline{z})]+\frac{1}{z-w}\frac{f(z)}{u(z)}[u,\begin{pmatrix}\varphi^{z_1}(\overline{z_0})\\ \overline{z_0}\varphi^{z_1}(\overline{z_0})\end{pmatrix}]_\pm-\frac{1}{z-w}[f,\varphi^{z_1}(\overline{w})].$$

Proceeding as in the above part of the proof yields the desired result.

□

Remark 4.2. If in the situation of Proposition 4.1, the interval Δ contains no critical point of $\tilde{A}$, then with the same proof we find

$$\tilde{E}(\Delta)f=\int_\Delta\frac{f(t)}{u(t)}(t-z_0)\,d\tilde{E}_tR_{z_0}^-u,\ \ f\in\mathfrak{P},$$

$$\tilde{E}(\Delta)R_wf=\int_\Delta\frac{f(t)}{u(t)}\frac{t-z_0}{t-w}\,d\tilde{E}_tR_{z_0}^-u,\ \ f\in\mathfrak{P}.$$

If $\beta\in\sigma(\tilde{A})\setminus\mathbb{R}$ and $u(\beta)\neq 0$, then

$$\tilde{E}_{\{\beta,\overline{\beta}\}}f=\sum_{l=0}^{\nu-1}\frac{1}{l!}\frac{d^l}{dz^l}(\frac{f(z)}{u(z)}(z-z_0))|_{z=\beta}A_{-l-1}R_{z_0}^-u+$$

$$+\sum_{l=0}^{\nu-1}\frac{1}{l!}\frac{d^l}{dz^l}(\frac{f(z)}{u(z)}(z-z_0))\big|_{z=\overline{\beta}}A_{-l-1}^*R_{z_0}^-u,\ \ f\in\mathfrak{P}. \tag{4.7}$$

Here A_{-l-1} are the coefficients in the Laurent expansion of R_z at β:

$$R_z = \frac{A_{-\nu}}{(z-\beta)^\nu} + \ldots + \frac{A_{-1}}{z-\beta} + \ldots .$$

The relation (4.7) is proved similar as Proposition 4.1 by integrating (4.6) along a curve which surrounds β and $\overline{\beta}$ and is such that no other spectral points or zeros of u lie on the curve or in its interior.

Note that in Proposition 4.1 the dependence on z_0 is not essential.

Corollary 4.3. *Assume that the interval $\Delta \subseteq \mathbb{R}$ contains no critical point of $\tilde{A}$, and that $u(t) \neq 0$ for $t \in \Delta$. Then*

$$\int_\Delta (t-z)\, d\tilde{E}_t R_z^- u = \int_\Delta (t-z_0)\, d\tilde{E}_t R_{z_0}^- u, \tag{4.8}$$

whenever $z \in \rho(\tilde{A}) \setminus \mathbb{R}$.

Proof : Let K be a rectangular path which contains Δ in its interior and is such that no zeros of u or nonreal spectral points of $\tilde{A}$ lie on K or in its interior. Choose a function $f \in \mathfrak{P}$ which has no zeros on K. By Remark 4.2 we have ($w \in K$):

$$\tilde{E}(\Delta)(\frac{u(z)f(w)-u(w)f(z)}{z-w}) = \int_\Delta \frac{u(t)f(w)-u(w)f(t)}{t-w}\frac{t-z_0}{u(t)}\, d\tilde{E}_t R_{z_0}^- u =$$

$$= f(w)\int_\Delta \frac{1}{t-w}\,(t-z_0) d\tilde{E}_t R_{z_0}^- u - u(w)\underbrace{\int_\Delta \frac{f(t)}{u(t)}\frac{t-z_0}{t-w}\, d\tilde{E}_t R_{z_0}^- u}_{=\tilde{E}(\Delta)R_w f}.$$

Hence the integral

$$\int_\Delta \frac{1}{t-w}\,(t-z_0) d\tilde{E}_t R_{z_0}^- u$$

does not depend on z_0 and (4.8) follows.

□

The assertion of Proposition 4.1 is the main tool for constructing embeddings of $\mathfrak{P}$. It enables us to prove

Proposition 4.4. *Let $\tilde{A} \subseteq \tilde{\mathfrak{P}}^2$, $\rho(\tilde{A}) \neq \emptyset$, be a selfadjoint extension of $\mathcal{S}$ which is $\mathfrak{P}$-minimal. If $u \in \mathfrak{P}_-$ satisfies*

(i) $u(t) \neq 0$ for $t \in \mathbb{R} \cup \sigma(\tilde{A})$,

(ii) if $\overline{\operatorname{dom} \tilde{A}} \neq \tilde{\mathfrak{P}}$, then $R_{z_0}^- u \notin \overline{\operatorname{dom} \tilde{A}}$,

then $\tilde{A}$ is $R_{z_0}^- u$-minimal. There exists an element $E_1(z) \in \mathfrak{P}_-$, with $\mathfrak{P}(E_1) = \mathfrak{P}(E)$ which satisfies (i) and (ii).

Proof : We set

$$\operatorname{cls}\{R_z^- u | z \in \rho(\tilde{A})\} =: \mathfrak{L}.$$

Since $R_z^- u = (I + (z - z_0)R_z)R_{z_0}^- u$ for $z \in \rho(\tilde{A}) \setminus \mathbb{R}$, the fact that $\tilde{A}$ is $R_{z_0}^- u$-minimal is equivalent to $\mathfrak{L} = \tilde{\mathfrak{P}}$.

We first show that for any interval $\Delta \subseteq \mathbb{R}$ whose endpoints are not critical points of $\tilde{A}$ and which contains exactly one critical point α, the spaces $\tilde{E}(\Delta)\mathfrak{P}$ and $\tilde{E}(\Delta)R_z\mathfrak{P}$ are contained in the above span. The integral terms on the right hand sides of (4.1) and (4.2) are clearly contained in $\mathfrak{L}$ (cf. [L]). The elements u_l are by construction given as $B_l R_{z_0}^- u$. Consider the relation (4.3): Since $R_z^{(0)} = R_z\tilde{E}(\Delta_0)$, we have $R_z^{(0)} R_{z_0}^- u \in \mathfrak{L}$, and the integral term on the right hand side of (4.3) also belongs to $\mathfrak{L}$. Hence for all $z \in \rho(\tilde{A})$,

$$\sum_{l=1}^{2\nu+1} \frac{1}{(z-\alpha)^l} B_l R_{z_0}^- u \in \mathfrak{L},$$

and we conclude that $B_l R_{z_0}^- u \in \mathfrak{L}$. By (4.1) it follows that $\tilde{E}(\Delta)\mathfrak{P} \subseteq \mathfrak{L}$ and by (4.2) that $\tilde{E}(\Delta)R_z\mathfrak{P} \subseteq \mathfrak{L}$.

Note that in view of Remark 4.2 the relations $\tilde{E}(\Delta)\mathfrak{P} \subseteq \mathfrak{L}$, $\tilde{E}(\Delta)R_z\mathfrak{P} \subseteq \mathfrak{L}$ also hold if Δ contains no critical points. Similarly $\tilde{E}_{\mathbb{C}\setminus\mathbb{R}}\mathfrak{P} \subseteq \mathfrak{L}$, $\tilde{E}_{\mathbb{C}\setminus\mathbb{R}}R_z\mathfrak{P} \subseteq \mathfrak{L}$.

If $\infty \notin \sigma_p(\tilde{A})$, we are already done since then the closed span $\mathfrak{E}$ of the spaces $\tilde{E}(\Delta)\mathfrak{P}$, $\tilde{E}(\Delta)R_z\mathfrak{P}$, where Δ runs through all finite intervalls with noncritical endpoints and $z \in \rho(\tilde{A}) \setminus \mathbb{R}$, and of the spaces $\tilde{E}_{\mathbb{C}\setminus\mathbb{R}}\mathfrak{P}$, $\tilde{E}_{\mathbb{C}\setminus\mathbb{R}}R_z\mathfrak{P}$, $z \in \rho(\tilde{A}) \setminus \mathbb{R}$ contains the span of all spaces of the from $(\tilde{E}(\Delta) + \tilde{E}_{\mathbb{C}\setminus\mathbb{R}})\tilde{\mathfrak{P}}$. But, since now $\tilde{A}$ is an operator, the latter span is dense in $\tilde{\mathfrak{P}}$.

Otherwise we employ the condition *(ii)* on u: It states that

$$\operatorname{span}\{R_{z_0}^- u\} + \overline{\operatorname{dom}\tilde{A}} = \tilde{\mathfrak{P}}. \tag{4.9}$$

Note that the space $\mathfrak{E}$ introduced above equals $\mathfrak{S}_\infty^\perp$, when $\mathfrak{S}_\infty$ denotes the spectral subspace corresponding to ∞. Since $\overline{\operatorname{dom}\tilde{A}} = \overline{\operatorname{ran} R_z}$, applying R_z several times to the relation (4.9) yields

$$\operatorname{span}\{R_{z_0}^- u, \dots, R_z^k R_{z_0}^- u\} + \overline{\operatorname{ran}(R_z^k)} = \tilde{\mathfrak{P}}.$$

If k exceeds the maximal lenght of a Jordan-chain at infinity, then $\overline{\operatorname{ran}(R_z^k)} = \mathfrak{S}_\infty^\perp$, and we conclude that $\mathfrak{P} \subseteq \mathfrak{L}$.

It remains to show that some function $E_1(z) = A_1(z) + iB_1(z)$ with $\mathfrak{P}(E_1) = \mathfrak{P}(E)$ actually satisfies *(i)* and *(ii)*. Recall that by Corollary I.6.2 all such functions E_1 can be obtained as $(E_1 = A_1 - iB_1)$

$$(A_1, B_1) = (A, B)U,$$

where U is a 2×2-matrix with real entries and $\det U = 1$. Clearly there is a choice of U, such that $E_1(z) \neq 0$ for $z \in \sigma(\tilde{A}) \setminus \mathbb{R}$. By our overall assumption $\mathfrak{d}(\mathfrak{P}) = 0$, we have $E_1(t) \neq 0$ for $t \in \mathbb{R}$ (compare Lemma I.5.4), hence E_1 satisfies *(i)*. Now consider the case that $\overline{\operatorname{dom}\tilde{A}} \neq \tilde{\mathfrak{P}}$. Choose $G \in \overline{\operatorname{dom}\mathcal{S}}$ with $G(z_0) = E_1(z_0)$, then by Lemma I.4.5

$$E_1(z) = (z - z_0)F(z) + G(z), \tag{4.10}$$

for some $F \in \mathfrak{P}$. It follows that

$$R_{z_0}^- E_1 = F + R_{z_0}^- G.$$

Assume on the contrary that $R_{z_0}^- E_1 \in \overline{\operatorname{dom} \tilde{A}}$. Since $R_{z_0}^- G \in \operatorname{dom} \tilde{A}$, also $F \in \overline{\operatorname{dom} \tilde{A}}$. Note that $\overline{\operatorname{dom} \tilde{A}} \cap \mathfrak{P} = \overline{\operatorname{dom} \mathcal{S}} \neq \mathfrak{P}$. It follows from (4.10), that $E_1 \in \operatorname{Ass}(\overline{\operatorname{dom} \mathcal{S}})$. Hence $A_1, B_1 \in \operatorname{Ass}(\overline{\operatorname{dom} \mathcal{S}})$, and we conclude that

$$\frac{B_1(z)\overline{A_1(w)} - A_1(z)\overline{B_1(w)}}{z - \overline{w}} \in \overline{\operatorname{dom} \mathcal{S}},$$

a contradiction.

□

Remark 4.5. It follows from the proof of Proposition 4.4 that

$$\mathfrak{S}_\infty^\perp \subseteq \operatorname{cls}\{R_z^- u | z \in \rho(\tilde{A})\},$$

if $u \in \mathfrak{P}_-$ only satisfies assumption *(i)* in Proposition 4.4. Let us also remark explicitly that the fact of $\tilde{A}$ being $R_{z_0}^- u$-minimal is equivalent to the relation

$$\tilde{\mathfrak{P}} = \operatorname{cls}\{R_z^- u | z \in \rho(\tilde{A})\}.$$

Let $\tilde{A} \subseteq \tilde{\mathfrak{P}}^2$, $\rho(\tilde{A}) \neq \emptyset$, be a selfadjoint extension of $\mathcal{S}$ which is $\mathfrak{P}$-minimal and let $u \in \mathfrak{P}_-$ satisfy *(i)* and *(ii)* of Proposition 4.4. By Proposition 3.4 there exists a triple $(\Pi(\phi), A_\phi, 1)$ isomorphic to $(\tilde{\mathfrak{P}}, \tilde{A}, R_{z_0}^- u)$. By this isomorphism Γ the space $\mathfrak{P}$ can be considered as a subspace of $\Pi(\phi)$.

Proposition 4.6. *With the above notation let $\Delta \subseteq \mathbb{R}$ be an interval whose endpoints are not contained in the set $s(\phi)$ and denote by χ_Δ the characteristic function of Δ. Then*

$$\Gamma(\tilde{E}(\Delta)f)(t) = \frac{f(t)}{u(t)}(t - z_0)\chi_\Delta(t), \;\; f \in \mathfrak{P}. \tag{4.11}$$

If V is a sufficiently small neighbourghhood of $\sigma(\tilde{A}) \setminus \mathbb{R}$, then

$$\Gamma(\tilde{E}_{\mathbb{C}\setminus\mathbb{R}} f)(z) = \frac{f(z)}{u(z)}(z - z_0)\chi_V(z), \;\; f \in \mathfrak{P}. \tag{4.12}$$

Proof : We show that (4.11) holds if Δ contains exactly one critical point α. If Δ contains no critical point, the assertion follows by the same proof, even with some simplifications. Also the relation (4.12) is proved in this way. For general intervals (4.11) follows by decomposing Δ into intervals of the considered types.

By Proposition 3.1 and Proposition 3.4 we know

$$\Gamma(\tilde{E}(\Delta)R_z^- u)(t) = (1 + (z - z_0)\frac{1}{t - z})\chi_\Delta(t), \tag{4.13}$$

hence

$$[\tilde{E}(\Delta)R_z^- u, R_w^- u] = [(1+\frac{z-z_0}{t-z})\chi_\Delta(t), 1+\frac{w-z_0}{t-w}]_\phi. \tag{4.14}$$

Using the relations

$$R_z^- u = (I + (z-z_0)R_z)R_{z_0}^- u,\ \ R_{\overline{w}}^- u = (I + (\overline{w}-z_0)R_{\overline{w}})R_{z_0}^- u,$$

and (4.3) we compute

$$[\tilde{E}(\Delta)R_z^- u, R_w^- u] = [\tilde{E}(\Delta)R_{z_0}^- u, R_{z_0}^- u] + \tag{4.15}$$

$$+\frac{(\overline{w}-\overline{z_0})(z_0-\overline{w})}{z-\overline{w}}\int_\Delta \left(\frac{1}{t-\overline{w}} + \sum_{l=1}^{2\nu}\frac{(t-\alpha)^{l-1}}{(\overline{w}-\alpha)^l}\right) d[\tilde{E}_t R_{z_0}^- u, R_{z_0}^- u] +$$

$$+\frac{(z-\overline{z_0})(z-z_0)}{z-\overline{w}}\int_\Delta \left(\frac{1}{t-z} + \sum_{l=1}^{2\nu}\frac{(t-\alpha)^{l-1}}{(z-\alpha)^l}\right) d[\tilde{E}_t R_{z_0}^- u, R_{z_0}^- u] +$$

$$+\sum_{l=1}^{2\nu+1}[B_l R_{z_0}^- u, R_{z_0}^- u]\left(\frac{(\overline{w}-\overline{z_0})(z_0-\overline{w})}{z-\overline{w}}\frac{1}{(\overline{w}-\alpha)^l} + \frac{(z-\overline{z_0})(z-z_0)}{z-\overline{w}}\frac{1}{(z-\alpha)^l}\right).$$

We decompose $\phi = \varphi_1 + \varphi_2 + \psi$ according to Lemma 2.3. It follows from Lemma 1.2 of [JLT] that we can represent φ_1 as

$$\varphi_1 \cdot f = \int_\Delta (f(t) - \sum_{j=0}^{2\nu'-1}(j!)^{-1}(t-\alpha)^j f^{(j)}(\alpha))d\sigma(t) +$$

$$+\sum_{j=0}^{2\nu'} c_j (j!)^{-1} f^{(j)}(\alpha),\ \ f \in C^\infty(\overline{\mathbb{R}}), \tag{4.16}$$

where $\nu' \in \mathbb{N}$, $c_0, \ldots, c_{2\nu'} \in \mathbb{C}$, and where σ is a positive measure on Δ. We compute

$$[(1+\frac{z-z_0}{t-z})\chi_\Delta(t), 1+\frac{w-z_0}{t-w}]_\phi = [\chi_\Delta, 1]_{\varphi_1} + (z-z_0)[\frac{1}{t-z}\chi_\Delta, 1]_{\varphi_1} +$$

$$+(\overline{w}-z_0)[\chi_\Delta, \frac{1}{t-w}]_{\varphi_1} + (z-z_0)(\overline{w}-\overline{z_0})[\frac{1}{t-z}\frac{1}{t-\overline{w}}\chi_\Delta, 1]_{\varphi_1} =$$

$$= [\chi_\Delta, 1]_{\varphi_1} + \frac{(z-\overline{z_0})(z-z_0)}{z-\overline{w}}[\frac{1}{t-z}\chi_\Delta, 1]_{\varphi_1} + \frac{(\overline{w}-\overline{z_0})(z_0-\overline{w})}{z-\overline{w}}[\frac{1}{t-\overline{w}}\chi_\Delta, 1]_{\varphi_1} =$$

$$= [\chi_\Delta, 1]_{\varphi_1} + \frac{(\overline{w}-\overline{z_0})(z_0-\overline{w})}{z-\overline{w}}\int_\Delta \left(\frac{1}{t-\overline{w}} + \sum_{l=1}^{2\nu'}\frac{(t-\alpha)^{l-1}}{(\overline{w}-\alpha)^l}\right) d\sigma +$$

$$-\sum_{l=0}^{2\nu'} c_l \left(\frac{(\overline{w}-\overline{z_0})(z_0-\overline{w})}{z-\overline{w}}\frac{1}{(\overline{w}-\alpha)^l} + \frac{(z-\overline{z_0})(z-z_0)}{z-\overline{w}}\frac{1}{(z-\alpha)^l}\right).$$

Comparing this relation with (4.15) and using the Stieltjes inversion formula it follows that $\nu = \nu'$, $c_l = -[B_{l+1}R_{z_0}^- u, R_{z_0}^- u]$, and $d\sigma(t) = d[\tilde{E}_t R_{z_0}^- u, R_{z_0}^- u]$ on Δ.

Since Γ is an isomorphism, relation (4.13) implies that for $f \in \mathfrak{P}$ we have

$$[\tilde{E}(\Delta)f, R_w^- u] = \phi \cdot (\Gamma(\tilde{E}(\Delta)f)\frac{t-\overline{z_0}}{t-\overline{w}}\chi_\Delta) = \varphi_1 \cdot (\Gamma(\tilde{E}(\Delta)f)\frac{t-\overline{z_0}}{t-\overline{w}}\chi_\Delta).$$

Comparing this relation with (4.2) and using (4.16), we conclude that

$$\varphi_1 \cdot (\Gamma(\tilde{E}(\Delta)f)\frac{t-\overline{z_0}}{t-\overline{w}}\chi_\Delta) = \varphi_1 \cdot (\frac{f(t)}{u(t)}(t-z_0)\chi_\Delta\frac{t-\overline{z_0}}{t-\overline{w}}).$$

Since the span of the elements $\frac{t-z_0}{t-w} \in \Pi(\phi)$ is dense in $\Pi(\phi)$, we obtain (4.11).

□

5 u-resolvents of entire matrix functions

Let a function $\tau \in \mathcal{N}_\nu$ and a 2×2-matrix valued function $W \in \mathcal{M}_\kappa^1$,

$$W(z) = \begin{pmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{pmatrix},$$

be given such that $-\tau \neq \frac{w_{22}}{w_{21}}$, and consider the function

$$q(z) = (W \circ \tau)(z) := \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_{21}(z)\tau(z) + w_{22}(z)}. \tag{5.1}$$

In this section we assume that the entries w_{ij} of W are real entire functions, and our aim is to relate properties of W to properties of q. Note that $q \in \mathcal{N}_\mu$ for some $\mu \leq \kappa + \nu$. This fact follows immediately from the relation

$$(w_{21}(z)\tau(z) + w_{22}(z))\frac{q(z) - \overline{q(w)}}{z - \overline{w}}\overline{(w_{21}(w)\tau(w) + w_{22}(w))} =$$

$$= (\tau(z), 1)\frac{W(z)JW(w)^* - J}{z - \overline{w}}\begin{pmatrix} \overline{\tau(w)} \\ 1 \end{pmatrix} + \frac{\tau(z) - \overline{\tau(w)}}{z - \overline{w}}. \tag{5.2}$$

First let us introduce some properties of a Nevanlinna function q and investigate some relations between them.

Definition 5.1. Let $q \in \mathcal{N}_\kappa$ and let ϕ be the unique distribution such that for some $\alpha \in \mathbb{R}$ the relation (3.5) holds. We say that the point ∞ is regular for q if $\infty \notin s(\phi)$ and $(\chi_U\phi)(\{\infty\}) = 0$, where U is a sufficiently small neighbourhood of ∞. The function q is finite at ∞ if it is regular at ∞ and if for a sufficiently small neighbourhood U of ∞ we have

$$\int_U t^2 d\phi < \infty.$$

Moreover, we call ∞ singular for q if $\infty \in s(\phi)$ and if for each sufficiently small neighbourhood U of infinity $(\chi_{U\setminus\{\infty\}}\phi)$ is a positive unbounded measure.

It follows immediately from (3.5) that if ∞ is not singular for q, then we can write $q(z) = q'(z) + p(z)$, where ∞ is regular for q' and p is a real polynomial.

Let $(\Pi(q), A_q, \gamma_q(z))$ be the (up to unitary equivalence) unique model space for the function $q(z)$ (see Corollary 3.5 and its proof). Then q is the Q-function of A_q and its symmetric restriction

$$S_q := \{(a;b) \in A_q | b - za \perp \gamma_q(\overline{z})\}.$$

Now we obtain by [JLT], [KL1], [L] and [KW2] the following

Lemma 5.2. *The point ∞ is regular for q if and only if $\mathcal{A}_q$ is an operator which is the case if and only if*

$$\lim_{y\to+\infty} \frac{1}{y} q(iy) = 0,$$

or, equivalently,

$$\liminf_{y\to+\infty} \frac{1}{y} |q(iy)| = 0.$$

The function q is finite at ∞ if and only if $\mathcal{A}_q$ is an operator and $\overline{\operatorname{dom} S_q} \neq \Pi(q)$. This is the case if and only if q can be written as ($\alpha \in \mathbb{R}$)

$$q(z) = \alpha + [(\mathcal{A}_q - z)^{-1}u, u],$$

with $u \in \Pi(q)$ and $\alpha = \lim_{y\to\infty} q(iy)$, or equivalently, if and only if

$$\lim_{y\to+\infty} \frac{1}{y} q(iy) = 0, \ \limsup_{y\to+\infty} y |\operatorname{Im} q(iy)| < \infty.$$

Moreover, q is singular at ∞ if and only if ∞ is a singular critical point of $\mathcal{A}_q$.

Proof : Except of the last statement all assertions follow directly from the cited papers. Concerning the last one we mention that, since $(\Pi(q), A_q, \gamma_q(z)) \cong (\Pi(\phi), A_\phi, \frac{t-z_0}{t-z})$, ∞ is a singular critical point of $\mathcal{A}_q$ if and only if it is a singular critical point of $\mathcal{A}_\phi$. Since by Proposition 3.1 $(\Pi(\phi), A_\phi, 1) \cong (\Pi(\operatorname{Inv}^*\phi), A^{-1}_{\operatorname{Inv}^*\phi}, 1)$, this happens if and only if 0 is a singular critical point of $A_{\operatorname{Inv}^*\phi}$. By [JLT] this is the case if and only if $\operatorname{Inv}^*\phi$ is singular at 0. Now we are done, since this is again equivalent to the fact that ϕ is singular at ∞.

□

Corollary 5.3. *Let $q_1 \in \mathcal{N}_\kappa$ be regular at ∞, let $\alpha \in \mathbb{C}$ and assume that*

$$q(z) := \frac{1}{(z-\alpha)(z-\overline{\alpha})} q_1(z) \in \mathcal{N}_\nu.$$

Then q is finite at ∞.

Proof : We have $\lim_{y\to+\infty} \frac{1}{y} q_1(iy) = 0$, hence

$$\lim_{y\to+\infty} yq(iy) = 0.$$

□

Lemma 5.4. *If q is finite at ∞, then there exists exactly one number $\alpha \in \mathbb{R}$ such that ∞ is not regular for the function $-\frac{1}{q(z)-\alpha}$. For all other numbers $\alpha \in \mathbb{R}$ this function is finite at ∞. If q is regular but not finite at ∞, then $-\frac{1}{q(z)-\alpha}$ is regular but not finite at ∞ for all $\alpha \in \mathbb{R}$.*

Since the proof of this lemma is almost the same as in the case $q \in \mathcal{N}_0$ (cf. [HLS]), we will not go into details.

Lemma 5.5. *Let $q \in \mathcal{N}_\kappa$, $\kappa \geq 0$, be given. Then there exists a number $c \geq 0$, such that*

$$q_1(z) = -\frac{1}{q(z) + cz} \tag{5.3}$$

is finite at ∞, $\lim_{y\to\infty} q_1(iy) = 0$, and q_1 is contained in $\mathcal{N}_\kappa$. If $q(z)$ is not regular at ∞, we can choose $c = 0$. If $q(z)$ is regular at ∞, we can choose $c > 0$, and in this case, we have $\lim_{y\to\infty} y q_1(iy) = \frac{i}{c}$.

Proof : In the case that

$$\liminf_{y\to+\infty} \frac{|q(iy)|}{y} = 0, \tag{5.4}$$

let c be any number $c > 0$ and put $q_0(z) = q(z) + cz$. If the limes inferior (5.4) is larger than zero let $q_0(z) = q(z)$. Then $q_0 \in \mathcal{N}_\kappa$ and

$$\limsup_{y\to+\infty} \frac{y}{|q_0(iy)|} < \infty.$$

We compute

$$\left| \operatorname{Im} \frac{-1}{q_0(z)} \right| = \frac{|\operatorname{Im} q_0(z)|}{|q_0(z)|^2} \leq \frac{1}{|q_0(z)|}.$$

Hence

$$\limsup_{y\to+\infty} y |\operatorname{Im} \frac{-1}{q_0(iy)}| < \infty,$$

and also

$$\frac{1}{y}\left|\frac{-1}{q_0(iy)}\right| \leq \frac{1}{y^2}\frac{y}{|q_0(iy)|} \longrightarrow 0,\ y \to +\infty.$$

It follows from Lemma 5.2 that q_1 as defined in (5.3) is finite at ∞. Clearly $q_1 \in \mathcal{N}_\kappa$, and the final assertions follow easily from elementary calculations.

□

If $W(z) \in \mathcal{M}_\kappa^1$ is a real entire 2×2-matrix function, we denote by E_W the entire function

$$E_W(z) := (0\ 1)W(z)\begin{pmatrix} 1 \\ -i \end{pmatrix} = A_W - iB_W,\ (A_W\ B_W) = (0,1)W(z).$$

Note the following fact:

Lemma 5.6. *If the entries in the lower row of W are linearly independent, then $E_W(z) \in \mathcal{HB}_\nu$ for some $\nu \leq \kappa$. The entries in the lower row of W are linearly dependent if and only if W is of one of the following forms ($\alpha, \beta \in \mathbb{R}$):*

$$W(z) = \begin{pmatrix} 1 & p(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta & 0 \\ \alpha & \frac{1}{\beta} \end{pmatrix}, \text{ or } W(z) = \begin{pmatrix} p(z) & \beta \\ -\frac{1}{\beta} & 0 \end{pmatrix}$$

with some polynomial $p(z)$.

Proof : The first assertion is obvious. To prove the second assertion let $W(z) = (w_{ij})_{i,j=1}^2 \in \mathcal{M}_\kappa^1$ be such that w_{21} and w_{22} are linearly dependent. Assume first that w_{22} does not vanish identically, then $w_{21} = \lambda w_{22}$. We multiply W from the right with the (iJ)-unitary matrix

$$\begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix},$$

then the left lower entry of the product matrix vanishes. Hence it suffices to consider the case $w_{21} = 0$. By (I.8.2) we have

$$H_W(w,z) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{-w_{11}(z)\overline{w_{22}(w)}+1}{z-\overline{w}} \\ 0 \end{pmatrix},$$

and we conclude from Corollary I.9.7 that $w_{11}(z)$ is a polynomial. Since $w_{11}w_{22} = 1$, the polynomial w_{11} has no zeros, hence is a constant $\omega \neq 0$. It also follows that $w_{22} = \frac{1}{\omega}$. Thus

$$H_W(z,w) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \omega\frac{w_{12}(z)-\overline{w_{12}(w)}}{z-\overline{w}} \\ 0 \end{pmatrix},$$

and we conclude that w_{12} is a polynomial, which proves that W is the of first mentioned form.

It remains to consider the case that w_{22} vanishes identically. Then $w_{21}w_{12} = -1$ and the same argumentation as above will show that W is of the second mentioned form.

□

Now let $W \in \mathcal{M}_\kappa^1$ and $\tau \in \mathcal{N}_\nu$, assume that W has real entire entries and define $q(z) := (W \circ \tau)(z)$ as in (5.1). Note that the space $\mathfrak{K}(W)$ is finite dimensional if and only if W is a matrix polynomial.

Theorem 5.7. *Let $W \in \mathcal{M}_\kappa^1$ and $\tau \in \mathcal{N}_\nu$ be given, $q := W \circ \tau$. Assume that W has real entire entries and is not a matrix polynomial. Then*

(i) The function q is finite at ∞ if and only if $\mathfrak{K}_-(W) = \mathfrak{K}(W)$ and $\mathfrak{K}(W)$ contains a nonzero constant.

(ii) The point ∞ is regular for q if and only if $\mathfrak{K}_-(W) = \mathfrak{K}(W)$.

(iii) There exists a real polynomial p such that ∞ is regular for $q+p$ if and only if $\mathrm{ind}_0\mathfrak{K}_-(W)=0$.

(iv) The point ∞ is singular for q if and only if $\mathrm{ind}_0\mathfrak{K}_-(W)\neq 0$.

Note that the conditions on the right hand sides of (i)-(iv) do not depend on the choice of the parameter function τ.

In the case that $W(z)$ is a matrix polynomial with real entries such that w_{21} and w_{22} are linearly independent, (i)-(iv) again hold if we additionally assume $\tau\neq -\frac{w_{22}}{w_{21}}$ and $\mathrm{ind}_-\mathfrak{K}(W)+\mathrm{ind}_-(\tau)=\mathrm{ind}_-(q)$.

Before we come to the proof of Theorem 5.7 we need some more lemmata. Let $\mathcal{A}\subseteq\mathfrak{P}^2$, $\rho(\mathcal{A})\neq\emptyset$, be a selfadjoint relation and let $\mathfrak{M}_0\subseteq\mathfrak{P}$. If we put

$$\mathfrak{M}:=\mathrm{cls}\,(\mathfrak{M}_0\cup\bigcup_{z\in\rho(\mathcal{A})}(\mathcal{A}-z)^{-1}\mathfrak{M}_0), \tag{5.5}$$

the relation $\mathcal{A}$ induces by restriction and factorization by $\mathfrak{M}^\circ$ (compare [HSW]) a selfadjoint relation $\mathcal{A}_\mathfrak{M}$ in $\mathfrak{M}/\mathfrak{M}^\circ$ with $\rho(\mathcal{A}_\mathfrak{M})\supseteq\rho(\mathcal{A})$. Denote in the following by $\mathfrak{S}_\infty$ the generalized eigenspace of $\mathcal{A}$ at ∞.

Lemma 5.8. *If $\mathfrak{M}_0\subseteq\mathfrak{S}_\infty^\perp$, the relation $\mathcal{A}_\mathfrak{M}$ is an operator.*

Proof : Put $\mathfrak{L}:=\mathfrak{S}_\infty^\perp$ and consider the relation $\mathcal{A}_\mathfrak{L}$. We show that $\mathcal{A}_\mathfrak{L}$ is an operator. Assume that for some element $x\in\mathfrak{L}$ we have $(\mathcal{A}-z)^{-1}x\in\mathfrak{L}^\circ=\mathfrak{S}_\infty^\circ$. Since $\mathfrak{S}_\infty$ is the generalized eigenspace of $(\mathcal{A}-z)^{-1}$ at 0, there exists a number k such that $(\mathcal{A}-z)^{-k}x=0$. Hence $x\in\mathfrak{S}_\infty$ and therefore in $\mathfrak{L}^\circ$.

Now consider the relation $\mathcal{A}_\mathfrak{M}$ with $\mathfrak{M}$ as in (5.5). Clearly, since $\mathfrak{M}_0\subseteq\mathfrak{S}_\infty^\perp$, also $\mathfrak{M}\subseteq\mathfrak{S}_\infty^\perp$. Thus $\mathcal{A}_\mathfrak{M}$ can be considered as a restriction and factorization of $\mathcal{A}_\mathfrak{L}$. The assertion of the lemma will follow if we show that the restriction and factorization of an operator is again an operator.

Let $\mathcal{B}\subseteq\mathfrak{P}^2$, $\rho(\mathcal{B})\neq\emptyset$, be a selfadjoint operator and let $\mathfrak{M}$ be a closed subspace of $\mathfrak{P}$ which is invariant under all resolvents $(\mathcal{B}-z)^{-1}$, $z\in\rho(\mathcal{B})$. Decompose the space $\mathfrak{P}$ as

$$\mathfrak{P}=(\mathfrak{M}+\mathfrak{M}^\perp)\dot{+}\mathfrak{M}',$$

and note that $\dim\mathfrak{M}'<\infty$. It follows that

$$\mathrm{dom}\,\mathcal{B}=(\mathcal{B}-z)^{-1}\mathfrak{P}=(\mathcal{B}-z)^{-1}(\mathfrak{M}+\mathfrak{M}^\perp)+(\mathcal{B}-z)^{-1}\mathfrak{M}',$$

hence, as $\dim\,(\mathcal{B}-z)^{-1}\mathfrak{M}'=\dim\mathfrak{M}'<\infty$ and as $\mathrm{dom}\,\mathcal{B}$ is dense in $\mathfrak{P}$ we get

$$\mathfrak{P}=\underbrace{\overline{(\mathcal{B}-z)^{-1}(\mathfrak{M}+\mathfrak{M}^\perp)}}_{\subseteq\mathfrak{M}+\mathfrak{M}^\perp}\dot{+}(\mathcal{B}-z)^{-1}\mathfrak{M}'.$$

We conclude that

$$\overline{(\mathcal{B}-z)^{-1}(\mathfrak{M}+\mathfrak{M}^\perp)}=\mathfrak{M}+\mathfrak{M}^\perp.$$

Let $x \in \mathfrak{M}$ be such that $(\mathcal{B}-z)^{-1}x \in \mathfrak{M}^\circ$, then for all $y \in \mathfrak{M}+\mathfrak{M}^\perp$ we have

$$[x, (\mathcal{B}-\overline{z})^{-1}y] = [(\mathcal{B}-z)^{-1}x, y] = 0,$$

hence $x \in \mathfrak{M}^\circ$.

□

For the notation which is used in the following compare [KW2], Section 7.

Let $\mathfrak{P}_1 \subseteq \mathfrak{P}_2 \subseteq \mathfrak{P}_3$ be Pontryagin spaces, $\mathcal{S}_1 \subseteq \mathfrak{P}_1^2$, $\mathcal{S}_2 \subseteq \mathfrak{P}_2^2$, be symmetric relations with defect index $(1,1)$ which satisfy $\mathcal{S}_1 \subseteq \mathcal{S}_2$. Construct a definite inner product $(.,.)$ on $\mathfrak{P}_2$ as in [KW2], Section 7, then the $[.,.]$-orthogonal projection P_{21} of $\mathfrak{P}_2$ onto $\mathfrak{P}_1$ is also $(.,.)$-orthogonal. If we take a definite inner product on $\mathfrak{P}_3$ in the same manner with respect to the already constructed inner product on $\mathfrak{P}_2$, then the $[.,.]$-orthogonal projections P_{32} and P_{31} are also $(.,.)$-orthogonal.

Assume that moreover a selfadjoint relation $\tilde{A} \supseteq \mathcal{S}_2$, $\rho(\tilde{A}) \neq \emptyset$, in $\mathfrak{P}_3$ is given, and let $\mathcal{S}_3$ be any symmetric restriction of $\tilde{A}$ with defect index $(1,1)$, such that $\mathcal{S}_3 \supseteq \mathcal{S}_2$. For $j=1,2,3$ let the spaces $\mathfrak{P}_{j,-}$ and $\mathfrak{P}_{j,+}$ and mappings $P'_{ij} : \mathfrak{P}_{j,-} \to \mathfrak{P}_{i,-}$ $(3 \geq i > j)$ be defined as in [KW2], Section 7.

Lemma 5.9. *Assume that $\mathfrak{P}_1 = \mathfrak{P}(E_1)$, $\mathfrak{P}_2 = \mathfrak{P}(E_2)$ are dB-Pontryagin spaces, $\mathfrak{P}_1 \subseteq \mathfrak{P}_2$, that $\tilde{A} \subseteq \tilde{\mathfrak{P}}_3^2$, $\rho(\tilde{A}) \neq \emptyset$, is an extension of $\mathcal{S}_2$ $(\subseteq \mathfrak{P}(E_2)^2)$, and that $u(z) \in \operatorname{Ass}\mathfrak{P}(E_1)$, i.e. $u \in \mathfrak{P}(E_1)_-$. Then*

$$(P'_{21}u)(w) = u(w),\ w \in \mathbb{C}.$$

If, for $j=1,2$,

$$\hat{\tilde{R}}_{j,z} : \mathfrak{P}(E_j)_- \longrightarrow \mathfrak{P}(E_j)_+$$

denotes the regularized generalized resolvent of $\tilde{A}$ considered as an extension of $\mathcal{S}_j$, then

$$[\hat{\tilde{R}}_{1,z}u, u]_{1,\pm} = [\hat{\tilde{R}}_{2,z}P'_{21}u, P'_{21}u]_{2,\pm}.$$

Proof : If $K_j(w,z)$ denotes the reproducing kernel of $\mathfrak{P}(E_j)$, we clearly have $P_{21}K_2(w,z) = K_1(w,z)$. Hence (cf. (0.7.2), (I.10.1))

$$(P'_{21}u)(w) = [P'_{21}u, \begin{pmatrix} K_2(w,z) \\ \overline{w}K_2(w,z) \end{pmatrix}]_{2,\pm} = [u, (P_{21} \oplus P_{21}) \begin{pmatrix} K_2(w,z) \\ \overline{w}K_2(w,z) \end{pmatrix}]_{1,\pm} =$$

$$= [u, \begin{pmatrix} K_1(w,z) \\ \overline{w}K_1(w,z) \end{pmatrix}]_{1,\pm} = u(w).$$

Now note that $P'_{31} = P'_{32}P'_{21}$. This follows from the relation ($v \in \mathfrak{P}_{1,-}$, $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathfrak{P}_{3,+}$)

$$[P'_{32}P'_{21}v, \begin{pmatrix} f \\ g \end{pmatrix}]_{3,\pm} = [P'_{21}v, (P_{32} \oplus P_{32}) \begin{pmatrix} f \\ g \end{pmatrix}]_{2,\pm} =$$

$$= [v, (P_{21} \oplus P_{21})(P_{32} \oplus P_{32}) \begin{pmatrix} f \\ g \end{pmatrix}]_{1,\pm} = [v, (P_{31} \oplus P_{31}) \begin{pmatrix} f \\ g \end{pmatrix}]_{1,\pm} =$$

$$= [P'_{31}v, \begin{pmatrix} f \\ g \end{pmatrix}]_{3,\pm}.$$

Denote by $\hat{R}_{3,z} : \mathfrak{P}_{3,-} \to \mathfrak{P}_{3,+}$ the regularized resolvent of $\tilde{A}$. Since, by Lemma 0.7.3, the relations

$$(P_{32} \oplus P_{32})\hat{R}_{3,z}P'_{32} = \hat{\tilde{R}}_{2,z},\ (P_{31} \oplus P_{31})\hat{R}_{3,z}P'_{31} = (P_{21} \oplus P_{21})\hat{R}_{2,z}P'_{21} = \hat{\tilde{R}}_{1,z}$$

hold, we find

$$[\hat{\tilde{R}}_{1,z}u, u]_{1,\pm} = [(P_{31} \oplus P_{31})\hat{R}_{3,z}P'_{31}u, u]_{1,\pm} = [\hat{R}_{3,z}P'_{31}u, P'_{31}u]_{3,\pm} =$$

$$= [\hat{R}_{3,z}P'_{32}P'_{21}u, P'_{32}P'_{21}u]_{3,\pm} = [\hat{\tilde{R}}_{2,z}P'_{21}u, P'_{21}u]_{2,\pm}.$$

□

Lemma 5.10. *Let $M_1 \in \mathcal{M}_{\kappa_1}$ and $M_2 \in \mathcal{M}_{\kappa_2}$, then $M_1M_2 \in \mathcal{M}_\kappa$ where $\kappa \leq \kappa_1 + \kappa_2$.*
Assume that M is a rational matrix function which is meromorphic in $\mathbb{C} \setminus \mathbb{R}$, invertible for some point $z_0 \in \mathbb{C}^+$ and takes J-unitary values on $\mathbb{R}$. Then the kernel

$$K(w, z) = \frac{M(z)JM(w)^* - J}{z - \overline{w}} \tag{5.6}$$

has a finite number of negative and positive squares. In fact the sum of negative and positive squares of the kernel (5.6) equals the so-called McMillan degree of M.
Proof : The first part of the assertion follows from the relation (compare [ADSR])

$$\frac{(M_1M_2)(z)J(M_1M_2)(w)^* - J}{z - \overline{w}} =$$

$$= M_1(z)\frac{M_2(z)JM_2(w)^* - J}{z - \overline{w}}M_1(w)^* + \frac{M_1(z)JM_1(w)^* - J}{z - \overline{w}}. \tag{5.7}$$

The assertion concerning rational matrix functions has been proved e.g. in [BGR].

□

Proof (of Theorem 5.7): First of all note that by [KW3] $\dim \mathfrak{K}(W) < \infty$ if and only if $\dim \mathfrak{P}(E) < \infty$ where $E := E_W = w_{21} - iw_{22}$, and this happens if and only if $W(z)$ is a matrix polynomial. Hence, as $\tau \in \mathcal{N}_\nu$ and as $\mathfrak{K}(\frac{w_{22}}{w_{21}}) \cong \mathfrak{P}(E)$ (cf. Lemma I.6.4), $\tau = -\frac{w_{22}}{w_{21}}$ can occur only if $W(z)$ is a matrix polynomial. But this possibility is excluded by our assumptions, so $W \circ \tau$ is in any case defined as a meromorphic function.

In the first step assume that $\mathfrak{K}_-(W) = \mathfrak{K}(W)$, i.e. that

$$\mathfrak{K}(W) \cong \mathfrak{P}(E).$$

By Proposition I.10.3 we have $1 \in \operatorname{Ass} \mathfrak{P}(E)$, and W is a 1-resolvent matrix of the operator $\mathcal{S}$ in $\mathfrak{P}(E)$. Hence there exists a $\mathfrak{P}(E)$-minimal selfadjoint extension $\tilde{\mathcal{A}}$ of $\mathcal{S}$ acting in a certain Pontryagin space $\tilde{\mathfrak{P}} \supseteq \mathfrak{P}(E)$, such that $\operatorname{ind}_- \tilde{\mathfrak{P}} = \operatorname{ind}_- \mathfrak{P}(E) + \nu$ and

$$q(z) = \beta + [\hat{\tilde{R}}_z 1, 1]_\pm,$$

where $\hat{\tilde{R}}_z$ denotes the regularized generalized resolvent of $\tilde{\mathcal{A}}$. Note that by the results of [KW2] changing the real constant β we can write R_z instead of $\hat{\tilde{R}}_z$ if $1 \in \mathfrak{P}(E)$.

If $\dim \mathfrak{K}(W) < \infty$, we obtain from Proposition I.8.3 that there is a constant nonzero function in $\mathfrak{K}(W)$. Since, $\mathfrak{K}_-(W) = \mathfrak{K}(W)$ we see that the second component of this constant function is nonzero. Applying π_- we obtain that $1 \in \mathfrak{P}(E)$. By our assumption on the negative squares of q we obtain that $\operatorname{ind}_-(\operatorname{cls}\{R_z 1 | z \in \rho(A)\}) = \operatorname{ind}_- \tilde{\mathfrak{P}}$. As $\operatorname{cls}\{R_z 1 | z \in \rho(A)\} \subseteq \overline{\operatorname{dom} \tilde{\mathcal{A}}}$, we see that $\tilde{\mathcal{A}}(0) = \overline{\operatorname{dom} \tilde{\mathcal{A}}}^\perp$ is positive. It follows that $R_z|_{\overline{\operatorname{dom} \tilde{\mathcal{A}}}}$ is the resolvent of a selfadjoint operator on $\overline{\operatorname{dom} \tilde{\mathcal{A}}}$. Moreover, we have

$$q(z) = \beta + [R_z|_{\overline{\operatorname{dom} \tilde{\mathcal{A}}}} \mathcal{P} 1, \mathcal{P} 1],$$

where $\mathcal{P}$ denotes the orthogonal projection of $\tilde{\mathfrak{P}}$ onto $\overline{\operatorname{dom} \tilde{\mathcal{A}}}$. It follows that q is finite and, in particular, regular at ∞.

If $W(z)$ is not a matrix polynomial we divide two cases (recall the notation of [KW3], Section 11):

1. For all $x \in M_{reg}$ the closure of $\{t \in M_{reg} | t < x\}$ does not contain x. Then M_{reg} consists of an infinite number of points which do not accumulate at 0. If $t_0 \in [-\infty, 0)$ is the largest accumulation point of M_{reg}, then $\{t \in M_{reg} | t > t_0\}$ is a monotonically decreasing sequence $(t_n)_{n \in \mathbb{N}}$ which tends to t_0. The number t_n corresponds to the dB-subspace $\overline{\operatorname{dom}(\mathcal{S}^n)}$ of $\mathfrak{P}(E)$.

2. There exists a point $x \in M_{reg}$ such that x belongs to the closure of $\{t \in M_{reg} | t < x\}$. Then there exists a dB-subspace $\mathfrak{P}_1 = \mathfrak{P}_x$ of $\mathfrak{P}(E)$ with $\mathfrak{d}(\mathfrak{P}_1) = \mathfrak{d}(\mathfrak{P}(E)) = 0$, such that (if $\mathcal{S}_1$ denotes the multiplication operator in $\mathfrak{P}_1$) the set $\operatorname{dom} \mathcal{S}_1$ is dense in $\mathfrak{P}_1$.

Consider the case 1. Since all but finitely many spaces $\overline{\operatorname{dom}(\mathcal{S}^n)}$ are nondegenerated (see Theorem I.11.6), we may choose $n \geq \dim \mathfrak{S}_\infty$ such that $\overline{\operatorname{dom}(\mathcal{S}^n)}$ is nondegenerated. From the fact that $\operatorname{dom}(\tilde{\mathcal{A}}^n) = \operatorname{ran}(\tilde{\mathcal{A}} - z)^{-n}$ and that $(\tilde{\mathcal{A}} - z)^{-1}$ is continuous, we conclude

$$\mathfrak{M} := \operatorname{cls}\Big(\operatorname{dom}(\mathcal{S}^n) \cup \bigcup_{z \in \rho(\tilde{\mathcal{A}})} (\tilde{\mathcal{A}} - z)^{-1} \operatorname{dom}(\mathcal{S}^n)\Big) \subseteq \operatorname{ran}(\tilde{\mathcal{A}} - z)^{-n} = \mathfrak{S}_\infty^\perp.$$

By Lemma 5.8 the restriction and factorization $\tilde{\mathcal{A}}_{\mathfrak{M}}$ is an operator. It follows from [dB7] that the fact $1 \in \operatorname{Ass} \mathfrak{P}(E)$ implies $1 \in \operatorname{Ass} \overline{\operatorname{dom}(\mathcal{S}^n)}$ and Lemma 5.9 shows

$$q(z) = \beta + [\hat{\tilde{R}}_{\mathfrak{M},z} 1, 1]_\pm,$$

where $\hat{\tilde{R}}_{\mathfrak{M},z}$ denotes the regularized generalized resolvent of $\tilde{\mathcal{A}}_{\mathfrak{M}}$. Now Lemma 5.2 and Proposition 0.4.5 imply that ∞ is regular for q. Moreover, again by Lemma 5.2, q is finite at

∞ if $1 \in \overline{\operatorname{dom}(\mathcal{S}^n)}$. By the results of [dB7] this condition is equivalent to $1 \in \mathfrak{P}(E)$. Since $\mathfrak{P}(E) = \pi_- \mathfrak{K}(W)$ and $\ker \pi_- = \{0\}$, a straightforward consideration (cf. Proposition I.8.3) yields that this is the case if and only if $\mathfrak{K}(W)$ contains a nonzero constant.

Consider the case 2. First note that it follows from Theorem I.11.6 that $\mathfrak{P}_t$ is nondegenerated for all but finitely many $t \in M_{reg},\ s_- < t < x$. Fix such a number t. Since $\overline{\operatorname{dom} S_1} \subseteq \operatorname{dom} \tilde{A} = \operatorname{ran}(\tilde{A}-z)^{-1}$, we have $\mathfrak{P}_1 \subseteq \overline{\operatorname{ran}(\tilde{A}-z)^{-1}}$, in particular $\overline{\operatorname{ran}(S_1 - z)} \subseteq \operatorname{ran}(\tilde{A}-z)^{-1}$. Applying $(\tilde{A}-z)^{-1}$ yields $\operatorname{dom} S_1 \subseteq \operatorname{ran}(\tilde{A}-z)^{-2}$, hence $\mathfrak{P}_1 \subseteq \overline{\operatorname{ran}(\tilde{A}-z)^{-2}}$, in particular $\operatorname{ran}(S_1 - z) \subseteq \overline{\operatorname{ran}(\tilde{A}-z)^{-2}}$. Proceeding inductively yields $\mathfrak{P}_1 \subseteq \mathfrak{S}_\infty^\perp$ and therefore ($\mathfrak{P}_t \subseteq \mathfrak{P}_1$)

$$\mathfrak{M} := \operatorname{cls}\Big(\mathfrak{P}_t \cup \bigcup_{z \in \rho(\tilde{A})} (\tilde{A}-z)^{-1}\mathfrak{P}_t\Big) \subseteq \mathfrak{S}_\infty^\perp.$$

By Lemma 5.8 the restriction and factorization $\tilde{\mathcal{A}}_{\mathfrak{M}}$ is an operator. Now we proceed as above. So we have proved the implications '*(ii)*: $\Leftarrow$' and '*(i)*: $\Leftarrow$'.

In the second step assume that $\operatorname{ind}_0 \mathfrak{K}_-(W) = 0$. Then by Proposition I.10.3 we have $1 \in \operatorname{Ass} \mathfrak{P}(E)$ and there exists an entire matrix function $W_1 \in \mathcal{M}^1_{\kappa'}$, such that

$$(0,1)W_1(z) = (w_{21}(z), w_{22}(z)),$$

and $\mathfrak{K}_-(W_1) = \mathfrak{K}(W_1)$. By Corollary I.9.8 there exists a polynomial $p(z)$ such that $W_1 = W_p W$, where

$$W_p(z) := \begin{pmatrix} 1 & p(z) \\ 0 & 1 \end{pmatrix}.$$

Note that the polynomial p is constant if and only if $\mathfrak{K}_-(W) = \mathfrak{K}(W)$. It follows from the already proved that ∞ is regular for the function

$$(W_1 \circ \tau)(z) = ((W_p W) \circ \tau)(z) = (W_p \circ (W \circ \tau))(z) = p(z) + q(z).$$

We have proved '*(iii)*: $\Leftarrow$'.

In the third step assume that $\mathfrak{K}_-(W) \neq \mathfrak{K}(W)$, i.e. $\ker \pi_- \neq \{0\}$, and we conclude from Corollary I.9.7 that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathfrak{K}(W).$$

By Corollary I.8.4 there is no other (linearly independent) constant contained in $\mathfrak{K}(W)$. Hence, again appealing to Corollary I.9.7, we find that for all $\alpha \in \mathbb{R}$ the relation $\mathfrak{K}_+(W_\alpha W) = \mathfrak{K}(W_\alpha W)$, where

$$W_\alpha := \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix},$$

holds. Step 1 of this proof applied to the matrix $-JW_\alpha WJ$ and to the parameter $-\frac{1}{\tau}$ shows that

$$(-JW_\alpha WJ) \circ (-\frac{1}{\tau}) = -\frac{1}{W_\alpha \circ (W \circ \tau)} = -\frac{1}{q+\alpha}$$

is finite at ∞. Lemma 5.4 implies that q is not regular at ∞. Hence we find that the implication '*(ii)*: $\Rightarrow$' holds.

We also conclude that '*(i)*: $\Rightarrow$' holds: For if q is finite at ∞, by *(ii)* we have $\mathfrak{K}_-(W) = \mathfrak{K}(W)$. By Lemma 5.4 for exactly one $\alpha \in \mathbb{R}$

$$-\frac{1}{q+\alpha} = (-JW_\alpha WJ) \circ (-\frac{1}{\tau})$$

is not regular at ∞, hence $\mathfrak{K}_+(W_\alpha W) \neq \mathfrak{K}(W_\alpha W)$, i.e. $\ker \pi_+ \neq \{0\}$. Corollary I.9.7 implies that

$$\begin{pmatrix} \alpha \\ 1 \end{pmatrix} \in \mathfrak{K}(W).$$

In the last step assume that there exists a polynomial p such that ∞ is regular for $q+p$. Put $W_1 := W_p W$, then by Lemma 5.10 and the already proved assertion *(ii)* we have $\mathfrak{K}_-(W_1) = \mathfrak{K}(W_1)$. Hence the space $\mathfrak{K}(W_1)$ contains no (nonzero) element whose second component vanishes. Since W_p is an upper triangular matrix also $W_p\mathfrak{K}(W_1)$ has this property. The relation

$$\frac{W_p(z)JW_p(w)^* - J}{z-\overline{w}} = \begin{pmatrix} \frac{p(z)-\overline{p(w)}}{z-\overline{w}} & 0 \\ 0 & 0 \end{pmatrix} \tag{5.8}$$

implies that (as a set)

$$\mathfrak{K}(W_p) = \{\begin{pmatrix} f(z) \\ 0 \end{pmatrix} | f \text{ polynomial}, \deg f < \deg p\}.$$

We conclude (compare [ADSR]) that the space $\mathfrak{K}(W)$ is the direct and orthogonal sum of an isomorphic copy of $\mathfrak{K}(W_1)$ (in fact of $W_p\mathfrak{K}(W_1)$) and of $\mathfrak{K}(W_p)$. In particular $\ker \pi_- = \mathfrak{K}(W_p)$ is orthocomplemented, and therefore $\operatorname{ind}_0 \mathfrak{K}_-(W) = 0$. This proves '*(iii)*: $\Rightarrow$' and '*(iv)*' follows from the fact that *(i)*, *(ii)*, *(iii)* hold.

□

We have used in the proof of Theorem 5.7 the fact that if $\operatorname{ind}_0 \mathfrak{K}_-(W) = 0$, then $1 \in \operatorname{Ass} \mathfrak{P}(E_W)$. Note the following converse:

Lemma 5.11. *If* $\operatorname{ind}_0 \mathfrak{K}_-(W) \neq 0$ *then* $1 \notin \operatorname{Ass} \mathfrak{P}(E_W)$.
Proof : Assume on the contrary that $1 \in \operatorname{Ass} \mathfrak{P}(E_W)$. By Proposition I.10.3 there exists a matrix $W_1(z)$ as in the proof of Theorem 5.7. Since again Corollary I.9.8 applies, we obtain relations of the form (5.7) and (5.8). The same argument as above yields that the subspace $\ker \pi_-$ of $\mathfrak{K}(W)$ is orthocomplemented, a contradiction.

□

Let $\mathfrak{P}(E)$, $E = A - iB$, be a dB-Pontryagin space and assume that for some $\psi \in [0, \pi)$ the function

$$S_\psi := \cos\psi A(z) + \sin\psi B(z) \tag{5.9}$$

belongs to $\mathfrak{P}(E)$. Denote by $\mathcal{A}_\psi$ the canonical selfadjoint extension corresponding to S_ψ.

Lemma 5.12. *Let* $W \in \mathcal{M}_\kappa^1$ *have real entire entries. Then* $W \circ \alpha \in \mathcal{N}_\mu$ *with* $\mu < \kappa$ *for at most one number* $\alpha \in \overline{\mathbb{R}}$.

If we additionally assume that $\mathfrak{K}_-(W) = \mathfrak{K}(W)$, then $W \circ \alpha$ is the regularized 1-resolvent of $\mathcal{S}$ induced by the canonical extension $\mathcal{A}_\psi$ with $\alpha = \cot \psi$ and $A = A_W = w_{21}$, $B = B_W = w_{22}$. We have $W \circ \alpha \in \mathcal{N}_\mu$ with $\mu < \kappa$ if and only if $S_\psi \in \mathfrak{P}(E)$ and $[S_\psi, S_\psi] \leq 0$.

Proof : Assume first that $\mathfrak{K}_-(W) = \mathfrak{K}(W)$ and note that by Proposition I.10.3 (exchanging W with $-JWJ$) W is a 1-resolvent matrix of $\mathcal{S}$. Clearly the generalized 1-resolvent $W \circ \alpha$ which is induced by a canonical selfadjoint extension $\mathcal{A}$ of $\mathcal{S}$ has a pole at a point t if and only if $A(t)\alpha + B(t) = 0$. Hereby $A(t)\infty + B(t)$ has to be read as $A(t)$. These points are exactly the spectral points of the extension $\mathcal{A}_\psi$ corresponding to S_ψ with $\cot \psi = \alpha$ (cf. Lemma I.6.4).

Now note that if $S_\psi \notin \mathfrak{P}(E)$, i.e. $\mathcal{A}_\psi$ is an operator, we find by Proposition 4.4 that $W \circ \alpha \in \mathcal{N}_\kappa$. If $[S_\psi, S_\psi] = 0$, which happens if and only if $S_\psi \in \overline{\operatorname{dom} \mathcal{S}} (= \overline{\operatorname{dom} \mathcal{A}_\psi})$, then, as $1 \in \operatorname{Ass}(\overline{\operatorname{dom} \mathcal{S}})$ (cf. [dB7]),

$$R_z^- 1(t) = \frac{1 - \frac{S_\psi(t)}{S_\psi(z)}}{t - z} \in \overline{\operatorname{dom} \mathcal{S}}$$

(cf. Lemma I.10.1) and we obtain

$$\operatorname{ind\_span} \{R_z^- 1\} \leq \operatorname{ind}_- \overline{\operatorname{dom} \mathcal{S}} < \kappa.$$

If $S_\psi \notin \overline{\operatorname{dom} \mathcal{S}}$ then $\overline{\operatorname{dom} \mathcal{S}}$ is a dB-Pontryagin space and, since $1, S_\psi \in \operatorname{Ass} \overline{\operatorname{dom} \mathcal{S}}$, we have $R_z^- 1 \in \overline{\operatorname{dom} \mathcal{S}}$. Moreover, $R_z^- 1 = ((\mathcal{A}_\psi|_{\overline{\operatorname{dom} \mathcal{S}}} - z)^{-1})^- 1$ (cf. Lemma 5.9), where $\mathcal{A}_\psi|_{\overline{\operatorname{dom} \mathcal{S}}}$ is an operator. Hence, by Proposition 4.4 we see that $\operatorname{ind}_-(W \circ \alpha) = \operatorname{ind\_span} \{R_z^- 1\} = \operatorname{ind}_- \overline{\operatorname{dom} \mathcal{S}}$, and the last assertion follows. Note that $S_\psi \in \mathfrak{P}(E)$, i.e. $\mathcal{A}_\psi$ is not an operator, occurs for at most one $\psi \in [0, \pi)$ (cf. [KW3]).

The first assertion is already proved if $\mathfrak{K}_-(W) = \mathfrak{K}(W)$. If $\mathfrak{K}_-(W) \neq \mathfrak{K}(W)$, which happens if and only if (cf. Corollary I.9.7)

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathfrak{K}(W),$$

we consider $\tilde{W} = -JWJ \in \mathcal{M}_\kappa$ instead of W. As $\mathfrak{K}(\tilde{W}) = J\mathfrak{K}(W)$ we obtain $\mathfrak{K}_-(\tilde{W}) = \mathfrak{K}(\tilde{W})$. Since $W \circ \alpha = -\frac{1}{\tilde{W} \circ \frac{-1}{\alpha}}$, the first assertion follows also in the general situation.

$\square$

Corollary 5.13. *Let $W \in \mathcal{M}_\kappa^1$ be a matrix polynomial with the same properties as in Theorem 5.7. Then there are two possibilities:*

(i) $\mathfrak{K}_-(W) = \mathfrak{K}(W)$ *and* $1 \in \mathfrak{P}(E_W)$.

(ii) $\mathfrak{K}_-(W) \neq \mathfrak{K}(W)$ *and* $\mathfrak{K}_-(W)^\circ = \{0\}$.

Proof : If $\mathfrak{K}_-(W) = \mathfrak{K}(W)$ then $1 \in \mathfrak{P}(E_W)$ follows in the same way as indicated in the proof of Theorem 5.7.

If $\mathfrak{K}_-(W) \neq \mathfrak{K}(W)$, let $\alpha \in \overline{\mathbb{R}}$ be such that $\operatorname{ind}_-(W \circ \alpha) = \operatorname{ind}_- W$ (cf. Lemma 5.12). Since q is rational, ∞ cannot be singular for q. Therefore, Theorem 5.7 yields $\mathfrak{K}_-(W)^\circ = \{0\}$.

□

The following corollary will be useful in the sequel.

Corollary 5.14. *Let $W_i \in \mathcal{M}^1_{\kappa_i}$, $i = 1, 2, 3$, have real entire entries, assume that W_1 and W_2 are entire but not polynomials and that $W_1 = W_2 W_3$. Then W_1 satisfies $\mathfrak{K}_-(W_1) = \mathfrak{K}(W_1)$ ($\mathfrak{K}_-(W_1) = \mathfrak{K}(W_1)$ and there exists a constant in $\mathfrak{K}(W_1)$, $\operatorname{ind}_0 \mathfrak{K}_-(W_1) = 0$, $\operatorname{ind}_0 \mathfrak{K}_-(W_1) \neq 0$) if and only if W_2 has the respective property. If we assume that $\kappa_1 = \kappa_2 + \kappa_3$, then the assertion also holds if W_1, W_2 are polynomials.*

Proof : By Lemma 5.12 we can choose $\alpha \in \mathbb{R}$ such that $\operatorname{ind}_-(W_1 \circ \alpha) = \kappa_1$. If $\kappa_1 = \kappa_2 + \kappa_3$, it follows that $\operatorname{ind}_-(W_2 \circ (W_3 \circ \alpha)) = \operatorname{ind}_- W_2 + \operatorname{ind}_-(W_3 \circ \alpha)$. The assertion now follows from Theorem 5.7.

□

Corollary 5.15. *Let $W_i \in \mathcal{M}_{\kappa_i}$, $i = 1, 2, 3$, have real entire entries such that $W_1 = W_2 W_3$, and assume that $\kappa_1 = \kappa_2 + \kappa_3$. Then $\mathfrak{K}(W_1)$ contains the constant function $(\cos\psi, \sin\psi)^T$ if and only if $\mathfrak{K}(W_2)$ contains this function. If $\dim \mathfrak{K}(W_2) > 1$ or if $\mathfrak{K}(W_2)$ and $\mathfrak{K}(W_3)$ do not contain the same constant function, the fact $(\cos\psi, \sin\psi)^T \in \mathfrak{K}(W_2)$ implies*

$$[\begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}, \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}]_{\mathfrak{K}(W_1)} = [\begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}, \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}]_{\mathfrak{K}(W_2)}. \tag{5.10}$$

Proof : Multiplying W_1 and W_2 from the left by

$$\begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}$$

we can assume that $\psi = 0$. By Corollary I.9.7 the fact $(1,0)^T \in \mathfrak{K}(W_i),\ i = 1, 2$ is equivalent to $\mathfrak{K}(W_i) \neq \mathfrak{K}_-(W_i),\ i = 1, 2$. By Corollary 5.14 we have $\mathfrak{K}(W_1) \neq \mathfrak{K}_-(W_1)$ if and only if $\mathfrak{K}(W_2) \neq \mathfrak{K}_-(W_2)$.

Since for a (iJ)-unitary matrix U we have $\mathfrak{K}(UW_i) = U\mathfrak{K}(W_i)$, a similar argument as above shows that we can assume that $\psi = \frac{\pi}{2}$. In particular, $\mathfrak{K}(W_i) = \mathfrak{K}_-(W_i) \cong \mathfrak{P}(E_{W_i}),\ i = 1, 2$. Assume $\dim \mathfrak{K}(W_2) > 1$. As $(A_{W_2}, B_{W_2})W_3 = (A_{W_1}, B_{W_1})$, it follws from Proposition I.13.5 that there is a nontrivial db-space $\mathfrak{P}(E_{W_2})^l$ which is isometrically contained in $\mathfrak{P}(E_{W_2})$ with codimension 1 and which is isometrically contained in $\mathfrak{P}(E_{W_1})$. From [dB7] we get $1 \in \mathfrak{P}(E_{W_2})^l$, and hence

$$[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}]_{\mathfrak{K}(W_1)} = [1,1]_{\mathfrak{P}(E_{W_1})} = [1,1]_{\mathfrak{P}(E_{W_2})} = [\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}]_{\mathfrak{K}(W_2)}.$$

If $\dim \mathfrak{K}(W_2) = 1$, and if $\mathfrak{K}(W_2)$ and $\mathfrak{K}(W_3)$ do not contain the same constant function, it follows from [ADSR] that $\mathfrak{K}(W_2)$ is isometrically contained in $\mathfrak{K}(W_1)$.

□

Lemma 5.16. *Let $W \in \mathcal{M}_\kappa^1$ be a not linear real entire matrix function, such that $\mathfrak{K}_-(W) = \mathfrak{K}(W)$ and that $\mathfrak{K}(W)$ contains a nonzero constant, say $\begin{pmatrix} \alpha \\ 1 \end{pmatrix}$. Moreover, let $\tau \in \mathcal{N}_\nu$ be such that $q(z) := W \circ \tau \in \mathcal{N}_{\kappa+\nu} \setminus \{\infty\}$. Then*

$$\lim_{y \to +\infty} y(q(iy) - \alpha) = i[1,1]_{\mathfrak{P}(E_W)}. \tag{5.11}$$

In particular the limit on the left hand side of (5.11) does not depend on τ.

If, in addition, $M \in \mathcal{M}_\mu^1$ is a real entire matrix function, such that $WM \in \mathcal{M}_{\kappa+\mu}^1$, then the limit in (5.11) with function of the form $(WM) \circ \tau$ is the same as with function $W \circ \tau$.

Proof : By Lemma I.8.6 we have $1 \in \mathfrak{P}(E_W)$. Since W is a generalized 1-resolvent matrix of $\mathcal{S}$, there exists a selfadjoint extension $\tilde{A}$ of $\mathcal{S}$ acting in some Pontryagin space $\tilde{\mathfrak{P}} \supseteq \mathfrak{P}(E_W)$, $\text{ind}_-\mathfrak{P}(E_W) = \kappa + \nu$, such that (for some $\alpha_\tau \in \mathbb{R}$)

$$q(z) - \alpha_\tau = [(\tilde{A} - z)^{-1}1, 1]_{\mathfrak{P}(E_W)}.$$

By Lemma 5.4 the number α_τ is uniquely determined by the property that $-\frac{1}{q(z)-\alpha_\tau}$ is not regular at ∞. Multiplying W from the left with

$$W_1 := \begin{pmatrix} 1 & -1 \\ 1 & -\alpha \end{pmatrix},$$

and using Theorem 5.7 we find that $\alpha_\tau = \alpha$.

If $\tilde{A}$ is 1-minimal, the relation (5.11) follows at once. If $\tilde{A}$ is not 1-minimal, we have by Proposition 4.4

$$(\tilde{A} - z)^{-1}1 \in \overline{\text{dom}\,\tilde{A}},\ z \in \rho(\tilde{A}).$$

Since $\text{ind}_-\text{cls}\,\{(\tilde{A} - z)^{-1}|z \in \rho(\tilde{A})\} = \kappa + \nu$, the space $\tilde{A}(0)$ is positive definit. Hence $\mathfrak{S}_\infty = \tilde{A}(0) = \overline{\text{dom}\,\tilde{A}}^{\perp}$. If we put $\mathfrak{M} := \mathfrak{S}_\infty^\perp$, Lemma 5.8 implies that $A_{\mathfrak{M}}$ is an operator. Since $1 \in \overline{\text{dom}\,\mathcal{S}} \subseteq \overline{\text{dom}\,\tilde{A}}$, we find again that (5.11) holds.

The final assertion easily follows, if we take for τ a nonexceptional constant for WM as in Lemma 5.12.

□

The following results give some conditions on $W \in \mathcal{M}_\kappa^1$ and $\tau \in \mathcal{N}_\nu$ in order to ensure $W \circ \tau \in \mathcal{N}_{\kappa+\nu}$. First note that, if $W \in \mathcal{M}_\kappa^1$ satisfies $\mathfrak{K}_-(W) = \mathfrak{K}(W)$, then w_{22} and w_{21} are linearly independent, hence we may identify $\mathfrak{K}(W)$ with $\mathfrak{P}(E_W)$.

Lemma 5.17. *Let $W \in \mathcal{M}_\kappa^1$ have real entire entries and assume that $\mathfrak{K}_-(W) = \mathfrak{K}(W)$. Moreover, assume that for the operator $\mathcal{S} \subseteq \mathfrak{P}(E)^2$, where $E = E_W = w_{21} - iw_{22}$, we have*

$$\text{ind}_-\overline{\text{dom}\,\mathcal{S}} = \kappa.$$

Then for all $\tau \in \mathcal{N}_0$ we have $(W \circ \tau)(z) \in \mathcal{N}_\kappa$. If even $\overline{\text{dom}\,\mathcal{S}} = \mathfrak{P}(E)$, then for all $\tau \in \mathcal{N}_\nu$ we have $(W \circ \tau)(z) \in \mathcal{N}_{\kappa+\nu}$. In this case for all $W_1 \in \mathcal{M}_\nu^1$ we have

$$\text{ind}_-\mathfrak{K}(WW_1) = \kappa + \nu.$$

Proof : The function $q = W \circ \tau$ is the regularized generalized 1-resolvent of a certain extension $\tilde{\mathcal{A}}$ of $\mathcal{S}$ acting in a Pontryagin space $\tilde{\mathfrak{P}}$ with $\operatorname{ind}_-\tilde{\mathfrak{P}} = \kappa$. If $\tilde{\mathcal{A}}$ is an operator, which is in particular the case if $\overline{\operatorname{dom}\mathcal{S}} = \mathfrak{P}(E)$, we obtain from Proposition 4.4 that

$$q(z) = \beta + [\hat{\tilde{R}}_z 1, 1]_\pm \in \mathcal{N}_{\kappa+\nu}.$$

If $\tilde{\mathcal{A}}$ is not an operator and $\tau \in \mathcal{N}_0$, the space $\tilde{\mathcal{A}}(0)$ positive since $\tilde{\mathcal{A}}(0) = (\operatorname{dom}\tilde{\mathcal{A}})^\perp \subseteq (\operatorname{dom}\mathcal{S})^\perp$ and $\operatorname{dom}\mathcal{S}$ contains a maximal negative subspace. Hence the space $\mathfrak{S}_\infty^\perp = \operatorname{dom}\tilde{\mathcal{A}}^\perp$ contains a maximal negative subspace. Again by Proposition 4.4 we conclude $q \in \mathcal{N}_\kappa$.

According to Lemma 5.12 we may choose $\alpha \in \mathbb{R}$, such that $W_1 \circ \alpha \in \mathcal{N}_\nu$. As $\mathcal{S} \subseteq \mathfrak{P}(E)^2$ is densely defined, i.e. admits only operator extensions, we conclude $W \circ (W_1 \circ \alpha) \in \mathcal{N}_{\kappa+\nu}$.

□

Lemma 5.18. *Let $W \in \mathcal{M}_\kappa^1$ be such that $\mathfrak{K}_-(W) = \mathfrak{K}(W)$. Moreover, let $\tau \in \mathcal{N}_\nu$ be regular but not finite at ∞. Then $W \circ \tau \in \mathcal{N}_{\kappa+\nu}$.*

Proof : By Theorem 0.6.5 the matrix W is the resolvent matrix of a certain symmetric relation $\mathcal{S}$ with defect index $(1,1)$ in a Pontryagin space $\mathfrak{P}$ with negative κ (in fact $\mathfrak{P} = \mathfrak{P}(E_W)$ and $\mathcal{S}$ is the multiplication operator). Hence there exists a $\mathfrak{P}$-minimal selfadjoint extension $\tilde{\mathcal{A}} \subseteq \tilde{\mathfrak{P}}$, $\rho(\tilde{\mathcal{A}}) \neq \emptyset$, $\operatorname{ind}_-\tilde{\mathfrak{P}} = \kappa + \nu$, such that $W \circ \tau$ is the regularized generalized 1-resolvent of $\tilde{\mathcal{A}}$. Note that, since τ is not constant, $\tilde{\mathcal{A}}$ is not a canonical extension of $\mathcal{S}$.

Denote as in Lemma 5.2 by $\mathfrak{P}_\tau$, A_τ and $\mathcal{S}_\tau$ the Pontryagin space, selfadjoint and symmetric relation representing τ as Q-function. Moreover, choose a canonical extension $\mathcal{A}$ of $\mathcal{S}$. It is proved in [HKS] that there exists a selfadjoint extension $B \subseteq (\mathfrak{P} \oplus \mathfrak{P}_\tau)^2$ of $\mathcal{S} \oplus \mathcal{S}_\tau$, such that the resolvents of $\tilde{\mathcal{A}}$ and B if compressed to $\mathfrak{P}$ coincide. In fact this is proved there only in the Hilbert space setting. But the same considerations hold in the Pontryagin space situation. Note that this implies that the $\mathfrak{P}$-minimal part of B is (up to unitary equivalence) equal to $\tilde{\mathcal{A}}$.

Assume that $x \in B(0)$, $x \neq 0$. By Lemma 5.2 the relation $\mathcal{S}_\tau$ is densely defined, hence $x \in \mathfrak{P}$ and clearly $x \perp \operatorname{dom}\mathcal{S}$. Hence B extends the selfadjoint relation $B_1 := \mathcal{S} \dot{+} \operatorname{span}\{(0;x)\} \subseteq \mathfrak{P}^2$. It follows that the $\mathfrak{P}$-minimal part of B is B_1, a contradiction since $\tilde{\mathcal{A}}$ is not canonical. It follows that B, and by Lemma 5.8 also $\tilde{\mathcal{A}}$ is an operator. By Proposition 4.4 the relation $\tilde{\mathcal{A}}$ is 1-minimal, hence $W \circ \tau \in \mathcal{N}_{\kappa+\nu}$.

□

The next lemmata are concerned with canonical extensions of $\mathcal{S}$. They supplement the results of [KW3], Sections 6 and 7. Let $\mathfrak{P}(E)$, $E = A - iB$, be a dB-Pontryagin space and assume that for some $\psi \in [0,\pi)$ the function $S_\psi = \cos\psi A(z) + \sin\psi B(z)$ belongs to $\mathfrak{P}(E)$. Let $\mathcal{A} = \mathcal{A}_\psi$ be the canonical selfadjoint extension corresponding to S_ψ. By Lemma I.7.1 we have

$$\mathfrak{S}_\infty = \{S_\psi, zS_\psi, \dots, z^n S_\psi\},$$

where n is such that $z^n S_\psi \in \mathfrak{P}(E)$ but $z^{n+1} S_\psi \notin \mathfrak{P}(E)$. Consider the chain of subspaces $\mathfrak{P}_t$, $t \in M_{reg}$, of $\mathfrak{P}(E)$ as in Proposition I.11.4.

Lemma 5.19. *We have*

$$\overline{\mathrm{dom}\,(\mathcal{S}^k)} = \overline{\mathrm{ran}\,(\mathcal{A}-z)^{-k}} = \mathrm{span}\,\{S_\psi, \ldots, z^{k-1}S_\psi\}^\perp,\ k = 1, \ldots, n+1. \tag{5.12}$$

The spaces $\overline{\mathrm{dom}\,(\mathcal{S}^k)}$, $k = 1, \ldots, n$, *are degenerated. The space* $\overline{\mathrm{dom}\,(\mathcal{S}^{n+1})}$ *is degenerated if and only if* $[S_\psi, z^n S_\psi] = 0$, *which is the case if and only if* $S_\psi \in \overline{\mathrm{dom}\,(\mathcal{S}^{n+1})}$. *In this case* $\overline{\mathrm{dom}\,(\mathcal{S}^{n+2})} = \overline{\mathrm{dom}\,(\mathcal{S}^{n+1})}$. *If* $S_\psi \notin \overline{\mathrm{dom}\,(\mathcal{S}^{n+1})}$, *then*

$$\overline{\mathrm{dom}\,(\mathcal{S}^k)} = \overline{\mathrm{dom}\,(\mathcal{S}^{n+1})} \dot{+} \mathrm{span}\,\{z^l S_\psi | l = 0, \ldots, n-k\},\ k = 0, \ldots, n+1.$$

Proof : We clearly have $S_\psi \in \mathrm{dom}\,(\mathcal{S}^n) \setminus \mathrm{dom}\,(\mathcal{S}^{n+1})$, in particular it follows that $\overline{\mathrm{dom}\,(\mathcal{S}^k)}$ is degenerated for $k = 1, \ldots, n$.

The second equality in (5.12) is obvious, whereas the first equality follows from the fact that $S_\psi \in \mathrm{dom}\,(\mathcal{S}^n)$: Using (I.4.7) for $(\mathcal{A}-z)^{-1}$ one can easily check that that $\overline{\mathrm{dom}\,(\mathcal{S}^k)} = \overline{\mathrm{ran}\,(\mathcal{A}-z)^{-k}}$ for $k \in \mathbb{N}$ with $S_\psi \in \overline{\mathrm{dom}\,(\mathcal{S}^{k-1})}$. Note that $\overline{\mathrm{dom}\,(\mathcal{S}^{n+1})} = \mathfrak{S}_\infty^\perp$ is degenerated if and only $S_\psi \in \overline{\mathrm{dom}\,(\mathcal{S}^{n+1})}$. In this case by the above argument

$$\overline{\mathrm{dom}\,(\mathcal{S}^{n+2})} = \overline{\mathrm{ran}\,((\mathcal{A}-z)^{-(n+2)})} = \mathfrak{S}_\infty^\perp = \overline{\mathrm{dom}\,(\mathcal{S}^{n+1})}.$$

If $\mathfrak{S}_\infty^\perp = \overline{\mathrm{dom}\,(\mathcal{S}^{n+1})}$ is nondegenerated, then the remaining assertion of the lemma follows from $\mathfrak{P}(E) = \mathfrak{S}_\infty \oplus \mathfrak{S}_\infty^\perp$.

□

Lemma 5.20. *Assume that in the situation of Lemma 5.19 we have* $S_\psi \in \overline{\mathrm{dom}\,(\mathcal{S}^{n+1})}$. *Let* $t_0 \in M_{reg}$ *be chosen such that* $\mathfrak{P}_{t_0} = \overline{\mathrm{dom}\,(\mathcal{S}^{n+1})}$. *If* $\mathcal{S}_t$ *denotes the operator of multiplication in the space* $\mathfrak{P}_t$, $t \in M_{reg}$, *then* $\mathcal{S}_{t_0}$ *is densely defined in* $\mathfrak{P}_{t_0}$, $t_0 = \sup\{t \in M_{reg} | t < 0, \mathrm{ind}_0(\mathfrak{P}_t) = 0\}$ *and*

$$\overline{\bigcup_{t \in M_{reg}, t < t_0} \mathrm{ran}\,(\mathcal{S}_t - w)} = \mathrm{ran}\,(\mathcal{S}_{t_0} - w).$$

Proof : Since by Lemma 5.19 $\overline{\mathrm{dom}\,(\mathcal{S}^{n+1})} = \overline{\mathrm{dom}\,(\mathcal{S}^{n+2})}$ we see that $\mathcal{S}_{t_0}$ is densely defined. Hence, $\bigcup_{t \in M_{reg}, t < t_0} \mathfrak{P}_t$ is dense in $\mathfrak{P}_{t_0}$. Moreover, again by Lemma 5.19 there is no $t \in M_{reg},\ t_0 < t < 0$ such that $\mathrm{ind}_0(\mathfrak{P}_t) = 0$. From Theorem I.11.6 we obtain that there are only finitly many $t \in M_{reg}$ such that $\mathrm{ind}_0(\mathfrak{P}_t) > 0$. Hence, there is a $t_1 < t_0,\ t_1 \in M_{reg}$ such that $\mathrm{ind}_0(\mathfrak{P}_t) = 0$ for all $t \in M_{reg},\ t_1 \le t < t_0$. This proves $t_0 = \sup\{t \in M_{reg} | t < 0, \mathrm{ind}_0(\mathfrak{P}_t) = 0\}$.

Let $F \in \mathrm{ran}\,(\mathcal{S}_{t_0} - w)$, i.e. $F \in \mathfrak{P}_{t_0}$ and $F(w) = 0$. Since $\bigcup_{t \in M_{reg}, t < t_0} \mathfrak{P}_t$ is dense in $\mathfrak{P}_{t_0}$, there exists a sequence $F_n \in \mathfrak{P}_{t_n}$ which converges to F in the norm of $\mathfrak{P}_{t_0}$, hence in particular $F_n(w) \to 0$. Choose $t_1 \in M_{reg},\ t_1 < t_0$ and a function $G \in \mathfrak{P}_{t_1}$ with $G(w) = 1$, then

$$G_n(z) := F_n(z) - F_n(w)G(z) \in \mathrm{ran}\,(\mathcal{S}_{t_n} - w)$$

and clearly $G_n \to F$ in the norm of $\mathfrak{P}_{t_0}$.

□

6 Isometric embeddings of dB-spaces

In Section 4 we have seen that if $\mathfrak{P} = \mathfrak{P}(E)$ is a dB-Pontryagin space, $\tilde{A} \subseteq \tilde{\mathfrak{P}}^2$, $\rho(\tilde{A}) \neq \emptyset$, is a selfadjoint extension of $\mathcal{S}$ and $u \in \mathfrak{P}_-$ satisfies *(i)* and *(ii)* of Proposition 4.4, the space $\mathfrak{P}$ can be embedded isometrically into a space $\Pi(\phi)$. In Proposition 4.6 the action of the embedding $\Gamma : \tilde{\mathfrak{P}} \to \Pi(\phi)$ was determined explicitly on subspaces of the form $\tilde{E}(\Delta)\mathfrak{P}$, where $\Delta \subseteq \mathbb{R}$ has no endpoint in $s(\phi)$. In this short section we shall investigate circumstances under which Γ can be explicitly determined on $\mathfrak{P}$.

We start with a corollary of Proposition 4.6. Let $\mathfrak{P}, \tilde{\mathfrak{P}}, \mathcal{S}, \tilde{A}$ and u be as above and let $\Pi(\phi)$ be the model space of $(\tilde{\mathfrak{P}}, \tilde{A}, R_z^- u)$. Recall that $\mathfrak{S}_\infty$ denotes the generalized eigenspace of $\tilde{A}$ at ∞.

Corollary 6.1. *Assume that u satisfies (i) and (ii) of Proposition 4.4. If $\mathfrak{S}_\infty = \{0\}$, the restriction Γ' of Γ to $\mathfrak{P}$ is given by*

$$\Gamma'(f)(t) = \frac{f(t)}{u(t)}(t - z_0), \; f \in \mathfrak{P}. \tag{6.1}$$

If $\mathfrak{S}_\infty \neq \{0\}$ and is positive definite, the mapping Γ' defined by (6.1) is a contraction of $\mathfrak{P}$ into $\Pi(\phi)$.

Proof : If $\mathfrak{S}_\infty = \{0\}$, the span of the spaces $\tilde{E}(\Delta)\tilde{\mathfrak{P}}$ and $\tilde{E}_{\mathbb{C}\setminus\mathbb{R}}\tilde{\mathfrak{P}}$ is dense in $\tilde{\mathfrak{P}}$, hence the first assertion follows from Proposition 4.6. To prove the second assertion note that the orthogonal projection P of $\tilde{\mathfrak{P}}$ onto $\mathfrak{S}_\infty^\perp$ is contractive and that Γ' defined by (6.1) can be written as $\Gamma' = P\Gamma$.

□

Lemma 6.2. *Let $\phi \in \mathcal{F}$ be such that $\infty \notin \sigma(\phi)$, and assume that $u \in \mathfrak{P}_-(= \operatorname{Ass}\mathfrak{P}(E))$ is such that $u(t) \neq 0$ for $t \in \sigma(\phi)$ and that (6.1) describes an isometry of $\mathfrak{P}$ into $\Pi(\phi)$. Then A_ϕ is a $\Gamma'\mathfrak{P}$-minimal extension of $\Gamma'\mathcal{S}$.*

Proof : Clearly A_ϕ is an extension of $\Gamma'\mathcal{S}$. Denote by $\mathfrak{L}$ the closed linear span of $\Gamma'\mathfrak{P} \cup \{(A_\phi - w)^{-1}\Gamma'\mathfrak{P} | w \in \rho(\phi)\}$. We have to show that $\mathfrak{L} = \Pi(\phi)$.

Let $\Delta \subseteq \mathbb{R}$ be a closed finite interval such that its endpoints do not belong to $s(\phi)$. First we show that there exists a function $f \in \mathfrak{P}$ with $f(t) \neq 0$ for $t \in \Delta \cap \sigma(\phi)$: In fact, since $u(t) \neq 0$ for $t \in \sigma(\phi)$ we have $\mathfrak{d}(\mathfrak{P}) \cap \sigma(\phi) = \emptyset$. Let $g(z)$ be any nonzero function in $\mathfrak{P}$ and denote by $t_1, \dots, t_n$ its zeros in $\Delta \cap \sigma(\phi)$. It follows that the function

$$f(z) := g(z)\frac{1}{(z - t_1)\dots(z - t_n)}$$

satisfies the desired properties.

If we had $E(\Delta)\mathfrak{L} \neq E(\Delta)\Pi(\phi)$, we would find an element $x \in E(\Delta)\Pi(\phi) = \Pi(\chi_\Delta\phi)$ with

$$[\frac{g(.)}{u(.)}(. - z_0), x]_{\chi_\Delta\phi} = 0, \; [\frac{g(.)}{u(.)}\frac{. - z_0}{. - w}, x]_{\chi_\Delta\phi} = 0, \; g \in \mathfrak{P}, \; w \in \rho(\phi).$$

Using the Tailor expansion of $\frac{\bar{f}(.)}{\bar{u}(.)}(. - z_0)$ and the fact that the multiplication operator in $\Pi(\chi_\Delta\phi)$ is continuous (compare [JLT]), we find that with x also $\frac{\bar{f}(.)}{\bar{u}(.)}(. - z_0)x$ belongs to $\Pi(\chi_\Delta\phi)$. We get in particular that

$$[1, \frac{\bar{f}(.)}{\bar{u}(.)}(. - z_0)x]_{\chi_\Delta\phi} = 0,\ [\frac{1}{. - w}, \frac{\bar{f}(.)}{\bar{u}(.)}(. - z_0)x]_{\chi_\Delta\phi} = 0,\ w \in \rho(\phi).$$

By Proposition 3.3 we see $\frac{\bar{f}(.)}{\bar{u}(.)}(. - z_0)x = 0$ in $\Pi(\chi_\Delta\phi)$. As $f(t) \neq 0$, $t \in \Delta \cap \sigma(\phi)$, one easily checks that this implies $x = 0$. Since Δ was arbitrary, and since by our assumption $(I - E_{\mathbb{C}\setminus\mathbb{R}})\Pi(\phi)$ is the closure of the linear span of all the spaces $E(\Delta)\Pi(\phi)$, we see $(I - E_{\mathbb{C}\setminus\mathbb{R}})\mathfrak{L} = (I - E_{\mathbb{C}\setminus\mathbb{R}})\Pi(\phi)$. Treating the nonreal spectrum $\sigma(\phi) \setminus \mathbb{R}$ similar, we finally get $\mathfrak{L} = \Pi(\phi)$.

□

In the sequel let $R_w = (A_\phi - w)^{-1}$.

Corollary 6.3. *Let $\phi \in \mathcal{F}$, Γ' and $u \in \mathfrak{P}_-$ be as in Lemma 6.2. The mapping Γ' induces an isomorphism from $\mathfrak{P}_-$ onto $(\Gamma'\mathfrak{P})_-$. We denote this isomorphism also by Γ'. If R_z^- denotes the (continued) resolvent of A_ϕ, we have*

$$R_{z_0}^-\Gamma' u = 1. \tag{6.2}$$

Put $\kappa := \text{ind}\_\mathfrak{P}$ and $\nu := \text{ind}\_\Pi(\phi) - \text{ind}\_\mathfrak{P}$. There exists a matrix $W \in \mathcal{M}_\kappa^u$ with $E = E_W$ and a parameter $\tau \in \mathcal{N}_\nu$ such that

$$\phi \cdot \left((\frac{1}{t - z} - \frac{t - \operatorname{Re} z_0}{|t - z_0|^2})|t - z_0|^2 \right) = W \circ \tau. \tag{6.3}$$

Proof : Denote by $\tilde{P}$ the orthogonal projection of $\Pi(\phi)$ onto $\Gamma'\mathfrak{P}$. Let $w \in \rho(\phi)$, and let $f \in \mathfrak{P}$ be such that $f(w) = u(w)$. Now consider $\tilde{R}_w^-(u - \iota f) = \tilde{P}R_w^-(u - \iota f)$ (cf. [KW2]). By (0.3.8) and Lemma I.10.1 we obtain

$$((\Gamma')^{-1}\tilde{R}_w^-\Gamma'(u - \iota f))(z) = \frac{u(z) - f(z)}{z - w}.$$

Hence

$$\tilde{R}_w^-\Gamma'(u - \iota f) = (\frac{. - z_0}{. - w} - \frac{f(.)}{u(.)}\frac{. - z_0}{. - w}) = (\tilde{P}\frac{. - z_0}{. - w}) - (\tilde{P}\frac{f(.)}{u(.)}\frac{. - z_0}{. - w}).$$

On the other hand we have

$$\tilde{R}_w^-\Gamma'(u - \iota f) = (\tilde{P}R_w^-\Gamma' u) - (\tilde{P}R_w\Gamma' f) =$$

$$= (\tilde{P}R_w^-\Gamma' u) - (\tilde{P}\frac{f(.)}{u(.)}\frac{. - z_0}{. - w}).$$

Comparing these relations we get

$$(\tilde{P}(I + (w - z_0)(A_\phi - w)^{-1})(R_{z_0}^-\Gamma' u - 1)) = (\tilde{P}(R_w^-\Gamma' u - \frac{. - z_0}{. - w})) = 0.$$

This means that there is a $(\tilde{A}_\phi - w)^{-1},\ w \in \rho(\phi)$ invariant subspace of $(I - \tilde{P})\Pi(\phi)$, which contradicts Lemma 6.2 if (6.2) would not hold.

To prove the remaining part take for W' a u-resolvent matrix of $\mathcal{S}$, then a certain parameter τ with $\operatorname{ind}_-\tau = \operatorname{ind}_-\Pi(\phi) - \operatorname{ind}_-\mathfrak{P}$ (cf. [KW2]) corresponds to the extension A_ϕ. If $\hat{\tilde{R}}_z : \mathfrak{P}_- \to \mathfrak{P}_+$ denotes the regularized generalized resolvent of A_ϕ, we find with $\beta \in \mathbb{R}$ using (0.4.1)

$$W' \circ \tau = [\hat{\tilde{R}}_z u, u]_\pm + \beta = (z - \operatorname{Re} z_0)[R_{z_0}^- u, R_{z_0}^- u] +$$
$$+ (z - z_0)(z - \overline{z_0})[(A_\phi - z)^{-1} R_{z_0}^- u, R_{z_0}^- u]_\phi + \beta =$$
$$= \phi \cdot \left((\frac{1}{t - z} - \frac{t - \operatorname{Re} z_0}{|t - z_0|^2})|t - z_0|^2 \right) + \beta.$$

Adding the $-\beta$ multiple of the second row to the first row of W' we obtain a matrix function $W(z)$ with the desired properties. (6.3) shows that $\operatorname{ind}_- W \circ \tau = \operatorname{ind}_-\Pi(\phi)$, and furthermore that $W \in \mathcal{M}_\kappa$.

□

In the following proposition we start with a space $\Pi(\phi)$ and construct dB-Pontryagin spaces which are contained isometrically in $\Pi(\phi)$ via Γ' starting from factorizations of

$$q(z) = \phi \cdot \left((\frac{1}{t - z} - \frac{t - \operatorname{Re} z_0}{|t - z_0|^2})|t - z_0|^2 \right). \tag{6.4}$$

Note that we we may expect a dB-Pontryagin space $\mathfrak{P}$ to be contained isometrically in a space $\Pi(\phi)$ via Γ' only if $\infty \notin \sigma(\phi)$.

Proposition 6.4. *Let $\Pi(\phi)$, $\infty \notin \sigma(\phi)$, be given and define q by (6.4). Assume that the function $q \in \mathcal{N}_\kappa$ is written as $q(z) = (W \circ \tau)(z)$ with*

$$W(z) = \begin{pmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{pmatrix} \in \mathcal{M}_\nu^1,$$

and $\tau \in \mathcal{N}_\mu$ for some ν and μ. Moreover, assume that W has real entire entries and either is not a polynomial or $\tau \neq -\frac{w_{22}}{w_{21}}$ and $\kappa = \nu + \mu$. Then $E_W \in \mathcal{HB}_\nu$ and $1 \in \operatorname{Ass}\mathfrak{P}(E_W)$. If 1 satisfies (ii) of Proposition 4.4, then $\mathfrak{P}(E_W)$ is contained isometrically in $\Pi(\phi)$ and $\operatorname{ind}_-\Pi(\phi) = \kappa = \nu + \mu$.

Proof : Since $W \circ \tau = q$ and ∞ is regular for q, Theorem 5.7 implies that $\mathfrak{K}_-(W) = \mathfrak{K}(W)$. Hence $\mathfrak{P}(E) \cong \mathfrak{K}(W)$ and $1 \in \operatorname{Ass}\mathfrak{P}(E)$. It follows that $W(z)$ is a resolvent matrix of the operator $\mathcal{S}$ in the space $\mathfrak{P}(E)$ associated with the element $1 \in \operatorname{Ass}\mathfrak{P}(E)$.

The parameter τ corresponds to a certain extension $\tilde{A}$ of $\mathcal{S}$ which acts in a Pontryagin space $\tilde{\mathfrak{P}}$ with $\operatorname{ind}_-\tilde{\mathfrak{P}} = \nu + \mu$ and for which $q(z) = [\hat{\tilde{R}}_z 1, 1]_\pm$. By *(ii)* of Proposition 4.4 and Proposition 4.4 the relation $\tilde{A}$ is $R_{z_0}^- 1$-minimal. This gives $\kappa = \operatorname{ind}_-\tilde{\mathfrak{P}}$. Moreover, there exists a distribution ϕ', such that

$$(\tilde{\mathfrak{P}}, \tilde{A}, R_{z_0}^- 1) \cong (\Pi(\phi'), A_{\phi'}, 1).$$

Clearly $q(z) = [\hat{\tilde{R}}(\phi')_z 1, 1]_\pm$, when $\hat{\tilde{R}}(\phi')_z^-$ denotes the regularized resolvent of $A_{\phi'}$. Since the distribution which represents an $\mathcal{N}_\kappa$ function is unique, we conclude that $\phi' = \phi$. From our assumption $\infty \notin \sigma(\phi)$, we obtain that $\tilde{A}$ is in fact an operator. Corollary 6.1 shows that $\mathfrak{P}$ is contained isometrically in $\Pi(\phi)$.

□

7 Matrix chains

In this and the following sections all considered matrices of class $\mathcal{M}_\kappa^1$ are assumed to have real and entire entries. Our first aim is to show that a given matrix $W \in \mathcal{M}_\kappa^1$ invents a whole chain of matrices $W_t \in \mathcal{M}_{\mu(t)}^1$, $t \leq 0$. If in addition a parameter function $\tau \in \mathcal{N}_0$ is given this chain can be extended to $t \geq 0$. Recall that for a matrix W we denote by $\mathfrak{t}(W)$ the trace of $W'(0)J$.

Theorem 7.1. *Let $W \in \mathcal{M}_\kappa^1$, $W(0) = 1$, have real and entire entries and be not constant. Then there exists a number $c_- \in [-\infty, 0)$, a set $\tilde{D}$ which is the union of a finite set $D \subseteq (c_-, 0)$ and at most κ intervals with both endpoints in D, and matrices $W_t(z)$ for $t \in D^c := (c_-, 0] \setminus \tilde{D}$, such that the family $(W_t)_{t \in D^c}$ has the following properties:*

(i) *The matrix W_t has real and entire entries with $W_t(0) = 1$.*

(ii) *$W_t \in \mathcal{M}_{\mu(t)}^1$ where $\mu(t)$ is is nondecreasing, constant on each connected component of D^c and takes different values on different components.*

(iii) *For all $s, t \in D^c$, $s < t$, there exists a real entire matrix function $W_{st} \in \mathcal{M}_{\mu(t)-\mu(s)}^1$, $W_{st}(0) = 1$, such that $W_t = W_s W_{st}$.*

(iv) *$W_0 = W$.*

(v) *The function $\mathfrak{t}(W_t)$ is continuous on D^c and strictly increasing on each connected component of D^c. If $t_0 \in D$ is a boundary point of D^c, then $\lim_{t \nearrow t_0} \mathfrak{t}(W_t) = +\infty$ ($\lim_{t \searrow t_0} \mathfrak{t}(W_t) = -\infty$) if $t_0 = \sup\{t \in D^c | t < t_0\}$ ($t_0 = \inf\{t \in D^c | t > t_0\}$).*

(vi) *The family $(W_t)_{t \in D^c}$ is maximal in the following sense: If $W = M_1 M_2$, where $M_1 \in \mathcal{M}_{\nu_1}^1$ and $M_2 \in \mathcal{M}_{\nu_2}^1$ are real and entire matrix functions and $\kappa = \nu_1 + \nu_2$, then there exists a number $t \in D^c$ such that $M_1 = W_t$ and $M_2 = W_{t0}$.*

The family $(W_t)_{t \in D^c}$ is uniquely determined by the listed properties up to reparametrizations of the form

$$W_t^\bullet = W_{t^\bullet(t)},\ t \in D^{\bullet c},$$

where $t^\bullet(t)$ is a continuous bijection of $D^{\bullet c}$ onto D^c.

The next lemmata are needed in the proof of Theorem 7.1. They supplement the results of [KW3], Sections 12 and 13. Certain linear matrices will play an important role: If

$l \in \mathbb{R}$, $l \neq 0$, and $\alpha \in [0, \pi)$ define

$$W_{(l,\alpha)}(z) := \begin{pmatrix} 1 - lz\sin\alpha\cos\alpha & lz\cos^2\alpha \\ -lz\sin^2\alpha & 1 + lz\sin\alpha\cos\alpha \end{pmatrix}. \tag{7.1}$$

The following relations will be used frequently:

$$W_{(l,\alpha)}W_{(k,\alpha)} = W_{(l+k,\alpha)},\ W_{(l,\alpha)}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix},\ W_{(l,\alpha)} \circ \cot\alpha = \cot\alpha.$$

Recall the following result (compare [dB7]):

Lemma 7.2. *The space $\mathfrak{K}(W_{(l,\alpha)})$ is given by*

$$\mathfrak{K}(W_{(l,\alpha)}) = \operatorname{span}\{\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}\},$$

$$[\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}]_{\mathfrak{K}(W_{(l,\alpha)})} = \frac{1}{l}.$$

If $c \in \mathbb{R}$, then $W_{(l,\alpha)}$ can be factorized as $W_{(l,\alpha)} = W_{(c,\alpha)}W_{(l-c,\alpha)}$. If c and $l - c$ are both positive (both negative), then this is the only factorization of $W_{(l,\alpha)}$ into a product of two matrices $W_i \in \mathcal{M}_0$ ($W_i^{-1} \in \mathcal{M}_0$), $W_i(0) = 1$, $i = 1, 2$, such that $\mathfrak{t}(W_1) = c$.

Lemma 7.3. *Let the dB-Pontryagin space $\mathfrak{P}(E_1)$ be contained isometrically in the dB-Pontryagin space $\mathfrak{P}(E_2)$. Assume that $W \in \mathcal{M}_\kappa$, $\kappa = \operatorname{ind}_-\mathfrak{P}(E_2) - \operatorname{ind}_-\mathfrak{P}(E_1)$ is the transfer matrix as in Theorem I.12.2. If $\cos\alpha A_2 + \sin\alpha B_2 \in \mathfrak{P}(E_2)$ for some $\alpha \in [0, \pi)$, then*

$$W\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \in \mathfrak{K}(W), \tag{7.2}$$

and

$$[W\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, W\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}]_{\mathfrak{K}(W)} = [\cos\alpha A_2 + \sin\alpha B_2, \cos\alpha A_2 + \sin\alpha B_2]_{\mathfrak{P}(E_2)}.$$

Proof : Assume that (7.2) does not hold. Then let $l > 0$ and consider $WW_{(l,\alpha)}$. Since $\mathfrak{K}_+(W) = \mathfrak{K}(W)$ or $\mathfrak{K}_-(W) = \mathfrak{K}(W)$, it follows from Section 13 of [KW3] that $\mathfrak{K}(WW_{(l,\alpha)}) = \mathfrak{K}(W) \oplus W\mathfrak{K}(W_{(l,\alpha)})$. By Corollary 5.15 and Theorem I.12.2 we obtain for $(A_3, B_3) := (A_1, B_1)WW_{(l,\alpha)}$ that $\mathfrak{P}(E_3) = \mathfrak{P}(E_1) \oplus (A_1, B_1)\mathfrak{K}(WW_{(l,\alpha)}) = \mathfrak{P}(E_1) \oplus (A_1, B_1)\mathfrak{K}(W) \oplus (A_1, B_1)W\mathfrak{K}(W_{(l,\alpha)}) = \mathfrak{P}(E_2) \oplus (A_2, B_2)\mathfrak{K}(W_{(l,\alpha)})$. So $\mathfrak{P}(E_2)$ is contained isometrically in $\mathfrak{P}(E_3)$. By Theorem I.12.2 this is only possible if $\cos\alpha A_2 + \sin\alpha B_2 \notin \mathfrak{P}(E_2)$.

The second relation follows immediately from $(A_1, B_1)\mathfrak{K}(W) \cong \mathfrak{K}(W)$.

□

Corollary 7.4. *Let $W \in \mathcal{M}_\kappa^1$ and assume that $\mathfrak{K}_-(W) = \mathfrak{K}(W)$. If $\alpha \in [0, \pi)$ is such that*

$$\pi_-(W\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}) \in \pi_-\mathfrak{K}(W), \tag{7.3}$$

then in fact

$$W\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}\in\mathfrak{K}(W). \tag{7.4}$$

With $E=E_W$ we have

$$[W\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix},W\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}]_{\mathfrak{K}(W)}=[\cos\alpha w_{21}+\sin\alpha w_{22},\cos\alpha w_{21}+\sin\alpha w_{22}]_{\mathfrak{P}(E)}.$$

Proof : Setting $B_1=1, A_1=0$ ($\mathfrak{P}(E_1)=\{0\}$), we can use the same proof as in Lemma 7.3.

□

Lemma 7.5. *Let $W\in\mathcal{M}^1_\kappa$. Then $\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}$ belongs to $\mathfrak{K}(W)$ and is not neutral if and only if $W=W_{(l,\alpha)}W_1$ with $W_1\in\mathcal{M}^1_{\kappa-\delta}$, $\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}\notin\mathfrak{K}(W_1)$, where*

$$l=\frac{1}{[\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix},\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}]_{\mathfrak{K}(W)}},$$

$$\delta=\begin{cases}0\,,\ l>0\\ 1\,,\ l<0\end{cases}.$$

Proof : Assume that for some matrix $\tilde{W}$ we have $\mathfrak{K}(\tilde{W})\subseteq\mathfrak{K}(W)$ isometrically. The kernel relation

$$\tilde{W}(z)^{-1}\frac{W(z)JW(w)^*-J}{z-\overline{w}}\tilde{W}(w)^{-*}+\frac{\tilde{W}(z)^{-1}J\tilde{W}(w)^{-*}-J}{z-\overline{w}}=$$
$$=\tilde{W}(z)^{-1}\left(\frac{W(z)JW(w)^*-J}{z-\overline{w}}-\frac{\tilde{W}(z)J\tilde{W}(w)^*-J}{z-\overline{w}}\right)\tilde{W}(w)^{-*}$$

shows that $\mathfrak{K}(\tilde{W}^{-1}W)$ is isomorphic to $\mathfrak{K}(W)\ominus\mathfrak{K}(\tilde{W})$ via the mapping $\begin{pmatrix}F_+\\ F_-\end{pmatrix}\mapsto\tilde{W}\begin{pmatrix}F_+\\ F_-\end{pmatrix}$. Taking $\tilde{W}=W_{(l,\alpha)}$ the assertion follows.

□

Lemma 7.6. *Let $\mathfrak{P}(E)$, $E=A-iB$ be a dB-Pontryagin space, and let $W\in\mathcal{M}^1_\kappa$ have real entire entries and be such that there is no constant function*

$$\begin{pmatrix}u\\ v\end{pmatrix}\in\mathfrak{K}(W)\ \textit{with}\ uA+vB\in\mathfrak{P}(E).$$

We set $(A_1,B_1)=(A,B)W$, *and* $E_1=A_1-iB_1$. *Then* $\mathfrak{K}(W)$ *contains a constant function if and only if*

$$\mathfrak{P}_0=\bigcap\{\mathfrak{P}|\mathfrak{P}\text{ is a dB-subspace of }\mathfrak{P}(E_1),\ \mathfrak{P}(E)\subsetneq\mathfrak{P}\}\supsetneq\mathfrak{P}(E).$$

In this case there exists a unique number $\psi\in[0,\pi)$ *such that*

$$\begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix}\in\mathfrak{K}(W),$$

and

$$[\begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix},\begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix}]_{\mathfrak{K}(W)}=[\cos\psi A+\sin\psi B,\cos\psi A+\sin\psi B]_{\mathfrak{P}(E_1)}.$$

Proof : Assume that $\mathfrak{K}(W)$ contains a constant function (cf. Corollary I.8.4):

$$\begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix}\in\mathfrak{K}(W),\ \ \psi\in[0,\pi).$$

Let $\mathfrak{P}'=\mathfrak{P}(E)\oplus\operatorname{span}(\cos\psi A+\sin\psi B)\subseteq\mathfrak{P}(E_1)$, where the second component is provided with the scalar product inherited from $(A,B)\mathfrak{K}(W)$. It is elementary to check that $\mathfrak{P}'$ is a dB-subspace of $\mathfrak{P}(E_1)$. Thus, $\mathfrak{P}'=\mathfrak{P}_0\supsetneq\mathfrak{P}(E)$.

Conversely, assume that $\mathfrak{P}_0\supsetneq\mathfrak{P}(E)$. Then $\mathfrak{P}_0$ is a dB-subspace of $\mathfrak{P}(E_1)$. Providing $\mathfrak{P}_0$ with a positive definite scalar product (cf. [KW3]) we obtain from [dB7] that $\mathfrak{P}(E)$ has codimension one in $\mathfrak{P}_0$ and $\mathfrak{P}(E)$ is the closure of the domain of the multiplication operator with the independent variable in $\mathfrak{P}_0$. Let $S(z)\in\mathfrak{P}_0\cap\mathfrak{P}(E)^\perp$. Since $\mathfrak{P}(E)$ is a dB-subspace we can choose $S(z)$ such that $S(z)=S^\#(z)$. Consider $\mathcal{R}_S(w)$, and note tha the range of $\mathcal{R}_S(w)$ is the domain of the multiplication operator with the independent variable. In particular, $\mathfrak{P}(E)$ is invariant under $\mathcal{R}_S(w)$, and hence $S\in\operatorname{Ass}\mathfrak{P}(E)$. Since $S\in\mathfrak{P}(E_1)=(A,B)\mathfrak{K}(W)$, there is a function

$$\begin{pmatrix}F_+\\ F_-\end{pmatrix}\in\mathfrak{K}(W),\ \text{such that}\ F_+A+F_-B=S.$$

Recall that for any matrix function with $M(z)JM(\overline{z})^*=S(z)JS^\#(z)$

$$M(z)\mathcal{R}_1(w)\begin{pmatrix}F_+(z)\\ F_-(z)\end{pmatrix}=\mathcal{R}_S(w)M(z)\begin{pmatrix}F_+(z)\\ F_-(z)\end{pmatrix}+$$

$$+\frac{M(z)JM(\overline{w})^*-S(z)JS^\#(w)}{z-w}\frac{JM(w)}{S(w)S^\#(w)}\begin{pmatrix}F_+(w)\\ F_-(w)\end{pmatrix}. \tag{7.5}$$

We use the second row of this relation for a generalized S-resolvent matrix M on $\mathfrak{P}(E)$, $(0,1)M=(A,B)$, and obtain $(0,1)M(z)\mathcal{R}_1(w)\begin{pmatrix}F_+(z)\\ F_-(z)\end{pmatrix}\in\mathfrak{P}(E)$. By Corollary I.12.3 this expression is also contained in $\mathfrak{P}(E_1)\ominus\mathfrak{P}(E)$, hence $\mathcal{R}_1(w)\begin{pmatrix}F_+(z)\\ F_-(z)\end{pmatrix}=0$, and we conclude that $\mathfrak{K}(W)$ contains a constant function.

The remaining assertions follow from Corollary I.8.4 and the fact that $(A,B)\mathfrak{K}(W)\cong\mathfrak{K}(W)$.

$\square$

Now we come to the proof of Theorem 7.1.

Proof (of Theorem 7.1): First we construct a chain $(W_t)_{t\in D^c}$ and verify *(i)*-*(iv)*. By Corollary I.8.4 not both $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ can be contained in $\mathfrak{K}(W)$. Together with Corollary I.9.7 we conclude that at least one of $\mathfrak{K}_-(W)=\mathfrak{K}(W)$ or $\mathfrak{K}_+(W)=\mathfrak{K}(W)$ holds. We may assume without loss of generality that $\mathfrak{K}_-(W)=\mathfrak{K}(W)$, otherwise consider the matrix $-JWJ$ instead. Note that, since W is not constant, Lemma 5.6 then implies that w_{21} and w_{22} are linearly independent. Hence the function $E(z)=A(z)-iB(z)=E_W(z),\ A=w_{21},\ B=w_{22}$ belongs to $\mathcal{HB}_\kappa$ and we may consider the dB-Pontryagin space $\mathfrak{P}(E)$.

By [KW3], Section 11, there exists a number $c_-<0$, a set $M_{reg}\subseteq(c_-,0]$, $\inf M_{reg}=c_-$, and a unique chain of dB-subspaces $\mathfrak{P}_t$, $t\in M_{reg}$. If $t\in M_{sing}$ we denote by

$$(t_-(t),t_+(t)),\ t_-(t),t_+(t)\in M_{reg}$$

the interval around t which is contained in M_{sing}. Let

$$D:=\{t\in M_{reg}|\mathrm{ind}_0\mathfrak{P}_t\neq 0\}\cup\{\frac{t_-(t)+t_+(t)}{2}|t\in M_{sing},$$

$$\mathrm{ind}_-\mathfrak{P}_{t_-(t)}<\mathrm{ind}_-\mathfrak{P}_{t_+(t)},\ \mathrm{ind}_0\mathfrak{P}_{t_-(t)}=\mathrm{ind}_0\mathfrak{P}_{t_+(t)}=0\},$$

then D is finite (cf. Theorem I.11.6), and for $t\in M_{reg}\setminus D$ we have $\mathfrak{P}_t=\mathfrak{P}(E_t)$ for some function $E_t\in\mathcal{HB}_{\mu(t)}$. By Corollary I.6.2 we can choose $E_t=A_t-iB_t$ such that the transfer matrix $W_{t0}\in\mathcal{M}^1_{\kappa-\mu(t)}$ in Theorem I.12.2 satisfies $W_{t0}(0)=1$. Hence $E_t(0)=-i$. Since $1\in\mathrm{Ass}\,\mathfrak{P}(E)$, it follows from [dB7] that $1\in\mathrm{Ass}\,\mathfrak{P}_t$ for all $t\in M_{reg}$. By Proposition I.10.3, Corollary I.10.4 (exchanging W and $-JWJ$) and Corollary I.9.8 there exists a matrix $W_t\in\mathcal{M}^1_{\mu(t)}$ having real entire entries with $\mathfrak{K}_-(W_t)=\mathfrak{K}(W_t)$ and $(0,1)W_t=(A_t,B_t)$. Such a matrix W_t is unique up to transformations of the form ($\alpha\in\mathbb{R}$)

$$\tilde{W}_t=\begin{pmatrix}1&\alpha\\0&1\end{pmatrix}W_t. \tag{7.6}$$

Since $E_t(0)=-i$, we may assume that $W_t(0)=1$. Now we have $W_tW_{t0}=W$: In fact, since $\mathfrak{P}(E_t)\cong\mathfrak{K}(W_t)$, $(A_t,B_t)\mathfrak{K}(W_{t0})\cong\mathfrak{K}(W_{t0})$ and $\mathfrak{P}(E)=\mathfrak{P}(E_t)\oplus(A_t,B_t)\mathfrak{K}(W_{t0})$ we obtain from some results concerning the sum of kernels given in [ADSR], that $\mathfrak{K}(W_tW_{t0})=\mathfrak{K}(W_t)\oplus W_t\mathfrak{K}(W_{t0})\cong\mathfrak{P}(E)$, and hence $\mathfrak{K}_-(W_tW_{t0})=\mathfrak{K}(W_tW_{t0})$. Thus, W and W_tW_{t0} differ only by a transformation of the same kind as in (7.6). Since $W(0)=1=W_t(0)W_{t0}(0)$, we have $W=W_tW_{t0}$. If $s,t\in M_{reg}\setminus D,\ s<t$, and if $W_{st}\in\mathcal{M}_{\mu(t)-\mu(s)}$ denotes the transfer matrix (cf. Theorem I.12.2) such that $(A_s,B_s)W_{st}=(A_t,B_t)$, then a similar reasoning shows that $W_sW_{st}=W_t$.

Now define $\tilde{D}$ as the union of D and those intervals contained in M_{sing} with both endpoints in $\{t\in M_{reg}|\mathrm{ind}_0\mathfrak{P}_t\neq 0\}$ and put $D^c:=(c_-,0]\setminus\tilde{D}$. By definition each connected component of D^c contains at least one point of M_{reg}, and elementary considerations using

Proposition I.11.11 show that $\mu(t),\ t\in M_{reg}$ is constant on those components. Moreover, $\mu(s)\neq\mu(t)$ if $s,t\in M_{reg}\cap D^c$ belong to different components of D^c. Therefore, there are at most $\kappa+1$ components of D^c, and $\tilde{D}$ is the union of at most κ disjoint intervals. It remains to define matrices W_t if $t\in D^c\cap M_{sing}$.

Assume first that $\mathrm{ind}_-\mathfrak{P}_{t_-(t)}=\mathrm{ind}_-\mathfrak{P}_{t_+(t)}=0$. Then by Theorem I.12.2 and Lemma 7.6 the transfer matrix $W_{t_-(t)t_+(t)}$ is of the form $W_{(l,\alpha)}$ for some $l\in\mathbb{R}$, $l\neq 0$, and $\alpha\in[0,\pi)$. We shall speak in this case of an indivisable interval of type α and weight l. If $l>0$, i.e. $\mu(t_-(t))=\mu(t_+(t))$, then by Lemma 7.2 there exists a unique factorization $W_{(l,\alpha)}=W_{t_-(t)t}W_{tt_+(t)}$ such that the function $\mathfrak{t}(W_{t_-(t)t})$ is linear and $W_{t_-(t)t},W_{tt_+(t)}\in\mathcal{M}_0^1$. In fact,

$$W_{t_-(t)t}=W_{(l\frac{t-t_-(t)}{t_+(t)-t_-(t)},\alpha)},\ W_{tt_+(t)}=W_{(l\frac{t_+(t)-t}{t_+(t)-t_-(t)},\alpha)}. \tag{7.7}$$

For $t\in(t_-,t_+)$ we define a matrix $W_t:=W_{t_-(t)}W_{t_-(t)t}$ and we easily see that $\mu(t_-(t))=\mu(t_+(t))=\mathrm{ind}_-\mathfrak{K}(W_t)=:\mu(t)$.

If $l<0$, i.e. $\mu(t_+(t))=\mu(t_-(t))+1$, we define with $t_0(t)=\frac{t_+(t)+t_-(t)}{2}\in D$ for $t_-(t)<t<t_0(t)$

$$W_{t_-(t)t}:=W_{(\tan(\frac{\pi}{2}\frac{t-t_-(t)}{t_0(t)-t_-(t)}),\alpha)}\in\mathcal{M}_0^1,$$

$$W_{tt_+(t)}:=W_{t_-(t)t}^{-1}W_{t_-(t)t_+(t)}=W_{(l-\tan(\frac{\pi}{2}\frac{t-t_-(t)}{t_0(t)-t_-(t)}),\alpha)}\in\mathcal{M}_1^1,$$

and for $t_0(t)<t<t_+(t)$

$$W_{tt_+(t)}:=W_{(\tan(\frac{\pi}{2}\frac{t_+(t)-t}{t_+(t)-t_0(t)}),\alpha)}\in\mathcal{M}_0^1,$$

$$W_{t_-(t)t}:=W_{t_-(t)t_+(t)}W_{tt_+(t)}^{-1}=W_{(l-\tan(\frac{\pi}{2}\frac{t_+(t)-t}{t_+(t)-t_0(t)}),\alpha)}\in\mathcal{M}_1^1,$$

and set $W_t:=W_{t_-(t)}W_{t_-(t)t}$. Since $\cos\alpha A_{t_-(t)}+\sin\alpha B_{t_-(t)}\notin\mathfrak{P}(E_{t_-(t)})$ (cf. Theorem I.12.2), we obtain that $\mu(t)=\mathrm{ind}_-\mathfrak{K}(W_t)=\mu(t_-(t))$ if $t<t_0(t)$, and $\mu(t)=\mu(t_+(t))$, otherwise.

Next assume that $\mathrm{ind}_-\mathfrak{P}_{t_-(t)}\neq 0$. Then $t_+(t)\in D^c$, $\cos\alpha A_{t_+(t)}+\sin\alpha B_{t_+(t)}\in\mathfrak{P}(E_{t_+(t)})$ for some $\alpha\in[0,\pi)$ and this element is neutral in $\mathfrak{P}(E_{t_+(t)})$ (cf. Lemma 5.19). By Corollary 7.4

$$W_{t_+(t)}\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}\in\mathfrak{K}(W_{t_+(t)})$$

is neutral in $\mathfrak{K}(W_{t_+(t)})$. Now we set $W_t:=W_{t_+(t)}W_{tt_+(t)}^{-1}$ where

$$W_{tt_+(t)}:=W_{(\tan(\frac{\pi}{2}\frac{t_+(t)-t}{t_+(t)-t_-(t)}),\alpha)}. \tag{7.8}$$

It follows from [ADSR] that the reproducing kernel space $\mathfrak{K}(W_t)$ is isomorphic to $\mathfrak{L}^\perp/\mathfrak{L}^\circ$, where $\mathfrak{L}$ is a certain subspace of $\mathfrak{K}(W_{t_+(t)})\oplus W_{t_+(t)}\mathfrak{K}(W_{tt_+(t)}^{-1})$:

$$\mathfrak{L}:=\{(F(z);-F(z))|F(z)\in\mathfrak{K}(W_{t_+(t)})\cap W_{t_+(t)}\mathfrak{K}(W_{tt_+(t)}^{-1})\}.$$

Since $\mathfrak{K}(W_{t_+(t)})\cap W_{t_+(t)}\mathfrak{K}(W_{tt_+(t)}^{-1})$ is one-dimensional, neutral as a subset of $\mathfrak{K}(W_{t_+(t)})$ and negative as a subset of $W_{t_+(t)}\mathfrak{K}(W_{tt_+(t)}^{-1})$, we see that

$$\mu(t):=\mathrm{ind}_-\mathfrak{K}(W_t)=\mathrm{ind}_-(\mathfrak{L}^\perp/\mathfrak{L}^\circ)=\mu(t_+(t))+\mathrm{ind}_-\mathfrak{K}(W_{tt_+(t)}^{-1})-1=\mu(t_+(t)).$$

Finally, if $\operatorname{ind}_-\mathfrak{P}_{t_+(t)} \neq 0$, then $t_-(t) \in D^c$, and by Lemma 7.6 we see that

$$\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \in \mathfrak{K}(W_{t_-(t)0}), \text{ for some } \alpha \in [0,\pi),$$

and by Theorem I.12.2 $\cos\alpha A_{t_-(t)} + \sin\alpha B_{t_-(t)} \notin \mathfrak{P}(E_{t_-(t)})$. Setting $W_t := W_{t_-(t)} W_{t_-(t)t}$ with

$$W_{t_-(t)t} := W_{(\tan(\frac{\pi}{2}\frac{t-t_-(t)}{t_+(t)-t_-(t)}),\alpha)},$$

we see that $\mathfrak{K}(W_t) = \mathfrak{K}(W_{t_-(t)}) \oplus W_{t_-(t)}\mathfrak{K}(W_{t_-(t)t})$. Hence, $\mu(t) := \operatorname{ind}_-\mathfrak{K}(W_t) = \mu(t_-(t))$.

We have defined a matrix chain $(W_t)_{t\in D^c}$ and we verified *(i)*, *(ii)* and *(iv)* completely and *(iii)* for those matrices $W_{st},\ s,t \in D^c, s<t$ which are already defined. Assume now that $s<t,\ s,t\in D^c$ and that W_{st} is not yet defined. Then not both of s,t belong to M_{reg}. Put $W_{st} := W_s^{-1}W_t$, then we have to show that $W_{st} \in \mathcal{M}_{\mu(t)-\mu(s)}$. If $s \in [t_-(t), t_+(t)]$ or $t \in [t_-(s), t_+(s)]$, the assertion follows from our definitions by elementary calculations. We can therefore assume that $s \notin [t_-(t), t_+(t)]$ and $t \notin [t_-(s), t_+(s)]$. For notational simplicity we set $t_-(r) = t_+(r) = r$ if $r \in M_{reg}$.

Now we reduce the problem to the case that $t \in M_{reg}$. If $t_-(t) \in D^c$ then we can write $W_{st} = W_{st_-(t)}W_{t_-(t)t}$, and hence

$$\operatorname{ind}_-\mathfrak{K}(W_{st}) \leq \operatorname{ind}_-\mathfrak{K}(W_{st_-(t)}) + \operatorname{ind}_-\mathfrak{K}(W_{t_-(t)t}).$$

As, by the above considerations, $\operatorname{ind}_-\mathfrak{K}(W_{t_-(t)t}) = \mu(t) - \mu(t_-(t))$, once we have shown $\operatorname{ind}_-\mathfrak{K}(W_{st_-(t)}) = \mu(t_-(t)) - \mu(s)$ we will get $\operatorname{ind}_-\mathfrak{K}(W_{st}) \leq \mu(t) - \mu(s)$. Here in fact equality will hold, since otherwise we would obtain $\operatorname{ind}_-\mathfrak{K}(W_t) \leq \operatorname{ind}_-\mathfrak{K}(W_s) + \operatorname{ind}_-\mathfrak{K}(W_{st}) < \mu(t)$. If $t_-(t) \notin D^c$, then $t_+(t) \in D^c$, and we write W_{st} as $W_{st_+(t)}W_{tt_+(t)}^{-1}$. It follows from Lemma 7.3 that

$$W_{st_+(t)}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

is contained in $\mathfrak{K}(W_{st_+(t)})$ for some $\alpha \in [0,\pi)$ as a neutral element. The same arguments that were applied when we considered (7.8) yield $\operatorname{ind}_-\mathfrak{K}(W_{st}) = \operatorname{ind}_-\mathfrak{K}(W_{st_+(t)})$. As $\mu(t) = \mu(t_+(t))$ it is enough to show that $\operatorname{ind}_-\mathfrak{K}(W_{st_+(t)}) = \mu(t_+(t)) - \mu(s)$.

So it is left to show that $\operatorname{ind}_-\mathfrak{K}(W_{st}) = \mu(t) - \mu(s)$ if $t \in M_{reg}$. If $t_+(s) \in D^c$ then the assertion follows in the same way as when we considered above the case $t_-(t) \in D^c$. If $t_+(t) \notin D^c$, then $t_-(s) \in D^c$, and we write $W_{st} = W_{t_-(s)s}^{-1}W_{t_-(s)t}$. By Lemma 7.6 and by the definition of $W_{t_-(s)s}$ the space $\mathfrak{K}(W_{t_-(s)t})$ contains the same constant function as $\mathfrak{K}(W_{t_-(s)s}^{-1})$. In $\mathfrak{K}(W_{t_-(s)t})$ this function is neutral (cf. Lemma 7.6), and the space $\mathfrak{K}(W_{t_-(s)s}^{-1})$ is spanned by this function and it has one negative square. The same reasoning as when we considered (7.8) yields the assertion. The proof for *(iii)* is complete.

Now we will prove *(v)*. The fact that $\mathfrak{t}(W_t)$ increases monotonically follows from $W_t'(0)J = W_s'(0)J + W_{st}'(0)J$ and from the fact that $W_{st} \in \mathcal{M}_0^1$, if s and t belong to the same component of D^c. The continuity follows from $W_{st} \in \mathcal{M}_0^1$, if s and t belong to the same component of D^c, and from the integral equation which is satisfied by W_{st}, when s is fixed and $t>s$ (cf. [dB7]).

Let $t_0 \in D$ and assume that D^c accumulates at t_0 from below. If there is a largest number $t \in D^c \cap M_{reg}$ which is smaller than t_0 then $t = t_-(t_0)$ and it follows from our definitions that $\lim_{s \nearrow t_0} \mathfrak{t}(W_s) = +\infty$. Otherwise, there is a monotonically increasing sequence $t_n \in D^c \cap M_{reg},\ n \in \mathbb{N}$, which is contained in the same component of D^c and converges to t_0. We assume on the contrary that $\mathfrak{t}(W_{t_n})$ is bounded from above. By [dB7] the matrix functions $W_{t_1 t_n}(z)$ are uniformly bounded on compact sets. Hence, there is a subsequence of $W_{t_1 t_n}$ which converges uniformly on compact sets to an entire matrix function $W_{t_1 \infty} \in \mathcal{M}_0^1$ with real entries. With no loss of generality we denote this subsequence again by W_{t_n}.

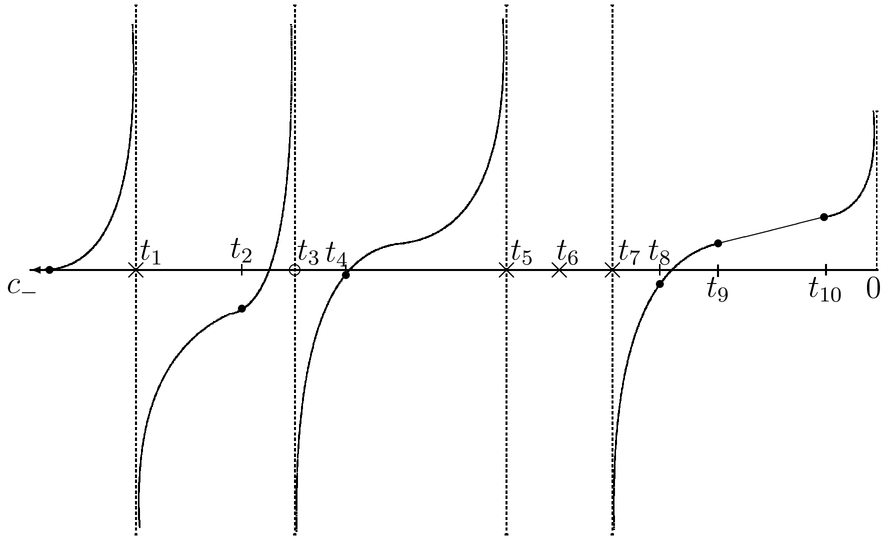
Now we set $W_{t_k \infty} = W_{t_1 t_k}^{-1} W_{t_1 \infty}$ and $W_\infty := W_{t_1} W_{t_1 \infty}$, and see $W_{t_k \infty} = \lim_{n \to \infty} W_{t_k t_n} \in \mathcal{M}_0^1,\ W_\infty \in \mathcal{M}_\nu^1,\ \nu \leq \mu(t_1)$. As $W_{t_1 t_n}$ converges, so does $W_{t_n 0} = W_{t_1 t_n}^{-1} W_{t_1 0} \in \mathcal{M}_{\kappa - \mu(t_1)}^1$. Its limit $W_{\infty 0}$ satisfies $W_{\infty 0} = W_{t_1 \infty}{}^{-1} W_{t_1 0} \in \mathcal{M}_\lambda^1,\ \lambda \leq \kappa - \mu(t_1)$. As $W_\infty W_{\infty 0} = W$, we obtain $\nu = \mu(t_1),\ \lambda = \kappa - \mu(t_1)$. Moreover, since $(w_{21}^\infty, w_{22}^\infty) = (A_{t_1}, B_{t_1}) W_{t_1 \infty},\ (w_{21}^\infty, w_{22}^\infty) W_{\infty 0} = (A, B)$, when $W_\infty(z) = (w_{i,j}^\infty(z))_{i,j=1,2}$, a similar argument shows $\operatorname{ind}_- \mathfrak{P}(w_{21}^\infty - i w_{22}^\infty) = \mu(t_1)$. Since $\mathfrak{P}(E_{t_n})$ is contained isometrically in $\mathfrak{P}(E)$, it follows from Corollary 5.15 and Theorem I.12.2 that $\mathfrak{P}(E_{t_n}),\ n \in \mathbb{N}$ is contained isometrically in $\mathfrak{P}(w_{21}^\infty - i w_{22}^\infty)$. If $\mathfrak{P}(w_{21}^\infty - i w_{22}^\infty)$ is contained isometrically in $\mathfrak{P}(E)$, we have obtained a contradiction to our assumption that there is no largest number $t \in D^c \cap M_{reg}$ smaller than t_0. If $\mathfrak{P}(w_{21}^\infty - i w_{22}^\infty)$ is not contained isometrically in $\mathfrak{P}(E)$ then it follows from Proposition I.13.5 that there is a dB-space $\mathfrak{P}(w_{21}^\infty - i w_{22}^\infty)^l$ which is contained isometrically in both, $\mathfrak{P}(E)$ and $\mathfrak{P}(w_{21}^\infty - i w_{22}^\infty)$, and has codimension one in the latter space. Since then $\mathfrak{P}(E_{t_1}) \subseteq \mathfrak{P}(w_{21}^\infty - i w_{22}^\infty)^l \subseteq \mathfrak{P}(w_{21}^\infty - i w_{22}^\infty)$ as Pontryagin spaces, we obtain that $\mathfrak{P}(w_{21}^\infty - i w_{22}^\infty)^l$ is nondegenerated, and we arrive at a contradiction again. So $\mathfrak{t}(W_t)$ cannot remain bounded when t approaches t_0, i.e. $\lim_{t \nearrow t_0} \mathfrak{t}(W_t) = +\infty$. If $t_0 \in D$ and D^c accumulates at t_0 from above, a similar argument show that $\lim_{t \searrow t_0} \mathfrak{t}(W_t) = -\infty$.

The uniqueness statement follows immediately from *(vi)*, hence it remains to show *(vi)*. So let $W = M_1 M_2$, where $M_1 = (m_{ij})_{i,j=1,2} \in \mathcal{M}_{\nu_1}$ and $M_2 \in \mathcal{M}_{\nu_2}$ are real and entire matrix functions such that $\kappa = \nu_1 + \nu_2$. Since $(A, B) = (m_{21}, m_{22}) M_2$, a comparison of the negative squares of the respective kernels yields $\mathfrak{P}(m_{21} - i m_{22}) \in \nu_1$. It follows from Proposition I.13.5 and its proof that there is a nondegenerated dB-space $\mathfrak{P}$ which is contained isometrically in $\mathfrak{P}(E)$ and which has the same number of negative squares as $\mathfrak{P}(m_{21} - i m_{22})$. Hence, there is a number $t \in D^c$ such that $\mathfrak{P}(E_t) = \mathfrak{P}$, and in particular, $\mu(t) = \nu_1$. Now *(vi)* immediately follows from *(v)* and Theorem I.13.1.

□

Remark 7.7. The definition of matrices W_t for $t \in M_{sing}$ is arbitrary, since there exist no isometrically contained spaces dB-spaces for such t. By our choice of W_t we ensure the maximality property *(vi)* and the uniqueness of the chain $(W_t)_{t \in D^c}$.

The phenomena occuring at the points of increase of $\mu(t)$ can be visualized by drawing the function $\mathfrak{t}(W_t)$. In the example shown in the following picture the point c_- may be $-\infty$, the points $t = t_1, t_5, t_6, t_7$ are those with $\operatorname{ind}_- \mathfrak{P}_t \neq 0$, the set D equals $\{t_1, t_3, t_5, t_6, t_7\}$, $\tilde{D} = D \cup (t_5, t_6) \cup (t_6, t_7)$, the interval (t_2, t_4) is indivisable with negative weight, (t_7, t_8) is indivisable, and (t_9, t_{10}) is indivisable with positive weight.



In the following lemma we study the behaviour of the above chain at a right endpoint of an indivisable interval.

Lemma 7.8. *Let $\mathfrak{P} = \mathfrak{P}(E)$, $E(0) = 1$, be a dB-Pontryagin space, $1 \in \operatorname{Ass}\mathfrak{P}(E)$, assume that E is not a polynomial and that $\overline{\operatorname{dom}\mathcal{S}} \neq \mathfrak{P}(E)$. If $S_\alpha = \cos\alpha A(z) + \sin\alpha B(z) \in \mathfrak{P}(E)$ and is not neutral, then the transfer matrix of $\overline{\operatorname{dom}\mathcal{S}}$ to $\mathfrak{P}(E)$ is $W_{(l,\alpha)}$ with*

$$l = \frac{1}{[S_\alpha, S_\alpha]}.$$

If S_α is neutral and satisfies $S_\alpha \in \operatorname{dom}(\mathcal{S}^{n-1}) \setminus \overline{\operatorname{dom}(\mathcal{S}^n)}$, which means that $\mathfrak{P}(E)$ has the chain

$$\mathfrak{P}_n = \overline{\operatorname{dom}\mathcal{S}^n} \subsetneq \overline{\operatorname{dom}\mathcal{S}^{n-1}} \subsetneq \ldots \subsetneq \mathfrak{P}(E)$$

of dB-subspaces where $n \geq 2$ is such that the space $\mathfrak{P}_n$ is the largest proper nondegenerated dB-subspace of $\mathfrak{P}$, $\mathfrak{P}_n = \mathfrak{P}(E_n)$, $E_n(0) = 1$, then the transfer matrix W of $\mathfrak{P}_n$ to $\mathfrak{P}$ has polynomial entries. Let W_0 and W_n, $\mathfrak{K}_-(W_0) = \mathfrak{K}(W_0)$, $\mathfrak{K}_-(W_n) = \mathfrak{K}(W_n)$, $W_0(0) = W_n(0) = 1$, be the 1-resolvent matrices of $\mathfrak{P}(E)$ and $\mathfrak{P}(E_n)$, respectively. For $k = 0, \ldots, n$ the relation

$$\overline{\operatorname{dom}(\mathcal{S}^k)} = \pi_-(W_n\{\begin{pmatrix} F_+ \\ F_- \end{pmatrix} \in \mathfrak{K}(W) | \deg F_\pm < n-k\})[\dot{+}]\mathfrak{P}(E_n) \tag{7.9}$$

holds. The constant $\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$ is contained in $\mathfrak{K}(W)$, is neutral and satisfies

$$W\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}. \tag{7.10}$$

Proof : First assume that $\overline{\operatorname{dom}\mathcal{S}}$ is nondegenerated and let W be the corresponding transfer matrix. By Corollary I.12.3 the space $\mathfrak{K}(W)$ is one-dimensional, hence by the results of [dB7] and Lemma 7.2 equal to $W_{(l,\alpha)}$.

Now consider the situation where S_α is neutral. The fact that $\mathfrak{P}_n$ is nondegenerated follows from Lemma 5.19. Again by Corollary I.12.3 the space $\mathfrak{K}(W)$ is finite dimensional, hence W is a polynomial. Since $\mathfrak{K}(W)$ contains exactly one (linearly independent) constant, the operator $\mathcal{R}_1(w)$ (cf. Proposition I.8.3) has exactly one Jordan chain at 0 which has length n

$$C_0 := 0, C_1, \dots, C_n,\ \mathcal{R}_1(w)C_k = C_{k-1},\ k = 1, \dots, n.$$

Clearly C_k is a polynomial vector and the maximum degree of its components is $k-1$.

Let $U \in \operatorname{Ass}\mathfrak{P}(E_n)$ and let $M_n \in \mathcal{M}_\nu^U$, $M_0 \in \mathcal{M}_\kappa^U$ be U-resolvent matrices as in Proposition I.10.3. Choose M_n and M_0 such that $\mathfrak{K}_-(M_0) = \mathfrak{K}(M_0)$, $\mathfrak{K}_-(M_n) = \mathfrak{K}(M_n)$, $M_0(0) = M_n(0) = 1$. Since by Theorem 0.7.4 the transfer matrix does not depend on the choice of U, we have $M_0 = M_n W$. Moreover,

$$\mathfrak{K}(M_0) = M_n\mathfrak{K}(W)[\dot{+}]\mathfrak{K}(M_n)$$

and $\pi_-\mathfrak{K}(M_n) = \mathfrak{P}(E_n)$.

Recall that for any matrix function with $M(z)JM(\overline{z})^* = U(z)JU^\#(z)$ the relation

$$M(z)\mathcal{R}_1(w)\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} = \mathcal{R}_U(w)M(z)\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} +$$

$$+\frac{M(z)JM(\overline{w})^* - U(z)JU^\#(w)}{z-w}\frac{JM(w)}{U(w)U^\#(w)}\begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix} \tag{7.11}$$

holds (see Theorem I.12.2). Consider in particular the matrix M_n and the element $S_\alpha \in \operatorname{Ass}\mathfrak{P}(E_n)$. The relation (7.11) applied $n-1$ times and the fact that $(A_n, B_n)\mathfrak{K}(W) \cap \mathfrak{P}(E_n) = \{0\}$ then shows that

$$\pi_-(M_n(z)C_n(z)) \notin \overline{\operatorname{dom}\mathcal{S}}.$$

Hence, again by (7.11), we obtain

$$\pi_-(M_n(z)C_{n-k}(z)) \in \overline{\operatorname{dom}(\mathcal{S}^k)} \setminus \overline{\operatorname{dom}(\mathcal{S}^{k+1})},\ k = 0, \dots, n-1,$$

since for such k we have $\mathcal{R}_{S_\alpha}(w)\overline{\operatorname{dom}(\mathcal{S}^k)} = \overline{\operatorname{dom}(\mathcal{S}^{k+1})}$. Since $(0,1)M_n = (0,1)W_n$ the relation (7.9) follows.

Corollary 7.4 implies that $W_0\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \in \mathfrak{K}(W_0)$. Clearly $W_0\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \perp \mathfrak{K}(W_n)$, hence for some $\begin{pmatrix} F_+ \\ F_- \end{pmatrix} \in \mathfrak{K}(W)$

$$W_0\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = W_n\begin{pmatrix} F_+ \\ F_- \end{pmatrix}. \tag{7.12}$$

Since $\pi_- W_0\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \in \overline{\operatorname{dom}(\mathcal{S}^{n-1})}$, the relation (7.9) implies that $\begin{pmatrix} F_+ \\ F_- \end{pmatrix}$ is constant and, since $W(0) = 1$, equal to $\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$. Moreover,

$$[\begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \begin{pmatrix} F_+ \\ F_- \end{pmatrix}]_{\mathfrak{K}(W)} = [W_n\begin{pmatrix} F_+ \\ F_- \end{pmatrix}, W_n\begin{pmatrix} F_+ \\ F_- \end{pmatrix}]_{\mathfrak{K}(W_0)} = [S_\alpha, S_\alpha]_{\mathfrak{P}(E)} = 0.$$

The relation (7.10) follows immediately from (7.12).

□

Recall from [dB7], [W1] that for any function $\tau \in \mathcal{N}_0$ there exists a unique chain $(W_{0t})_{t\geq 0}$ of real entire matrices $W_{0t} \in \mathcal{M}_0^1$, $W_{0t}(0) = 1$ and $\mathfrak{t}(W_{0t}) = t$, such that for any function $\theta \in \mathcal{N}_0$ the relation

$$\lim_{t\to\infty} (W_{0t} \circ \theta)(z) = \tau(z)$$

holds. We have seen in Theorem 7.1 that a matrix $W \in \mathcal{M}_\kappa^1$ determines a chain of matrices which goes downwards. In the following we show how these chains can be linked to a chain $(\tilde{W}_t)$ on an interval (c_-, c_+) such that for each $s \in (c_-, c_+)$ the chain $(\tilde{W}_t)_{c_- < t \leq s}$ conincides (up to reparametrization) with the chain constructed in Theorem 7.1 for $W = \tilde{W}_s$. Moreover,

$$\lim_{t\to c_+} \mathfrak{t}(W_t) = \infty.$$

We construct matrices $\tilde{W}_t$ from the given chains $(W_t)_{t\in D^c}$ and $(W_{0t})_{t\geq 0}$. By a similar reasoning as in the beginning of the proof of Theorem 7.1 we can assume that $\mathfrak{K}_-(W) = \mathfrak{K}(W)$. We also use the notation introducede in the proof of Theorem 7.1. First consider the case that 0 is not at the same time right endpoint of an indivisable interval of type α, in the chain $(W_t)_{t\in D^c}$ and left endpoint of an indivisable interval of the same type α in $(W_{0t})_{t\geq 0}$. Then define $\tilde{D}^c = D^c \cup [0, \infty)$ and

$$\tilde{W}_t := \begin{cases} W_t \ , \ t \in D^c \\ W_0 W_{0t} \ , \ t \geq 0 \end{cases} . \qquad (7.13)$$

It follows from Theorem I.12.2 applied to $\mathfrak{P}(E)$ that $\operatorname{ind}_-\mathfrak{K}(\tilde{W}_t) = \operatorname{ind}_-\mathfrak{K}(W) = \mu(0)$, $t \geq 0$.

Now assume that 0 is right endpoint of an indivisable interval $(t_-, 0)$, $t_- \in M_{reg}$, and left endpoint of an indivisable interval $(0, s_+)$, $0 < s_+ \leq \infty$ of length s_+ and of the same type α. The fact $s_+ = \infty$ just means that $W_{0s} = W_{(s,\alpha)}$ for all $s > 0$. If $(t_-, 0)$ has positive weight, i.e. $t_- \in D^c$ and $\mu(t_-) = \mu(0)$, we define $\tilde{D}^c = D^c \cup [0, \infty)$ and $\tilde{W}_t$ as in (7.13). If $(t_-, 0)$ has negative weight, i.e. $t_- \in D^c$ and $\mu(t_-) = \mu(0) - 1$, we have $W_{t_-0} = W_{(l,\alpha)}$, where $-\infty < l < 0$. If $l + s_+ = 0$, cancel the interval (t_-, s_+) and proceed linking the remaining chains. If $s_+ < \infty$, $l + s_+ > 0$ then set $\tilde{D}^c = D^c \cup [t_-, \infty)$ and

$$\tilde{W}_t := \begin{cases} W_t \ , \ t \in D^c \cap (-\infty, t_-] \\ W_{t_-} W_{((l+s_+)\frac{t-t_-}{s_+-t_-},\alpha)} \ , \ t \in [t_-, s_+] \\ W_0 W_{0t} \ , \ t \geq s_+ \end{cases} . \qquad (7.14)$$

If $s_+ = \infty$, we set $\tilde{D}^c = D^c \cup [t_-, \infty)$ and $\tilde{W}_t := W_t$, $t \in D^c \cap (-\infty, t_-]$ and $\tilde{W}_t = W_{t_-} W_{(t-t_-,\alpha)}$, $t \geq t_-$. In all above cases it follows from Theorem I.12.2 applied to $\mathfrak{P}(E_{t_-})$ that $\operatorname{ind}_-\mathfrak{K}(\tilde{W}_t) = \operatorname{ind}_-\mathfrak{K}(W) = \mu(t_-)$, $t \geq t_-$.

If $s_+ < \infty$, $l + s_+ < 0$ then set $r_0 := \frac{t_- + s_+}{2}$ and $\tilde{D}^c := (D^c \cup [t_-, \infty)) \setminus \{r_0\}$:

$$\tilde{W}_t := \begin{cases} W_t \ , \ t \in D^c \cap (-\infty, t_-] \\ W_{t_-} W_{(\tan(\frac{\pi}{2}\frac{t-t_-}{r_0-t_-}),\alpha)} \ , \ t \in [t_-, r_0) \\ W_{t_-} W_{(l+s_+-\tan(\frac{\pi}{2}\frac{s_+-t}{s_+-r_0}),\alpha)} \ , \ t \in (r_0, s_+] \\ W_0 W_{0t} \ , \ t \geq s_+ \end{cases} . \qquad (7.15)$$

It follows from Theorem I.12.2 applied to $\mathfrak{P}(E_{t_-})$ that $\text{ind}_-\mathfrak{K}(\tilde{W}_t) = \text{ind}_-\mathfrak{K}(W) = \mu(0), t > r_0$, and $\text{ind}_-\mathfrak{K}(\tilde{W}_t) = \mu(t_-),\ t \in [t_-, r_0)$.

If $t_- \notin D^c$, i.e. $\text{ind}_0 \mathfrak{P}_{t_-} \neq 0$, we devide two cases. If $s_+ < \infty$, we define $\tilde{D}^c = D^c \cup [0, \infty)$ and $\tilde{W}_t$ as in (7.13). Since, in this case the function

$$W(z) \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

belongs to $\mathfrak{K}(W)$ and is neutral in this space (cf. Corollary 7.4), and since $(\cos\alpha, \sin\alpha)^T$ spans the Hilbert space $\mathfrak{K}(W_{0t})$ for $0 < t \leq s_+$, we obtain from [ADSR] that $\text{ind}_-\mathfrak{K}(\tilde{W}_t) = \mu(0),\ t > 0$. If $s_+ = \infty$, we cut off the chain $(W_t)_{t \in D^c}$ at the point $t_0 := \sup\{t \in D^c | t < t_-\}$:

$$\tilde{W}_t := W_t,\ t \in D^c \cap (-\infty, t_0) =: \tilde{D}^c.$$

In any case we easily see that $\text{ind}_-\mathfrak{K}(\tilde{W}_{st}) = \text{ind}_-\mathfrak{K}(\tilde{W}_t) - \text{ind}_-\mathfrak{K}(\tilde{W}_s),\ s, t \in \tilde{D}^c,\ s < t$. It follows from Theorem I.13.1 and Theorem 7.1 that for each $s \in \tilde{D}^c$ the chain $(\tilde{W}_t)_{t \leq s,\ t \in \tilde{D}^c}$ conincides (up to reparametrization) with the chain constructed in Theorem 7.1 for $W = \tilde{W}_s$. We proved the following

Lemma 7.9. *Let $(\tilde{W}_t)$ be the chain constructed above and let $s \in \tilde{D}^c$. The chain $(\tilde{W}_t)_{t \leq s, t \in \tilde{D}^c}$ satisfies (i)-(vi) of Theorem 7.1 with $W = \tilde{W}_s$. Moreover, we have with $c_+ := \sup \tilde{D}^c$*

$$\lim_{t \to c_+} \mathfrak{t}(\tilde{W}_t) = \infty.$$

Remark 7.10. Note that the linked chain $(\tilde{W})$ need not contain the matrix W we started with, e.g. whenever $W \circ \tau \in \mathcal{N}_{\kappa'}$ with $\kappa' < \kappa$. If $\kappa' = \kappa$, then W is not contained in the chain $(\tilde{W}_t)$ if and only if for some $s'_+ > 0$ and $\alpha \in [0, \pi)$, we have $W \circ (W_{(s'_+, \alpha)} \circ \tau) \in \mathcal{N}_{\kappa''},\ \kappa'' < \kappa$.

8 Weyl coefficients of matrix chains

In this section we consider truncated chains of matrices.

Definition 8.1. Denote by $\mathfrak{C}$ the set of all matrix chains $(W_t)_{t > t_0}$ with the properties

(i) W_t has real entire entries and $W_t(0) = 1,\ W_t \neq 1$.

(ii) For some $\kappa \in \mathbb{N} \cup \{0\}$ we have $W_t \in \mathcal{M}^1_\kappa,\ t > t_0$.

(iii) $\mathfrak{t}(W_t) = t$.

(iv) For $t_0 < s < t$ there exists a matrix $W_{st} \in \mathcal{M}^1_0$, such that $W_t = W_s W_{st}$.

Thereby chains $(W_t)_{t > t_0}$ and $(\tilde{W}_t)_{t > \tilde{t}_0}$ which differ only up to a certain point, i.e. satisfy $W_t = \tilde{W}_t$ for $t \geq t_1$, are identified. The number κ is called the index of negativity of the chain $(W_t)_{t > t_0}$.

Note that by Theorem 7.1 two equivalent chains differ only in their lenght. From a chain contained in $\mathfrak{C}$ we construct its so called Weyl coefficient. Recall that for notational convenience the function $\tau(z) \equiv \infty$ belongs to $\mathcal{N}_0$.

Lemma 8.2. *Let $(W_t)_{t>t_0} \in \mathfrak{C}$ be given and let κ be its index of negativity. For any family $(\tau^t)_{t>t_0}$, $\tau^t \in \mathcal{N}_0$, the limit*

$$\lim_{t\to\infty} (W_t \circ \tau^t)(z) =: q(z) \tag{8.1}$$

exists in $\mathbb{C}\cup\{\infty\}$ locally uniformly on $\mathbb{C}\setminus\mathbb{R}$ if we provide $\mathbb{C}\cup\{\infty\}$ with the spherical metric. It is independent from the choice of $(\tau^t)_{t>t_0}$. Moreover, $q \in \mathcal{N}_{\kappa'}$ for some $\kappa' \leq \kappa$.

For each $t > t_0$ there exists a unique parameter $\tau_t \in \mathcal{N}_0$, such that

$$(W_t \circ \tau_t)(z) = q(z). \tag{8.2}$$

Proof : If $\kappa = 0$ the assertion of the lemma is the well known fact that $\mathfrak{t}(W_t) \to \infty$ implies the so called limit point case (compare e.g. [W1], [W3], [dB7]). If $\kappa > 0$ choose any number $t_1 > t_0$ and consider the chain $(\tilde{W}_t)_{t>t_1}$ where $\tilde{W}_t := W_{t_1t}$. By the above remark

$$\lim_{t\to\infty} (\tilde{W}_t \circ \tau^t)(z) =: q_{t_1}(z)$$

exists locally uniformly on $\mathbb{C}\setminus\mathbb{R}$ and is contained in $\mathcal{N}_0$. The function q_{t_1} is independent from the functions τ^t (cf. [W3]).

Assume that $q_{t_1} \not\equiv \infty$, and let $z \in \mathbb{C}\setminus\mathbb{R}$ and assume first that $(0,1)W_{t_1}\begin{pmatrix} q_{t_1}(z) \\ 1 \end{pmatrix} = 0$. Since $\det W_{t_1}(z) = 1$, we have $(1,0)W_{t_1}\begin{pmatrix} q_{t_1}(z) \\ 1 \end{pmatrix} \neq 0$, hence

$$\lim_{t\to\infty} (W_{t_1} \circ (\tilde{W}_t \circ \tau^t))(z) = \infty = (W_{t_1} \circ q_{t_1})(z).$$

Now let $z \in \mathbb{C}\setminus\mathbb{R}$ be such that $(0,1)W_{t_1}\begin{pmatrix} q_{t_1}(z) \\ 1 \end{pmatrix} \neq 0$, then clearly $\lim_{t\to\infty}(W_{t_1} \circ (\tilde{W}_t \circ \tau^t))(z) = (W_{t_1} \circ q_{t_1})(z)$. We see that $q(z) \in \mathbb{C}\cup\{\infty\}$ exists for all $z \in \mathbb{C}\setminus\mathbb{R}$. If $q_{t_1} \equiv \infty$ similar arguments show the existence of $q(z)$.

By the above considerations we also see that the convergence takes place locally uniformly on $\{z \in \mathbb{C}\setminus\mathbb{R} | W_{t_1} \circ q_{t_1} \neq \infty\}$. By the same reasoning we see that $(JW_t)\circ\tau^t = \frac{-1}{W_t\circ\tau^t}$ converges locally uniformly on $\{z \in \mathbb{C}\setminus\mathbb{R} | (JW_{t_1}) \circ q_{t_1} \neq \infty\}$ to $\frac{-1}{q(z)}$. Since the union these two sets is $\mathbb{C}\setminus\mathbb{R}$ we see the limit $q(z)$ exists locally uniformly on $\mathbb{C}\setminus\mathbb{R}$ when $\mathbb{C}\cup\{\infty\}$ is provided with the spherical metric.

Now an elementry consideration using property *(iv)* shows that $q(z)$ is independent from t_1 and that the relation (8.2) holds.

□

The function q in (8.1) is called the Weyl coefficient of the chain $(W_t)_{t>t_0}$. Clearly two chains which are equivalent in the sense of Definition 8.1 have the same Weyl coefficient.

Remark 8.3. The importance of the condition *(iii)* of Definition 8.1 is that it ensures $\lim_{t\to\infty} \mathfrak{t}(W_t) = \infty$. Since reparametrizations of chains with continuous increasing functions $t^\bullet(t)$ do not change the behaviour of the limit (8.1), we could also work with chains $(W_t)_{t_0<t<c_+}$ and assume instead of *(iii)* that $\lim_{t\to c_+} \mathfrak{t}(W_t) = \infty$.

Note that the existence and independence of the limit (8.1) of $\tau^t \in \mathcal{N}_0$ is even equivalent to the fact $\mathfrak{t}(W_t) \to \infty$. This is seen as follows: If $\lim_{t\to c_+} \mathfrak{t}(W_t) < \infty$, then by [dB7] the chain $(W_t)_{t_0<t<c_+}$ can be continuously extended to c_+. Then, even if $\tau^t = \tau \in \mathcal{N}_0$,

$$\lim_{t\to c_+} W_t \circ \tau = W_{c_+} \circ \tau$$

cleary depends on τ.

It turns out to be useful to note that there exists a maximal chain $(W_t)_{t>c_-}$ in each equivalence class of $\mathfrak{C}$.

Lemma 8.4. *Let $(W_t)_{t>t_0} \in \mathfrak{C}$, then $(W_t)_{t>t_0}$ can be continued to a maximal chain $(W_t')_{t>c_-}$ which is in the same equivalence class as $(W_t)_{t>t_0}$. If the negativity index κ of $(W_t)_{t>t_0}$ is zero, then $c_- = 0$. Otherwise, $c_- = -\infty$.*

Proof : If $\kappa = 0$ the assertion is well known (cf. [dB7], [W3]). For $\kappa > 0$ the assertion follows immediately from Theorem 7.1.

□

Lemma 8.5. *Let a chain $(W_t)_{t>c_-} \in \mathfrak{C}$ be given, denote by κ its index of negativity, assume that $\kappa > 0$ and let q be its Weyl coefficient. Then $q \in \mathcal{N}_{\kappa'}$ with $\kappa' < \kappa$ if and only if the matrix $W_{t_1 t_2}$ is linear for all $t_2 > t_1 > c_-$.*

Proof : Assume first that there exists a matrix $W_{t_1 t_2}$ which is not linear. Consider the chain going downwards from W_{t_2}. Since the Weyl coefficient of $(-JW_tJ)_{t>c_-}$ is $\frac{-1}{q}$, we can assume that $\mathfrak{K}_-(W_{t_2}) = \mathfrak{K}(W_{t_2})$. By Corollary 5.14 we have $\mathfrak{K}_-(W_t) = \mathfrak{K}(W_t)$ for $t \leq t_2$. We use again the notation from the proof of Theorem 7.1. The interval (t_1, t_2) is not contained in M_{sing}, hence there exists a space $\mathfrak{P}(E_t)$, $t \in (t_1, t_2)$ which is a proper isometrically contained subspace of $\mathfrak{P}(E_{t_2})$ (cf. Theorem I.12.2 and Corollary 5.15). In particular, for the operator $\mathcal{S} \subseteq \mathfrak{P}(E_{t_2})^2$, we have $\mathfrak{P}(E_t) \subseteq \overline{\operatorname{dom} \mathcal{S}}$. Since $W_{t_2} \circ \tau_t$ is a generalized 1-resolvent of $\mathcal{S}$, Lemma 5.17 shows that $q \in \mathcal{N}_\kappa$.

Assume now that all matrices $W_{t_1 t_2}$, $t_2 > t_1 > c_-$, are linear, then we have for some $\alpha \in [0, \pi)$

$$W_{t_1 t_2} = W_{(t_2 - t_1, \alpha)}.$$

Choose $t_0 > c_-$ and consider the chain $(\tilde{W}_t)_{t\in D^c}$ going downwards from W_{t_0} constructed in Theorem 7.1. Clearly, $W_t = \tilde{W}_{t^\bullet(t)}$, where $t^\bullet$ is continuous and monotone function from $(-\infty, t_0]$ into D^c with $t^\bullet(t_0) = 0$. We assume again $\mathfrak{K}_-(W_{t_0}) = \mathfrak{K}(W_{t_0})$. It follows that for $t_- := \sup\{t \in M_{reg} | t \leq \lim_{t\searrow-\infty} t^\bullet(t)\}$ we have $(t_-, 0) \subseteq M_{sing}$, and either $\operatorname{ind}_0 \mathfrak{P}_{t_-} \neq 0$ or $(t_-, 0)$ is the right half of an indivisable interval with negative weight. It follows from the construction of $(\tilde{W}_t)$ that $S_\alpha = \cos\alpha A_{W_{t_0}} + \sin\alpha B_{W_{t_0}} \in \mathfrak{P}(E_{W_{t_0}})$. Moreover, $[S_\alpha, S_\alpha] \leq 0$ by Lemma 5.19. Since $q(z) = W_{t_0} \circ (\lim_{t\nearrow\infty} W_{t_0 t} \circ \tau) = W_{t_0} \circ \cot\alpha$, it follows from Lemma

5.12 that $q \in \mathcal{N}_{\kappa'}$ with $\kappa' < \kappa$.

□

Note the following result:

Proposition 8.6. *Let $W \in \mathcal{M}^1_\kappa$, $W(0) = 1$, and $\tau \in \mathcal{N}_0$ be given. The chain $(\tilde{W}_t)$ constructed in the previous section has the Weyl coefficient $W \circ \tau$.*

Proof : We can assume again $\mathfrak{K}(W) = \mathfrak{K}_-(W)$. Let $(\tilde{W}_t)_{t\in\tilde{D}^c}$ be the chain defined in the end of Section 7. With the same notation as there the assertion is obvious except of the last case: (W_t) ends with an indivisible interval of type α, $t_- \notin D^c$ and $W_{0t} = W_{(t,\alpha)}$, $t > 0$. In particular, $\tau = \cot\alpha$, $\alpha \in [0, \pi)$. By Lemma 5.12 the function $W \circ \tau$ is the regularized 1-resolvent of the selfadjoint extension $\mathcal{A}_\alpha$ induced by the (neutral) element $S_\alpha \in \mathfrak{P}(w_{22}+iw_{21})$. Note that $\mathfrak{P}(w_{22} + iw_{21}) = \mathfrak{P}(w_{21} - iw_{22})$.

Let $n + 1 = \dim \mathfrak{S}_\infty$ and assume first that $S_\alpha \notin \overline{\operatorname{dom}(\mathcal{S}^{n+1})}$. Then $\overline{\operatorname{dom}(\mathcal{S}^{n+1})} = \mathfrak{P}(E_{t_1})$ and the restriction of $\mathcal{A}_\alpha$ to $\mathfrak{P}(E_{t_1})$ is again selfadjoint. By Lemma 5.9 the regularized 1-resolvent of this restriction coincides with $W \circ \alpha$. Since by the construction of (W_t) in Theorem 7.1 we have $W_t = W_{t_1} W_{(l,\alpha)}$, $t \in \tilde{D}^c \cap [t_1, \infty)$ for some $l > 0$, and since $W_{(l,\alpha)} \circ \cot\alpha = \cot\alpha$, we obtain that $W \circ \alpha$ is the Weyl coefficient of $(\tilde{W}_t)_{t\in\tilde{D}^c}$.

Now assume that $S_\alpha \in \overline{\operatorname{dom}(\mathcal{S}^{n+1})} =: \mathfrak{P}_{t_0}$. Since $\lim_{t\nearrow t_0} \mathfrak{t}(W_t) = \infty$ the chain $(W_t)_{t<t_0}$ has a Weyl coefficient $q(z)$. The selfadjoint relation $\mathcal{A}_\alpha \subseteq \mathfrak{P}_0^2$ is an extension of the operator $\mathcal{S}_t \subseteq \mathfrak{P}_t^2$ for all $t < t_0$. If $t < t_0$ is such that $\mathfrak{P}_t$ is nondegenerated, which is the case for t sufficiently near to t_0 (cf. Lemma 5.20, Theorem I.11.6), we have by Lemma 5.9 that the function $W \circ \tau$ is a 1-resolvent of $\mathcal{S}_t$, when 1 is understood as an element of $\mathfrak{P}_{t,-}$. As W_t is a generalized 1-resolvent matrix of $\mathcal{S}_t$, there exists a parameter $\tau_t \in \mathcal{N}_\nu$ such that $W \circ \tau = W_t \circ \tau_t$. The number ν is the negative index of the extending space of the $\mathfrak{P}_t$-minimal part of $\mathcal{A}_\alpha$ (cf. [KW2]). Since the resolvent of $\mathcal{A}_\alpha$ leaves $\mathfrak{P}_{t_0}$ invariant, the mentioned $\mathfrak{P}_t$-minimal part of $\mathcal{A}_\alpha$ is contained in $\mathfrak{P}_{t_0}$. Now note that, since the union of the spaces $\mathfrak{P}_t$, $t < t_0$, is dense in $\mathfrak{P}_{t_0}$, the extending space of $\mathfrak{P}_t$ is positive if t is sufficiently near to t_0. We conclude that $\tau_t \in \mathcal{N}_0$ for such t, hence

$$W \circ \tau = \lim_{t\nearrow t_0} W_t \circ \tau_t = q(z).$$

□

The main result of this paper is a converse of Lemma 8.2.

Theorem 8.7. *Let $q \in \mathcal{N}_\kappa$ be given. Then there exists a unique chain $(W_t)_{t>t_0} \in \mathfrak{C}$ with index κ of negativity, whose Weyl coefficient is q.*

The proof of Theorem 8.7, which is carried out in Section 11, will use induction on κ. Recall that in the case $\kappa = 0$ the analogue of Theorem 8.7 has been proved in [dB7], [W1]. In order to carry out the induction step we use some transformation formulas of chains, i.e. mappings on the set $\mathfrak{C}$. These transformations are closely related to some results of [W1], [W2]. We also employ a transformation of a Nevanlinna function. These tools will be introduced in the following two sections.

Remark 8.8. The assumption that the negative index of $(W_t)_{t>c_-}$ equals the number of negative squares of q in Theorem 8.7 is necessary in order to ensure uniqueness. This is seen by considering the relational extension of $\mathcal{S} \subseteq \mathfrak{P}(E)^2$ where $\mathfrak{P}(E)$ is such that $\overline{\operatorname{dom} \mathcal{S}}$ is degenerated (compare Lemma 5.12, Remark 7.10 and Lemma 8.5).

9 A transformation of Nevanlinna functions

The aim of this section is to prove the following:

Proposition 9.1. *Assume that the function $q \in \mathcal{N}_\kappa$, $\kappa \geq 0$, is finite at ∞ and let $\alpha \in \mathbb{C}^+ \cup \mathbb{R}$. Then there exist unique numbers $c, d \in \mathbb{R}$, such that*

$$q_1(z) := (z-\alpha)(z-\overline{\alpha})(q(z)+d)+cz$$

is regular at infinity. In fact $d = -\lim_{y\to+\infty} q(iy)$ and $c = -i \lim_{y\to+\infty} y(q(iy)+d)$. The function q_1 is contained in $\mathcal{N}_{\kappa'}$ with $\kappa - 1 \leq \kappa' \leq \kappa$. Thereby $\kappa' = \kappa - 1$ if and only if $\alpha \notin \mathbb{R}$ and q has a pole at α or $\alpha \in \mathbb{R}$ and is a point of negative type for q. If $\kappa > 0$ there exists a choice of α, such that $\kappa' = \kappa - 1$.

The proof of this result is split into several lemmata. We make use of the integral representation of the function q as given in [KL1].

Lemma 9.2. *Let $\alpha \in \mathbb{R}$, $\rho \in \mathbb{N}$, let $\Delta \subseteq \mathbb{R}$ be a finite interval which contains α and let σ be a finite positive measure on Δ. Then we have*

$$(z-\alpha)^2 \int_\Delta \left(\frac{1}{t-z} + \sum_{r=1}^{2\rho} \frac{(t-\alpha)^{r-1}}{(z-\alpha)^r} \right) \frac{(t^2+1)^\rho}{(t-\alpha)^{2\rho}} \, d\sigma(t) =$$

$$= \int_\Delta \left(\frac{1}{t-z} + \sum_{r=1}^{2(\rho-1)} \frac{(t-\alpha)^{r-1}}{(z-\alpha)^r} \right) \frac{(t^2+1)^{\rho-1}}{(t-\alpha)^{2(\rho-1)}} \, d\tilde{\sigma}(t),$$

where $d\tilde{\sigma}(t) = (1+t^2)d\sigma(t)$.

Proof : A computation using (compare [KL1])

$$\left(\frac{1}{t-z} + \sum_{r=1}^{\nu} \frac{(t-\alpha)^{r-1}}{(z-\alpha)^r} \right) \frac{1}{(t-\alpha)^\nu} = \frac{1}{(t-z)(z-\alpha)^\nu} \tag{9.1}$$

will show that

$$(z-\alpha)^2 \left(\frac{1}{t-z} + \sum_{r=1}^{2\rho} \frac{(t-\alpha)^{r-1}}{(z-\alpha)^r} \right) \frac{(t^2+1)^\rho}{(t-\alpha)^{2\rho}} = \frac{(t^2+1)^\rho}{(t-z)(z-\alpha)^{2(\rho-1)}} =$$

$$= \left(\frac{1}{t-z} + \sum_{r=1}^{2(\rho-1)} \frac{(t-\alpha)^{r-1}}{(z-\alpha)^r} \right) \frac{(t^2+1)^{\rho-1}}{(t-\alpha)^{2(\rho-1)}} (1+t^2).$$

□

Lemma 9.3. *Let $\alpha \in \mathbb{C}^+ \cup \mathbb{R}$, $\beta \in \mathbb{R}$, $\alpha \neq \beta$, $\rho \in \mathbb{N}$, let $\Delta \subseteq \mathbb{R}$ be a finite interval which contains β and let σ be a finite positive measure on Δ. Then we have*

$$(z-\alpha)(z-\overline{\alpha}) \int_\Delta \left(\frac{1}{t-z} + \sum_{r=1}^{2\rho} \frac{(t-\beta)^{r-1}}{(z-\beta)^r} \right) \frac{(t^2+1)^\rho}{(t-\beta)^{2\rho}} \, d\sigma(t) =$$

$$= \int_\Delta \left(\frac{1}{t-z} + \sum_{r=1}^{2\rho} \frac{(t-\beta)^{r-1}}{(z-\beta)^r} \right) \frac{(t^2+1)^\rho}{(t-\beta)^{2\rho}} \, d\tilde{\sigma}(t) + \frac{c_1}{(z-\beta)^{2\rho-1}} + \frac{c_0}{(z-\beta)^{2\rho}},$$

where $d\tilde{\sigma}(t) = |t-\alpha|^2 d\sigma(t)$ and $c_0, c_1 \in \mathbb{R}$.

Proof : We use the identities (9.1) and

$$(z-\alpha)(z-\overline{\alpha}) = (z-\beta)(z-\overline{\beta}) + (\beta-\alpha)(z-\overline{\beta}) +$$

$$+ (\overline{\beta}-\overline{\alpha})(z-\beta) + |\beta-\alpha|^2, \tag{9.2}$$

which holds for arbitrary numbers $z, \alpha, \beta \in \mathbb{C}$. Since $\beta \in \mathbb{R}$ it follows that

$$(z-\alpha)(z-\overline{\alpha}) \left(\frac{1}{t-z} + \sum_{r=1}^{2\rho} \frac{(t-\beta)^{r-1}}{(z-\beta)^r} \right) \frac{1}{(t-\beta)^{2\rho}} =$$

$$= \frac{1}{t-z} \left(\frac{1}{(z-\beta)^{2\rho-2}} + \frac{2\,\mathrm{Re}\,(\beta-\alpha)}{(z-\beta)^{2\rho-1}} + \frac{|\beta-\alpha|^2}{(z-\beta)^{2\rho}} \right) =$$

$$= \left(\frac{1}{t-z} + \sum_{r=1}^{2\rho-2} \frac{(t-\beta)^{r-1}}{(z-\beta)^r} \right) \frac{1}{(t-\beta)^{2\rho-2}} +$$

$$+ \left(\frac{1}{t-z} + \sum_{r=1}^{2\rho-1} \frac{(t-\beta)^{r-1}}{(z-\beta)^r} \right) \frac{2\,\mathrm{Re}\,(\beta-\alpha)}{(t-\beta)^{2\rho-1}} +$$

$$+ \left(\frac{1}{t-z} + \sum_{r=1}^{2\rho} \frac{(t-\beta)^{r-1}}{(z-\beta)^r} \right) \frac{|\beta-\alpha|^2}{(t-\beta)^{2\rho}} =$$

$$= \left(\frac{1}{t-z} + \sum_{r=1}^{2\rho} \frac{(t-\beta)^{r-1}}{(z-\beta)^r} \right) \frac{(t-\alpha)(t-\overline{\alpha})}{(t-\beta)^{2\rho}} - \frac{1}{(z-\beta)^{2\rho-1}} -$$

$$- \frac{(t-\beta) + 2\,\mathrm{Re}\,(\beta-\alpha)}{(t-\beta)^{2\rho}}.$$

□

Lemma 9.4. *Let $\alpha \in \mathbb{C}^+ \cup \mathbb{R}$, $\Delta \subseteq \mathbb{R}$ be a finite (possibly empty) interval and let σ be a finite positive measure on $\mathbb{R} \setminus \Delta$. Then we have*

$$(z-\alpha)(z-\overline{\alpha}) \int_{\mathbb{R}\setminus\Delta} \frac{1}{t-z}\, d\sigma(t) =$$

$$= \int_{\mathbb{R}\setminus\Delta} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\tilde{\sigma}(t) + c_1 z + c_0,$$

where $d\tilde{\sigma}(t) = |t-\alpha|^2 d\sigma(t)$ and $c_0, c_1 \in \mathbb{R}$.

Proof : We use the identity

$$\frac{1}{t-z} - \frac{t}{1+t^2} = \frac{1+tz}{t-z} \frac{1}{t^2+1}$$

to find

$$\left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) |t-\alpha|^2 - \frac{(z-\alpha)(z-\overline{\alpha})}{t-z} =$$

$$= \frac{1}{1+t^2} \left(z(1+t^2) + t(1-|\alpha|^2) - 2 \operatorname{Re} \alpha \right).$$

The assertion follows with

$$c_0 = \int_{\mathbb{R}\setminus\Delta} \frac{t(|\alpha|^2-1) + 2\operatorname{Re}\alpha}{1+t^2}\, d\sigma(t),$$

$$c_1 = -\int_{\mathbb{R}\setminus\Delta} d\sigma(t).$$

□

Lemma 9.5. *Let $\alpha, \beta \in \mathbb{C}^+ \cup \mathbb{R}$ and let $p(z)$ be a polynomial of degree $\rho \geq 1$, $p(0) = 0$. Then*

$$(z-\alpha)(z-\overline{\alpha}) \left(p(\frac{1}{z-\beta}) + \overline{p(\frac{1}{\overline{z}-\beta})} \right) =$$

$$= \left(\tilde{p}(\frac{1}{z-\beta}) + \overline{\tilde{p}(\frac{1}{\overline{z}-\beta})} \right) + c_1 z + c_0,$$

where $c_0, c_1 \in \mathbb{R}$ and $\tilde{p}$ is a polynomial with $p(0) = 0$. The degree of $\tilde{p}$ is

(i) at most $\rho - 2$ if $\alpha = \beta$ and $\alpha \in \mathbb{R}$ and equal to $\rho - 2$ if additionally p has real coefficients.

(ii) equal to $\rho - 1$ if $\alpha = \beta$ and $\alpha \notin \mathbb{R}$.

(iii) equal to ρ is $\alpha \neq \beta$.

Proof : Write the polynomial p as

$$p(z) = \sum_{i=1}^{\rho} d_i z^i, \; d_i \in \mathbb{C}.$$

Then we compute

$$(z-\alpha)(z-\overline{\alpha}) \sum_{i=1}^{\rho} \left(d_i \frac{1}{(z-\alpha)^i} + \overline{d_i} \frac{1}{(z-\overline{\alpha})^i} \right) =$$

$$= \sum_{i=1}^{\rho} \left(d_i \frac{z-\overline{\alpha}}{(z-\alpha)^{i-1}} + \overline{d_i} \frac{z-\alpha}{(z-\overline{\alpha})^{i-1}} \right) =$$

$$= \sum_{i=0}^{\rho-1} \left(d_{i+1}(\alpha-\overline{\alpha}) \frac{1}{(z-\alpha)^i} + \overline{d_{i+1}}(\overline{\alpha}-\alpha) \frac{1}{(z-\overline{\alpha})^i} \right) +$$

$$+ \sum_{i=-1}^{\rho-2} \left(d_{i+2} \frac{1}{(z-\alpha)^i} + \overline{d_{i+2}} \frac{1}{(z-\overline{\alpha})^i} \right),$$

and the assertions *(i)* and *(ii)* follow. In order to prove *(iii)* we use the identity (9.2) and a similar computation as above shows that the degree cannot increase. Computing explicitly the leading coefficients we find that the degree actually equals ρ.

□

Proof (of Proposition 9.1): Clearly the existence of the limit $\lim_{y\to+\infty} \frac{1}{y} q_1(iy)$ is equivalent to the existence of $\lim_{y\to+\infty} y q(iy)$. Hence we have to put $d = -\lim_{y\to+\infty} q(iy)$. Then the requirement $\lim_{y\to+\infty} \frac{1}{y} q_1(iy) = 0$ determines c uniquely, in fact $c = -\lim_{y\to+\infty}(iy)(q(iy)+d)$.

Consider the integral representation of q as given in [KL1]. Note that, if $q \in \mathcal{N}_\kappa$ with $\kappa \geq 1$, there exists at least one nonreal pole or one regularized integral term.

Consider the function $(z-\alpha)(z-\overline{\alpha})(q(z)+d)$ where $\alpha \in \mathbb{C}^+$. By Lemma 9.5, except the addition of a linear polynomial, the degree of the rational summand with poles $\alpha, \overline{\alpha}$ decreases by one, whereas the degree of those rational summands with poles different from $\alpha, \overline{\alpha}$ do not change. By Lemma 9.3 and Lemma 9.4 also the integral terms do not change their structure and their degree of regularization remains the same. By the results of [KL1] we conclude that, with the real constant c as above

$$(z-\alpha)(z-\overline{\alpha})(q(z)+d) + cz \in \mathcal{N}_{\kappa'}$$

where $\kappa' = \kappa - 1$ or $\kappa' = \kappa$ depending whether q has a pole at α or not.

Let $\alpha \in \mathbb{R}$ and consider the function $(z-\alpha)^2(q(z)+d)$. By Lemma 9.2 and Lemma 9.5, the degree of the integral regularization at α as well as the degree of a possible rational summand with pole α decreases by two. By Lemma 9.3, Lemma 9.4 and Lemma 9.5 the remaining summands in the integral representation of q retain their form, except a possibly addition of a linear polynomial. Again appealing to [KL1] we conclude that for the number c as above

$$(z-\alpha)^2(q(z)+d) + cz \in \mathcal{N}_{\kappa'}$$

where $\kappa' = \kappa - 1$ or $\kappa' = \kappa$ depending whether there exists an integral term regularized at α (a pole at α) or not.

□

10 Some transformations of chains

In this section we study some mappings of the set $\mathfrak{C}$ into itself.

Lemma 10.1. *Let $K > 0$ be given. The mappings*

$$\mathcal{T}_{\cdot K} : (W_t) : (W_t)_{t>t_0} \mapsto (W_{\frac{1}{K}t}(Kz))_{t>t_0},$$

$$\mathcal{T}_J : (W_t)_{t>t_0} \mapsto (-JW_tJ)_{t>t_0}$$

define a bijection of $\mathfrak{C}$ onto itself and satisfy $\mathcal{T}_{\cdot K}^{-1} = \mathcal{T}_{\cdot(-K)}$, $(\mathcal{T}_J)^2 = \mathrm{id}_{\mathfrak{C}}$. The Weyl coefficients q, $q_{\cdot K}$ and q_J of $(W_t)_{t>t_0}$, $\mathcal{T}_{\cdot K}(W_t)$ and $\mathcal{T}_J(W_t)$ are related by $q_{\cdot K}(z) = q(Kz)$ and $q_J(z) = -\frac{1}{q(z)}$. The mappings $\mathcal{T}_{\cdot K}$ and $\mathcal{T}_J$ preserve the index of negativity.

Proof : Clearly the conditions *(i)* and *(ii)* of Definition 8.1 with the same number κ hold. An elementary computation shows that also *(iii)* and *(iv)* hold. The relation $q_{\cdot K}(z) = q(Kz)$ and $q_J = -\frac{1}{q}$ are seen by a straightforward argument.

□

Lemma 10.2. *Let $\alpha \in \mathbb{R}$ be given. The mappings*

$$\mathcal{T}_\alpha : (W_t)_{t>t_0} \mapsto (\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} W_t \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix})_{t>t_0},$$

$$\mathcal{T}^\alpha : (W_t)_{t>t_0} \mapsto (W_{t^\bullet}(z+\alpha)W_{t^\bullet}(\alpha)^{-1})_{t^\bullet > t_0^\bullet},$$

where $t^\bullet$ is a certain continuous increasing function of t with $\lim_{t\to\infty} t^\bullet(t) = \infty$, define bijections of $\mathfrak{C}$ onto itself and satisfy $(\mathcal{T}_\alpha)^{-1} = \mathcal{T}_{-\alpha}$, $(\mathcal{T}^\alpha)^{-1} = \mathcal{T}^{-\alpha}$. The Weyl coefficients q, q_α and q^α are related by $q_\alpha(z) = q(z) + \alpha$, $q^\alpha(z) = q(z+\alpha)$. Moreover, $\mathcal{T}_\alpha$ and $\mathcal{T}^\alpha$ preserve the index of negativity.

Proof : Let $(W_t)_{t>t_0}$ be given and consider the chain

$$\tilde{W}_t := \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} W_t \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}.$$

The properties *(i)*, *(ii)* and *(iv)* and $\tilde{W}_t(0) = 1$ are clearly satisfied. Since $\mathrm{tr}(CD) = \mathrm{tr}(DC)$ for any matrices C, D, we have $t = \mathfrak{t}(W_t) = \mathrm{tr}(W_t'(0)J) = \mathrm{tr}(\tilde{W}_t'(0)J) = \mathfrak{t}(\tilde{W}_t)$, and hence *(iii)* holds. The facts that $\mathcal{T}_\alpha$ is well defined on $\mathfrak{C}$ and that $\mathcal{T}_\alpha \circ \mathcal{T}_{-\alpha} = \mathrm{id}_{\mathfrak{C}}$ are obvious.

Now consider the chain

$$\tilde{W}_t(z) := W_t(z+\alpha)W_t(\alpha)^{-1}.$$

Clearly, $\mathfrak{t}(\tilde{W}_t)$ depends continuously of t and is strictly increasing (cf. [W1], [W2]). We define $t^\bullet(t) := \mathfrak{t}(\tilde{W}_t)$ and the reparametrization $\tilde{W}_{t^\bullet(t)} = \tilde{W}_t$. Again *(i)*, *(ii)*, *(iv)* and $\tilde{W}_t(0) = 1$ are obvious. It is left to show that

$$\lim_{t\to\infty} \mathfrak{t}(\tilde{W}_t) = \infty.$$

As $\mathfrak{t}(W_{t_0 t}) \to \infty$ we have for any family $\tau^t \in \mathcal{N}_0$

$$\lim_{t\to\infty} W_{t_0 t}(z+\alpha)\circ \tau^t(z) = W_{t_0}(z+\alpha)^{-1}\circ q(z+\alpha).$$

Since $\tau^t := W_t(\alpha)^{-1}\circ\tau \in \mathcal{N}_0$ for $\tau\in\mathcal{N}_0$, we conclude that in fact

$$\lim_{t\to\infty} \big(W_{t_0}(\alpha)W_{t_0 t}(z+\alpha)W_t(\alpha)^{-1}\big)\circ\tau(z) = \big(W_{t_0}(\alpha)W_{t_0}(z+\alpha)^{-1}\big)\circ q(z+\alpha).$$

As $\tau\in\mathcal{N}_0$ was arbitrarily chosen [W2] (compare also [W1]) shows

$$\lim_{t\to\infty} \mathfrak{t}(W_{t_0}(\alpha)W_{t_0 t}(z+\alpha)W_t(\alpha)^{-1}) = \infty.$$

Since $\tilde{W}_t = (W_{t_0}(z+\alpha)W_{t_0}(\alpha)^{-1})(W_{t_0}(\alpha)W_{t_0 t}(z+\alpha)W_t(\alpha)^{-1})$, we obtain $\mathfrak{t}(\tilde{W}_t)\to\infty$. The facts that $\mathcal{T}^\alpha$ is well defined on $\mathfrak{C}$ and that $\mathcal{T}^\alpha\circ\mathcal{T}^{-\alpha} = \mathrm{id}_{\mathfrak{C}}$ are obvious.

□

Lemma 10.3. *Let $l\in\mathbb{R}$ be given. The mapping*

$$\mathcal{T}_{lz} : (W_t)_{t>t_0} \mapsto (\begin{pmatrix} 1 & lz \\ 0 & 1\end{pmatrix} W_{t-l})_{t>t_1+l},$$

where $t_1\geq t_0$ is a certain number, defines a bijection of $\mathfrak{C}$ onto itself and satisfies $(\mathcal{T}_{lz})^{-1} = \mathcal{T}_{-lz}$. The Weyl coefficients q and q_{lz} are related by $q_{lz}(z) = q(z)+lz$. Let $\gamma=0$ if $l>0$, $\gamma=1$ if $l<0$, and let κ be the index of negativity of (W_t). If $\mathfrak{K}_-(W_t)=\mathfrak{K}(W_t)$, the index of negativity of $\mathcal{T}_{lz}(W_t)$ is $\kappa+\gamma$.

If $t>t_1$, $\mathfrak{K}_-(W_t)\neq\mathfrak{K}(W_t)$ and $\mathrm{ind}_- q=\kappa$, then the index of negativity of $\mathcal{T}_{lz}(W_t)$ is $\kappa+\gamma-\delta$, where $\delta=1$ if $\frac{1}{l}+[(1,0)^T,(1,0)^T]_{\mathfrak{K}(W_t)}\leq 0$ and $\delta=0$ otherwise. The number $[(1,0)^T,(1,0)^T]_{\mathfrak{K}(W_t)}$ does not depend on $t>t_1$.

Proof : Assume first that $\mathfrak{K}_-(W_t)=\mathfrak{K}(W_t)$ (cf. Corollary 5.14). Since

$$\mathfrak{K}(\begin{pmatrix} 1 & lz \\ 0 & 1\end{pmatrix}) = \mathrm{span}\,\{\begin{pmatrix} 1 \\ 0\end{pmatrix}\},$$

we conclude by [ADSR] that

$$\mathrm{ind}_-\mathfrak{K}(\begin{pmatrix} 1 & lz \\ 0 & 1\end{pmatrix} W_t) = \mathrm{ind}_-\mathfrak{K}(W_t) + \frac{\mathrm{sgn}(-l)+1}{2}.$$

With $t_1=t_0$ the properties *(i)*-*(iv)* and the relation $q_{lz}=q+lz$ are easily checked.

If $\mathfrak{K}_-(W_t) \neq \mathfrak{K}(W_t)$, then $(1,0)^T \in \mathfrak{K}(W_t)$. If $\mathfrak{K}(W_t)$ is one dimensional for all $t > t_0$, we set $t_1 = t_0$ if $l > 0$ and $t_1 = t_0 - l$ if $l < 0$. In this case it is easy to see that the assertions of the lemma hold.

If $\dim \mathfrak{K}(W_t) > 1$, $t > t_1$ for some $t_1 \geq t_0$ it follows from Corollary 5.15 that $[(1,0)^T, (1,0)^T]_{\mathfrak{K}(W_t)}$ coincides for all $t > t_1$. It follows from [ADSR] that the reproducing kernel space $\mathfrak{K}(W_{(l,0)}W_{t-l})$ is isomorphic to $\mathfrak{L}^\perp/\mathfrak{L}^\circ$, where $\mathfrak{L}$ is a subspace of $\mathfrak{K}(W_{(l,0)}) \oplus W_{(l,0)}\mathfrak{K}(W_{t-l})$:

$$\mathfrak{L} = \{(F(z); -F(z)) | F(z) \in \mathfrak{K}(W_{(l,0)}) \cap W_{(l,0)}\mathfrak{K}(W_{t-l})\}.$$

Since $\mathfrak{L}$ is spanned by $((1,0)^T; (-1,0)^T)$ and

$$[((1,0)^T; (-1,0)^T), ((1,0)^T; (-1,0)^T)]_{\mathfrak{L}} = \frac{1}{l} + [(1,0)^T, (1,0)^T]_{\mathfrak{K}(W_t)},$$

the numbers $\operatorname{ind}_-\mathfrak{L}$ and $\operatorname{ind}_0\mathfrak{L}$ are independent from t. Moreover, $\operatorname{ind}_-\mathfrak{L} + \operatorname{ind}_0\mathfrak{L} = \delta$, where $\delta = 1$ if $\frac{1}{l} + [(1,0)^T, (1,0)^T]_{\mathfrak{K}(W_t)} \leq 0$ and $\delta = 0$ if $\frac{1}{l} + [(1,0)^T, (1,0)^T]_{\mathfrak{K}(W_t)} > 0$. Hence $\mathfrak{K}(W_{(l,0)}W_{t-l})$ has the same number $\operatorname{ind}_-\mathfrak{K}(W_t) + \operatorname{ind}_-\mathfrak{K}(W_{(l,0)}) - \delta$ of negative squares for all $t > t_1 + l$. Now the properties *(i)-(iv)* and the relation $q_{lz} = q + lz$ are easily checked.

□

Note that

Lemma 10.4. *Assume that $\mathfrak{K}_-(W) = \mathfrak{K}(W)$. The operator of multiplication by z in the space $\mathfrak{P}(E_W)$ is densely defined if and only if the function $\frac{B_W}{A_W}$ is regular but not finite at infinity.*

Proof : This follows from Lemma 5.2 since $\frac{B_W}{A_W}$ is a Q-function of $\mathcal{S}$ and the selfadjoint extension $\mathcal{A}_0$ corresponding to the element $A_W \in \operatorname{Ass} \mathfrak{P}(E_W)$.

□

In the following denote by W_t° the matrix

$$W_t^\circ(z) := \begin{pmatrix} \cos(tz) & \sin(tz) \\ -\sin(tz) & \cos(tz) \end{pmatrix}. \tag{10.1}$$

It is easy to check that $(W_t^\circ)_{t>0}$ belongs to $\mathfrak{C}$ and has index of negativity 0. Since the entries w_{ij}^t, $i,j = 1,2$ of W_t° are of exponential type t, it follows from [dB7] that (W_t°) does not contain an indivisible intervals. In particular, for $t > 0$, the space $\mathfrak{K}(W_t^\circ)$ does not contain a constant function, $\mathfrak{K}(W_t^\circ) = \mathfrak{K}_-(W_t^\circ)$, and the multiplication operator by the independent variable has dense domain in $\mathfrak{P}(E_{W_t^\circ})$.

Now we introduce some more special transformations of chains. Let $c \in \mathbb{R}$ be fixed, write

$$W(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} \in \mathcal{M}_\kappa^1,\ W(0) = 1,$$

and put $\alpha = A'(0)B'(0) - 2cA'(0) - \frac{B''(0)}{2}$. We consider the transformation $\mathcal{T}$ defined if $B'(0) \neq c$ by

$$\mathcal{T}(W) := \begin{pmatrix} \frac{1}{z^2} & -c\frac{1}{z} \\ 0 & 1 \end{pmatrix} W(z) \begin{pmatrix} 0 & (c - B'(0))z \\ -\frac{1}{c-B'(0)}z & 1 - \frac{\alpha}{c-B'(0)}z \end{pmatrix}.$$

First of all note that, since $A'(0)=-D'(0)$, we have

$$\mathcal{T}(W)(z)=\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}\frac{1}{z^2}+\begin{pmatrix}0&-c\\0&0\end{pmatrix}\frac{1}{z}+\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)\Big(\sum_{j=0}^\infty\frac{W(0)^{(j)}}{j!}z^n\Big)\cdot$$

$$\cdot\left(\begin{pmatrix}0&0\\0&1\end{pmatrix}+\begin{pmatrix}0&(c-B'(0))\\-\frac{1}{c-B'(0)}&-\frac{\alpha}{c-B'(0)}\end{pmatrix}z\right)=$$

$$=\begin{pmatrix}1&0\\0&0\end{pmatrix}W'(0)\begin{pmatrix}0&(c-B'(0))\\-\frac{1}{c-B'(0)}&-\frac{\alpha}{c-B'(0)}\end{pmatrix}+\begin{pmatrix}1&0\\0&0\end{pmatrix}\frac{1}{2}W''(0)\begin{pmatrix}0&0\\0&1\end{pmatrix}+$$

$$+\begin{pmatrix}0&-c\\0&0\end{pmatrix}\begin{pmatrix}0&(c-B'(0))\\-\frac{1}{c-B'(0)}&-\frac{\alpha}{c-B'(0)}\end{pmatrix}+\begin{pmatrix}0&-c\\0&0\end{pmatrix}W'(0)\begin{pmatrix}0&0\\0&1\end{pmatrix}+$$

$$+\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}+\sum_{j=1}^\infty\frac{\mathcal{T}(W)(0)^{(j)}}{j!}z^n=I+\sum_{j=1}^\infty\frac{\mathcal{T}(W)(0)^{(j)}}{j!}z^n,$$

and we see that $\mathcal{T}(W)(z)$ is an entire function with $\mathcal{T}(W)(0)=I$. Also, by Lemma 5.10, we have $\mathcal{T}(W)\in\mathcal{M}_\nu^1$ for some $\nu\geq 0$.

Lemma 10.5. *Let a chain $(W_t)_{t>c_-}\in\mathfrak{C}$ with index κ of negativity be given, assume that its Weyl coefficient q belongs to $\mathcal{N}_\kappa$ and is regular at ∞. Moreover, assume that*

$$q^\bullet(z):=\begin{pmatrix}\frac{1}{z^2}&-c\frac{1}{z}\\0&1\end{pmatrix}\circ q(z)=\frac{1}{z^2}(q(z)-cz)\in\mathcal{N}_{\kappa+1}.$$

Then there exists a number T_0, $c_-\leq T_0<\infty$, such that $W_t\in\operatorname{dom}\mathcal{T}$ for $t>T_0$. The limit $l^\bullet:=\lim_{t\to\infty}\mathfrak{t}(\mathcal{T}(W_t))$ exists in $\mathbb{R}\cup\{+\infty\}$. The chain $(W_t^\bullet)$ defined by

$$W^\bullet_{\mathfrak{t}(\mathcal{T}(W_t))}:=\mathcal{T}(W_t),\ t>T_0,$$

and, in case $l^\bullet<\infty$,

$$W_x^\bullet:=(W_{l^\bullet}^\bullet)W_{(x-l^\bullet,0)},\ x>l^\bullet,$$

belongs to $\mathfrak{C}$, has index $\kappa+1$ of negativity and its Weyl coefficient is $q^\bullet$.

Proof : We start with the investigation of some properties of $\mathcal{T}$. Note that whenever $\mathcal{T}(W_t)$ is defined we have

$$\mathcal{T}(W_t)\circ\theta=\begin{pmatrix}\frac{1}{z^2}&-c\frac{1}{z}\\0&1\end{pmatrix}\circ(W_t\circ\tau),\tag{10.2}$$

where θ and τ are related by

$$\tau(z)=T(z,W_t)\circ\theta(z)=\frac{c-B_t'(0)}{\frac{\theta(z)}{B_t'(0)-c}+\frac{1}{z}-\frac{\alpha_t}{c-B_t'(0)}},\tag{10.3}$$

or, equivalently,

$$\theta(z)=T(z,W_t)^{-1}\circ\tau=(B_t'(0)-c)^2\frac{-1}{\tau}+\alpha_t+(B_t'(0)-c)\frac{-1}{z},$$

where

$$T(z,W_t) := \begin{pmatrix} 0 & (c-B_t'(0))z \\ -\frac{1}{c-B_t'(0)}z & 1-\frac{\alpha_t}{c-B_t'(0)}z \end{pmatrix}.$$

One easily checks that $T(z,W_t) \in \mathcal{M}_\delta^S$, $T(z,W_t)^{-1} \in \mathcal{M}_{1-\delta}^S$, where $S(z)=z$ and $\delta=0$ if $c > B_t'(0)$ and $\delta = 1$ if $c < B_t'(0)$. For some $\tau_t \in \mathcal{N}_0$ we have $W_t \circ \tau_t = q$ and thus $\mathcal{T}(W_t)\circ\theta_t = q^\bullet$ where τ_t and θ_t are connected by (10.3). By Lemma 5.9 and [KW2] we have $\theta_t \in \mathcal{N}_0$ for $c < B_t'(0)$ and $\theta_t \in \mathcal{N}_0 \cup \mathcal{N}_1$ otherwise.

Now we compute $\mathrm{ind}\_\mathfrak{K}(\mathcal{T}(W_t))$. First note that by Theorem 5.7 $\mathfrak{K}(W_t) = \mathfrak{K}_-(W_t)$. For $\psi \in [0,\pi)$ we set $\theta_\psi := \cot\psi$ and $\tau_\psi(z) := T(z,W_t)\circ\cot\psi$ (cf. (10.2)). A straightforward calculation yields $\tau_\psi(z) = W_{(l_\psi,\phi_\psi)} \circ d_\psi$ where $\cot\phi_\psi = \frac{-(c-B_t'(0))^2}{\cot\psi+\alpha_t}$ and $l_\psi \in \mathbb{R}\setminus\{0\}$, $d_\psi \neq \cot\phi_\psi$ for $\psi \neq 0$. Hereby

$$\frac{-\mathrm{sgn}(l_\psi)+1}{2} = \frac{-\mathrm{sgn}(c-B_t'(0))+1}{2} = \mathrm{ind}\_\tau_\psi = \mathrm{ind}\_\mathfrak{K}(W_{(l_\psi,\phi_\psi)}).$$

Note in this place that any linear fractional transformation $\frac{\alpha z+\beta}{\gamma z+1}$ can be written as $W_{(l,\varphi)}\circ\beta$ with $\cot\varphi = \frac{\alpha}{\gamma}$ and $l = \frac{\alpha^2+\beta^2}{\alpha-\gamma\beta}$. We choose $\psi \neq 0$ such that $\cot\psi$ is not the exceptional number for $\mathcal{T}(W_t)$ in Lemma 5.12 and such that $\cot\phi_\psi$ is not the exceptional number for W_t. It follows from Lemma 5.12 that $\mathrm{ind}\_\mathfrak{K}(\mathcal{T}(W_t)) = \mathrm{ind}\_\mathcal{T}(W_t)\circ\cot\psi$ and from Lemma 5.12 and Theorem I.12.2 that

$$\mathrm{ind}\_\mathfrak{K}(W_t) + \mathrm{ind}\_\tau_\psi = \mathrm{ind}\_\mathfrak{K}(W_t) + \mathrm{ind}\_\mathfrak{K}(W_{(l_\psi,\phi_\psi)}) =$$

$$= \mathrm{ind}\_\mathfrak{K}(W_t W_{(l_\psi,\phi_\psi)}) = \mathrm{ind}\_W_t W_{(l_\psi,\phi_\psi)}\circ d_\psi = \mathrm{ind}\_W_t\circ\tau_\psi.$$

Consider the relation $\frac{(W_t\circ\tau_\psi)(z)-cz}{z^2} = (\mathcal{T}(W_t)\circ\cot\psi)(z)$. Since $\mathcal{T}(W_t)(0) = I$, we see that $\mathcal{T}(W_t)\circ\cot\psi$ is analytic at 0 whenever $\psi\neq 0$. It follows from Corollary 5.3 and Proposition 9.1 that $\mathrm{ind}\_W_t\circ\tau_\psi = \mathrm{ind}\_\mathcal{T}(W_t)\circ\cot\psi$ for all but at most three exceptional values of $\psi\in[0,\pi)$. Thus, $\mathrm{ind}\_\mathfrak{K}(\mathcal{T}(W_t)) = \mathrm{ind}\_\mathfrak{K}(W_t) + \frac{-\mathrm{sgn}(c-B_t'(0))+1}{2}$, and we showed that $\mathrm{ind}\_\mathfrak{K}(\mathcal{T}(W_t)) = \kappa+1$ for $c < B_t'(0)$ and $\mathrm{ind}\_\mathfrak{K}(\mathcal{T}(W_t)) = \kappa$ for $c > B_t'(0)$. Since $\mathcal{T}(W_t)\circ\cot\psi$ is finite at ∞ (cf. Corollary 5.3) and as $\mathrm{ind}\_\mathfrak{K}(\mathcal{T}(W_t)) = \mathrm{ind}\_\mathcal{T}(W_t)\circ\cot\psi$ for sufficiently many $\psi\in[0,\pi)$, Theorem 5.7 shows that $\mathfrak{K}(\mathcal{T}(W_t)) = \mathfrak{K}_-(\mathcal{T}(W_t))$ and that $1 \in \mathfrak{P}(E_{\mathcal{T}(W_t)})$. Since $q^\bullet \in \mathcal{N}_{\kappa+1}$, we obtain that $\theta_t \in \mathcal{N}_1$ if $c > B_t'(0)$.

Next we show that there exists a number T_0 such that $B_t'(0) > c$ for $t > T_0$. Assume the contrary. Then, because $B_t'(0)$ is nondecreasing, there are two possibilities: The first is that $B_t'(0) < c$ for all t. In this case let $\theta(z) = u \in \overline{\mathbb{R}}$, then $\tau_t(z) := T(z,W_t)\circ u \in \mathcal{N}_0$, hence the right hand side of (10.2) tends to $q^\bullet$ if $t\to\infty$. By the above consideration the left hand side of (10.2) is contained in $\mathcal{N}_{\leq\kappa} := \bigcup_{\nu\leq\kappa}\mathcal{N}_\nu$. But this contradicts the fact that a limit of $\mathcal{N}_{\leq\kappa}$ functions belongs to $\mathcal{N}_{\leq\kappa}$.

The second possibility is that $B_t'(0) = c$ for all $t\in\mathbb{R}$, $t\geq T_1$. Let T_1 be minimal with respect to this property. Since $W_{t_2}'(0) = W_{t_2}'(0) + W_{t_1t_2}'(0)$, we see that then $W_{t_1t_2} = W_{(t_2-t_1,\frac{\pi}{2})}$ for all $t_1,t_2\in\mathbb{R}, T_1\leq t_1\leq t_2$. Since by the assumption of the lemma and Lemma 8.5 $\dim\mathfrak{K}(W_{t_1t_2}) > 1$ for some $c_- < t_1 < t_2$, we have $c_- < T_1$, and see that $c > B_t'(0)$ for $t < T_1$. Now we get

$$q(z) = W_{T_1}\circ 0 = \lim_{t\nearrow T_1} W_t\circ 0.$$

If $\tau(z)=0$ then $\theta(z)=\infty$, hence for $c_- < t < T_1$ we have $\mathcal{T}(W_t)\circ\infty\in\mathcal{N}_{\leq\kappa}$, and it follows that

$$q^\bullet(z)=\lim_{t\nearrow T_1}\frac{1}{z^2}(W_t\circ 0-cz)=\lim_{t\nearrow T_1}\mathcal{T}(W_t)\circ\infty$$

belongs to $\mathcal{N}_{\leq\kappa}$. But this contradicts our assumption that $q^\bullet\in\mathcal{N}_{\kappa+1}$.

Thus we have shown that there exists a number $T_0\geq c_-$ such that $B_t'(0)>c$, in particular $W_t\in\operatorname{dom}\mathcal{T}$ and $\mathcal{T}(W_t)\in\mathcal{M}^1_{\kappa+1}$ for $t>T_0$. We take T_0 minimal with respect to this property.

In order to prove that for $t_1,t_2>T_0$ we have $\mathcal{T}(W_{t_1})^{-1}\mathcal{T}(W_{t_2})\in\mathcal{M}_0^1$, we first note that, as $\mathcal{T}(W_{t_1})^{-1}\mathcal{T}(W_{t_2})=T(z,W_{t_1})^{-1}W_{t_1t_2}T(z,W_{t_2})$, this matrix function belongs to $\mathcal{M}_0^1\cup\mathcal{M}_1^1$. Now we show that the domain of the multiplication operator in $\mathfrak{P}(E_{\mathcal{T}(W_t)})$ is dense if and only if the domain of the multiplication operator in $\mathfrak{P}(E_{W_t})$ is dense. This follows from Lemma 10.4, since (with the notation of Lemma 10.4)

$$\frac{B_{\mathcal{T}(W_t)}}{A_{\mathcal{T}(W_t)}}=(B_t'(0)-c)^2\frac{-1}{\frac{B_{W_t}}{A_{W_t}}}+(B_t'(0)-c)\frac{-1}{z}+\alpha_t,$$

and since by Lemma 5.4 with $\frac{B_{W_t}}{A_{W_t}}$ also the function $\frac{-1}{\frac{B_{W_t}}{A_{W_t}}}$ is regular but not finite at ∞. If the domain of the multiplication operator is dense in $\mathfrak{P}(E_{\mathcal{T}(W_{t_1})})$, it follows from Lemma 5.17 that $\mathcal{T}(W_{t_1})^{-1}\mathcal{T}(W_{t_2})\in\mathcal{M}_0^1$. If this domain is not dense, let $0<\epsilon<t_1-T_0$ and define a new chain $(W_t^\epsilon)_{t>T_0}$:

$$W_t^\epsilon=W_t \text{ for } t\leq t_1-\epsilon,\ W_t^\epsilon=W_{t_1-\epsilon}W^\circ_{t-(t_1-\epsilon)} \text{ for } t_1-\epsilon<t<t_1,$$

$$W_t^\epsilon=W_{t_1-\epsilon}W_\epsilon^\circ W_{t_1t} \text{ for } t_1\leq t.$$

Since $\mathfrak{K}(W_\epsilon^\circ)$ contains no constant function (cf. (10.1)), we easily see that $(W_t^\epsilon)_{t>T_0}\in\mathfrak{C}$ has index of negativity κ. Let B_t^ϵ be the right upper entry of W_t^ϵ. As $(B_t^\epsilon)'(0)$ is nondecreasing, we find $(B_t^\epsilon)'(0)>c,\ t>T_0$. Corollary 5.14 shows that $\mathfrak{K}(W_t^\epsilon)=\mathfrak{K}_-(W_t^\epsilon)$. Since $(B_t^\epsilon)'(0)>c$, the same argumentation as in the second paragraph of this proof applied to W_t^ϵ shows that $\mathcal{T}(W_t^\epsilon)$ is well defined, belongs to $\mathcal{M}^1_{\kappa+1}$ and satisfies $\mathfrak{K}(\mathcal{T}(W_t^\epsilon))=\mathfrak{K}_-(\mathcal{T}(W_t^\epsilon))$. By Lemma 7.3 we see that the domain of the multiplication operator in $\mathfrak{P}(E_{W^\epsilon_{t_1}})$ is dense. Hence, as already proved above $\mathcal{T}(W^\epsilon_{t_1})^{-1}\mathcal{T}(W^\epsilon_{t_2})\in\mathcal{M}_0^1$. Since $\mathcal{T}(W^\epsilon_{t_1})^{-1}\mathcal{T}(W^\epsilon_{t_2})=\mathcal{T}(W_{t_1})^{-1}\mathcal{T}(W_{t_2})$, we obtain $\mathcal{T}(W_{t_1})^{-1}\mathcal{T}(W_{t_2})\in\mathcal{M}_0^1$ in any case.

An elementary calculation shows that $\mathcal{T}(W_{t_1t_2})\neq I$ for $t_2>t_1>T_0$, hence $\mathfrak{t}(\mathcal{T}(W_t))$ is strictly increasing. In particular $l^\bullet=\lim_{t\to\infty}\mathfrak{t}(\mathcal{T}(W_t))$ exists.

If $l^\bullet=\infty$ we have proved that the chain $(W_t^\bullet)$ belongs to $\mathfrak{C}$ and has index of negativity $\kappa+1$. Since for $t>T_0$ both functions $\theta_t(z)$ and $\tau_t(z)$ belong to $\mathcal{N}_0$, the Weyl coefficient of $(W_t^\bullet)$ is $q^\bullet$.

If $l^\bullet<\infty$ choose $x_0<l^\bullet,\ x_0>\inf\{\mathfrak{t}(\mathcal{T}(W_t))|t>T_0\}$ and continue the family $W^\bullet_{x_0x}$, $x<l^\bullet$ to $x=l^\bullet$ which is possible by [dB7] and set $W^\bullet_{l^\bullet}=W^\bullet_{x_0}W^\bullet_{x_0l^\bullet}$. Moreover, $W^\bullet_{x_0l^\bullet}\in\mathcal{M}_0^1$. Hence our definition of $W_x^\bullet$ for $x>l^\bullet$ is meaningful. We obtain for $x_1>l^\bullet$

$$q^\bullet(z)=\frac{1}{z^2}(\lim_{t\nearrow\infty}(W_t\circ 0)(z)-cz)=\lim_{x\nearrow l^\bullet}(W_x^\bullet\circ\infty)(z)=$$

$$= (W^\bullet_{l^\bullet} \circ \infty)(z) = (W^\bullet_{x_1} \circ \infty)(z), \tag{10.4}$$

hence all matrices $W^\bullet_x$, $x \geq l^\bullet$, are contained in $\mathcal{M}^1_{\kappa+1}$. It follows that the constructed chain belongs to $\mathfrak{C}$. Moreover, again by (10.4) the Weyl coefficient of $(W^\bullet_t)$ is $q^\bullet$.

□

Now we consider the inverse transformation of $\mathcal{T}$. Let $c \in \mathbb{R}$ be fixed, write

$$V(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} \in \mathcal{M}^1_\kappa,\ V(0) = 1,$$

and put $\beta := \frac{c''(0)}{2} - c'(0)d'(0)$. We consider the transformation

$$\mathcal{E}(V) := \begin{pmatrix} 1 & c\frac{1}{z} \\ 0 & \frac{1}{z^2} \end{pmatrix} V(z) \begin{pmatrix} 1 - \frac{\beta}{c'(0)}z & \frac{1}{c'(0)}z \\ -c'(0)z & 0 \end{pmatrix},$$

defined whenever $c'(0) \neq 0$. Since $\det \mathcal{E}(V) = 1$ we know by Lemma 5.10 that $\mathcal{E}(V)$ belongs to $\mathcal{M}^1_\nu$ for some $\nu \geq 0$. A similar computation as for $\mathcal{T}$ shows that $\mathcal{E}(V)$ is an entire matrix function.

Lemma 10.6. *The transformations $\mathcal{T}$ and $\mathcal{E}$ are inverses of each other in the sense that $\mathcal{E}\mathcal{T}(W) = W$ ($\mathcal{T}\mathcal{E}(W) = W$) whenever $W \in \operatorname{dom}\mathcal{T}$ ($W \in \operatorname{dom}\mathcal{E}$).*
Proof : Let $c \in \mathbb{R}$ be given and let

$$W(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix},\ W(0) = 1,$$

be such that $\gamma := c - B'(0) \neq 0$, i.e. that $W \in \operatorname{dom}\mathcal{T}$. Moreover, let α be as in the definition of $\mathcal{T}(W)$ and put

$$\mathcal{T}(W) =: \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}.$$

An elementary computation using the definition of $\mathcal{T}(W)$ shows that

$$\beta = \frac{c''(0)}{2} - c'(0)d'(0) = -\frac{\alpha}{\gamma^2}.$$

Using this relation and the fact $c'(0) = -\frac{1}{\gamma} \neq 0$, which in particular implies that $\mathcal{T}(W) \in \operatorname{dom}\mathcal{E}$, a straightforward computation shows that $\mathcal{E}(\mathcal{T}(W)) = W$.

Now assume that $W \in \operatorname{dom}\mathcal{E}$, i.e. that $C'(0) \neq 0$. If we put

$$\mathcal{E}(W) =: \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$

a computation shows that $b(z) = \frac{1}{C'(0)}(zA(z) + cC(z))$, hence

$$b'(0) - c = \frac{1}{C'(0)} \neq 0,$$

i.e. $\mathcal{E}(W) \in \operatorname{dom}\mathcal{T}$. It follows from the already proved that $\mathcal{E}\mathcal{T}\mathcal{E}(W) = \mathcal{E}(W)$ and since clearly $\mathcal{E}$ is injective we find $\mathcal{T}(\mathcal{E}(W)) = W$.

$\square$

Lemma 10.7. *Let a chain $(V_t)_{t>c_-} \in \mathfrak{C}$ with index $\kappa+1$, $\kappa \geq 0$, of negativity be given and assume that its Weyl coefficient q belongs to $\mathcal{N}_{\kappa+1}$ and is such that the limit $\lim_{y\to+\infty} yq(iy)$ exists. Let $c \in \mathbb{R}$ be the unique number such that*

$$q^\bullet(z) := z^2 q(z) + cz$$

is regular at ∞ and assume that $q^\bullet \in \mathcal{N}_\kappa$. Then there exists a number T_1, $c_- < T_1 \leq \infty$, such that $W_t \in \operatorname{dom}\mathcal{E}$ if and only if $c_- < t < T_1$. The chain

$$V^\bullet_{\mathfrak{t}(\mathcal{E}(V_t))} := \mathcal{E}(V_t),\ t < T_1,$$

belongs to $\mathfrak{C}$, has index κ of negativity and its Weyl coefficient is $q^\bullet$.

Proof : First note that whenever $\mathcal{E}(V_t)$ is defined we have

$$\mathcal{E}(V_t) \circ \vartheta = \begin{pmatrix} 1 & \frac{c}{z} \\ 0 & \frac{1}{z^2} \end{pmatrix} \circ (V_t \circ \sigma),$$

where

$$\sigma(z) = S(z, V_t) \circ \vartheta(z) = -\frac{1}{c_t'(0)^2 \vartheta(z)} - \frac{1}{c_t'(0) z} + \frac{\beta_t}{c_t'(0)^2}, \tag{10.5}$$

or, equivalently,

$$\vartheta(z) = S(z, V_t)^{-1} \circ \sigma(z) = \frac{-1}{c_t'(0)^2 \sigma(z) + \frac{c_t'(0)}{z} - \beta}.$$

Herby

$$S(z, V_t) = \begin{pmatrix} 1 - \frac{\beta_t}{c_t'(0)} z & \frac{1}{c_t'(0)} z \\ -c_t'(0) z & 0 \end{pmatrix}.$$

One easily checks that $S(z, V_t) \in \mathcal{M}^S_\delta$, $S(z, V_t)^{-1} \in \mathcal{M}^S_{1-\delta}$, where $S(z) = z$, $\delta = 0$ if $c_t'(0) > 0$ and $\delta = 1$ if $c_t'(0) < 0$. For some $\sigma_t \in \mathcal{N}_0$ we have $V_t \circ \sigma_t = q$ and thus $\mathcal{E}(V_t) \circ \vartheta_t = q^\bullet$ where σ_t and ϑ_t are connected by (10.5). By Lemma 5.9 and [KW2] we have $\vartheta_t \in \mathcal{N}_0$ for $c_t'(0) < 0$ and $\vartheta_t \in \mathcal{N}_0 \cup \mathcal{N}_1$ otherwise.

In the following let $c_r \geq c_-$ be the supremum of all numbers $t > c_-$ such that V_t is linear. By Lemma 8.5 we know that $c_r < \infty$. Note also that by Theorem 5.7 $\mathfrak{K}(V_t) = \mathfrak{K}_-(V_t)$, $t > c_-$.

Now we compute $\operatorname{ind}_-\mathfrak{K}(\mathcal{E}(V_t))$, $t > c_r$. For $\psi \in [0, \pi)$ we set $\vartheta_\psi = \cot\psi$ and $\sigma_\psi(z) = S(z, V_t) \circ \cot\psi$ (cf. (10.5)). A straightforward calculation yields $\sigma_\psi(z) = W_{(l_\psi, \phi_\psi)} \circ d_\psi$ where $\cot\phi_\psi = \frac{\beta_t}{c_t'(0)^2} - \frac{1}{\cot\psi c_t'(0)^2}$ and $l_\psi \in \mathbb{R}\setminus\{0\}$, $d_\psi \neq \cot\phi_\psi$ for $\psi \neq \frac{\pi}{2}$. Hereby $\frac{-\operatorname{sgn}(l_\psi)+1}{2} = \frac{-\operatorname{sgn}(c_t'(0))+1}{2} = \operatorname{ind}_-\sigma_\psi = \operatorname{ind}_-\mathfrak{K}(W_{(l_\psi, \phi_\psi)})$. We choose $\psi \neq \frac{\pi}{2}$ such that $\cot\psi$ is not the exceptional number for $\mathcal{E}(V_t)$ in Lemma 5.12 and such that $\cot\phi_\psi$ is not the exceptional number for V_t. It follows from Lemma 5.12 that $\operatorname{ind}_-\mathfrak{K}(\mathcal{E}(V_t)) = \operatorname{ind}_-\mathcal{E}(V_t) \circ \cot\psi$ and from Lemma 5.12 and Theorem I.12.2 that

$$\operatorname{ind}_-\mathfrak{K}(V_t) + \operatorname{ind}_-\sigma_\psi = \operatorname{ind}_-\mathfrak{K}(V_t) + \operatorname{ind}_-\mathfrak{K}(W_{(l_\psi,\phi_\psi)}) =$$

$$= \operatorname{ind}_-\mathfrak{K}(V_t W_{(l_\psi,\phi_\psi)}) = \operatorname{ind}_- V_t W_{(l_\psi,\phi_\psi)} \circ d_\psi = \operatorname{ind}_- V_t \circ \sigma_\psi.$$

Consider the relation $z^2(V_t \circ \sigma_\psi)(z) + cz = (\mathcal{E}(V_t) \circ \cot\psi)(z)$. It follows from $t > c_r$, Lemma 5.16 and Proposition 9.1 that $\mathcal{E}(V_t) \circ \cot\psi$ is regular at ∞ if ψ does not correspond to the exceptional value of Lemma 5.12. Since $\mathcal{E}(V_t)(0) = I$, we see that $(\mathcal{E}(V_t) \circ \cot\psi)(0) \neq 0$ whenever $\psi \neq \frac{\pi}{2}$. It follows from Proposition 9.1 that $\operatorname{ind}_- V_t \circ \sigma_\psi = \operatorname{ind}_-\mathcal{E}(V_t) \circ \cot\psi + 1$ for all but at most four excepted values of ψ in $[0,\pi)$. Thus, $\operatorname{ind}_-\mathfrak{K}(\mathcal{E}(V_t)) = \operatorname{ind}_-\mathfrak{K}(V_t) + \frac{-\operatorname{sgn}(c_t'(0))+1}{2} - 1$, and we showed that $\operatorname{ind}_-(\mathcal{E}(V_t)) = \kappa+1$ for $c_t' < 0,\ t > c_r$ and $\operatorname{ind}_-(\mathcal{E}(V_t)) = \kappa$ for $c_t' > 0,\ t > c_r$. Since $\mathcal{E}(V_t) \circ \cot\psi$ is regular at ∞ and as $\operatorname{ind}_-\mathfrak{K}(\mathcal{E}(V_t)) = \operatorname{ind}_-\mathcal{E}(V_t) \circ \cot\psi$ for almost all values of ψ in $[0,\pi)$, Theorem 5.7 shows that $\mathfrak{K}(\mathcal{E}(V_t)) = \mathfrak{K}_-(\mathcal{E}(V_t)),\ t > c_r$.

Now we will show that

$$\mathcal{E}(V_{t_1})^{-1}\mathcal{E}(V_{t_2}) \in \mathcal{M}_0^1,$$

whenever $c_r < t_1 \leq t_2$ are such that $c_{t_1}'(0)$ and $c_{t_2}'(0)$ are both not zero and have the same sign. First note that, as $\mathcal{E}(V_{t_1})^{-1}\mathcal{E}(V_{t_2}) = S(z, V_{t_1})^{-1} V_{t_1 t_2} S(z, V_{t_2})$, this matrix function belongs to $\mathcal{M}_0^1 \cup \mathcal{M}_1^1$. Now we show that the domain of the multiplication operator in $\mathfrak{P}(E_{\mathcal{E}(V_t)})$ is dense if and only if the domain of the multiplication operator in $\mathfrak{P}(E_{V_t})$ is dense. This follows from Lemma 5.4 and Lemma 10.4, since (with the notation of Lemma 10.4)

$$\frac{B_{\mathcal{E}(V_t)}}{A_{\mathcal{E}(V_t)}} = \frac{-1}{-\frac{c_t'(0)}{z} + \beta_t + c_t'(0)^2 \frac{B_{V_t}}{A_{V_t}}}.$$

If the domain of the multiplication operator is dense in $\mathfrak{P}(E_{\mathcal{E}(V_{t_1})})$, it follows from Lemma 5.17 that $\mathcal{E}(V_{t_1})^{-1}\mathcal{E}(V_{t_2}) \in \mathcal{M}_0^1$. If this domain is not dense, let $0 < \epsilon < t_1 - c_r$ and define a new chain $(V_t^\epsilon)_{t > c_r}$:

$$V_t^\epsilon = V_t \text{ for } t \leq t_1 - \epsilon,\ V_t^\epsilon = V_{t_1-\epsilon} W_{t-(t_1-\epsilon)}^\circ \text{ for } t_1 - \epsilon < t < t_1,$$

$$V_t^\epsilon = V_{t_1-\epsilon} W_\epsilon^\circ V_{t_1 t} \text{ for } t_1 \leq t.$$

Since $\mathfrak{K}(W_\epsilon^\circ)$ contains no constant function (cf. (10.1)), we easily see that $(V_t^\epsilon)_{t > c_r} \in \mathfrak{C}$ has index of negativity $\kappa + 1$. It is easy to see that for sufficiently small ϵ $(c_{t_1}^\epsilon)'(0)$ and $(c_{t_2}^\epsilon)'(0)$ are both not zero and have the same sign as $(c_{t_1})'(0)$ and $(c_{t_2})'(0)$, where c_t^ϵ is the left lower entry of V_t^ϵ. Corollary 5.14 shows that $\mathfrak{K}(V_t^\epsilon) = \mathfrak{K}_-(V_t^\epsilon)$. Now the same argumentation as in the second paragraph of this proof applied to V_t^ϵ shows that $\mathcal{E}(V_t^\epsilon)$ is well defined, belongs to $\mathcal{M}_{\kappa+\delta}^1$, where $\delta = \frac{-\operatorname{sgn}(c_t'(0))+1}{2}$, and satisfies $\mathfrak{K}(\mathcal{E}(V_t^\epsilon)) = \mathfrak{K}_-(\mathcal{E}(V_t^\epsilon))$. The only thing which has to be noted additionally is that $z^2 V_t^\epsilon \circ \sigma_\psi + cz = \mathcal{E}(V_t^\epsilon) \circ \cot\psi$ is regular at ∞ because of Lemma 5.16 since $V_t^\epsilon = V_t,\ t \leq t_1 - \epsilon$. By Lemma 7.3 we see that the domain of the multiplication operator in $\mathfrak{P}(E_{V_{t_1}^\epsilon})$ is dense. As already proved above $\mathcal{E}(V_{t_1}^\epsilon)^{-1}\mathcal{E}(V_{t_2}^\epsilon) \in \mathcal{M}_0^1$. Since $\mathcal{E}(V_{t_1}^\epsilon)^{-1}\mathcal{E}(V_{t_2}^\epsilon) = \mathcal{E}(V_{t_1})^{-1}\mathcal{E}(V_{t_2})$, we obtain $\mathcal{E}(V_{t_1})^{-1}\mathcal{E}(V_{t_2}) \in \mathcal{M}_0^1$ in any case.

The function $c_t'(0)$ is continuous and nonincreasing, hence the set (c_-,∞) is devided into three, possibly empty, intervals: namely (c_-,T_1) ($[T_1,T_2]$, (T_2,∞)), where $c_t'(0)$ is positive (zero, negative). We shall show that $T_1>c_-$ and that already $T_2=\infty$, i.e. that $c_t'(0)$ is always nonnegative.

Assume on the contrary that for some t we have $c_t'(0)<0$, i.e. $T_2<\infty$. We show that for some numbers $t_2>t_1>T_2$ the space $\mathfrak{K}(\mathcal{E}(V_{t_1})^{-1}\mathcal{E}(V_{t_2}))$ is not one-dimensional. Since the right factor in the definition of $\mathcal{E}$ is linear in z, an elementary consideration shows that $\mathcal{E}(V_{t_1})^{-1}\mathcal{E}(V_{t_2})$ is linear if and only $V_{t_1}^{-1}V_{t_2}$ is linear. Assume on the contrary that for all $t_2>t_1>T_2$ we have $V_{t_1t_2}=W_{(t_2-t_1,\phi)}$, i.e. that the interval $[T_2,\infty)$ is indivisable in the chain (V_t). Then q is the 1-resolvent of the extension of the multiplication operator determined by a number $\phi\in[0,\pi)$. Since $q^\bullet\in\mathcal{N}_\kappa$ Proposition 9.1 yields that q has a pole at 0 and we conclude from Lemma 5.12 and Lemma I.6.4 that $\phi=0$. But then $c_t'(0)$ is constant on (T_2,∞). If $T_2>-\infty$, we have a contradiction since $c_{T_2}'(0)=0$. Otherwise, by Lemma 8.5 we have a contradiction as we assumed $q\in\mathcal{N}_{\kappa+1}$. Hence $\mathcal{E}(V_{t_1})^{-1}\mathcal{E}(V_{t_2})$ is not linear for some choice of $t_2>t_1>T_2$, and there exists a smallest number $s_0>T_2$, $s_0\leq\infty$, such that $V_{st}=W_{(t-s,0)},\ s,t\in\mathbb{R},\ t\geq s\geq s_0$. Clearly, $c_r\leq s_0$. Assume first $s_0=c_r$. As V_{c_r} is linear, as $q=V_{c_r}\circ\infty$, and as we assumed $\operatorname{ind}_-q^\bullet=\operatorname{ind}_-q-1$, we have $q(z)=\frac{-c}{z}$, where $\frac{1}{c_r}=c<0$. This gives $V_{c_r}=W_{(\frac{1}{c},\frac{\pi}{2})}$ and $c_{c_r}'(0)=-c_r>0$, which contradicts $s_0=c_r>T_2$. If $s_0>c_r$, we find numbers t_1,t_2, $\max(c_r,T_0)<t_1<t_2$, such that $\mathcal{E}(V_{t_1})^{-1}\mathcal{E}(V_{t_2})\in\mathcal{M}_0^1$ is not linear. Using $\vartheta_t\in\mathcal{N}_0$ we conclude as in in the first part of the proof of Lemma 8.5 that $q^\bullet\in\mathcal{N}_{\kappa+1}$, which again is a contradiction.

If $T_1=c_-$, i.e. $c_t'(0)=0$ for all $t>c_-$, then $V_{t_1t_2}=W_{(t_2-t_1,0)}$ for all $t_2>t_1>c_-$. This contradicts Lemma 8.5.

If $T_1<\infty$, then $\lim_{t\nearrow T_1}\mathfrak{t}(\mathcal{E}(V_t))=\infty$. This follows from the fact that

$$(1,0)\mathcal{E}(V_t)'(0)\begin{pmatrix}0\\1\end{pmatrix}=\frac{1}{c_t'(0)}+c.$$

Since $V_{t_1t_2}=W_{(t_2-t_1,0)}$ for $t_2>t_1\geq T_1$, we have $\sigma_t=\infty,\ t\geq T_1$ and hence

$$\lim_{t\nearrow T_1}\mathcal{E}(V_t)\circ 0=\begin{pmatrix}1&c\frac{1}{z}\\0&\frac{1}{z^2}\end{pmatrix}\circ(\lim_{t\to T_1}(V_t\circ\infty))=z^2(V_{T_1}\circ\infty)(z)+cz=q^\bullet.$$

Thus, the chain $(V_t^\bullet)$ has Weyl coefficient $q^\bullet$. If $T_1=\infty$ note that for any $\theta\in\mathcal{N}_0$ also the function σ^t which corresponds to θ by (10.5) is contained in $\mathcal{N}_0$. Hence, by Lemma 8.2 we obtain

$$\lim_{t\to\infty}\mathcal{E}(V_t)\circ\theta=\begin{pmatrix}1&c\frac{1}{z}\\0&\frac{1}{z^2}\end{pmatrix}\circ(\lim_{t\to\infty}(V_t\circ\sigma^t))=q^\bullet.$$

By Remark 8.3 we conclude that $\mathfrak{t}(\mathcal{E}(V_t))\to\infty$ and that the chain $(V_t^\bullet)$ has Weyl coefficient $q^\bullet$.

□

Let $c\in\mathbb{R}$ be fixed and let

$$W(z)=\begin{pmatrix}A(z)&B(z)\\C(z)&D(z)\end{pmatrix}\in\mathcal{M}_\kappa^1,\ W(0)=1.$$

Assume that the numbers

$$R = \operatorname{Re}\left(\frac{B(i) - icD(i)}{A(i) - icC(i)}\right),\ J = \operatorname{Im}\left(\frac{B(i) - icD(i)}{A(i) - icC(i)}\right), \tag{10.6}$$

are finite, i.e. that $A(i) - icC(i) \neq 0$, and that $J \neq 0$. Then we define a transformation

$$\mathcal{F}(W) := \begin{pmatrix} \frac{1}{z^2+1} & -cz\frac{1}{z^2+1} \\ 0 & 1 \end{pmatrix} W(z) \begin{pmatrix} 1 - z\frac{R}{J} & -z(J + \frac{R^2}{J}) \\ z\frac{1}{J} & 1 + z\frac{R}{J} \end{pmatrix}.$$

Write

$$\mathcal{F}(W)(z) = \begin{pmatrix} \hat{A}(z) & \hat{B}(z) \\ \hat{C}(z) & \hat{D}(z) \end{pmatrix} \in \mathcal{M}_\kappa^1,$$

then we find

$$\hat{A}(z) = \frac{1}{z^2+1}((1 - z\frac{R}{J})(A(z) - czC(z)) + z\frac{1}{J}(B(z) - czD(z))),$$

$$\hat{B}(z) = \frac{1}{z^2+1}(z(-J - \frac{R^2}{J})(A(z) - czC(z)) + (1 + z\frac{R}{J})(B(z) - czD(z))),$$

$$\hat{C}(z) = (1 - z\frac{R}{J})C(z) + z\frac{1}{J}D(z),$$

$$\hat{D}(z) = -z(J + \frac{R^2}{J})C(z) + (1 + z\frac{R}{J})D(z).$$

Since $\det \mathcal{F}(W) = 1$ we know by Lemma 5.10 that $\mathcal{F}(W)$ belongs to $\mathcal{M}_\nu^1$ for some $\nu \geq 0$. Moreover, we see that the functions

$$(1 - z\frac{R}{J})(A(z) - czC(z)) + z\frac{1}{J}(B(z) - czD(z))$$

and

$$z(-J - \frac{R^2}{J})(A(z) - czC(z)) + (1 + z\frac{R}{J})(B(z) - czD(z))$$

are zero at i and $-i$. Hence $\mathcal{F}(W)$ is an entire matrix function.

Lemma 10.8. *Let a chain $(W_t)_{t>c_-} \in \mathfrak{C}$ with index κ of negativity be given and assume that its Weyl coefficient q belongs to $\mathcal{N}_\kappa$ and is regular at ∞. Moreover, assume that*

$$q^\bullet(z) := \begin{pmatrix} \frac{1}{z^2+1} & -cz\frac{1}{z^2+1} \\ 0 & 1 \end{pmatrix} \circ q(z) = \frac{1}{z^2+1}(q(z) - cz) \in \mathcal{N}_{\kappa+1}.$$

Then there exists a number T_0, $c_- \leq T_0 < \infty$, such that $W_t \in \operatorname{dom} \mathcal{F}$ for $t > T_0$. The limit $l^\bullet := \lim_{t\to\infty} \mathfrak{t}(\mathcal{F}(W_t))$ exists in $\mathbb{R} \cup \{+\infty\}$. For a certain number $\Phi \in [0, \pi)$, the chain $(W_t^\bullet)$ defined by

$$W^\bullet_{\mathfrak{t}(\mathcal{F}(W_t))} := \mathcal{F}(W_t),\ t > T_0,$$

and, in case $l^\bullet < \infty$,

$$W_x^\bullet := W_{l^\bullet}^\bullet W_{(x-l^\bullet,\Phi)},\ x > l^\bullet,$$

belongs to $\mathfrak{C}$, has index $\kappa+1$ of negativity and its Weyl coefficient is $q^\bullet$.
Proof : Let

$$p_t := \frac{(A(-i)+icC(-i))(B(i)-icD(i))-(A(i)-icC(i))(B(-i)+icD(-i))}{2i}.$$

We see that $p_t = J_t|A(i)-icC(i)|^2$. Since $J_t = -\operatorname{Im}[(W_t^{-1}\circ cz)(i)]$, a short calculation using (5.2) shows that p_t is nondecreasing. Note that $p_t \neq 0$ if and only if $A(i)-icC(i)\neq 0$ and $J_t \neq 0$, or equivalently $W_t \in \operatorname{dom}\mathcal{F}$.

Note also that there exists a number $T_0 > c_-$ such that $p_t \neq 0$ for $t > T_0$. Assume the contrary: Then there exists a sequence (t_n), $t_n \to \infty$, such that for each $n\in\mathbb{N}$ either $A_{t_n}(i) - icC_{t_n}(i) = 0$ or $J_{t_n} = 0$. We put $\tau^n = \infty$ if $A_{t_n}(i)-icC_{t_n}(i) = 0$ and $\tau^n = -R_{t_n}$ otherwise, and obtain

$$q(i) = \lim_{n\to\infty}(W_{t_n}\circ\tau^n)(i) = ic,$$

which is a contradiction since, by Proposition 9.1, $q^\bullet \in \mathcal{N}_{\kappa+1}$ implies $q(i)\neq ic$. We choose T_0 minimal with respect to this property.

Whenever $\mathcal{F}(W_t)$ is defined we have

$$\mathcal{F}(W_t)\circ\theta = \begin{pmatrix} \frac{1}{z^2+1} & -cz\frac{1}{z^2+1} \\ 0 & 1 \end{pmatrix} \circ (W_t\circ\tau), \tag{10.7}$$

where

$$\tau(z) = F(z,W_t)\circ\theta(z) = \frac{\theta(z)(\frac{J_t}{z}-R_t)-(J_t^2+R_t^2)}{\theta(z)+\frac{J_t}{z}+R_t}, \tag{10.8}$$

or, equivalently,

$$\theta(z) = F(z,W_t)^{-1}\circ\tau(z) = -\frac{\tau(z)(R_t+\frac{J_t}{z})+(J_t^2+R_t^2)}{\tau(z)+(R_t-\frac{J_t}{z})}, \tag{10.9}$$

where

$$F(z,W_t) = \begin{pmatrix} 1-z\frac{R_t}{J_t} & -z(J_t+\frac{R_t^2}{J_t}) \\ z\frac{1}{J_t} & 1+z\frac{R_t}{J_t} \end{pmatrix}.$$

One easily checks that $F(z,W_t)\in\mathcal{M}_\delta^S$, $F(z,W_t)^{-1}\in\mathcal{M}_{1-\delta}^S$, where $S(z) = z-i$ and $\delta = 0$ if $J_t < 0$ and $\delta = 1$ if $J_t > 0$. For some $\tau_t \in \mathcal{N}_0$ we have $W_t\circ\tau_t = q$ and thus $\mathcal{F}(W_t)\circ\theta_t = q^\bullet$ where τ_t and θ_t are connected by (10.8). By Lemma 5.9 and [KW2] we have $\theta_t\in\mathcal{N}_0$ for $J_t > 0$ and $\theta_t \in \mathcal{N}_0\cup\mathcal{N}_1$ otherwise.

Now we compute $\operatorname{ind}_-\mathfrak{K}(\mathcal{F}(W_t))$. First note that by Theorem 5.7 $\mathfrak{K}(W_t) = \mathfrak{K}_-(W_t)$. For $\psi\in[0,\pi)$ we set $\theta_\psi = \cot\psi$ and $\tau_\psi(z) = F(z,W_t)\circ\cot\psi$ (cf. (10.7)). A straightforward calculation yields $\tau_\psi(z) = W_{(l_\psi,\phi_\psi)}\circ d_\psi$ where $\cot\phi_\psi = -R_t - \frac{J_t^2}{\cot\psi+R_t}$ and $l_\psi\in\mathbb{R}\setminus\{0\}$, $d_\psi \neq \cot\phi_\psi$. Hereby $\frac{-\operatorname{sgn}(l_\psi)+1}{2} = \frac{\operatorname{sgn}(J_t)+1}{2} = \operatorname{ind}_-\tau_\psi = \operatorname{ind}_-\mathfrak{K}(W_{(l_\psi,\phi_\psi)})$. We choose ψ such that $\cot\psi$ is not the exceptional number for $\mathcal{F}(W_t)$ in Lemma 5.12 and such that $\cot\phi_\psi$ is not the exceptional number for W_t. It follows from Lemma 5.12 that $\operatorname{ind}_-\mathfrak{K}(\mathcal{F}(W_t)) = \operatorname{ind}_-\mathcal{F}(W_t)\circ\cot\psi$ and from Lemma 5.12 and Theorem I.12.2 that

$$\operatorname{ind}_-\mathfrak{K}(W_t) + \operatorname{ind}_-\tau_\psi = \operatorname{ind}_-\mathfrak{K}(W_t) + \operatorname{ind}_-\mathfrak{K}(W_{(l_\psi,\phi_\psi)}) =$$

$$= \operatorname{ind}_{-}\mathfrak{K}(W_t W_{(l_\psi,\phi_\psi)}) = \operatorname{ind}_{-} W_t W_{(l_\psi,\phi_\psi)} \circ d_\psi = \operatorname{ind}_{-} W_t \circ \tau_\psi.$$

Consider the relation $\frac{(W_t \circ \tau_\psi)(z) - cz}{z^2+1} = (\mathcal{F}(W_t) \circ \cot\psi)(z)$. Since $\det \mathcal{F}(W_t)(i) = 1$, we see that $\mathcal{F}(W_t) \circ \cot\psi$ is analytic at i for all $\psi \in [0,\pi)$ with one possible exception. It follows from Corollary 5.3 and Proposition 9.1 that $\operatorname{ind}_{-} W_t \circ \tau_\psi = \operatorname{ind}_{-}\mathcal{F}(W_t) \circ \cot\psi$ for all but at most three excepted $\psi \in [0,\pi)$. Thus, $\operatorname{ind}_{-}\mathfrak{K}(\mathcal{F}(W_t)) = \operatorname{ind}_{-}\mathfrak{K}(W_t) + \frac{\operatorname{sgn}(J_t)+1}{2}$, and we showed that $\operatorname{ind}_{-}\mathfrak{K}(\mathcal{F}(W_t)) = \kappa + 1$ for $J_t > 0$ and $\operatorname{ind}_{-}\mathfrak{K}(\mathcal{F}(W_t)) = \kappa$ for $J_t < 0$. Since $\mathcal{F}(W_t) \circ \cot\psi$ is finite at ∞ (cf. Corollary 5.3) and as $\operatorname{ind}_{-}\mathfrak{K}(\mathcal{F}(W_t)) = \operatorname{ind}_{-}\mathcal{F}(W_t) \circ \cot\psi$ for sufficiently many $\psi \in [0,\pi)$, Theorem 5.7 shows that $\mathfrak{K}(\mathcal{F}(W_t)) = \mathfrak{K}_{-}(\mathcal{F}(W_t))$ and that $1 \in \mathfrak{P}(E_{\mathcal{F}(W_t)})$. Since $q^\bullet \in \mathcal{N}_{\kappa+1}$, we obtain that $\theta_t \in \mathcal{N}_1$ if $J_t < 0$.

We show that $J_t > 0$ for $t > T_0$. Assume the contrary: Since p_t is nondecreasing we have $J_t < 0$ for all $t > T_0$. Choose $\psi \in [0,\pi)$, then the function $\mathcal{F}(W_t) \circ \cot\psi$ is contained in $\mathcal{N}_{\leq\kappa}$. Since $\tau_\psi \in \mathcal{N}_0$, the right hand side of (10.7) tends to $q^\bullet \in \mathcal{N}_{\kappa+1}$ if $t \to \infty$. But this contradicts the fact that a limit of $\mathcal{N}_{\leq\kappa}$ functions belongs to $\mathcal{N}_{\leq\kappa}$. In the following we choose T_0 minimal with respect to the property that $p_t > 0$ for all $t > T_0$.

For $t_2 \geq t_1 > T_0$ we consider $\mathcal{F}(W_{t_1})^{-1}\mathcal{F}(W_{t_2})$. First note that $\mathcal{F}(W_{t_1})^{-1}\mathcal{F}(W_{t_2}) = F(z, W_{t_1})^{-1} W_{t_1 t_2} F(z, W_{t_2}) \in \mathcal{M}_0^1 \cup \mathcal{M}_1^1$. Since

$$\frac{B_{\mathcal{F}(W_t)}(z)}{A_{\mathcal{F}(W_t)}(z)} = \Big(\frac{J_t}{z} + R_t - \frac{J_t^2}{z^2}\frac{1}{\frac{D_t(z)}{C_t(z)} + \frac{J_t}{z} - R_t}\Big) - \frac{J_t^2}{\frac{B_{W_t}(z)}{A_{W_t}(z)} + \frac{J_t}{z} - R_t}, \tag{10.10}$$

it follows from Lemma 10.4, that the domain of the multiplication operator in $\mathfrak{P}(E_{\mathcal{F}(W_t)})$ is dense if and only if the domain of the multiplication operator in $\mathfrak{P}(E_{W_t})$ is dense. If this happens, it follows from Lemma 5.17 that $\mathcal{F}(W_{t_1})^{-1}\mathcal{F}(W_{t_2}) \in \mathcal{M}_0^1$. If the domain of the multiplication operator is not dense, the same considerations introducing W_t^ϵ as in the proof of Lemma 10.5 show that $\mathcal{F}(W_{t_1})^{-1}\mathcal{F}(W_{t_2}) \in \mathcal{M}_0^1$. An elementary calculation yields that $\mathcal{F}(W_{t_1 t_2}) \neq I$ for $t_2 > t_1 > T_0$, hence $\mathfrak{t}(\mathcal{F}(W_{t_1 t_2}))$ is strictly increasing.

If $l^\bullet = \lim_{t\to\infty} \mathfrak{t}(\mathcal{F}(W_t)) = \infty$, the chain $W^\bullet_{\mathfrak{t}(\mathcal{F}(W_t))} := (\mathcal{F}(W_t))$, $t > T_0$, is contained in $\mathfrak{C}$ and has index $\kappa + 1$. Put $\tau(z) = u \in \overline{\mathbb{R}}$, and let $\theta_{u,t} = F(z, W_t)^{-1} \circ u$ be the function corresponding to τ in (10.9). As $J_t > 0$ the function $\theta_{u,t}$ belongs to $\mathcal{N}_0$. By Lemma 8.2 the Weyl coefficient of the chain $(W_x^\bullet)$ is $q^\bullet$.

Now consider the case that $l^\bullet < \infty$. In the same way as in Lemma 10.5 we extend $W_x^\bullet$ by continuity to $x = l^\bullet$. Since $\theta_t \in \mathcal{N}_0$ for $t > T_0$ and since $q^\bullet \in \mathcal{N}_{\kappa+1}$, we see that $W_{l^\bullet}^\bullet \in \mathcal{M}_{\kappa+1}$. We have to construct the number $\Phi \in [0,\pi)$ which has to be used in the definition of $W_x^\bullet$, $x > l^\bullet$. Consider the relation

$$q^\bullet(z) = \frac{1}{z^2+1}\big(\lim_{t\to\infty} (W_t \circ u^t)(z) - cz\big)$$

for $u^t \in \overline{\mathbb{R}}$. Since the corresponding functions $\theta_{u^t,t}$ are contained in $\mathcal{N}_0$ and $\mathcal{N}_0$ is compact with respect to locally uniform convergence on $\mathbb{C} \setminus \mathbb{R}$, we find for each sequence $(s_n)_{n\in\mathbb{N}}$, $s_n > T_0$, tending to ∞ a subsequence $(t_n)_{n\in\mathbb{N}}$ tending to ∞ such that $\theta_{u^{t_n},t_n}(z)$ converges to some function $\theta \in \mathcal{N}_0$. It follows that

$$q^\bullet(z) = \frac{1}{z^2+1}\big(\lim_{t\to\infty} (W_t \circ u^t)(z) - cz\big) = \lim_{n\to\infty} (\mathcal{F}(W_{t_n}) \circ \theta_{u^{t_n},t_n})(z) = (W_{l^\bullet}^\bullet \circ \theta)(z).$$

In this equation the left side does not depend on $(s_n)_{n\in\mathbb{N}}$ and not on the family $(u^t)_{t>T_0}$. Hence θ is independent from the sequence $(s_n)_{n\in\mathbb{N}}$ and from the family $(u^t)_{t>T_0}$. This shows that the limit

$$\lim_{t\to\infty} \theta_{u^t,t} = \theta,$$

exists locally uniformly and does not depend on $(u^t)_{t>T_0}$. If $|R_t + iJ_t|$ tends to infinity, we see from the definition of $\theta_{u,t}$ (cf. (10.9)) that $\theta = \infty$. Otherwise there is a sequence $(t_n)_{n\in\mathbb{N}}$, $t_n > T_0$, tending to infinity such that $R_{t_n} + iJ_{t_n}$ converges to a certain limit $R + iJ$. Now we set $u^t = \infty$ and obtain that $\theta(z) = -R - \frac{J}{z}$. Setting $u^t = -R_t$ we get $\theta(z) = -R + Jz$. Thus $J = 0$ and $\theta = -R$.

In either case let $\Phi \in [0,\pi)$ be such that $\cot\Phi := \theta$ and consider the chain $(W_x^\bullet)$ defined in the assertion of the lemma. Since the relation

$$W_x^\bullet \circ \theta = W_{l^\bullet}^\bullet \circ \theta = q^\bullet,\ x \geq l^\bullet, \tag{10.11}$$

holds, we have $\mathrm{ind}\_\mathfrak{K}(W_x^\bullet) \geq \kappa + 1$. In any case $\mathrm{ind}\_\mathfrak{K}(W_x^\bullet) \leq \kappa + 1$ and we conclude that $(W_x^\bullet) \in \mathfrak{C}$ and has index $\kappa + 1$ of negativity. The relation (10.11) shows that the Weyl coefficient of $(W_x^\bullet)$ is $q^\bullet$.

□

In the sequel we consider the inverse transformation of $\mathcal{F}$. Let

$$V(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} \in \mathcal{M}_\kappa^1,\ V(0) = 1,$$

and define

$$r := \mathrm{Re}\,(\frac{d(i)}{c(i)}),\ j = -\,\mathrm{Im}\,(\frac{d(i)}{c(i)}),$$

if $c(i) \neq 0$. If additionally $j \neq 0$ we define a transformation

$$\mathcal{D}(V) := \begin{pmatrix} 1 & c\frac{z}{z^2+1} \\ 0 & \frac{1}{z^2+1} \end{pmatrix} V(z) \begin{pmatrix} 1 + \frac{r}{j}z & (j + \frac{r^2}{j})z \\ -\frac{1}{j}z & 1 - \frac{r}{j}z \end{pmatrix}.$$

Since $\det \mathcal{D}(V)(z) = 1$ we know by Lemma 5.10 that $\mathcal{D}(V)$ belongs to $\mathcal{M}_\nu^1$ for some $\nu \geq 0$. As for $\mathcal{F}$ we see that $\mathcal{D}(V)$ is an entire matrix function.

Lemma 10.9. *The transformations $\mathcal{F}$ and $\mathcal{D}$ are inverses of each other in the sense that $\mathcal{D}\mathcal{F}(W) = W$ ($\mathcal{F}\mathcal{D}(W) = W$) whenever $W \in \mathrm{dom}\,\mathcal{F}$ ($W \in \mathrm{dom}\,\mathcal{D}$).*
Proof : Let $c \in \mathbb{R}$ be given and let

$$W(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix},$$

be such that R and J as defined in (10.6) are finite and $J \neq 0$. Put

$$\mathcal{F}(W) =: \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}.$$

Using the relation (10.10) we find

$$r = \operatorname{Re}\left(\frac{d(i)}{c(i)}\right) = R, \ j = -\operatorname{Im}\left(\frac{d(i)}{c(i)}\right) = J,$$

hence $\mathcal{F}(W) \in \operatorname{dom}\mathcal{D}$ and a computation will prove that $\mathcal{D}(\mathcal{F}(W)) = W$. If $W \in \operatorname{dom}\mathcal{D}$ and we put

$$\mathcal{D}(W) =: \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix},$$

we obtain that

$$\frac{b(i) - icd(i)}{a(i) - icc(i)} = r + ij,$$

hence $\mathcal{D}(W) \in \operatorname{dom}\mathcal{F}$. It follows from the already proved that $\mathcal{DFD}(W) = \mathcal{D}(W)$ and since clearly $\mathcal{D}$ is injective we find $\mathcal{F}(\mathcal{D}(W)) = W$.

□

Lemma 10.10. *Let a chain $(V_t)_{t>c_-} \in \mathfrak{C}$ with index $\kappa + 1$, $\kappa \geq 0$, of negativity be given, assume that its Weyl coefficient q belongs to $\mathcal{N}_{\kappa+1}$ and is such that the limit $\lim_{y\to+\infty} yq(iy)$ exists. Let $c \in \mathbb{R}$ be the unique number such that*

$$q^\bullet(z) := (z + i)(z - i)q(z) + cz$$

is regular at ∞ and assume that $q^\bullet \in \mathcal{N}_\kappa$. Then there exists a number T_1, $c_- < T_1 \leq \infty$, such that $W_t \in \operatorname{dom}\mathcal{D}$ if and only if $c_- < t < T_1$. The chain

$$V^\bullet_{\mathrm{t}(\mathcal{D}(V_t))} := \mathcal{D}(V_t), \ t < T_1,$$

belongs to $\mathfrak{C}$, has index κ of negativity and its Weyl coefficient is $q^\bullet$.

Proof : Let

$$q_t := -\frac{d_t(i)\overline{c_t(i)} - c_t(i)\overline{d_t(i)}}{2i}.$$

We see that $q_t = j_t|c_t(i)|^2$. Since for $t_2 \geq t_1$ (cf. (I.12.3))

$$-q_{t_2} = -q_{t_1} + \begin{pmatrix} c_{t_1}(i) & d_{t_1}(i) \end{pmatrix} H_{V_{t_1 t_2}}(i,i)\overline{\begin{pmatrix} c_{t_1}(i) \\ d_{t_1}(i) \end{pmatrix}}, \tag{10.12}$$

q_t is nonincreasing. Note that $q_t \neq 0$ if and only if $c_t(i) \neq 0$ and $j_t \neq 0$, or equivalently $V_t \in \operatorname{dom}\mathcal{F}$.

Whenever $\mathcal{D}(V_t)$ is defined the following relation holds:

$$\mathcal{D}(V_t) \circ \vartheta = \begin{pmatrix} 1 & cz\frac{1}{z^2+1} \\ 0 & \frac{1}{z^2+1} \end{pmatrix} \circ (V_t \circ \sigma),$$

where

$$\sigma(z) = R(z, V_t) \circ \vartheta = \frac{-\vartheta(z)(\frac{j_t}{z} + r_t) - (j_t^2 + r_t^2)}{\vartheta(z) - \frac{j_t}{z} + r_t}, \tag{10.13}$$

or, equivalently,

$$\vartheta(z) = R(z,V_t)^{-1}\circ\sigma = \frac{\sigma(z)(\frac{j_t}{z}-r_t)-(j_t^2+r_t^2)}{\sigma(z)+(\frac{j_t}{z}+r_t)},$$

Herby

$$R(z,V_t) = \left(\begin{array}{cc} 1+\frac{r}{j}z & (j+\frac{r^2}{j})z \\ -\frac{1}{j}z & 1-\frac{r}{j}z \end{array}\right).$$

One easily checks that $R(z,V_t)\in\mathcal{M}_\delta^S,\ R(z,V_t)^{-1}\in\mathcal{M}_{1-\delta}^S$, where $S(z)=z-i$ and $\delta=0$ if $j_t>0$ and $\delta=1$ if $j_t<0$. For some $\sigma_t\in\mathcal{N}_0$ we have $V_t\circ\sigma_t=q$ and thus $\mathcal{D}(V_t)\circ\vartheta_t=q^\bullet$ where σ_t and ϑ_t are connected by (10.13). By Lemma 5.9 and [KW2] we have $\vartheta_t\in\mathcal{N}_0$ for $j_t<0$ and $\vartheta_t\in\mathcal{N}_0\cup\mathcal{N}_1$ otherwise.

In the following let $c_r\geq c_-$ be the supremum of all numbers $t>c_-$ such that V_t is linear. By Lemma 8.5 we know that $c_r<\infty$. Note also that by Theorem 5.7 $\mathfrak{K}(V_t)=\mathfrak{K}_-(V_t),\ t>c_-$.

Now we compute $\mathrm{ind}_-\mathfrak{K}(\mathcal{D}(V_t)),\ t>c_r$. For $\psi\in[0,\pi)$ we set $\vartheta_\psi=\cot\psi$ and $\sigma_\psi(z)=R(z,V_t)\circ\cot\psi$ (cf.(10.5)). A straightforward calculation yields $\sigma_\psi(z)=W_{(l_\psi,\phi_\psi)}\circ d_\psi$ where $\cot\phi_\psi=-\frac{\cot\psi r_t+j_t^2+r_t^2}{\cot\psi-r_t}$ and $l_\psi\in\mathbb{R}\setminus\{0\},\ d_\psi\neq\cot\phi_\psi$. Hereby $\frac{-\mathrm{sgn}(l_\psi)+1}{2}=\frac{-\mathrm{sgn}(j_t)+1}{2}=\mathrm{ind}_-\sigma_\psi=\mathrm{ind}_-\mathfrak{K}(W_{(l_\psi,\phi_\psi)})$. We choose ψ such that $\cot\psi$ is not the exceptional number for $\mathcal{D}(V_t)$ in Lemma 5.12 and such that $\cot\phi_\psi$ is not the exceptional number for V_t. It follows from Lemma 5.12 that $\mathrm{ind}_-\mathfrak{K}(\mathcal{D}(D_t))=\mathrm{ind}_-\mathcal{D}(V_t)\circ\cot\psi$ and from Lemma 5.12 and Theorem I.12.2 that

$$\mathrm{ind}_-\mathfrak{K}(V_t)+\mathrm{ind}_-\sigma_\psi=\mathrm{ind}_-\mathfrak{K}(V_t)+\mathrm{ind}_-\mathfrak{K}(W_{(l_\psi,\phi_\psi)})=$$

$$=\mathrm{ind}_-\mathfrak{K}(V_tW_{(l_\psi,\phi_\psi)})=\mathrm{ind}_-V_tW_{(l_\psi,\phi_\psi)}\circ d_\psi=\mathrm{ind}_-V_t\circ\sigma_\psi.$$

Consider the relation $(z^2+1)(V_t\circ\sigma_\psi)(z)+cz=(\mathcal{D}(V_t)\circ\cot\psi)(z)$. It follows from $t>c_r$, Lemma 5.16 and Proposition 9.1 that $\mathcal{D}(V_t)\circ\cot\psi$ is regular at ∞ if ψ does not correspond to the exceptional value of Lemma 5.16. Since $\det\mathcal{D}(V_t)(i)=1$, we see that $(\mathcal{D}(V_t)\circ\cot\psi)(i)\neq 0$ for all $\psi\in[0,\pi)$ with one possible exception. It follows from Proposition 9.1 that $\mathrm{ind}_-V_t\circ\sigma_\psi=\mathrm{ind}_-\mathcal{D}(V_t)\circ\cot\psi+1$ for all but at most four excepted values of ψ in $[0,\pi)$. Thus, $\mathrm{ind}_-\mathfrak{K}(\mathcal{D}(V_t))=\mathrm{ind}_-\mathfrak{K}(V_t)+\frac{-\mathrm{sgn}(j_t)+1}{2}-1$, and we showed that $\mathrm{ind}_-(\mathcal{D}(V_t))=\kappa+1$ for $j_t<0,\ t>c_r$ and $\mathrm{ind}_-(\mathcal{D}(V_t))=\kappa$ for $j_t>0,\ t>c_r$. Since $\mathcal{D}(V_t)\circ\cot\psi$ is regular at ∞ and as $\mathrm{ind}_-\mathfrak{K}(\mathcal{D}(V_t))=\mathrm{ind}_-\mathcal{D}(V_t)\circ\cot\psi$ for almost all values of ψ in $[0,\pi)$, Theorem 5.7 shows that $\mathfrak{K}(\mathcal{D}(V_t))=\mathfrak{K}_-(\mathcal{D}(V_t)),\ t>c_r$.

Now we show that

$$\mathcal{D}(V_{t_1})^{-1}\mathcal{D}(V_{t_2})\in\mathcal{M}_0^1,$$

whenever $c_r<t_1\leq t_2$ are such that q_{t_1} and q_{t_2} are both not zero and have the same sign. First note that, as $\mathcal{D}(V_{t_1})^{-1}\mathcal{D}(V_{t_2})=R(z,V_{t_1})^{-1}V_{t_1t_2}R(z,V_{t_2})$, this matrix function belongs to $\mathcal{M}_0^1\cup\mathcal{M}_1^1$. As

$$\frac{B_{\mathcal{D}(V_t)}(z)}{A_{\mathcal{D}(V_t)}(z)}=(-\frac{j_t}{z}+r_t-\frac{j_t^2}{z^2}\frac{1}{\frac{B_{V_t}(z)}{A_{V_t}(z)}-\frac{j_t}{z}-r_t})-\frac{j_t^2}{\frac{B_{V_t}(z)}{A_{V_t}(z)}-\frac{j_t}{z}-r_t},$$

the domain of the multiplication operator in $\mathfrak{P}(E_{\mathcal{D}(V_t)})$ is dense if and only if the domain of the multiplication operator in $\mathfrak{P}(E_{V_t})$ is dense (cf. Lemma 10.4). If this happens, it follows from Lemma 5.17 that $\mathcal{D}(V_{t_1})^{-1}\mathcal{D}(V_{t_2}) \in \mathcal{M}_0^1$. If the domain of the multiplication operator is not dense, the same considerations introducing V_t^ϵ as in the proof of Lemma 10.7 show that $\mathcal{D}(V_{t_1})^{-1}\mathcal{D}(V_{t_2}) \in \mathcal{M}_0^1$. An elementary calculation shows that $\mathcal{D}(V_{t_1t_2}) \neq I$ for $t_2 > t_1$, hence $\mathfrak{t}(\mathcal{D}(V_{t_1t_2}))$ is strictly increasing.

The function q_t is continuous and nonincreasing, hence the set (c_-, ∞) is devided into three, possibly empty, intervals: namely (c_-, T_1) ($[T_1, T_2]$, (T_2, ∞)), where q_t is positive (zero, negative). We shall show that $T_1 > c_-$ and that already $T_2 = \infty$, i.e. that q_t is always nonnegative.

Now assume that for some t we have $c_t \neq 0$ and $j_t < 0$, i.e. $T_2 < \infty$. We show that for each $t_1 > T_2$ there is some numbers $t_2 > t_1$ such that the space $\mathfrak{K}(\mathcal{D}(V_{t_1})^{-1}\mathcal{D}(V_{t_2}))$ is not one-dimensional. Since the right factor in the definition of $\mathcal{D}$ is linear in z, an elementary consideration shows that $\mathcal{D}(V_{t_1})^{-1}\mathcal{D}(V_{t_2})$ is linear if and only $V_{t_1}^{-1}V_{t_2}$ is linear. Assume on the contrary that there is a $t_1 > T_2$ such that for all $t_2 > t_1$ we have $V_{t_1t_2} = W_{(t_2-t_1,\phi)}$, i.e. that the interval $[t_1, \infty)$ is indivisable in the chain (V_t). Then q is the 1-resolvent of the extension of the multiplication operator determined by a $\phi \in [0, \pi)$. Since $q^\bullet \in \mathcal{N}_\kappa$ Proposition 9.1 yields that q has a pole at i and we conclude from Lemma 5.12 and Lemma I.6.4 that $\frac{d_t(i)}{c_t(i)} = -\cot\phi \in \overline{\mathbb{R}}$. Since we assumed $j_t < 0$, this is impossible. We obtained, in particular, that there is a number $t_1 > \max(T_2, c_r)$ such that $\mathcal{E}(V_{t_1})^{-1}\mathcal{E}(V_{t_2})$ is not linear for some choice of $t_2 > t_1$ and belongs to $\mathcal{M}_0^1$. Using $\vartheta_t \in \mathcal{N}_0$ in the case $j_t < 0$ we conclude as in the first part of the proof of Lemma 8.5 that $q^\bullet \in \mathcal{N}_{\kappa+1}$, which is a contradiction again.

To prove $T_1 > c_-$ note that if $q_t = 0$ for two numbers $t = t_1, t_2$ with $t_1 < t_2$ then it follows from (10.12) and from (I.8.2) that $V_{t_1t_2}$ is of the form $W_{(t_2-t_1,\varphi)}$. Thus, if we had $T_1 = c_-$, Lemma 8.5 would we violated for the chain $(V_t)_{t>c_-}$.

We show that $\lim_{t\nearrow T_1} \mathfrak{t}(\mathcal{D}(V_t)) = \infty$. Consider first the case that $T_1 = \infty$. Let $\vartheta \in \mathcal{N}_0$, $t < T_1$ and denote by σ^t the function corresponding to ϑ by (10.13). Since $R(t,z) \in \mathcal{M}_0^S$ if $j_t > 0$, we have $\sigma^t \in \mathcal{N}_0$. Hence the limit

$$\lim_{t\to\infty} \mathcal{D}(V_t) \circ \vartheta = \begin{pmatrix} 1 & cz\frac{1}{z^2+1} \\ 0 & \frac{1}{z^2+1} \end{pmatrix} \circ (V_t \circ \sigma^t) = q^\bullet$$

does not depend on ϑ. By Remark 8.3 we have $\mathfrak{t}(\mathcal{D}(V_t)) \to \infty$. Moreover, we see that the chain $(V_t^\bullet)$ has Weyl coefficient $q^\bullet$.

Now consider the case that $T_1 < \infty$. An elementary calculation yields $\mathfrak{t}(\mathcal{D}(V_t)) = \frac{1}{j_t} + j_t + \frac{r_t^2}{j_t}$. If there is a monotone sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \nearrow T_1$ and $j_{t_n} \to 0$, then we conclude from this formula for $\mathfrak{t}(\mathcal{D}(V_t))$ and from the fact that $\mathfrak{t}(\mathcal{D}(V_t))$ is increasing that $\mathfrak{t}(\mathcal{D}(V_t)) \to \infty$. If there is no such sequence, we obtain $j_t > \epsilon$ for a fixed $\epsilon > 0$, and from the continuity of j_t at points where $c_t(i) \neq 0$ that $c_{T_1}(i) = 0$. Hence $r_t^2 + j_t^2 = |\frac{d_t(i)}{c_t(i)}|^2 \to \infty$ as $t \nearrow T_1$. Since $\mathfrak{t}(\mathcal{D}(V_t)) = \frac{r_t^2+j_t^2}{j_t} + \frac{1}{j_t}$, we obtain again $\mathfrak{t}(\mathcal{D}(V_t)) \to \infty$.

Since by the same argument as above $V_{t_1t_2} = W_{(t_2-t_1,\varphi)}$ for $t_2 > t_1 \geq T_1$ with some fixed $\varphi \in [0, \pi)$, we have $\sigma_t = \cot\varphi$, $t \geq T_1$. As q has a pole at i, we obtain $\cot\varphi = -\frac{d_{T_1}(i)}{c_{T_1}(i)}$ which coincides with $-r_{T_1}$ if $\cot\varphi \neq \infty$. Now for $t < T_1$ set $\vartheta^t = -r_t$ and let $\sigma^t = R(t,z) \circ \vartheta^t$.

Then $\sigma^t = -r_t + j_t z$ belongs to $\mathcal{N}_0$. If $\cot\varphi \neq \infty$, we see that $\sigma^t \to -r_{T_1}$ as $t \nearrow T_1$ for $z \in \mathbb{C} \setminus \mathbb{R}$. If $\cot\varphi = \infty$, we see that $|\sigma^t(z)| \to \infty$ at least for $z \in i\mathbb{R}^+$. In any case we have for $z \in i\mathbb{R}^+$, $\cot\varphi \neq \infty$

$$\lim_{t\nearrow T_1} (\mathcal{D}(V_t) \circ \vartheta^t)(z) = \begin{pmatrix} 1 & cz\frac{1}{z^2+1} \\ 0 & \frac{1}{z^2+1} \end{pmatrix} \circ (\lim_{t\nearrow T_1} (V_t \circ \sigma^t)(z)) =$$

$$= (z^2+1)(V_{T_1} \circ \cot\varphi)(z) + cz = q^\bullet(z).$$

Hence, we again see that the chain $(V_t^\bullet)$ has Weyl coefficient $q^\bullet$.

We conclude that the chain $(V_t^\bullet)$ is contained in $\mathfrak{C}$ has index κ of negativity and its Weyl coefficient is $q^\bullet$.

□

11 Proof of the converse theorem

In this section we complete the proof of Theorem 8.7 by carrying out the induction step.

Assume that Theorem 8.7 has already been proved for the index κ of negativity and let a function $q \in \mathcal{N}_{\kappa+1}$ be given. By Lemma 5.5 there exists a number $c_1 \geq 0$ such that

$$q_1(z) := -\frac{1}{q(z) + c_1 z}$$

is finite at ∞, contained in $\mathcal{N}_{\kappa+1}$ and $\lim_{y\to+\infty} q_1(iy) = 0$. If $q(z)$ is not regular at ∞ we can choose $c_1 = 0$. Otherwise we can choose $c_1 > 0$; note that then $\lim_{y\to\infty} yq_1(iy) = \frac{i}{c_1}$. By Proposition 9.1 there exist numbers $\alpha \in \mathbb{C}^+ \cup \mathbb{R}$ and $c \in \mathbb{R}$ such that the function

$$(z-\alpha)(z-\overline{\alpha})q_1(z) + cz$$

is regular at ∞ and contained in $\mathcal{N}_\kappa$. We distinguish the cases that $\alpha \in \mathbb{R}$ or $\alpha \in \mathbb{C}^+$.

Case 1, $\alpha \in \mathbb{R}$: Define functions $q_2(z) := q_1(z+\alpha)$ and $q_3(z) := z^2 q_2(z) + cz$. Then $q_2 \in \mathcal{N}_{\kappa+1}$ and q_3 is regular at ∞ and contained in $\mathcal{N}_\kappa$. By the inductive hypothesis there exists a chain $(W_t) \in \mathfrak{C}$ such that the Weyl coefficient of (W_t) is q_3. By Lemma 10.5 we may apply the transformation $\mathcal{T}$ and obtain a chain $(W_t^{\bullet\bullet}) \in \mathfrak{C}$ whose Weyl coefficient is q_2 and which has index $\kappa+1$ of negativity. By Lemma 10.2 the chain $(\mathcal{T}^{-\alpha}(W_t^{\bullet\bullet})_s)$ has the Weyl coefficient q_1 and index of negativity $\kappa+1$. If q is not regular at ∞, we have $c_1 = 0$ and Lemma 10.1 and Lemma 10.2 show that the chain

$$(V_r) := \mathcal{T}_J(\mathcal{T}^{-\alpha}(W_t^{\bullet\bullet})_r)$$

is contained in $\mathfrak{C}$, has index $\kappa+1$ of negativity and its Weyl coefficient is q. If q is regular at ∞, note first that by Lemma 5.16

$$i[1,1]_{\mathfrak{P}(E_{\mathcal{T}^{-\alpha}(W_t^{\bullet\bullet})_s})} = \lim_{y\to\infty} yq_1(iy) = \frac{i}{c_1},$$

for sufficiently large s (cf. Lemma 8.5). Since $\mathfrak{P}(E_{\mathcal{T}^{-\alpha}(W_{t^\bullet}^\bullet)_s})$ is isometrically isomorphical to $\mathfrak{K}(\mathcal{T}^{-\alpha}(W_{t^\bullet}^\bullet)_s)$, and this space is isometrically isomorphical to $\mathfrak{K}(-J\mathcal{T}^{-\alpha}(W_{t^\bullet}^\bullet)_sJ)$ we see that $[1,1]_{\mathfrak{P}(E_{\mathcal{T}^{-\alpha}(W_{t^\bullet}^\bullet)_s})} = [(1,0)^T,(1,0)^T]_{-J\mathcal{T}^{-\alpha}(W_{t^\bullet}^\bullet)_sJ}$. Hence by Lemma 10.1, Lemma 10.2 and Lemma 10.3 the chain

$$(V_r) := (\mathcal{T}_{-c_1z}\mathcal{T}_J(\mathcal{T}^{-\alpha}(W_{t^\bullet}^\bullet)_s)_r)$$

is contained in $\mathfrak{C}$, has index $\kappa+1$ of negativity and its Weyl coefficient is q.

To prove uniqueness assume that a chain $(\tilde{V}_r) \in \mathfrak{C}$ is given such that the Weyl coefficient of $(\tilde{V}_r)$ is q and the index of negativity of $(\tilde{V}_r)$ is $\kappa+1$. Consider the same functions q_1, q_2 and q_3 as above. Then by Lemma 10.3, Lemma 10.1, Lemma 10.2 and Lemma 10.7 the chain

$$(\tilde{W}_t) := (\mathcal{E}(\mathcal{T}^{\alpha}(\mathcal{T}_J\mathcal{T}_{c_1z}(\tilde{V}_r)_s)_{t^\bullet})_t)$$

is contained in $\mathfrak{C}$, has index κ of negativity and Weyl coefficient q_3. By the inductive hypothesis we have $(\tilde{W}_t) = (W_t)$, hence using the fact that all used transformations are bijections (of their respective domains) we obtain $\tilde{V}_r = V_r$.

<u>**Case 2, $\alpha \in \mathbb{C}^+$:**</u> Define a function $q_2(z) := q_1(z\operatorname{Im}\alpha + \operatorname{Re}\alpha)$, then $q_3(z) := (z-i)(z+i)q_2(z) + cz$ is regular at ∞ and contained in $\mathcal{N}_\kappa$. The same argument as in the first case but using the transformation

$$(V_r) := (\mathcal{T}_{-c_1z}(\mathcal{T}_J\mathcal{T}^{-\operatorname{Re}\alpha}(\mathcal{T}_{\cdot\frac{1}{\operatorname{Im}\alpha}}(\mathcal{F}(W_t)_{t^\bullet})_{s^\bullet})_s)_r)$$

yields existence of a chain in $\mathfrak{C}$ with Weyl coefficient q and index $\kappa+1$ of negativity. Uniqueness is proved similar as in Case 1.

The proof of Theorem 8.7 is now complete.

Note that the uniqueness statement of Theorem 8.7 can be slightly strengthened:

Corollary 11.1. *Assume that $W \in \mathcal{M}_\kappa^1$ and $\tau \in \mathcal{N}_\nu$ are given and that $q := W \circ \tau \in \mathcal{N}_{\kappa+\nu}$. Let $(W_t)_{t>c_-}$ be the unique chain in $\mathfrak{C}$ which has Weyl coefficient q and continue this chain downwards. Then $W = W_t$ for some t.*

Proof : Let (M_t) be the unique chain in $\mathfrak{C}$ with index ν of negativity which has τ as its Weyl coefficient, and consider the chain (WM_t). Since $\lim_{t\to\infty} WM_t \circ \alpha = q \in \mathcal{N}_{\kappa+\nu}$, $M_t \in \mathcal{M}_\nu^1$ and $W \in \mathcal{M}_\kappa^1$, for sufficiently large t we must have $WM_t \in \mathcal{M}_{\kappa+\nu}^1$. After an appropriate reparametrization we have $(WM_t) \in \mathfrak{C}$, hence $WM_t = W_t$. Fix t_0 and continue the chain (W_t) downwards from W_{t_0}. By *(vi)* of Theorem 7.1 we have $W = W_t$ for some t.

□

As another corollary of Theorem 8.7 we prove a version of the ordering theorem Proposition I.11.3 for dB-Pontryagin spaces which are isometrically contained in a space $\Pi(\phi)$.

Corollary 11.2. *Let $\phi \in \mathcal{F}$ and assume that $\infty \notin \sigma(\phi)$. Moreover, let $\mathfrak{P}_1 = \mathfrak{P}(E_1)$ and $\mathfrak{P}_2 = \mathfrak{P}(E_2)$ be dB-Pontryagin spaces, such that*

$$\Gamma' : \left\{ \begin{array}{ccc} \mathfrak{P}_i & \to & \Pi(\phi) \\ F(z) & \mapsto & F(z)(z-z_0) \end{array} \right.$$

is an isometry of $\mathfrak{P}_i$ *into* $\Pi(\phi)$. *Then either* $\mathfrak{P}_1 \subseteq \mathfrak{P}_2$ *or* $\mathfrak{P}_2 \subseteq \mathfrak{P}_1$.

Proof : By Corollary 6.3 there exist matrices $W_i \in \mathcal{M}^1_{\kappa_i}$, $\kappa_i = \text{ind}\_\mathfrak{P}(E_i)$, and functions $\tau_i \in \mathcal{N}_{\nu_i}$, $\nu_i = \text{ind}\_\Pi(\phi) - \kappa_i$, such that

$$W_1 \circ \tau_1 = W_2 \circ \tau_2 = \phi \cdot \left((\frac{1}{t-z} - \frac{t - \operatorname{Re} z_0}{|t-z_0|^2})|t-z_0|^2 \right).$$

Without loss of generality we may assume that $W_1(0) = W_2(0) = 1$. By Corollary 11.1 there exists either a matrix $W_{12} \in \mathcal{M}^1_{\kappa_2-\kappa_1}$ with $W_2 = W_1 W_{12}$ or a matrix $W_{21} \in \mathcal{M}^1_{\kappa_1-\kappa_2}$ with $W_1 = W_2 W_{21}$. Since both spaces $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are contained isometrically in $\Pi(\phi)$, the assertion follows from Theorem I.12.2 and Proposition I.13.5.

□

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