Some interpolation problems of Nevanlinna-Pick type. The Kreĭn-Langer method.

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Dedicated to Heinz Langer on the occasion of his 60th birthday

The method of M.G. Kreĭn and H. Langer to solve interpolation problems of Nevanlinna-Pick type is explored. The classical Nevanlinna-Pick problem and a version involving derivatives are treated. The data give rise to an indefinite inner product space and a symmetric operator in it. In general, the inner product space is degenerate.

1. Introduction

In this paper we consider some interpolation problems of Nevanlinna-Pick type with data which are not necessarily positive definite. An approach to such problems was proposed by M.G. Kreĭn and H. Langer [15], who adapted a construction for the case of positive definite data by B. Sz.-Nagy and A. Koranyi [16, 17]. The method consists of constructing an indefinite inner product space and a symmetric linear operator or relation in it, so that the solutions of the particular Nevanlinna-Pick problem correspond to selfadjoint extensions of the symmetric operator. The construction of the indefinite inner product space can be given abstractly [13, 14, 15, 16, 17] or via reproducing kernel spaces [3]. Several papers have appeared, where this method was applied to similar situations under different conditions on the data [1, 2, 7, 8, 9, 18]. The aim of our paper is expository: we show in detail the basic ideas of the method in conjunction with some interpolation problems. A similar approach with the Nevanlinna class on the upper half plane replaced by the Schur class on the unit disk and with selfadjoint relations replaced by unitary operators was discussed by J.A. Ball [5].

Some preliminary material about selfadjoint relations in Pontryagin spaces can be found in Section 2, cf. [10, 11]. In Sections 3 and 4 the basic constructions associated with such selfadjoint relations are presented [13, 14]. In Sections 5 and 6 we consider the classical indefinite Nevanlinna-Pick problem and a version involving derivatives. For each problem we associate a model, i.e. an indefinite inner product space and a symmetric operator or relation to the prescribed data. There are no restrictions on the data, so that the model spaces may be degenerate. We show that the solutions of these interpolation problems are in one-to-one correspondence with the selfadjoint relations which extend the model operator. In a sequel to this paper we will give parametrizations of the solutions in terms of resolvent matrices. Also certain special types of solutions will be considered and a connection of our results to the theory of Q-functions will be established. If the model space is nondegenerate or even a Hilbert space solutions of the Nevanlinna-Pick interpolation problems exist. In general, this need not be true and the existence of solutions depends on the structure of the model.

A central role in this paper is played by functions belonging to the so-called generalized Nevanlinna class. In order to define this class, we will introduce some terminology and notations. Let $\mathfrak{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$. Let $f:\mathbb{C}\setminus\mathbb{R}\to\mathbf{L}(\mathfrak{H})$ be a meromorphic function; its domain of holomorphy in $\mathbb{C}\setminus\mathbb{R}$ is denoted by $\rho(f)$. The function f is called real, if for each $z\in\rho(f)$ also $\bar z\in\rho(f)$ and $f(\bar z)=f(z)^*$. Let $\nu,\pi\in\overline{\mathbb{N}}=\mathbb{N}\cup\{0\}\cup\{\infty\}$, not both equal to ∞. Then $\mathbf{N}_\nu^\pi(\mathfrak{H})$ is the set of all real meromorphic functions $f:\mathbb{C}\setminus\mathbb{R}\to\mathbf{L}(\mathfrak{H})$, such that the Nevanlinna kernel

$$\mathsf{N}_f(z,w)=\frac{f(z)-f(w)^*}{z-\bar w},\quad z,w\in\rho(f),\quad z\neq\bar w,$$

$$\mathsf{N}_f(z,\bar z)=f'(z),\quad z\in\rho(f),$$

has precisely ν negative and π positive squares. In other words, for each $n\in\mathbb{N}$ and each choice of $z_1,\dots,z_n\in\rho(f)$ and $x_1,\dots,x_n\in\mathfrak{H}$, the quadratic form

$$\sum_{i,j=1}^{n}(\mathsf{N}_f(z_i,z_j)x_i,x_j)\xi_i\bar\xi_j$$

has at most ν negative and π positive squares, and there is an $n\in\mathbb{N}$ and a choice of $z_1,\dots,z_n\in\rho(f)$ and $x_1,\dots,x_n\in\mathfrak{H}$, such that if $\nu<\infty$ then the quadratic form has precisely ν negative and if $\pi<\infty$ it has precisely π positive squares. In this definition we may restrict ourselves to Hilbert spaces. For if $\mathfrak{K}$ is a Kreĭn space with fundamental symmetry J and $\mathfrak{H}$ is the associated Hilbert space, then J gives a bijective correspondence $f\to fJ$ between $\mathbf{N}_\nu^\pi(\mathfrak{K})$ and $\mathbf{N}_\nu^\pi(\mathfrak{H})$.

2. Selfadjoint relations in Pontryagin spaces

The indices (ν,π) of a Kreĭn space $\mathfrak{P}$ are the maximal dimensions of a negative and of a positive subspace of $\mathfrak{P}$. We will always assume that one of the indices is finite, in which case we speak of a Pontryagin space. Let $\mathfrak{P}$ be a Pontryagin space and let A be a selfadjoint relation in $\mathfrak{P}$. In general, the resolvent set $\rho(A)$ may be empty due to the structure of the multivalued part of A; if A is an operator, it is nonempty. In the sequel we will consider only selfadjoint relations A whose resolvent set is nonempty. In that case $\mathbb{C}\setminus\mathbb{R}\subset\rho(A)$ with a possible exception of finitely many points which are normal eigenvalues of A and which lie symmetrically with respect to the real axis, cf. [10, 11]. Two selfadjoint relations A_1, A_2 in Pontryagin spaces $\mathfrak{P}_1$, $\mathfrak{P}_2$ with nonempty resolvent sets are unitarily equivalent if there exists a unitary operator U from $\mathfrak{P}_1$ onto $\mathfrak{P}_2$, such that $(A_2 - z)^{-1}U = U(A_1 - z)^{-1}$, $z \in \rho(A_1) \cap \rho(A_2)$. In this case, for $z \in \rho(A_1) \cap \rho(A_2)$,

$$\begin{aligned} \{\, \{U(A_1 - z)^{-1}h, U(I + z(A_1 - z)^{-1})x\} : x \in \mathfrak{P} \,\} \\ = \{\, \{(A_2 - z)^{-1}Uh, (I + z(A_2 - z)^{-1})Ux\} : x \in \mathfrak{P} \,\}, \end{aligned}$$

which leads to $\rho(A_1) = \rho(A_2)$. Now we will discuss the reduction of a selfadjoint relation and the construction of a selfadjoint relation via a symmetric operator or relation in an indefinite inner product space.

Let $\mathfrak{M}$ be a subspace of $\mathfrak{P}$, not necessarily closed. The selfadjoint relation A induces a closed linear subspace $\mathfrak{L}_{\mathfrak{M}}$ of $\mathfrak{P}$ defined by

$$\mathfrak{L}_{\mathfrak{M}} = \overline{\operatorname{span}}\, \{\, (I + (z - z_0)(A - z)^{-1})a : a \in \mathfrak{M},\ z \in \rho(A) \,\}, \quad z_0 \in \rho(A).$$

Clearly, $\mathfrak{M} \subset \mathfrak{L}_{\mathfrak{M}}$. It follows from the resolvent identity and the continuity of $(A - w)^{-1}$, that

$$(A - w)^{-1}\mathfrak{L}_{\mathfrak{M}} \subset \mathfrak{L}_{\mathfrak{M}}, \quad w \in \rho(A). \tag{2.1}$$

The invariant subspace $\mathfrak{L}_{\mathfrak{M}}$ may be a proper subspace of $\mathfrak{P}$; it can even be degenerate. However, after factorization the invariant subspace $\mathfrak{L}_{\mathfrak{M}}$ and the selfadjoint relation A give rise to a Pontryagin space $\mathfrak{P}_{\mathfrak{M}}$ and a "minimal" selfadjoint relation $A_{\mathfrak{M}}$ in $\mathfrak{P}_{\mathfrak{M}}$ in the following way. The invariance property (2.1) implies that

$$\{\, \{(A - z)^{-1}x, (I + z(A - z)^{-1})x\} : x \in \mathfrak{L}_{\mathfrak{M}} \,\} \subset A \cap \mathfrak{L}_{\mathfrak{M}}^2.$$

Conversely, each element in $A \cap \mathfrak{L}_{\mathfrak{M}}^2$ is contained in the left side. Hence for each $z \in \rho(A)$,

$$A \cap \mathfrak{L}_{\mathfrak{M}}^2 = \{\, \{(A - z)^{-1}x, (I + z(A - z)^{-1})x\} : x \in \mathfrak{L}_{\mathfrak{M}} \,\}. \tag{2.2}$$

Let $\mathfrak{L}_{\mathfrak{M}}^0 = \mathfrak{L}_{\mathfrak{M}} \cap \mathfrak{L}_{\mathfrak{M}}^{\perp}$ be the isotropic part of $\mathfrak{L}_{\mathfrak{M}}$. Then the factor space

$$\mathfrak{P}_{\mathfrak{M}} = \mathfrak{L}_{\mathfrak{M}}/\mathfrak{L}_{\mathfrak{M}}^0$$

is a Pontryagin space, cf. [4, p.69] and [6, Theorem 2.6]. In $\mathfrak{P}_{\mathfrak{M}}$ we define the relation $A_{\mathfrak{M}}$ by $A_{\mathfrak{M}} = (A \cap \mathfrak{L}_{\mathfrak{M}}^2)/(\mathfrak{L}_{\mathfrak{M}}^0)^2$ or, more explicitly, by

$$A_{\mathfrak{M}} = \{\, \{\hat{x}, \hat{y}\} : \{x, y\} \in A \cap \mathfrak{L}_{\mathfrak{M}}^2 \,\}.$$

It follows from (2.1) that $\mathfrak{L}_{\mathfrak{M}}^{\perp}$, hence also $\mathfrak{L}_{\mathfrak{M}}^0$, is invariant under $(A - w)^{-1}$, $w \in \rho(A)$. Therefore, the resolvent $(A - z)^{-1}$ induces a bounded linear mapping in $\mathfrak{P}_{\mathfrak{M}}$, which we denote by R_z. The identity $(A - z)^{-*} = (A - \bar{z})^{-1}$ implies that $R_z^* = R_{\bar{z}}$. It follows from the definition and (2.2) that

$$A_{\mathfrak{M}} = \{\, \{R_z x, (I + zR_z)x\} : x \in \mathfrak{P}_{\mathfrak{M}} \,\}, \quad z \in \rho(A).$$

These observations give the following result.

Theorem 2.1. *Let $\mathfrak{M}$ be a subspace of the Pontryagin space $\mathfrak{P}$ and let A be a selfadjoint relation in $\mathfrak{P}$ with a nonempty resolvent set. Then the relation $A_{\mathfrak{M}}$ is selfadjoint in $\mathfrak{P}_{\mathfrak{M}}$ and $\rho(A) \subset \rho(A_{\mathfrak{M}})$, so that the resolvent set of $A_{\mathfrak{M}}$ is nonempty. Moreover, the resolvent operator $(A_{\mathfrak{M}} - z)^{-1}$ coincides with the mapping induced by $(A - z)^{-1}$ in $\mathfrak{P}_{\mathfrak{M}}$.*

The selfadjoint relation A is called minimal with respect to $\mathfrak{M}$ if $\mathfrak{L}_{\mathfrak{M}} = \mathfrak{P}$, in which case $A_{\mathfrak{M}} = A$. Clearly, the relation $A_{\mathfrak{M}}$ is minimal with respect to the image of $\mathfrak{M}$ in $\mathfrak{P}_{\mathfrak{M}}$. We will call $A_{\mathfrak{M}}$ the minimal part of A.

Let $\mathfrak{L}$ be a linear space with inner product $[\cdot, \cdot]$. The indices (ν, π) of $\mathfrak{L}$ are the maximal dimensions of a negative and of a positive subspace of $\mathfrak{L}$. Assume that either ν or π is finite. A sequence $\{(a_n)\}_1^\infty$ of elements in $\mathfrak{L}$ is said to converge to an element $a \in \mathfrak{L}$ if

(i) $[a_n, b] \to [a, b]$ for all $b \in \mathfrak{L}$;

(ii) $[a_n, a_n] \to [a, a]$.

A linear subspace $\mathfrak{A}$ of $\mathfrak{L}$ is dense if every element of $\mathfrak{L}$ can be approximated in this sense by a sequence of elements of $\mathfrak{A}$. Since $\mathfrak{L}$ may be degenerate, i.e. the isotropic part $\mathfrak{L}^0 = \mathfrak{L} \cap \mathfrak{L}^\perp$ may be nontrivial, limits of sequences are not uniquely determined. If $a_n \to a$, then also $a_n \to a + h$ for any $h \in \mathfrak{L}^0$. Conversely, if $a_n \to a$ and $a_n \to a'$, then clearly $a - a' \in \mathfrak{L}^0$. The completion of the factor space $\mathfrak{L}/\mathfrak{L}^0$ is a Pontryagin space $\mathfrak{P}$ with indices (ν, π) in which the above notion of convergence is preserved [12, 13].

Theorem 2.2. *Let $\mathfrak{L}$ be an inner product space with indices (ν, π), where $\nu, \pi \in \overline{\mathbb{N}}$, not both equal to ∞. Let S be a symmetric relation in $\mathfrak{L}$, such that* $\operatorname{ran}(S - z)$ *is dense in $\mathfrak{L}$ for some $z \in \mathbb{C}^+$ and some $z \in \mathbb{C}^-$. Then*

$$A = \overline{\operatorname{span}}\, \{\, \{\hat{x}, \hat{y}\} \in \mathfrak{P}^2 : \{x, y\} \in S \,\},$$

is a selfadjoint relation in the Pontryagin space $\mathfrak{P}$ with a nonempty resolvent set.

Proof. It is easy to see that

$$A_1 = \operatorname{span}\, \{\, \{\hat{x}, \hat{y}\} \in \mathfrak{P}^2 : \{x, y\} \in S \,\},$$

is a symmetric linear relation in $\mathfrak{P}$. Moreover, since $\operatorname{ran}(S - z)$ is dense in $\mathfrak{L}$, also $\operatorname{ran}(A_1 - z)$ is dense in $\mathfrak{P}$. As the closure of (the graph of) A_1, A is symmetric and $\operatorname{ran}(A - z) = \mathfrak{P}$. The symmetry of A implies that $\ker(A - \bar{z}) = \{0\}$. We conclude that A is selfadjoint and has a nonempty resolvent set [10, 11]. □

In various forms of the Nevanlinna-Pick interpolation problem we will encounter an indefinite inner product space $\mathfrak{G}$, a Pontryagin space $\mathfrak{P}$, and an isometry Φ

from $\mathfrak{G}$ to $\mathfrak{P}$. Then a relation T in $\mathfrak{G}$ can be lifted to the relation $\Phi \circ T$ in $\mathfrak{P}$ by

$$\Phi \circ T = \{\, \{\Phi x, \Phi y\} : \{x, y\} \in T\,\}.$$

If $\mathfrak{G}$ is degenerate, then the isometry Φ may have a nontrivial kernel $\ker \Phi \subset \mathfrak{G}^0$, in which case $\Phi \circ T$ may be a proper linear relation. Clearly, if T is a symmetric relation in $\mathfrak{G}$, then $\Phi \circ T$ is a symmetric relation in $\mathfrak{P}$.

3. Induced functions

Let $\mathfrak{P}$ be a Pontryagin space with inner product $[\cdot,\cdot]$ and let A be a selfadjoint relation in $\mathfrak{P}$ with a nonempty resolvent set. Let $\mathfrak{H}$ be a Hilbert space and assume that $\Gamma \in \mathbf{L}(\mathfrak{H}, \mathfrak{P})$. Fix $z_0 \in \rho(A)$ and let $B \in \mathbf{L}(\mathfrak{H})$ such that $\operatorname{Im} B = (\operatorname{Im} z_0)\, \Gamma^* \Gamma$. Define the function f_A by

$$f_A(z) = B^* + (z - \bar{z}_0)\Gamma^*(I + (z - z_0)(A - z)^{-1})\Gamma, \quad z \in \rho(A), \tag{3.1}$$

where the adjoints are taken in the corresponding spaces. Note that $f(z) \in \mathbf{L}(\mathfrak{H})$ and $f_A(z_0) = B$. We will consider f_A as a function defined on its domain of holomorphy $\rho(f_A)$ within $\mathbb{C} \setminus \mathbb{R}$. Clearly, $\rho(f_A) \supseteq \rho(A) \cap (\mathbb{C} \setminus \mathbb{R})$, so that $\rho(f_A)$ contains $\mathbb{C} \setminus \mathbb{R}$ with the possible exception of finitely many points. Using

$$B^* + (z - \bar{z}_0)\Gamma^*\Gamma = \operatorname{Re} B + (z - \operatorname{Re} z_0)\Gamma^*\Gamma,$$

we see that the function f_A also has the representation

$$f_A(z) = \operatorname{Re} B + (z - \operatorname{Re} z_0)\Gamma^*\Gamma + (z - z_0)(z - \bar{z}_0)\Gamma^*(A - z)^{-1}\Gamma. \tag{3.2}$$

Hence f_A is real. Moreover, by using the resolvent identity we see that

$$\mathsf{N}_f(z, w) = [(I + (z - z_0)(A - z)^{-1})\Gamma x, (I + (w - z_0)(A - w)^{-1})\Gamma y]. \tag{3.3}$$

This relation implies the following result.

Theorem 3.1. *Let $\mathfrak{P}$ be a Pontryagin space with indices (ν, π). Let A be a selfadjoint relation in $\mathfrak{P}$ with a nonempty resolvent set and let $\Gamma \in \mathbf{L}(\mathfrak{H}, \mathfrak{P})$. Then $f_A \in \mathbf{N}_{\nu'}^{\pi'}(\mathfrak{H})$ where $\nu' \le \nu$ and $\pi' \le \pi$. Moreover, if A is Γ-minimal, then $\nu' = \nu$ and $\pi' = \pi$.*

Let A_1 and A_2 be selfadjoint relations in Pontryagin spaces $\mathfrak{P}_1$, $\mathfrak{P}_2$, whose resolvent sets are nonempty, and let $\Gamma_1 \in \mathbf{L}(\mathfrak{H}, \mathfrak{P}_1)$ and $\Gamma_2 \in \mathbf{L}(\mathfrak{H}, \mathfrak{P}_2)$. The relations A_1 and A_2 are called Γ-unitarily equivalent, if A_1 and A_2 are unitarily equivalent and the associated unitary operator $U : \mathfrak{P}_1 \to \mathfrak{P}_2$ satisfies $U\Gamma_1 = \Gamma_2$.

Theorem 3.2. *If A_1 and A_2 are Γ-unitarily equivalent then $f_{A_1} = f_{A_2}$. Conversely, if A_1 is Γ_1-minimal, A_2 is Γ_2-minimal and $f_{A_1} = f_{A_2}$, then A_1 and A_2 are Γ-unitarily equivalent.*

Proof. Let A_1 and A_2 be Γ-unitarily equivalent and let U be the associated unitary operator. Due to $U\Gamma_1 = \Gamma_2$ we have $\Gamma_1^* U^* = \Gamma_2^*$. These identities together with $U^*U = I$ and $(A_2 - z)^{-1}U = U(A_1 - z)^{-1}$ immediately show that $f_{A_1}(z) = f_{A_2}(z)$ for all $z \in \rho(A_1) \cap \rho(A_2)$. (Note that we use the normalization $f_{A_1}(z_0) = f_{A_2}(z_0) = B$.) Since $(\mathbb{C} \setminus \mathbb{R}) \setminus (\rho(A_1) \cap \rho(A_2))$ is finite we conclude that $\rho(f_{A_1}) = \rho(f_{A_2})$, i.e. $f_{A_1} = f_{A_2}$.

To prove the converse part, let A_1 and A_2 be Γ-minimal selfadjoint relations and suppose that $f_{A_1} = f_{A_2}$. It follows from (3.3) that the relation

$$V(I + (z - z_0)(A_1 - z)^{-1})\Gamma_1 y = (I + (z - z_0)(A_2 - z)^{-1})\Gamma_2 y, \quad y \in \mathfrak{H},$$

defines a linear mapping from the dense subspace $\mathfrak{L}_{\operatorname{ran}\Gamma_1} \subset \mathfrak{P}_1$ onto the dense subspace $\mathfrak{L}_{\operatorname{ran}\Gamma_2} \subset \mathfrak{P}_2$, which is isometric. Moreover, $V\Gamma_1 = \Gamma_2$. It is also clear that $V(A_1 - z)^{-1}\Gamma_1 = (A_2 - z)^{-1}\Gamma_2$. We extend this identity from the range of Γ_1 by means of the resolvent identity as follows

$$\begin{aligned} &V(A_1 - w)^{-1}(I + (z - z_0)(A_1 - z)^{-1})\Gamma_1 \\ &\quad = (A_2 - w)^{-1}(I + (z - z_0)(A_2 - z)^{-1})\Gamma_2 \\ &\quad = (A_2 - w)^{-1}V(I + (z - z_0)(A_1 - z)^{-1})\Gamma_1. \end{aligned}$$

Then, by linearity the relation

$$V(A_1 - w)^{-1}y = (A_2 - w)^{-1}Vy, \quad w \in \rho(A_1) \cap \rho(A_2),$$

holds for all $y \in \mathfrak{L}_{\operatorname{ran}\Gamma_1}$. Sinve V maps a dense set from the Pontryagin space $\mathfrak{P}_1$ onto a dense subset of the Pontryagin space $\mathfrak{P}_2$, it has a unitary continuation U from $\mathfrak{P}_1$ onto $\mathfrak{P}_2$. Then clearly A_1 and A_2 are Γ-unitarily equivalent under U. $\square$

In order to study the derivatives of the function f_A in (3.1) it is convenient to introduce the function ϕ_A by

$$\phi_A(z) = (I + (z - z_0)(A - z)^{-1})\Gamma, \quad z \in \rho(A). \tag{3.4}$$

Then the right side of (3.3) can be written as $[\phi_A(z)x, \phi_A(w)y]$. Hence when $z, w \in \rho(A)$, $x, y \in \mathfrak{H}$, and $k, l \geq 0$, this gives

$$[\phi_A^{(k)}(z)x, \phi_A^{(l)}(w)y] = \left(\frac{\partial^k}{\partial z^k} \frac{\partial^l}{\partial \bar{w}^l} \frac{f_A(z) - f_A(\bar{w})}{z - \bar{w}} x, y \right), \tag{3.5}$$

and differentiation of the function in the right side leads to

$$\begin{aligned} &\sum_{h=0}^{k} \binom{k}{h} \frac{(-1)^{k-h}(k + l - h)!}{(z - \bar{w})^{k+l+1-h}} (f_A^{(h)}(z)x, y) \\ &\qquad + \sum_{h=0}^{l} \binom{l}{h} \frac{(-1)^{l-h}(k + l - h)!}{(\bar{w} - z)^{k+l+1-h}} (f_A^{(h)}(\bar{w})x, y). \end{aligned}$$

Lemma 3.3. *Let $z_1, \dots, z_n \in \rho(A)$ and $x_1, \dots, x_n \in \mathfrak{H}$, then*

$$\{\textstyle\sum_{i=1}^n \phi_A(z_i)x_i, \sum_{i=1}^n z_i\phi_A(z_i)x_i\} \in A \text{ if } \sum_{i=1}^n x_i = 0. \tag{3.6}$$

Let $z \in \rho(A)$, $x \in \mathfrak{H}$, and $k \geq 1$, then

$$\left\{\phi_A^{(k)}(z)x, z\phi_A^{(k)}(z)x + k\phi_A^{(k-1)}(z)x\right\} \in A. \tag{3.7}$$

Let $z, w \in \rho(A)$, $x, y \in \mathfrak{H}$, and $k, l \geq 1$, then

$$(f_A^{(k)}(z)x, y) = (z - \bar{w})[\phi_A^{(k)}(z)x, \phi_A(w)y] + k[\phi_A^{(k-1)}(z)x, \phi_A(w)y], \tag{3.8}$$

and

$$\begin{aligned}(z - \bar{w})&[\phi_A^{(k)}(z)x, \phi_A^{(l)}(w)y] \\ &= l[\phi_A^{(k)}(z)x, \phi_A^{(l-1)}(w)y] - k[\phi_A^{(k-1)}(z)x, \phi_A^{(l)}(w)y].\end{aligned} \tag{3.9}$$

Proof. The resolvent identity implies that

$$(A - z_0)^{-1}\phi(z) = (A - z)^{-1}\Gamma,$$

which leads to the inclusion (3.6). By differentiation of (3.4) we obtain

$$\phi_A^{(k)}(z) = (z - z_0)\frac{d^k}{dz^k}(A - z)^{-1}\Gamma + k\frac{d^{k-1}}{dz^{k-1}}(A - z)^{-1}\Gamma.$$

With

$$\frac{d^k}{dz^k}(A - z)^{-1} = k!(A - z)^{-(k+1)},$$

this gives

$$\phi_A^{(k)}(z) = k!(z - z_0)(A - z)^{-(k+1)}\Gamma + k!(A - z)^{-k}\Gamma = k!(A - z)^{-k}\phi_A(z). \tag{3.10}$$

The inclusion (3.7) can be seen from (3.10). Since

$$f_A(z) = f_A(z_0)^* + \Gamma^*(z - \bar{z}_0)\phi_A(z),$$

it follows for $k \geq 1$ that,

$$(f_A^{(k)}(z)x, y) = \left((z - \bar{z_0})\phi_A^{(k)}(z)x + k\phi_A^{(k-1)}(z)x, \Gamma y\right), \tag{3.11}$$

so that (3.8) holds for $w = z_0$. The general case follows from the fact that the element in the left side of (3.7) belongs to A and that, according to (3.6), also the element

$$\{\phi_A(w)y - \phi_A(z_0)y, w\phi_A(w)y - z_0\phi_A(z_0)y\}$$

belongs to A. Since A is selfadjoint, these elements are adjoint which in conjunction with (3.11) implies (3.8). Similarly, (3.9) follows from (3.7) and

$$\left\{\phi_A^{(l)}(w)y, w\phi_A^{(l)}(w)y + l\phi_A^{(l-1)}(w)y\right\} \in A.$$

□

4. Induced models

Let $f \in \mathbf{N}_\nu^\pi(\mathfrak{H})$ where $\nu, \pi \in \overline{\mathbb{N}}$, not both equal to ∞. We will show that there exist a Pontryagin space $\mathfrak{P}$, a selfadjoint relation A in $\mathfrak{P}$ with a nonempty resolvent set, and a mapping $\Gamma \in \mathbf{L}(\mathfrak{H}, \mathfrak{P})$ such that $f = f_A$ as in (3.1). Define a linear space $\mathfrak{L}_f$ of formal finite sums by

$$\mathfrak{L}_f = \{ \textstyle\sum_{z\in\rho(f)} x_z e_z : x_z \in \mathfrak{H},\, x_z = 0 \text{ for almost all } z \},$$

and provide $\mathfrak{L}_f$ with an inner product given by

$$[xe_z, ye_w] = (\mathsf{N}_f(z,w)x, y), \quad z, w \in \rho(f), \quad x, y \in \mathbb{C}.$$

Then $\mathfrak{L}_f$ is an inner product space with ν negative and π positive squares. Convergence in $\mathfrak{L}_f$ is defined as in Section 2. Define the (graph of the) operator S_f in $\mathfrak{L}_f$ by

$$S_f = \{ \{\textstyle\sum_{z\in\rho(f)} x_z e_z, \sum_{z\in\rho(f)} z x_z e_z\} : \sum_{z\in\rho(f)} x_z = 0 \},$$

so that S_f stands for pointwise multiplication.

Lemma 4.1. *The operator S_f in $\mathfrak{L}_f$ is symmetric without eigenvalues in $\rho(f)$. For each $z \in \rho(f)$, the range* $\operatorname{ran}(S_f - z)$ *is dense in $\mathfrak{L}_f$. Moreover, for all $z, z_0 \in \rho(f)$ and all $x, y \in \mathfrak{H}$:*

(4.1) $$(f(z)x, y) = (f(z_0)^* x, y) + (z - \bar{z}_0)[(I + (z - z_0)(S_f - z)^{-1})xe_{z_0}, ye_{z_0}].$$

Proof. The operator S_f is symmetric, since it follows from $\sum_{z\in\rho(f)} x_z = 0$ and $\sum_{w\in\rho(f)} y_w = 0$, that

$$\begin{aligned}
&[\textstyle\sum_{z\in\rho(f)} z x_z e_z, \sum_{w\in\rho(f)} y_w e_w] - [\sum_{z\in\rho(f)} x_z e_z, \sum_{w\in\rho(f)} w y_w e_w] \\
&\quad = \textstyle\sum_{z,w\in\rho(f)} (z - \bar{w})[x_z e_z, y_w e_w] \\
&\quad = \textstyle\sum_{z,w\in\rho(f)} ((f(z)x_z, y_w) - (x_z, f(w)y_w)) \\
&\quad = (\textstyle\sum_{z\in\rho(f)} f(z)x_z, \sum_{w\in\rho(f)} y_w) - (\sum_{z\in\rho(f)} x_z, \sum_{w\in\rho(f)} f(w)y_w) \\
&\quad = 0.
\end{aligned}$$

The operator $S_f - z$, $z \in \rho(f)$, is injective, as no single component e_z is included in $\operatorname{dom} S_f$. It follows directly from the definition that

(4.2) $$(S_f - z)^{-1} xe_w = \frac{xe_z - xe_w}{z - w}, \quad z \neq w.$$

The relation (4.2) implies that for $z, z_0 \in \rho(f)$, $z \neq \bar{z}_0$,

$$[(I + (z - z_0)(S_f - z)^{-1})xe_{z_0}, ye_{z_0}] = [xe_z, ye_{z_0}] = \left(\frac{f(z) - f(z_0)^*}{z - \bar{z}_0} x, y \right).$$

This gives (4.1). Furthermore, we obtain for each $z_0 \in \rho(f)$,

$$\mathfrak{L}_f = \operatorname{span}\{\, xe_{z_0}, xe_{z_0} + (z-z_0)(S_f - z)^{-1} xe_{z_0} :\, z \in \rho(f) \setminus \{z_0\}, x \in \mathfrak{H}\,\}. \tag{4.3}$$

We now show that $\operatorname{ran}(S_f - z)$ is dense in $\mathfrak{L}_f$. It follows from (4.2) that

$$\operatorname{ran}(S_f - z) = \operatorname{span}\{\, xe_w :\, w \in \rho(f) \setminus \{z\}, x \in \mathfrak{H}\,\}. \tag{4.4}$$

Let $z_n, z \in \rho(f)$, $z_n \to z$ in $\mathbb{C}$ and let $x_n, x \in \mathfrak{H}$, $x_n \to x$ in the norm of $\mathfrak{H}$. We claim that

$$x_n e_{z_n} \to x e_z \text{ in } \mathfrak{L}_f. \tag{4.5}$$

Then (4.5) and (4.4) imply that $\operatorname{ran}(S_f - z)$ is dense in $\mathfrak{L}_f$. To see (4.5), first note that $z_n \to z$ implies that

$$\frac{f(z_n) - f(z_0)^*}{z_n - \bar{z}_0} \to \frac{f(z) - f(z_0)^*}{z - \bar{z}_0} \text{ or } f'(\bar{z}_0),$$

depending on $z \neq \bar{z}_0$ or $z = \bar{z}_0$, respectively. Moreover, this sequence is uniformly bounded in the operator norm. Therefore, we conclude

$$\begin{aligned} [x_n e_{z_n}, y e_{z_0}] &= \left(\frac{f(z_n) - f(z_0)^*}{z_n - \bar{z}_0}\, x, y\right) + \left(\frac{f(z_n) - f(z_0)^*}{z_n - \bar{z}_0}\, (x_n - x), y\right) \\ &\to [xe_z, ye_{z_0}], \end{aligned}$$

and

$$\begin{aligned} [x_n e_{z_n}, x_n e_{z_n}] &= \left(\frac{f(z_n) - f(z_n)^*}{z_n - \bar{z}_n}\, x, x\right) \\ &\quad + \left(\frac{f(z_n) - f(z_n)^*}{z_n - \bar{z}_n}\, (x_n - x), x\right) \\ &\quad + \left(\frac{f(z_n) - f(z_n)^*}{z_n - \bar{z}_n}\, x, x_n - x\right) \\ &\quad + \left(\frac{f(z_n) - f(z_n)^*}{z_n - \bar{z}_n}\, (x_n - x), x_n - x\right) \\ &\to \left(\frac{f(z) - f(z)^*}{z - \bar{z}} x, x\right) \\ &= [xe_z, xe_z]. \end{aligned}$$

According to Section 2 these limiting relations imply (4.5). □

Theorem 4.2. *Let $f \in \mathbf{N}_\nu^\pi(\mathfrak{H})$ where $\nu, \pi \in \overline{\mathbb{N}}$, not both equal to ∞. Then there exist a Pontryagin space $\mathfrak{P}_f$ with indices (ν, π), a mapping $\Gamma \in \mathbf{L}(\mathfrak{H}, \mathfrak{P}_f)$, and a Γ-minimal selfadjoint relation A_f in $\mathfrak{P}_f$ with $\rho(A_f) \cap (\mathbb{C} \setminus \mathbb{R}) = \rho(f)$, such that*

$$f(z) = f(z_0)^* + (z - \bar{z}_0)\Gamma^*(I + (z - z_0)(A_f - z)^{-1})\Gamma, \quad z, z_0 \in \rho(f). \tag{4.6}$$

Proof. Denote the completion of $\mathfrak{L}_f$ by $\mathfrak{P}_f$. Since $f \in \mathbf{N}_\nu^\pi(\mathfrak{H})$, the indices of $\mathfrak{P}_f$ are given by (ν, π). By Lemma 4.1, S_f is a symmetric operator in $\mathfrak{L}_f$ and its range $\operatorname{ran}(S_f - z)$ is dense in $\mathfrak{L}_f$ for each $z \in \rho(f)$. Hence by Theorem 2.2, S_f in $\mathfrak{L}_f$ induces a selfadjoint relation A_f in $\mathfrak{P}_f$. The relation A_f has a nonempty resolvent set; in fact, $\rho(f) \subseteq \rho(A_f)$. Define the mapping $\Gamma : \mathfrak{H} \to \mathfrak{L}_f$ by

$$\Gamma x = x e_{z_0}, \quad x \in \mathfrak{H}. \tag{4.7}$$

For $x, y \in \mathfrak{H}$ it follows that

$$[\Gamma x, \Gamma x] = [x e_{z_0}, x e_{z_0}] = (\mathsf{N}_f(z_0, z_0)x, x), \qquad [\Gamma x, y e_w] = (\mathsf{N}_f(z_0, w)x, y).$$

Hence the operator Γ is bounded and by

$$\Gamma x = \widehat{x e}_{z_0}, \quad x \in \mathfrak{H},$$

it can be viewed as $\Gamma \in \mathbf{L}(\mathfrak{H}, \mathfrak{P})$. Due to (4.3), A_f is Γ-minimal. We have

$$[a, \widehat{y e}_{z_0}] = [a, \Gamma y] = (\Gamma^* a, y), \quad y \in \mathfrak{H}, \quad a \in \mathfrak{P}_f,$$

and, in particular,

$$\frac{\operatorname{Im} f(z_0)}{\operatorname{Im} z_0} = \frac{f(z_0) - f(z_0)^*}{z - \bar{z}_0} = \Gamma^* \Gamma.$$

Now the identity (4.6) follows from the corresponding identity (4.1). In other words, $f = f_{A_f}$ and we have $\rho(A_f) \cap (\mathbb{C} \setminus \mathbb{R}) = \rho(f)$. □

Note that if a selfadjoint relation A is Γ-minimal, then $\rho(f_A) = \rho(A) \cap (\mathbb{C} \setminus \mathbb{R})$.

5. A classical Nevanlinna-Pick interpolation problem

The classical Nevanlinna-Pick interpolation problem which we consider here is to find all solutions $f \in \mathbf{N}_\nu^\pi(\mathfrak{H})$ of

$$z_i \in \rho(f), \quad f(z_i) = W_i, \quad i = 1, \dots, n, \tag{5.1}$$

when for some $n \in \mathbb{N}$ the data $z_1, \dots, z_n \in \mathbb{C}^+$ and $W_1, \dots, W_n \in \mathbf{L}(\mathfrak{H})$ are given. In order to describe the solutions of this problem we will follow the approach used by Kreĭn and Langer [15]. Define the linear space $\mathfrak{G}$ of formal finite sums by

$$\mathfrak{G} = \{ \textstyle\sum_{i=1}^n x_i e_i : x_i \in \mathfrak{H} \},$$

and provide $\mathfrak{G}$ with an inner product given by

$$[x e_i, y e_j] = \left(\frac{W_i - W_j^*}{z_i - \bar{z}_j} x, y \right), \quad x, y \in \mathfrak{H}, \quad i, j = 1, \dots, n.$$

Define the (graph of the) operator T in $\mathfrak{G}$ by

$$T = \{ \{\sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} z_i x_i e_i\} : \sum_{i=1}^{n} x_i = 0 \}.$$

Clearly, T is a symmetric operator in $\mathfrak{G}$, since it follows from $\sum_{i=1}^{n} x_i = 0$ and $\sum_{j=1}^{n} y_j = 0$, that

$$\begin{aligned}
&[\sum_{i=1}^{n} z_i x_i e_i, \sum_{j=1}^{n} y_j e_j] - [\sum_{i=1}^{n} x_i e_i, \sum_{j=1}^{n} z_j y_j e_j] \\
&\quad = \sum_{i,j=1}^{n} (z_i - \bar{z}_j)[x_i e_i, y_j e_j] \\
&\quad = \sum_{i,j=1}^{n} ((W_i - W_j^*) x_i, y_j) \\
&\quad = (\sum_{i=1}^{n} W_i x_i, \sum_{j=1}^{n} y_j) - (\sum_{i=1}^{n} x_i, \sum_{j=1}^{n} W_j y_j) \\
&\quad = 0.
\end{aligned}$$

Theorem 5.1. *Let $\nu, \pi \in \overline{\mathbb{N}}$, not both equal to ∞. The function $f \in \mathbf{N}_\nu^\pi(\mathfrak{H})$ satisfies* (5.1) *if and only if there exist a Pontryagin space $\mathfrak{P}$, a selfadjoint relation A in $\mathfrak{P}$, and an isometry $\Phi : \mathfrak{G} \to \mathfrak{P}$ with the following properties:*

(i) *the indices of $\mathfrak{P}$ are (ν, π);*

(ii) *$z_1, \dots, z_n \in \rho(A)$;*

(iii) *A is Γ-minimal, when $\Gamma \in \mathbf{L}(\mathfrak{H}, \mathfrak{P})$ and $\Gamma x = \Phi(x e_1)$, $x \in \mathfrak{H}$;*

(iv) *A extends $\Phi \circ T$,*

such that

$$f(z) = W_1^* + (z - \bar{z}_1)\Gamma^*(I + (z - z_1)(A - z)^{-1})\Gamma. \tag{5.2}$$

Proof. Let $\mathfrak{P}$, A, and Φ be given, such that (i)–(iv) hold. Hence the function f defined by (5.2) is of the form (3.1) with $z_0 = z_1$: $f(z) = f_A(z)$. It follows from (i), (ii), (iii), and Theorem 3.1 that $f \in \mathbf{N}_\nu^\pi(\mathfrak{H})$ with $z_1, \dots, z_n \in \rho(f)$. The equality (5.1) is equivalent to

$$\frac{W_i - W_1^*}{z_i - \bar{z}_1} = \Gamma^*(I + (z_i - z_1)(A - z_i)^{-1})\Gamma, \quad i = 1, \dots, n. \tag{5.3}$$

For $i = 1$, the identity (5.3) follows from

$$\left(\frac{W_1 - W_1^*}{z_1 - \bar{z}_1} x, y\right) = [xe_1, ye_1] = [\Phi(xe_1), \Phi(ye_1)] = [\Gamma x, \Gamma y] = (\Gamma^* \Gamma x, y),$$

when $x, y \in \mathfrak{H}$. For $i > 1$, the definition of T implies that

$$\left\{ xe_1, \frac{xe_i - xe_1}{z_i - z_1} \right\} \in (T - z_i)^{-1}.$$

It follows from (iv), that

$$\left\{\Gamma x, \Phi\left(\frac{xe_i - xe_1}{z_i - z_1}\right)\right\} = \left\{\Phi(xe_1), \Phi\left(\frac{xe_i - xe_1}{z_i - z_1}\right)\right\} \in (A - z_i)^{-1},$$

and, therefore, that

$$\Gamma^*(I + (z_i - z_1)(A - z_i)^{-1})\Gamma x = \Gamma^*\Phi(xe_i). \tag{5.4}$$

For all $y \in \mathfrak{H}$,

$$(\Gamma^*\Phi(xe_i), y) = [\Phi(xe_i), \Gamma y] = [\Phi(xe_i), \Phi(ye_1)] = [xe_i, ye_1] = \left(\frac{W_i - W_1^*}{z_i - \bar{z}_1}\, x, y\right),$$

so that (5.4) implies (5.3). Hence $f = f_A$ satisfies (5.1).

Conversely, let $f \in \mathbf{N}_\nu^\pi(\mathfrak{H})$ be a function with $z_1, \ldots, z_n \in \rho(f)$. Let $\mathfrak{P}$, A, and Γ be as in Theorem 4.2, so that $f = f_A$ with $z_0 = z_1$ in (3.1), cf. (3.2). This establishes (i), (ii), and the first statement of (iii), that A is Γ-minimal. Define $\Phi : \mathfrak{G} \to \mathfrak{P}$ by

$$\Phi(xe_i) = \widehat{xe}_{z_i},$$

so that

$$[\Phi(xe_i), \Phi(ye_j)] = [\widehat{xe}_{z_i}, \widehat{ye}_{z_j}] = [xe_{z_i}, ye_{z_j}] = \left(\frac{f(z_i) - f(z_j)^*}{z_i - \bar{z}_j}\, x, y\right).$$

Hence, if in addition f satisfies (5.1), the right side equals $[xe_i, ye_j]$ and Φ is an isometry from $\mathfrak{G}$ to $\mathfrak{P}$. Moreover, from (4.7) with $z_0 = z_1$ it follows that $\Gamma x = \Phi(xe_1)$, which completes the proof of (iii). The definition of T and A imply (iv). □

6. A multiple point Nevanlinna-Pick interpolation problem

The multiple point Nevanlinna-Pick interpolation problem is to find all solutions $f \in \mathbf{N}_\nu^\pi(\mathfrak{H})$ of

$$z_i \in \rho(f), \quad f^{(k)}(z_i) = W_{ik}, \quad i = 1, \ldots, n, \quad k = 0, \ldots, k_i, \tag{6.1}$$

when for some $n \in \mathbb{N}$ the data $z_1, \ldots, z_n \in \mathbb{C}^+$, $k_1, \ldots, k_n \in \mathbb{N} \cup \{0\}$, and $W_{ik} \in \mathbf{L}(\mathfrak{H})$, $i = 1, \ldots, n$, $k = 0, \ldots, k_i$, are given. Closely connected to (6.1) is the following classical Nevanlinna-Pick problem: to find all solutions $f \in \mathbf{N}_\nu^\pi(\mathfrak{H})$ of

$$z_i \in \rho(f), \quad f(z_i) = W_{i0}, \quad i = 1, \ldots, n. \tag{6.2}$$

In order to solve (6.2) we define the linear space $\mathfrak{G}_0$ as the set of all finite sums

$$\mathfrak{G}_0 = \left\{ \textstyle\sum_{i=1}^n x_{i0} e_{i0} : x_{i0} \in \mathfrak{H} \right\},$$

provided with an inner product by

$$[x_{i0}e_{i0}, y_{j0}e_{j0}] = \left(\frac{W_{i0} - W_{j0}^*}{z_i - \bar{z}_j} \, x_{i0}, y_{j0} \right).$$

In $\mathfrak{G}_0$ the operator T_0 defined by

$$T_0 = \{ \{ \textstyle\sum_{i=1}^n x_{i0}e_{i0}, \sum_{i=1}^n z_i x_{i0} e_{i0} \} : \sum_{i=1}^n x_{i0} = 0 \}$$

is symmetric. In order to solve (6.1) we extend the space $\mathfrak{G}_0$ to the linear space $\mathfrak{G}$ of formal finite sums by

$$\mathfrak{G} = \left\{ \textstyle\sum_{i=1}^n \sum_{k=0}^{k_i} x_{ik} e_{ik} \, : \, x_{ik} \in \mathfrak{H} \right\},$$

and provide $\mathfrak{G}$ with the inner product inductively defined by

$$[xe_{ik}, ye_{jl}] = \frac{l[xe_{ik}, ye_{j,l-1}] - k[xe_{i,k-1}, ye_{jl}]}{z_i - \bar{z}_j}, \quad k, l \geq 1,$$

$$[xe_{ik}, ye_{j0}] = \frac{(W_{ik}x, y) - k[xe_{i,k-1}, ye_{j0}]}{z_i - \bar{z}_j}, \quad k \geq 1,$$

and

$$[xe_{i0}, ye_{jl}] = \frac{l[xe_{i0}, ye_{j,l-1}] - (x, W_{jl}y)}{z_i - \bar{z}_j}, \quad l \geq 1.$$

In $\mathfrak{G}$ we extend T_0 to the operator T by

$$\left\{ \left\{ \textstyle\sum_{i=1}^n \sum_{k=0}^{k_i} x_{ik} e_{ik}, \sum_{i=1}^n \sum_{k=0}^{k_i} (z_i x_{ik} e_{ik} + k x_{i,k-1} e_{i,k-1}) \right\} : \sum_{i=1}^n x_{i0} = 0 \right\},$$

where we formally put $x_{i,-1} = 0$, so that

$$T(\textstyle\sum_{i=1}^n x_{i0} e_{i0}) = \sum_{i=1}^n z_i x_{i0} e_{i0} \text{ if } \sum_{i=1}^n x_{i0} = 0,$$

and

$$T(x_{ik}e_{ik}) = z_i x_{ik} e_{ik} + k x_{ik} e_{i,k-1}, \quad k \geq 1.$$

The operator T is symmetric, since for $k, l \geq 1$,

$$\begin{aligned} &[T(x_{ik}e_{ik}), y_{jl}e_{jl}] - [x_{ik}e_{ik}, T(y_{jl}e_{jl})] \\ &\quad = (z_i - \bar{z}_j)[x_{ik}e_{ik}, y_{jl}e_{jl}] + k[xe_{i,k-1}, y_{jl}e_{jl}] - l[x_{ik}e_{ik}, y_{jl}e_{j,l-1}] \\ &\quad = 0, \end{aligned}$$

while, for instance for $l \geq 1$, $\sum_{i=1}^n x_{i0} = 0$ implies that

$$\begin{aligned} &[\, T(\textstyle\sum_{i=1}^n x_{i0}e_{i0}), y_{jl}e_{jl} \,] - [\, \sum_{i=1}^n x_{i0}e_{i0}, T(y_{jl}e_{jl}) \,] \\ &\quad = \textstyle\sum_{i=1}^n \left((z_i - \bar{z}_j)[x_{i0}e_{i0}, y_{jl}e_{jl}] - [x_{i0}e_{i0}, l y_{jl} e_{j,l-1}] \right) \\ &\quad = -(\textstyle\sum_{i=1}^n x_{i0}, W_{jl} y_{jl}) \\ &\quad = 0, \end{aligned}$$

due to the definition of the inner product. Finally, note that the explicit form of the inner product $[\cdot,\cdot]$ is given by

$$
\begin{aligned}
(6.3)\quad [xe_{ik}, ye_{jl}] \;=\; & \sum_{h=0}^{k} \binom{k}{h} \frac{(-1)^{k-h}(k+l-h)!}{(z_i-\bar z_j)^{k+l+1-h}}\,(W_{ih}x,y) \\
& + \sum_{h=0}^{l} \binom{l}{h} \frac{(-1)^{l-h}(k+l-h)!}{(\bar z_j-z_i)^{k+l+1-h}}\,(W_{jh}^*x,y), \quad x,y\in\mathfrak{H}.
\end{aligned}
$$

Theorem 6.1. *Let $\nu,\pi \in \overline{\mathbb{N}}$, not both equal to ∞. The function $f \in \mathbf{N}_\nu^\pi(\mathfrak{H})$ satisfies* (6.1) *if and only if there exist a Pontryagin space $\mathfrak{P}$, a selfadjoint relation A in $\mathfrak{P}$, and an isometry $\Phi : \mathfrak{G} \to \mathfrak{P}$ with the following properties:*

(i) *the indices of $\mathfrak{P}$ are (ν,π);*

(ii) *$z_1,\dots,z_n \in \rho(A)$;*

(iii) *A is Γ-minimal, when $\Gamma \in \mathbf{L}(\mathfrak{H},\mathfrak{P})$ and $\Gamma x = \Phi(xe_{10})$, $x\in\mathfrak{H}$;*

(iv) *A extends $\Phi\circ T$,*

such that

$$
(6.4)\qquad f(z) = W_{10}^* + (z-\bar z_1)\Gamma^*(I+(z-z_1)(A-z)^{-1})\Gamma.
$$

Proof. Let A, $\mathfrak{P}$, and Φ be given such that (i)–(iv) hold. Hence the function f defined by (6.4) is of the form (3.1) with $z_0 = z_1$: $f(z) = f_A(z)$. It follows from (i), (ii), (iii), and Theorem 3.1 that $f \in \mathbf{N}_\nu^\pi(\mathfrak{H})$ with $z_1,\dots,z_n \in \rho(f)$. The inclusion $T_0 \subset T$ and Theorem 5.1 imply that (6.2) holds, i.e. (6.1) holds for $k=0$. Now assume that $k \geq 1$. By the definition of T,

$$
\{xe_{ik}, z_i xe_{ik} + kxe_{i,k-1}\} \in T,
$$

and it follows from (iv) that

$$
\{\Phi(kxe_{i,k-1}), \Phi(xe_{ik})\} \in (A-z_i)^{-1}.
$$

Hence for all $k \geq 1$,

$$
k(A-z_i)^{-1}\Phi(xe_{i,k-1}) = \Phi(xe_{ik}),
$$

which implies that

$$
(6.5)\qquad k!(A-z_i)^{-k}\Phi(xe_{i,0}) = \Phi(xe_{ik}).
$$

Since $T_0 \subset T$, it follows as in (5.4) that $(I+(z_i-z_1)(A-z_i)^{-1})\Gamma x = \Phi(xe_{i,0})$. Substituting this into (6.5) and using (3.10) we observe,

$$
\phi_A^{(k)}(z_i)x = \Phi(xe_{ik}).
$$

Hence according to (3.11) with $z = z_i$ and $z_0 = z_1$,

$$\begin{aligned}(f_A^{(k)}(z_i)x, y) &= (z_i - \bar{z}_1)[\phi_A^{(k)}(z_i)x, \Gamma y] + k[\phi_A^{(k-1)}(z_i)x, \Gamma y] \\ &= (z_i - \bar{z}_1)[\Phi(xe_{ik}), \Phi(ye_{10})] + k[\Phi(xe_{i,k-1}), \Phi(ye_{10})] \\ &= (z_i - \bar{z}_1)[xe_{ik}, ye_{10}] + k[xe_{i,k-1}, ye_{10}].\end{aligned}$$

Therefore, using the definition of the inner product in $\mathfrak{G}$, we conclude that

$$(f_A^{(k)}(z_i)x, y) = (W_{ik}x, y).$$

Hence $f = f_A$ satisfies (6.1).

Conversely, assume that a function $f \in \mathbf{N}_\nu^\pi(\mathfrak{H})$ satisfies (6.1) with $z_1, \dots, z_n \in \rho(f)$. Let $\mathfrak{P}$, A, and Γ be as in Theorem 4.2, so that $f = f_A$ with $z_0 = z_1$ in (3.1), cf. (3.2). This establishes (i), (ii), and the first statement of (iii), that A is Γ-minimal. Define the corresponding function $\phi_A(z)$ as in (3.4) and define $\Phi : \mathfrak{G} \to \mathfrak{P}$ by

$$\Phi(xe_{ik}) = \phi_A^{(k)}(z_i)x.$$

Since f is a solution of (6.1), it follows from Lemma 3.3 with $z = z_i$ and $w = z_j$ and the definition of the inner product in $\mathfrak{G}$, that

$$[\phi_A^{(k)}(z_i)x, \phi_A^{(l)}(z_j)y] = [xe_{ik}, ye_{jl}],$$

and hence Φ is isometric. Clearly, $\Gamma x = \Phi(xe_{10})$, which completes the proof of (iii). Since, in particular, f satisfies (6.2), Theorem 5.1 shows that $\Phi \circ T_0 \subset A$. Now let $k \geq 1$ and let $\{xe_{ik}, z_i xe_{ik} + kxe_{i,k-1}\} \in T$. Then

$$\{\Phi(xe_{ik}), \Phi(z_i xe_{ik} + kxe_{i,k-1})\} = \{\phi_A^{(k)}x, z_i\phi_A^{(k)}(z_i)x + k\phi_A^{(k-1)}(z_i)x\} \in A,$$

by (3.7) of Lemma 3.3. By the definition of T we conclude that $\Phi \circ T \subset A$, showing (iv). □

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1991 Mathematics Subject Classification. Primary 30E05, 47A57; Secondary 47B25, 47B50

Received Date inserted by the Editor