Resolvents of Symmetric Operators and the Degenerated Nevanlinna-Pick Problem

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Let $\mathcal{H}$ be an n-dimensional space, which is equipped with a positive semidefinite inner product with a one-dimensional isotropic part. Consider a symmetric operator S in $\mathcal{H}$ with defect index $(1,1)$. We give a parametrization of the family of Štraus relations extending S and acting in a Pontryagin space $\mathcal{P}_c \supseteq \mathcal{H}$ of dimension $n+1$ and of negative index 1, and a parametrization of the corresponding set of u-resolvents of S. These results are applied to a classical Nevanlinna-Pick interpolation problem for which the Pick matrix is positive semidefinite but singular: We obtain explicit formulas for the solutions of this interpolation problem belonging to the class $\mathcal{N}_1$ (see Section 4).

1. Introduction

In various extension, interpolation or moment problems the following question arises: Given a symmetric operator S in an inner product space $\mathcal{H}$ and an element $u \in \mathcal{H}$. Describe the so-called u-resolvents of S, i.e. find a formula which parametrizes the family of expressions of the form $[(A-z)^{-1}u, u]$, where A runs through the selfadjoint extensions of S in Pontryagin spaces $\mathcal{P}$ extending $\mathcal{H}$.

If S is a densely defined operator with defect $(1,1)$ and $\mathcal{H}$ is a Pontryagin space, such a description was given e.g. in [6]. There the so-called resolvent matrix was introduced and studied. This matrix defines a fractional linear transformation which parametrizes the family of u-resolvents.

In [7] these results were applied to an extension problem for positive definite functions with an accelerant. There a description of all the extensions is given under the assumption that the endpoint of the original interval is not singular. If the endpoint is singular, the results of [6] cannot be applied, as the inner product on the original space $\mathcal{H}$ is degenerated. It is this situation which is considered here for the particular case that the space $\mathcal{H}$ is finite dimensional. That is, in this note we consider the case that $\mathcal{H}$ is a positive semidefinite, finite dimensional inner product space and we assume that $\dim \mathcal{H}^\circ = 1$, $\mathcal{H}^\circ = \langle h^\circ \rangle$, where $\mathcal{H}^\circ$ denotes the isotropic part of $\mathcal{H}$: $\mathcal{H}^\circ = \mathcal{H} \cap \mathcal{H}^\perp$. Let S be a symmetric operator defined on a hyperplane $\mathcal{D}(S)$ of $\mathcal{H}$, $\dim \mathcal{D}(S) = \dim \mathcal{H} - 1$, and assume that h° is not an eigenvector of S. Let $u \in \mathcal{H}$ and let A run through the selfadjoint extensions of S acting in a Pontryagin space $\mathcal{P}$ with negative index 1 (a π_1-space) extending $\mathcal{H}$. As in the classical case we obtain a description of the family of u-resolvents of S by means of a fractional linear transformation involving a parameter function.

However, the family of parameters occurring here is more complicated than in the classical case.

The present paper is divided into two parts. In the first part, consisting of Sections 2 and 3, a solution of the problem described above is given. The second part, Sections 4 and 5, deals with an application of the derived formulas to an interpolation problem of Nevanlinna-Pick type.

Here is a short summary of the contents of the sections. In Section 2 the Štraus extension associated with a selfadjoint extension of S is considered. In this way we reduce the problem of the description of the u-resolvents to a finite dimensional problem. In Section 3 we give a formula for the family of all u-resolvents. There it is convenient to distinguish between the cases if $\mathcal{D}(S)$ is degenerated or not. In Section 4 we formulate a Nevanlinna-Pick type interpolation problem, recall some results of [11] and discuss the unique positive definite (in some sense minimal) solution. Finally, in Section 5, explicit interpolation formulas are derived.

The notation used in this article is similar to that of [3] and [4].

2. Štraus extensions

Let $(\mathcal{H}, [\cdot\,,\cdot])$ be a finite dimensional inner product space. Assume that $[\cdot\,,\cdot]$ is positive semidefinite and that $\dim \mathcal{H}^\circ = 1$, say $\mathcal{H}^\circ = \langle h^\circ \rangle$; here $\langle\,\cdot\,\rangle$ denotes the linear span of the elements between the brackets. If $\mathcal{H}_0$ is a nondegenerated hyperplane of $\mathcal{H}$ then $\mathcal{H} = \mathcal{H}_0[\dot{+}]\langle h^\circ \rangle$; here $\dot{+}$ denotes the direct sum and $[\dot{+}]$ denotes the $[\cdot\,,\cdot]$-orthogonal and direct sum. Further, let h^1 be a formal element, $h^1 \notin \mathcal{H}$.

Definition 2.1. Let $\mathcal{H}_0$ be a nondegenerated hyperplane of $\mathcal{H}$. The vector space

$$\mathcal{P}_c = \mathcal{H}_0[\dot{+}](\langle h^\circ \rangle \dot{+} \langle h^1 \rangle) \tag{2.1}$$

endowed with the inner product

$$[x, y] = [x_0, y_0] + \xi_0 \overline{\eta_1} + \xi_1 \overline{\eta_0}, \tag{2.2}$$

where $x = x_0 + \xi_0 h^\circ + \xi_1 h^1$, $y = y_0 + \eta_0 h^\circ + \eta_1 h^1$, $x_0, y_0 \in \mathcal{H}_0$, $\xi_0, \xi_1, \eta_0, \eta_1 \in \mathbb{C}$, is called a canonical extension of $\mathcal{H}$.

Note that, as a vector space, $\mathcal{P}_c$ is isomorphic to $\mathcal{H}\dot{+}\mathbb{C}$. Any nondegenerated hyperplane $\mathcal{H}_0$ is isometrically isomorphic to $\mathcal{H}/\mathcal{H}^\circ$, thus $\mathcal{P}_c$ is unique up to isometric isomorphisms.

Proposition 2.2. *Let $\mathcal{P}$ be a π_1-space extending $\mathcal{H}$ and let $\mathcal{P}_c$ be a canonical extension of $\mathcal{H}$. Then there is an isometric embedding of $\mathcal{P}_c$ into $\mathcal{P}$.*

Proof. As $\mathcal{H}_0$ is a nondegenerated closed subspace of $\mathcal{P}$ it has an orthogonal complement in $\mathcal{P}$: $\mathcal{H}_0[\dot{+}]\mathcal{H}_0^\perp = \mathcal{P}$. The space $\mathcal{H}_0^\perp$ is itself a Pontryagin space and contains h°. Thus we can find an element $h \in \mathcal{H}_0^\perp$ which is skewly linked with h°. Due to the definition (2.2) of the inner product of $\mathcal{P}_c$, the mapping which acts as the identity on $\mathcal{H}$ and maps h^1 onto h is an isometric embedding of $\mathcal{P}_c$ into $\mathcal{P}$. □

Fixing such an embedding in the following, we can assume that $\mathcal{P}_c \subseteq \mathcal{P}$. Evidently, the orthogonal complement of $\mathcal{P}_c$ in $\mathcal{P}$ is a Hilbert space.

Let S be a symmetric operator in $\mathcal{H}$, defined on a hyperplane $\mathcal{D}(S) \subseteq \mathcal{H}$, $\dim \mathcal{D}(S) = \dim \mathcal{H} - 1$. If A is a selfadjoint relation extending S with $\rho(A) \neq \emptyset$ acting in a π_1-space $\mathcal{P} \supseteq \mathcal{H}$ and $\mathcal{P}_c$ is a canonical extension of $\mathcal{H}$, $\mathcal{P}_c \subseteq \mathcal{P}$, then A is also an extension of S considered in $\mathcal{P}_c$. Evidently, $\dim \mathcal{P}_c/\mathcal{D}(S) = 2$.

Denote the orthogonal projection of $\mathcal{P}$ onto $\mathcal{P}_c$ by P_c. The operator function

$$R(z) = P_c(A-z)^{-1}|_{\mathcal{P}_c}, \qquad z \in \rho(A),$$

is called a generalized resolvent of S, the function

$$T(z) = R(z)^{-1} + z, \qquad z \in \rho(A) \tag{2.3}$$

is called the Štraus extension of S associated with A. Observe that the values of the Štraus extension are in general linear relations in $\mathcal{P}_c$.

We have to adapt some results of [2] to the situation considered here. First let us recall some notation. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are Pontryagin spaces we denote by $\mathcal{S}_0(\mathcal{P}_1, \mathcal{P}_2)$ the set of all meromorphic functions

$$\vartheta : \mathbb{D} = \{z \in \mathbb{C} :\ |z| < 1\} \to \mathcal{L}(\mathcal{P}_1, \mathcal{P}_2),$$

such that the matrix kernel

$$S_\vartheta(z,w) = \begin{pmatrix} \frac{I-\vartheta(w)^+\vartheta(z)}{1-z\overline{w}} & \frac{\vartheta(\overline{z})^+-\vartheta(w)^+}{z-\overline{w}} \\ \frac{\vartheta(z)-\vartheta(\overline{w})}{z-\overline{w}} & \frac{I-\vartheta(\overline{w})\vartheta(\overline{z})^+}{1-z\overline{w}} \end{pmatrix}, \qquad |z|, |w| < 1,$$

is positive semidefinite. For a Pontryagin space $\mathcal{P}$ we call a linear subspace of $\mathcal{P}^2$ a linear relation. Let $\mathcal{N}_0^r(\mathcal{P})$ denote the set of all functions T in $\mathbb{C}^+$ with values in the set of linear relations in $\mathcal{P}$ and such that:

(i) for $z_i \in \mathbb{C}^+$ and $(f_i; g_i) \in T(z_i)$, $i = 1, \dots, n$, the matrix

$$\left(\frac{[g_i, f_j] - [f_i, g_j]}{z_i - \overline{z_j}} \right)_{i=1}^n$$

is positive semidefinite,

(ii) there exists a $z_0 \in \mathbb{C}^+$ and a neighbourhood $U(z_0) \subseteq \mathbb{C}^+$ of z_0, such that for $z \in U(z_0)$ the Cayley transform

$$C_{z_0}(T(z)) = \{(g - z_0 f; g - \overline{z_0} f) :\ (f; g) \in T(z)\}$$

is a bounded operator in $\mathcal{P}$ and depends holomorphically on z.

If, in particular, $T(z)$ is a matrix function $T(z) : \mathbb{C}^+ \to \mathbb{C}^{n\times n}$ such that the Nevanlinna kernel

$$N_T(z,w) = \frac{T(z) - T(w)^*}{z - \overline{w}}$$

is positive semidefinite, we write $T \in \mathcal{N}_0(\mathbb{C}^n)$.

If $\mathcal{P}_1$ and $\mathcal{P}_2$ have the same negative index, which is the case in the following considerations, it is proved in [3] (compare also [2]) that a function $\vartheta \in \mathcal{S}_0(\mathcal{P}_1, \mathcal{P}_2)$ is uniquely determined by its values on any nonempty open subset of $\mathbb{D}$, and if it is extended to $\mathbb{C} \setminus \partial\mathbb{D}$ by

$$\vartheta(\frac{1}{\overline{z}}) = \vartheta(z)^+$$

the kernel S_ϑ remains positive semidefinite. Furthermore, ϑ is in fact holomorphic. Similarly, a function $T \in \mathcal{N}_0^r(\mathcal{P})$ is uniquely determined by its values on any open subset of $\mathbb{C}^+$, and if it is extended to $\mathbb{C} \setminus \mathbb{R}$ by

$$T(\overline{z}) = T(z)^+$$

the kernel N_T remains positive semidefinite.

Let $\mathcal{P}_1 \subseteq \mathcal{P}_2$ be Pontryagin spaces and assume that the extending space $\mathcal{P}_1^\perp$ is positive. Denote by P_1 the orthogonal projection of $\mathcal{P}_2$ onto $\mathcal{P}_1$. If A is a selfadjoint relation in $\mathcal{P}_2$ with $\rho(A) \neq \emptyset$ we denote by $R(z)$ the compressed resolvent of A:

$$R(z) = P_1(A - z)^{-1}|_{P_1}, \qquad z \in \rho(A),$$

and by $T(z)$, defined as in (2.3), the Štraus extension associated with A. Recall that A is called $\mathcal{P}_1$-minimal if

$$\mathcal{P}_2 = \overline{\langle x, (A - z)^{-1}x : \ x \in \mathcal{P}_1, z \in \rho(A) \rangle}.$$

Lemma 2.3. *Let $\mathcal{P}_1$, $\mathcal{P}_2$, A and $T(z)$ be as above. Then*

$$T(z)(0) := \{x \in \mathcal{P}_1 : \ (0; x) \in T(z)\} = A(0) \cap \mathcal{P}_1.$$

Furthermore,

$$\overline{\mathcal{D}(T(z))} = (A(0) \cap \mathcal{P}_1)^\perp,$$

where the orthogonal companion on the right hand side is considered in $\mathcal{P}_1$.

If $\mathcal{P}_1$ is a Hilbert space the proof of this lemma can be found in [3]. It can be generalized immediately to the situation considered here.

Lemma 2.3 applies to the situation described in the beginning of this section with $\mathcal{P}_1 = \mathcal{P}_c$, the π_1-space $\mathcal{P}_2 = \mathcal{P}$ and a selfadjoint extension A of S.

For the convenience of the reader we formulate a result which is a particular case of [2], Theorem 2.1.

Proposition 2.4. *Let $\mathcal{P}_c$, $\mathcal{P}$, S and A be as before, and let $z_0 \in \rho(A) \setminus \mathbb{R}$ be such that $\mathcal{R}(S - z_0)$ and $\mathcal{R}(S - \overline{z_0})$ are nondegenerated. Then there exists a* *neighbourhood $U(z_0)$ of z_0 with $\Im(z)\Im(z)_0 > 0$ for $z \in U(z_0)$, such that the Štraus extension $T(z)$ associated with A is of the form*

$$T(z) = S\dot{+}\{((I - \vartheta(\zeta(z)))f, (\overline{z_0} - z_0\vartheta(\zeta(z)))f) : \ f \in \mathcal{R}\,(S - z_0)^{\perp}\}, \tag{2.4}$$

where $\zeta(z) = \frac{z-z_0}{z-\overline{z_0}}$ and $\vartheta \in \mathcal{S}_0(\mathcal{R}\,(S - z_0)^{\perp}, \mathcal{R}\,(S - \overline{z_0})^{\perp})$.

Conversely, if $\vartheta \in \mathcal{S}_0(\mathcal{R}\,(S - z_0)^{\perp}, \mathcal{R}\,(S - \overline{z_0})^{\perp})$, then there exists a selfadjoint extension A of S in a π_1-space $\mathcal{P}$ with $z_0 \in \rho(A)$ such that its corresponding Štraus extension is given by (2.4).

In the situation of Proposition 2.4 the Cayley transform

$$F(\zeta) := C_{z_0}(T(z(\zeta))) = (T(z(\zeta)) - z_0)(T(z(\zeta)) - \overline{z_0})^{-1},$$

where $z(\zeta) = \frac{\overline{z_0}\zeta - z_0}{\zeta - 1}$, is given by

$$F(\zeta) = \begin{pmatrix} C_{z_0}(S) & 0 \\ 0 & \vartheta(\zeta) \end{pmatrix} : \begin{pmatrix} \mathcal{R}\,(S - z_0) \\ [\dot{+}] \\ \mathcal{R}\,(S - z_0)^{\perp} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{R}\,(S - \overline{z_0}) \\ [\dot{+}] \\ \mathcal{R}\,(S - \overline{z_0})^{\perp} \end{pmatrix}. \tag{2.5}$$

As $\mathcal{P}_c$ is finite dimensional, $\mathcal{D}(T(z)) = \overline{\mathcal{D}(T(z))}$ and therefore, according to Lemma 2.3, $T(z)(0)$ as well as $\mathcal{D}(T(z))$ are independent of z. We denote $\mathcal{D} := \mathcal{D}(T(z))$ and $T_\infty := T(z)(0) = \ker(F(\zeta(z)) - I)$. The main result of this section is the following representation of the Štraus extension.

Theorem 2.5. *Let $\mathcal{H}$, S be as above, let $\mathcal{P}$ be a π_1-space , $\mathcal{P}_c \subseteq \mathcal{P}$ a canonical extension of $\mathcal{H}$ and let A be a selfadjoint extension of S with $\rho(A) \neq \emptyset$. Denote by $T(z)$, $z \in \rho(A)$, the Štraus extension given by* (2.3). *Then there exists an operator valued meromorphic function*

$$T_0 : \mathbb{C} \setminus \mathbb{R} \to \mathcal{L}(\mathcal{D}, \mathcal{P}_c),$$

such that

(i) *there exists a $z_0 \in \mathbb{C}^+$ with $\mathcal{R}\,(T(z_0) - z_0) + \mathcal{D}^{\perp} = \mathcal{P}_c$,*

(ii) *$-T_0 \in \mathcal{N}_0^r(\mathcal{P}_c)$,*

and for $z \in \rho(A)$,

(iii) *$S \subseteq T_0(z)$,*

(iv) *$T_0(z) \subseteq T_0(\overline{z})^+$,*

(v) *$T(z) = T_0(z)\dot{+}(0 \times T_\infty)$.*

Conversely, if $\mathcal{D} \supseteq \mathcal{D}(S)$ is a subspace of $\mathcal{P}_c$ and $T_0(z) : \mathbb{C} \setminus \mathbb{R} \to \mathcal{L}(\mathcal{D}, \mathcal{P}_c)$ is meromorphic and satisfies (i)–(v)*, then there exists a selfadjoint extension A of S in a π_1-space $\mathcal{P}$ with $\rho(A) \neq \emptyset$ such that its corresponding Štraus extension is given by*

$$T(z) = T_0(z) \dot{+} (0 \times \mathcal{D}^{\perp}).$$

Proof. Let $z_0 \in \rho(A) \cap \mathbb{C}^+$ be such that $\mathcal{R}(S - z_0)$ and $\mathcal{R}(S - \overline{z_0})$ are nondegenerated. Then Proposition 2.4 shows that there exists a neighbourhood $U(z_0)$ of z_0 such that $T(z)$ and its Cayley transform $F(\zeta)$ are given by (2.4) and (2.5), respectively. We have

$$\mathcal{D}(C_{z_0}(S)) \cap \ker(F(\zeta) - I) = \{0\}.$$

Indeed, if $x \in \mathcal{D}(C_{z_0}(S))$ we have $F(\zeta)x = C_{z_0}(S)x$ and thus $x \in \ker(F(\zeta) - I)$ shows that $x \in \ker(C_{z_0}(S) - I)$. As S is an operator ($S(0) = \{0\}$) we conclude that $x = 0$.
Recall that $\ker(F(\zeta) - I) = T_\infty$. We decompose $\mathcal{P}_c$ as

$$\mathcal{P}_c = \mathcal{D}_1 \dot{+} T_\infty$$

where $\mathcal{D}_1$ is chosen such that $\mathcal{D}(C_{z_0}(S)) \subseteq \mathcal{D}_1$. Observing that $\mathcal{D} = \mathcal{D}(T(z)) = \mathcal{R}(F(\zeta) - I)$, the operator $F(\zeta) - I$ induces a bijection

$$G(\zeta) = (F(\zeta) - I)|_{\mathcal{D}_1}$$

from $\mathcal{D}_1$ onto $\mathcal{D}$. As $F(\zeta)$ depends holomorphically on ζ we find that $G(\zeta)^{-1}$ and thus also

$$T_0(z) = (z_0 F(\zeta) - \overline{z_0}) G(\zeta)^{-1}$$

depend holomorphically on ζ and on $z = \frac{\overline{z_0}\zeta - z_0}{\zeta - 1}$, respectively.
The inclusions $\mathcal{D}(C_{z_0}(S)) \subseteq \mathcal{D}_1$ and $G(\zeta) \subseteq F(\zeta) - I$ imply $S \subseteq T_0(z) \subseteq T(z)$. Furthermore, $T(z) = T(\overline{z})^+$, which shows that $T_0(z) \subseteq T_0(\overline{z})^+$. From the relation

$$F(\zeta) - I = G(\zeta) \dot{+} (T_\infty \times \{0\}),$$

where $\dot{+}$ has to be understood as the sum of subspaces of $\mathcal{P}_c^2$, we find

$$T(z) = (z_0 F(\zeta) - \overline{z_0})(F(\zeta) - I)^{-1} = T_0(z) \dot{+} (\{0\} \times T_\infty).$$

It is shown in [3] that the kernels $S_{F(\zeta)}$ and N_{-T} have the same number of negative squares. As $\vartheta(\zeta) \in \mathcal{S}_0(\mathcal{R}(S - z_0)^{\perp}, \mathcal{R}(S - \overline{z_0})^{\perp})$, obviously $C_{z_0}(T_0(z))$ depends holomorphically on z and thus $T_0(z) \in \mathcal{N}_0^r(\mathcal{P}_c)$. Clearly,

$$\mathcal{R}(T_0(z) - z) + \mathcal{D}^{\perp} = \mathcal{D}_1 + T_\infty = \mathcal{P}_c.$$

Conversely, suppose that $T_0(z)$ is given. Choose z_0 as in (i). Then the relation

$$\dim \mathcal{P}_c = \dim \mathcal{R}(T_0(z_0) - z_0) + \dim(\mathcal{D}^{\perp} / \mathcal{R}(T_0(z_0) - z_0) \cap \mathcal{D}^{\perp})$$

$$\le \dim \mathcal{P}_c - \dim \mathcal{D}^{\perp} + \dim (\mathcal{D}^{\perp}/\mathcal{R}\,(T_0(z_0) - z_0) \cap \mathcal{D}^{\perp})$$

implies that $\mathcal{R}\,(T_0(z_0) - z_0) \cap \mathcal{D}^{\perp} = \{0\}$. Thus, in a sufficiently small neighbourhood $U(z_0)$ of z_0, $C_{z_0}(T_0(z))$ can be extended to $\mathcal{P}_c$ by

$$F(\zeta)x = \begin{cases} C_{z_0}(T_0(z))x & \text{if } x \in \mathcal{R}\,(T_0(z) - z)\,, \\ x & \text{if } x \in \mathcal{D}^{\perp}. \end{cases}$$

Clearly, $F(\zeta) = C_{z_0}(T_0(z) \dot{+} (\{0\} \times \mathcal{D}^{\perp}))$ and $F(\zeta) \in \mathcal{S}_0(\mathcal{P}_c, \mathcal{P}_c)$. Thus (as in [3] for the Hilbert space case) $T(z) = T_0(z) \dot{+} (\{0\} \times \mathcal{D}^{\perp})$ is the Štraus extension of some selfadjoint extension A of S with $z_0 \in \rho(A)$. □

Remark 2.6. If T_∞ is nondegenerated, Theorem 2.5 can be deduced immediately from the fact that $\mathcal{D}[\dot{+}]T_\infty = \mathcal{P}_c$. In the situation considered here it may actually occur that T_∞ is degenerated.

3. The u-resolvents of S

Let $\mathcal{H}$ and S be as in Section 2 and fix an element $u \in \mathcal{H}$. Let A be any selfadjoint extension of S in a π_1-space $\mathcal{P}$, containing $\mathcal{H}$, and thus containing a canonical extension $\mathcal{P}_c$. In this section we consider the corresponding u-resolvent of S:

$$[(A - z)^{-1}u, u], \qquad z \in \rho(A), \tag{3.1}$$

and find a description of all such u-resolvents. Observe that with the Štraus extension $T(z)$ corresponding to A the u-resolvent (3.1) can be written as

$$[(A - z)^{-1}u, u] = [(T(z) - z)^{-1}u, u].$$

As $T(z) = T_0(z) \dot{+} (0 \times T_\infty)$, in order to find $(T(z) - z)^{-1}u$, it suffices to solve the equation

$$(T_0(z) - z)x = u + b \tag{3.2}$$

for $x \in \mathcal{D}$ and $b \in T_\infty$.

We consider the cases where $\mathcal{D}(S)$ is degenerated or nondegenerated separately.

I. If $\mathcal{D}(S)$ is nondegenerated, we use the decomposition

$$\mathcal{P}_c = \mathcal{D}(S)\,[\dot{+}](\langle h^\circ \rangle \dot{+} \langle h^1 \rangle) \tag{3.3}$$

with skewly linked neutral elements h°, h^1 as in (2.1), $[h^\circ, h^1] = 1$. For $x \in \mathcal{D}(S)$ we write

$$Sx = S_d x + [x, a]h^\circ, \tag{3.4}$$

where $a \in \mathcal{D}(S)$, and decompose u as

$$u = u_d + \eta_0 h^\circ. \tag{3.5}$$

Now let A be a selfadjoint extension of S as above and let $T(z)$ be the corresponding Štraus extension:

$$T(z) = T_0(z)\dot{+}(\{0\} \times T_\infty).$$

Note that $T_\infty \subseteq \langle h^\circ, h^1 \rangle$. We shall treat the cases

(i) $T_\infty = \{0\}$,

(ii) $\dim T_\infty = 1, \quad h^\circ \notin T_\infty$,

(iii) $\dim T_\infty = 1, \quad h^\circ \in T_\infty$,

(iv) $\dim T_\infty = 2$

separately. As the proofs of the respective results in the cases (i)–(iv) are quite similar we will go into detail only in case (i).

Case (i): Write $T_0(z)$ as a block matrix with respect to the decomposition (3.3):

$$(3.6) \quad T_0(z) = \left(\begin{array}{c|cc} S_d & 0 & a \\ \hline [\,\cdot\,, a] & t_{11}(z) & t_{12}(z) \\ 0 & t_{21}(z) & t_{22}(z) \end{array} \right) : \begin{pmatrix} \mathcal{D}(S) \\ + \\ \langle h^\circ \rangle \\ + \\ \langle h^1 \rangle \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{D}(S) \\ + \\ \langle h^\circ \rangle \\ + \\ \langle h^1 \rangle \end{pmatrix}.$$

Here the form of the first row and column of the matrix (3.6) follows from the properties (iii) and (iv) of Theorem 2.5. Furthermore, (ii) implies that

$$\Phi(z) = -\begin{pmatrix} t_{12}(z) & t_{11}(z) \\ t_{22}(z) & t_{21}(z) \end{pmatrix} \in \mathcal{N}_0(\mathbb{C}^2).$$

Lemma 3.1. *If* $T_\infty = \{0\}$ *and* $T_0(z)$ *and* u *are given by* (3.6) *and* (3.5), *respectively, we have*

$$(3.7) \quad [(A-z)^{-1}u, u] = [(S_d - z)^{-1}u_d, u_d] - \frac{(\eta_0 - [(S_d - z)^{-1}u_d, a])(\overline{\eta_0} - [(S_d - z)^{-1}a, u_d])}{\tau(z) + [(S_d - z)^{-1}a, a]},$$

where

$$(3.8) \quad \tau(z) = \frac{(t_{11}(z) - z)(t_{22}(z) - z)}{t_{21}(z)} - t_{12}(z).$$

If $t_{21} = 0$ *then* $\tau(z) = \infty$, *i.e.*

$$[(A-z)^{-1}u, u] = [(S_d - z)^{-1}u_d, u_d].$$

Conversely, given a function $\Phi(z) \in \mathcal{N}_0(\mathbb{C}^2)$, *the right hand side of* (3.7) *defines a* u*-resolvent of* S *if* $\tau \neq -[(S_d - z)^{-1}a, a]$.

Proof. In order to compute $[(A-z)^{-1}u, u]$ we will explicitly solve the equation $(A-z)x = u$, which, for $x = x_d + \xi_0 h^\circ + \xi_1 h^1$, in vector form can be written as

$$\begin{pmatrix} S_d - z & 0 & a \\ [\,\cdot\,, a] & t_{11}(z) - z & t_{12}(z) \\ 0 & t_{21}(z) & t_{22}(z) - z \end{pmatrix} \begin{pmatrix} x_d \\ \xi_0 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} u_d \\ \eta_0 \\ 0 \end{pmatrix}. \tag{3.9}$$

Consider first the case that $t_{21} \neq 0$. The third row of (3.9) implies

$$t_{21}(z)\xi_0 + (t_{22}(z) - z)\xi_1 = 0,$$

i.e.

$$\xi_0 = -\frac{t_{22}(z) - z}{t_{21}(z)}\xi_1. \tag{3.10}$$

The first row gives $(S_d - z)x_d + \xi_1 a = u_d$, or

$$x_d = (S_d - z)^{-1}u_d - \xi_1 (S_d - z)^{-1} a.$$

This implies

$$[x_d, a] = [(S_d - z)^{-1}u_d, a] - \xi_1 [(S_d - z)^{-1}a, a]. \tag{3.11}$$

From the second row of (3.9) we obtain $[x_d, a] + \xi_0(t_{21}(z) - z) + \xi_1 t_{12}(z) = \eta_0$, and by substituting the expressions (3.10) and (3.11) into this relation we find

$$\xi_1 = \frac{[(S_d - z)^{-1}u_d, a] - \eta_0}{\tau(z) + [(S_d - z)^{-1}a, a]}$$

and thus

$$x_d = (S_d - z)^{-1}u_d + \frac{\eta_0 - [(S_d - z)^{-1}u_d, a]}{\tau(z) + [(S_d - z)^{-1}a, a]}(S_d - z)^{-1}a.$$

Using these expressions we compute

$$\begin{aligned} [(A-z)^{-1}u, u] &= \left[\begin{pmatrix} x_d \\ \xi_0 \\ \xi_1 \end{pmatrix}, \begin{pmatrix} u_d \\ \eta_0 \\ 0 \end{pmatrix} \right] = [x_d, u_d] + \xi_1 \overline{\eta_0} \\ &= [(S_d - z)^{-1}u_d, u_d] \\ &\quad - \frac{(\eta_0 - [(S_d - z)^{-1}u_d, a])(\overline{\eta_0} - [(S_d - z)^{-1}a, u_d])}{\tau(z) + [(S_d - z)^{-1}a, a]}. \end{aligned}$$

Suppose now that $t_{21}(z) \equiv 0$. Then, as $\Phi(z) \in \mathcal{N}_0(\mathbb{C}^2)$, $t_{11}(z) - z \not\equiv 0$ and $t_{22}(z) - z \not\equiv 0$. Thus it follows from (3.9) that $\xi_1 = 0$, $x_d = (S_d - z)^{-1}u_d$ and ζ_0 can be computed from the second row of (3.9). We obtain

$$[(A-z)^{-1}u, u] = [(S_d - z)^{-1}u_d, u_d].$$

Consider the converse part of the lemma. It is obvious from the definition (3.6) of $T_0(z)$, that conditions (ii)–(iv) of Theorem 2.5 are satisfied. Note that condition (i) holds for some z_0 if and only if $T_0(z_0)-z_0$ is injective. Assume that for each z we have $\ker(T_0(z)-z)\neq\{0\}$, say

$$0\neq\begin{pmatrix} x_d\\ \xi_0\\ \xi_1\end{pmatrix}\in\ker(T_0(z)-z).$$

If $t_{21}\neq 0$ we do the same calculations as above (with $u_d=0$, $\eta_0=0$) up to (3.11), then the second row of (3.9) yields

$$\xi_1(\tau(z)+[(S_d-z)^{-1}a,a])=0.$$

In case $\xi_1=0$ we obtain $x_d=0$ and $\xi_0=0$, a contradiction. Consequently, $\tau+[(S_d-z)^{-1}a,a]=0$. If $t_{21}(z)\equiv 0$, again $t_{11}(z)-z\not\equiv 0$ and $t_{22}(z)-z\not\equiv 0$, and therefore (3.9) ($u_d=0$, $\eta_0=0$) shows that $\xi_1=0$, $x_d=0$ and $\xi_0=0$. □

Remark 3.2. Note that as a parameter in (3.7) the entries of the matrix $\Phi(z)$ do not appear separately, but only in form of the scalar function $\tau(z)$. The same is true for the representation formulas below.

Case (ii): Let $T_\infty=\langle b\rangle$ with $b=b_0h^\circ+h^1\in\mathcal{D}(S)^\perp$. Then

$$T_0(z)=\left(\begin{array}{c|c} S_d & a\\ \hline [\,\cdot\,,a] & t_1(z)\\ 0 & t_2(z)\end{array}\right):\begin{pmatrix}\mathcal{D}(S)\\ +\\ \langle d\rangle\end{pmatrix}\longrightarrow\begin{pmatrix}\mathcal{D}(S)\\ +\\ \langle h^\circ\rangle\\ +\\ \langle h^1\rangle\end{pmatrix},\tag{3.12}$$

where $d=-\overline{b_0}h^\circ+h^1$. If $f=f_d+\lambda_f d, h=h_d+\lambda_h d\in\mathcal{D}(T_0(z))$, we find

$$\begin{aligned}\left[\begin{pmatrix}0 & 0\\ 0 & N_{t_1}(z,w)\\ 0 & N_{t_2}(z,w)\end{pmatrix}f,h\right] &= \left[\begin{pmatrix}0\\ N_{t_1}(z,w)\lambda_f\\ N_{t_2}(z,w)\lambda_f\end{pmatrix},\lambda_h\begin{pmatrix}0\\ -\overline{b_0}\\ 1\end{pmatrix}\right]\\ &= N_{t_1-b_0t_2}(z,w)\lambda_f\overline{\lambda_h},\end{aligned}$$

which shows with (ii) of Theorem 2.5 that

$$\varphi(z)=-(t_1(z)-b_0t_2(z))\in\mathcal{N}_0(\mathbb{C}).$$

Lemma 3.3. *If* $\dim T_\infty=1$ *and* $h^\circ\notin T_\infty$*, we have*

$$\begin{aligned}[(A-z)^{-1}u,u] &= [(S_d-z)^{-1}u_d,u_d]\\ &\quad-\frac{(\eta_0-[(S_d-z)^{-1}u_d,a])(\overline{\eta_0}-[(S_d-z)^{-1}a,u_d])}{\tau(z)+[(S_d-z)^{-1}a,a]},\end{aligned}\tag{3.13}$$

where

$$\tau(z) = -2\Re(b_0)z + \varphi(z), \tag{3.14}$$

with $\varphi(z) = -(t_1(z) - b_0 t_2(z)) \in \mathcal{N}_0(\mathbb{C})$. *Conversely, given a function* $\varphi(z) \in \mathcal{N}_0(\mathbb{C})$ *and* $b_0 \in \mathbb{C}$, *the right hand side of* (3.13) *defines a u-resolvent of* S *if* $\tau(z) \not\equiv -[(S_d - z)^{-1}a, a]$.

Proof. We have to solve the equation

$$\begin{pmatrix} S_d - z & a \\ [\,\cdot\,, a] & t_1 + \overline{b_0} z \\ 0 & t_2 - z \end{pmatrix} \begin{pmatrix} x_d \\ \xi_2 \end{pmatrix} = \begin{pmatrix} u_d \\ \eta_0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ b_0 \\ 1 \end{pmatrix}$$

for $x = x_d + \xi_2 d$ and λ. This is done by similar computations as in the proof of Lemma 3.1 and yields (3.13). The converse statement is proved analogously as well. □

Case (iii): Let $T_\infty = \langle h^\circ \rangle$. Then

$$T_0(z) = \begin{pmatrix} S_d & 0 \\ [\,\cdot\,, a] & t_1(z) \\ 0 & t_2(z) \end{pmatrix} : \begin{pmatrix} \mathcal{D}(S) \\ + \\ \langle h^\circ \rangle \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{D}(S) \\ + \\ \langle h^\circ \rangle \\ + \\ \langle h^1 \rangle \end{pmatrix}.$$

Lemma 3.4. *If* $\dim T_\infty = 1$ *and* $h^\circ \in T_\infty$, *we have*

$$[(A - z)^{-1}u, u] = [(S_d - z)^{-1}u_d, u_d]. \tag{3.15}$$

Conversely, the right hand side of (3.15) *defines a u-resolvent of* S.

In order to prove the Lemma 3.4, we have to consider the equation

$$\begin{pmatrix} S_d - z & 0 \\ [\,\cdot\,, a] & t_1 - z \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} x_d \\ \xi_2 \end{pmatrix} = \begin{pmatrix} u_d \\ \eta_0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Case (iv): Let $\dim T_\infty = 2$. Then $T_0(z) = S$.

Lemma 3.5. *If* $\dim T_\infty = 2$, *the formula* (3.15) *holds.*

In order to prove Lemma 3.5, consider the equation

$$\begin{pmatrix} S_d - z \\ [\,\cdot\,, a] \\ 0 \end{pmatrix} x_d = \begin{pmatrix} u_d \\ \eta_0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Definition 3.6. Let $\mathcal{T}_n$ denote the set of all functions of the form

$$\tau(z) = \frac{(t_{11}(z) - z)(t_{22}(z) - z)}{t_{21}(z)} - t_{12}(z),$$

where

$$\Phi(z) = -\begin{pmatrix} t_{12}(z) & t_{11}(z) \\ t_{22}(z) & t_{21}(z) \end{pmatrix} \in \mathcal{N}_0(\mathbb{C}^2),$$

or $\tau(z) = -b_1 z + \varphi(z)$ where $\varphi(z) \in \mathcal{N}_0(\mathbb{C})$ and $b_1 \in \mathbb{R}$, or $\tau(z) \equiv \infty$.

Note that all functions of $\mathcal{T}_n$ can be obtained as limits of functions of the first kind, if certain entries of the matrix $\Phi(z)$ tend to infinity.

¿From the Lemmas 3.1 to 3.5 we obtain the following result.

Theorem 3.7. *Let $\mathcal{D}(S)$ be nondegenerated. Then the set of u-resolvents of S is parametrized by the formula*

$$[(A-z)^{-1}u, u] = [(S_d - z)^{-1}u_d, u_d] - \frac{(\eta_0 - [(S_d - z)^{-1}u_d, a])(\overline{\eta_0} - [(S_d - z)^{-1}a, u_d])}{\tau(z) + [(S_d - z)^{-1}a, a]},$$

where $\tau \in \mathcal{T}_n$.

Remark 3.8. The correspondence between u-resolvents and parameters is bijective, whereas different extensions may yield the same u-resolvent. Consider for example Lemma 3.1 with $t_{21} = 0$.

II. If $\mathcal{D}(S)$ is degenerated we have $h^\circ \in \mathcal{D}(S)$. Assume for the moment that $\dim \mathcal{H} \geq 3$. We choose a decomposition $\mathcal{D}(S) = \mathcal{D}_1[\dot{+}]\langle h^\circ \rangle$ of $\mathcal{D}(S)$ where $\mathcal{D}_1$ is nondegenerated. Let $g \in \mathcal{H}$ be such that $g \perp \mathcal{D}(S)$ and $[g, g] = 1$. Then we can choose $\mathcal{H}_0 = \langle g \rangle [\dot{+}] \mathcal{D}_1$ for the construction of a canonical extension $\mathcal{P}_c$:

$$\mathcal{P}_c = \langle g \rangle [\dot{+}] \underbrace{\mathcal{D}_1 [\dot{+}] (\langle h^\circ \rangle}_{\mathcal{D}(S)} \dot{+} \langle h^1 \rangle) \tag{3.16}$$

with skewly linked elements h°, h^1. For $x = x_d + \xi_0 h^\circ \in \mathcal{D}(S)$ let, according to the decomposition (3.16),

$$Sx = ([x_d, a] + c_1 \xi_0) g + S_d x_d + ([x_d, b] + c_2) h^\circ, \tag{3.17}$$

and

$$u = \eta_2 g + u_d + \eta_0 h^\circ. \tag{3.18}$$

Recall our assumption that h° is not an eigenvector of S, thus $c_1 \neq 0$. Similar as in the preceeding discussion we distinguish the cases

(i) $T_\infty = \{0\}$,

(ii) $\dim T_\infty = 1, \quad h^\circ \notin T_\infty$,

(iii) $\dim T_\infty = 1, \quad h^\circ \in T_\infty$,

(iv) $\dim T_\infty = 2$.

Again the details will be carried out only in the first case.

Case (i): $T_0(z)$ can be written as a block matrix with respect to the decomposition (3.16) of the space $\mathcal{P}_c$:

$$T_0(z) = \left(\begin{array}{c|cc|c} t_{11}(z) & [\,\cdot\,,a] & c_1 & t_{12}(z) \\ \hline a & S_d & 0 & b \\ t_{21}(z) & [\,\cdot\,,b] & c_2 & t_{22}(z) \\ \hline \overline{c_1} & 0 & 0 & \overline{c_2} \end{array}\right) : \begin{pmatrix} \langle g\rangle \\ + \\ \mathcal{D}_1 \\ + \\ \langle h^\circ\rangle \\ + \\ \langle h^1\rangle \end{pmatrix} \longrightarrow \begin{pmatrix} \langle g\rangle \\ + \\ \mathcal{D}_1 \\ + \\ \langle h^\circ\rangle \\ + \\ \langle h^1\rangle \end{pmatrix}. \tag{3.19}$$

The form of the second and third column and the second and fourth row of the block matrix (3.19) follows from (iii) and (iv) of Theorem 2.5, (ii) implies that

$$\Phi(z) = -\begin{pmatrix} t_{11}(z) & t_{12}(z) \\ t_{21}(z) & t_{22}(z) \end{pmatrix} \in \mathcal{N}_0(\mathbb{C}^2).$$

Remark 3.9. Note that in this case the operator S_d acts in a space of dimension $\dim\mathcal{H} - 2$, whereas in the case $h^\circ \notin \mathcal{D}(S)$ the operator S_d acts in a space of dimension $\dim\mathcal{H} - 1$.

Lemma 3.10. *If* $\dim T_\infty = 0$ *and* $T_0(z)$ *and* u *are given by* (3.19) *and* (3.18), *respectively, we have*

$$[(A-z)^{-1}u,u] = [(S_d-z)^{-1}u_d,u_d] \tag{3.20}$$
$$-\frac{((c_2-z)\eta_2 - c_1\eta_0 + [(S_d-z)^{-1}u_d, c_{\overline{z}}])((\overline{c_2}-z)\overline{\eta_2} - \overline{c_1\eta_0} - [(S_d-z)^{-1}c_z,u_d])}{\tau(z) + z(c_2-z)(\overline{c_2}-z) + [(S_d-z)^{-1}c_z, c_{\overline{z}}]}$$

where $c_z = \overline{c_1}b - (\overline{c_2}-z)a$ *and*

$$\begin{aligned} \tau(z) \;=\; & -t_{11}(z)z^2 - 2(c_1t_{21}(z) + \overline{c_1}\overline{t_{12}(\overline{z})} - (c_2+\overline{c_2})t_{11}(z))z \\ & -(|c_2|^2 t_{11}(z) + |c_1|^2 t_{22}(z) - 2(c_1\overline{c_2}t_{21}(z) + \overline{c_1}c_2\overline{t_{12}(\overline{z})})). \end{aligned} \tag{3.21}$$

Conversely, given a function $\Phi(z) \in \mathcal{N}_0(\mathbb{C}^2)$, *the right hand side of* (3.20) *defines a* u*-resolvent of* S *if*

$$\tau(z) \not\equiv -z(c_2-z)(\overline{c_2}-z) + [(S_d-z)^{-1}c_z, c_{\overline{z}}]. \tag{3.22}$$

Proof. We have to solve the equation

$$\left(\begin{array}{c|cc|c} t_{11}(z)-z & [\,\cdot\,,a] & c_1 & t_{12}(z) \\ \hline a & S_d-z & 0 & b \\ t_{21}(z) & [\,\cdot\,,b] & c_2-z & t_{22}(z) \\ \hline \overline{c_1} & 0 & 0 & \overline{c_2}-z \end{array}\right)\begin{pmatrix}\xi_2\\ x_d\\ \xi_0\\ \xi_1\end{pmatrix}=\begin{pmatrix}\eta_2\\ u_d\\ \eta_0\\ 0\end{pmatrix}. \tag{3.23}$$

The last row of (3.23) implies that

$$\overline{c_1}\xi_2+(\overline{c_2}-z)\xi_1=0. \tag{3.24}$$

The second row gives $\xi_2 a+(S_d-z)x_d+\xi_1 b=u_d$, i.e.

$$x_d=(S_d-z)^{-1}u_d-\xi_2(S_d-z)^{-1}c_z' \tag{3.25}$$

where $c_z'=\frac{\overline{c_1}}{\overline{c_2}-z}b-a$. We find

$$[x_d,a]=[(S_d-z)^{-1}u_d,a]+\xi_2[(S_d-z)^{-1}c_z',a],$$
$$[x_d,b]=[(S_d-z)^{-1}u_d,b]+\xi_2[(S_d-z)^{-1}c_z',b].$$

Substituting these expressions into the relation obtained from the first row of (3.23) we get

$$\xi_2\left(t_{11}(z)-z-\frac{\overline{c_1}}{\overline{c_2}-z}t_{12}(z)+[(S_d-z)^{-1}c_z',a]\right)+\xi_0c_1 = \eta_2-[(S_d-z)^{-1}u_d,a], \tag{3.26}$$

and, substituted into the relation obtained from the third row,

$$\xi_2\left(t_{21}(z)-\frac{\overline{c_1}}{\overline{c_2}-z}t_{22}(z)+[(S_d-z)^{-1}c_z',b]\right)+\xi_0(c_2-z) = \eta_0-[(S_d-z)^{-1}u_d,b]. \tag{3.27}$$

From the linear system of equations (3.26) and (3.27) we obtain, using Cramer's rule and (3.24),

$$\xi_0=\frac{\Delta_0}{\Delta},\qquad \xi_2=z\frac{\Delta_2}{\Delta},$$

where Δ, Δ_0 and Δ_2 are given by

$$\Delta = \begin{vmatrix} t_{11}(z)-z-\frac{\overline{c_1}}{\overline{c_2}-z}t_{12}(z)+[(S_d-z)^{-1}c_z,a] & c_1 \\ t_{21}(z)-\frac{\overline{c_1}}{\overline{c_2}-z}t_{22}(z)+[(S_d-z)^{-1}c_z,b] & c_2-z \end{vmatrix},$$

$$\Delta_0 = \begin{vmatrix} t_{11}(z)-z-\frac{\overline{c_1}}{\overline{c_2}-z}t_{12}(z)+[(S_d-z)^{-1}c_z,a] & \eta_2-[(S_d-z)^{-1}u_d,a] \\ t_{21}(z)-\frac{\overline{c_1}}{\overline{c_2}-z}t_{22}(z)+[(S_d-z)^{-1}c_z,b] & \eta_0-[(S_d-z)^{-1}u_d,b] \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} \eta_2-[(S_d-z)^{-1}u_d,a] & c_1 \\ \eta_0-[(S_d-z)^{-1}u_d,b] & c_2-z \end{vmatrix}.$$

Finally, we find

$$\xi_1 = -\frac{\overline{c_1}}{\overline{c_2} - z}\frac{\Delta_2}{\Delta},$$

and by (3.25)

$$x_d = (S_d - z)^{-1}u_d + \frac{\Delta_2}{\Delta}(S_d - z)^{-1}c_z'.$$

Using these expressions we compute

$$\begin{aligned}
&[(A - z)^{-1}u, u] \\
&= \left[\begin{pmatrix} \xi_2 \\ x_d \\ \xi_0 \\ \xi_1 \end{pmatrix}, \begin{pmatrix} \eta_2 \\ u_d \\ \eta_0 \\ 0 \end{pmatrix}\right] = \xi_2\overline{\eta_2} + [x_d, u_d] + \xi_1\overline{\eta_0} \\
&= [(S_d - z)^{-1}u_d, u_d] + \frac{\Delta_2}{\Delta}\left(\overline{\eta_2} + [(S_d - z)^{-1}c_z, u_d] - \frac{\overline{c_1}}{\overline{c_2} - z}\right) \\
&= [(S_d - z)^{-1}u_d, u_d] \\
&\quad - \frac{((c_2 - z)\eta_2 - c_1\eta_0 + [(S_d - z)^{-1}u_d, c_{\overline{z}}])((\overline{c_2} - z)\overline{\eta_2} - \overline{c_1\eta_0} - [(S_d - z)^{-1}c_z, u_d])}{\tau(z) + z(c_2 - z)(\overline{c_2} - z) + [(S_d - z)^{-1}c_z, c_{\overline{z}}]}.
\end{aligned}$$

Consider the converse part of the lemma. As $\Delta \neq 0$ if and only if (3.22) is satisfied, we find, by similar computations as above, that $T_0(z)$ is injective and thus surjective, i.e. condition (i) of Theorem 2.5 is satisfied. All other conditions of Theorem 2.5 follow immediately from the construction (3.19) of $T_0(z)$. □

Remark 3.11. Again not all the entries of the matrix $\Phi(z)$ occur separately in (3.20), but only the scalar function τ.

Case (ii): We can write $T_\infty = \langle b \rangle$ with $b = g + b_0 h^\circ$, and

$$T_0(z) = \begin{pmatrix} [\,\cdot\,, a] & c_1 & t_1(z) \\ S_d & 0 & b - \overline{b_0}a \\ [\,\cdot\,, b] & c_2 & t_2(z) \\ 0 & 0 & -\overline{b_0} \end{pmatrix} : \begin{pmatrix} \mathcal{D}_1 \\ + \\ \langle h^\circ \rangle \\ + \\ \langle d \rangle \end{pmatrix} \longrightarrow \begin{pmatrix} \langle g \rangle \\ + \\ \mathcal{D}(S) \\ + \\ \langle h^\circ \rangle \\ + \\ \langle h^1 \rangle \end{pmatrix}.$$

Lemma 3.12. *If* $\dim T_\infty = 1$ *and* $h^\circ \notin T_\infty$*, we have*

$$[(A - z)^{-1}u, u] = [(S_d - z)^{-1}u_d, u_d]. \tag{3.28}$$

Conversely, the right hand side of (3.28) *defines a u-resolvent of* S.

In order to prove Lemma 3.12, consider the equation

$$\begin{pmatrix} [\cdot\,,a] & c_1 & t_1(z)+z\overline{b_0} \\ S_d - z & 0 & b-\overline{b_0}a \\ [\cdot\,,b] & c_2 - z & t_2(z) \\ 0 & 0 & -\overline{b_0}-z \end{pmatrix} \begin{pmatrix} x_d \\ \xi_0 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \eta_2 \\ u_d \\ \eta_0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ b_0 \\ 0 \end{pmatrix}.$$

Case (iii): We have

$$T_0(z) = \begin{pmatrix} t_1(z) & [\cdot\,,a] & c_1 \\ a & S_d & 0 \\ t_2(z) & [\cdot\,,b] & c_2 \\ \overline{c_1} & 0 & 0 \end{pmatrix} : \begin{pmatrix} \langle g\rangle \\ + \\ \mathcal{D}_1 \\ + \\ \langle h^\circ\rangle \end{pmatrix} \longrightarrow \begin{pmatrix} \langle g\rangle \\ + \\ \mathcal{D}_1 \\ + \\ \langle h^\circ\rangle \\ + \\ \langle h^1\rangle \end{pmatrix}.$$

Lemma 3.13. *If* $\dim T_\infty = 1$ *and* $h^\circ \in T_\infty$*, the formula* (3.28) *holds.*

In order to prove Lemma 3.13, consider the equation

$$\begin{pmatrix} t_1(z)-z & [\cdot\,,a] & c_1 \\ a & S_d - z & 0 \\ t_2(z) & [\cdot\,,b] & c_2 - z \\ \overline{c_1} & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_2 \\ x_d \\ \xi_0 \end{pmatrix} = \begin{pmatrix} \eta_2 \\ u_d \\ \eta_0 \\ \eta_1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{3.29}$$

with $\eta_1 = 0$. To prove the converse part note that (3.29) remains solvable if $\eta_1 \in \mathbb{C}$ is arbitrary, and therefore condition (i) of Theorem 2.5 is satisfied.

Case (iv): In this case $T_0(z) = S$ and $\mathcal{D}^\perp = \langle g, h^\circ\rangle$. Thus

$$\mathcal{R}\,(T_0(z)-z) + \mathcal{D}^\perp \subseteq \mathcal{H} \neq \mathcal{P}_c$$

and (i) of Theorem 2.5 cannot be satisfied.

Remark 3.14. Similarly to the case $h^\circ \notin \mathcal{D}(S)$ different extensions may have the same u-resolvent.

For the sake of completeness we consider the case $\dim \mathcal{H} = 2$. Then the component $\mathcal{D}_1$ in the decomposition (3.16) of $\mathcal{P}_c$ does not appear. With similar calculations as in the preceding lemmata we obtain that the formulas developed above remain valid if only the terms involving S_d are deleted.

Definition 3.15. Let $\mathcal{T}_d$ denote the set of all functions τ of the form

$$\begin{aligned} \tau(z) \;=\; & -t_{11}(z)z^2 - 2(c_1 t_{21}(z) + \overline{c_1}\overline{t_{12}(\overline{z})} - (c_2+\overline{c_2})t_{11}(z))z \\ & -(|c_2|^2 t_{11}(z) + |c_1|^2 t_{22}(z) - 2(c_1\overline{c_2}t_{21}(z) + \overline{c_1}c_2\overline{t_{12}(\overline{z})})) \end{aligned}$$

with c_1, c_2 given by (3.17), where

$$\Phi(z) = -\begin{pmatrix} t_{11}(z) & t_{12}(z) \\ t_{21}(z) & t_{22}(z) \end{pmatrix} \in \mathcal{N}_0(\mathbb{C}^2),$$

or $\tau(z) = \infty$.

¿From the above lemmata we obtain the following result.

Theorem 3.16. *Let $\mathcal{D}(S)$ be degenerated. Then the set of all u-resolvents of S is parametrized by the formula*

$$[(A-z)^{-1}u, u] = [(S_d - z)^{-1}u_d, u_d]$$

$$-\frac{((c_2 - z)\eta_2 - c_1\eta_0 + [(S_d - z)^{-1}u_d, c_{\overline{z}}])((\overline{c_2} - z)\overline{\eta_2} - \overline{c_1\eta_0} + [(S_d - z)^{-1}c_z, u_d])}{\tau(z) + z(c_2 - z)(\overline{c_2} - z) + [(S_d - z)^{-1}c_z, c_{\overline{z}}]},$$

where $\tau \in \mathcal{T}_d$.

4. The degenerated Nevanlinna-Pick problem

By $\mathcal{N}_\kappa$ we denote the class of all complex functions f which are meromorphic in $\mathbb{C}^+$, and such that the kernel

$$N_f(z, w) = \frac{f(z) - \overline{f(w)}}{z - \overline{w}}, \qquad z, w \in \rho(f),$$

has exactly κ negative squares. Here $\rho(f)$ denotes the domain of holomorphy of the function f.

Recall that the classical Nevanlinna-Pick interpolation problem for the upper half plane can be formulated as follows:

Given $n \in \mathbb{N}$, $z_1, \ldots, z_n \in \mathbb{C}^+$ and $w_1, \ldots, w_n \in \mathbb{C}^+$. Find conditions such that there exist functions $f \in \mathcal{N}_0$ which satisfy

$$f(z_i) = w_i, \qquad i = 1, \ldots, n. \tag{4.1}$$

It was shown in [9] that such functions exist if and only if the corresponding so-called Pick matrix

$$\mathbb{P} = \left(\frac{w_j - \overline{w_i}}{z_j - \overline{z_i}} \right)_{i,j=1}^{n}$$

is positive semidefinite. In fact the problem has a unique solution $f \in \mathcal{N}_0$ if and only if the Pick matrix $\mathbb{P}$ is degenerated, otherwise it has infinitely many solutions, which can be described e.g. by a fractional linear transformation with a parameter running through the set $\mathcal{N}_0$.

Here we consider the following problem: Given $n \in \mathbb{N}$, $z_1, \dots, z_n \in \mathbb{C}^+$ and $w_1, \dots, w_n \in \mathbb{C}^+$, and suppose that the corresponding Pick matrix $\mathbb{P}$ is positive semidefinite and degenerate, having zero as a simple eigenvalue. Hence the Nevanlinna-Pick interpolation problem has a unique solution $f \in \mathcal{N}_0$. Describe all functions $f \in \mathcal{N}_1$ which are holomorphic in $z_1, \dots, z_n$ and which are such that (4.1) holds. It was shown in [11] that there exist infinitely many solutions f of this problem. They correspond to the selfadjoint extensions A of a symmetric operator S, where S acts in a finite dimensional space $\mathcal{H}$ with a positive semidefinite inner product, having a one dimensional isotropic subspace, and where A is a selfadjoint extension of S in a π_1-space $\mathcal{P} \supseteq \mathcal{H}$ such that $z_i \in \rho(A)$ for $i = 1, \dots, n$.

Recall (see [11]) that $\mathcal{H}$ and S can be constructed as follows: $\mathcal{H}$ is the linear space of all formal sums

$$\sum_{i=1}^{n} \xi_i e_i, \qquad \xi_i \in \mathbb{C},$$

equipped with the inner product given by

$$[e_i, e_j] = \frac{w_i - \overline{w_j}}{z_i - \overline{z_j}}, \qquad i, j = 1, \dots, n,$$

and S is the operator in $\mathcal{H}$ with domain

$$\mathcal{D}(S) = \left\{ \sum_{i=1}^{n} \xi_i e_i : \ \sum_{i=1}^{n} \xi_i = 0 \right\}$$

which acts as

$$S\left(\sum_{i=1}^{n} \xi_i e_i\right) = \sum_{i=1}^{n} z_i \xi_i e_i \ \text{ for } \ \sum_{i=1}^{n} \xi_i e_i \in \mathcal{D}(S)\,.$$

By a straightforward calculation it can be shown that S is symmetric and has no eigenvalues.

The following result is a particular case of Theorem 1 of [11]:

Proposition 4.1. *The family of solutions $f \in \mathcal{N}_1$ of the interpolation problem* (4.1) *corresponds to the family of selfadjoint (relational) extensions A of S in Pontryagin spaces $\mathcal{P} \supseteq \mathcal{H}$ with negative index* 1 *which contain the points $z_1, \dots, z_n$ in their resolvent set and are e_1-minimal. This correspondence is established by the formula $f = Q_A$ with*

$$Q_A(z) = \overline{w_1} + \frac{\Im(w_1)}{\Im(z_1)}(z - \overline{z_1}) + (z - \overline{z_1})(z - z_1)[(A - z)^{-1} e_1, e_1] \tag{4.2}$$

and becomes bijective if we do not distinguish between e_1-unitarily equivalent extensions of S.

There an extension A of S in a Pontryagin space $\mathcal{P}$ with $z_1 \in \rho(A)$ is called e_1-minimal if

$$\overline{\langle e_1, (A-z)^{-1}e_1 : \ z \in \rho(A) \rangle} = \mathcal{P}.$$

Two extensions A_1 and A_2 acting in Pontryagin spaces $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, are called e_1-unitarily equivalent if there exists a unitary operator $U : \mathcal{P}_1 \to \mathcal{P}_2$ with $Ue_1 = e_1$, such that $UA_1 = A_2U$.

Observe that in the relation (4.2) an e_1-resolvent appears. Moreover, the space $\mathcal{H}$ and the operator S satisfy the assumptions of Sections 2 and 3. Hence, in the results obtained there, we can take $u = e_1$ and decompose this element according to (3.3) or (3.16):

$$e_1 = u_d + \eta_0 h^\circ \quad \text{or} \quad e_1 = \eta_2 g + u_d + \eta_0 h^\circ,$$

respectively, where again $\langle h^\circ \rangle = \mathcal{H}^\circ$. With

$$h^\circ = \sum_{i=1}^{n} \chi_i^\circ e_i$$

it follows that $\mathcal{D}(S)$ degenerates if and only if $\sum_{i=1}^n \chi_i^\circ = 0$. Let S_d be defined as in (3.4) or (3.17). Denote in the following by Q_{S_d} the function

$$Q_{S_d}(z) = \overline{w_1} + \frac{\Im(w_1)}{\Im(z_1)}(z-\overline{z_1}) + (z-\overline{z_1})(z-z_1)[(S_d-z)^{-1}u_d, u_d]. \tag{4.3}$$

Then the results of the preceding sections imply:

If $\sum_{i=1}^n \chi_i^\circ \neq 0$ the solutions of the interpolation problem (4.1) within $\mathcal{N}_0 \cup \mathcal{N}_1$ are given by

$$f(z) = Q_{S_d}(z) - (z-z_1)(z-\overline{z_1}) \cdot \frac{(\eta_0 - [(S_d-z)^{-1}u_d, a])(\overline{\eta_0} - [(S_d-z)^{-1}a, u_d])}{\tau(z) + [(S_d-z)^{-1}a, a]}, \tag{4.4}$$

where the parameter τ runs through the set $\mathcal{T}_n$. If $\sum_{i=1}^n \chi_i^\circ = 0$ and c_1, c_2 are as in (3.17), the solutions of (4.1) within $\mathcal{N}_0 \cup \mathcal{N}_1$ are given by

$$f(z) = Q_{S_d}(z) - (z-z_1)(z-\overline{z_1}) \cdot \frac{((c_2-z)\eta_2 - c_1\eta_0 + [(S_d-z)^{-1}u_d, a])((\overline{c_2}-z)\eta_2 - \overline{c_1\eta_0} + [(S_d-z)^{-1}a, u_d])}{\tau(z) + z(c_2-z)(\overline{c_2}-z) + [(S_d-z)^{-1}c_z, c_{\overline{z}}]}, \tag{4.5}$$

where the parameter τ runs through the set $\mathcal{T}_d$.

Parameters which correspond to selfadjoint extensions of S which contain some data points in their spectrum have to be excluded. We will see in Section 5 that this is the case if and only if the denominator of (4.4) (or (4.5), respectively) vanishes at a data point.

In both cases the unique solution $f \in \mathcal{N}_0$ of (4.1) corresponds to the parameter $\tau(z) \equiv \infty$, that is:

Proposition 4.2. *The function Q_{S_d} in (4.3) is the unique solution of the interpolation problem (4.1) in $\mathcal{N}_0$.*

Proof. As the operator S_d acts in the positive definite inner product space $\mathcal{D}(S)$ or $\mathcal{D}_1$, respectively, we have $Q_{S_d} \in \mathcal{N}_0$ (compare [11]).
As S_d is not an extension of S we cannot apply Proposition 4.1 in order to show that Q_{S_d} is a solution, we actually have to compute the values of Q_{S_d} at $z_1, \ldots, z_n$. As the points z_i are nonreal and S_d is selfadjoint, we have $z_i \in \rho(S_d)$ for $i = 1, \ldots, n$, and $Q_{S_d}(z_1) = w_1$. If $\mathcal{D}(S)$ is nondegenerated, then for $i = 2, \ldots, n$

$$\begin{aligned} u_d + \eta_0 h^\circ &= u = e_1 = (S - z)\left(\frac{e_i - e_1}{z_i - z_1}\right) \\ &= (S_d - z)\left(\frac{e_i - e_1}{z_i - z_1}\right) + \left[\left(\frac{e_i - e_1}{z_i - z_1}\right), a\right] h^\circ \end{aligned}$$

holds. If $\mathcal{D}(S)$ is degenerated we find for $i \in \{2, \ldots, n\}$,

$$\begin{aligned} \eta_2 g + u_d + \eta_0 h^\circ &= u = e_1 = (S - z)\left(\frac{e_i - e_1}{z_i - z_1}\right) \\ &= (S_d - z)\left(\frac{e_i - e_1}{z_i - z_1} - \lambda h^\circ\right) + \left[\left(\frac{e_i - e_1}{z_i - z_1} - \lambda h^\circ\right), b\right] h^\circ \\ &\quad + \left[\left(\frac{e_i - e_1}{z_i - z_1} - \lambda h^\circ\right), a\right] g + \lambda c_1 g + \lambda c_2 h^\circ, \end{aligned}$$

where $\left(\frac{e_i - e_1}{z_i - z_1} - \lambda h^\circ\right) + \lambda h^\circ$ is the decomposition of $\frac{e_i - e_1}{z_i - z_1}$ with respect to $\mathcal{D}(S) = \mathcal{D}_1 [\dot{+}] \langle h^\circ \rangle$.
In the first case we have

$$(S_d - z)^{-1} u_d = \frac{e_i - e_1}{z_i - z_1},$$

whereas in the second case

$$(S_d - z)^{-1} u_d = \frac{e_i - e_1}{z_i - z_1} - \lambda h^\circ.$$

A straightforward computation shows that in both cases the relations $Q_{S_d}(z_i) = w_i$, $i = 1, \ldots, n$ hold. □

Remark 4.3. Proposition 4.2 gives another proof of G. Pick's result, that a solution of the interpolation problem (4.1) within the class $\mathcal{N}_0$ actually exists.

Proposition 4.4. *The unique solution Q_{S_d} of* (4.1) *in* $\mathcal{N}_0$ *is holomorphic at* ∞ *if and only if* $\mathcal{D}(S)$ *is nondegenerated, and it has a simple pole at* ∞ *if and only if* $\mathcal{D}(S)$ *is degenerated.*

Proof. Let f be the unique solution of (4.1) in $\mathcal{N}_0$. Then f is a rational function and $\deg f = n-1$. From [10] we know that f is given by

$$f(z) = \frac{\sum_{i=1}^n \overline{\chi_i^\circ} \overline{w_i} \prod_{j=1, j\neq i}^n (z - \overline{z_j})}{\sum_{i=1}^n \overline{\chi_i^\circ} \prod_{j=1, j\neq i}^n (z - \overline{z_j})} = \frac{p(z)}{q(z)} \tag{4.6}$$

where $\sum_{i=1}^n \chi_i^\circ e_i = h^\circ$. As $\deg f = n-1$ (see [10]) the numerator $p(z)$ and denominator $q(z)$ in (4.6) are relatively prime.

Suppose that $\mathcal{D}(S)$ is nondegenerated, i.e. $h^\circ \notin \mathcal{D}(S)$. Equivalently, $\sum_{i=1}^n \chi_i^0 \neq 0$, which implies that $\deg q = n-1$. As $\deg f = \max(\deg p, \deg q) = n-1$ we find that f is analytic at ∞.

If $\mathcal{D}(S)$ is degenerated, then S_d acts in a $(n-2)$-dimensional space, i.e. it can be written as an $(n-2)\times(n-2)$-matrix. Thus

$$(S_d - z)^{-1} = \frac{1}{|S_d - z|} A^T$$

where A is the matrix of algebraic complements of $S_d - z$. The determinant $|S_d - z|$ is a polynomial of degree $n-2$, whereas each entry of A is a polynomial of degree $n-3$. Thus

$$[(S_d - z)^{-1} u_d, u_d] = \frac{p_1(z)}{q_1(z)} \tag{4.7}$$

where $\deg p_1 \le n-3$ and $q_1(z) = |S_d - z|$, i.e. $\deg q_1 = n-2$. Substituting (4.7) into the expression (4.3) for Q_{S_d} we find that

$$Q_{S_d}(z) = \frac{p(z)}{q_1(z)}$$

where $\deg p \le n-1$ and $\deg q_1 = n-2$. As f interpolates we must have $\deg f = n-1$ and therefore $\deg p = n-1$. Thus f has a simple pole at ∞. □

5. Explicit formulas

I. Assume that $h^\circ \notin \mathcal{D}(S)$. If $h^\circ = \sum_{i=1}^n \chi_i^\circ e_i$, we can assume without loss of generality that $\sum_{i=1}^n \chi_i^\circ = 1$.

We introduce the functions

$$N_1(z) = (\eta_0 - [(S_d - z)^{-1} u_d, a])(\overline{\eta_0} - [(S_d - z)^{-1} a, u_d])$$

and

$$M_1(z) = [(S_d - z)^{-1} a, a],$$

where a, η_0 and u_d are as in (3.4) and (3.5). As the space $\mathcal{H}$ is finite dimensional and S_d is selfadjoint, N_1 and M_1 are real rational functions.

Lemma 5.1. *Let*

$$S = \begin{pmatrix} S_d \\ [\,\cdot\,, a] \end{pmatrix} : \mathcal{D}(S) \to \mathcal{H}$$

be the matrix representation of the operator S with respect to the decomposition $\mathcal{H} = \mathcal{D}(S)\,[\dot{+}]\langle h^\circ \rangle$. Then $a = \sum_{i=1}^n \alpha_i e_i$ where the numbers α_i are the (unique) solutions of the system of linear equations

$$\sum_{i=1}^n \alpha_i \left(\frac{w_1 - \overline{w_j}}{z_1 - \overline{z_j}} - \frac{w_1 - \overline{w_1}}{z_1 - \overline{z_1}} \right) = \overline{z_j} - \overline{z_1}, \quad j = 2, \dots, n, \qquad \sum_{i=1}^n \alpha_i = 0. \tag{5.1}$$

If we put $u = e_1 = u_d + \eta_0 h^\circ$, we have

$$\eta_0 = 1 \quad \text{and} \quad u_d = e_1 - h^\circ.$$

Proof. We have to find $a \in \mathcal{D}(S)$ such that

$$Sx = S_d x + [x, a] h^\circ, \qquad x \in \mathcal{D}(S)\,.$$

Let $x = \sum_{i=1}^n \xi_i e_i \in \mathcal{D}(S)$. Then

$$Sx = \sum_{i=1}^n z_i \xi_i e_i = \left(\sum_{i=1}^n z_i \xi_i e_i - \left(\sum_{i=1}^n z_i \xi_i \right) h^\circ \right) + \left(\sum_{i=1}^n z_i \xi_i \right) h^\circ,$$

i.e. $[x, a] = \sum\limits_{i=1}^n z_i \xi_i$. Setting $x = e_j - e_1$ for $j \in \{2, \dots, n\}$ we obtain $[e_j - e_1, a] = z_j - z_1$ or

$$\sum_{i=1}^n \overline{\alpha_i} \left(\frac{w_j - \overline{w_1}}{z_j - \overline{z_1}} - \frac{w_j - \overline{w_1}}{z_j - \overline{z_1}} \right) = z_j - z_1, \qquad j = 2, \dots, n,$$

where additionally $\sum_{i=1}^n \alpha_i = 0$ holds as $a \in \mathcal{D}(S)$. Due to the fact that $\mathcal{D}(S)$ is nondegenerated this system of linear equations has a unique solution $\alpha_1, \dots, \alpha_n$. As we have assumed that $\sum_{i=1}^n \chi_i^\circ = 1$, the decomposition

$$u = u_d + \eta_0 h^\circ \ \text{ with } \ u_d = e_1 - h^\circ, \qquad \eta_0 = 1$$

holds. □

Recall that if a rational function is written as a quotient of two polynomials which are relatively prime, then its degree is defined as the maximum of the degrees of the numerator and the denominator. The degree of a rational function equals the total multiplicity of its poles or its zeros (including ∞).

Proposition 5.2. *The rational function $N_1(z)$ is of degree $2n-2$, and*

$$\lim_{z\to\infty} N_1(z) = 1. \tag{5.2}$$

The zeros of $N_1(z)$ are the points $z_2,\dots,z_n$ and $\overline{z_2},\dots,\overline{z_n}$. The poles of $N_1(z)$ are the solutions of the equation

$$\sum_{i=1}^{n} \frac{\chi_i^\circ}{z_i - z} = 0,$$

and they are all real. Therefore

$$N_1(z) = \frac{\prod\limits_{i=2}^{n}(z-z_i)(z-\overline{z_i})}{\left[\sum\limits_{i=1}^{n}\chi_i^\circ \prod\limits_{\substack{j=1\\ j\neq i}}^{n}(z-z_j)\right]^2}. \tag{5.3}$$

The function $M_1(z)$ is of degree at most $n-1$, and

$$\lim_{z\to\infty} z^l M_1(z) = -\sum_{i=1}^{n} z_i^l \alpha_i \neq 0 \tag{5.4}$$

where $l \in \{1,\dots,n-1\}$ is such that $S^{l-1}a \in \mathcal{D}(S)$ but $S^l a \notin \mathcal{D}(S)$. The zeros of $M_1(z)$ are the solutions of the equation

$$\sum_{i=1}^{n} \frac{\alpha_i}{z_i - z} = 0.$$

The poles of $M_1(z)$ coincide with the poles of $N_1(z)$, the multiplicity of a pole in $M_1(z)$ is half the multiplicity of the corresponding pole of $N_1(z)$. Therefore

$$M_1(z) = (-1)^{n-1} \frac{\sum\limits_{i=1}^{n}\alpha_i \prod\limits_{\substack{j=1\\ j\neq i}}^{n}(z-z_j)}{\sum\limits_{i=1}^{n}\chi_i^\circ \prod\limits_{\substack{j=1\\ j\neq i}}^{n}(z-z_j)}.$$

Proof. As S_d acts on $\mathcal{D}(S)$ we have $(S_d - z)^{-1} = \frac{L(z)}{q(z)}$, where $\deg q = \dim \mathcal{D}(S) = n-1$ and $\deg L \leq n-2$. Note that $q(z) = |S_d - z|$ is a real polynomial. Thus

$$\begin{aligned} N_1(z) &= \left(1 - \frac{[L(z)u_d, a]}{q(z)}\right)\left(1 - \frac{[L(z)a, u_d]}{q(z)}\right) \\ &= \frac{q(z)^2 - q(z)([L(z)u_d, a] + [L(z)a, u_d]) + [L(z)u_d, a][L(z)a, u_d]}{q(z)^2}. \end{aligned}$$

As $\deg q^2 = 2n-2$ and the degree of the remaining terms in the numerator is at most $2n-3$ we find that $\deg N_1 \le 2n-2$ and (5.2) holds.
From [11] we know that there exist parameters τ such that the corresponding function Q_A interpolates. Thus, as the function Q_{S_d} also interpolates, we must have $N_1(z_i) = 0$ for $i = 2, \dots, n$. As N_1 is real it has also the zeros $\overline{z_2}, \dots, \overline{z_n}$. In particular, the degree of N_1 is in fact equal to $2n-2$.
The poles of $N_1(z)$ are obviously the eigenvalues of S_d, and therefore real. A value z is an eigenvalue of S_d if and only if there exists a vector $x = \sum_{i=1}^n \xi_i e_i \in \mathcal{D}(S)$, $x \neq 0$, such that $(S_d - z)x = 0$. We find

$$0 = (S_d - z)x = P(S-z)x = P\left(\sum_{i=1}^n (z_i - z)\xi_i e_i\right),$$

where P denotes the projection onto $\mathcal{D}(S)$ with kernel $\langle h^\circ \rangle$. Thus

$$\sum_{i=1}^n (z_i - z)\xi_i e_i = \mu h^\circ \quad \text{for some } \mu \in \mathbb{C}.$$

Furthermore, as we can restrict our attention to real values of z, we must have $\mu \neq 0$. Due to this fact we find that z is an eigenvalue of S_d if and only if there exist numbers ξ_i, not all zero, such that

$$\sum_{i=1}^n (z_i - z)\xi_i e_i = h^\circ, \qquad \sum_{i=1}^n \xi_i = 0.$$

The first equation has the unique solution (observe that $z \in \mathbb{R}$ and thus $z_i - z \neq 0$)

$$\xi_i = \frac{\chi_i^\circ}{z_i - z}, \qquad i = 1, \dots, n.$$

Thus z is an eigenvalue of S_d if and only if

$$\sum_{i=1}^n \xi_i = \sum_{i=1}^n \frac{\chi_i^\circ}{z_i - z} = 0.$$

We have

$$\sum_{i=1}^n \frac{\chi_i^\circ}{z_i - z} = \frac{\sum\limits_{i=1}^n \chi_i^\circ \prod\limits_{\substack{j=1\\ j\neq i}}^n (z_j - z)}{\prod\limits_{i=1}^n (z_i - z)},$$

and therefore $\sum_{i=1}^n \frac{\chi_i^\circ}{z_i - z} = 0$ if and only if

$$\sum_{i=1}^n \chi_i^\circ \prod_{\substack{j=1\\ j\neq i}}^n (z_j - z) = 0.$$

Now it is clear that (5.3) holds.
The assertion concerning the poles of $M_1(z)$ is obvious. Also the degree of the numerator of $M_1(z)$ is clearly at most $n-2$, whereas the degree of its denominator is $n-1$ and thus $\deg M_1 \leq n-1$.
We show that z is a zero of $M_1(z)$ if and only if $a \in \mathcal{R}(S-z)$: Put $x(z) = (S_d - z)^{-1}a$. Then

$$a = (S_d - z)x(z) = (S - z)x(z) - [x(z), a]h^\circ.$$

If $M_1(z) = 0$ we have $a = (S-z)x(z)$, i.e. $a \in \mathcal{R}(S-z)$. Suppose, conversely, that $a \in \mathcal{R}(S-z)$, i.e. let $a = (S-z)x$ with some $x \in \mathcal{D}(S)$. Then

$$a = (S - z)x = (S_d - z)x + [x, a]h^\circ.$$

As $\mathcal{D}(S) \cap \mathcal{H}^\circ = \{0\}$ we find $[x, a] = 0$ and therefore $a = (S_d - z)x$, i.e. $x = (S_d - z)^{-1}a$. We conclude that $M_1(z) = 0$.
In order to determine those values z for which $a \in \mathcal{R}(S-z)$, consider the equation

$$(S - z)x = a, \qquad x \in \mathcal{D}(S)\,.$$

With $x = \sum_{i=1}^n \xi_i e_i$ and $a = \sum_{i=1}^n \alpha_i e_i$ this equation becomes

$$(z_i - z)\xi_i = \alpha_i, \quad i = 1, \ldots, n, \qquad \sum_{i=1}^n \xi_i = 0.$$

Hence $\xi_i = \frac{\alpha_i}{z_i - z}$ if $z \neq z_1, \ldots, z_n$. Thus in this case $a \in \mathcal{R}(S-z)$ if and only if

$$\sum_{i=1}^n \frac{\alpha_i}{z_i - z} = 0.$$

If $z = z_i$ for some $i \in \{1, \ldots, n\}$ we have $a \in \mathcal{R}(S-z)$ if and only if $\alpha_i = 0$. Thus the zeros of $M_1(z)$ coincide with those of the polynomial

$$\sum_{i=1}^n \alpha_i \prod_{\substack{j=1 \\ j \neq i}}^n (z_j - z).$$

It remains to prove the relation (5.4). Let $l \in \{1, \ldots, n-1\}$ be such that $S^{l-1}a \in \mathcal{D}(S)$ but $S^l \notin \mathcal{D}(S)$. Such a number l exists as $S^k a = \sum_{i=1}^n z_i^k \alpha_i e_i$ and

$$\begin{vmatrix} 1 & z_1 & z_1^2 & \ldots & z_1^{n-1} \\ 1 & z_2 & z_2^2 & \ldots & z_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & z_n & z_n^2 & \ldots & z_n^{n-1} \end{vmatrix} \neq 0.$$

Thus $S^0 a \in \mathcal{D}(S)$, but it is impossible that $S^k a \in \mathcal{D}(S)$ for $k = 0, \dots, n-1$. Consider the power series expansion of $(S_d - z)^{-1}$ outside of a sufficiently large disc:

$$(S_d - z)^{-1} = -\sum_{k=0}^{\infty} \frac{1}{z^{k+1}} S_d^k.$$

It follows that

$$M_1(z) = [(S_d - z)^{-1} a, a] = -\sum_{k=0}^{\infty} \frac{1}{z^{k+1}} [S_d^k a, a].$$

Note that $S_d^k a = S^k a$ for $k = 0, \dots, l-1$: For $k = 0$ this assertion is trivial. For $k = 1 \le l - 1$ we have

$$Sa = S_d a + [a, a] h^\circ,$$

and, as $[a, a] = \sum_{i=1}^n z_i \alpha_i = 0$ we find $Sa = S_d a$. Using induction we obtain for $k \le l - 1$,

$$S^k a = S(S^{k-1} a) = S_d(S^{k-1} a) + [S^{k-1} a, a] h^\circ = S_d(S_d^{k-1} a) = S_d^k a,$$

as $[S^{k-1} a, a] = \sum_{i=1}^n z_i^k \alpha_i = 0$. Thus

$$\begin{aligned} M_1(z) &= -\sum_{k=0}^{l-1} \frac{1}{z^{k+1}} [S^k a, a] - \sum_{k=l}^{\infty} \frac{1}{z^{k+1}} [S_d^k a, a] \\ &= -\frac{1}{z^l} \left(\sum_{i=1}^n z_i^l \alpha_i \right) - \frac{1}{z^{l+1}} \sum_{k=l}^{\infty} \frac{1}{z^{k-l}} [S_d^k a, a], \end{aligned}$$

and we find

$$\lim_{z \to \infty} z^l M_1(z) = -\sum_{i=1}^n z_i^l \alpha_i.$$

The proof is complete. □

Remark 5.3. The function $N_1(z)$ has only simple zeros.

Let $\mathcal{T}_n$ be as in Definition 3.6. From Proposition 4.1 and Proposition 5.2 we now get:

Theorem 5.4. *Let $n \in \mathbb{N}$, $z_1, \dots, z_n \in \mathbb{C}^+$ and $w_1, \dots, w_n \in \mathbb{C}^+$ be given such that the corresponding Pick matrix $\mathbb{P}$ is positive semidefinite,* $\operatorname{rank} \mathbb{P} = n - 1$ *and $\mathbb{P} h^\circ = 0$ for a vector $h^\circ = \sum_{i=1}^n \chi_i^\circ e_i \neq 0$ with $\sum_{i=1}^n \chi_i^\circ = 1$. Let $\alpha_1, \dots, \alpha_n$ be the solution of the system* (5.1).
The solutions f of the interpolation problem (4.1) *in $\mathcal{N}_0 \cup \mathcal{N}_1$ are given by the*

formula

$$f_\tau(z) = \frac{\sum\limits_{i=1}^{n} \chi_i^\circ w_i \prod\limits_{\substack{j=1\\ j\neq i}}^{n} (z-z_j)}{\sum\limits_{i=1}^{n} \chi_i^\circ \prod\limits_{\substack{j=1\\ j\neq i}}^{n} (z-z_j)} - \frac{\prod\limits_{i=1}^{n} (z-z_i)(z-\overline{z_i})}{\sum\limits_{i=1}^{n} \chi_i^\circ \prod\limits_{\substack{j=1\\ j\neq i}}^{n} (z-z_j) \left[\sum\limits_{i=1}^{n} \alpha_i \prod\limits_{\substack{j=1\\ j\neq i}}^{n} (z-z_j) + \tau(z) \sum\limits_{i=1}^{n} \chi_i^\circ \prod\limits_{\substack{j=1\\ j\neq i}}^{n} (z-z_j) \right]}.$$

The parameter τ runs through those functions of $\mathcal{T}_n$ for which

$$\tau(z_k) \neq -\frac{\sum\limits_{i=1}^{n} \alpha_i \prod\limits_{\substack{j=1\\ j\neq i}}^{n} (z_k-z_j)}{\sum\limits_{i=1}^{n} \chi_i^\circ \prod\limits_{\substack{j=1\\ j\neq i}}^{n} (z_k-z_j)}, \qquad k=1,\dots,n,$$

holds. The function f_∞ is the unique solution in $\mathcal{N}_0$.

II. Consider now the case $h^\circ \in \mathcal{D}(S)$. Assume that $\mathbb{P}h^\circ = 0$ for $h^\circ = \sum_{i=1}^{n} \chi_i^\circ e_i \neq 0$ with $\sum_{i=1}^{n} \chi_i^\circ = 0$. Let $\gamma_1, \dots, \gamma_n$ be the solutions of

$$\sum_{i=1}^{n} \gamma_i \left(\frac{w_i - \overline{w_j}}{z_i - \overline{z_j}} - \frac{w_i - \overline{w_2}}{z_i - \overline{z_2}} \right) = 0, \qquad j = 3, \dots, n, \qquad \gamma_1 = 0,$$

which satisfy additionally

$$\sum_{i,j=1}^{n} \frac{w_i - \overline{w_j}}{z_i - \overline{z_j}} \gamma_i \overline{\gamma_j} = 1, \qquad \sum_{i=1}^{n} \gamma_i \in \mathbb{R},$$

and let $\alpha_1, \dots, \alpha_n$ be the solutions of

$$\sum_{i=1}^{n} \alpha_i \left(\frac{w_i - \overline{w_j}}{z_i - \overline{z_j}} - \frac{w_i - \overline{w_2}}{z_i - \overline{z_2}} \right) = \frac{1}{\sum\limits_{i=1}^{n} \gamma_i} (\overline{z_j} - \overline{z_2}), \qquad j = 3, \dots, n,$$

$$\alpha_1 = 0, \qquad \sum_{i=2}^{n} \alpha_i = 0.$$

Moreover, let

$$c_1 = \frac{1}{\sum\limits_{i=1}^{n} \gamma_i} \sum_{i=1}^{n} z_i \chi_i^\circ, \qquad c_2 = z_1.$$

Let $\mathcal{T}_d$ be as in Definition 3.15 with c_1, c_2 from above. With similar computations as in the case where $\mathcal{D}(S)$ is nondegenerated we obtain:

Theorem 5.5. *Let $n \in \mathbb{N}$, $z_1, \ldots, z_n \in \mathbb{C}^+$ and $w_1, \ldots, w_n \in \mathbb{C}^+$ be given such that the corresponding Pick matrix $\mathbb{P}$ is semidefinite,* $\operatorname{rank} \mathbb{P} = n-1$ *and that $\mathbb{P}h^\circ = 0$ for $h^\circ = \sum_{i=1}^n \chi_i^\circ e_i \neq 0$ with $\sum_{i=1}^n \chi_i^\circ = 0$.*
The solutions of the interpolation problem (4.1) *in $\mathcal{N}_0 \cup \mathcal{N}_1$ are given by the formula*

$$f_\tau(z) = \frac{\sum\limits_{i=1}^n \chi_i^\circ w_i \prod\limits_{\substack{j=1\\ j\neq i}}^n (z-z_j)}{\sum\limits_{i=1}^n \chi_i^\circ \prod\limits_{\substack{j=1\\ j\neq i}}^n (z-z_j)} - \frac{1}{\sum\limits_{i=2}^n \gamma_i \prod\limits_{\substack{j=1\\ j\neq i}}^n (z-z_j)}$$

$$\cdot \frac{\prod\limits_{i=2}^n (z-z_i)(z-\overline{z_i})}{(z-z_1)(z-\overline{z_1})(z\sum\limits_{i=2}^n \gamma_i \prod\limits_{\substack{j=1\\ j\neq i}}^n (z-z_j) + \sum\limits_{i=2}^n \alpha_i \prod\limits_{\substack{j=1\\ j\neq i}}^n (z-z_j)) + \tau(z) \sum\limits_{i=2}^n \gamma_i \prod\limits_{\substack{j=1\\ j\neq i}}^n (z-z_j)},$$

The parameter τ runs through those functions of $\mathcal{T}_d$ for which

$$\tau(z_k) \neq -\frac{(z_k-z_1)(z_k-\overline{z_1})(z_k\sum\limits_{i=2}^n \gamma_i \prod\limits_{\substack{j=1\\ j\neq i}}^n (z_k-z_j) + \sum\limits_{i=2}^n \alpha_i \prod\limits_{\substack{j=1\\ j\neq i}}^n (z_k-z_j))}{\sum\limits_{i=2}^n \gamma_i \prod\limits_{\substack{j=1\\ j\neq i}}^n (z_k-z_j)}$$

holds for $k = 1, \ldots, n$. The function f_∞ is the unique solution in $\mathcal{N}_0$.

References

[1] Bruinsma, P.: Interpolation problems for Schur and Nevanlinna pairs; Doctoral Dissertation, Rijksuniversiteit Groningen 1991.

[2] Dijksma, A., Langer, H., de Snoo, H.S.V.: Generalized coresolvents of standard isometric operators and generalized resolvents of standard symmetric relations in Krein spaces; Operator Theory: Adv. Appl. 48 (1990), 261–274.

[3] Dijksma, A., Langer, H., de Snoo, H.S.V.: Selfadjoint Π_κ-extensions of symmetric subspaces: An abstract approach to boundary problems with spectral parameter in the boundary conditions; Integral Equations Operator Theory 7 (1984), 459–515.

[4] Iohvidov, I., Krein, M.G., Langer, H.: Spectral theory in spaces with an indefinite metric; Akademie Verlag, Berlin 1982.

[5] Krein, M.G., Langer, H.: Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume Π_κ zusammenhängen. I. Einige Funktionenklassen und ihre Darstellungen; Math. Nachr. 77 (1977), 187–236.

[6] Krein, M.G., Langer, H.: Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume Π_κ zusammenhängen. II. Verallgemeinerte Resolventen, u-Resolventen und ganze Operatoren; J. Funct. Anal. 30 (1978), 390–447.

[7] Krein, M.G., Langer, H.: On some continuation problems which are closely related to the theory of operators in spaces Π_κ. IV; J. Operator Theory 13 (1985), 299–417.

[8] Nevanlinna, R.: Über beschränkte analytische Funktionen; Ann. Acad. Sci. Fenn. A 22.7 (1929), 1–75.

[9] Pick, G.: Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden; Math. Ann. 77 (1916), 7–23.

[10] Woracek, H.: Nevanlinna-Pick interpolation: The degenerated case; Linear Algebra Applications 252 (1997), 141–158.

[11] Woracek, H.: An operator theoretic approach to degenerated Nevanlinna-Pick interpolation; Math. Nachr. 176 (1995), 335–350.

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1991 Mathematics Subject Classification. 47A57, 47B50, 30E05

Received May 28, 1996