GENERALIZED RESOLVENT MATRICES AND SPACES OF ANALYTIC FUNCTIONS

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We introduce triplet spaces for symmetric relations with defect index $(1,1)$ in a Pontryagin space. Representations of Pontryagin spaces by spaces of vector-valued analytic functions are investigated. These concepts are used to study 2×2-matrix valued analytic functions which satisfy a certain kernel condition.

1 Introduction

If $\mathfrak{P}$ is a Pontryagin space, S is a densely defined symmetric operator in $\mathfrak{P}$ with defect index $(1,1)$, A is a selfadjoint extension of S acting in some Pontryagin space extending $\mathfrak{P}$, and u is an element of $\mathfrak{P}$, then the function

$$r(z) = [(A-z)^{-1}u, u], \; z \in \rho(A), \tag{1.1}$$

is called the u-resolvent of S induced by A. It is proved in [KL2] that the set of all u-resolvents of S can be parametrized by

$$r(z) = \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_{21}(z)\tau(z) + w_{22}(z)}, \tag{1.2}$$

with a parameter function $\tau(z)$ and certain analytic functions $w_{11}, w_{12}, w_{21}, w_{22}$. A matrix

$$W(z) = \begin{pmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{pmatrix}$$

such that (1.2) parametrizes the u-resolvents of S is called a u-resolvent matrix of S. It is shown in [KL2] that W is uniquely determined up to some simple transformations. Moreover, if W is a u-resolvent matrix, the kernel

$$\frac{W(z)JW(w)^* - J}{z - \overline{w}} \tag{1.3}$$

has finitely many negative squares. If the entries of $W(z)$ satisfy certain growth conditions, then in [KL2] also a converse result is shown.

In this paper we show that the above results hold without additional assumptions, if we allow u to be a so-called generalized element and S (A) to be a symmetric (selfadjoint) relation. Since we use generalized elements u, we have to consider regularized resolvents in (1.1). Hence we will speak of generalized u-resolvents and, correspondingly, of generalized u-resolvent matrices.

With this more general notion we show in particular that any analytic 2×2-matrix function $W(z)$, such that the kernel (1.3) has finitely many negative squares is a generalized u-resolvent matrix for some choice of $\mathfrak{P}$, S and u.

If $\mathfrak{P}$ is a Hilbert space and S is a densely defined operator, similar results are obtained e.g. in [GG], [HS], [S2].

In Section 2 we collect some preliminary material which is well known and has been extensively studied in particular situations. It will however be needed in a general setting. In Section 3 we introduce the notions of so called triplet spaces and of generalized elements. Also the action of resolvents on generalized elements is studied; in particular a version of the Krein formula is obtained.

Section 4 is devoted to the study of generalized u-resolvents and generalized u-resolvent matrices. We show that, for given $\mathfrak{P}$, S and u, there exists a generalized u-resolvent matrix $W(z)$ and that, if S satisfies a certain regularity condition, $W(z)$ is uniquely determined up to some simple transformations.

In Section 5 we construct, for given $\mathfrak{P}$, S and u, an isomorphism of $\mathfrak{P}$ onto a space $\mathfrak{P}_u$ which consists of pairs of analytic functions. Thereby the relation S corresponds to a certain symmetric relation, which is closely related to the operator of multiplication by the independent variable. The space $\mathfrak{P}_u$ is identified as the reproducing kernel Pontryagin space with the matrix kernel

$$\frac{W(z)JW(w)^* - J}{z - \overline{w}},$$

where $W(z)$ is an appropriately chosen generalized u-resolvent matrix, and

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We show that the difference quotient operator defined by

$$\mathcal{R}_1(a) : \mathbf{f}(z) \mapsto \frac{\mathbf{f}(z) - \mathbf{f}(a)}{z - a}, \ \mathbf{f} \in \mathfrak{P}_u,$$

acts as a bounded linear operator on $\mathfrak{P}_u$. If $u \in \mathfrak{P}$ and S is a densely defined symmetric operator, some of the results of Sections 4 and 5 can be found in [KL2]. Compare also the book [GG] and the reference to some work of M.G.Krein given there.

Section 6 is concerned with the study of arbitrary analytic 2×2-matrix functions $W(z)$, which are such that the kernel (1.3) has finitely many negative squares. We show that any such matrix function $W(z)$ can be represented as a generalized u-resolvent matrix. This result implies that the assertions stated for the spaces $\mathfrak{P}_u$ hold for any reproducing kernel space generated by a kernel of the form (1.3). In particular any such reproducing kernel space is closed under application of $\mathcal{R}_1(a)$.

Finally, in Section 7, a relation between generalized u-resolvent matrices of S and of a symmetric extension S_1 of S is investigated. More precisely, we consider the following situation: let two Pontryagin spaces $\mathfrak{P}$ and $\mathfrak{P}_1$ and two symmetric relations $S \subseteq \mathfrak{P}^2$ and $S_1 \subseteq \mathfrak{P}_1^2$, both with defect index $(1,1)$, be given and assume that $\mathfrak{P} \subseteq \mathfrak{P}_1$ and $S \subseteq S_1$. If u is a generalized element of $\mathfrak{P}$, we show that u can be considered as a generalized element of $\mathfrak{P}_1$. Hence it makes sense to speak of generalized u-resolvent matrices $W(z)$ and $W_1(z)$ of S and S_1, respectively. We show that there exists a matrix $M(z)$, which satisfies

$$W_1(z) = W(z)M(z),$$

and for which the kernel (1.3) has finitely many negative squares. The matrix $M(z)$ does not depend on the choice of u.

We will use some results concerning the geometry of Pontryagin spaces and their symmetric (selfadjoint) relation, which are provided in [DS] and [IKL]. Also our notation is similar to that of these references. Further standard references are [ADSR] and [KL2].

The results considered in this paper applied to entire operators in the sense of M.G.Krein are closely connected with the theory of Hilbert spaces of entire functions developed in [dB]. Since our results are valid for Pontryagin spaces, this connection leads us to a generalization of L.de Branges theory of spaces of entire functions to indefinite inner product spaces (see [KW]).

2 Some preliminary results

In this section we collect some results concerning symmetric relations in Pontryagin spaces, which will be used in the sequel. In particular we give the Krein formula on the description of generalized resolvents in a general setting.

In various special situations these results are well known. Compare for example [DLS1], [DLS2] or [GG].

Let $(\mathfrak{P}, [.,.])$ be a Pontryagin space and let S be a closed symmetric relation in $\mathfrak{P}$, i.e. a closed subspace of $\mathfrak{P}^2$ with the property

$$[f_1, g_2] - [g_1, f_2] = 0,\ (f_1; g_1), (f_2; g_2) \in S.$$

We call a point $z \in \mathbb{C}$ a point of regular type of S, if for some $\gamma > 0$

$$\|f\| \leq \gamma \|g - zf\|,\ (f; g) \in S,$$

holds. Here $\|.\|$ denotes a positive definite norm on $\mathfrak{P}$ which is induced by a fundamental symmetry. Denote by $r(S)$ the set of all points $z \in \mathbb{C}$, such that z and $\overline{z}$ are points of regular type of S. It is proved in [DS] that $r(S)$ contains $\mathbb{C} \setminus \mathbb{R}$ with exception of those points z, such that z or $\overline{z}$ is an eigenvalue of S.

Throughout the following we assume $r(S) \neq \emptyset$. It is well known that the number $\operatorname{def}(S - z)$, i.e. the codimension of $\operatorname{ran}(S - z)$ is constant on each connected component of $r(S)$. Since the upper (lower) half plane with exception of the eigenvalues of S is contained in a single component of $r(S)$ the number $\operatorname{def}(S - z)$ is constant on the upper (lower) half plane with possible exception of finitely many points, which must be eigenvalues of S. Denote this number by n_+ (n_-), the so called defect numbers of S. The pair (n_+, n_-) is called the defect index of S. For an inner product space $\mathfrak{A}$, denote by $\mathfrak{A}^\circ$ its isotropic part.

Lemma 2.1. *Assume that S has finite and equal defect numbers and let $z \in \mathbb{C}$. Then $z \in r(S)$ if and only if there exists a canonical selfadjoint extension $A \subseteq \mathfrak{P}^2$ of S with $z \in \rho(A)$.*

Proof : If A is a canonical selfadjoint extension of S such that $z \in \rho(A)$, then clearly $z \in r(S)$.

Assume now that $z \in r(S) \setminus \mathbb{R}$. Then also $\overline{z} \in r(S)$, the numbers z and $\overline{z}$ are not eigenvalues of S, and $\operatorname{ran}(S-z)$ and $\operatorname{ran}(S-\overline{z})$ are closed subspaces of $\mathfrak{P}$. The Cayley transform $V = I + (z-\overline{z})(S-z)^{-1}$ of S (compare [DS]) is an injective isometric operator of $\operatorname{ran}(S-z)$ onto $\operatorname{ran}(S-\overline{z})$. Hence V maps $\operatorname{ran}(S-z)^\circ$ onto $\operatorname{ran}(S-\overline{z})^\circ$, in particular $\dim\operatorname{ran}(S-z)^\circ = \dim\operatorname{ran}(S-\overline{z})^\circ$. Decompose $\mathfrak{P}$ as

$$\mathfrak{P} = \mathfrak{P}_1[\dot{+}](\mathfrak{P}_2\dot{+}\mathfrak{P}_2')[\dot{+}]\mathfrak{P}_3,$$

where $\mathfrak{P}_2 = \operatorname{ran}(S-z)^\circ$, $\mathfrak{P}_1 + \mathfrak{P}_2 = \operatorname{ran}(S-z)$ and $\mathfrak{P}_2'$ is skewly linked with $\mathfrak{P}_2$ (compare [IKL]). Put $\mathfrak{Q}_1 = V\mathfrak{P}_1$, $\mathfrak{Q}_2 = V\mathfrak{P}_2 = \operatorname{ran}(S-\overline{z})^\circ$, and decompose $\mathfrak{P}$ as

$$\mathfrak{P} = \mathfrak{Q}_1[\dot{+}](\mathfrak{Q}_2\dot{+}\mathfrak{Q}_2')[\dot{+}]\mathfrak{Q}_3,$$

where $\mathfrak{Q}_2'$ is skewly linked with $\mathfrak{Q}_2$. Then $\mathfrak{Q}_1 + \mathfrak{Q}_2 = \operatorname{ran}(S-\overline{z})$. Since S has equal defect numbers $\dim\mathfrak{P}_3 = \dim\mathfrak{Q}_3$, and since V is isometric $\operatorname{Ind}_-\mathfrak{P}_3 = \operatorname{Ind}_-\mathfrak{Q}_3$. Choose an isometry V_3 of $\mathfrak{P}_3$ onto $\mathfrak{Q}_3$ and extend $V|_{\mathfrak{P}_2}$ to an isometry V_2 of $\mathfrak{P}_2\dot{+}\mathfrak{P}_2'$ onto $\mathfrak{Q}_2\dot{+}\mathfrak{Q}_2'$. The existence of such a mapping V_2 is seen by choosing appropriate bases of $\mathfrak{P}_2\dot{+}\mathfrak{P}_2'$ and $\mathfrak{Q}_2\dot{+}\mathfrak{Q}_2'$. The mapping

$$U = V|_{\mathfrak{P}_1}[\dot{+}]V_2[\dot{+}]V_3$$

is a unitary operator of $\mathfrak{P}$ onto itself which extends V. Its inverse Cayley transform is selfadjoint, extends S and contains z in its resolvent set.

For $z \in r(S) \cap \mathbb{R}$ consider the symmetric and bounded operator $(S-z)^{-1}$, which is defined on a domain with codimension one. Let G be a fundamental symmetry of $\mathfrak{P}$. Then $G(S-z)^{-1}$ is a bounded symmetric operator in the Hilbert space $(\mathfrak{P}, [G.,.])$. It is well known (see [AG]) that there exists a bounded operator B, which is selfadjoint with respect to the inner product $[G.,.]$, and which extends $G(S-z)^{-1}$. Thus $(GB)^{-1} + z$ is a selfadjoint relation with respect to the inner product $[.,.]$, which extends S and contains z in its resolvent set.

□

The set $\Delta(S)$ is the set of those points $z \in r(S)$, such that $\operatorname{ran}(S-z)$ is degenerated. It is proved in [DS] that $\Delta(S)$ lies symmetric with respect to the real axis and that, if $\Delta(S) \neq r(S)$, it contains no interior point and its complement $\mathbb{C} \setminus \Delta(S)$ is open. If $\operatorname{dom} S$ contains a maximal negative subspace, then $\Delta(S)$ is contained in a strip around the real axis (see [So]). The relation S is called a standard symmetric relation if $\Delta(S) \neq r(S)$.

Lemma 2.2. *Assume that S has defect index $(1,1)$ and let $z \in r(S)$. If $z \notin \Delta(S) \cup \mathbb{R}$, then $z \in \rho(A)$ for each canonical selfadjoint extension A of S. If $z \in \Delta(S) \cup \mathbb{R}$, then there exists exactly one canonical selfadjoint extension A of S with $z \in \sigma(A)$. Moreover, if $\Delta(S) \neq r(S)$, then this extension has nonempty resolvent set.*

Proof : Since $z \in r(S)$ and S has defect index $(1,1)$, we have $\operatorname{def}(S-z) = 1$. Assume that $z \in \sigma(A)$ for a canonical selfadjoint extension A of S. Since $z \in r(S)$, we have $z \in \sigma_p(A)$. By

$$\ker(A-z) = \operatorname{ran}(A-\overline{z})^{\perp} \subseteq \operatorname{ran}(S-\overline{z})^{\perp}, \tag{2.1}$$

we find that $\dim \ker(A-z) = 1$ and that $\ker(A-z)$ is uniquely determined by S and z.

Let $f \in \ker(A-z)$, $f \neq 0$, then $(f; zf) \in A \setminus S$. Since S has defect index $(1,1)$, it follows by considering the linear isomorphism $B \mapsto (B-z)^{-1}$ of the set of all subspaces of $\mathfrak{P}^2$ onto itself, that $\dim A/S = 1$. Hence $A = S \dot{+} \operatorname{span}\{(f; zf)\}$, and is uniquely determined by S and z. If $z \notin \mathbb{R}$, $\ker(A-z)$ is neutral and by (2.1) we have $z \in \Delta(S)$.

If $z \in \Delta(S) \cup \mathbb{R}$, and $f \in \operatorname{ran}(S-z)^{\perp}$, then the relation defined by $A := S \dot{+} \operatorname{span}\{(f; \overline{z}f)\}$ is selfadjoint. Assume that its resolvent set is empty. The results of [DS] imply that we have $(\lambda f; \mu f) \in A$ for all $\lambda, \mu \in \mathbb{C}$. Moreover, the element f is neutral. Since, for $z \in r(S)$, we have $\operatorname{def}(S-z) = 1$, the relation (2.1) shows that $\Delta(S) = r(S)$.

□

Assume in the following that S has defect index (n,n), $n \in \mathbb{N}$. If A is a canonical selfadjoint extension of S, the mapping

$$U_{zw} = I + (z-w)(A-z)^{-1}, \ z, w \in \rho(A),$$

maps $\operatorname{ran}(S-z)$ onto $\operatorname{ran}(S-w)$. Hence $U_{zw}^* = U_{\overline{zw}}$ maps $\operatorname{ran}(S-w)^{\perp}$ onto $\operatorname{ran}(S-z)^{\perp}$. Here the adjoint and the orthogonal companion have to be understood with respect to the inner product $[.,.]$. Note that $U_{z_2z_3}U_{z_1z_2} = U_{z_1z_3}$, in particular U_{zw} is a bijection.

Let $z_0 \in r(S)$ and choose a selfadjoint extension A of S with $z_0 \in \rho(A)$. Let $\{\varphi_1(z_0), \ldots, \varphi_n(z_0)\}$ be a basis of the defect space at $\overline{z_0}$, i.e. let

$$\operatorname{span}\{\varphi_1(z_0), \ldots, \varphi_n(z_0)\} = \operatorname{ran}(S-\overline{z_0})^{\perp}.$$

For notational reasons we denote in the following by $\varphi(z_0)$ the vector

$$\varphi(z_0) = (\varphi_1(z_0), \ldots, \varphi_n(z_0))^T.$$

All formulas involving $\varphi(z_0)$ have to be interpreted as vector- (matrix-) formulas. Let $\varphi(z)$ be defined by

$$\varphi(z) = (I + (z-z_0)(A-z)^{-1})\varphi(z_0), \ z \in \rho(A). \tag{2.2}$$

Then $\varphi(z)$ parametrizes the defect spaces of S, i.e. satisfies

$$\operatorname{span}\{\varphi_1(z), \ldots, \varphi_n(z)\} = \operatorname{ran}(S-\overline{z})^{\perp}.$$

A $n \times n$-matrix function $Q(z)$ which satisfies

$$\frac{Q(z) - Q(w)^*}{z - \overline{w}} = [\varphi(z), \varphi(w)], \ z, w \in \rho(A), \tag{2.3}$$

is called a Q-function of S and A. By the requirement (2.3), Q is uniquely determined up to an additive constant hermitian matrix. If S is not standard, Q is a real constant.

Recall the notion of a generalized Nevanlinna function: Let $\kappa \in \mathbb{N} \cup \{0\}$. Then a $n \times n$-matrix function Q, which is analytic in some open set O and which is not a constant hermitian matrix, belongs to the class $\mathcal{N}_\kappa^n$ if $Q(\overline{z}) = Q(z)^*$ whenever $z, \overline{z} \in O$, and if the matrix kernel

$$\frac{Q(z) - Q(w)^*}{z - \overline{w}}, \ z, w \in O,$$

has κ negative squares. If $n = 1$ we write $\mathcal{N}_\kappa$ instead of $\mathcal{N}_\kappa^1$. Moreover, let $\widetilde{\mathcal{N}}_0$ be the union of $\mathcal{N}_0$ and of all constant functions $Q(z) = t \in \mathbb{R} \cup \{\infty\}$, and put $\widetilde{\mathcal{N}}_\kappa = \mathcal{N}_\kappa$ for $\kappa > 0$.

A selfadjoint extension $\tilde{A}$ of S which acts in a Pontryagin space $\widetilde{\mathfrak{P}} \supseteq \mathfrak{P}$ is called $\mathfrak{P}$-minimal, if the closed linear span

$$\operatorname{cls}\left(\mathfrak{P} \cup \{(\tilde{A} - z)^{-1} f : z \in \rho(\tilde{A}), f \in \mathfrak{P}\}\right) = \widetilde{\mathfrak{P}}.$$

Recall that an arbitrary extension of S can always be reduced to a minimal one.

Denote in the following by $\widetilde{P}$ the orthogonal projection of $\widetilde{\mathfrak{P}}$ onto $\mathfrak{P}$ and by $\operatorname{Ind}_- \widetilde{\mathfrak{P}}$ the negative index of $\widetilde{\mathfrak{P}}$. We will use the Krein formula for the description of generalized resolvents in the following form:

Proposition 2.3. *Let S be a closed symmetric relation in the Pontryagin space $\mathfrak{P}$ with defect index $(1, 1)$. Choose a canonical selfadjoint extension A of S with $\rho(A) \neq \emptyset$, and let $Q(z)$ and $\varphi(z)$ be defined by (2.3) and (2.2). The relation*

$$\widetilde{P}(\tilde{A} - z)^{-1}|_{\mathfrak{P}} = (A - z)^{-1} - \frac{[., \varphi(\overline{z})]}{Q(z) + \tau(z)} \varphi(z) \tag{2.4}$$

establishes a bijective correspondence between the $\mathfrak{P}$-minimal selfadjoint extensions $\tilde{A} \subseteq \widetilde{\mathfrak{P}}^2$, $\rho(\tilde{A}) \neq \emptyset$ of S and the functions $\tau \in \widetilde{\mathcal{N}}_{\tilde{\kappa}-\kappa} \setminus \{-Q\}$. Here $\kappa = \operatorname{Ind}_- \mathfrak{P}$ and $\tilde{\kappa} = \operatorname{Ind}_- \widetilde{\mathfrak{P}}$. Moreover, in (2.4) $\tilde{A}$ is a canonical extension of S if and only if $\tau(z) = t \in \mathbb{R} \cup \{\infty\}$.

Usually in the literature the relation S in Proposition 2.3 is required to be standard. Recently H.de Snoo gave a proof of the Krein formula which, although formulated only for the case of Hilbert spaces, immediately carries over to the Pontryagin space situation, and which does not require S to be standard. For the convenience of the reader we outline this proof.

Proof : [of Proposition 2.3] First note that, by the considerations of [DLS1], a analytic operator valued function $R(z)$ is a generalized resolvent of S if and only if $R(\overline{z}) = R(z)^*$, the kernel

$$\frac{R(z) - R(w)^*}{z - \overline{w}} - R(w)^* R(z)$$

has finitely many negative squares and $R(z)|_{\operatorname{ran}(S-z)} = (S - z)^{-1}$.

Let $\tau(z) \in \widetilde{\mathcal{N}}_{\tilde{\kappa}-\kappa} \setminus \{-Q(z)\}$ and let $R(z)$ be defined by the right hand side of (2.4). A computation shows that for $f, g \in \mathfrak{P}$

$$[\left(\frac{R(z) - R(w)^*}{z - \overline{w}} - R(w)^* R(z)\right) f, g] =$$

$$= \overline{\left(\frac{[g,\varphi(\overline{w})]}{q(w)+\tau(w)}\right)}\frac{\tau(z)-\overline{\tau(w)}}{z-\overline{w}}\left(\frac{[f,\varphi(\overline{z})]}{q(z)+\tau(z)}\right). \tag{2.5}$$

Since by the definition of $R(z)$

$$R(z)|_{\mathrm{ran}\,(S-z)} = (A-z)^{-1}|_{\mathrm{ran}\,(S-z)} = (S-z)^{-1},$$

the operator function $R(z)$ defines a generalized resolvent of S.

Conversely, let $R(z) = \widetilde{P}(\tilde{A}-z)^{-1}|_{\mathfrak{P}}$ be a generalized resolvent of S. Then

$$\mathrm{ran}\,(S-z) \subseteq \ker\,(R(z)-(A-z)^{-1}). \tag{2.6}$$

Hence $\mathrm{ran}\,(R(z)-(A-z)^{-1}) \subseteq \mathrm{ran}\,(S-\overline{z})^{\perp}$, and we find

$$R(z)f = (A-z)^{-1}f + c(f,z)\varphi(z).$$

Clearly, for fixed z, the term $c(f,z)$ is a bounded linear functional on $\mathfrak{P}$, its kernel containes $\mathrm{ran}\,(S-z)$, and, for fixed f, $c(f,z)$ depends analytically on z. These facts imply that $R(z)$ can be written as

$$R(z) = (A-z)^{-1} + \psi(z)[.,\varphi(\overline{z})]\varphi(z),$$

where $\psi(z)$ is analytic on $\rho(A)\cap\rho(\tilde{A})$. If we put $\tau(z) = -q(z) - \dfrac{1}{\psi(z)}$, we obtain a representation (2.4) of $R(z)$. By (2.5) the assertion follows.

□

3 Triplet spaces related to a symmetric operator

Let $(\mathfrak{P},[.,.])$ be a Pontryagin space with negative index κ, and denote by $(.,.)$ a Hilbert space inner product on $\mathfrak{P}$ induced by a fundamental symmetry. Let S be a closed symmetric relation in $\mathfrak{P}$ with defect index $(1,1)$, and denote by S^* the adjoint relation of S with respect to $[.,.]$. Write

$$S^*(0) = \{g \in \mathfrak{P} : \begin{pmatrix} 0 \\ g \end{pmatrix} \in S^*\},\ S^*_\infty = \{0\}\times S^*(0).$$

In the following we construct two Hilbert spaces $\mathfrak{P}_+$ and $\mathfrak{P}_-$ from $\mathfrak{P}$ and S. Provide $\mathfrak{P}^2$ with the positive definite inner product $(.,.)_+$ defined by

$$\left(\begin{pmatrix} f_1 \\ g_1 \end{pmatrix},\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right)_+ = (f_1,f_2)+(g_1,g_2),\ \begin{pmatrix} f_1 \\ g_1 \end{pmatrix},\begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \in \mathfrak{P}^2,$$

and denote by $\mathfrak{P}_+$ the space

$$\mathfrak{P}_+ = (S^*,(.,.)_+).$$

Let π be the mapping of $\mathfrak{P}_+$ into $\mathfrak{P}$ defined by $\pi\begin{pmatrix} f \\ g \end{pmatrix} = f$. By π^* we denote the adjoint of π with respect to the inner products $(.,.)_+$ on $\mathfrak{P}_+$ and $[.,.]$ on $\mathfrak{P}$. We provide $\mathfrak{P}/\ker\pi^*$ with a positive definite inner product:

$$(f,g)_- = (\pi^*f,\pi^*g)_+,\ f,g \in \mathfrak{P}/\ker\pi^*.$$

Note here that the action of π^* on $\mathfrak{P}/\ker\pi^*$ is well defined. Then $(\mathfrak{P}/\ker\pi^*, (.,.)_-)$ is a Pre-Hilbert space and has a completion $(\mathfrak{D}_-, (.,.)_-)$. Let $S^*(0)'$ be an isomorphic copy of $S^*(0) \subseteq (\mathfrak{P}, (.,.))$, and denote by $\mathfrak{P}_-$ the space

$$\mathfrak{P}_- = \mathfrak{D}_- \oplus S^*(0)',$$

provided with an inner product $(.,.)_-$ in the natural way. Here and in the following, the symbol $\mathfrak{A} \oplus \mathfrak{B}$ denotes the direct and orthogonal sum of the inner product spaces $\mathfrak{A}$ and $\mathfrak{B}$.

We also consider the mapping $V' : \mathfrak{P}/\ker\pi^* \oplus S^*(0)' \to \mathfrak{P}_+$ defined by

$$V'(f + x) = \pi^* f + \begin{pmatrix} 0 \\ x \end{pmatrix}, \ f \in \mathfrak{P}/\ker\pi^*, \ x \in S^*(0)'.$$

By the construction of $\mathfrak{P}_-$ the factor space $\mathfrak{P}/\ker\pi^*$ can be canonically as a linear subspace of $\mathfrak{P}_-$. We denote by ι the canonical mapping of $\mathfrak{P}$ into $\mathfrak{P}_-$. The chain

$$(\mathfrak{P}_+, (.,.)_+) \xrightarrow{\pi} (\mathfrak{P}, [.,.]) \xrightarrow{\iota} (\mathfrak{P}_-, (.,.)_-)$$

is called the space triplet associated with the symmetry S.

If $(\widetilde{\mathfrak{P}}, [.,.])$ is a Pontryagin space extending $\mathfrak{P}$, consider the spaces

$$\widetilde{\mathfrak{P}}_+ = \mathfrak{P}_+ \oplus (\widetilde{\mathfrak{P}} \ominus \mathfrak{P})^2, \ \widetilde{\mathfrak{P}}_- = \mathfrak{P}_- \oplus (\widetilde{\mathfrak{P}} \ominus \mathfrak{P})^2,$$

provided with inner products $(.,.)_+$ and $(.,.)_-$ in a natural way. Here, and in the following, the symbol $\mathfrak{A} \ominus \mathfrak{B}$ denotes the orthogonal companion of $\mathfrak{B}$ in the inner product space $\mathfrak{A}$. Extend V' by

$$\widetilde{V}' = V' \oplus I,$$

where I denotes the identity operator in the space $(\widetilde{\mathfrak{P}} \ominus \mathfrak{P})^2$. Note that $\mathfrak{P}_+ \oplus (\widetilde{\mathfrak{P}} \ominus \mathfrak{P})^2$ coincides as a set with the adjoint relation of S in $\widetilde{\mathfrak{P}}$.

Lemma 3.1. *We have*

$$\ker\pi = S^*_\infty, \ \operatorname{ran}\pi = \operatorname{dom} S^*,$$

and

$$\ker\pi^* = (\operatorname{dom} S^*)^{[\perp]} = S(0), \ \overline{\operatorname{ran}\pi^*} = (S^*_\infty)^{(\perp)_+}.$$

Moreover, the mapping V' is isometric, and $V'\iota = \pi^*$.

Proof : The first part of the lemma follows immediately from the above definitions and general properties of adjoints.

In order to show that V' is isometric let $u, v \in \mathfrak{P}/\ker\pi^* \oplus S^*(0)'$, $u = f + x$, $v = g + y$, and compute

$$(u,v)_- = (f,g)_- + (x,y) = (\pi^* f, \pi^* g)_+ + \left(\begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right)_+ = (V'u, V'v)_+.$$

The relation $V'\iota = \pi^*$ follows from the definition of V'.

□

Since $\mathfrak{P}/\ker\pi^* \oplus S^*(0)'$ is dense in $\mathfrak{P}_-$ and $\operatorname{ran}\pi^* + S^*_\infty$ is dense in $\mathfrak{P}_+$, Lemma 3.1 has the following corollary:

Corollary 3.2. *The mappings V' and $\widetilde{V}'$ can be extended to unitary mappings V of $\mathfrak{P}_-$ onto $\mathfrak{P}_+$ and $\widetilde{V}$ of $\widetilde{\mathfrak{P}}_-$ onto $\widetilde{\mathfrak{P}}_+$, respectively.*

Now we are in position to define a duality $[.,.]_\pm$ between the spaces $\widetilde{\mathfrak{P}}_+$ and $\widetilde{\mathfrak{P}}_-$:

$$\left[\begin{pmatrix} f \\ g \end{pmatrix}, u\right]_\pm = \left(\begin{pmatrix} f \\ g \end{pmatrix}, \widetilde{V}u\right)_+, \quad \begin{pmatrix} f \\ g \end{pmatrix} \in \widetilde{\mathfrak{P}}_+, \ u \in \widetilde{\mathfrak{P}}_-.$$

For notational convenience we set

$$\left[u, \begin{pmatrix} f \\ g \end{pmatrix}\right]_\pm = \overline{\left[\begin{pmatrix} f \\ g \end{pmatrix}, u\right]_\pm}, \quad \begin{pmatrix} f \\ g \end{pmatrix} \in \widetilde{\mathfrak{P}}_+, \ u \in \widetilde{\mathfrak{P}}_-.$$

The mapping ι is the adjoint of π with respect to the duality $[.,.]_\pm$.

Lemma 3.3. *If $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathfrak{P}_+$ and $h \in \mathfrak{P}$, we have*

$$\left[\begin{pmatrix} f \\ g \end{pmatrix}, \iota h\right]_\pm = \left[\pi \begin{pmatrix} f \\ g \end{pmatrix}, h\right].$$

Proof : Using the definition of $[.,.]_\pm$ we obtain

$$\left[\begin{pmatrix} f \\ g \end{pmatrix}, \iota h\right]_\pm = \left(\begin{pmatrix} f \\ g \end{pmatrix}, V\iota h\right)_+ = \left(\begin{pmatrix} f \\ g \end{pmatrix}, \pi^* h\right)_+ = \left[\pi \begin{pmatrix} f \\ g \end{pmatrix}, h\right].$$

□

Consider a selfadjoint extension A of S with $\rho(A) \neq \emptyset$ which acts in a Pontryagin space $\widetilde{\mathfrak{P}} \supseteq \mathfrak{P}$. Since $\widetilde{\mathfrak{P}}_+$ can be understood as the adjoint relation of S in $\widetilde{\mathfrak{P}}$, we have $A \subseteq \widetilde{\mathfrak{P}}_+$. For $z \in \rho(A)$ let mappings $R_z^+ : \widetilde{\mathfrak{P}} \to \widetilde{\mathfrak{P}}_+$ and $R_z^- : \widetilde{\mathfrak{P}}_- \to \widetilde{\mathfrak{P}}$ be defined by

$$R_z^+ f = \begin{pmatrix} (A-z)^{-1} f \\ (I + z(A-z)^{-1}) f \end{pmatrix}, \quad f \in \widetilde{\mathfrak{P}},$$

and

$$R_z^- = (R_{\overline{z}}^+)^* \widetilde{V}.$$

Here $(R_{\overline{z}}^+)^*$ is the adjoint of $R_{\overline{z}}^+$ with respect to the inner products $(.,.)_+$ on $\widetilde{\mathfrak{P}}_+$ and $[.,.]$ on $\widetilde{\mathfrak{P}}$.

Denote in the following by $\widetilde{P}$ and $\widetilde{P}^+$ the orthogonal projections of $\widetilde{\mathfrak{P}}$ onto $\mathfrak{P}$ and of $\widetilde{\mathfrak{P}}_+$ onto $\mathfrak{P}_+$, respectively, and put

$$\widetilde{R}_z^+ = \widetilde{P}^+ R_z^+|_{\mathfrak{P}}, \ \widetilde{R}_z^- = \widetilde{P} R_z^-|_{\mathfrak{P}_-}.$$

Let

$$T(z) = \left\{ \begin{pmatrix} \widetilde{P}(A-z)^{-1} f \\ (I + z\widetilde{P}(A-z)^{-1}) f \end{pmatrix} : f \in \mathfrak{P} \right\}$$

be the so called Štraus relation associated with A. It is well known (compare [DLS2]) that $S \subseteq T(z) \subseteq S^* = \mathfrak{P}_+$ and that $\operatorname{codim}_{S^*} T(z) = 1$.

Note that, if A is a canonical extension of S, i.e. if $\widetilde{\mathfrak{P}} = \mathfrak{P}$, we have $\widetilde{R}_z^+ = R_z^+$, $\widetilde{R}_z^- = R_z^-$ and $T(z) = A$.

Lemma 3.4. *For the mappings R_z^+ and R_z^- the following identities hold:*

$$R_z^+ - R_w^+ = (z-w)R_z^+(A-w)^{-1},\ z,w \in \rho(A), \tag{3.1}$$

$$R_z^- - R_w^- = (z-w)(A-z)^{-1}R_w^-,\ z,w \in \rho(A). \tag{3.2}$$

They satisfy

$$[R_z^+ f, u]_\pm = [f, R_{\overline{z}}^- u],\ f \in \widetilde{\mathfrak{P}}, u \in \widetilde{\mathfrak{P}}_-. \tag{3.3}$$

Moreover, for $z \in \rho(A)$, we have

$$\ker R_z^+ = \{0\},\ \operatorname{ran} R_z^+ = A,\ \ker R_z^- = \widetilde{V}^{-1}(\widetilde{\mathfrak{P}}_+ \ominus A),\ \operatorname{ran} R_z^- = \widetilde{\mathfrak{P}},$$

and

$$\ker \widetilde{R}_z^+ = \{0\},\ \operatorname{ran} \widetilde{R}_z^+ = T(z),\ \operatorname{codim}_{\mathfrak{P}_+} \operatorname{ran} \widetilde{R}_z^+ = 1,$$
$$\ker \widetilde{R}_z^- = V^{-1}(\mathfrak{P}_+ \ominus T(\overline{z})),\ \operatorname{ran} \widetilde{R}_z^- = \mathfrak{P}. \tag{3.4}$$

The mapping $\widetilde{R}_z^-$ extends the generalized resolvent $\widetilde{P}(A-z)^{-1}|_{\mathfrak{P}}$ in the sense that

$$\widetilde{R}_z^- \iota = \widetilde{P}(A-z)^{-1}|_{\mathfrak{P}}. \tag{3.5}$$

Proof : We compute

$$(z-w)R_z^+(A-w)^{-1} = \begin{pmatrix} (z-w)(A-z)^{-1}(A-w)^{-1} \\ (z-w)(I + z(A-z)^{-1})(A-w)^{-1} \end{pmatrix} =$$

$$= \begin{pmatrix} (A-z)^{-1} - (A-w)^{-1} \\ (z-w)(A-w)^{-1} + z((A-z)^{-1} - (A-w)^{-1}) \end{pmatrix} =$$

$$= \begin{pmatrix} (A-z)^{-1} - (A-w)^{-1} \\ z(A-z)^{-1} - w(A-w)^{-1} \end{pmatrix} = R_z^+ - R_w^+,$$

which proves (3.1). The relation (3.2) follows by taking adjoints and multiplying with $\widetilde{V}$ from the right.

Let $u \in \widetilde{\mathfrak{P}}_-$ and $f \in \widetilde{\mathfrak{P}}$, then

$$[f, R_{\overline{z}}^- u] = [f, (R_z^+)^* \widetilde{V} u] = (R_z^+ f, \widetilde{V} u)_+ = [R_z^+ f, u]_\pm.$$

Assume that $f \in \ker R_z^+$, then $(A-z)^{-1} f = 0$ and $f + z(A-z)^{-1} f = 0$, hence $f = 0$. It is elementary to show that $\operatorname{ran} R_z^+ = A$, and therefore $\ker (R_{\overline{z}}^+)^* = \widetilde{\mathfrak{P}}_+ \ominus A$. This yields

$$\ker R_z^- = \widetilde{V}^{-1}(\widetilde{\mathfrak{P}}_+ \ominus A).$$

Since $\ker R_{\overline{z}}^{+} = 0$, we have $\overline{\operatorname{ran}(R_{\overline{z}}^{+})^*} = \widetilde{\mathfrak{P}}$. The closed range theorem and the fact that $\widetilde{V}$ is onto show that $\operatorname{ran} R_z^- = \widetilde{\mathfrak{P}}$. The assertions concerning $\widetilde{R}_z^+$ and $\widetilde{R}_z^-$ follow by similar arguments and the facts concerning the Štraus relation which have been mentioned above.

Relation (3.5) follows from

$$\widetilde{R}_z^- \iota f = \widetilde{P}(R_{\overline{z}}^+)^* \widetilde{V} \iota f = \widetilde{P}(R_{\overline{z}}^+)^* \pi^* f = \widetilde{P}(\pi R_{\overline{z}}^+)^* f =$$

$$= \widetilde{P}((A-\overline{z})^{-1})^* f = \widetilde{P}(A-z)^{-1} f, \ f \in \mathfrak{P}.$$

□

Let A be a canonical selfadjoint extension of S. If $\varphi(z)$ are defect elements of S according to (2.2), clearly $\begin{pmatrix} \varphi(z) \\ z\varphi(z) \end{pmatrix} \in \mathfrak{P}_+$.

Lemma 3.5. *Assume that S is minimal, i.e. $\mathfrak{P} = \operatorname{cls}\{\varphi(z) : z \in \rho(A)\}$. Then*

$$\mathfrak{P}_+ = \operatorname{cls}\{\begin{pmatrix} \varphi(z) \\ z\varphi(z) \end{pmatrix} : z \in \rho(A)\}. \tag{3.6}$$

Proof : Choose $w \in \rho(A)$. First note that $\varphi(w) \in (S^* - w)^{-1}(0)$. Since $\dim S^*/A = 1$, we have

$$(S^* - w)^{-1} = (A-w)^{-1} \dot{+} \operatorname{span}\{(0; \varphi(w))\}.$$

Since S is minimal and $(A-w)^{-1}$ is a bounded operator, we find

$$\operatorname{cls}\{(\varphi(z); \frac{\varphi(w)-\varphi(z)}{w-z}) : z \in \rho(A) \setminus \{w\}\} = (A-w)^{-1}.$$

Hence $(S^* - w)^{-1} = \operatorname{cls}\{((w-z)\varphi(z); -\varphi(z)) : z \in \rho(A)\}$ which implies (3.6).

□

The Krein formula can be rewritten in terms of $\widetilde{R}_z^+$ and $\widetilde{R}_z^-$.

Lemma 3.6. *Let A be a fixed canonical selfadjoint extension of S. Then for each selfadjoint extension $\tilde{A}$ of S the mappings $\widetilde{R}_z^+$ and $\widetilde{R}_z^-$ can be written as*

$$\widetilde{R}_z^+ = R_z^+ - \frac{[., \varphi(\overline{z})]}{q(z)+\tau(z)} \begin{pmatrix} \varphi(z) \\ z\varphi(z) \end{pmatrix}, \ z \in \rho(\tilde{A}), \tag{3.7}$$

and

$$\widetilde{R}_z^- = R_z^- - \frac{\left[., \begin{pmatrix} \varphi(\overline{z}) \\ \overline{z}\varphi(\overline{z}) \end{pmatrix} \right]_{\pm}}{q(z)+\tau(z)} \varphi(z), \ z \in \rho(\tilde{A}), \tag{3.8}$$

respectively. Here the elements $\varphi(z)$ parametrize the defect spaces of S according to (2.2), q is a Q-function of A and S as in (2.3), and τ is the parameter function associated with $\tilde{A}$ by the Krein formula.

Proof : Relation (3.7) follows immediately from (2.4) and the definition of R_z^+. Taking adjoints in (3.7) and taking $\overline{z}$ instead of z we obtain

$$(\widetilde{R}_{\overline{z}}^+)^* = (R_{\overline{z}}^+)^* - \frac{\left(., \begin{pmatrix} \varphi(\overline{z}) \\ \overline{z}\varphi(\overline{z}) \end{pmatrix} \right)_+}{q(z)+\tau(z)} \varphi(z).$$

By the definiton of $[.,.]_\pm$ and $\widetilde{R}_z^-$ we obtain (3.8).

□

4 Generalized u-resolvents and u-resolvent matrices

Throughout this and the following sections we assume that $r(S) \neq \emptyset$. Let $\tilde{A}$ be a selfadjoint extension of S with nonempty resolvent set acting in a Pontryagin space $\widetilde{\mathfrak{P}} \supseteq \mathfrak{P}$, and choose $z_0 \in \rho(\tilde{A}) \cap \mathbb{C}^+$. We define a regularized generalized resolvent $\hat{\widetilde{R}}_z : \mathfrak{P}_- \to \mathfrak{P}^2$ by

$$\hat{\widetilde{R}}_z = \begin{pmatrix} \widetilde{R}_z^- - \frac{\widetilde{R}_{z_0}^- + \widetilde{R}_{\overline{z_0}}^-}{2} \\ z\widetilde{R}_z^- - \frac{z_0\widetilde{R}_{z_0}^- + \overline{z_0}\widetilde{R}_{\overline{z_0}}^-}{2} \end{pmatrix}.$$

If A is a canonical selfadjoint extension of S we speak of a regularized resolvent $\hat{R}_z$. Using (3.2) a straightforward calculation gives

$$\hat{\widetilde{R}}_z = (z - \operatorname{Re} z_0)\widetilde{P}^+ R_{z_0}^+ R_{\overline{z_0}}^-|_{\mathfrak{P}_-} + (z-z_0)(z-\overline{z_0})\widetilde{P}^+ R_{z_0}^+ (\tilde{A}-z)^{-1} R_{\overline{z_0}}^-|_{\mathfrak{P}_-}. \qquad (4.1)$$

This relation shows that $\hat{\widetilde{R}}_z$ is in fact a mapping of $\mathfrak{P}_-$ into $\mathfrak{P}_+$.

The following result is an analogue to Lemma 3.6 for regularized resolvents.

Proposition 4.1. *Let A be a fixed canonical selfadjoint extension of S with the regularized resolvent $\hat{R}_z$, and let the functions τ, $q(z)$ and $\varphi(z)$ have the same meaning as in* Lemma 3.6.

Then for each selfadjoint extension $\tilde{A}$ the regularized generalized resolvent $\hat{\widetilde{R}}_z$ can be written as

$$\hat{\widetilde{R}}_z = \hat{R}_z - \frac{\left[., \begin{pmatrix} \varphi(\overline{z}) \\ \overline{z}\varphi(\overline{z}) \end{pmatrix} \right]_\pm}{q(z)+\tau(z)} \begin{pmatrix} \varphi(z) \\ z\varphi(z) \end{pmatrix} + \qquad (4.2)$$

$$+\frac{\left[., \begin{pmatrix} \varphi(\overline{z_0}) \\ \overline{z_0}\varphi(\overline{z_0}) \end{pmatrix} \right]_\pm}{2(q(z_0)+\tau(z_0))} \begin{pmatrix} \varphi(z_0) \\ z_0\varphi(z_0) \end{pmatrix} + \frac{\left[., \begin{pmatrix} \varphi(z_0) \\ z_0\varphi(z_0) \end{pmatrix} \right]_\pm}{2(q(\overline{z_0})+\tau(\overline{z_0}))} \begin{pmatrix} \varphi(\overline{z_0}) \\ \overline{z_0}\varphi(\overline{z_0}) \end{pmatrix}.$$

Proof : By Lemma 3.6 we have

$$\hat{\widetilde{R}}_z = \begin{pmatrix} \widetilde{R}_z^- - \frac{\widetilde{R}_{z_0}^- + \widetilde{R}_{\overline{z_0}}^-}{2} \\ z\widetilde{R}_z^- - \frac{z_0\widetilde{R}_{z_0}^- + \overline{z_0}\widetilde{R}_{\overline{z_0}}^-}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} R_z^- - \frac{R_{z_0}^- + R_{\overline{z_0}}^-}{2} \\ zR_z^- - \frac{z_0 R_{z_0}^- + \overline{z_0} R_{\overline{z_0}}^-}{2} \end{pmatrix} - \frac{\left[\cdot, \begin{pmatrix} \varphi(\overline{z}) \\ \overline{z}\varphi(\overline{z}) \end{pmatrix} \right]_\pm}{q(z)+\tau(z)} \begin{pmatrix} \varphi(z) \\ z\varphi(z) \end{pmatrix} +$$

$$+ \frac{\left[\cdot, \begin{pmatrix} \varphi(\overline{z_0}) \\ \overline{z_0}\varphi(\overline{z_0}) \end{pmatrix} \right]_\pm}{2(q(z_0)+\tau(z_0))} \begin{pmatrix} \varphi(z_0) \\ z_0\varphi(z_0) \end{pmatrix} + \frac{\left[\cdot, \begin{pmatrix} \varphi(z_0) \\ z_0\varphi(z_0) \end{pmatrix} \right]_\pm}{2(q(\overline{z_0})+\tau(\overline{z_0}))} \begin{pmatrix} \varphi(\overline{z_0}) \\ \overline{z_0}\varphi(\overline{z_0}) \end{pmatrix}.$$

□

Definition 4.2. Let $u \in \mathfrak{P}_-$. If $\alpha \in \mathbb{R}$ and $\tilde{A}$ is a selfadjoint extension of S with nonempty resolvent set, the function

$$\tilde{r}(z) = \alpha + [\hat{\tilde{R}}_z u, u]_\pm,\ z \in \rho(\tilde{A}),$$

is called the generalized u-resolvent of S induced by $\tilde{A}$. If $\tilde{A}$ is $\mathfrak{P}$-minimal and $\operatorname{Ind}_-\widetilde{\mathfrak{P}} = \tilde{\kappa}$, we say that the generalized u-resolvent is of index $\tilde{\kappa}$.

Note that by (3.3) and (4.1) the function $\tilde{r}(z)$ is real, i.e. satisfies $\overline{\tilde{r}(\overline{z})} = \tilde{r}(z)$, $z \in \rho(\tilde{A})$. In the following we will identify the set of generalized u-resolvents of S as the set of Q-functions corresponding to selfadjoint extensions $\tilde{A}$ of S and certain symmetric restrictions of $\tilde{A}$ (which are in general different from S).

Lemma 4.3. *Let $u \in \mathfrak{P}_-$, and put $\gamma(z) = R_z^- u$. Then*

$$\gamma(z) = (I + (z-w)(\tilde{A}-z)^{-1})\gamma(w),\ z, w \in \rho(\tilde{A}). \tag{4.3}$$

The elements $\gamma(z)$ vanish identically if and only if $\tilde{A}$ is a canonical extension of S and $\widetilde{V}u \perp \tilde{A}$.

If $\gamma(z)$ is a family of elements of $\widetilde{\mathfrak{P}}$ which satisfies (4.3) *for some extension $\tilde{A}$ of S, then there exists an element $u \in \widetilde{\mathfrak{P}}_-$, such that $\gamma(z) = R_z^- u$, $z \in \rho(\tilde{A})$.*

Proof : The first assertion follows immediately from (3.2). Assume now that $\gamma(z)$ vanishes identically, i.e. $u \in \ker R_z^-$, $z \in \rho(\tilde{A})$. Then $u \in \ker \widetilde{R}_z^-$, and by (3.4) we find $Vu \perp T(\bar{z})$ for all $z \in \rho(\tilde{A})$. As the Štraus relation $T(z)$ has codimension one in $\mathfrak{P}_+$ we obtain that $T(z)$ does not depend on z. Then $\tilde{A}$ is canonical and $T(z) = \tilde{A}$. Conversely, if $\tilde{A}$ is canonical and $\widetilde{V}u \perp \tilde{A}$ then, by (3.4), $\gamma(z) = 0$ for all $z \in \rho(\tilde{A})$.

Let a family $\gamma(z)$ be given, and choose $z_0 \in \rho(\tilde{A})$. By Lemma 3.4, there exists an element $u \in \widetilde{\mathfrak{P}}_-$, such that $\gamma(z_0) = R_{z_0}^- u$. Since both families $\gamma(z)$ and $R_z^- u$ satsify (4.3), they coincide for all z.

□

Corollary 4.4. *If in Lemma 4.3, $\tilde{A}$ is a canonical extension of S, then by the requirement $\gamma(z) = R_z^- u$, the element u is uniquely determined up to summands $h \in V^{-1}(\mathfrak{P}_+ \ominus A)$. For* *any element* $h \in V^{-1}(\mathfrak{P}_+ \ominus A)$*,* $h \neq 0$*, we have*

$$\left[\left(\begin{array}{c}\varphi(z)\\ z\varphi(z)\end{array}\right), h\right]_\pm = \beta,\ z \in \rho(A),$$

for some constant $\beta \neq 0$*. Any number* $\beta \in \mathbb{C} \setminus \{0\}$ *occurs in this relation for a suitable choice of* h*.*

Proof : Let $h \in V^{-1}(\mathfrak{P}_+ \ominus A)$, $h \neq 0$, be given. For $z \in \rho(A)$ the element $\left(\begin{array}{c}\varphi(z)\\ z\varphi(z)\end{array}\right)$ does not belong to A, and hence $\left[\left(\begin{array}{c}\varphi(z)\\ z\varphi(z)\end{array}\right), h\right]_\pm \neq 0$. Since

$$\left(\begin{array}{c}(A-z)^{-1}\varphi(w)\\ (I+z(A-z)^{-1})\varphi(w)\end{array}\right) \in A,$$

the latter scalar product is in fact independent from z:

$$\left[\left(\begin{array}{c}\varphi(z)\\ z\varphi(z)\end{array}\right), h\right]_\pm = \left[\left(\begin{array}{c}\varphi(w)\\ w\varphi(w)\end{array}\right) + (z-w)\left(\begin{array}{c}(A-z)^{-1}\varphi(w)\\ (I+z(A-z)^{-1})\varphi(w)\end{array}\right), h\right]_\pm =$$

$$= \left[\left(\begin{array}{c}\varphi(w)\\ w\varphi(w)\end{array}\right), h\right]_\pm .$$

□

Proposition 4.5. *Let* $u \in \mathfrak{P}_-$*, and let* $\tilde{r}(z)$ *be a generalized* u*-resolvent of* S *induced by the selfadjoint extension* $\tilde{A}$*. Then* $\tilde{r}(z)$ *is a Q-function of* $\tilde{A}$ *and its restriction*

$$S_u = \left\{\left(\begin{array}{c}f\\ g\end{array}\right) \in \tilde{A} : \left[\left(\begin{array}{c}f\\ g\end{array}\right), u\right]_\pm = 0\right\}.$$

Any Q-function of a canonical selfadjoint extension $\tilde{A}$ *of* S *and some symmetric restriction* S_1 *of* $\tilde{A}$ *can be represented in this way.*

Proof : Assume first that a generalized u-resolvent $\tilde{r}(z)$ is given. By (3.3)

$$[g - \overline{z}f, R_z^- u] = [R_z^+(g - \overline{z}f), u]_\pm = \left[\left(\begin{array}{c}f\\ g\end{array}\right), u\right]_\pm ,$$

hence, by (4.3), the family $\gamma(z) = R_z^- u$ is an appropriate parametrization of the defect spaces of S_u.

To prove that $\tilde{r}(z)$ is a Q-function of S_u and $\tilde{A}$, we consider the Nevanlinna kernel of $\tilde{r}(z)$ and calculate using (4.1):

$$\frac{\tilde{r}(z) - \tilde{r}(w)}{z - w} = [\gamma(\overline{z_0}), \gamma(\overline{z_0})] +$$

$$+\frac{(z-z_0)(z-\overline{z_0})[(\tilde{A}-z)^{-1}\gamma(\overline{z_0}),\gamma(\overline{z_0})]-(w-z_0)(w-\overline{z_0})[(\tilde{A}-w)^{-1}\gamma(\overline{z_0}),\gamma(\overline{z_0})]}{z-w}=$$

$$=\frac{(z-z_0)[\gamma(z),\gamma(\overline{z_0})]-(w-z_0)[\gamma(w),\gamma(\overline{z_0})]}{z-w}=$$

$$=\frac{(z-z_0)[\gamma(z),\gamma(\overline{z_0})]-(w-z_0)[(I+(z-w)(A-w)^{-1})\gamma(z),\gamma(\overline{z_0})]}{z-w}=[\gamma(z),\gamma(\overline{w})].$$

Now let $\tilde{A}$ be a canonical selfadjoint extension of S. Choose an appropriate parametrization $\gamma(z)$ of the defect spaces of S_1. Due to Lemma 4.3, we may represent $\gamma(z)$ by

$$\gamma(z)=R_z^- u,\ \ z\in\rho(\tilde{A}),$$

for some element $u\in\mathfrak{P}_-$. Since a Q-function is uniquely determined by its Nevanlinna kernel up to additive real constants, the previous part of the proof yields the assertion.

□

Corollary 4.6. *Let $\tilde{r}(z)$ be a generalized u-resolvent of S induced by the selfadjoint extension $\tilde{A}$. Assume that $\tilde{A}$ acts in a Pontryagin space $\widetilde{\mathfrak{P}}$ with negative index $\tilde{\kappa}$. Then $\tilde{r}(z)\in\widetilde{\mathcal{N}}_{\kappa'}$ for some $\kappa'\le\tilde{\kappa}$.*

Let $W(z)$ be a 2×2-matrix valued function and let $\tau(z)$ be a scalar function, both meromorphic in some open set. If

$$W(z)=\begin{pmatrix} w_{11}(z) & w_{12}(z)\\ w_{21}(z) & w_{22}(z)\end{pmatrix},$$

we denote by $W\circ\tau$ the scalar function

$$(W\circ\tau)(z)=\frac{w_{11}(z)\tau(z)+w_{12}(z)}{w_{21}(z)\tau(z)+w_{22}(z)}.$$

Clearly, $W\circ\tau$ is meromorphic unless $w_{21}\tau+w_{22}$ vanishes identically. A straightforward computation shows that

$$W_1\circ(W_2\circ\tau)=(W_1W_2)\circ\tau.$$

If $u\in\mathfrak{P}_-$ is given, denote by $r_u(S)$ the set of all $z\in r(S)$ such that

$$\left[u,\begin{pmatrix} f\\ \bar{z}f\end{pmatrix}\right]_{\pm}\neq 0,$$

when $f\neq 0$, $f\in\ker(S^*-\bar{z})$.

Definition 4.7. Let $u\in\mathfrak{P}_-$, $u\neq 0$, be such that $r_u(S)\neq\emptyset$. A 2×2-matrix valued function $W(z)$, analytic on $r_u(S)$, is called a u-resolvent matrix of S if for each $\tau\in\widetilde{\mathcal{N}}_0$ the function

$$r(z)=(W\circ\tau)(z)$$

is a generalized u-resolvents of S of index $\kappa = \mathrm{Ind}_-\mathfrak{P}$, and each such generalized u-resolvent can be obtained in this way up to a real additive constant.

First we note that, if $r_u(S) \cap \mathbb{C}^+ = \emptyset$ or $r_u(S) \cap \mathbb{C}^- = \emptyset$, and if we choose a canonical selfadjoint extension A of S, we have by (4.2) for each selfadjoint extension $\tilde{A}$ of S

$$[\hat{\tilde{R}}_z u, u]_\pm = [\hat{R}_z u, u]_\pm, \; z \in \rho(A) \cap \rho(\tilde{A}),$$

i.e. there exists, up to real additive constants, exactly one generalized u-resolvent.

Definition 4.8. Let $u \in \mathfrak{P}_-$ with $r_u(S) \neq \emptyset$, and let A be a canonical selfadjoint extension of S. Choose a generalized u-resolvent $r(z)$ of S induced by A, and a Q-function $q(z)$ of A and S. Define a matrix valued function

$$W(z) = \begin{pmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{pmatrix}, \; z \in r_u(S) \cap \rho(A),$$

by

$$w_{11}(z) = \frac{r(z)}{\left[u, \begin{pmatrix} \varphi(\bar{z}) \\ \bar{z}\varphi(\bar{z}) \end{pmatrix}\right]_\pm}, \tag{4.4}$$

$$w_{12}(z) = \frac{r(z)q(z) - \left[u, \begin{pmatrix} \varphi(\bar{z}) \\ \bar{z}\varphi(\bar{z}) \end{pmatrix}\right]_\pm \left[\begin{pmatrix} \varphi(z) \\ z\varphi(z) \end{pmatrix}, u\right]_\pm}{\left[u, \begin{pmatrix} \varphi(\bar{z}) \\ \bar{z}\varphi(\bar{z}) \end{pmatrix}\right]_\pm}, \tag{4.5}$$

$$w_{21}(z) = \frac{1}{\left[u, \begin{pmatrix} \varphi(\bar{z}) \\ \bar{z}\varphi(\bar{z}) \end{pmatrix}\right]_\pm}, \tag{4.6}$$

$$w_{22}(z) = \frac{q(z)}{\left[u, \begin{pmatrix} \varphi(\bar{z}) \\ \bar{z}\varphi(\bar{z}) \end{pmatrix}\right]_\pm}. \tag{4.7}$$

The matrix $W(z)$ depends on the choice of A, $q(z)$ and $r(z)$. If we choose another generalized u-resolvent induced by A: $r'(z) = r(z) + \beta$, $\beta \in \mathbb{R}$, then the matrix $W'(z)$ defined correspondingly satisfies

$$W'(z) = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} W(z), \; z \in r_u(S) \cap \rho(A). \tag{4.8}$$

If we choose another Q-function: $q'(z) = q(z) + \beta$, $\beta \in \mathbb{R}$, then

$$W'(z) = W(z) \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \; z \in r_u(S) \cap \rho(A). \tag{4.9}$$

Moreover, it follows from Proposition 4.1 by a straightforward computation, that if A' is another canonical selfadjoint extension of S, $r'(z)$ is a conveniently chosen generalized u-resolvent of S induced by A', $q'(z)$ is a Q-function of A' and S, and $W'(z)$ is defined correspondingly, then

$$W'(z) = W(z) \begin{pmatrix} \frac{\tau}{q(\overline{z_0})+\tau} & -(q(z_0)+\tau) \\ \frac{1}{q(\overline{z_0})+\tau} & 0 \end{pmatrix}, \; z \in r_u(S) \cap \rho(A) \cap \rho(\tilde{A}). \tag{4.10}$$

Here $\tau \in \mathbb{R}$ is the parameter corresponding to A' in the Krein formula (2.4).

By Lemma 2.1 this implies that for any choice of A and $r(z)$, the matrix function $W(z)$ defined above has an analytic continuation to $r_u(S)$. We will always understand $W(z)$ as an analytic function on $r_u(S)$. Note that, if $z, \overline{z} \in r_u(S)$, we have

$$W(z) J W(\overline{z})^* = J,$$

with

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{4.11}$$

and that

$$\det W(z) = \frac{\left[\begin{pmatrix} \varphi(z) \\ z\varphi(z) \end{pmatrix}, u \right]_\pm}{\left[u, \begin{pmatrix} \varphi(\bar{z}) \\ \bar{z}\varphi(\bar{z}) \end{pmatrix} \right]_\pm}.$$

In particular $\det W(z) \neq 0$ if $z, \bar{z} \in r_u(S)$, and $\det W$ vanish identically if and only if $r_u(S)$ is contained in one half plane.

Theorem 4.9. *The matrix $W(z)$ given in Definition 4.8 is a generalized u-resolvent matrix of S. Let $\tilde{\kappa} \geq \kappa$, then for any $\tau \in \tilde{\mathcal{N}}_{\tilde{\kappa}-\kappa}$*

$$r(z) = (W \circ \tau)(z)$$

is a generalized u-resolvent of index $\tilde{\kappa}$. Any generalized u-resolvent of index $\tilde{\kappa}$ can be represented in this way up to a real additive constant.

Proof : In the definition of $W(z)$ write $r(z) = \gamma + [\hat{R}_z u, u]_\pm$. Consider the generalized u-resolvent $[\hat{\tilde{R}} u, u]_\pm$ of S induced by a $\mathfrak{P}$-minimal extension $\tilde{A}$, and denote by $\tau(z)$ the parameter corresponding to $\tilde{A}$ by the Krein formula. Due to (4.2) we have for $z \in \rho(A) \cap \rho(\tilde{A}) \cap r_u(S)$

$$[\hat{\tilde{R}}_z u, u]_\pm = [\hat{R}_z u, u]_\pm - \frac{\left[u, \begin{pmatrix} \varphi(\bar{z}) \\ \bar{z}\varphi(\bar{z}) \end{pmatrix} \right]_\pm \left[\begin{pmatrix} \varphi(z) \\ z\varphi(z) \end{pmatrix}, u \right]_\pm}{q(z) + \tau(z)} + \beta(\tilde{A}) =$$

$$= \frac{r(z)\tau(z) + r(z)q(z) - \left[u, \begin{pmatrix} \varphi(\bar{z}) \\ \bar{z}\varphi(\bar{z}) \end{pmatrix} \right]_\pm \left[\begin{pmatrix} \varphi(z) \\ z\varphi(z) \end{pmatrix}, u \right]_\pm}{q(z) + \tau(z)} + \beta(\tilde{A}) - \gamma =$$

$$= \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_{21}(z)\tau(z) + w_{22}(z)} + \beta(\tilde{A}) - \gamma$$

where

$$\beta(\tilde{A}) = \frac{\left[u, \begin{pmatrix} \varphi(\overline{z_0}) \\ \overline{z_0}\varphi(\overline{z_0}) \end{pmatrix}\right]_\pm}{2(q(z_0) + \tau(z_0))} \left[\begin{pmatrix} \varphi(z_0) \\ z_0\varphi(z_0) \end{pmatrix}, u\right]_\pm +$$

$$+ \frac{\left[u, \begin{pmatrix} \varphi(z_0) \\ z_0\varphi(z_0) \end{pmatrix}\right]_\pm}{2(q(\overline{z_0}) + \tau(\overline{z_0}))} \left[\begin{pmatrix} \varphi(\overline{z_0}) \\ \overline{z_0}\varphi(\overline{z_0}) \end{pmatrix}, u\right]_\pm .$$

□

A 2×2-matrix is called (iJ)-unitary if

$$UJU^* = J. \tag{4.12}$$

Note that with U also U^* is (iJ)-unitary. Moreover, it follows from the fact that a matrix U is (iJ)-unitary if and only if $|\det U| = 1$ and the fractional linear transformation $U \circ z$ maps the upper half plane onto itself, that the relation $\sigma = U \circ \tau$ establishes a bijection of $\mathcal{N}_\kappa$ onto itself. Hence the matrix $W(z)U$ is a generalized u-resolvent matrix whenever $W(z)$ has this property. Note that the matrices occuring on the right hand sides of (4.9) and (4.10) are (iJ)-unitary. Clearly, for any generalized u-resolvent matrix $W(z)$ and number $\beta \in \mathbb{R}$, the matrix defined by the right hand side of (4.8) also is a generalized u-resolvent matrix. Under a certain additional assumption also a converse result holds. Before we prove this converse, we provide a lemma which can be proved by an elementary calculation.

Lemma 4.10. *Consider a matrix*

$$W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \in \mathbb{C}^{2\times 2},$$

and assume that $\det W \neq 0$ *and* $w_{21} \neq 0$. *The fractional linear transformation* $W \circ z$ *maps the real line onto the circle with radius*

$$r = \left| \frac{\det W}{w_{21}\overline{w_{22}} - \overline{w_{21}}w_{22}} \right|,$$

and center

$$C = \frac{w_{11}\overline{w_{22}} - w_{12}\overline{w_{21}}}{w_{21}\overline{w_{22}} - \overline{w_{21}}w_{22}}.$$

This circle is in fact a straight line if and only if $w_{21}\overline{w_{22}} - \overline{w_{21}}w_{22} = 0$. *Otherwise, the upper half plane is mapped onto the interior of this circle if and only if*

$$\operatorname{Im} \frac{w_{22}}{w_{21}} > 0.$$

Theorem 4.11. *Let* $W_1(z)$ *and* $W_2(z)$ *be generalized* u*-resolvent matrices of* S. *Assume* $r_u(S) \cap \mathbb{C}^+ \neq \emptyset$, $r_u(S) \cap \mathbb{C}^- \neq \emptyset$ *and that there exists a positive defect element of* S. *Then there* *exist an (iJ)-unitary matrix U, a number $\beta \in \mathbb{R}$, and a function $\gamma(z)$ which is meromorphic and not identically zero on $r_u(S)$, such that*

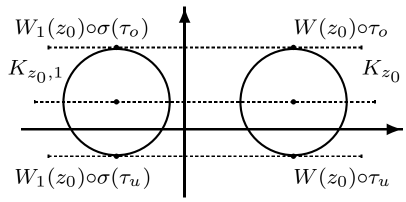
$$W_2(z) = \gamma(z) \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} W_1(z)U, \; z \in r_u(S).$$

Proof : Let $W_1(z)$ be a generalized u-resolvent matrix of S and let $W(z)$ be as in Definition 4.8. Then for each $\tau \in \widetilde{\mathcal{N}}_0$ there exists a function $\sigma(\tau) \in \widetilde{\mathcal{N}}_0$ and a number $\alpha(\tau) \in \mathbb{R}$, such that

$$W_1(z) \circ \sigma(\tau) = W(z) \circ \tau + \alpha(\tau), \; z \in r_u(S). \tag{4.13}$$

The fractional linear transformation $W(z) \circ w$ maps $\mathbb{C}^+$ onto the interior (exterior) of some circle K_z. By our assumtions on S there exists a number $z_0 \in r_u(S) \cap \mathbb{C}^+$, such that K_{z_0} has a finite nonzero radius and $\mathbb{C}^+$ is mapped onto the interior of K_{z_0}. Let $O \neq \emptyset$ be an open set containing z_0 such that K_z has finite and nonzero radius for all $z \in O$. If the radius of K_z is not constant on O, assume moreover that the radius of K_{z_0} is not a local minimum of the radii in O.

In (4.13) we can choose $\tau(z) = \tau \in \mathbb{C}^+ \cup \mathbb{R}$ and, conversely, any constant function $\sigma(z) = \sigma \in \mathbb{C}^+ \cup \mathbb{R}$ can be obtained by some choice of $\tau(z)$. Since the number α in (4.13) is real, each point of K_z lies on the same horizontal line as some point of $K_{z,1}$. Hence the transformation $W_1(z) \circ w$ maps $\mathbb{C}^+ \cup \mathbb{R}$ onto the interior of a circle $K_{z,1}$. For $z = z_0$ we have the following picture:



Here $\tau_o, \tau_u \in \mathbb{R}$ are constants. The corresponding functions $\sigma(\tau_o)$ and $\sigma(\tau_u)$ assume a real value at z_0. Hence they are also real constants. Clearly $\alpha(\tau_o) = \alpha(\tau_u)$.

Consider the transformation $VW(z) \circ w$ with

$$V = \begin{pmatrix} 1 & -\alpha(\tau_o) \\ 0 & 1 \end{pmatrix},$$

and let $K_{z,2}$ be the image of $\mathbb{C}^+ \cup \mathbb{R}$ under this transformation. Then $K_{z_0,1} = K_{z_0,2}$. Since $\alpha(\tau)$ does not depend on z the points $p_1(z) = VW(z) \circ \tau_o$ and $p_2(z) = VW(z) \circ \tau_u$ are located on the boundary of all circles $K_{z,1}$ and $K_{z,2}$ for $z \in O$. Since, by (4.13), the circles $K_{z,1}$ and $K_{z,2}$ have the same radius and their centers lie on a horizontal line, they must coincide unless the line segment connecting $p_1(z)$ and $p_2(z)$ is vertical.

Assume on the contrary that they do not coincide on O. Since by analyticity the function $p_1(z) - p_2(z)$ is constant, we find that the diameter of $K_{z,2}$ cannot be smaller than $|p_1(z_0) - p_2(z_0)|$. By the choice of z_0 this implies that the diameter of $K_{z,2}$ is constant, in fact equal to $|p_1(z_0) - p_2(z_0)|$. Since a circle is uniquely determined by two antipodal points, the circles $K_{z,2}$ and $K_{z,1}$ must coincide, a contradiction.

We arrive at the conclusion that $K_{z,1} = K_{z,2}$ for $z \in O$, hence the fractional linear transformation $[VW(z)]^{-1}W_1(z) \circ w$ maps $\mathbb{C}^+$ onto itself and depends analytically on z. This implies that it is of the form $\gamma(z)U$ with some (iJ)-unitary matrix U and an analytic function $\gamma(z),\ z \in O$ (compare [S1]).

Since $r_u(S) \cap \mathbb{C}^+$ is connected, it follows by analyticity that $\gamma(z)$ can be extended meromorphically to $r_u(S) \cap \mathbb{C}^+$ and that

$$W_1(z) = \gamma(z)VW(z)U,\ z \in r_u(S) \cap \mathbb{C}^+.$$

Let $z \in r_u(S) \cap \mathbb{C}^-$, then

$$W_1(z) \circ \tau = r(z) = \overline{r(\overline{z})} = \overline{W_1(\overline{z})} \circ \tau,$$

where $\overline{W_1}$ denotes the conjugate (but not transpose) of the matrix W_1. This implies that the fractional linear transformations $W_1(z) \circ w$ and $\overline{W_1(\bar{z})} \circ w$ are identical, hence $W_1(z) = \alpha(z)\overline{W_1(\bar{z})}$, where $\alpha(z)$ depends analytically on $z \in \mathbb{C}^- \cap r_u(S)$.

From this relation, the definition of $W(z)$ and the fact that V and U are (iJ)-unitary, we find

$$W_1(z) = \alpha(z)\overline{W_1(\overline{z})} = \alpha(z)\overline{\gamma(\overline{z})VW(\overline{z})U} =$$
$$= \left(\lambda\alpha(z)\overline{\gamma(\overline{z})}\frac{[u, \varphi(\overline{z})]}{[\varphi(z), u]}\right) VW(z)U,\ z \in r_u(S) \cap \mathbb{C}^-.$$

Here λ is such that $\overline{U} = \lambda U$. The existence of such a number λ is proved by an elementary consideration.

□

5 Representations by spaces of analytic functions

Choose an element $u \in \mathfrak{P}_-$, such that $r_u(S) \neq \emptyset$. Let A be a canonical selfadjoint extension of S, $\rho(A) \neq \emptyset$, and denote by $\varphi(z)$ defect elements of S connected with A by (4.3). Denote by $\mathcal{P}(z)$, $z \in r_u(S) \cap \rho(A)$, the linear functional on $\mathfrak{P}$ defined by

$$\mathcal{P}(z)f = \frac{[f, \varphi(\bar{z})]}{\left[u, \begin{pmatrix} \varphi(\bar{z}) \\ \bar{z}\varphi(\bar{z}) \end{pmatrix}\right]_{\pm}},\ f \in \mathfrak{P}. \tag{5.1}$$

Moreover, denote by $\mathcal{Q}(z)$, $z \in r_u(S) \cap \rho(A)$ the linear functional on $\mathfrak{P}$ defined by

$$\mathcal{Q}(z)f = [R_z^+ f, u]_{\pm} - (\mathcal{P}(z)f)r(z),\ f \in \mathfrak{P}, \tag{5.2}$$

where $r(z)$ denotes a fixed generalized u-resolvent induced by A. Clearly, $\mathcal{P}(z)$ and $\mathcal{Q}(z)$ do not depend on the choice of the defect elements $\varphi(z)$. Moreover, if we choose another generalized u-resolvent induced by A, then $\mathcal{Q}(z)$ is changed only by adding a real multiple of $\mathcal{P}(z)$.

Lemma 5.1. *The functionals $\mathcal{P}(z)$ and $\mathcal{Q}(z)$ have an analytic continuation to $r_u(S)$. If $\tilde{A}$ is another canonical selfadjoint extension of S and $\widetilde{\mathcal{P}}(z)$ and $\widetilde{\mathcal{Q}}(z)$, $z \in r_u(S) \cap \rho(\tilde{A})$, are defined similar as $\mathcal{P}$ and $\mathcal{Q}$ with $\tilde{A}$ instead of A, and with the generalized u-resolvent $\tilde{r}(z)$ induced by $\tilde{A}$, such that*

$$\tilde{r}(z) = r(z) - \frac{\left[u, \begin{pmatrix} \varphi(\bar{z}) \\ \bar{z}\varphi(\bar{z}) \end{pmatrix}\right]_\pm \left[\begin{pmatrix} \varphi(z) \\ z\varphi(z) \end{pmatrix}, u\right]_\pm}{q(z) + \tau},$$

where $\tau = \tau(z)$ and $q(z)$ are as in the Krein formula, then

$$\widetilde{\mathcal{P}}(z)f = \mathcal{P}(z)f, \;\; f \in \mathfrak{P}, z \in r_u(S) \cap \rho(A) \cap \rho(\tilde{A}),$$

and

$$\widetilde{\mathcal{Q}}(z)f = \mathcal{Q}(z)f, \;\; f \in \mathfrak{P}, z \in r_u(S) \cap \rho(A) \cap \rho(\tilde{A}). \tag{5.3}$$

Proof : Clearly $\mathcal{P}(z)$ and $\mathcal{Q}(z)$ are analytic functions on $r_u(S) \cap \rho(A)$. Hence, the first assertion will follow once we have proved that $\mathcal{P}$ and $\mathcal{Q}$ do not essentially depend on the choice of A.

Let $f \in \mathfrak{P}$. Since $\mathcal{P}(z)f$ is the unique number, such that

$$\iota f - (\mathcal{P}(z)f)u \in \ker\left(\left[., \begin{pmatrix} \varphi(\bar{z}) \\ \bar{z}\varphi(\bar{z}) \end{pmatrix}\right]_\pm\right),$$

and the kernel of the functional $\left[., \begin{pmatrix} \varphi(\bar{z}) \\ \bar{z}\varphi(\bar{z}) \end{pmatrix}\right]_\pm$ does not depend on the choice of A, also $\mathcal{P}(z)f$ does not depend on A.

By the Krein formula (2.4), there corresponds to $\tilde{A}$ a parameter $\tau(z) = \tau \in \mathbb{R}$. By (3.7) and (4.2) we find

$$\widetilde{\mathcal{Q}}(z)f = [R_z^+ f, u] - (\mathcal{P}(z)f)r(z) - \frac{[f, \varphi(\bar{z})]\left[\begin{pmatrix} \varphi(z) \\ z\varphi(z) \end{pmatrix}, u\right]_\pm}{q(z) + \tau} +$$
$$+(\mathcal{P}(z)f)\frac{\left[u, \begin{pmatrix} \varphi(\bar{z}) \\ \bar{z}\varphi(\bar{z}) \end{pmatrix}\right]_\pm \left[\begin{pmatrix} \varphi(z) \\ z\varphi(z) \end{pmatrix}, u\right]_\pm}{q(z) + \tau} = \mathcal{Q}(z)f. \tag{5.4}$$

□

We will always consider $\mathcal{P}(z)$ and $\mathcal{Q}(z)$ as analytic functions on $r_u(S)$. In the following we assume that for $u \in \mathfrak{P}_-$ the minimality condition

$$\text{cls}\left(\{\varphi(z) : z \in \rho(A)\} \cup \{R_z^- u : z \in \rho(A)\}\right) = \mathfrak{P}. \tag{5.5}$$

is satisfied. Note that by (3.8) this condition does not depend on the choice of A.

Let Φ be the mapping, which assings to each $f \in \mathfrak{P}$ the analytic vector function

$$\Phi(f) = \mathbf{f}(z) = \begin{pmatrix} -\mathcal{Q}(z)f \\ \mathcal{P}(z)f \end{pmatrix}, \; z \in r_u(S).$$

If $r_u(S)$ has a nonvoid intersection with both halfplanes, then by (5.5) the mapping Φ is injective.

Definition 5.2. Let $u \in \mathfrak{P}_-$ be such that $r_u(S) \cap \mathbb{C}^+ \neq \emptyset$, $r_u(S) \cap \mathbb{C}^- \neq \emptyset$ and assume that (5.5) holds. Denote by $\mathfrak{P}_u$ the Pontryagin space of vector functions analytic on $r_u(S)$

$$\mathfrak{P}_u = \Phi(\mathfrak{P}) = \left\{ \mathbf{f}(z) = \begin{pmatrix} -\mathcal{Q}(z)f \\ \mathcal{P}(z)f \end{pmatrix} : f \in \mathfrak{P} \right\},$$

endowed with the inner product

$$[\mathbf{f}(z), \mathbf{g}(z)]_{\mathfrak{P}_u} = [f, g]_{\mathfrak{P}}.$$

Clearly, the mapping Φ is an isomorphism from $\mathfrak{P}$ onto $\mathfrak{P}_u$. Let π_- be the projection of $\mathfrak{P}_u$ onto its second component. In the space $\mathfrak{P}_u$ define a linear relation S_m by

$$S_m = \left\{ (\mathbf{f}(z); \mathbf{g}(z)) \in \mathfrak{P}_u^2 : (\pi_- \mathbf{g})(z) = z(\pi_- \mathbf{f})(z) \right\}. \tag{5.6}$$

Let J be as in (4.11), and denote by $\mathcal{R}_1(a)$ the difference quotient

$$\mathcal{R}_1(a)\mathbf{f}(z) = \frac{\mathbf{f}(z) - \mathbf{f}(a)}{z - a}.$$

Theorem 5.3. *Assume that $r_u(S)$ has a nonvoid intersection with both halfplanes. Then via the isomorphism Φ, the relation S corresponds to the subset of S_m determined by*

$$(\Phi(S) - a)^{-1} = \mathcal{R}_1(a)|_{\{\mathbf{f} \in \mathfrak{P}_u : (\pi_- \mathbf{f})(a) = 0\}}.$$

If $a \in r_u(S)$, then

$$\mathcal{R}_1(a)\mathbf{f}(z) \in \mathfrak{P}_u, \;\; \mathbf{f}(z) \in \mathfrak{P}_u.$$

The operator $\mathcal{R}_1(a) : \mathfrak{P}_u \to \mathfrak{P}_u$ is the resolvent operator of a certain linear relation extending S_m. In particular it depends continuously on $\mathbf{f} \in \mathfrak{P}_u$ and (in the operator norm of $\mathfrak{P}_u$) analytically on $a \in r_u(S)$. For $\mathbf{f}, \mathbf{g} \in \mathfrak{P}_u$, $a, b \in r_u(S)$ the following identity holds:

$$\mathbf{g}(b)^* J \mathbf{f}(a) = [\mathbf{f}(z), \mathcal{R}_1(b)\mathbf{g}(z)] - [\mathcal{R}_1(a)\mathbf{f}(z), \mathbf{g}(z)] + (a - \bar{b})[\mathcal{R}_1(a)\mathbf{f}(z), \mathcal{R}_1(b)\mathbf{g}(z)].$$

Proof : We first show that $\mathfrak{P}_u$ is invariant under forming difference quotients. Assume that $a \in \rho(A)$, and consider the element $R_a^-(\iota f - (\mathcal{P}(a)f)u) \in \mathfrak{P}$. Since

$$R_a^+ \varphi(z) = \begin{pmatrix} \frac{\varphi(z) - \varphi(a)}{z - a} \\ \frac{z\varphi(z) - a\varphi(a)}{z - a} \end{pmatrix}, \tag{5.7}$$

we have

$$\mathcal{P}(z)(R_a^-(\iota f-(\mathcal{P}(a)f)u))=\frac{[R_a^-(\iota f-(\mathcal{P}(a)f)u),\varphi(\bar z)]}{\left[u,\begin{pmatrix}\varphi(\bar z)\\ \bar z\varphi(\bar z)\end{pmatrix}\right]_\pm}=$$

$$=\frac{\left[\iota f-(\mathcal{P}(a)f)u,\begin{pmatrix}\frac{\varphi(\bar z)-\varphi(\bar a)}{\bar z-\bar a}\\ \frac{\bar z\varphi(\bar z)-\bar a\varphi(\bar a)}{\bar z-\bar a}\end{pmatrix}\right]_\pm}{\left[u,\begin{pmatrix}\varphi(\bar z)\\ \bar z\varphi(\bar z)\end{pmatrix}\right]_\pm}=\frac{(\mathcal{P}(z)f)-(\mathcal{P}(a)f)}{z-a}.$$

The relation (3.2) implies that

$$R_z^+R_a^-=\frac{1}{z-a}\begin{pmatrix}R_z^--R_a^-\\ (I+zR_z^-)-(I+aR_a^-)\end{pmatrix}=\frac{\hat R_z-\hat R_a}{z-a}\tag{5.8}$$

and, since $R_z^-\iota=(A-z)^{-1}$, we find

$$\mathcal{Q}(z)(R_a^-(\iota f-(\mathcal{P}(a)f)u))=$$

$$=[R_z^+R_a^-(\iota f-(\mathcal{P}(a)f)u),u]_\pm-\mathcal{P}(z)(R_a^-(\iota f-(\mathcal{P}(a)f)u))r(z)=$$

$$=\frac{[R_z^+f,u]_\pm-[R_a^+f,u]_\pm-(\mathcal{P}(a)f)r(z)+(\mathcal{P}(a)f)r(a)}{z-a}-$$

$$-\frac{(\mathcal{P}(z)f)-(\mathcal{P}(a)f)}{z-a}r(z)=\frac{(\mathcal{Q}(z)f)-(\mathcal{Q}(a)f)}{z-a}.$$

This shows that

$$\mathcal{R}_1(a)\mathbf{f}(z)=\frac{\mathbf{f}(z)-\mathbf{f}(a)}{z-a}=\Phi(R_a^-(\iota f-(\mathcal{P}(a)f)u)),$$

in particular the difference quotient is contained in $\mathfrak{P}_u$ and depends continuously on a. If $a\in r_u(S)\setminus\rho(A)$, the assertion follows from the already proved fact, by a convenient choice of another canonical selfadjoint extension $\tilde A$ and use of Lemma 5.1. Clearly, $\mathcal{R}_1$ satisfies the resolvent identity. Hence it is the resolvent of a certain linear relation which obviously extends $\Phi(S)$.

To prove the asserted identity for difference quotients we calculate $(a,b\in r_u(S)\cap\rho(A))$

$$[f,R_b^-(\iota g-(\mathcal{P}(b)g)u)]-[R_a^-(\iota f-(\mathcal{P}(a)f)u),g]+$$

$$+(a-\bar b)[R_a^-(\iota f-(\mathcal{P}(a)f)u),R_b^-(\iota g-(\mathcal{P}(b)g)u)]=$$

$$=[((A-\bar b)^{-1}-(A-a)^{-1})f,g]+(a-\bar b)[(A-\bar b)^{-1}(A-a)^{-1}f,g]-$$

$$-\overline{(\mathcal{P}(b)g)}[f,R_b^-u]+(\mathcal{P}(a)f)[R_a^-u,g]-(a-\bar b)(\mathcal{P}(a)f)[(A-\bar b)^{-1}R_a^-u,g]-$$

$$-(a-\bar b)\overline{(\mathcal{P}(b)f)}[(A-a)^{-1}R_b^+f,u]_\pm+(a-\bar b)(\mathcal{P}(a)f)\overline{(\mathcal{P}(b)f)}[R_{\bar b}^+R_a^-u,u]_\pm=$$

$$=(\mathcal{P}(a)f)([R_{\bar b}^-u,g]-\overline{(\mathcal{P}(b)f)r(b)})-\overline{(\mathcal{P}(b)f)}([f,R_a^-u]-(\mathcal{P}(a)f)r(a))=$$

$$= \begin{pmatrix} -\mathcal{Q}(b) \\ \mathcal{P}(b) \end{pmatrix}^* J \begin{pmatrix} -\mathcal{Q}(a) \\ \mathcal{P}(a) \end{pmatrix}.$$

For arbitrary $a, b \in r_u(S)$ the identity now follows by continuity.

Finally we show that S corresponds to a subset of S_m via Φ. Let $f \in \mathfrak{P}$ and $a \in r_u(S) \cap \rho(A)$, then $f \in \operatorname{ran}(S-a)$ if and only if $\mathcal{P}(a)f = 0$. Hence

$$R_a^-(\iota f - (\mathcal{P}(a)f)u) = R_a^- \iota f = (S-a)^{-1} f.$$

The first part of this proof shows that

$$\Phi(S-a)^{-1} = \{(\mathbf{f}(z); \frac{\mathbf{f}(z) - \mathbf{f}(a)}{z-a}) \in \mathfrak{P}_u^2 : (\pi_- f)(a) = 0\},$$

which yields the desired result.

□

In the sequel we will show that the space $\mathfrak{P}_u$ is a reproducing kernel space, and that its reproducing kernel is in fact given by

$$\frac{W(z)JW(w)^* - J}{z - \bar{w}}, \; z, w \in r_u(S), \tag{5.9}$$

where $W(z)$ is as in Definition 4.8.

Theorem 5.4. *For $z, w \in r_u(S)$ we have*

$$\frac{W(z)JW(w)^* - J}{z - \bar{w}} = \begin{pmatrix} -\mathcal{Q}(z) \\ \mathcal{P}(z) \end{pmatrix} (-\mathcal{Q}(w)^*, \mathcal{P}(w)^*). \tag{5.10}$$

Moreover, if $r_u(S)$ has a nonvoid intersection with both halfplanes, then for each vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2$ and each number $w \in r_u(S)$ the function

$$\frac{W(z)JW(w)^* - J}{z - \bar{w}} \begin{pmatrix} x \\ y \end{pmatrix}, \; z \in r_u(S),$$

belongs to $\mathfrak{P}_u$, and for all $\mathbf{f} \in \mathfrak{P}_u$ we have

$$\left[\mathbf{f}(z), \frac{W(z)JW(w)^* - J}{z - \bar{w}} \begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} x \\ y \end{pmatrix}^* \mathbf{f}(w), \; z, w \in r_u(S). \tag{5.11}$$

Proof : For $w \in r_u(S)$ the functions

$$f \mapsto \mathcal{P}(w) f, \; f \mapsto \mathcal{Q}(w) f, \; f \in \mathfrak{P},$$

are bounded linear functionals on $\mathfrak{P}$. Their adjoints are given by

$$\mathcal{P}(w)^* \xi = \xi \frac{1}{\left[\begin{pmatrix} \varphi(\bar{w}) \\ \bar{w}\varphi(\bar{w}) \end{pmatrix}, u\right]_\pm} \varphi(\bar{w}), \; \xi \in \mathbb{C},$$

$$\mathcal{Q}(w)^*\xi = \xi(R^-_{\bar w}u - \frac{r(\bar w)}{\left[\begin{pmatrix}\varphi(\bar w)\\ \bar w\varphi(\bar w)\end{pmatrix}, u\right]_\pm}\varphi(\bar w)),\ \xi\in\mathbb{C}.$$

First we prove (5.10). In fact by (5.7) and (5.8)

$$\mathcal{Q}(z)\mathcal{Q}(w)^* = \frac{r(\bar w)r(z)[\varphi(\bar w),\varphi(\bar z)]}{\left[u,\begin{pmatrix}\varphi(\bar z)\\ \bar z\varphi(\bar z)\end{pmatrix}\right]_\pm\left[\begin{pmatrix}\varphi(\bar w)\\ \bar w\varphi(\bar w)\end{pmatrix}, u\right]_\pm}+$$

$$+[R^+_z R^-_{\bar w}u,u] - \frac{r(\bar w)\,[R^+_z\varphi(\bar w),u]_\pm}{\left[\begin{pmatrix}\varphi(\bar w)\\ \bar w\varphi(\bar w)\end{pmatrix}, u\right]_\pm} - \frac{r(z)[u,R^+_w\varphi(\bar z)]_\pm}{\left[u,\begin{pmatrix}\varphi(\bar z)\\ \bar z\varphi(\bar z)\end{pmatrix}\right]_\pm} =$$

$$= \frac{r(\bar w)r(z)q(z) - r(\bar w)r(z)q(\bar w)}{(z-\bar w)\left[u,\begin{pmatrix}\varphi(\bar z)\\ \bar z\varphi(\bar z)\end{pmatrix}\right]_\pm\left[\begin{pmatrix}\varphi(\bar w)\\ \bar w\varphi(\bar w)\end{pmatrix}, u\right]_\pm} - \frac{r(\bar w)\left[\begin{pmatrix}\varphi(z)\\ z\varphi(z)\end{pmatrix}, u\right]_\pm}{(z-\bar w)\left[\begin{pmatrix}\varphi(\bar w)\\ \bar w\varphi(\bar w)\end{pmatrix}, u\right]}+$$

$$+\frac{r(z)\left[u,\begin{pmatrix}\varphi(w)\\ w\varphi(w)\end{pmatrix}\right]_\pm}{(z-\bar w)\left[u,\begin{pmatrix}\varphi(\bar z)\\ \bar z\varphi(\bar z)\end{pmatrix}\right]} = \frac{w_{12}(z)\overline{w_{11}(w)} - w_{11}(z)\overline{w_{12}(w)}}{z-\bar w},$$

$$-\mathcal{P}(z)\mathcal{Q}(w)^* = -\frac{[R^-_{\bar w}u,\varphi(\bar z)]}{\left[u,\begin{pmatrix}\varphi(\bar z)\\ \bar z\varphi(\bar z)\end{pmatrix}\right]} + \frac{r(\bar w)[\varphi(\bar w),\varphi(\bar z)]}{\left[u,\begin{pmatrix}\varphi(\bar z)\\ \bar z\varphi(\bar z)\end{pmatrix}\right]_\pm\left[\begin{pmatrix}\varphi(\bar w)\\ \bar w\varphi(\bar w)\end{pmatrix}, u\right]_\pm} =$$

$$= -\frac{1}{z-\bar w} + \frac{\left[u,\begin{pmatrix}\varphi(w)\\ w\varphi(w)\end{pmatrix}\right]}{(z-\bar w)\left[u,\begin{pmatrix}\varphi(\bar z)\\ \bar z\varphi(\bar z)\end{pmatrix}\right]}+$$

$$+\frac{r(\bar w)q(z) - r(\bar w)q(\bar w)}{(z-\bar w)\left[u,\begin{pmatrix}\varphi(\bar z)\\ \bar z\varphi(\bar z)\end{pmatrix}\right]_\pm\left[\begin{pmatrix}\varphi(\bar w)\\ \bar w\varphi(\bar w)\end{pmatrix}, u\right]_\pm} =$$

$$= \frac{w_{22}(z)\overline{w_{11}(w)} - w_{21}(z)\overline{w_{12}(w)} - 1}{z-\bar w},$$

$$-\mathcal{Q}(z)\mathcal{P}(w)^* = -\frac{[R^+_z\varphi(\bar w),u]}{\left[\begin{pmatrix}\varphi(\bar w)\\ \bar w\varphi(\bar w)\end{pmatrix}, u\right]} + \frac{r(z)[\varphi(\bar w),\varphi(\bar z)]}{\left[u,\begin{pmatrix}\varphi(\bar z)\\ \bar z\varphi(\bar z)\end{pmatrix}\right]_\pm\left[\begin{pmatrix}\varphi(\bar w)\\ \bar w\varphi(\bar w)\end{pmatrix}, u\right]_\pm} =$$

$$= \frac{1}{z-\bar w} - \frac{\left[\begin{pmatrix}\varphi(z)\\ z\varphi(z)\end{pmatrix}, u\right]}{(z-\bar w)\left[\begin{pmatrix}\varphi(\bar w)\\ \bar w\varphi(\bar w)\end{pmatrix}, u\right]}+$$

$$+\frac{r(z)q(z)-r(z)q(\bar w)}{(z-\bar w)\left[u,\begin{pmatrix}\varphi(\bar z)\\ \bar z\varphi(\bar z)\end{pmatrix}\right]_{\pm}\left[\begin{pmatrix}\varphi(\bar w)\\ \bar w\varphi(\bar w)\end{pmatrix},u\right]_{\pm}}=$$

$$=\frac{w_{12}(z)\overline{w_{21}(w)}-w_{11}(z)\overline{w_{22}(w)}+1}{z-\bar w},$$

$$\mathcal{P}(z)\mathcal{P}(w)^*=\frac{[\varphi(\bar w),\varphi(\bar z)]}{\left[u,\begin{pmatrix}\varphi(\bar z)\\ \bar z\varphi(\bar z)\end{pmatrix}\right]_{\pm}\left[\begin{pmatrix}\varphi(\bar w)\\ \bar w\varphi(\bar w)\end{pmatrix},u\right]_{\pm}}=$$

$$=\frac{w_{22}(z)\overline{w_{21}(w)}-w_{21}(z)\overline{w_{22}(w)}}{z-\bar w}.$$

As for each $w\in r_u(S)$ and $x,y\in\mathbb{C}$ the elements $\mathcal{P}(w)^*x$ and $\mathcal{Q}(w)^*y$ belong to $\mathfrak{P}$, relation (5.10) implies that

$$\frac{W(z)JW(w)^*-J}{z-\bar w}\begin{pmatrix}x\\ y\end{pmatrix},\ w\in r_u(S),$$

belongs to $\mathfrak{P}_u$ as a function of z. Relation (5.10) also shows that (5.11) holds.

□

6 Matrices of the class $\mathcal{M}_\kappa$

Definition 6.1. Let $\kappa\in\mathbb{N}\cup\{0\}$. A 2×2-matrix function $M(z)$ defined and analytic in some open set O, which is not constant, is said to belong to the class $\mathcal{M}_\kappa$ if it satisfies

$$M(z)JM(\bar z)^*=J,\ z,\bar z\in O, \tag{6.1}$$

and if the kernel

$$\frac{M(z)JM(w)^*-J}{z-\bar w},\ z,w\in O, \tag{6.2}$$

has exactly κ negative squares.

If U is (iJ)-unitary, then $M(z)U$ and $UM(z)$ belong to $\mathcal{M}_\kappa$ whenever $M(z)$ has this property.

Due to (5.10) the matrix function $W(z)$ given in Definition 4.8, hence also $VW(z)U$ with (iJ)-unitary matrices U and V, belongs to a class $\mathcal{M}_{\kappa'}$ where $\kappa'\leq \mathrm{Ind}\_\mathfrak{P}$. Note that from the already proved and (5.10) it follows that, if $r_u(S)$ has a nonvoid intersection with both halfplanes, in fact $\kappa'=\mathrm{Ind}\_\mathfrak{P}$. It will follow from Theorem 6.5 that $\kappa'=\mathrm{Ind}\_\mathfrak{P}$ in general.

We shall show in the sequel that every matrix of the class $\mathcal{M}_\kappa$ can be written as $VW(z)U$ with W as in Definition 4.8.

In order to study matrices of the class $\mathcal{M}_\kappa$ it is convenient to consider the so called Potapov-Ginzburg transfomation. Recall that, if $M(z)$ is a 2×2-matrix function

$$M(z)=\begin{pmatrix}m_{11}(z)&m_{12}(z)\\ m_{21}(z)&m_{22}(z)\end{pmatrix},$$

and $m_{21}(z)$ does not vanish identically, its Potapov-Ginzburg transform $\Psi(M)(z)$ is defined by

$$\Psi(M)(z) = \begin{pmatrix} \frac{m_{11}(z)}{m_{21}(z)} & \frac{m_{11}(z)m_{22}(z)-m_{21}(z)m_{12}(z)}{m_{21}(z)} \\ \frac{1}{m_{21}(z)} & \frac{m_{22}(z)}{m_{21}(z)} \end{pmatrix}.$$

The following result has been proved e.g. in [Br].

Lemma 6.2. *The Potapov-Ginzburg transformation satisfies* $(\Psi \circ \Psi)(M) = M$. *The function* $M(z)$ *satisfies* (6.1) *if and only if*

$$\Psi(M)(\bar{z}) = [\Psi(M)(z)]^*.$$

Moreover, the kernel relation

$$\frac{M(z)JM(w)^* - J}{z - \bar{w}} = \tag{6.3}$$

$$\begin{pmatrix} -1 & m_{11}(z) \\ 0 & m_{21}(z) \end{pmatrix} \frac{\Psi(z) - \Psi(w)^*}{z - \bar{w}} \begin{pmatrix} -1 & m_{11}(w) \\ 0 & m_{21}(w) \end{pmatrix}^*$$

holds. Hence $M \in \mathcal{M}_\kappa$ *if and only if* $\Psi(M)(z) \in \mathcal{N}_\kappa^2$.

Lemma 6.3. *Let* $M \in \mathcal{M}_\kappa$ *be given. Then there exist* (iJ)*-unitary matrices* U *and* V, *such that the left lower entry of* $V^{-1}M(z)U^{-1}$ *does not vanish identically and that not both off-diagonal entries and the lower right entry of* $\Psi(V^{-1}MU^{-1})(z)$ *are constant.*

Proof : First we arrange, by multiplying $M(z)$ from the left and right with appropriate (iJ)-unitary matrices U_1 and U_2, that the left lower entry of $U_1M(z)U_2$ does not vanish identically. If already $m_{21} \neq 0$ put $U_1 = U_2 = I$. If $m_{21} = 0$ and $m_{22} \neq 0$ choose

$$U_1 = I \text{ and } U_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

If $m_{21} = m_{22} = 0$ choose

$$U_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and $U_2 = I$ or $U_2 = U_1$ depending whether $m_{11} \neq 0$ or $m_{11} = 0$. Remember that, since M is not constant some entry does not vanish identically.

Next we arrange that at least one of the off-diagonal and lower right entries of the Potapov-Ginzburg transform $\Psi(U_1MU_2)(z)$ is not constant. If this is already the case we set $U_3 = I$. If these entries of $\Psi(U_1MU_2)$ are constant, we conclude by the fact that M itself is not constant, that m_{11} is not constant. Choose

$$U_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and consider the matrix $U_1M(z)U_2U_3$. It follows that the upper right entry of $\Psi(U_1MU_2U_3)(z)$ is not constant.

Finally we set $V = U_1^{-1}$ and $U = (U_2U_3)^{-1}$ which proves the assertion of the lemma.

□

We apply the Potapov-Ginzburg transformation in particular to the matrix function $W(z)$ of Definition 4.8.

Lemma 6.4. *The matrix function $W(z)$ given in Definition 4.8 is the Potapov-Ginzburg transformation of*

$$Q(z) = \begin{pmatrix} r(z) & \left[\begin{pmatrix} \varphi(z) \\ z\varphi(z) \end{pmatrix}, u\right]_\pm \\ \left[u, \begin{pmatrix} \varphi(\bar{z}) \\ \bar{z}\varphi(\bar{z}) \end{pmatrix}\right]_\pm & q(z) \end{pmatrix}, \; z \in \rho(A).$$

The Nevanlinna kernel of $Q(z)$ can be written as

$$\frac{Q(z) - Q(w)^*}{z - \bar{w}} = \begin{pmatrix} [\gamma(z), \gamma(w)] & [\varphi(z), \gamma(w)] \\ [\gamma(z), \varphi(w)] & [\varphi(z), \varphi(w)] \end{pmatrix}, \; z, w \in \rho(A), \tag{6.4}$$

where $\gamma(z) = R_z^- u$.

Proof : It is obvious that $W(z)$ is the Potapov-Ginzburg transformation of $Q(z)$. We calculate the off-diagonal entries of the Nevanlinna kernel of $Q(z)$ using (5.7):

$$\frac{\left[\begin{pmatrix} \varphi(z) \\ z\varphi(z) \end{pmatrix}, u\right]_\pm - \left[\begin{pmatrix} \varphi(\bar{w}) \\ \bar{w}\varphi(\bar{w}) \end{pmatrix}, u\right]_\pm}{z - \bar{w}} = [R_{\bar{w}}^+ \varphi(z), u]_\pm = [\varphi(z), \gamma(w)],$$

$$\frac{\left[u, \begin{pmatrix} \varphi(\bar{z}) \\ \bar{z}\varphi(\bar{z}) \end{pmatrix}\right]_\pm - \left[u, \begin{pmatrix} \varphi(w) \\ w\varphi(w) \end{pmatrix}\right]_\pm}{z - \bar{w}} = [u, R_{\bar{z}}^+ \varphi(w)]_\pm = [\gamma(z), \varphi(w)].$$

The assertion concerning the diagonal entries follows from Proposition 4.5 and (2.3).

□

Theorem 6.5. *Let a 2×2-matrix valued function*

$$M(z) = \begin{pmatrix} m_{11}(z) & m_{12}(z) \\ m_{21}(z) & m_{22}(z) \end{pmatrix}, \; z \in O,$$

be given, and assume that $M(z) \in \mathcal{M}_\kappa$. Then there exists a Pontryagin space $\mathfrak{P}$ with negative index κ, a symmetric relation $S \subseteq \mathfrak{P}^2$ with defect index $(1,1)$, a canonical selfadjoint extension A of S with $\rho(A) \neq \emptyset$, an element $u \in \mathfrak{P}_-$, and (iJ)-unitary matrices U and V, such that $r_u(S) \cap O \neq \emptyset$ and

$$M(z) = VW(z)U, \; z \in r_u(S) \cap O, \tag{6.5}$$

where $W(z)$ is a generalized u-resolvent matrix as in Definition 4.8.

If m_{21} or m_{22} does not vanish identically, we can choose $V = I$. If $m_{21} \neq 0$ and one of m_{21}, m_{22}, $\det M$ is not constant, we can choose $V = U = I$. In this case, if $\mathfrak{P}$ is chosen minimal in the sense of (5.5)*, then $\mathfrak{P}$, S, A and u are uniquely determined up to unitary equivalence by the requirement $M(z) = W(z)$.*

Proof : Choose (iJ)-unitary matrices U and V as in Lemma 6.3, and consider the Potapov-Ginzburg transform $Q(z) = \Psi(V^{-1}MU^{-1})(z)$. By Lemma 6.2 we have $Q(z) \in \mathcal{N}_\kappa^2$. It is well known (see [HSW]) that there exists a Pontryagin space $\mathfrak{P}$ with negative index κ, a selfadjoint relation A with nonempty resolvent set, and functions

$$\gamma, \varphi : \rho(A) \to \mathfrak{P}$$

satisfying (2.2) such that

$$\frac{Q(z) - Q(w)^*}{z - \bar{w}} = \begin{pmatrix} [\gamma(z), \gamma(w)] & [\varphi(z), \gamma(w)] \\ [\gamma(z), \varphi(w)] & [\varphi(z), \varphi(w)] \end{pmatrix}, \; z, w \in \rho(A). \tag{6.6}$$

Moreover, the space $\mathfrak{P}$ can be chosen minimal in the sense that

$$\operatorname{cls}\left(\{\varphi(z) : z \in \rho(A)\} \cup \{\gamma(z) : z \in \rho(A)\}\right) = \mathfrak{P}.$$

Then the space $\mathfrak{P}$, the relation A, and the mappings $\gamma(z)$ and $\varphi(z)$ are determined uniquely by $Q(z)$ up to unitary equivalence.

The functions $\varphi(z)$ and $\gamma(z)$ do not vanish identically. Let S be the symmetric restriction of A with defect index $(1, 1)$ given by

$$S = \{(f; g) \in A : g - \overline{z} f \perp \varphi(z)\}.$$

By Lemma 4.3 we can choose $u' \in \mathfrak{P}_-$ such that $R_z^- u' = \gamma(z)$. By Lemma 6.4 the matrix function

$$Q'(z) = \begin{pmatrix} r'(z) & \left[\begin{pmatrix} \varphi(z) \\ z\varphi(z) \end{pmatrix}, u'\right]_\pm \\ \left[u', \begin{pmatrix} \varphi(\bar{z}) \\ \bar{z}\varphi(\bar{z}) \end{pmatrix}\right]_\pm & q'(z) \end{pmatrix}, \; z \in \rho(A),$$

where $r'(z)$ is a generalized u-resolvent induced by A, and $q'(z)$ is a Q-function of A and S and satisfies (6.6).

Since (6.6) determines $Q(z)$ up to an additive constant hermitian matrix, we may write by Corollary 4.4 for some $h \in V^{-1}(\mathfrak{P}_+ \ominus A)$ and $u = u' + h$, some generalized u-resolvent $r(z)$ and some Q-function $q(z)$

$$Q(z) = \begin{pmatrix} r(z) & \left[\begin{pmatrix} \varphi(z) \\ z\varphi(z) \end{pmatrix}, u\right]_\pm \\ \left[u, \begin{pmatrix} \varphi(\bar{z}) \\ \bar{z}\varphi(\bar{z}) \end{pmatrix}\right]_\pm & q(z) \end{pmatrix}, \; z \in \rho(A).$$

Since M is analytic in O, the left lower entry of $Q = \Psi(V^{-1}MU^{-1})$ cannot vanish for any $z \in O$. Hence $r_u(S)$ contains $O \setminus \sigma(A)$. The relation (6.5) holds with $W(z)$ as in Definition 4.8, and U and V as in Lemma 6.3.

If $U = V = I$ and if the minimality condition (5.5) is satisfied, then, as mentioned above, the space $\mathfrak{P}$, the relation A, and the mappings $\gamma(z)$ and $\varphi(z)$ are uniquely determined up to unitary equivalence by $Q(z)$, and hence by $M(z)$. Moreover, S is uniquely determined by A and $\varphi(z)$, and the element u is uniquely determined by $\gamma(z)$ and the requirement that $\left[u, \begin{pmatrix} \varphi(\bar z) \\ \bar z \varphi(\bar z) \end{pmatrix}\right]_\pm$ is the left lower entry of $\Psi(M)(z)$.

□

Corollary 6.6. *Let a* 2×2*-matrix valued function* $M(z)$*,* $z \in O$*, be given, and assume that* $M(z) \in \mathcal{M}_\kappa$*. If* $O \cap \mathbb{C}^+ \neq \emptyset$ *(*$O \cap \mathbb{C}^- \neq \emptyset$*), the function* $M(z)$ *has an analytic continuation to* $\mathbb{C}^+$ *(*$\mathbb{C}^-$*) with possible exception of an isolated set. This continuation still belongs to* $\mathcal{M}_\kappa$*. The function* $M(z)$ *can in fact be extended to* $\mathbb{C} \setminus \mathbb{R}$ *(with possible exception of an isolated set) maintaining the property to belong to* $\mathcal{M}_\kappa$*, if and only if its determinant does not vanish identically.*

For a matrix function $M \in \mathcal{M}_\kappa$ we denote by $\rho(M)$ its maximal domain of analyticity. Consider the reproducing kernel space $\mathfrak{K}(M)$ which is generated by the kernel (6.2). The space $\mathfrak{K}(M)$ is a Pontryagin space with negative index κ. The elements of $\mathfrak{K}(M)$ are pairs of functions analytic on $\rho(M)$. Together with the results of the previous section we find:

Corollary 6.7. *Let* $M \in \mathcal{M}_\kappa$ *be given, assume that* $\det M$ *does not vanish identically and that one of* $m_{21}, m_{22}, \det M$ *is not constant. Then we can choose in Theorem* 6.5 *for* $\mathfrak{P}$ *and* S *the space* $\mathfrak{K}(M)$ *and the relation* S *defined by*

$$(S-a)^{-1} = \mathcal{R}_1(a)|_{\{\mathbf{f}\in\mathfrak{K}(M):(\pi_-\mathbf{f})(a)=0\}}.$$

It follows that the difference quotient operator

$$\mathcal{R}_1(a)\mathbf{f}(z) = \frac{\mathbf{f}(z)-\mathbf{f}(a)}{z-a}, \ \mathbf{f} \in \mathfrak{K}(M),$$

leaves $\mathfrak{K}(M)$ *invariant, and in fact is the resolvent operator of a relation extending* S*. In particular it depends continuously on* $\mathbf{f}$ *and (in the operator norm of* $\mathfrak{K}(M)$*) analytically on* $a \in \rho(M)$*.*

7 Generalized resolvent matrices of symmetric extensions

In this section we consider the following situation: Let $(\mathfrak{P}, [.,.])$ and $(\mathfrak{P}_1, [.,.]_1)$ be Pontryagin spaces and let $S \subseteq \mathfrak{P}^2$ and $S_1 \subseteq \mathfrak{P}_1^2$ be symmetric relations, both with defect index $(1,1)$. Assume that $\mathfrak{P} \subseteq \mathfrak{P}_1$ (including $[.,.]_1|_{\mathfrak{P}^2} = [.,.]$) and $S \subseteq S_1$.

Choose a maximal negative subspace $\mathfrak{Q}$ of $\mathfrak{P}$ and denote by $(.,.)$ the positive definite inner product induced by the fundamental symmetry associated with $\mathfrak{Q}$. Let $\mathfrak{R}$ be a maximal negative subspace of the Pontryagin space $\mathfrak{P}_1 \ominus_{[.,.]_1} \mathfrak{P}$, and put $\mathfrak{Q}_1 = \mathfrak{Q}[\dot{+}]\mathfrak{R}$. Then $\mathfrak{Q}_1$ is a maximal negative subspace of $\mathfrak{P}_1$, hence gives rise to a fundamental symmetry of $\mathfrak{P}_1$ which in turn induces a positive definite inner product $(.,.)_1$.

Construct Hilbert spaces $\mathfrak{P}_+, \mathfrak{P}_-, \mathfrak{P}_{1,+}$ and $\mathfrak{P}_{1,-}$ as in Section 3, using the inner products just introduced. Moreover, denote by V, V_1, π, π_1 etc. the respective mappings for the spaces $\mathfrak{P}$ and $\mathfrak{P}_1$. Some basic properties of the introduced notions are collected in the following lemma. Denote by P_1 the $[.,.]_1$-orthogonal projection of $\mathfrak{P}_1$ onto $\mathfrak{P}$.

Lemma 7.1. *The inner product $(.,.)_1$ extends $(.,.)$. The projection P_1 is orthogonal with respect to $(.,.)_1$, i.e.*

$$\mathfrak{P}_1 \ominus_{(.,.)_1} \mathfrak{P} = \mathfrak{P}_1 \ominus_{[.,.]_1} \mathfrak{P}.$$

We have

$$(P_1 \oplus P_1)\mathfrak{P}_{1,+} \subseteq \mathfrak{P}_+. \tag{7.1}$$

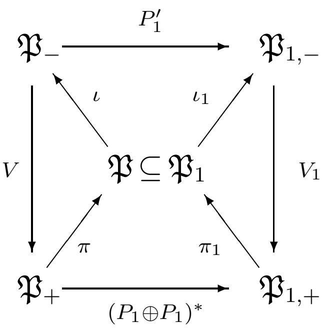
Put $P_1' = V_1^{-1}(P_1 \oplus P_1)^* V$ where the adjoint has to be understood with respect to the inner products of $\mathfrak{P}_{1,+}$ and $\mathfrak{P}_+$. If $h \in \mathfrak{P}$, $u \in \mathfrak{P}_-$ and $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathfrak{P}_{1,+}$, we have

$$[P_1' u, \begin{pmatrix} f \\ g \end{pmatrix}]_{1,\pm} = [u, (P_1 \oplus P_1)\begin{pmatrix} f \\ g \end{pmatrix}]_{\pm}, \tag{7.2}$$

and

$$[P_1' \iota h, \begin{pmatrix} f \\ g \end{pmatrix}]_{1,\pm} = [h, f]_1. \tag{7.3}$$

In order to visualize the introduced spaces and mappings consider the following diagram:



Proof : [of Lemma 7.1] The first assertion follows immediately from the fact that the fundamental symmetry of $\mathfrak{P}_1$ used above leaves $\mathfrak{P}$ invariant. The relation (7.1) holds, as

$$S_1^* \subseteq S^*[\dot{+}](\mathfrak{P}_1 \ominus_{[.,.]_1} \mathfrak{P})^2. \tag{7.4}$$

Note that $S^*[\dot{+}](\mathfrak{P}_1 \ominus_{[.,.]_1} \mathfrak{P})^2$ is the adjoint relation of S in $\mathfrak{P}_1^2$.

If $u \in \mathfrak{P}_-$ and $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathfrak{P}_{1,+}$ we compute

$$[P_1' u, \begin{pmatrix} f \\ g \end{pmatrix}]_{1,\pm} = [(V_1^{-1}(P_1 \oplus P_1)^* V)u, \begin{pmatrix} f \\ g \end{pmatrix}]_{1,\pm} = ((P_1 \oplus P_1)^* V u, \begin{pmatrix} f \\ g \end{pmatrix})_{1,+} =$$

$$= (Vu, (P_1 \oplus P_1) \begin{pmatrix} f \\ g \end{pmatrix})_+ = [u, (P_1 \oplus P_1) \begin{pmatrix} f \\ g \end{pmatrix}]_\pm .$$

If $h \in \mathfrak{P}$, then

$$[P_1'\iota h, \begin{pmatrix} f \\ g \end{pmatrix}]_{1,\pm} = [\iota h, (P_1 \oplus P_1) \begin{pmatrix} f \\ g \end{pmatrix}]_\pm = [h, P_1 f] = [h, f]_1 .$$

□

Denote by $r(S, S_1)$ the set

$$r(S, S_1) = \{z \in r(S_1) : \mathfrak{P} \not\subseteq \mathrm{ran}\,(S_1 - z)\}.$$

For a selfadjoint extension $A_1 \subseteq \mathfrak{P}_1^2$, $\rho(A_1) \neq \emptyset$, denote by $\varphi_1(z)$ the defect elements of S_1 connected with A_1 by (2.2). Note in this place that $r(S) \supseteq r(S_1)$.

Lemma 7.2. *If $r(S, S_1) \cap \mathbb{C}^+ \neq \emptyset$, then $r(S, S_1)$ contains $r(S_1) \cap \mathbb{C}^+$ with possible exception of a set isolated in $r(S_1) \cap \mathbb{C}^+$. The same assertion holds with $\mathbb{C}^+$ replaced by $\mathbb{C}^-$. For $z \in r(S, S_1)$ we have*

$$P_1(\mathrm{ran}\,(S_1 - z)^{[\perp]}) = \mathrm{ran}\,(S - z)^{[\perp]}. \tag{7.5}$$

Moreover, if $u \in \mathfrak{P}_1$, then

$$r_u(S) \cap r(S, S_1) = r_{P_1'u}(S_1) \cap r(S, S_1). \tag{7.6}$$

Proof : Assume that $r(S_1) \setminus r(S, S_1)$ has an accumulation point z_0 in $r(S_1) \cap \mathbb{C}^+$. Choose a selfadjoint extension A_1 of S_1 with $z_0 \in \rho(A_1)$ and let $f \in \mathfrak{P}$. The function $[f, \varphi_1(\overline{z})]$ is analytic on $\rho(A_1)$ and vanishes on a set with accumulation point z_0. Hence it vanishes identically on $\rho(A_1) \cap \mathbb{C}^+$. Since $f \in \mathfrak{P}$ was arbitrary, we obtain $\mathfrak{P} \subseteq \mathrm{ran}\,(S_1 - z)$ for all $z \in \rho(A_1) \cap \mathbb{C}^+$. By a convenient choice of other selfadjoint extensions of S_1 we find that $\mathfrak{P} \subseteq \mathrm{ran}\,(S_1 - z)$, even for $z \in r(S_1) \cap \mathbb{C}^+$.

Let $z \in r(S, S_1) \cap \rho(A_1)$, then $\varphi_1(\overline{z}) \notin \mathfrak{P} \ominus_{[.,.]_1} \mathfrak{P}$, i.e. $P_1 \varphi_1(\overline{z}) \neq 0$. By (7.1) we have

$$0 \neq P_1 \varphi_1(\overline{z}) \in \ker\,(S^* - \overline{z}) = \mathrm{ran}\,(S - z)^{[\perp]}.$$

The relation (7.6) follows from (7.2) applied to the particular case $\begin{pmatrix} f \\ \overline{z} f \end{pmatrix} \in \mathfrak{P}_{1,+}$ with $z \in r(S, S_1)$ and $f \in \mathrm{ran}\,(S_1 - z)^{[\perp]}$.

□

We assume throughout the following that $r(S, S_1)$ has a nonvoid intersection with both halfplanes.

Let A and A_1 be canonical selfadjoint extensions of S and S_1, respectively, which have nonempty resolvent set. Note that, by the Krein formula, there corresponds a certain parameter function $\tau(z)$ to A_1, if A_1 is considered as an extension of S. Let $\hat{\tilde{R}}_z$ be the regularized generalized resolvent of S induced by A_1 mapping $\mathfrak{P}_-$ into $\mathfrak{P}_+$, and let $\hat{R}_{1,z}$ be the regularized resolvent of S_1 induced by A_1 mapping $\mathfrak{P}_{1,-}$ into $\mathfrak{P}_{1,+}$. Now $\hat{R}_{1,z}$ and $\hat{\widetilde{R}}_z$ are connected as follows:

Lemma 7.3. *With the above notation we have*

$$(P_1 \oplus P_1)\hat{R}_{1,z}P_1' = \hat{\widetilde{R}}_z.$$

Proof : As in Section 3 we define $R_z^+ : \widetilde{\mathfrak{P}} \to \widetilde{\mathfrak{P}}_+$, $R_z^- : \widetilde{\mathfrak{P}}_- \to \widetilde{\mathfrak{P}}$, and $\widetilde{R}_z^+ : \mathfrak{P} \to \mathfrak{P}_+$, $\widetilde{R}_z^- : \mathfrak{P}_- \to \mathfrak{P}$ for the selfadjoint extension $\tilde{A} = A_1$ of S acting in the space $\widetilde{\mathfrak{P}} = \mathfrak{P}_1$. Similarly, let $R_{1,z}^+ : \mathfrak{P}_1 \to \mathfrak{P}_{1,+}$ and $R_{1,z}^- : \mathfrak{P}_{1,-} \to \mathfrak{P}_1$ be as in Section 3 with S (A, $\mathfrak{P}$) replaced by S_1 (A_1, $\mathfrak{P}_1$).

Note, that then $jR_{1,z}^+ = R_z^+$. Here j is the inclusion map from $\mathfrak{P}_{1,+}$ into $\widetilde{\mathfrak{P}}_+$.

Since P_1 is the adjoint of the inclusion map $\mathfrak{P} \to \mathfrak{P}_1$, we have

$$P_1 R_{1,z}^- V_1^{-1}(P_1 \oplus P_1)^* V = P_1 (R_{1,\bar{z}}^+)^* (P_1 \oplus P_1)^* V =$$

$$= ((P_1 \oplus P_1) R_{\bar{z}}^+|_{\mathfrak{P}})^* V = (\widetilde{R}_{\bar{z}}^+)^* V = \widetilde{R}_z^-.$$

Hence

$$(P_1 \oplus P_1)\hat{R}_{1,z}P_1' = \begin{pmatrix} P_1 R_{1,z}^- - \frac{P_1 R_{1,z_0}^- + P_1 R_{1,\overline{z_0}}^-}{2} \\ z P_1 R_{1,z}^- - \frac{z_0 P_1 R_{1,z_0}^- + \overline{z_0} P_1 R_{1,\overline{z_0}}^-}{2} \end{pmatrix} V_1^{-1}(P_1 \oplus P_1)^* V =$$

$$= \begin{pmatrix} \widetilde{R}_z^- - \frac{\widetilde{R}_{z_0}^- + \widetilde{R}_{\overline{z_0}}^-}{2} \\ z\widetilde{R}_z^- - \frac{z_0 \widetilde{R}_{z_0}^- + \overline{z_0} \widetilde{R}_{\overline{z_0}}^-}{2} \end{pmatrix} = \hat{\widetilde{R}}_z.$$

□

Let $u \in \mathfrak{P}_-$. In the sequel let $r(z)$ ($r_1(z)$) be a generalized u- ($P_1'u$-) resolvent of S (S_1). By Lemma 7.3 and Proposition 4.1 $r(z)$ and $r_1(z)$ are connected by

$$r_1(z) = r(z) - \frac{\left[u, \begin{pmatrix} \varphi(\bar{z}) \\ \bar{z}\varphi(\bar{z}) \end{pmatrix}\right]_\pm \left[\begin{pmatrix} \varphi(z) \\ z\varphi(z) \end{pmatrix}, u\right]_\pm}{q(z) + \tau(z)} + \beta,$$

for some $\beta \in \mathbb{R}$. Let $\mathcal{P}(z)$ and $\mathcal{Q}(z)$ ($\mathcal{P}_1(z)$ and $\mathcal{Q}_1(z)$) be defined by (5.1) and (5.2), where the generalized u- ($P_1'u$-) resolvent $r(z)$ ($r_1(z)$) is used. Moreover, let $W(z)$ ($W_1(z)$) be the generalized u- ($P_1'u$-) resolvent matrix of S (S_1) introduced in Definition 4.8 using $r(z)$ and $r_1(z)$.

Theorem 7.4. *Let $\mathfrak{P} \subseteq \mathfrak{P}_1$ and $S \subseteq S_1$ be given and assume that $r(S, S_1) \cap \mathbb{C}^+ \neq \emptyset$ and $r(S, S_1) \cap \mathbb{C}^- \neq \emptyset$. Let $u \in \mathfrak{P}_-$ be given and assume that $r_u(S) \cap \mathbb{C}^+ \neq \emptyset$ and $r_u(S) \cap \mathbb{C}^- \neq \emptyset$. With the above notation we have*

$$\mathcal{P}_1(z)f = \mathcal{P}(z)f, \;\; \mathcal{Q}_1(z)f = \mathcal{Q}(z)f - \beta\mathcal{P}(z)f, \;\; f \in \mathfrak{P}, z \in r_u(S) \cap r(S, S_1).$$

If $\beta = 0$, then there exists a matrix function $M(z) \in \mathcal{M}_{\kappa_1 - \kappa}$, analytic on $r(S, S_1)$, such that

$$W_1(z) = W(z)M(z), \;\; z \in r(S, S_1).$$

The matrix function $M(z)$ does not depend on the choice of u.

Before we prove Theorem 7.4 we need another lemma. Denote by $\varphi(z)$ ($\varphi_1(z)$) the defect elements of S (S_1) connected with A (A_1).

Lemma 7.5. *Let $u, v \in \mathfrak{P}_-$ and let $z \in r(S, S_1) \cap r_u(S) \cap r_v(S) \cap \rho(A) \cap \rho(A_1)$. Then*

$$\frac{\left[u, \begin{pmatrix} \varphi(\bar z) \\ \bar z \varphi(\bar z) \end{pmatrix}\right]_\pm}{\left[P_1' u, \begin{pmatrix} \varphi_1(\bar z) \\ \bar z \varphi_1(\bar z) \end{pmatrix}\right]_{1,\pm}} = \frac{\left[v, \begin{pmatrix} \varphi(\bar z) \\ \bar z \varphi(\bar z) \end{pmatrix}\right]_\pm}{\left[P_1' v, \begin{pmatrix} \varphi_1(\bar z) \\ \bar z \varphi_1(\bar z) \end{pmatrix}\right]_{1,\pm}}. \tag{7.7}$$

Proof : The above assumptions imply that

$$\operatorname{span} u \dot{+} \ker \left[\cdot, \begin{pmatrix} \varphi(\bar z) \\ \bar z \varphi(\bar z) \end{pmatrix}\right]_\pm = \mathfrak{P}_-,$$

$$\operatorname{span} v \dot{+} \ker \left[\cdot, \begin{pmatrix} \varphi(\bar z) \\ \bar z \varphi(\bar z) \end{pmatrix}\right]_\pm = \mathfrak{P}_-.$$

Hence, there exists a unique number $\alpha(z) \in \mathbb{C} \setminus \{0\}$, such that $u = \alpha(z)v + m(z)$, for some $m(z) \in \ker \left[\cdot, \begin{pmatrix} \varphi(\bar z) \\ \bar z \varphi(\bar z) \end{pmatrix}\right]_\pm$. Then

$$\left[u, \begin{pmatrix} \varphi(\bar z) \\ \bar z \varphi(\bar z) \end{pmatrix}\right]_\pm = \alpha \left[v, \begin{pmatrix} \varphi(\bar z) \\ \bar z \varphi(\bar z) \end{pmatrix}\right]_\pm$$

and by (7.2) and (7.5) we find

$$\left[P_1' u, \begin{pmatrix} \varphi_1(\bar z) \\ \bar z \varphi_1(\bar z) \end{pmatrix}\right]_{1,\pm} = \alpha \left[P_1' v, \begin{pmatrix} \varphi_1(\bar z) \\ \bar z \varphi_1(\bar z) \end{pmatrix}\right]_{1,\pm}.$$

□

Proof : [of Theorem 7.4] The relation $\mathcal{P}_1(z)f = \mathcal{P}(z)f$ follows immediately from Lemma 7.5, if we note that the complement of $r(S, S_1) \cap r_u(S) \cap r_v(S) \cap \rho(A) \cap \rho(A_1)$ in $r_u(S)$ is isolated. The relation $\mathcal{Q}_1(z)f = \mathcal{Q}(z)f - \beta\mathcal{P}(z)f$ follows by a similar computation as in (5.4).

We define a matrix function $M(z)$, analytic on the nonempty set $\{z \in r_u(S) \cap r(S, S_1) : \det W(z) \neq 0\}$, by

$$M(z) = W(z)^{-1} W_1(z).$$

By the first part of this proof the space $\mathfrak{P}_u$ is contained in $\mathfrak{P}_{1,P_1'u}$. Hence the orthogonal projection of $\mathfrak{P}_{1,P_1'u}$ onto $\mathfrak{P}_u$ maps the reproducing kernel of $\mathfrak{P}_{1,P_1'u}$ onto the reproducing kernel of $\mathfrak{P}_u$. Since

$$\frac{W_1(z)JW_1(w)^* - J}{z - \bar w} \begin{pmatrix} x \\ y \end{pmatrix} - \frac{W(z)JW(w)^* - J}{z - \bar w} \begin{pmatrix} x \\ y \end{pmatrix} =$$

$$= W(z)\frac{M(z)JM(w)^*-J}{z-\bar{w}}W(w)^*\begin{pmatrix} x \\ y \end{pmatrix},\ x,y\in\mathbb{C},$$

the matrix kernel

$$W(z)\frac{M(z)JM(w)^*-J}{z-\bar{w}}W(w)^*,\ z,w\in\{z\in r_u(S)\cap r(S,S_1):\det W(z)\neq 0\}$$

is a reproducing kernel of $\mathfrak{P}_{1,P_1'u}\ominus\mathfrak{P}_u$. Hence $M(z)\in\mathcal{M}_{\kappa_1-\kappa}$.

It remains to prove that $M(z)$ does not depend on u. If

$$M(z)=\begin{pmatrix} m_{11}(z) & m_{12}(z) \\ m_{21}(z) & m_{22}(z) \end{pmatrix},\ W(z)=\begin{pmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{pmatrix},$$

we find from

$$M(z)=W(z)^{-1}W_1(z)=\frac{\left[u,\begin{pmatrix}\varphi(\bar z)\\ \bar z\varphi(\bar z)\end{pmatrix}\right]_\pm}{\left[\begin{pmatrix}\varphi(z)\\ z\varphi(z)\end{pmatrix},u\right]_\pm}\begin{pmatrix} w_{22}(z) & -w_{12}(z) \\ -w_{21}(z) & w_{11}(z) \end{pmatrix}W_1(z),$$

that

$$m_{11}(z)=\frac{\tau(z)}{q(z)+\tau(z)}\frac{\left[u,\begin{pmatrix}\varphi(\bar z)\\ \bar z\varphi(\bar z)\end{pmatrix}\right]_\pm}{\left[P_1'u,\begin{pmatrix}\varphi_1(\bar z)\\ \bar z\varphi_1(\bar z)\end{pmatrix}\right]_\pm},$$

$$m_{12}(z)=\frac{\tau(z)q_1(z)}{q(z)+\tau(z)}\frac{\left[u,\begin{pmatrix}\varphi(\bar z)\\ \bar z\varphi(\bar z)\end{pmatrix}\right]_\pm}{\left[P_1'u,\begin{pmatrix}\varphi_1(\bar z)\\ \bar z\varphi_1(\bar z)\end{pmatrix}\right]_\pm}-q(z)\frac{\left[\begin{pmatrix}\varphi_1(z)\\ z\varphi_1(z)\end{pmatrix},P_1'u\right]_\pm}{\left[\begin{pmatrix}\varphi(z)\\ z\varphi(z)\end{pmatrix},u\right]_\pm},$$

$$m_{21}(z)=\frac{1}{q(z)+\tau(z)}\frac{\left[u,\begin{pmatrix}\varphi(\bar z)\\ \bar z\varphi(\bar z)\end{pmatrix}\right]_\pm}{\left[P_1'u,\begin{pmatrix}\varphi_1(\bar z)\\ \bar z\varphi_1(\bar z)\end{pmatrix}\right]_\pm},$$

$$m_{22}(z)=\frac{q_1(z)}{q(z)+\tau(z)}\frac{\left[u,\begin{pmatrix}\varphi(\bar z)\\ \bar z\varphi(\bar z)\end{pmatrix}\right]_\pm}{\left[P_1'u,\begin{pmatrix}\varphi_1(\bar z)\\ \bar z\varphi_1(\bar z)\end{pmatrix}\right]_\pm}+\frac{\left[\begin{pmatrix}\varphi_1(z)\\ z\varphi_1(z)\end{pmatrix},P_1'u\right]_\pm}{\left[\begin{pmatrix}\varphi(z)\\ z\varphi(z)\end{pmatrix},u\right]_\pm}.$$

By Lemma 7.5 $M(z)$ does not depend on u. Since for all $z\in r(S,S_1)$ there exists $u\in\mathfrak{P}_-$ such that $z,\overline{z}\in r_u(S)$, $M(z)$ has an analytic continuation to $r(S,S_1)$.

□

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