THE KREIN FORMULA FOR GENERALIZED RESOLVENTS IN DEGENERATED INNER PRODUCT SPACES

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Abstract

Let S be a symmetric operator with defect index $(1,1)$ in a Pontryagin space $\mathcal{H}$. The Krein formula establishes a bijective correspondence between the generalized resolvents of S and the set of Nevanlinna functions as parameters. We give an analogue of the Krein formula in the case that $\mathcal{H}$ is a degenerated inner product space. The set of parameters is determined by a kernel condition. These results are applied to some classical interpolation problems with singular data.

1 Introduction

Let $\mathcal{H}$ be a, possibly degenerated, inner product space such that $\mathcal{H}/(\mathcal{H}\cap\mathcal{H}^\perp)$ is a Pontryagin space, and let S be a symmetric operator in $\mathcal{H}$ with defect index $(1,1)$. If $\mathcal{P}$ is a Pontryagin space containing $\mathcal{H}$ as a singular subspace, and A is a selfadjoint relation in $\mathcal{P}$ extending S we call the set of analytic functions

$$[(A-z)^{-1}x,y],\ x,y\in\mathcal{H} \tag{1.1}$$

a generalized resolvent of S.

Note that, if $\mathcal{H}$ is nondegenerated, there exists an orthogonal projection P of $\mathcal{P}$ onto $\mathcal{H}$, hence a generalized resolvent (1.1) can be viewed as an operator valued function $P(A-z)^{-1}|_{\mathcal{H}}$. If $\mathcal{H}$ is degenerated such a projection does not exist and we have to use the form (1.1).

Denote in the following by $\langle\,.\,\rangle$ the linear span of the elements between the brackets. If A is $\mathcal{H}$-minimal in the sense that

$$\overline{\langle\mathcal{H},(A-z)^{-1}\mathcal{H}|z\in\varrho(A)\rangle}=\mathcal{P},$$

the corresponding generalized resolvent is called minimal. In this case, if $\mathcal{P}$ has index κ of negativity ($\kappa=\operatorname{ind}_-\mathcal{P}$), the generalized resolvent is said to have index κ.

If τ is a complex valued meromorphic function, denote by $\varrho(\tau)$ its domain of holomorphy. For $\kappa \in \mathbb{N}_0$ let the set $\mathcal{N}_\kappa$ of generalized Nevanlinna functions τ be defined as follows: τ is meromorphic in $\mathbb{C} \setminus \mathbb{R}$, $\tau(\overline{z}) = \overline{\tau(z)}$ for $z \in \varrho(\tau)$, and the Nevanlinna kernel $(z, w \in \varrho(\tau))$

$$N_\tau(z,w) = \begin{cases} \frac{\tau(z)-\tau(\overline{w})}{z-\overline{w}} & , z \neq \overline{w} \\ \tau'(z) & , z = \overline{w} \end{cases}$$

has κ negative squares.

If $\mathcal{H}$ is nondegenerated, i.e. a Pontryagin space, Krein's formula (see [8] or [2])

$$[(A-z)^{-1}x, y] = [(\mathring{A}-z)^{-1}x, y] - [x, \chi(\overline{z})]\frac{1}{\tau(z)+q(z)}[\chi(z), y], \tag{1.2}$$

establishes a bijective correspondence between the minimal generalized resolvents of index κ of S and the set $\mathcal{N}_{\kappa-\kappa_0}$ ($\kappa_0 = \operatorname{ind}_- \mathcal{H}$) of parameters τ. Here $\mathring{A} \subseteq \mathcal{H}^2$ is a fixed selfadjoint extension of S, $\chi(z)$ are certain defect elements of S and $q(z)$ is a corresponding Q-function, which means that the relation $N_q(z,w) = [\chi(z), \chi(w)]$ holds.

In this paper we consider the case that $\mathcal{H}$ is actually degenerated, $\dim \mathcal{H}^\circ = \Delta > 0$. It turns out that the relation (1.2) still holds, where $\mathring{A}$, $\chi(z)$ and $q(z)$ have a similar meaning. However, the parameter τ does not run through the Nevanlinna class $\mathcal{N}_{\kappa-\kappa_0}$, but through a different set of functions $\mathcal{K}^\Delta_{\kappa-\kappa_0}$ which is defined as follows:

Definition 1 *For $\nu, \Delta \in \mathbb{N}_0$, denote by $\mathcal{K}^\Delta_\nu$ the set of all complex valued functions $\tau(z)$, meromorphic in $\mathbb{C} \setminus \mathbb{R}$, which satisfy $\tau(\overline{z}) = \overline{\tau(z)}$ for $z \in \varrho(\tau)$, and are such that the maximal number of the negative squares of the quadratic forms ($m \in \mathbb{N}_0$, $z_1, \ldots, z_m \in \varrho(\tau)$)*

$$Q(\xi_1, \ldots, \xi_m; \eta_0, \ldots, \eta_{\Delta-1}) = \sum_{i,j=1}^m N_\tau(z_i, z_j)\xi_i\overline{\xi_j} + \sum_{k=0}^{\Delta-1}\sum_{i=1}^m \operatorname{Re}\left(z_i^k \xi_i \overline{\eta_k}\right) \tag{1.3}$$

is ν.

Note that $\mathcal{K}^0_\nu = \mathcal{N}_\nu$.

In Section 2 we collect some results on the defect spaces of a symmetric relation S. Our analogue of Krein's formula (1.2) is proved in Section 3. This formula enables us to treat some interpolation problems with singular data. We consider the extension problem of a hermitian function with a finite number of negative squares (Section 4), and the indefinite Nevanlinna-Pick interpolation problem (Section 5) in the case of degenerated datas. This means for each problem that there exists a unique solution with minimal index of negativity. Our parametrization of generalized resolvents leads to a parametrization of the solutions with higher index of negativity.

We use the notation and some results on symmetric and selfadjoint relations in Pontryagin spaces provided in [3]. For elementary facts concerning the geometry of Pontryagin spaces we refer to [5].

2 Defect elements

Let $\mathcal{H}$ be a degenerated inner product space and assume that

$$\mathcal{H} = \mathcal{H}_r[\dot{+}]\mathcal{H}^\circ, \tag{2.1}$$

where $1 \leq \dim \mathcal{H}^\circ = \Delta < \infty$ and where $\mathcal{H}_r$ is a Pontryagin space. The condition (2.1) ensures that $\mathcal{H}$ can be embedded canonically into a Pontryagin space $\mathcal{P}_c$ with negative index $\operatorname{ind}_- \mathcal{P}_c = \operatorname{ind}_- \mathcal{H} + \dim \mathcal{H}^\circ$. To see this define

$$\mathcal{P}_c = \mathcal{H}_r[\dot{+}](\mathcal{H}^\circ \dot{+} \mathcal{H}^1), \tag{2.2}$$

where $\mathcal{H}^1$ is a neutral space and $\mathcal{H}^\circ$ and $\mathcal{H}^1$ are skewly linked (see [5]).

A linear relation $S \subseteq \mathcal{H}^2 \subseteq \mathcal{P}_c^2$ is called closed if it is closed in the topology of $\mathcal{P}_c^2$.

Let $S \subseteq \mathcal{H}^2$ be a closed symmetric relation. It is shown similar as in the classical (nondegenerated) situation that the number $\operatorname{codim} \mathcal{R}(S - z)$ is constant on the upper (lower) half plane with possible exception of finitely many points. Hence we may speak of the defect index of the symmetry S in the usual sense (see [3]).

Throughout this paper we assume that S is a closed symmetric relation with defect index $(1, 1)$ which satisfies the following regularity conditions: For each $h \in \mathcal{H}^\circ$

$$S \cap (\langle h \rangle \times \langle h \rangle) = \{0\}, \tag{2.3}$$

and for some $z \in \mathbb{C}^+$ and some $z \in \mathbb{C}^-$

$$\mathcal{R}(S - z) + \mathcal{H}^\circ = \mathcal{H}. \tag{2.4}$$

Remark 1 Condition (2.4) implies that the relation $S/\mathcal{H}^\circ$ acting in the Pontryagin space $\mathcal{H}/\mathcal{H}^\circ$ is selfadjoint and has nonempty resolvent set. This follows from the considerations in [3], in particular from Proposition 4.4 and Theorem 4.6 with its corollary. It also follows that (2.4) holds for all $z \in \varrho(S/\mathcal{H}^\circ)$.

Proposition 1 *The relation S can be decomposed (as a subspace of $\mathcal{H}^2$) as $S = S_0 \dot{+} S_1$, where $S_0 = S \cap (\mathcal{H}^\circ)^2$ and S_1 is a closed symmetric relation such that $\mathcal{R}(S_1 - z)$ is nondegenerated for $z \in \varrho(S/\mathcal{H}^\circ)$. There exists a basis $\{h_0, \dots, h_{\Delta-1}\}$ of $\mathcal{H}^\circ$, such that $S_0 = 0 \in (\mathcal{H}^\circ)^2$ if $\Delta = 1$, and*

$$S_0 = \langle (h_i; h_{i+1}) | i = 0, \dots, \Delta - 2 \rangle \tag{2.5}$$

otherwise.

Proof : Let $S_1 = S \cap (S_0)^{(\perp)}$, where the orthogonal complement has to be understood with respect to a definite inner product on $\mathcal{P}_c^2$. Then S_1 is symmetric, satisfies $S = S_0 \dot{+} S_1$ and is closed.

Choose a decomposition (2.1) and denote by P the projection of $\mathcal{H}$ onto $\mathcal{H}_r$ with kernel $\mathcal{H}^\circ$. Define $S_P = (P \times P)S$, then we have

$$\mathcal{H}_r \cong \mathcal{H}/\mathcal{H}^\circ \text{ and } S_P \cong S/\mathcal{H}^\circ.$$

Since $S_0 = S \cap \ker(P \times P)$ we have $S_1 \cap \ker(P \times P) = \{0\}$, and therefore the restriction $(P \times P)|_{S_1}$ maps S_1 bijectively onto S_P. Hence there exists an inverse mapping $\Phi : S_P \to S_1$. Clearly $\Phi - I$ maps S_P into $(\mathcal{H}^\circ)^2$.

We show that $\mathcal{R}(S_1 - z) \cap \mathcal{H}^\circ = \{0\}$ if $z \in \varrho(S/\mathcal{H}^\circ)$. Assume the opposite. Then there exists a pair $(a; b) \in S_1$, such that $b - za \neq 0$ and $b - za \in \mathcal{H}^\circ$. We have $(a; b) = \Phi(x; y)$ for some element $(x; y) \in S_P$. If we put $(x'; y') = (\Phi - I)(x; y)$, we have $(x'; y') \in (\mathcal{H}^\circ)^2$ and

$$b - za = (y - zx) + (y' - zx').$$

Since the left hand side as well as the second term on the right hand side of the above relation are elements of $\mathcal{H}^\circ$, $y - zx \in \mathcal{H}_r$, and $\mathcal{H}_r \cap \mathcal{H}^\circ = \{0\}$, we find $y - zx = 0$. As $z \in \varrho(S/\mathcal{H}^\circ) = \varrho(S_P)$, this shows that $x = y = 0$, hence $a = b = 0$.

Since $\mathcal{R}(S_0 - z) \subseteq \mathcal{H}^\circ$ and $\mathcal{R}(S - z) = \mathcal{R}(S_0 - z) + \mathcal{R}(S_1 - z)$, we find due to (2.4) and Remark 1, for $z \in \varrho(S/\mathcal{H}^\circ)$

$$\mathcal{R}(S_1 - z) \dot{+} \mathcal{H}^\circ = \mathcal{R}(S - z) + \mathcal{H}^\circ = \mathcal{H}.$$

This shows that $\mathcal{R}(S_1 - z)$ is nondegenerated for $z \in \varrho(S/\mathcal{H}^\circ)$.

It remains to show that S_0 is of the form (2.5). The condition (2.3) asserts that S_0 has no eigenvalue, in particular S_0 is (the graph of) an operator. Since S has defect index $(1,1)$, and S_0 acts in a Δ-dimensional space we find that $\dim \mathcal{R}(S_0 - z) = \Delta - 1$ and hence

$$\dim \mathcal{D}(S_0) = \Delta - 1.$$

Consider the domain of S_0^2. Since $S_0\mathcal{D}(S_0^2) = \mathcal{D}(S_0) \cap \mathcal{R}(S_0)$ we have either

$$\dim \mathcal{D}\left(S_0^2\right) = \dim \mathcal{D}(S_0) \text{ or } \dim \mathcal{D}\left(S_0^2\right) = \dim \mathcal{D}(S_0) - 1.$$

If $\dim \mathcal{D}(S_0^2) = \dim \mathcal{D}(S_0)$, i.e. $\mathcal{D}(S_0^2) = \mathcal{D}(S_0)$, this space is an invariant subspace of S_0. Hence S_0 has a nonzero eigenvector in $\mathcal{D}(S_0^2)$, a contradiction to condition (2.3). We find (by an inductive procedure) that

$$\mathcal{H}^\circ \supsetneq \mathcal{D}(S_0) \supsetneq \mathcal{D}\left(S_0^2\right) \supsetneq \ldots \supsetneq \mathcal{D}\left(S_0^{\Delta-1}\right) \supsetneq \{0\},$$

and that at each step the dimension decreases by 1.

Let $h_0 \in \mathcal{H}^\circ$ be such that $\mathcal{D}\left(S_0^{\Delta-1}\right) = \langle h_0 \rangle$ and put $h_i = S_0^i h_0$, $i = 1, \ldots, \Delta - 1$. A straightforward argument shows that $\{h_0, \ldots, h_{\Delta-1}\}$ is a basis of $\mathcal{H}^\circ$. Clearly $S_0 = \langle (h_i; h_{i+1}) | i = 0, \ldots, \Delta - 2 \rangle$.

□

Corollary 1 *The element h_0 is uniquely determined (up to constant multiples) by the properties*

$$h_0 \in \mathcal{D}\left(S^{\Delta-1}\right) \cap \mathcal{H}^\circ \text{ and } S^l h_0 \in \mathcal{H}^\circ,\ l = 0, \ldots, \Delta - 1.$$

Proof : Assume that $h \in \mathcal{D}\left(S^{\Delta-1}\right) \cap \mathcal{H}^\circ$ and $S^l h \in \mathcal{H}^\circ$ for $l = 0, \ldots, \Delta - 1$. Write, with respect to the basis $\{h_0, \ldots, h_{\Delta-1}\}$ of $\mathcal{H}^\circ$ constructed in Proposition 1,

$$h = \sum_{k=0}^{m} \eta_k h_k,\ \eta_m \neq 0$$

and assume on the contrary that $m > 0$. Then

$$S^{\Delta-m} h = \sum_{k=0}^{m} \eta_k S^{\Delta-m} h_k = \sum_{k=0}^{m-1} \eta_k h_{k+\Delta-m} + \eta_m S h_{\Delta-1},$$

hence $Sh_{\Delta-1} \in \mathcal{H}^\circ$, a contradiction.

□

Corollary 2 *Let $k \in \{0, \ldots, \Delta - 1\}$ and $z \neq 0$. Then $z \in \sigma(S/\mathcal{H}^\circ)$ if and only if*

$$h_k \in \mathcal{R}(S - z).$$

Proof : This follows from Remark 1, together with the fact that for $z \neq 0$

$$\langle h_k \rangle \dot{+} \mathcal{R}\,(S_0 - z) = \mathcal{H}^\circ.$$

□

Due to the fact that S has defect index $(1,1)$ we have for $z \in \varrho(S/\mathcal{H}^\circ)$

$$\mathcal{R}\,(S-z)^{\perp_{\mathcal{P}_c}} = \mathcal{H}^\circ \dot{+} \langle \chi \rangle$$

where χ is an element of $\mathcal{R}\,(S-z)^{\perp_{\mathcal{P}_c}} \setminus \mathcal{H}$.

Choose $\chi(z_0) \in \mathcal{R}\,(S-\overline{z_0})^{\perp_{\mathcal{P}_c}} \setminus \mathcal{H}$. If $\mathring{A}$ is a selfadjoint extension of S in $\mathcal{P}_c$, denote by $\chi(z)$ the element

$$\chi(z) = (I + (z-z_0)(\mathring{A}-z)^{-1})\chi(z_0).$$

Lemma 1 *The relation*

$$\mathcal{R}\,(S-\overline{z})^{\perp_{\mathcal{P}_c}} = \mathcal{H}^\circ \dot{+} \langle \chi(z) \rangle$$

holds for $z \in \varrho(\mathring{A}) \cap \varrho(S/\mathcal{H}^\circ)$ with possible exception of an isolated set.

Proof : Similar as in the classical case (see e.g. [8]) we find that $\chi(z) \perp \mathcal{R}\,(S-\overline{z})$. It remains to show that $\chi(z) \notin \mathcal{H}$. Choose a basis $\{h_0, \ldots, h_{\Delta-1}\}$ of $\mathcal{H}^\circ$. Since $\mathcal{H} = (\mathcal{H}^\circ)^{\perp_{\mathcal{P}_c}}$, we have $\chi(z) \in \mathcal{H}$ if and only if

$$[\chi(z), h_i] = 0 \text{ for } i = 0, \ldots, \Delta - 1.$$

The element $\chi(z)$ depends analytically on $z \in \varrho(\mathring{A})$ and $\chi(z_0) \notin \mathcal{H}$, hence for some i the function $[\chi(z), h_i]$ does not vanish identically. Its zeros are therefore isolated in $\varrho(\mathring{A})$, hence the points z with $\chi(z) \in \mathcal{H}$ are also isolated in $\varrho(\mathring{A})$.

□

Proposition 2 *There exists a selfadjoint extension $\mathring{A} \subseteq \mathcal{P}_c^2$ of S with $\varrho(\mathring{A}) \neq \emptyset$, such that for $z \in \varrho(\mathring{A}) \cap \varrho(S/\mathcal{H}^\circ)$*

$$(\mathring{A}-z)^{-1}\mathcal{H} \subseteq \mathcal{H},\ (\mathring{A}-z)^{-1}h_0 = 0 \tag{2.6}$$

and

$$[\chi(z), h_i] = z^i,\ i = 0, \ldots, \Delta - 1, \tag{2.7}$$

holds.

Proof : Consider the relation

$$S' = S \dot{+} \langle (0; h_0) \rangle.$$

Then S' is symmetric and $\mathcal{R}\,(S' - z) = \mathcal{H}$ for all $z \in \varrho(S/\mathcal{H}^\circ)$.

We show that $\ker\,(S' - z) = \{0\}$ if $z \in \varrho(S/\mathcal{H}^\circ)$: Assume that $h \in \ker\,(S' - z)$ then, since $\mathcal{R}\,(S' - \overline{z}) = \mathcal{H}$ (see Proposition 1) and $\mathcal{R}\,(S' - \overline{z})^\perp = \ker\,(S' - z)$, we find $h \in \mathcal{H}^\circ$. The relation S' decomposes according to Proposition 1 as

$$S' = S_0 \dot{+} S_1 \dot{+} \langle (0; h_0) \rangle.$$

Since $S_1 \cap (\mathcal{H}^\circ)^2 = \{0\}$, we may write, with respect to this decomposition

$$(h; zh) = (\sum_{i=0}^{\Delta-2} \lambda_i h_i; \sum_{i=1}^{\Delta-1} \lambda_{i-1} h_i) + (0; \lambda h_0).$$

Hence

$$\lambda h_0 + \sum_{i=1}^{\Delta-1} \lambda_{i-1} h_i = z \sum_{i=0}^{\Delta-2} \lambda_i h_i,$$

which implies that $h = 0$. Note that the resolvent set of the relation $S_0 \dot{+} \langle (0; h_0) \rangle$ is $\mathbb{C}$.

Now choose $z_0 \in \varrho(S/\mathcal{H}^\circ)$ and consider the Cayley transform C of S'. Then C is an isometric operator in $\mathcal{P}_c$, and

$$\operatorname{codim} \mathcal{D}\,(C) = \operatorname{codim} \mathcal{R}\,(C) = \Delta.$$

Hence C may be extended to a unitary operator U. The inverse Cayley transform $\mathring{A}$ of U is a selfadjoint relation extending S'. Since $\varrho(U) \neq \emptyset$ we find $\varrho(\mathring{A}) \neq \emptyset$, and for $z \in \varrho(\mathring{A}) \cap \varrho(S/\mathcal{H}^\circ)$ we have

$$(\mathring{A} - z)^{-1} \mathcal{H} = (\mathring{A} - z)^{-1} \mathcal{R}\,(S' - z) \subseteq \mathcal{H}.$$

To define defect elements choose $z_0 \in \varrho(\mathring{A}) \cap \varrho(S/\mathcal{H}^\circ)$, and let $\mathcal{P}_c$ be decomposed as

$$\mathcal{P}_c = \mathcal{R}\,(S_1 - z_0)\,[\dot{+}](\mathcal{H}^\circ \dot{+} \mathcal{H}'), \tag{2.8}$$

where $\mathcal{H}^\circ$ and $\mathcal{H}'$ are skewly linked. Let $\{h'_0, \ldots, h'_{\Delta-1}\}$ be a basis of $\mathcal{H}'$ which is skewly linked to the basis $\{h_0, \ldots, h_{\Delta-1}\}$ of $\mathcal{H}^\circ$ given by Proposition 1, i.e. let $[h_i, h'_j] = \delta_{ij}$. A short computation shows that we may choose

$$\chi(z_0) = h'_0 + z_0 h'_1 + \ldots + z_0^{\Delta-1} h'_{\Delta-1}.$$

Then $\chi(z) = (I + (z - z_0)(\mathring{A} - z)^{-1})\chi(z_0)$ are defect elements (see Lemma 1).

We compute $(i = 0, \dots, \Delta - 1)$

$$[\chi(z), h_i] = [(I + (z - z_0)(\mathring{A} - z)^{-1})\chi(z_0), h_i] =$$

$$= [\chi(z_0), h_i] + (z - z_0)[\chi(z_0), (\mathring{A} - \overline{z})^{-1} h_i].$$

Since $\mathring{A}$ extends S' we have $(\mathring{A} - z)^{-1} h_i = (S' - z)^{-1} h_i$. A straightforward computation shows that

$$(S' - z)^{-1} h_i = z^{i-1} h_0 + z^{i-2} h_1 + \dots + h_{i-1},\ i = 1, \dots, \Delta - 1.$$

Moreover $(S' - z)^{-1} h_0 = 0$, hence

$$[\chi(z), h_i] = z_0^i + (z - z_0)(z^{i-1} + z^{i-2} z_0 + \dots + z_0^{i-1}) = z^i,\ i = 0, \dots, \Delta - 1.$$

□

Corollary 3 *For any set $M \subseteq \varrho(\mathring{A}) \cap \varrho(S/\mathcal{H}^\circ)$ with $|M| \geq \Delta$ we have*

$$\mathcal{H}^\circ \cap \bigcap_{z \in M} \mathcal{R}(S - z) = \{0\}.$$

Proof : Let $h = \sum_{k=0}^{\Delta-1} \eta_k h_k$ and assume that $h \in \bigcap_{z \in M} \mathcal{R}(S - z)$. Then

$$[h, \chi(\overline{z})] = \sum_{k=0}^{\Delta-1} \eta_k z^k = 0,\ z \in M.$$

As a nonzero polynomial of degree $\Delta - 1$ has at most $\Delta - 1$ different zeros, we find $\eta_0 = \dots = \eta_{\Delta-1} = 0$, i.e. $h = 0$.

□

Remark 2 Note that each selfadjoint extension of S satisfying (2.6) induces the same generalized resolvent of S. For if π denotes the canonical projection of $\mathcal{H}$ onto $\mathcal{H}/\mathcal{H}^\circ$, we have for $x, y \in \mathcal{H}$ and $z \in \varrho(\mathring{A}) \cap \varrho(S/\mathcal{H}^\circ)$

$$[(\mathring{A} - z)^{-1} x, y] = [(S/\mathcal{H}^\circ - z)^{-1} \pi x, \pi y].$$

If $\chi(z)$ are defect elements as in Proposition 2 we define $q(z)$ by

$$\frac{q(z)-q(\overline{w})}{z-\overline{w}} = [\chi(z),\chi(w)]. \tag{2.9}$$

The function q is determined by (2.9) up to a real constant (see [8]).

In order to determine the generalized resolvents of S, the defect elements $\chi(z)$ play (similar as in the classical situation) an important role.

Proposition 3 *Let $A \subseteq \mathcal{P}^2$ be a selfadjoint extension of S acting in a Pontryagin space $\mathcal{P} \supseteq \mathcal{H}$, with $\varrho(A) \neq \emptyset$. Assume that the relation $\mathring{A}$ is chosen according to Proposition 2. Then there exists a function $\Psi(z)$ analytic on $\varrho(A) \cap \varrho(\mathring{A}) \cap \varrho(S/\mathcal{H}^\circ)$, with $\Psi(\overline{z}) = \overline{\Psi(z)}$ such that for $x, y \in \mathcal{H}$*

$$[(A-z)^{-1}x,y] = [(\mathring{A}-z)^{-1}x,y] + [x,\chi(\overline{z})]\Psi(z)[\chi(z),y]. \tag{2.10}$$

In fact $\Psi(z) = [(A-z)^{-1}h_0,h_0]$.

Proof : We will define $\Psi(z)$ for $z \in \varrho(A) \cap \varrho(\mathring{A}) \cap \varrho(S/\mathcal{H}^\circ)$. As for such z we have $[\chi(z),h_0] = 1$, we can decompose the space $\mathcal{H}$ as $\mathcal{H} = \mathcal{R}\,(S-z) \dotplus \langle h_0\rangle$. Due to (2.7) we can write

$$x = x_z + [x,\chi(\overline{z})]h_0 \text{ and } y = y_{\overline{z}} + [y,\chi(z)]h_0,$$

with $x_z \in \mathcal{R}\,(S-z)$ and $y_{\overline{z}} \in \mathcal{R}\,(S-\overline{z})$. Since

$$(A-z)^{-1}x = (\mathring{A}-z)^{-1}x \text{ for } x \in \mathcal{R}\,(S-z)\,,$$

we find

$$[((A-z)^{-1}-(\mathring{A}-z)^{-1})x,y] = \Big[((A-z)^{-1}-(\mathring{A}-z)^{-1})[x,\chi(\overline{z})]h_0,[y,\chi(z)]h_0\Big] =$$

$$= [x,\chi(\overline{z})]\Big[((A-z)^{-1}-(\mathring{A}-z)^{-1})h_0,h_0\Big][\chi(z),y].$$

Due to (2.6) we have $[(\mathring{A}-z)^{-1}h_0,h_0] = 0$, hence the assertion follows.

$\square$

3 Parametrization of the generalized resolvents

In this section we will prove the main result of this paper, which is the following

Theorem 1 *Let S be a closed symmetric relation in the degenerated inner product space $\mathcal{H}$, assume that S has defect index $(1,1)$ and satisfies (2.3) and (2.4). Let $\kappa_0 = \operatorname{ind}_- \mathcal{H}$ and $\Delta = \dim \mathcal{H}^\circ > 0$. Then for $z \in \varrho(A) \cap \varrho(S/\mathcal{H}^\circ)$ and $x, y \in \mathcal{H}$ the relation*

$$[(A-z)^{-1}x, y] = [(S/\mathcal{H}^\circ - z)^{-1}\pi x, \pi y] - [x, \chi(\overline{z})] \frac{1}{\tau(z) + q(z)} [\chi(z), y] \tag{3.1}$$

establishes a bijective correspondence between the generalized resolvents of S of index κ and the set $(\mathcal{K}^\Delta_{\kappa-\kappa_0} \setminus \{-q\})$ of parameters τ. Here $\chi(z)$ are the defect elements of S chosen according to Proposition 2, $q(z)$ is as in (2.9) and by π we denote the canonical projection of $\mathcal{H}$ onto $\mathcal{H}/\mathcal{H}^\circ$. The unique generalized resolvent $[(S/\mathcal{H}^\circ - z)^{-1}\pi x, \pi y]$ corresponds to the parameter $\tau = \infty$.

Remark 3 If we set $\sigma(z) = -\frac{1}{\tau(z)}$ for $\tau \neq 0$, the quadratic form (1.3) rewrites

$$\sum_{i,j=1}^{m} N_\sigma(z_i, z_j) \xi_i \overline{\xi_j} + \sum_{k=0}^{\Delta-1} \sum_{i=1}^{m} \operatorname{Re}\left(z_i^k \sigma(z_i) \xi_i \overline{\eta_k} \right)$$

Note that, since the quadratic form (1.3) is the same for each constant function $\tau(z) = \lambda$, $\lambda \in \mathbb{R}$, the case $\tau = 0$ is also covered.

By the results of [7] we find that $\tau \in \mathcal{K}^\Delta_{\kappa-\kappa_0}$ if and only if $\sigma = -\frac{1}{\tau}$ has a representation

$$\sigma(z) = [(B-z)^{-1}h, h],$$

where B is a selfadjoint relation in a Pontryagin space with negative index $\kappa - \kappa_0$, which extends a shift operator S_0 with $(\Delta - 1)$-dimensional domain, acting in a Δ-dimensional neutral space $\mathcal{L}$, where $\langle h \rangle = \mathcal{D}\left(S_0^{\Delta-1}\right)$, and where B is $\mathcal{L}$-minimal. In particular, for $\Delta \leq \kappa - \kappa_0$ the sets $\mathcal{K}^\Delta_{\kappa-\kappa_0}$ contain infinitely many elements.

We also see that the set $\mathcal{K}^\Delta_{\kappa-\kappa_0}$ is empty, if $\kappa - \kappa_0 < \Delta$. This corresponds to the fact that the negative index of a Pontryagin space extending $\mathcal{H}$ must be at least $\kappa_0 + \Delta$.

Theorem 1 is proved in several steps.

Lemma 2 *Let A be a selfadjoint extension of S and assume that the generalized resolvent $[(A-z)^{-1}x, y]$ does not coincide with $[(S/\mathcal{H}^\circ - z)^{-1}\pi x, \pi y]$. Then there exists a $\mathcal{H}$-minimal extension A_1 of S with*

$$[(A-z)^{-1}x, y] = [(A_1 - z)^{-1}x, y], \ z \in \varrho(A) \cap \varrho(A_1), \ x, y \in \mathcal{H}.$$

Proof : Consider the subspace

$$\mathcal{M} = \overline{\langle \mathcal{H}, (A-z)^{-1}\mathcal{H} | z \in \varrho(A) \rangle}.$$

The relation A induces a selfadjoint relation A_1 in the Pontryagin space $\mathcal{P}_1 = \mathcal{M}/\mathcal{M}^\circ$ which is $\mathcal{H}$-minimal. If $\mathcal{M}^\circ \cap \mathcal{H} = \{0\}$, we can assume that $\mathcal{H} \subseteq \mathcal{P}_1$. Then A_1 extends S and clearly

$$[(A_1 - z)^{-1}x, y] = [(A-z)^{-1}x, y], \; x, y \in \mathcal{H}.$$

If $h \in \mathcal{M}^\circ \cap \mathcal{H}$, $h \neq 0$, then $h \in \mathcal{H}^\circ$ and Corollary 3 implies

$$\mathcal{R}\,(S-z) + \langle h \rangle = \mathcal{H}$$

for all but finitely many $z \in \varrho(\mathring{A}) \cap \varrho(S/\mathcal{H}^\circ)$. Hence

$$[(A-z)^{-1}x, y] = [(S/\mathcal{H}^\circ - z)^{-1}\pi x, \pi y].$$

$\square$

Assume now that a selfadjoint extension A of S is given, where $A \subseteq \mathcal{P}^2$ and $\varrho(A) \neq \emptyset$, and put $R_z = (A-z)^{-1}$. If $\Psi = 0$, then $\tau = \infty$. Otherwise we may assume due to Lemma 2 that A is $\mathcal{H}$-minimal.

Let $\mathring{A}$ be chosen according to Proposition 2 and put $\mathring{R}_z = (\mathring{A} - z)^{-1}$. We define the operator valued function R'_z by ($x \in \mathcal{P}_c$)

$$R'_z x = \mathring{R}_z x + [x, \chi(\overline{z})]\Psi(z)\chi(z), \tag{3.2}$$

where $\Psi(z)$ is given by Proposition 3.

Remark 4 For $x, y \in \mathcal{H}$ we have

$$[R_z x, y] = [R'_z x, y]. \tag{3.3}$$

If either x or y does not belong to $\mathcal{H}$, this relation need not hold.

Let a function τ be defined by $\Psi(z) = -\frac{1}{\tau(z)+q(z)}$, i.e. let

$$\tau(z) = -\frac{1}{\Psi(z)} - q(z).$$

Note that, since $\Psi \neq \infty$, we have $\tau \neq -q$. Denote by $P_{\mathcal{H}}$ the projection of $\mathcal{P}_c$ onto $\mathcal{H}$ with kernel $\mathcal{H}^1$.

Let $m \in \mathbb{N}_0$, $z_1, \ldots, z_m \in \varrho(A) \cap \varrho(S/\mathcal{H}^\circ)$ and $a, a_1, \ldots, a_m, b, b_1, \ldots, b_m \in \mathcal{H}^\circ$. Consider the expression

$$U = \sum_{i,j=1}^{m} \Big([R_{z_i} a_i, R_{z_j} b_j] - [R'_{z_i} a_i, R'_{z_j} b_j] \Big) + \sum_{i=1}^{m} [R_{z_i} a_i, b] + \sum_{j=1}^{m} [a, R_{z_j} b_j].$$

Then U can be written in two different ways.

Lemma 3 *With the above notation we have*

$$U = [a + \sum_{i=1}^{m} (R_{z_i} - P_{\mathcal{H}} R'_{z_i}) a_i, b + \sum_{j=1}^{m} (R_{z_j} - P_{\mathcal{H}} R'_{z_j}) b_j], \tag{3.4}$$

and

$$U = \sum_{i,j=1}^{m} \Psi(z_i) [a_i, \chi(\overline{z_i})] N_\tau(z_i, z_j) [\chi(\overline{z_j}), b_j] \Psi(\overline{z_j}) +$$

$$+ \sum_{i=1}^{m} \Psi(z_i) [a_i, \chi(\overline{z_i})] [\chi(z_i), b] + \sum_{j=1}^{m} [a, \chi(z_j)] [\chi(\overline{z_j}), b_j] \Psi(\overline{z_j}). \tag{3.5}$$

Proof : First we show that the relation (3.4) holds. Note that $\mathcal{R}(I - P_{\mathcal{H}}) = \mathcal{H}^1$ is a neutral subspace. Due to (3.3) we have for $x \in \mathcal{H}$, $y \in \mathcal{P}_c$

$$[(R_z - R'_z)x, y] = [(R_z - R'_z)x, (I - P_{\mathcal{H}})y].$$

We compute

$$U = \sum_{i,j=1}^{m} \Big([(R_{z_i} - R'_{z_i}) a_i, (R_{z_j} - R'_{z_j}) b_j] + [(R_{z_i} - R'_{z_i}) a_i, R'_{z_j} b_j] +$$

$$+ [R'_{z_i} a_i, (R_{z_j} - R'_{z_j}) b_j] \Big) + \sum_{i=1}^{m} [R_{z_i} a_i, b] + \sum_{j=1}^{m} [a, R_{z_j} b_j] =$$

$$= \sum_{i,j=1}^{m} \Big([(R_{z_i} - R'_{z_i}) a_i, (R_{z_j} - R'_{z_j}) b_j] + [(R_{z_i} - R'_{z_i}) a_i, (I - P_{\mathcal{H}}) R'_{z_j} b_j] +$$

$$+ [(I - P_{\mathcal{H}}) R'_{z_i} a_i, (R_{z_j} - R'_{z_j}) b_j] \Big) + \sum_{i=1}^{m} [(R_{z_i} - P_{\mathcal{H}} R'_{z_i}) a_i, b] + \sum_{j=1}^{m} [a, (R_{z_j} - P_{\mathcal{H}} R'_{z_j}) b_j] =$$

$$= \sum_{i,j=1}^{m} [(R_{z_i} - P_{\mathcal{H}} R'_{z_i}) a_i, (R_{z_j} - P_{\mathcal{H}} R'_{z_j}) b_j] + \sum_{i=1}^{m} [(R_{z_i} - P_{\mathcal{H}} R'_{z_i}) a_i, b] +$$

$$+ \sum_{j=1}^{m} [a, (R_{z_j} - P_{\mathcal{H}} R'_{z_j}) b_j] = [a + \sum_{i=1}^{m} (R_{z_i} - P_{\mathcal{H}} R'_{z_i}) a_i, b + \sum_{j=1}^{m} (R_{z_j} - P_{\mathcal{H}} R'_{z_j}) b_j].$$

Hence (3.4) is proved. On the other hand use the definition (3.2) and the relation (2.10) to compute U. We find, due to the resolvent identity which holds for R_z (recall that $a_i, b_j \in \mathcal{H}^\circ$)

$$[R_{z_i}a_i, R_{z_j}b_j] = [\frac{R_{z_i} - R_{\overline{z_j}}}{z_i - \overline{z_j}}a_i, b_j] = [\frac{\mathring{R}_{z_i} - \mathring{R}_{\overline{z_j}}}{z_i - \overline{z_j}}a_i, b_j]+$$

$$+\frac{[a_i, \chi(\overline{z_i})]\Psi(z_i)[\chi(z_i), b_j] - [a_i, \chi(z_j)]\Psi(\overline{z_j})[\chi(\overline{z_j}), b_j]}{z_i - \overline{z_j}},$$

and

$$[R'_{z_i}a_i, R'_{z_j}b_j] = [\mathring{R}_{z_i}a_i, \mathring{R}_{z_j}b_j] + [\mathring{R}_{z_i}a_i, [b_j, \chi(\overline{z_j})]\Psi(z_j)\chi(z_j)]+$$

$$+[[a_i, \chi(\overline{z_i})]\Psi(z_i)\chi(z_i), \mathring{R}_{z_j}b_j] + [[a_i, \chi(\overline{z_i})]\Psi(z_i)\chi(z_i), [b_j, \chi(\overline{z_j})]\Psi(z_j)\chi(z_j)]].$$

Using the resolvent identity for $\mathring{R}_z$ and the relations

$$\mathring{R}_{\overline{z_i}}\chi(z_j) = \frac{\chi(\overline{z_i}) - \chi(z_j)}{\overline{z_i} - z_j}, \ \mathring{R}_{\overline{z_j}}\chi(z_i) = \frac{\chi(\overline{z_j}) - \chi(z_i)}{\overline{z_j} - z_i},$$

we find

$$[R'_{z_i}a_i, R'_{z_j}b_j] = [\frac{\mathring{R}_{z_i} - \mathring{R}_{\overline{z_j}}}{z_i - \overline{z_j}}a_i, b_j]+$$

$$+[\chi(\overline{z_j}), b_j]\overline{\Psi(z_j)}[a_i, \frac{\chi(\overline{z_i}) - \chi(z_j)}{\overline{z_i} - z_j}] + [a_i, \chi(\overline{z_i})]\Psi(z_i)[\frac{\chi(\overline{z_j}) - \chi(z_i)}{\overline{z_j} - z_i}, b_j]+$$

$$+[a_i, \chi(\overline{z_i})]\Psi(z_i)[\chi(z_i), \chi(z_j)]\overline{\Psi(z_j)}[\chi(\overline{z_j}), b_j].$$

Due to (2.6) we have

$$[\mathring{R}_{z_i}a_i, b] = [a, \mathring{R}_{z_j}b_j] = 0,$$

hence the last two terms in the definition of U compute as

$$[R_{z_i}a_i, b] = [a_i, \chi(\overline{z_i})]\Psi(z_i)[\chi(z_i), b]$$

$$[a, R_{z_j}b_j] = [a, \chi(z_j)]\overline{\Psi(z_j)}[\chi(\overline{z_j}), b_j].$$

We find

$$U = \sum_{i,j=1}^{m}[a_i, \chi(\overline{z_i})]\left(\frac{\Psi(z_i) - \overline{\Psi(z_j)}}{z_i - \overline{z_j}} - \Psi(z_i)[\chi(z_i), \chi(z_j)]\overline{\Psi(z_j)}\right)[\chi(\overline{z_j}), b_j]+$$

$$+\sum_{i=1}^{m}[a_i, \chi(\overline{z_i})]\Psi(z_i)[\chi(z_i), b] + \sum_{j=1}^{m}[a, \chi(z_j)]\overline{\Psi(z_j)}[\chi(\overline{z_j}), b_j].$$

If we substitute $\Psi(z) = -\frac{1}{\tau(z)+q(z)}$, where $q(z)$ is defined by (2.9), we obtain

$$\begin{aligned} U &= \sum_{i,j=1}^{m} [a_i, \chi(\overline{z_i})]\Psi(z_i) \left(\frac{\tau(z_i) + q(z_i) - \tau(\overline{z_j}) - q(\overline{z_j})}{z_i - \overline{z_j}} - \right. \\ & \left. -N_q(z_i, z_j) \right) \Psi(\overline{z_j})[\chi(\overline{z_j}), b_j] + \sum_{i=1}^{m} [a_i, \chi(\overline{z_i})]\Psi(z_i)[\chi(z_i), b] + \\ & \sum_{j=1}^{m} [a, \chi(z_j)]\Psi(\overline{z_j})[\chi(\overline{z_j}), b_j], \end{aligned}$$

hence (3.5) follows.

□

Lemma 4 *For any nonzero element $h \in \mathcal{H}^\circ$, we have*

$$\overline{\langle \mathcal{H}^\circ, (R_z - P_{\mathcal{H}} R'_z)h | z \in \varrho(A) \cap \varrho(S/\mathcal{H}^\circ) \rangle} = \mathcal{H}_r^\perp.$$

Proof : Since $(R_z - P_{\mathcal{H}} R'_z)h \perp \mathcal{H}_r$, the linear space

$$\mathcal{L} = \langle \mathcal{H}^\circ, (R_z - P_{\mathcal{H}} R'_z)h | z \in \varrho(A) \rangle$$

is dense in $\mathcal{H}_r^\perp$ if and only if the linear space

$$\mathcal{L}_1 = \langle \mathcal{H}, (R_z - P_{\mathcal{H}} R'_z)h | z \in \varrho(A) \rangle$$

is dense in $\mathcal{P}$. Since for all $z \in \mathbb{C} \setminus \mathbb{R}$, with possible exception of finitely many, the relation $\mathcal{R}(S - z) \dot{+} \langle h \rangle = \mathcal{H}$ holds, and since $R_z \mathcal{R}(S - z) \subseteq \mathcal{H}$, we find that

$$\mathcal{L}_1 = \langle \mathcal{H}, R_z h | z \in \varrho(A) \rangle = \langle \mathcal{H}, R_z \mathcal{H} | z \in \varrho(A) \rangle.$$

By the assumption that A is $\mathcal{H}$-minimal, the last space is dense in $\mathcal{P}$.

□

If we put in Lemma 3

$$a = b = \sum_{k=0}^{\Delta-1} \eta_k h_k, \ a_i = b_i = \xi_i h_0,$$

we obtain with (2.7)

$$U = \sum_{i,j=1}^{m} \Psi(z_i)\xi_i N_\tau(z_i, z_j)\overline{\xi_j}\Psi(\overline{z_j}) + \sum_{k=0}^{\Delta-1} \left(\sum_{i=1}^{m} \Psi(z_i)\xi_i z_i^k \overline{\eta_k} + \sum_{j=1}^{m} \overline{z_j}^k \eta_k \overline{\xi_j} \Psi(\overline{z_j}) \right).$$

Due to Lemma 4 this quadratic form in the variables $\xi_1, \ldots, \xi_m; \eta_0, \ldots, \eta_{\Delta-1}$ has $\mathrm{ind}_- \mathcal{H}_r^\perp = \kappa - \kappa_0$ negative squares. After the change of variables

$$\Psi(z_i)\xi_i \to \xi_i,\ 2\eta_k \to \eta_k$$

and use of the fact that $\Psi(z)$ vanishes only on an isolated set, we obtain the form (1.3), hence $\tau(z) \in \mathcal{K}^\Delta_{\kappa-\kappa_0}$. We have shown that the generalized resolvent $[R_z x, y]$ has the representation (3.1) with the parameter $\tau \in \mathcal{K}^\Delta_{\kappa-\kappa_0}$.

It remains to prove the converse implication of Theorem 1. Let $\mathring{A}$ and $\chi(z)$ be as in Proposition 2 and assume that a parameter $\tau \in \mathcal{K}^\Delta_{\kappa-\kappa_0}$, $\tau \neq -q$, is given. We will construct a selfadjoint extension A of S, such that for $x, y \in \mathcal{H}$

$$[(A-z)^{-1}x, y] = [(S/\mathcal{H}^\circ - z)^{-1}\pi x, \pi y] - [x, \chi(\overline{z})]\frac{1}{\tau(z)+q(z)}[\chi(z), y]. \tag{3.6}$$

Put $\psi(z) = -\frac{1}{\tau(z)+q(z)}$. Let $\mathcal{H}$ be decomposed as in (2.1) and consider the linear space

$$\mathcal{L} = \mathcal{H}_r[\dot{+}](\mathcal{H}^\circ \dot{+} \langle f_z | z \in \mathbb{C}\setminus\mathbb{R}, \tau(z)+q(z) \neq 0\rangle),$$

where f_z are formal elements, equipped with the inner product

$$[x, y]_\mathcal{L} = [x, y]_\mathcal{H},\ x, y \in \mathcal{H}_r,$$

$$\begin{aligned}
&[\sum_{k=0}^{\Delta-1}\eta_k h_k + \sum_{i=1}^m \xi_i f_{z_i}, \sum_{k=0}^{\Delta-1}\eta'_k h_k + \sum_{j=1}^m \xi'_j f_{z_j}]_\mathcal{L} = \\
&= \sum_{i,j=1}^m \psi(z_i)\xi_i N_\tau(z_i, z_j)\overline{\xi'_j}\psi(\overline{z_j}) + \\
&+ \sum_{k=0}^{\Delta-1}\left(\sum_{i=1}^m \psi(z_i)\xi_i z_i^k \overline{\eta'_k} + \sum_{j=1}^m \overline{z_j}^k \eta_k \overline{\xi'_j}\psi(\overline{z_j})\right),
\end{aligned} \tag{3.7}$$

$$\left[x, \sum_{k=0}^{\Delta-1}\eta_k h_k + \sum_{i=1}^m \xi_i f_{z_i}\right]_\mathcal{L} = 0,\ x \in \mathcal{H}_r.$$

The $\tau \in \mathcal{K}^\Delta_{\kappa-\kappa_0}$ implies

$$\mathrm{ind}_- \mathcal{L} = \mathrm{ind}_- \mathcal{H}_r + (\kappa - \kappa_0) = \kappa.$$

Hence we obtain from $\mathcal{L}$ by factorization with respect to its isotropic part and completion a Pontryagin space $\mathcal{P}$ with $\mathrm{ind}_- \mathcal{P} = \kappa$: $\mathcal{P} = \widehat{\mathcal{L}/\mathcal{L}^\circ}$.

Lemma 5 *We have $\mathcal{H} \subseteq \mathcal{P}$.*

Proof : Since $\mathcal{H}_r$ is nondegenerated it suffices to show that $\mathcal{H}^\circ \cap \mathcal{L}^\circ = \{0\}$. Assume that $\sum\limits_{k=0}^{\Delta-1} \eta_k h_k \in \mathcal{L}^\circ$. Choose Δ disjoint numbers $z_1, \ldots, z_\Delta \in \varrho(\psi)$, such that $\psi(\overline{z_i}) \neq 0$ for $i = 1, \ldots, \Delta$. Then, for arbitrary numbers ξ_i, we have

$$0 = [\sum_{k=0}^{\Delta-1} \eta_k h_k, \sum_{i=1}^{\Delta} \xi_i f_{z_i}] = \sum_{k=0}^{\Delta-1} \sum_{i=1}^{\Delta} \overline{z_i}^k \eta_k \overline{\xi_i} \psi(\overline{z_i}) =$$

$$= (\eta_0, \ldots, \eta_{\Delta-1}) \left(\overline{z_i}^k\right)_{\substack{i=1,\ldots,\Delta \\ k=0,\ldots,\Delta-1}} \begin{pmatrix} \psi(\overline{z_1})\overline{\xi_1} \\ \vdots \\ \psi(\overline{z_\Delta})\overline{\xi_\Delta} \end{pmatrix}.$$

Since the matrix $(\overline{z_i}^k)_{\substack{i=1,\ldots,\Delta \\ k=0,\ldots,\Delta-1}}$ is regular this relation implies $\eta_0 = \ldots = \eta_{\Delta-1} = 0$.

□

The space $\mathcal{P}_c$ can be viewed as a subspace of $\mathcal{P}$: $\mathcal{P}_c = \mathcal{H}_r[\dot{+}](\mathcal{H}^\circ \dot{+} \mathcal{H}')$ with $\mathcal{H}' \subseteq \overline{\mathcal{H}^\circ \dot{+} \langle f_z | z \in \mathbb{C} \setminus \mathbb{R}, \tau(z) + q(z) \neq 0 \rangle}$. Let R'_z be defined by $(x \in \mathcal{P}_c)$

$$R'_z x = \overset{\circ}{R}_z x + [x, \chi(\overline{z})] \psi(z) \chi(z).$$

Note that, due to Proposition 2,

$$R'_z h_0 = \psi(z) \chi(z). \tag{3.8}$$

Define elements

$$e_z = f_z + P_{\mathcal{H}} R'_z h_0.$$

Clearly $\mathcal{P} = \mathcal{H} + \overline{\langle e_z | z \in \varrho(\psi) \rangle}$.

Lemma 6 *The inner product of expressions involving elements e_z is given by*

$$[e_z, e_w] = N_\psi(z, w), \;\; z, w \in \varrho(\psi),$$

$$[e_z, x] = [\chi(z), x] \psi(z), \;\; z \in \varrho(\psi), x \in \mathcal{H}.$$

Proof : We compute

$$[e_z, e_w] = [f_z, f_w] + [f_z, P_{\mathcal{H}} R'_w h_0] + [P_{\mathcal{H}} R'_z h_0, f_w] + +[P_{\mathcal{H}} R'_z h_0, P_{\mathcal{H}} R'_w h_0]. \tag{3.9}$$

Denote by $\{h'_0, \ldots, h'_{\Delta-1}\}$ a basis of $\mathcal{H}'$ skewly linked to $\{h_0, \ldots, h_{\Delta-1}\}$. Since $[\chi(z), \chi(w)] = N_q(z, w)$, $\mathcal{H}'$ is a neutral subspace, and (3.8) holds, the last term on the right hand side of (3.9) computes as

$$[P_{\mathcal{H}} R'_z h_0, P_{\mathcal{H}} R'_w h_0] = \psi(z)(N_q(z, w) -$$

$$-\sum_{k=0}^{\Delta-1}(z^k[h'_k,\chi(w)]+\overline{w}^k[\chi(z),h'_k])\overline{\psi(w)}.$$

The definition (3.7) of the inner product $[.,.]$ shows that

$$[P_{\mathcal{H}}\chi(z),f_w]=\sum_{k=0}^{\Delta-1}\overline{w}^k[\chi(z),h'_k]\overline{\psi(w)}.$$

We find

$$[e_z,e_w]=\psi(z)\left(N_\tau(z,w)+N_q(z,w)\right)\psi(\overline{w})=N_\psi(z,w).$$

Let $x\in\mathcal{H}$ be decomposed as $x=x_r+\sum\limits_{k=0}^{\Delta-1}\eta_k h_k$ with $x_r\in\mathcal{H}_r$. Then, due to (3.7) and (3.8), we find

$$[e_z,x]=[f_z,x]+[P_{\mathcal{H}}R'_z h_0,x]=\sum_{k=0}^{\Delta-1}z^k\psi(z)\overline{\eta_k}+\psi(z)[\chi(z),x_r]=\psi(z)[\chi(z),x].$$

$\square$

Let $A\subseteq\mathcal{P}^2$ be defined as

$$A=\overline{\langle S,(e_z;h_0+ze_z)|z\in\varrho(\psi)\rangle}.$$

Lemma 7 *The relation A is selfadjoint and has a nonempty resolvent set. In fact*

$$\varrho(A)\supseteq\varrho(\psi)\cap\varrho(S/\mathcal{H}^\circ).$$

Moreover, A is $\mathcal{H}$-minimal.

Proof : We first show that A is symmetric. To see this it suffices to note that for $(a;b)\in S$ and $z,w\in\varrho(\psi)$ the relations

$$[e_z,h_0+we_w]-[h_0+ze_z,e_w]=\psi(z)+\overline{w}N_\psi(z,w)-\overline{\psi(w)}-zN_\psi(z,w)=0,$$

$$[e_z,b]-[h_0+ze_z,a]=[\chi(z),b]\psi(z)-z[\chi(z),a]\psi(z)=$$
$$=[\chi(z),b-\overline{z}a]\psi(z)=0$$

hold.

It remains to prove that $\mathcal{R}\,(A-z)$ is dense in $\mathcal{P}$ if $z\in\varrho(\psi)\cap\varrho(S/\mathcal{H}^\circ)$, since then, by the same argument as in Remark 1 (using [3]), the assertion follows.

Let $z\in\varrho(\psi)\cap\varrho(S/\mathcal{H}^\circ)$, then

$$(e_z;h_0)\in A-z \tag{3.10}$$

$$(\frac{e_z - e_w}{z-w}; e_w) \in A - z,\ w \neq z.$$

Since $\mathcal{R}(S-z) \subseteq \mathcal{R}(A-z)$ and $\mathcal{R}(S-z) \dot{+} \langle h_0 \rangle = \mathcal{H}$, we obtain

$$\langle \mathcal{H}, e_w | w \in \varrho(\psi) \setminus \{z\} \rangle \subseteq \mathcal{R}(A-z).$$

Hence $\mathcal{R}(A-z)$ is dense in $\mathcal{P}$.

The relation (3.10) shows that

$$\overline{\langle \mathcal{H}, (A-z)^{-1} h_0 | z \in \varrho(A) \rangle} = \mathcal{P}.$$

In particular A is $\mathcal{H}$-minimal.

□

By Proposition 3 we have

$$\Psi(z) = [R_z h_0, h_0] = [e_z, h_0] = \psi(z) = -\frac{1}{\tau(z) + q(z)}.$$

Hence the generalized resolvent corresponding to the relation A has the representation (3.6) with the prescribed parameter τ.

All assertions of Theorem 1 are proved.

In the classical case $\dim \mathcal{H}^\circ = 0$, the canonical extensions of S, i.e. those acting the space $\mathcal{H}$ itself, correspond to the parameter functions $\tau(z) = t \in \mathbb{R}$.

The space $\mathcal{P}_c$ is the minimal Pontryagin space extending $\mathcal{H}$. Hence we will refer to an extension $A \subseteq \mathcal{P}_c^2$ of S as a canonical extension. The parameters corresponding to such canonical extensions are of more complicated structure. However, they can be identified by making use of Remark 3. The following statement deals with operator extensions ($A(0) = \{0\}$) of S. For proper relations a similar argument can be applied if the spectrum is transformed conveniently.

Remark 5 Let B be a minimal selfadjoint operator in the space $\mathbb{C}^{2\Delta}$, endowed with the inner product induced by the Gram matrix

$$J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Assume that B extends a shift operator with $(\Delta - 1)$-dimensional domain as in Remark 3. Then, with respect to a convenient basis, the operator B has the matrix representation

$$B = \left(\begin{array}{cccc|cccc} 0 & \cdots & 0 & c_1 & a_{11} & \cdots & a_{1,\Delta-1} & a_{1\Delta} \\ 1 & \ddots & \vdots & \vdots & a_{21} & \cdots & a_{2,\Delta-1} & a_{2\Delta} \\ \vdots & \ddots & 0 & c_{\Delta-1} & \vdots & & \vdots & \vdots \\ 0 & \cdots & 1 & c_\Delta & a_{\Delta 1} & \cdots & a_{\Delta,\Delta-1} & a_{\Delta\Delta} \\ \hline 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & c & \overline{c_1} & \cdots & \overline{c_{\Delta-1}} & \overline{c_\Delta} \end{array}\right),$$

where $c, c_1, \dots, c_\Delta \in \mathbb{C}$, $c \neq 0$, and where $(a_{rs})_{r,s=1}^{\Delta}$ is a complex hermitian matrix.

The function $\tau(z) = -\frac{1}{[(B-z)^{-1}h,h]}$, which runs by Remark 3 through the class $\mathcal{K}_\Delta^\Delta$ (with exception of those functions coming from a proper relation), can be written as

$$\tau(z) = \frac{1}{c} p(z)\overline{p(\overline{z})} - \sum_{r,s=1}^{\Delta} a_{rs} z^{r+s-2},$$

with $p(z) = -z^\Delta + \sum_{j=1}^{\Delta} c_j z^{j-1}$.

4 A continuation problem for hermitian functions

Let $0 < a \leq \infty$. A hermitian ($f(-t) = \overline{f(t)}$) function f defined and continuous on the interval $(-2a, 2a)$ is said to be in the class $\mathcal{P}_{\kappa,a}$ if the kernel $f(t-s)$, $t, s \in (-a, a)$, has κ negative squares. Explicitly this means that for each choice of $m \in \mathbb{N}$ and $t_1, \dots, t_m \in (-a, a)$ the quadratic form

$$\sum_{i,j=1}^{m} f(t_j - t_i)\xi_i \overline{\xi_j}$$

has at most κ negative squares, and for at least one choice of m and $t_1, \dots, t_m$ this form has exactly κ negative squares.

It was proved in [4] that a function $f \in \mathcal{P}_{\kappa_0,a}$ has at least one extension to the whole real axis $\tilde{f} \in \mathcal{P}_{\kappa_0,\infty}$. If there exists more than one extension, the extensions $\tilde{f} \in \mathcal{P}_{\kappa_0,\infty}$ are parametrized by the formula

$$i \int_0^\infty e^{izt} \tilde{f}(t)\, dt = \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_{21}(z)\tau(z) + w_{22}(z)}, \ \ \mathrm{Im}\, z > h,$$

where $w_{ij}(z)$ are certain analytic functions and the parameter $\tau(z)$ runs through the set $\mathcal{N}_0 \cup \{\infty\}$. The number $h \in \mathbb{R},\ h \geq 0$ depends on the parameter $\tau(z)$.

We consider the case that there is only one extension of f in $\mathcal{P}_{\kappa_0,\infty}$. In [6] the existence of a number $\Delta \in \mathbb{N} \cup \{\infty\}$ was proved, such that there are no extensions of f in $\mathcal{P}_{\kappa,\infty}$ if $\kappa_0 < \kappa < \kappa_0 + \Delta$ and (if $\Delta < \infty$) there is at least one extension in $\mathcal{P}_{\kappa_0+\Delta,\infty}$. We assume in the following that $\Delta < \infty$. In [6] a model space $\mathcal{H}$ and an operator S is associated to the function f. The model space $\mathcal{H}$ is an inner product space with $\operatorname{ind}_- \mathcal{H} = \kappa_0$ and $\dim \mathcal{H}^\circ = \Delta$. The extensions $\tilde{f} \in \mathcal{P}_{\kappa,\infty}$ with $\kappa \geq \kappa_0 + \Delta$ correspond to the minimal selfadjoint extensions A of S acting in a Pontryagin space $\mathcal{P} \supseteq \mathcal{H}$ with $\operatorname{ind}_- \mathcal{P} = \kappa$ by

$$i \int_0^\infty e^{izt} \tilde{f}(t)\, dt = [(A - z)^{-1} f_0, f_0],\ \ \operatorname{Im} z > h.$$

Here f_0 is a certain element of $\mathcal{H}$ and $h > 0$.

It follows from Proposition 7 of [6] that the model operator S satisfies the regularity conditions (2.3) and (2.4). Hence the results of the preceding section can be applied, and we obtain

Theorem 2 *Let $f \in \mathcal{P}_{\kappa_0,a}$, assume that $0 < \Delta < \infty$ and let $\kappa \geq \kappa_0 + \Delta$. There exist four analytic functions $w_{ij}(z)$ $(i,j = 1,2)$, such that the extensions $\tilde{f} \in \mathcal{P}_{\kappa,\infty}$ of f are parametrized by the formula*

$$i \int_0^\infty e^{izt} \tilde{f}(t)\, dt = \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_{21}(z)\tau(z) + w_{22}(z)},\ \ \operatorname{Im} z > h.$$

The parameter $\tau(z)$ runs through the set $\mathcal{K}^\Delta_{\kappa-\kappa_0}$, and the number $h \in \mathbb{R},\ h \geq 0$ depends on $\tau(z)$. The functions $w_{ij}(z)$ are given by

$$w_{11}(z) = [(S/\mathcal{H}^\circ - z)^{-1} \pi f_0, \pi f_0],\ \ w_{21}(z) = 1,$$

$$w_{12}(z) = q(z)[(S/\mathcal{H}^\circ - z)^{-1} \pi f_0, \pi f_0] - [f_0, \chi(\overline{z})][\chi(z), f_0],\ \ w_{22}(z) = q(z).$$

Note that in particular f actually has infinitely many extensions in $\mathcal{P}_\kappa$ for $\kappa \geq \kappa_0 + \Delta$.

5 The indefinite Nevanlinna-Pick interpolation problem

The Nevanlinna-Pick interpolation problem can be formulated as follows: Given $N \in \mathbb{N}$, points $z_1, \ldots, z_N \in \mathbb{C}^+$ and values $w_1, \ldots, w_N \in \mathbb{C}$, describe the functions $f \in \mathcal{N}_\kappa$ $(\kappa \in \mathbb{N}_0)$ which satisfy

$$f(z_i) = w_i,\ i = 1, \ldots, N. \tag{5.1}$$

With the interpolation data the so called Pick matrix $\mathbb{P}$ is associated:

$$\mathbb{P} = \left(\frac{w_j - \overline{w_i}}{z_j - \overline{z_i}} \right)_{i,j=1}^{N}.$$

If $\mathbb{P}$ is regular, a description of the solutions of (5.1) can be given in several ways, e.g. using an operator theoretic approach or the theory of reproducing kernel spaces (see [1]). It is shown in [11] that the operator theoretic approach also works in the case of singular data, i.e. if $\mathbb{P}$ is singular.

Denote by $\mathcal{H}$ the inner product space of all (formal) sums

$$\mathcal{H} = \{\sum_{i=1}^{N} \xi_i e_i | \xi_i \in \mathbb{C}\}$$

endowed with the inner product given by

$$[e_i, e_j] = \frac{w_i - \overline{w_j}}{z_i - \overline{z_j}}, \; i, j = 1, \ldots, N.$$

Note that $\mathcal{H}$ is simply $\mathbb{C}^N$ with the inner product

$$[\varepsilon_i, \varepsilon_j] = (\mathbb{P}\varepsilon_i, \varepsilon_j), \; i, j = 1, \ldots, N,$$

when ε_i denotes the i-th canonical basis vector of $\mathbb{C}^N$, and $(.,.)$ is the usual inner product. On $\mathcal{H}$ we define an operator S by

$$\mathcal{D}(S) = \{\sum_{i=1}^{N} \xi_i e_i \in \mathcal{H} | \sum_{i=1}^{N} \xi_i = 0\},$$

$$S(\sum_{i=1}^{N} \xi_i e_i) = \sum_{i=1}^{N} \xi_i z_i e_i, \; \sum_{i=1}^{N} \xi_i = 0.$$

It is checked by a straightforward calculation that S is symmetric and has no eigenvalues. The following result was proved in [11]:

Proposition 4 *Let $N \in \mathbb{N}$, $z_1, \ldots, z_N \in \mathbb{C}^+$ and $w_1, \ldots, w_N \in \mathbb{C}$ be given. The formula*

$$f_A(z) = \overline{w_1} + \frac{\operatorname{Im} w_1}{\operatorname{Im} z_1}(z - \overline{z_1}) + (z - \overline{z_1})(z - z_1)[(A - z)^{-1} e_1, e_1]$$

establishes a bijective correspondence between the solutions $f \in \mathcal{N}_\kappa$ ($\kappa > \kappa_0$) of (5.1) and selfadjoint relations $A \supseteq S$ with $z_1, \ldots, z_N \in \varrho(A)$ which act in a Pontryagin space with negative index κ and are e_1-minimal.

We assume in the following that $\mathbb{P}$ is singular, i.e. $\dim \mathcal{H}^\circ = \Delta > 0$. The regularity conditions (2.3) and (2.4) are satisfied: (2.3) clearly follows from the fact that S has no eigenvalues, wheras (2.4) follows from the fact that a fixed element $x = \sum_{i=1}^N \xi_i e_i \in \mathcal{H}$ is contained in $\mathcal{R}(S-z)$ if and only if z is a solution of the equation

$$\sum_{i=1}^N \xi_i \prod_{\substack{j=1\\ j\neq i}}^N (z_j - z) = 0.$$

Hence in fact for any set $M \subseteq \mathbb{C}$ with $|M| \geq N$

$$\bigcap_{z\in M} \mathcal{R}(S-z) = \{0\}.$$

Due to Corollary 1 the element h_0 defined in Proposition 1 is given by $h_0 = \sum_{i=1}^N \chi_i e_i$ where the numbers χ_i are the (up to a common factor) unique solutions of the linear equations

$$\mathbb{P}\begin{pmatrix} \chi_1 & & 0 \\ & \ddots & \\ 0 & & \chi_N \end{pmatrix} \begin{pmatrix} 1 & z_1 & \cdots & z_1^{\Delta-1} \\ \vdots & \vdots & & \vdots \\ 1 & z_N & \cdots & z_N^{\Delta-1} \end{pmatrix} = 0,$$

$$(\chi_1, \dots, \chi_N)\begin{pmatrix} 1 & z_1 & \cdots & z_1^{\Delta-2} \\ \vdots & \vdots & & \vdots \\ 1 & z_N & \cdots & z_N^{\Delta-2} \end{pmatrix} = 0. \tag{5.2}$$

Assume that the data points are enumerated such that

$$\left| \frac{w_j - \overline{w_i}}{z_j - \overline{z_i}} \right|_{i,j=1}^{N-\Delta} \neq 0,$$

then $\mathcal{H}^\circ \cap \langle e_1, \dots, e_{N-\Delta}\rangle = \{0\}$. Since $\langle h_0, \dots, S^{\Delta-1}h_0\rangle = \mathcal{H}^\circ$ we must have $\chi_i \neq 0$ for $i = N-\Delta+1, \dots, N$.

It is well known (see [10]) that, if $\kappa_0 = \operatorname{ind}_- \mathcal{H}$, there is at most one solution $f \in \mathcal{N}_{\kappa_0}$, and no solutions $f \in \mathcal{N}_\kappa$ if $\kappa_0 < \kappa < \kappa_0 + \Delta$.

Lemma 8 *There exists a unique solution $f \in \mathcal{N}_{\kappa_0}$ of (5.1) if and only if*

$$z_1, \dots, z_N \in \varrho(S/\mathcal{H}^\circ),$$

or, equivalently, $\chi_i \neq 0,\ i = 1, \ldots, N$. *In this case the unique solution* f *equals* $f_{S/\mathcal{H}^\circ}$ *and is given explicitly by*

$$f_{S/\mathcal{H}^\circ}(z) = \frac{\sum\limits_{i=1}^{N} \chi_i w_i \prod_{\substack{j=1\\ j\neq i}}^{N} (z - z_j)}{\sum\limits_{i=1}^{N} \chi_i \prod_{\substack{j=1\\ j\neq i}}^{N} (z - z_j)}. \tag{5.3}$$

Proof : If $z_1, \ldots, z_N \in \varrho(S/\mathcal{H}^\circ)$ a computation similar as in Proposition 4 of [11] shows that $f_{S/\mathcal{H}^\circ}$ is a solution of (5.1). Clearly $f_{S/\mathcal{H}^\circ} \in \mathcal{N}_{\kappa_0}$. Conversely, if there exists a unique solution, it follows from Theorem 2 of [10] that $\chi_i \neq 0$ for $i = 1, \ldots, N - \Delta$. It is proved in [10] that, if the unique solution exists, it is given by (5.3).

□

In the considered case it is possible to determine explicitly the expressions $[x, \chi(\overline{z})]$ for $x \in \mathcal{H}$.

Lemma 9 *Let* $x = \sum\limits_{i=1}^{N} \xi_i e_i \in \mathcal{H}$ *and* $h \in \mathcal{H}^\circ$, $h = \sum\limits_{i=1}^{N} \gamma_i e_i$. *With possible exception of finitely many values of* $z \in \mathbb{C}$ *we can decompose* x *with respect to* $\mathcal{H} = \mathcal{R}\,(S - z) \dot{+} \langle h \rangle$*:*

$$x = \sum_{i=1}^{N} (z_i - z)\eta_i e_i + \lambda h,\ \sum_{i=1}^{N} \eta_i = 0.$$

The numbers η_k *and* λ *are given as*

$$\lambda = \frac{\sum\limits_{i=1}^{N} \xi_i \prod_{\substack{j=1\\ j\neq i}}^{N} (z_j - z)}{\sum\limits_{i=1}^{N} \gamma_i \prod_{\substack{j=1\\ j\neq i}}^{N} (z_j - z)},$$

$$\eta_k = \frac{\sum\limits_{i=1}^{N} (\xi_k \gamma_i - \xi_i \gamma_k) \prod_{\substack{j=1\\ j\neq i,k}}^{N} (z_j - z)}{\sum\limits_{i=1}^{N} \gamma_i \prod_{\substack{j=1\\ j\neq i}}^{N} (z_j - z)},\ k = 1, \ldots, N.$$

Proof : Solving the system of linear equations

$$\xi_k = (z_k - z)\eta_k + \lambda \gamma_k,\ k = 1, \ldots, N,\ \sum_{i=1}^{N} \eta_i = 0,$$

yields the assertion of the lemma.

□

If we use Lemma 9 with $h = h_0$ and the fact that $[\chi(z), h_0] = 1$, we obtain

Corollary 4 *Let $x \in \mathcal{H}$, $x = \sum_{i=1}^{N} \xi_i e_i$, then*

$$[x, \chi(\overline{z})] = \frac{\sum_{i=1}^{N} \xi_i \prod_{\substack{j=1\\j\neq i}}^{N} (z_j - z)}{\sum_{i=1}^{N} \chi_i \prod_{\substack{j=1\\j\neq i}}^{N} (z_j - z)}.$$

Let the function $q(z)$ be defined by (2.9). Theorem 1 implies the following result:

Theorem 3 *Let $N \in \mathbb{N}$, $z_1, \dots, z_N \in \mathbb{C}^+$ and $w_1, \dots, w_N \in \mathbb{C}$ be given. Assume that the Pick matrix $\mathbb{P}$ is singular, $\Delta = N - \operatorname{rank} \mathbb{P} > 0$. Let $\chi_1, \dots, \chi_N$ be given by (5.2) and assume that $\chi_i \neq 0$ for $i = 1, \dots, N$, i.e. assume that there exists a unique solution of (5.1) in $\mathcal{N}_{\kappa_0}$ ($\kappa_0 = \operatorname{ind}_- \mathbb{P}$). The solutions $f \in \mathcal{N}_\kappa$ of (5.1) with $\kappa \geq \kappa_0 + \Delta$ are parametrized by*

$$f(z) = \frac{\sum_{i=1}^{N} \chi_i w_i \prod_{\substack{j=1\\j\neq i}}^{N} (z - z_j)}{\sum_{i=1}^{N} \chi_i \prod_{\substack{j=1\\j\neq i}}^{N} (z - z_j)} + \frac{\prod_{j=1}^{N} (z - z_j)(z - \overline{z_j})}{(\sum_{i=1}^{N} \chi_i \prod_{\substack{j=1\\j\neq i}}^{N} (z - z_j))(\tau(z) + q(z))(\sum_{i=1}^{N} \overline{\chi_i} \prod_{\substack{j=1\\j\neq i}}^{N} (z - \overline{z_j}))},$$

where $\tau \in \mathcal{K}^{\Delta}_{\kappa-\kappa_0}$ and satisfies

$$\tau(z_i) \neq -q(z_i),\ i = 1, \dots, N.$$

In the case $\Delta = 1$ also the function q can be determined explicitly: Consider the proof of Proposition 2. Since $\chi_N \neq 0$, the subspace

$$\mathcal{R}(S - z_N) = \langle e_1, \dots, e_{N-1} \rangle$$

of $\mathcal{H}$ is nondegenerated, so we may choose $z_0 = z_N$ in (2.8). There exists an element $a = \sum_{i=1}^{N-1} \alpha_i e_i \in \mathcal{R}(S - z_N)$ which satisfies

$$[(S - z_N)x, a] = [x, h_0'],\ x \in \mathcal{D}(S). \tag{5.4}$$

Setting $x = e_k - e_N$ in (5.4) for $k = 1, \dots, N-1$, we obtain by Corollary 4 that the numbers α_i are the (unique) solution of the system of equations

$$\sum_{i=1}^{N-1} \overline{\alpha_i} \frac{w_k - \overline{w_i}}{z_k - \overline{z_i}} = -\frac{1}{\chi_N (z_k - z_N)},\ k = 1, \dots, N-1.$$

From the defining property (5.4) of a we obtain by a short calculation that the relation

$$\mathring{A} = S\dot{+}\langle(0;h_0)\rangle\dot{+}\langle(a;h_0' + \overline{z_N}a)\rangle, \tag{5.5}$$

is selfadjoint. Hence $\mathring{A}$ in Proposition 2 may be choosen as in (5.5). Let y be the solution of $a = (S-z)y + \lambda h_0$, i.e. let $y = \sum_{i=1}^{N} \eta_i e_i$ with

$$\eta_k = \frac{\sum_{i=1}^{N} (\alpha_k\chi_i - \alpha_i\chi_k) \prod_{\substack{j=1\\ j\neq i,k}}^{N} (z_j - z)}{\sum_{i=1}^{N} \gamma_i \prod_{\substack{j=1\\ j\neq i}}^{N} (z_j - z)}, \ k = 1, \ldots, N.$$

The definition of $\mathring{A}$ shows that

$$(a - (\overline{z_N} - z)y; h_0') \in \mathring{A} - z,$$

hence

$$[(\mathring{A} - z)^{-1}h_0', h_0'] = (z - \overline{z_N})\frac{\sum_{i=1}^{N} \eta_i \prod_{\substack{j=1\\ j\neq i}}^{N} (z_j - z)}{\sum_{i=1}^{N} \chi_i \prod_{\substack{j=1\\ j\neq i}}^{N} (z_j - z)}.$$

Now consider the definition (2.9) of q. As $\chi(z_N) = h_0'$ is neutral, we find $\operatorname{Im} q(z_N) = 0$. Since we can add to q a real constant without changing its defining property, we may choose $q(z_N) = 0$. Then

$$q(z) = (z - \overline{z_N})[(\mathring{A} - z)^{-1}h_0', h_0'] = (z - \overline{z_N})(z - \overline{z_N})\frac{\sum_{i=1}^{N} \eta_i \prod_{\substack{j=1\\ j\neq i}}^{N} (z_j - z)}{\sum_{i=1}^{N} \chi_i \prod_{\substack{j=1\\ j\neq i}}^{N} (z_j - z)}.$$

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