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On representations of matrix valued Nevanlinna functions by u-resolvents

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Abstract. We show that every matrix valued generalized Nevanlinna function can be represented as a u-resolvent of a certain selfadjoint relation acting in a Pontryagin space. The negative index of this Pontryagin space may be larger than the number of negative squares of the given function. The minimal index of negative squares which is needed to obtain such a representation is determined. In the case of scalar functions, the results presented give rise to some new classes of generalized Nevanlinna functions.

1. Introduction

It is well known (see, e.g., [KL2]) that a matrix valued generalized Nevanlinna function $q : \varrho(q) \to \mathbb{C}^{n\times n}$, $\varrho(q) \subseteq \mathbb{C}\setminus\mathbb{R}$, with domain of holomorphy $\varrho(q)$, admits a representation (for $z_0 \in \varrho(q)$) as

$$q(z) = q(z_0)^* + (z-\overline{z_0})\Gamma_q^*(I+(z-z_0)(A_q-z)^{-1})\Gamma_q,\ z\in\varrho(q), \tag{1.1}$$

where A_q is a certain selfadjoint relation acting in a Pontryagin space $\mathcal{P}_q$ and where Γ_q is a linear mapping of $\mathbb{C}^n$ into $\mathcal{P}_q$. Moreover, the negative index of $\mathcal{P}_q$ equals the number of negative squares of Nevanlinna kernel

$$N_q(z,w) = \begin{cases} \frac{q(z)-q(w)^*}{z-\overline{w}} & , z\neq\overline{w} \\ q'(z) & , z=\overline{w}\end{cases}$$

of q. It is shown in this note that there exists a representation of the form

$$q(z) = \Gamma^*(A-z)^{-1}\Gamma,\ z\in\varrho(q), \tag{1.2}$$

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where A is a selfadjoint relation in a Pontryagin space $\mathcal{P}$. Here the negative index of $\mathcal{P}$ is possibly larger than the number of negative squares of q.

To make these explanations more precise, recall that for $n \in \mathbb{N}$ and $\kappa \in \mathbb{N} \cup \{0\}$ the set $\mathcal{N}_\kappa^{n\times n}$ consists of all $n \times n$-matrix valued functions q, meromorphic in $\mathbb{C} \setminus \mathbb{R}$, which satisfy $q(\overline{z}) = q(z)^*$ for z in their domain of holomorphy $\varrho(q)$, and are such that the Nevanlinna kernel N_q has κ negative squares on $\varrho(q)$. The kernel condition means that for each $m \in \mathbb{N}$, $z_1, \dots, z_m \in \varrho(q)$ and $x_1, \dots, x_m \in \mathbb{C}^n$ the quadratic form

$$Q(\xi_1, \dots, \xi_m) = \sum_{i,j=1}^{m} (N_q(z_i, z_j)x_i, x_j)\xi_i\overline{\xi_j}$$

has at most κ negative squares, and that for some choice of m, $z_1, \dots, z_m$ and $x_1, \dots, x_m$ this upper bound is attained. Here $(.,.)$ denotes the usual inner product on $\mathbb{C}^n$. The functions in $\mathcal{N}_\kappa^{n\times n}$ are called generalized Nevanlinna functions. If $n = 1$, i.e. if q is a scalar function, we will write $\mathcal{N}_\kappa$ instead of $\mathcal{N}_\kappa^{1\times 1}$.

If $\mathcal{P}$ is a Pontryagin space we denote by $\operatorname{ind}_- \mathcal{P}$ ($\operatorname{ind}_+ \mathcal{P}$) the negative (positive) index of $\mathcal{P}$, i.e., the dimension of a maximal negative (positive) subspace of $\mathcal{P}$. For a subspace $\mathcal{L}$ of $\mathcal{P}$ let $\operatorname{ind}_0 \mathcal{L} = \dim \mathcal{L}^\circ$, where $\mathcal{L}^\circ$ denotes the isotropic part of $\mathcal{L}$: $\mathcal{L}^\circ = \mathcal{L} \cap \mathcal{L}^\perp$.

If $q \in \mathcal{N}_\kappa^{n\times n}$ the space $\mathcal{P}_q$ occurring in the representation (1.1) has negative index κ. A result of M.G.Krein and H.Langer (see [KL1], Satz 1.5) gives, under certain additional assumptions, a necessary and sufficient condition for a scalar function $q \in \mathcal{N}_\kappa$, to admit a representation (1.2) with $\operatorname{ind}_- \mathcal{P} = \kappa$. It follows from our considerations that for all $q \in \mathcal{N}_\kappa$ there exists a representation (1.2) if we allow $\operatorname{ind}_- \mathcal{P} = \kappa + 1$. Note that in the scalar case (1.2) is just a representation of q by a so called u-resolvent (we put $u = \Gamma(1)$):

(1.3) $$q(z) = [(A - z)^{-1}u, u],\ z \in \varrho(q).$$

In some generalizations of Krein's formula concerning the description of generalized resolvents of a symmetric operator, u-resolvents occur as a class of parameters, and, more generally, functions of the form (1.2) where $\mathcal{R}(\Gamma)$ is a neutral subspace of $\mathcal{P}$ also occur.

In this note we determine the minimal negative index ν, such that a given function $q \in \mathcal{N}_\kappa^{n\times n}$ admits a representation (1.2), where the selfadjoint relation A acts in a Pontryagin space with negative index ν and where $\mathcal{R}(\Gamma)$ is neutral. It turns out that such a representation always exists and that $\nu \le \kappa + n$.

In Section 2 we define a number κ_m and show that there exists a representation (1.2) where $\operatorname{ind}_- \mathcal{P} = \kappa_m$. Moreover, it is proved that any (in some sense) minimal representation is unitarily equivalent to the representation constructed. Hence, the number κ_m is actually the minimal negative index such that q admits a representation (1.2), i.e., $\kappa_m = \nu$. In Section 3 we show that the number κ_m is given by a kernel condition on the function q. Moreover, we determine the number κ_m, under certain additional assumptions, via analytic properties of q. Finally, in Section 4, we introduce and study some new classes of generalized Nevanlinna functions which are represented by u-resolvents of extensions of a finite dimensional shift operator.

For elementary facts concerning the geometry of Pontryagin spaces we refer to [IKL]. We also use some results of [DS] on symmetric relations. Our notation in this respect is also similar to [DS] and [IKL].

Representations of Nevanlinna functions as u-resolvents can also be obtained using the theory of so-called triplet spaces (see e.g. [Be], [KW], [DLZ]). This approach differs from the method presented here.

2. A representation of q

In the following let $q \in \mathcal{N}_\kappa^{n\times n}$ be fixed. Let us recall from [KL2] that in the representation (1.1) the space $\mathcal{P}_q$ is the Pontryagin space obtained by factorization with respect to its isotropic part and completion from the linear space of all finite formal sums

$$\mathcal{L}_q = \{\sum_i a_i e_{z_i} | a_i \in \mathbb{C}^n, z_i \in \varrho(q)\},$$

endowed with the inner product defined by

$$[ae_z, be_w]_{\mathcal{L}_q} = (N_q(z,w)a, b),\ \ a, b \in \mathbb{C}^n, z, w \in \varrho(q).$$

We will drop indices at inner products whenever no confusion can occur. The relation $A_q \subseteq \mathcal{P}_q^2$ is defined as the closure of

$$S_q = \langle (\sum_i a_i e_{z_i}; \sum_i z_i a_i e_{z_i}) \in \mathcal{L}_q^2 | \sum_i a_i = 0 \rangle,$$

where $\langle \dots \rangle$ denotes the linear span of the elements between the brackets.

For $h \in \mathbb{C}^n$ consider the linear functional $F_h : \mathcal{L}_q \to \mathbb{C}$, defined by

$$F_h(ae_z) = (q(z)a, h),\ \ a \in \mathbb{C}^n, z \in \varrho(q).$$

If F_h induces a well-defined continuous functional on $\mathcal{P}_q$, there exists a unique element $u(h) \in \mathcal{P}_q$, such that the relation

$$(q(z)a, h) = [ae_z, u(h)],\ \ a \in \mathbb{C}^n, z \in \varrho(q)$$

holds. Clearly the mapping $h \mapsto u(h)$ is linear.

Let $\mathcal{G} \subseteq \mathbb{C}^n$ be the space of all elements h, such that F_h induces a well-defined continuous functional on $\mathcal{P}_q$. Define an inner product on $\mathcal{G}$ by

$$[a, b]_{\mathcal{G}} = [u(a), u(b)],\ \ a, b \in \mathcal{G}. \tag{2.1}$$

The space $\mathcal{G}$ can be characterized by means of the relation A_q.

Lemma 2.1. *Let $h \in \mathbb{C}^n$. Then $h \in \mathcal{G}$ if and only if for some $z_0 \in \varrho(q)$ we have $he_{z_0} \in \mathcal{D}(A_q)$ and there exists an element u with $(he_{z_0}; u) \in A_q - z_0$, such that*

$$(q(z_0)a, h) = [ae_{z_0}, u]. \tag{2.2}$$

In this case (2.2) holds for all $z_0 \in \varrho(q)$, i.e. $u = u(h)$.

P r o o f. The representation (1.1) of q shows that

$$(q(z)a,h) = (q(\overline{z_0})a,h) + (z-\overline{z_0})[(I+(z-z_0)(A_q-z)^{-1})ae_{z_0}, he_{z_0}].$$

Assume that $he_{z_0} \in \mathcal{D}(A_q)$ and let $(he_{z_0}; u) \in A_q - z_0$, then

$$(q(z)a,h) = (q(\overline{z_0})a,h) +$$
$$+(z-\overline{z_0})[(A_q-\overline{z_0})^{-1}(I+(z-z_0)(A_q-z)^{-1})ae_{z_0}, u] =$$
$$= ((q(z_0)a,h) - [ae_{z_0}, u]) + [ae_z, u]. \tag{2.3}$$

Conversely, for $h \in \mathcal{G}$, let $(q(z)a,h) = [ae_z, u(h)]$. We show that $(he_{z_0}; u(h)+z_0he_{z_0}) \in A_q = A_q^*$:

$$[ae_z - ae_{z_0}, u(h)+z_0he_{z_0}] - [zae_z - z_0ae_{z_0}, he_{z_0}] =$$
$$= (q(z)a,h) - (q(z_0)a,h) + \overline{z_0}(\frac{q(z)-q(\overline{z_0})}{z-\overline{z_0}}a,h) - \overline{z_0}(\frac{q(z_0)-q(\overline{z_0})}{z_0-\overline{z_0}}a,h) -$$
$$-z(\frac{q(z)-q(\overline{z_0})}{z-\overline{z_0}}a,h) + z_0(\frac{q(z_0)-q(\overline{z_0})}{z_0-\overline{z_0}}a,h) = 0.$$

Hence $he_{z_0} \in \mathcal{D}(A_q)$ and the assertion follows. □

Put $\mathcal{H}_0 = \mathcal{G}^\perp$, where the orthogonal complement has to be understood with respect to the usual inner product $(.,.)$ of $\mathbb{C}^n$, and let $\mathcal{H}_1$ be an isomorphic copy of $\mathcal{H}_0$. If $n' = \dim \mathcal{H}_0 = \dim \mathcal{H}_1$, then as a linear space $\mathcal{H}_0 \cong \mathcal{H}_1 \cong \mathbb{C}^{n'}$.

Definition 2.2. Let $\mathcal{L}$ be the linear space

$$\mathcal{L} = \mathcal{L}_q \dot{+} \mathcal{H}_0 \dot{+} \mathcal{H}_1$$

endowed with the inner product defined by ($a,b \in \mathbb{C}^n$, $z,w \in \varrho(q)$, $h_0,h_0' \in \mathcal{H}_0$, $h_1,h_1' \in \mathcal{H}_1$)

$$[ae_z + h_0 + h_1, be_w + h_0' + h_1']_{\mathcal{L}} = (N_q(z,w)a,b)_{\mathbb{C}^n} + (q(z)a,h_0')_{\mathbb{C}^n} +$$
$$+(h_0, q(w)b)_{\mathbb{C}^n} + (h_1,h_0')_{\mathbb{C}^{n'}} + (h_0,h_1')_{\mathbb{C}^{n'}}.$$

In particular $[.,.]_{\mathcal{L}}|_{Lt^2} = [.,.]_{\mathcal{L}_q}$, $\mathcal{L}_q \perp \mathcal{H}_1$, and $\mathcal{H}_0$ and $\mathcal{H}_1$ are skewly linked neutral spaces. Denote by $\mathcal{P}$ the Pontryagin space obtained from $\mathcal{L}$ by factorization with respect to its isotropic part and completion:

$$\mathcal{P} = \widehat{\mathcal{L}/\mathcal{L}^\circ}.$$

Note that $\overline{\mathcal{L}_q}/(\overline{\mathcal{L}_q})^\circ \cong \mathcal{P}_q$ and $\mathcal{P} = \overline{\mathcal{L}_q} + \mathcal{H}_0 + \mathcal{H}_1$. Here $\overline{}$ denotes the closure in the space $\mathcal{P}$. In the following denote by P the orthogonal projection of $\mathcal{P}$ onto $(\mathcal{H}_0 \dot{+} \mathcal{H}_1)^\perp$.

Lemma 2.3. *The restriction $P|_{\overline{\mathcal{L}_q}}$ is an isometry of $\overline{\mathcal{L}_q}$ onto $(\mathcal{H}_0 \dot{+} \mathcal{H}_1)^\perp$. In fact*

$$[Pf,g] = [f,g], \; f,g \in \overline{\mathcal{L}_q}. \tag{2.4}$$

P r o o f.Let $f \in \overline{\mathcal{L}_q}$, then $\overline{\mathcal{L}_q} \perp \mathcal{H}_1$ shows that $(I-P)f \perp \mathcal{H}_1$. As $\mathcal{H}_1$ is a hypermaximal neutral subspace of $\mathcal{H}_0 \dot{+} \mathcal{H}_1$ it follows that $(I-P)f \in \mathcal{H}_1$. Thus, for $f, g \in \overline{\mathcal{L}_q}$

$$[f,g] = [Pf,g] + [(I-P)f,g] = [Pf,g],$$

in particular $P|_{\overline{\mathcal{L}_q}}$ is an isometry.

If $f_1 \in (\mathcal{H}_0 \dot{+} \mathcal{H}_1)^\perp$ we can write $f_1 = f + h_0 + h_1$ with $f \in \overline{\mathcal{L}_q}$, $h_0 \in \mathcal{H}_0$ and $h_1 \in \mathcal{H}_1$. Therefore $f_1 = Pf_1 = Pf$. □

Since $\operatorname{ind}_- \mathcal{L}_q = \kappa$ and $\operatorname{ind}_- (\mathcal{H}_0 \dot{+} \mathcal{H}_1) = \dim \mathcal{H}_0$, Lemma 2.3 implies that

$$\operatorname{ind}_- \mathcal{P} = \kappa + \dim \mathcal{H}_0. \tag{2.5}$$

Lemma 2.3 also enables us to determine $(\overline{\mathcal{L}_q})^\circ$.

Lemma 2.4. *The isotropic part $(\overline{\mathcal{L}_q})^\circ$ of $\overline{\mathcal{L}_q}$ coincides with $\mathcal{H}_1$.*

P r o o f.Clearly $\mathcal{H}_1 \cap \overline{\mathcal{L}_q} \subseteq (\overline{\mathcal{L}_q})^\circ$. The reverse inclusion also holds: Since P is the orthogonal projection onto the regular subspace $(\mathcal{H}_0 \dot{+} \mathcal{H}_1)^\perp$ we have

$$(\overline{\mathcal{L}_q})^\circ \subseteq \ker\left(P|_{\overline{\mathcal{L}_q}}\right) \subseteq \mathcal{R}\left((I-P)|_{\overline{\mathcal{L}_q}}\right) \subseteq \mathcal{H}_1.$$

It remains to show that $\mathcal{H}_1 \subseteq \overline{\mathcal{L}_q}$. Assume on the contrary that $\mathcal{H}_1 \cap \overline{\mathcal{L}_q} \subsetneq \mathcal{H}_1$. Then there exists an element $h \in \mathcal{H}_0$, $h \neq 0$, such that $h \perp (\mathcal{H}_1 \cap \overline{\mathcal{L}_q}) = \mathcal{L}_q^\circ$. Hence, the functional

$$[ae_z, h] = (q(z)a, h) = F_h(ae_z), \; a \in \mathbb{C}^n, z \in \varrho(q)$$

induces a well-defined continuous functional on $\overline{\mathcal{L}_q}/(\overline{\mathcal{L}_q})^\circ = \mathcal{P}_q$. This is easily seen by decomposing h with respect to the decomposition (compare [IKL])

$$\mathcal{P} = (\overline{\mathcal{L}_q})_n [\dot{+}] \left((\overline{\mathcal{L}_q})^\circ \dot{+} \overline{\mathcal{L}_q}^{-1}\right) [\dot{+}] (\overline{\mathcal{L}_q})_r$$

of $\mathcal{P}$, where $(\overline{\mathcal{L}_q})^\circ$ and $\overline{\mathcal{L}_q}^{-1}$ are skewly linked, and where $(\overline{\mathcal{L}_q})_n$ is nondegenerated and satisfies $(\overline{\mathcal{L}_q})_n [\dot{+}] (\overline{\mathcal{L}_q})^\circ = \overline{\mathcal{L}_q}$. We find $h \in \mathcal{G}$, which contradicts $h \in \mathcal{H}_0$. □

Note that Lemma 2.4 implies that P maps $\overline{\mathcal{L}_q}$ into itself, hence $\mathcal{P}$ can be decomposed as

$$\mathcal{P} = \underbrace{P\overline{\mathcal{L}_q} [\dot{+}] (\mathcal{H}_1}_{=\overline{\mathcal{L}_q}} \dot{+} \mathcal{H}_0). \tag{2.6}$$

Let $a \in \mathcal{G}$. Since $\mathcal{P}_q \cong \overline{\mathcal{L}_q}/(\overline{\mathcal{L}_q})^\circ \cong P\overline{\mathcal{L}_q}$ we can consider $u(a)$ as an element of $P\overline{\mathcal{L}_q}$. Denote by Γ the linear mapping of $\mathbb{C}^n$ into $\mathcal{P}$ defined by

$$\Gamma a = \begin{cases} u(a), & a \in \mathcal{G} \\ a, & a \in \mathcal{H}_0 \end{cases}.$$

Due to the definition of the inner product Γ has the property

$$[ae_z, \Gamma b] = (q(z)a, b), \; a, b \in \mathbb{C}^n, z \in \varrho(q). \tag{2.7}$$

Definition 2.5. Let $A \subseteq \mathcal{P}^2$ be the closure of the relation

$$S = \langle (0; h_1), (ae_z; \Gamma a + zae_z) | h_1 \in \mathcal{H}_1, a \in \mathbb{C}^n, z \in \varrho(q) \rangle.$$

Note that $\overline{\mathcal{D}(A)} = \overline{\mathcal{L}_q}$. If $\mathcal{G} \neq \mathbb{C}^n$, A is a proper relation and $A(0)$ is degenerated.

Proposition 2.6. *The relation A is selfadjoint and satisfies $\varrho(A) \supseteq \varrho(q)$. Moreover, we have $q(z) = \Gamma^*(A-z)^{-1}\Gamma$, i.e.*

$$(q(z)a, b) = [(A-z)^{-1}\Gamma a, \Gamma b], \ a, b \in \mathbb{C}^n, z \in \varrho(q).$$

Proof.We start with showing that S is symmetric. Since $\mathcal{H}_1 \perp \mathcal{L}_q$ we need not consider the pairs $(0; h_1)$. For $z \neq \overline{w}$ we have

$$[ae_z, \Gamma b + wbe_w] - [\Gamma a + zae_z, be_w] = [ae_z, \Gamma b] - [\Gamma a, be_w] +$$

$$+ (\overline{w} - z)[ae_z, be_w] = (q(z)a, b) - (q(\overline{w})a, b) +$$

$$+ (\overline{w} - z)\left(\frac{q(z) - q(\overline{w})}{z - \overline{w}} a, b\right) = 0.$$

If $z = \overline{w}$

$$[ae_z, \Gamma b + \overline{z} be_{\overline{z}}] - [\Gamma a + zae_z, be_{\overline{z}}] = [ae_z, \Gamma b] - [\Gamma a, be_{\overline{z}}] =$$

$$= (q(z)a, b) - (a, q(\overline{z})b) = 0.$$

Let $z \in \varrho(q)$, then

$$(\frac{ae_z - ae_w}{z - w}; ae_w) \in S - z, \ w \in \varrho(q), w \neq z,$$

$$(he_z; h) \in S - z, \ h \in \mathcal{H}_0,$$

$$(0; h) \in S - z, \ h \in \mathcal{H}_1.$$

Since, for $w_0 \in \varrho(q)$

$$\lim_{w \to w_0} ae_w = ae_{w_0},$$

in the norm of $\mathcal{P}$, we find that $\mathcal{R}(S - z)$ is dense in $\mathcal{P}$ for $z \in \varrho(q)$. It is proved in [DS] that then A is a selfadjoint relation, and

$$\varrho(A) \supseteq \varrho(q).$$

Since $(ae_z; \Gamma a) \in A - z$, we find due to (2.7)

$$[(A-z)^{-1}\Gamma a, \Gamma b] = [ae_z, \Gamma b] = (q(z)a, b), \ z \in \varrho(q).$$

□

In order to make $\mathcal{R}(\Gamma)$ a neutral subspace, we possibly have to enlarge the space $\mathcal{P}$. Let

(2.8) $$\kappa_m = \kappa + \dim \mathcal{H}_0 + \operatorname{ind}_+ \mathcal{G} = \kappa + n - \operatorname{ind}_- \mathcal{G} - \operatorname{ind}_0 \mathcal{G},$$

where the index of $\mathcal{G}$ has to be understood with respect to the inner product $[.,.]_{\mathcal{G}}$.

Proposition 2.7. *There exists a Pontryagin space $\mathcal{P}_1$ with $\operatorname{ind}_- \mathcal{P}_1 = \kappa_m$, a selfadjoint relation $A_1 \subseteq \mathcal{P}_1^2$ with $\varrho(A_1) \supseteq \varrho(q)$, and a mapping Γ_1 of $\mathbb{C}^n$ onto a neutral subspace of $\mathcal{P}_1$, such that*

$$q(z) = \Gamma_1^*(A_1 - z)^{-1}\Gamma_1,\ z \in \varrho(q).$$

Proof.Let $\mathcal{G}$ be decomposed as $\mathcal{G} = \mathcal{G}_+[\dot{+}]\mathcal{G}_-[\dot{+}]\mathcal{G}^\circ$, where $\mathcal{G}_+$ ($\mathcal{G}_-$) is a maximal positive (negative) subspace of $\mathcal{G}$. Let $\mathcal{G}'_\pm$ be isomorphic copies of $\mathcal{G}_\pm$, endowed with the inner product $[.,.]_{\mathcal{G}'_\pm} = -[.,.]_{\mathcal{G}_\pm}$ and define $\mathcal{P}_1$ as

$$\mathcal{P}_1 = \mathcal{P}[\dot{+}]\mathcal{G}'_+[\dot{+}]\mathcal{G}'_-. \tag{2.9}$$

Clearly $\operatorname{ind}_- \mathcal{P}_1 = \kappa_m$. Denote by $P_1 : \mathcal{G} \to \mathcal{G}'_+\dot{+}\mathcal{G}'_-$ the projection of $\mathcal{G}$ onto $\mathcal{G}_+\dot{+}\mathcal{G}_-$ with kernel $\mathcal{G}^\circ$, where the image of an element is considered as an element of $\mathcal{G}'_+\dot{+}\mathcal{G}'_-$.

For $a \in \mathcal{G}$ define Γ_1 as

$$\Gamma_1 a = \Gamma a + P_1 a,$$

for $a \in \mathcal{H}_0$ let $\Gamma_1 a = \Gamma a = a$. We have for $a \in \mathcal{G}$, $b \in \mathcal{H}_0$

$$[\Gamma_1 a, \Gamma_1 b] = [P\Gamma a, b] + [P_1 a, b] = 0,$$

and, due to Lemma 2.3, for $a, b \in \mathcal{G}$

$$[\Gamma_1 a, \Gamma_1 b] = [P\Gamma a, P\Gamma b] + [P_1 a, P_1 b] = [\Gamma a, \Gamma b] - [a, b]_{\mathcal{G}} = 0.$$

Since $\mathcal{H}_0$ is neutral, $\mathcal{R}(\Gamma_1)$ is a neutral subspace of $\mathcal{P}_1$.

Define a relation

$$A_1 = A + \langle (0; a') | a' \in \mathcal{G}'_\pm \rangle \subseteq \mathcal{P}_1^2.$$

It is easy to see that A_1 is a selfadjoint relation, $\varrho(A_1) \supseteq \varrho(A) \supseteq \varrho(q)$, and that $(A_1 - z)^{-1}(\mathcal{G}'_\pm) = 0$, $z \in \varrho(A_1)$. By Proposition 2.6 we obtain the desired result. □

Note that $\mathcal{G}'_+ + \mathcal{G}'_- + \mathcal{H}_0 \subseteq \overline{\mathcal{L}_q} + \mathcal{R}(\Gamma_1)$. Hence, the relation A_1 is Γ_1-minimal in the sense that

$$\mathcal{P}_1 = \overline{\langle \mathcal{R}(\Gamma_1), (A_1 - z)^{-1}\mathcal{R}(\Gamma_1) | a \in \mathbb{C}^n, z \in \varrho(A_1) \rangle}.$$

Let $q \in \mathcal{N}_\kappa^{n\times n}$ have the representation

$$q(z) = \Gamma_2^*(A_2 - z)^{-1}\Gamma_2,\ z \in \varrho(q), \tag{2.10}$$

with a selfadjoint relation A_2 acting in a Pontryagin space $\mathcal{P}_2$, where $\mathcal{R}(\Gamma_2)$ is neutral. We want to show that $\operatorname{ind}_- \mathcal{P}_2 \geq \kappa_m$.

We may assume without loss of generality that the relation A_2 is Γ_2-minimal, i.e. that

$$\overline{\langle \mathcal{R}(\Gamma_2), (A_2 - z)^{-1}\mathcal{R}(\Gamma_2) | z \in \varrho(A_2) \rangle} = \mathcal{P}_2.$$

Otherwise we could consider the Γ_2-minimal part of A_2: If we restrict the relation A_2 to

$$\mathcal{M} = \overline{\langle \mathcal{R}(\Gamma_2), (A_2 - z)^{-1}\mathcal{R}(\Gamma_2) | z \in \varrho(A_2) \rangle},$$

and factorize with respect to the isotropic part $\mathcal{M}^\circ$, we obtain a selfadjoint relation $A_\mathcal{M}$ with $\varrho(A_\mathcal{M}) \supseteq \varrho(A)$. Clearly, $A_\mathcal{M}$ also represents q, is Γ_2-minimal and $\operatorname{ind}_- \mathcal{M}/\mathcal{M}^\circ \leq \operatorname{ind}_- \mathcal{P}_2$.

Define a linear mapping $U : \mathcal{L}_q + \mathcal{R}(\Gamma_1) \to \mathcal{P}_2$ by $(a \in \mathbb{C}^n,\ z \in \varrho(q))$

$$U : \begin{cases} ae_z \mapsto (A_2 - z)^{-1}\Gamma_2 a \\ \Gamma_1 a \mapsto \Gamma_2 a \end{cases} .$$

Using (2.7) and the fact that $\mathcal{R}(\Gamma_2)$ is neutral, it is checked by a straightforward computation that U is isometric. The domain of U is dense in $\mathcal{P}_1$, and the range of U is, due to our assumption that A_2 is Γ_2-minimal, dense in $\mathcal{P}_2$. Hence (see [IKL]) U extends to a unitary operator $\tilde{U}$ from $\mathcal{P}_1$ onto $\mathcal{P}_2$. A short calculation shows that

$$\tilde{U} \circ A_1 = A_2 \text{ and } \tilde{U} \circ \Gamma_1 = \Gamma_2,$$

i.e. the representation (2.10) is unitarily equivalent to the representation constructed in Proposition 2.7.

Theorem 2.8. *Let $q(z) \in \mathcal{N}_\kappa^{n\times n}$, let κ_m be as in (2.8) and let $\kappa' \in \mathbb{N} \cup \{0\}$. Then q admits a representation*

$$q(z) = \Gamma^*(A - z)^{-1}\Gamma,\ \ z \in \varrho(q), \tag{2.11}$$

with a selfadjoint relation A acting in a Pontryagin space $\mathcal{P}$ with $\operatorname{ind}_- \mathcal{P} = \kappa'$, and with a mapping $\Gamma : \mathbb{C}^n \to \mathcal{P}$ where $\mathcal{R}(\Gamma)$ is a neutral subspace of $\mathcal{P}$, if and only if $\kappa' \geq \kappa_m$.

If A is Γ-minimal we have $\operatorname{ind}_- \mathcal{P} = \kappa_m$. Each two Γ-minimal representations are unitarily equivalent.

P r o o f. Proposition 2.6 shows that there exists a representation (2.11) where $\operatorname{ind}_- \mathcal{P} = \kappa_m$. If we extend $\mathcal{P}$ by a purely negative subspace, which is taken into the relational part of the representing relation, we obtain representations (2.11) with $\operatorname{ind}_- \mathcal{P} > \kappa_m$. The remaining assertions follow from the preceeding considerations. □

Remark 2.9. Every function $q \in \mathcal{N}_\kappa^{n\times n}$ has a representation (2.11) where $\operatorname{ind}_- \mathcal{P} = \kappa + n$.

Remark 2.10. Let $q \in \mathcal{N}_\kappa^{n\times n}$ have the representation (2.11) with a Γ-minimal relation A. Then the part at infinity $A(0)$ of A is in general degenerated. More precisely

$$\operatorname{ind}_0 A(0) = n - \dim \mathcal{G}.$$

For a function $q \in \mathcal{N}_{\kappa'}^{n\times n}$ let $\mathcal{G}_q$ be the corresponding inner product space defined in (2.1). By reversing the formulation of Theorem 2.8 we obtain the following corollary.

Corollary 2.11. *Let $\kappa \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$ be given. The set of all $n \times n$-matrix valued functions which admit a representation (2.11) where $\operatorname{ind}_- \mathcal{P} = \kappa$, is*

$$\bigcup_{\kappa'=0}^{\kappa} \{q \in \mathcal{N}_{\kappa'}^{n\times n} | \operatorname{ind}_- \mathcal{G}_q + \operatorname{ind}_0 \mathcal{G}_q \geq n + \kappa' - \kappa\}.$$

3. Characterizations of the number κ_m

In this section we determine κ_m in different ways. First via a kind of kernel condition on the function q, then via analytic properties of q where a certain additional condition on q is imposed, or where q is a scalar function.

Denote by $\{d_1, \dots, d_n\}$ a basis of $\mathbb{C}^n$.

Proposition 3.1. *Let $q \in \mathcal{N}_\kappa^{n\times n}$. The number κ_m is the maximal number of negative squares a quadratic form ($m \in \mathbb{N}$, $z_1, \dots, z_m \in \varrho(q)$, $a_1, \dots, a_m \in \mathbb{C}^n$)*

$$Q(\xi_1, \dots, \xi_m; \zeta_1, \dots, \zeta_n) = \sum_{i,j=1}^{m} (N_q(z_i, z_j)a_i, a_j)\xi_i\overline{\xi_j} + \sum_{i=1}^{m}\sum_{l=1}^{n} \operatorname{Re}\left((q(z_i)a_i, d_l)\xi_i\overline{\zeta_l}\right) \tag{3.1}$$

attains.

This follows from the fact that in Proposition 2.7

$$\overline{\mathcal{L}_q} + \mathcal{R}(\Gamma_1) = \mathcal{P}_1,$$

and that for $m \in \mathbb{N}$, $z_1, \dots, z_m \in \varrho(q)$, $a, a_1, \dots, a_m \in \mathbb{C}^n$ the relation

$$[\sum_{i=1}^{m} a_i e_{z_i} + \Gamma_1 a, \sum_{j=1}^{m} a_j e_{z_j} + \Gamma_1 a] = \sum_{i,j=1}^{m} (N_q(z_i, z_j)a_i, a_j) + \sum_{i=1}^{m} 2\operatorname{Re}(q(z_i)a_i, a)$$

holds.

Denote by $\mathcal{G}_\kappa^{0,n\times n}$ the set of all $n \times n$-matrix valued functions q, meromorphic in $\mathbb{C} \setminus \mathbb{R}$, such that the maximal number of negative squares of forms (3.1) equals κ. Proposition 3.1 has the following corollary:

Corollary 3.2. *We have $q \in \mathcal{G}_\kappa^{0,n\times n}$ if and only if $q(z)$ has a representation (1.2) where $\mathcal{R}(\Gamma)$ is neutral, A acts in a Pontryagin space with negative index κ and is Γ-minimal.*

In [KL2] and [KL1] there is made an additional assumptions on the function $q(z)$, which is equivalent to the fact that the selfadjoint relation A_q in (1.1) is an operator. To be more precise, it is assumed that

$$\operatorname*{w-lim}_{y\to+\infty} \frac{q(iy)}{y} = 0. \tag{3.2}$$

Throughout the next part of this section we shall also assume that (3.2) holds. We will prove the following

Proposition 3.3. *Assume that $q \in \mathcal{N}_\kappa^{n\times n}$ satisfies (3.2). Then $h \in \mathcal{G}$ if and only if*

(i) $\lim_{y\to+\infty} q(iy)h = 0$,

(ii) $\lim_{y\to+\infty} y(\operatorname{Im} q(iy)h, h)$ *exists.*

In this case the inner product $[.,.]_{\mathcal{G}}$ *is given by*

$$[a,b]_{\mathcal{G}} = \lim_{y\to+\infty} y(\operatorname{Im} q(iy)a,b),\ \ a,b\in\mathcal{G}. \tag{3.3}$$

The proof of Proposition 3.3 is split into several lemmata.

Lemma 3.4. *Assume that* $h\in\mathbb{C}^n$ *satisfies (i) and (ii). Then the limit*

$$\lim_{y\to+\infty} yhe_{iy} \tag{3.4}$$

exists in the norm of $\mathcal{P}_q$.

Proof.We compute

$$[yhe_{iy}-y'he_{iy'},ae_z] = y\left(\frac{q(iy)-q(\overline{z})}{iy-\overline{z}}h,a\right) - y'\left(\frac{q(iy')-q(\overline{z})}{iy'-\overline{z}}h,a\right) =$$

$$= \left(\frac{y'}{iy'-\overline{z}}-\frac{y}{iy-\overline{z}}\right)(q(\overline{z})h,a) + \frac{y}{iy-\overline{z}}(q(iy)h,a) - \frac{y'}{iy'-\overline{z}}(q(iy')h,a). \tag{3.5}$$

Each term on the right hand side of (3.5) tends to zero if y and y' tend to infinity. Moreover

$$[yhe_{iy}-y'he_{iy'},yhe_{iy}-y'he_{iy'}] = y^2(\frac{1}{y}\operatorname{Im} q(iy)h,h)+$$

$$+y'^2(\frac{1}{y'}\operatorname{Im} q(iy')h,h) - yy'(\frac{q(iy)-q(\overline{iy'})}{iy+iy'}h,h) - y'y(\frac{q(iy')-q(\overline{iy})}{iy'+iy}h,h) =$$

$$= \frac{y-y'}{y+y'}\left[y(\operatorname{Im} q(iy)h,h) - y'(\operatorname{Im} q(iy')h,h)\right]. \tag{3.6}$$

The right hand side of (3.6) tends to zero if y and y' tend to infinity. Hence (see [IKL]) the limit (3.4) exists. □

Lemma 3.5. *Assume that* $h\in\mathbb{C}^n$ *satisfies (i) and (ii). Then* $h\in\mathcal{G}$ *and*

$$u(h) = -i\lim_{y\to+\infty} yhe_{iy}.$$

Proof.Let $u=-i\lim_{y\to+\infty} yhe_{iy}$. We have

$$[ae_z,u+iyhe_{iy}] = [ae_z,u] - iy(\frac{q(z)-q(\overline{iy})}{z-\overline{iy}}a,h) =$$

$$= \left([ae_z,u] - \frac{iy}{z+iy}(q(z)a,h)\right) + \frac{iy}{z+iy}(a,q(iy)h). \tag{3.7}$$

Since the left hand side and the second term on the right hand side of (3.7) tend to zero, $[ae_z,u]=(q(z)a,h)$. □

Lemma 3.6. *Assume that* $h\in\mathcal{G}$. *Then* h *satisfies (i) and (ii). Moreover, the relation (3.3) holds.*

Proof.As in Lemma 2.1 we find

$$(A_q - z)^{-1}u(h) = he_z.$$

Due to our assumption (3.2), A_q is an operator. Hence,

$$-i \lim_{y\to+\infty} yhe_{iy} = \lim_{y\to+\infty}(-iy)(A_q - iy)^{-1}u(h) = u(h). \tag{3.8}$$

The relation (3.7) shows that

$$\lim_{y\to+\infty}(q(iy)h, a) = 0,$$

hence *(i)* holds. Let $h, h' \in \mathcal{G}$, then

$$[u(h) + iyhe_{iy}, u(h') + iyh'e_{iy}] = [u(h), u(h')] + iy([he_{iy}, u(h')] - [u(h), h'e_{iy}])+$$

$$+y^2[he_{iy}, h'e_{iy}] = [u(h), u(h')] - y(\operatorname{Im} q(iy)h, h'). \tag{3.9}$$

Since the left hand side of (3.9) tends to zero, we find that the relation (3.3) holds. In particular h satisfies *(ii)*. □

All assertions of Proposition 3.3 are proved.

Remark 3.7. The condition (3.2) has been used only in Lemma 3.6 to show that $-i \lim_{y\to+\infty} yhe_{iy} = u(h)$. Hence, also if q does not satisfy (3.2), the set of all elements $h \in \mathbb{C}^n$ which satisfy *(i)* and *(ii)*, endowed with the inner product (3.3), is a subspace of $\mathcal{G}$.

Denote by P the (with respect to the usual inner product) orthogonal projection of $\mathbb{C}^n$ onto $\mathcal{G}$.

Remark 3.8. Since $(q(z)a, b) = [(A_q - z)^{-1}u(a), u(b)]$ for $a, b \in \mathcal{G}$ and q satisfies (3.2), we find that $\lim_{y\to+\infty} yPq(iy)P$ exists, and that (beside (3.3))

$$[a, b]_{\mathcal{G}} = -i \lim_{y\to+\infty} y(q(iy)a, b),\ \ a, b \in \mathcal{G}$$

holds.

In the last part of this section we assume that q is a scalar function, i.e. $n = 1$. Then $\kappa_m = \kappa$ or $\kappa_m = \kappa + 1$. The space $\mathcal{G}$ is either $\{0\}$ or $\mathbb{C}$, in the second case it is either positive, negative or neutral. We have $\kappa_m = \kappa$ if and only if $\mathcal{G}$ equals $\mathbb{C}$ and is nonpositive, whereas in the remaining cases $\kappa_m = \kappa + 1$. Note that, if $\mathcal{G} = \mathbb{C}$, Lemma 2.1 shows that A_q is an operator, hence in this case the limit relation (3.8) holds. Due to these facts we have

Proposition 3.9. *Let $q \in \mathcal{N}_\kappa$, then q admits a representation as a u-resolvent*

$$q(z) = [(A - z)^{-1}u, u],\ \ z \in \varrho(q),$$

with a neutral element u and a u-minimal selfadjoint relation $A \subseteq \mathcal{P}^2$, where $\operatorname{ind}_- \mathcal{P} = \kappa$ *or* $\operatorname{ind}_- \mathcal{P} = \kappa + 1$. *The first case occurs if and only if*

$$(3.10) \qquad 0 \leq i \lim_{y \to +\infty} yq(iy) < \infty.$$

P r o o f. Assume that $\mathcal{G} = \mathbb{C}$ and is nonpositive. Lemma 2.1 shows that A_q is an operator, hence the results of the preceeding part of the present section are applicable. Remark 3.8 shows that (3.10) holds.

Conversely let (3.10) be valid, then in particular $\lim_{y\to\infty} \frac{1}{y} q(iy) = 0$. Again the results of the preceeding part of this section apply, and we find that $\mathcal{G}$ is nonpositive. □

4. Some classes of Nevanlinna functions

We study the following classes of generalized Nevanlinna functions.

Definition 4.1. Let $\mathcal{G}_\kappa^n$ $(\kappa, n \in \mathbb{N} \cup \{0\})$ be the set of all scalar functions $q \neq 0$, meromorphic in $\mathbb{C} \setminus \mathbb{R}$ which satisfy $q(\overline{z}) = \overline{q(z)}$ on their domain of holomorphy $\varrho(q)$, and are such that the maximal number of negative squares of the quadratic forms $(m \in \mathbb{N}, z_1, \ldots, z_m \in \varrho(q))$

$$(4.1) \quad Q(\xi_1, \ldots, \xi_m; \eta_1, \ldots, \eta_n) = \sum_{i,j=1}^{m} N_q(z_i, z_j)\xi_i\overline{\xi_j} + \sum_{k=0}^{n}\sum_{i=1}^{m} \operatorname{Re}(z_i^k q(z_i)\xi_i\overline{\eta_k})$$

is κ.

By Proposition 3.1 of Section 3. the class $\mathcal{G}_\kappa^0$ is the set of all functions q, such that $\kappa_m(q) = \kappa$. Note that each function q of $\mathcal{G}_\kappa^n$ is contained in some Nevanlinna class $\mathcal{N}_{\kappa'}$ with $\kappa' \leq \kappa$.

Denote by S_0 the shift operator in $\mathbb{C}^{n+1}$, i.e. let $\mathbb{C}^{n+1} = \langle h_0, \ldots, h_n \rangle$ and let S_0 be defined by

$$S_0 h_i = h_{i+1}, \; i = 0, \ldots, n-1.$$

Let $\mathcal{H}$ be the (neutral) inner product space $\mathcal{H} = \langle \mathbb{C}^{n+1}, [.,.] \rangle$ with $[x, y] = 0$. The aim of this section is to prove the following theorem.

Theorem 4.2. *The function q is an element of $\mathcal{G}_\kappa^n$ if and only if there exist a Pontryagin space $\mathcal{P}$ and a selfadjoint relation $A \subseteq \mathcal{P}^2$, such that* $\operatorname{ind}_- \mathcal{P} = \kappa$, *$\mathcal{P} \supseteq \mathcal{H}$, $A \supseteq S_0$, $\varrho(A) \neq \emptyset$, A is $\mathcal{H}$-minimal in the sense that*

$$(4.2) \qquad \mathcal{P} = \overline{\langle \mathcal{H}, (A - z)^{-1}\mathcal{H} | z \in \varrho(A) \rangle},$$

and q has the representation

$$(4.3) \qquad q(z) = [(A - z)^{-1} h_0, h_0], \; z \in \varrho(q).$$

Here S_0 and h_0 are as introduced above.

This result generalizes Corollary 3.2 in case of a scalar function.

Remark 4.3. Note that S_0 could be replaced by any operator S_0' in $\mathcal{H}$, such that $\dim \mathcal{D}(S_0') = n$ which has no eigenvalue, since each such operator is (algebraically) equivalent to the shift operator.

Before we prove Theorem 4.2, we state another lemma.

Lemma 4.4. *Let the inner product* $(.,.)$ *on* $\mathcal{H}$ *be defined by*

$$(h_i, h_j) = \delta_{ij},\ i, j = 0, \dots, n,$$

and, for $z \in \mathbb{C}$, *denote by* $\chi(z)$ *the element*

$$\chi(z) = z^n h_n + z^{n-1} h_{n-1} + \dots + h_0.$$

Then $\mathcal{R}(S_0 - \overline{z})^{(\perp)} = \langle \chi(z) \rangle$. *Moreover, each element* h *of* $\mathcal{H}$ *can be decomposed as*

$$h = h'(z) + (h, \chi(\overline{z}))h_0, \tag{4.4}$$

where $h'(z) \in \mathcal{R}(S_0 - z)$.

Proof.Since

$$\mathcal{R}(S_0 - z) = \langle h_{i+1} - z h_i | i = 0, \dots, n-1 \rangle,$$

and $(h_0, \chi(z)) = 1$, the assertion follows. □

Proof.**(of Theorem 4.2)** Assume first that q has the representation (4.3). Note that

$$\overline{\langle \mathcal{H}, (A-z)^{-1} h_0 | z \in \varrho(A) \rangle} = \overline{\langle \mathcal{H}, (A-z)^{-1} \mathcal{H} | z \in \varrho(A) \rangle} = \mathcal{P}.$$

Hence, the maximal number of negative squares of quadratic forms ($m \in \mathbb{N}, z_1, \dots, z_m \in \varrho(q)$)

$$[\sum_{k=0}^{n} \eta_k h_k + \sum_{i=1}^{m} \xi_i (A - z_i)^{-1} h_0, \sum_{k=0}^{n} \eta_k h_k + \sum_{j=1}^{m} \xi_j (A - z_j)^{-1} h_0] \tag{4.5}$$

equals κ. Since A extends S_0 we have

$$(A - z)^{-1} \mathcal{R}(S_0 - z) = \mathcal{D}(S_0) \subseteq \mathcal{H}.$$

Hence, as $\mathcal{H}$ is neutral, we have according to (4.4)

$$[(A-z)^{-1} h_0, h_k] = [(A-z)^{-1} h_0, h_k'(\overline{z}) + [h_k, \chi(z)] h_0] =$$

$$= [\chi(z), h_k][(A-z)^{-1} h_0, h_0] = z^k q(z). \tag{4.6}$$

Using the resolvent identity and (4.6), the form (4.5) can be written as

$$[\sum_{k=0}^{n} \eta_k h_k + \sum_{i=1}^{m} \xi_i (A - z_i)^{-1} h_0, \sum_{k=0}^{n} \eta_k h_k + \sum_{j=1}^{m} \xi_j (A - z_j)^{-1} h_0] =$$

$$= \sum_{i,j=1}^{m} N_q(z_i, z_j)\xi_i\overline{\xi_j} + \sum_{k=0}^{n}\left(\sum_{i=1}^{m} z_i^k q(z_i)\xi_i\overline{\eta_k} + \sum_{j=1}^{m}\overline{z_j^k q(z_j)\xi_j}\eta_k\right) =$$

$$= \sum_{i,j=1}^{m} N_q(z_i, z_j)\xi_i\overline{\xi_j} + \sum_{k=0}^{n}\sum_{i=1}^{m} \operatorname{Re}\left(z_i^k q(z_i)\xi_i\overline{(2\eta_k)}\right).$$

Replacing $2\eta_k$ by η_k we obtain that the maximal number of negative squares of the forms (4.1) equals κ.

Conversely let $q \in \mathcal{G}_\kappa^n$ be given. Consider the inner product space

$$\mathcal{L} = \langle \mathcal{H}, e_z | z \in \varrho(q)\rangle$$

endowed with the inner product defined by

$$[e_z, e_w] = N_q(z,w),\ z,w \in \varrho(q),$$

$$[e_z, h_i] = z^i q(z),\ z \in \varrho(q), i = 0,\dots,n,$$

$$[h_i, h_j] = 0,\ i,j = 0,\dots,n.$$

Let $S \subseteq \mathcal{L}^2$ be the linear relation

$$S = \langle S_0, (e_z; h_0 + z e_z) | z \in \varrho(q)\rangle.$$

Consider the Pontryagin space $\mathcal{P}$ obtained from $\mathcal{L}$ by factorization with respect to its isotropic part and completion, and the relation $A = \overline{S/\mathcal{L}^\circ} \subseteq \mathcal{P}^2$. The fact that $q \in \mathcal{G}_\kappa^n$ implies that $\operatorname{ind}_- \mathcal{P} = \kappa$. Since

$$[e_z, \sum_{i=0}^{n} \lambda_i h_i] = q(z)\sum_{i=0}^{n}\overline{\lambda_i} z^i,$$

we find $\mathcal{H} \cap \mathcal{L}^\circ = \{0\}$ as long as q does not vanish identically. Hence $\mathcal{P} \supseteq \mathcal{H}$ and $A \supseteq S_0$. By the same reasoning as in Proposition 2.6 the relation A is selfadjoint, $\varrho(A) \supseteq \varrho(q)$, and

$$q(z) = [(A-z)^{-1}h_0, h_0],\ z \in \varrho(q).$$

Clearly A is minimal in the sense of (4.2). □

The classes $\mathcal{G}_\kappa^n$ are related to the Nevanlinna classes $\mathcal{N}_\kappa$ by

Corollary 4.5. *Let $\kappa, n \in \mathbb{N}\cup\{0\}$, then*

$$\mathcal{N}_\kappa \subseteq \mathcal{G}_\kappa^n \cup \ldots \cup \mathcal{G}_{\kappa+n+1}^n,$$

and

$$\mathcal{G}_\kappa^n \subseteq \mathcal{N}_{\max(0,\kappa-n-1)} \cup \ldots \cup \mathcal{N}_\kappa.$$

Proof.Consider the hermitian matrix corresponding to a quadratic form (4.1). This matrix is obtained by adding $n+1$ rows and columns to the matrix

$$(N_q(z_i, z_j))_{i,j=1}^m.$$

Hence the number of negative eigenvalues increases by at most $n+1$. □

Corollary 4.6. *Let $\kappa, n \in \mathbb{N}\cup\{0\}$. For $n \geq \kappa$ we have $\mathcal{G}_\kappa^n = \emptyset$. If $n < \kappa$ the class $\mathcal{G}_\kappa^n$ contains infinitely many elements. In fact, $\bigcap_{n=0}^{\kappa-1} \mathcal{G}_\kappa^n \neq \emptyset$.*

Proof.The first assertion follows immediately from Theorem 4.2.

If $q \in \mathcal{G}_\kappa^n$ and $\lambda \in \mathbb{R}$, $\lambda > 0$, also $\lambda q \in \mathcal{G}_\kappa^n$. In order to show that $\mathcal{G}_\kappa^n$ contains infinitely many elements, it thus suffices to show $\mathcal{G}_\kappa^n \neq \emptyset$.

Let $\mathcal{H}$ and $\mathcal{H}'$ be neutral κ-dimensional spaces and put

$$\mathcal{P} = \mathcal{H} \dot{+} \mathcal{H}',$$

where $\mathcal{H}$ and $\mathcal{H}'$ are skewly linked. Let $\{h_0, \ldots, h_{\kappa-1}\}$ and $\{h_0', \ldots, h_{\kappa-1}'\}$ be skewly linked bases of $\mathcal{H}$ and $\mathcal{H}'$, respectively, i.e. let

$$[h_i, h_j'] = \delta_{ij},\ i,j = 1, \ldots, \kappa-1.$$

Consider the relation

$$A = \langle (h_j; h_{j+1}), (h_{j+1}'; h_j'), (h_{\kappa-1}; h_{\kappa-1}'), (h_0'; h_0) | j = 0, \ldots, \kappa-2 \rangle.$$

It is easily checked that A is a selfadjoint operator in $\mathcal{P}$. Its resolvent set is nonempty. Moreover, A has no nontrivial invariant subspace which contains h_0.

Now let $\kappa \geq 1$ be given and let $n < \kappa$. Denote by S_0 the shift operator in $\mathbb{C}^{n+1}$. We consider $\mathbb{C}^{n+1}$ as the subspace $\langle h_0, \ldots, h_n \rangle$ of $\mathcal{P}$, and S_0 as the operator

$$S_0 = \langle (h_j; h_{j+1}) | j = 0, \ldots, n-1 \rangle.$$

Then, clearly, $S_0 \subseteq A$. The space

$$\langle h_j, (A-z)^{-1} h_j | j = 0, \ldots, n \rangle$$

is an invariant subspace for each resolvent $(A-z)^{-1}$ and hence for A. Since it contains h_0, it must equal $\mathcal{P}$. By Theorem 4.2 the function

$$q(z) = [(A-z)^{-1} h_0, h_0] = \frac{1}{1 - z^{2\kappa}}$$

is contained in $\mathcal{G}_\kappa^n$. □

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