SELFADJOINT EXTENSIONS OF SYMMETRIC OPERATORS IN DEGENERATED INNER PRODUCT SPACES

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In this paper we give an analogue of Krein's formula on the description of generalized resolvents of a symmetry S in the case that the S acts in a degenerated inner product space. These results are applied to the extension problem of positive definite functions.

1 Introduction

If $\mathcal{H}$ is a Pontryagin space and S is a symmetric operator in $\mathcal{H}$ with defect numbers $(1,1)$, Krein's formula

$$[(A-z)^{-1}u,v] = [(A_0-z)^{-1}u,v] - [u,\chi(\overline{z})]\frac{1}{\tau(z)+q(z)}[\chi(z),v],\ \ u,v\in\mathcal{H}, \tag{1.1}$$

establishes a correspondence between all selfadjoint extensions A of S and parameters $\tau(z)$, when τ runs through a Nevanlinna class. Here A_0 is a fixed (canonical) selfadjoint extension of S, $\chi(z)$ parametrizes the defect spaces of S, and $q(z)$ is a Q-function of S (see [9]).

In various applications, e.g. interpolation or extension problems, it is possible to obtain parametrizations of solutions via Krein's formula. Thereby an inner product space $\mathcal{H}$ and a symmetric operator S is assigned to the given data (see [10], [12]). However, there are exceptional cases where the classical theory cannot be applied. This happens if the model space $\mathcal{H}$ degenerates, which takes plase e.g. in the so called singular points in [12] (see also [1], [2], [13]).

In this paper we are concerned with the case that the degeneration of $\mathcal{H}$ is one dimensional, and give an analogue of Krein's formula (1.1). The parameter τ does not run through a whole Nevanlinna class, but through a proper subclass $\mathcal{T}$. This class is determined by analytic properties. Our approach differs from the classical method: We introduce a graph perturbation of the given symmetry. Therefore it is convenient to use the notation of linear relations, instead of operators. We shall demonstrate our results on an extension problem for a hermitian function with one negative square.

In the (preliminary) Section 2 we show that the classical definition of the defect numbers of a symmetry (see [3], [5]) remains meaningful, even if $\mathcal{H}$ is degenerated. The graph perturbation which is the main tool of this work is introduced and studied in Sections 3 and 4. In Section 5 we prove our analogue of Krein's formula (1.1), which is a formula of the same type, but with a different set of parameters. It turns out that the expressions A_0, $\chi(z)$ and $q(z)$ can be viewed as canonical extension, parametrization of defect spaces and Q-function, respectively, of a certain (relational) symmetric extension of S. The Sections 6 and 7 deal with a characterization of the class $\mathcal{T}$ of parameters by their analytic properties. Finally, in Section 8, an example is given.

The notation used in this article is similar to [3] concerning linear relations and to [8] concerning the theory of Pontrjagin spaces. For abbreviation we will call a Pontrjagin space with negative index κ a π_κ-space.

2 Defect indices in degenerated inner product spaces

In this preliminary section we show that the notion of defect numbers can be carried over to degenerated inner product spaces.

Let $\mathcal{H}$ be an inner product space with a one dimensional degeneration, i.e. let

$$\mathcal{H} = \mathcal{H}_n[\dot{+}]\langle h_0\rangle \tag{2.1}$$

where $\mathcal{H}_n$ is a Pontrjagin space and the isotropic part $\mathcal{H}^\circ$ of $\mathcal{H}$ equals $\langle h_0\rangle$.

The space $\mathcal{H}$ can be embedded canonically into a Pontryagin space $\mathcal{P}_c$: Let

$$\mathcal{P}_c = \underbrace{\mathcal{H}_n[\dot{+}](\langle h_0\rangle}_{=\mathcal{H}} \dot{+}\langle h_1\rangle),$$

where h_1 is neutral and $[h_0, h_1] = 1$.

The norm defined on $\mathcal{H}$ as a subspace $\mathcal{P}_c$ is equivalent to the norm $\|.\|$ defined as

$$\|f\|^2 = \|f_n + \varphi_0 h_0\|^2 = \|f_n\|^2_{\mathcal{H}_n} + |\varphi_0|^2$$

when $f = f_n + \varphi_0 h_0$ is the decomposition of f with respect to (2.1). We will refer to the topology on $\mathcal{H}$ induced by this norm as the canonical topology.

We call a linear relation $T \subseteq \mathcal{H}^2$ closed if it is a closed subset of $\mathcal{H}^2$ with respect to the canonical topology. Equivalently T is a closed relation regarded as a relation in $\mathcal{P}_c$: $T \subseteq \mathcal{P}_c^2$.

Let T be a closed symmetric relation in $\mathcal{H}$. Assume that for some $z \in \mathbb{C}^+$ and some $z \in \mathbb{C}^-$

$$\ker(S - z) = \{0\}, \tag{2.2}$$

and that for some $z \in \mathbb{C}^+$ and some $z \in \mathbb{C}^-$

$$h_0 \notin \mathcal{R}(S - z)\,. \tag{2.3}$$

Then, regarding T as a relation in $\mathcal{P}_c$, the dimension of $\mathcal{R}(T - z)^\perp$ is constant on the upper (lower, respectively) half plane with possible exception of a finite set. These dimensions are the so called defect indices of T (see [5] if T is an operator, [3] in the case of relations).

Let T be a closed symmetric relation. It is shown in [3] that, if the conditions (2.2) and (2.3) hold for one $z \in \mathbb{C}^+$ ($\mathbb{C}^-$), they are satisfied for all but finitely many $z \in \mathbb{C}^+$ ($\mathbb{C}^-$). This follows from the fact that $T/\langle h_0\rangle^2 \subseteq (\mathcal{H}/\langle h_0\rangle)^2$ is again closed and symmetric.

Proposition 1 *Let T be a closed symmetric relation in $\mathcal{H}$ satisfying (2.2) and (2.3). For all values of $z \in \mathbb{C} \setminus \mathbb{R}$ with possible exception of a finite set we have*

$$\dim \mathcal{R}\,(T-z)^{\perp_{\mathcal{H}}} = \dim \mathcal{R}\,(T-z)^{\perp_{\mathcal{P}_c}} - 1 \tag{2.4}$$

as long as the right hand side of (2.4) is finite. In this case the relation (2.4) holds if and only if $h_0 \notin \mathcal{R}\,(T-z)$. Otherwise $\dim \mathcal{R}\,(T-z)^{\perp_{\mathcal{H}}} = \dim \mathcal{R}\,(T-z)^{\perp_{\mathcal{P}_c}}$.

Proof : Put $R = \mathcal{R}\,(T-z)$. As $\mathcal{H} \subseteq \mathcal{P}_c$ we have $R^{\perp_{\mathcal{H}}} \subseteq R^{\perp_{\mathcal{P}_c}}$. Let $f, g \in R^{\perp_{\mathcal{P}_c}}$, $f = f_n + \varphi_0 h_0 + \varphi_1 h_1$ and $g = g_n + \gamma_0 h_0 + \gamma_1 h_1$, then there exist numbers λ, μ not both zero, such that $\lambda\xi_1 + \mu\eta_1 = 0$. This shows that $\lambda f + \mu g \in R^{\perp_{\mathcal{H}}}$. Thus

$$\dim R^{\perp_{\mathcal{P}_c}} \leq \dim R^{\perp_{\mathcal{H}}} + 1. \tag{2.5}$$

By the comment before Proposition 1 it suffices to prove that in (2.5) equality holds if and only if $h_0 \notin R$.

If $h_0 \in R$ and $f = f_n + \varphi_0 h_0 + \varphi_1 h_1 \in R^{\perp_{\mathcal{P}_c}}$, then $0 = [f, h_0] = \varphi_1$ and therefore $f \in \mathcal{H}$, i.e. $R^{\perp_{\mathcal{P}_c}} = R^{\perp_{\mathcal{H}}}$.

If $h_0 \notin R$ $(= (R^{\perp_{\mathcal{P}_c}})^{\perp_{\mathcal{P}_c}})$ then there is an element $f = f_n + \varphi_0 h_0 + \varphi_1 h_1 \in R^{\perp_{\mathcal{P}_c}}$, such that $[h_0, f] \neq 0$. Thus $\varphi_1 \neq 0$ and $f \notin R^{\perp_{\mathcal{P}_c}}$.

□

Due to Proposition 1 the dimension of $\mathcal{R}\,(T-z)^{\perp_{\mathcal{H}}}$ is constant on the upper (lower, respectively) half plane with possible exception of finitely many points. Thus we may give the following

Definition 1 *Let T be a closed symmetric relation in $\mathcal{H}$. The numbers* $\dim \mathcal{R}\,(T-z)^{\perp}$ *for z in the upper (lower) half plane are called the defect indices of T.*

For a symmetric operator T ($T(0) = \{0\}$) the condition (2.2) is always satisfied.

Lemma 1 *Let T be a symmetric operator in $\mathcal{H}$. Condition (2.3) is not satisfied if and only if $T^j h_0 \in \mathcal{D}\,(T)$ for all $j = 0, 1, 2, \ldots$.*

Proof : Let $h_0 = (T-z)f$ and $h_0 = (T-w)g$ for $z, w \in \mathbb{C}^+$, then

$$0 = [h_0, g] = [((T-\overline{w}) + (\overline{w}-z))f, g] = [f, \underbrace{(T-w)g}_{=h_0}] + (\overline{w}-z)[f,g] = (\overline{w}-z)[f,g].$$

This shows that the elements contained in the inverse images $(T-z)^{(-1)}h_0$ for $z \in \mathbb{C}^+$ span a neutral subspace. Therefore their span is finite dimensional.

Assume that there are infinitely many points $z_i \in \mathbb{C}^+$, such that $h_0 = (T-z_i)f_i$. The elements f_i must be linearly dependent. After a possible renumeration of the points z_i we can assume that for some $n \geq 2$

$$0 = \sum_{i=1}^{n} \lambda_i f_i \tag{2.6}$$

is a vanishing linear combination of the elements f_i of minimal length, in particular that $\lambda_i \neq 0$ for $i = 1, \ldots, n$. Applying T to (2.6) we obtain

$$0 = \sum_{i=1}^{n} \lambda_i T f_i = (\sum_{i=1}^{n} \lambda_i) h_0 + \sum_{i=1}^{n} \lambda_i z_i f_i. \tag{2.7}$$

If $\sum_{i=1}^{n} \lambda_i = 0$ we find

$$0 = \sum_{i=1}^{n} \lambda_i f_i - \frac{1}{z_n} \sum_{i=1}^{n} \lambda_i z_i f_i = \sum_{i=1}^{n-1} \lambda_i (1 - \frac{z_i}{z_n}) f_i$$

a contradiction, as (2.6) is of minimal length. Thus $\sum_{i=1}^{n} \lambda_i \neq 0$ and (2.7) shows that $h_0 \in \mathcal{D}(T)$. By repeatedly applying T to (2.7) we conclude by induction that $T^j h_0 \in \mathcal{D}(T)$ for each $j \in \mathbb{N}_0$.

Assume conversely that $T^j h_0 \in \mathcal{D}(T)$ for $j = 0, 1, 2, \ldots$. It is easily seen by induction that in fact $T^j h_0 \in \mathcal{D}(T)^\circ$. Therefore the subspace $\langle T^j h_0 | j = 0, 1, \ldots \rangle$ is neutral and hence finite dimensional. It is an invariant subspace for T and we may consider the restriction $T_1 = T|_{\langle T^j h_0 | j=0,1,\ldots \rangle}$. Clearly $T_1 - z$ is injective and therefore bijective for all but finitely many $z \in \mathbb{C}$. Thus

$$h_0 \in \mathcal{R}(T_1 - z) \subseteq \mathcal{R}(T - z)$$

for all but finitely many values of $z \in \mathbb{C}$.

□

Note that (2.3) implies $T \cap \langle h_0 \rangle^2 = \{0\}$.

Remark 1 Assume that T has defect $(1,1)$. Then, due to the assumption $\dim \mathcal{H}^\circ = 1$, (2.2) and (2.3) are both satisfied if and only if T is a so called standard symmetric relation in $\mathcal{P}_c$ and admits selfadjoint extensions with nonempty resolvent set.

3 A perturbation formula

Let $S \subseteq \mathcal{H}^2$ be a symmetric relation with defect $(1,1)$ which satisfies (2.2) and (2.3). Moreover choose a decomposition (2.1). In this section we show that a certain range perturbation of S is selfadjoint in the Pontryagin space $\mathcal{H}_n$.

Denote by P the projection of $\mathcal{H}$ onto $\mathcal{H}_n$ with kernel $\langle h_0 \rangle$.

Lemma 2 *The projection P has the properties $P^2 = P$, $P^+ = P$ and*

$$[Pf, g] = [f, g] \text{ for } f, g \in \mathcal{H}.$$

Lemma 2 follows immediately from the fact that the kernel of P is isotropic.

Definition 2 *Denote by $S_P \subseteq \mathcal{H}_n^2$ the relation*

$$S_P = \{(Pf; Pg) \in \mathcal{H}_n^2 | (f, g) \in S\}.$$

Since S is closed also the finite dimensional extension $S + \langle h_0 \rangle^2$ is closed. This shows that

$$S_P = (S + \langle h_0 \rangle^2) \cap \mathcal{H}_n^2$$

is a closed relation.

Proposition 2 *Let S be a closed symmetric relation in $\mathcal{H}$ with defect $(1,1)$ satisfying (2.2) and (2.3). Then S_P is selfadjoint and has nonempty resolvent set. In fact*

$$\sigma(S_P) \setminus \mathbb{R} \subseteq \{z \in \mathbb{C} \setminus \mathbb{R} | h_0 \in \mathcal{R}(S-z)\}.$$

Proof : Let $(S_P - z)f = 0$, i.e. let $(f; zf) \in S_P$, then there exists a pair $(f'; g') \in S$, such that $f = f' + \varphi h_0$ and $zg = g' + \gamma h_0$. This shows that

$$(f'; zf' + (\gamma - z\varphi)h_0) \in S,$$

which means $(f'; (\gamma - z\varphi)h_0) \in S - z$, i.e. $h_0 \in \mathcal{R}(S-z)$ or $\ker(S-z) \neq \{0\}$. This is possible only for finitely many $z \in \mathbb{C} \setminus \mathbb{R}$.

The projection P maps $\mathcal{H}$ onto $\mathcal{H}_n$. As for all z (with exception of a finite set) $\langle h_0 \rangle + \mathcal{R}(S-z) = \mathcal{H}$ holds, we have

$$P\mathcal{R}(S-z) = \mathcal{H}_n.$$

A straightforward computation shows that $P\mathcal{R}(S-z) = \mathcal{R}(S_P - z)$ and thus $\dim \mathcal{H}_n / \mathcal{R}(S_P - z) = 0$ for all but finitely many $z \in \mathbb{C} \setminus \mathbb{R}$.

Theorem 4.6 of [3] together with its corollary shows that the assertions of Proposition 2 hold.

□

Remark 2 The space $\mathcal{H}_n$ is isomorphic to the factor space $\mathcal{H}/\langle h_0 \rangle$ and the relation S_P is isomorphic to $S/\langle h_0 \rangle^2$.

If (2.3) is not satisfied S_P is a closed symmetric relation in $\mathcal{H}_n$ with defect $(1,1)$.

Remark 3 The assumptions of Proposition 2, in particular the conditions (2.2) and (2.3) cannot be weakened: If S_P is selfadjoint and has nonempty resolvent set, then S has defect $(1,1)$ and satisfies (2.2) and (2.3).

Since (2.3) implies $S \cap \langle h_0 \rangle^2 = \{0\}$ the mapping $P \times P : S \to S_P$, $(f; g) \mapsto (Pf; Pg)$ is injective. Denote its inverse by Ψ, i.e. let $\Psi(f; g) = (f'; g')$ for $(f; g) \in S_P$ be such that $(f'; g') \in S$ and $f = Pf'$, $g = Pg'$ holds. Clearly Ψ is a bijective linear mapping of S_P onto S.

Lemma 3 *The mapping Ψ is continuous in the canonical topology on $\mathcal{H}^2$.*

Proof : The projection P is the orthogonal projection of $\mathcal{P}_c$ onto $\mathcal{H}_n$ restricted to $\mathcal{H}$. Thus the mapping $P \times P : S \to S_P$ is continuous, even if S is endowed with the canonical topology (as a subspace of $\mathcal{H}^2 \subseteq \mathcal{P}_c^2$). Finally the open mapping theorem applies and shows that its inverse Ψ is also continuous.

□

Lemma 4 *Denote by S_P° the isotropic part of S_P with respect to the inner produt ($(f; g), (f'; g') \in \mathcal{H}^2$)*

$$[(f; g), (f'; g')] = [f, f'] + [g, g']. \tag{3.1}$$

If $\pm i \notin \sigma_p(S_P)$ we have $S_P^\circ = \{0\}$.

Proof : Let $(f;g) \in S_P^\circ$, then

$$[f,f'] + [g,g'] = 0 \text{ for } (f';g') \in S_P.$$

Hence $(-g;f) \in S_P^* = S_P$. We find that $(f;-f) \in S_P^2$ and $(g;-g) \in S_P^2$. By the spectral mapping theorem this implies $f = g = 0$.

□

Since the nonreal spectrum of S_P is finite, there exists a number $\lambda > 0$, such that

$$\pm i \notin \sigma_p(\lambda S_P).$$

In order to describe the generalized resolvents of S, it suffices to describe the generalized resolvents of λS. In the final formula (of the type (1.1)) we only have to replace z by $\frac{z}{\lambda}$ and multiply by λ. Hence we may assume in the following that $\pm i \notin \sigma_p(S_P)$.

Proposition 3 *There exist elements $(c_0;d_0),(c_1;d_1) \in S_P$, such that Ψ admits the representation*

$$\Psi(f;g) = (f + ([f,c_0]+[g,d_0])h_0; g + ([f,c_1]+[g,d_1])h_0). \tag{3.2}$$

Proof : Due to Lemma 4 the space S_P is nondegenerated in the inner product (3.1). Consider the mapping

$$\Psi_1 = \Psi - I : S_P \to \langle h_0 \rangle^2 \cong \mathbb{C}^2.$$

As Ψ is continuous, the mapping Ψ_1 is also continuous. Thus the theorem of Riesz applies and we find elements $(c_0;d_0),(c_1;d_1) \in S_P$, such that

$$\Psi_1(f;g) = ([(f;g),(c_0;d_0)]h_0; [(f;g),(c_1;d_1)]h_0). \tag{3.3}$$

This implies that the representation (3.2) holds.

□

4 Symmetric extensions

In this section we will show that Ψ induces a correspondence between selfadjoint extensions of S and S_P.

Let $\mathcal{P}$ be a Pontrjagin space extending $\mathcal{H}$. Making use of the elements $(c_0;d_0)$ and $(c_1;d_1)$ given by Proposition 3 we can extend Ψ to $\mathcal{P}^2$. We will denote this extension again by using Ψ.

Definition 3 *Let $\Psi : \mathcal{P}^2 \to \mathcal{P}^2$ be defined as follows:*

$$\Psi(f;g) = (f + ([f,c_0]+[g,d_0])h_0 - [f,h_0]d_1 + [g,h_0]d_0;$$

$$; g + ([f,c_1]+[g,d_1])h_0 + [f,h_0]c_1 - [g,h_0]c_0).$$

In order to study symmetric relations the inner product

$$\langle (f;g),(f';g')\rangle = [f,g'] - [g,f'] \text{ for } (f;g),(f';g') \in \mathcal{P}$$

is introduced. We recall some properties of $\langle .,.\rangle$ which are proved e.g. in [4].

Lemma 5 *A relation T is symmetric (selfadjoint) if and only if it is a neutral (hypermaximal neutral) subspace of $(\mathcal{P}, i\langle .,.\rangle)$. If T is symmetric, then T^+/T endowed with the inner product $i\langle .,.\rangle$ is a Krein space. If T has finite defect T^+/T in fact is finite dimensional.*

As $\Psi(S_P) = S$ the mapping $\tilde{\psi} : S_P^+/S_P \to \mathcal{P}^2/S$ (where S_P^+ denotes the adjoint of S_P in $\mathcal{P}$) given by

$$\tilde{\psi} : (f;g) + S_P \mapsto \Psi(f;g) + S \text{ for } (f;g) \in S_P^+$$

is well defined.

Theorem 1 *The mapping $\tilde{\psi}$ is a bijective isometry of S_P^+/S_P onto S^+/S with respect to the inner product $\langle .,.\rangle$. Thus $T \mapsto \Psi(T)$ establishes a bijective correspondence between symmetric extensions of S_P acting in $\mathcal{P}$ and symmetric extensions of S acting in $\mathcal{P}$. In this correspondence selfadjoint extensions of S_P correspond to selfadjoint extensions of S.*

Proof : We first show that in fact Ψ is an isometry with respect to $\langle .,.\rangle$ of $\mathcal{P}^2$ into itself. Let $(f;g),(f';g') \in \mathcal{P}^2$, then

$$\langle (f;g),(f';g')\rangle = [f,g'] - [g,f'].$$

We have

$$\begin{aligned}\langle \Psi(f;g), \Psi(f';g')\rangle = [&f + ([f,c_0] + [g,d_0])h_0 - [f,h_0]d_1 + [g,h_0]d_0, \\ &, g' + ([f',c_1] + [g',d_1])h_0 + [f',h_0]c_1 - [g',h_0]c_0] - \\ &-[g + ([f,c_1] + [g,d_1])h_0 + [f,h_0]c_1 - [g,h_0]c_0, \\ &, f' + ([f',c_0] + [g',d_0])h_0 - [f',h_0]d_1 + [g',h_0]d_0].\end{aligned}$$

As $(c_0;d_0),(c_1;d_1) \in S_P = S_P^+$ we have

$$[c_0,d_0] = [d_0,c_0],\ [c_1,d_1] = [d_1,c_1] \text{ and } [c_0,d_1] = [d_0,c_1].$$

Now a straightforward computation shows that

$$\langle (f;g),(f';g')\rangle = \langle \Psi(f;g), \Psi(f';g')\rangle. \tag{4.1}$$

As S_P^+/S_P is nondegenerated Ψ is injective. It remains to prove that Ψ is surjective, the rest of the assertion will follow from Lemma 5.

First note that $S_P^+ = S_P[\dot{+}](\mathcal{H}_n^\perp)^2$. Decompose $\mathcal{P}$ as

$$\mathcal{P} = \mathcal{H}_n[\dot{+}](\langle h_0\rangle \dot{+} \langle h_1\rangle)[\dot{+}]\mathcal{H}_2.$$

Then

$$\Psi(S_P^+) = \Psi(S_P) + \Psi(\langle h_0\rangle^2) + \Psi(\langle h_1\rangle^2) + \Psi(\mathcal{H}_2^2),$$

and $\Psi(S_P) = S$ and $\Psi(\mathcal{H}_2^2) = \mathcal{H}_2^2$. As S^+ in $\mathcal{P}$ equals $S^* + \mathcal{H}_2^2$ where S^* is computed in $\mathcal{P}_c = \mathcal{H}_n[\dot{+}](\langle h_0\rangle \dot{+} \langle h_1\rangle)$ ($\mathcal{P}_c$ is defined as in Section 2), we have to show that

$$S^* = S \dot{+} \Psi(\langle h_0\rangle^2) \dot{+} \Psi(\langle h_1\rangle^2). \tag{4.2}$$

From (4.1) it follows that $\Psi(S_P^*) \subseteq S^*$ and therefore the right hand side of (4.2) is contained in S^*. We have

$$\Psi(\langle h_0\rangle^2) = \langle h_0\rangle^2 \text{ and } \Psi(\langle h_1\rangle^2) = \langle (h_1 - d_1; c_1)\rangle + \langle (d_0; h_1 - c_0)\rangle.$$

Let $(f;g) \in S^*$, $f = f' + \varphi_0 h_0 + \varphi_1 h_1$, $g = g' + \gamma_0 h_0 + \gamma_1 h_1$ with $f', g' \in \mathcal{H}_n$. Then

$$(f;g) - (\varphi_0 h_0; \gamma_0 h_0) - \varphi_1(h_1 - d_1; c_1) - \gamma_1(d_0; h_1 - c_0) =$$

$$= (f' + \varphi_1 d_1 - \gamma_1 d_0; g' - \varphi_1 c_1 + \gamma_1 c_0) \in S^*. \tag{4.3}$$

Observe that $S^* \cap \mathcal{H}_n^2 \subseteq S_P^* \cap \mathcal{H}_n^2 = S_P$. Thus the pair (4.3) is an element of $S_P \subseteq S + \langle h_0\rangle^2$ and therefore

$$(f;g) \in S + \langle h_0\rangle^2 + \langle (h_1 - d_1; c_1)\rangle + \langle (d_0; h_1 - c_0)\rangle.$$

□

In Theorem 1 selfadjoint extensions of S_P with empty resolvent set need not correspond to selfadjoint extensions of S with empty resolvent set (compare Corollary 1 in the following section).

5 Generalized resolvents

If $\mathcal{H}$ is a Pontryagin space, $\mathcal{H} \subseteq \mathcal{P}$ and A is a selfadjoint extension of S in $\mathcal{P}$ with nonempty resolvent set, the expression

$$P(A - z)^{-1}|_{\mathcal{H}},$$

where P denotes the orthogonal projection of $\mathcal{P}$ onto $\mathcal{H}$, is called a generalized resolvent of S. Clearly a generalized resolvent is determined by the expressions

$$[(A - z)^{-1}u, v], \ u, v \in \mathcal{H}. \tag{5.1}$$

In our case, i.e. if $\mathcal{H}$ is degenerated, a orthogonal projection of $\mathcal{P}$ onto $\mathcal{H}$ does not exist. But still the expressions (5.1) are meaningful, hence we will also speak of a generalized resolvent.

Let $u, v \in \mathcal{H}$ and write $u = u_n + \mu_0 h_0$, $v = v_n + \nu_0 h_0$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_n[\dot{+}]\langle h_0\rangle$. If $u \in \mathcal{R}(S - z)$ for all $z \in \mathbb{C} \setminus \mathbb{R}$, we have

$$[(A - z)^{-1}u, v] = [(S - z)^{-1}u, v] = [(S_P - z)^{-1}u_n, v_n].$$

Hence, for determining the action of generalized resolvents we may restrict ourselves to elements u, such that $u \notin \mathcal{R}(S - z)$ for some $z \in \mathbb{C}^+$ and some $z \in \mathbb{C}^-$. Equivalently

$$\mathcal{R}(S - z) \dot{+} \langle u\rangle = \mathcal{H},$$

i.e. u is a so called module element of S (compare [11]).

Put $R_P(z) = (S_P - z)^{-1}$, let $(c_0; d_0)$ and $(c_1; d_1)$ be as in Proposition 3, and denote by $a(z)$ and $b(z)$ the expressions

$$a(z) = (c_1 - zc_0) + z(d_1 - zd_0),$$

$$b(z) = R_P(z)a(z) + (d_1 - zd_0).$$

Furthermore let $\chi(z)$ and $q(z)$ be given as

$$\chi(z) = h_1 - b(z) - [b(z), c_0 + \overline{z}d_0]h_0$$

and

$$q(z) = -[a(z), \chi(\overline{z})] = [a(z), b(\overline{z})],$$

respectively.

Let the set $\mathcal{T}$ of parameters be defined as the set of all functions $\tau(z)$, that admit a representation

$$\tau(z) = [(B - z)^{-1}h, h],$$

with a selfadjoint (not necessarily minimal) relation in a π_1-space and a neutral element h.

In the subsequent sections the class $\mathcal{T}$ will be characterized by analytic properties, in fact it turns out that

$$\mathcal{T} = \mathcal{N}_0 \cup \{f \in \mathcal{N}_1 | 0 \leq \lim_{\eta \to \infty} i\eta f(i\eta) < \infty\}. \tag{5.2}$$

Here $\mathcal{N}_\kappa$ denotes the Nevanlinna class with κ negative squares, i.e. the set of all functions τ meromorphic in $\mathbb{C} \setminus \mathbb{R}$ with $\tau(\overline{z}) = \overline{\tau(z)}$, such that the Nevanlinna kernel

$$N_\tau(z, w) = \frac{\tau(z) - \overline{\tau(w)}}{z - \overline{w}}$$

has exactly κ negative squares. We understand the (formal) function $\tau(z) = \infty$ as an element of $\mathcal{N}_0$.

Theorem 2 *The formula*

$$[(A - z)^{-1}u, v] = [(S_P - z)^{-1}u_n, v_n] - [u, \chi(\overline{z})]\frac{1}{-\frac{1}{\tau(z)} + q(z)}[\chi(z), v] \text{ for } \tau(z) \in \mathcal{T} \tag{5.3}$$

establishes a bijective correspondence between the generalized resolvents of S and the set $(\mathcal{T} \cup \{\infty\}) \setminus \{\frac{1}{q(z)}\}$ of parameters.

Proof : Let A be a selfadjoint extension of S and let $A = \Psi(A_P)$, where A_P is a selfadjoint extension of S_P. Assume first that $\varrho(A_P) \neq \emptyset$ and that the functions $[(A_P - z)^{-1}h_0, h_0]$ and $\frac{1}{q(z)}$ do not coincide, i.e. that

$$[(A_P - z)^{-1}h_0, h_0] \neq \frac{1}{q(z)}$$

for $z \in M$ where $M \subseteq \mathbb{C} \setminus \mathbb{R}$ is such that $(\mathbb{C} \setminus \mathbb{R}) \setminus M$ has no accumulation point in $\mathbb{C} \setminus \mathbb{R}$. As S_P itself is selfadjoint we may decompose in $\mathcal{P}^2$ the relation A_P as $A_P = S_P[\dot{+}]A'_P$ where A'_P is a selfadjoint relation in the π_1-space $\mathcal{H}_n^\perp$. Hence $(A_P - z)^{-1}$ decomposes as

$$(A_P - z)^{-1} = \begin{pmatrix} R_P(z) & 0 \\ 0 & R'_P(z) \end{pmatrix} : \begin{matrix} \mathcal{H}_n \\ [\dot{+}] \\ \mathcal{H}_n^\perp \end{matrix} \rightarrow \begin{matrix} \mathcal{H}_n \\ [\dot{+}] \\ \mathcal{H}_n^\perp \end{matrix} ,$$

where $R_P(z) = (S_P - z)^{-1}$ and $R'_P(z) = (A'_P - z)^{-1}$.

The resolvent $R(z) = (A - z)^{-1}$ can be written as

$$R(z) = \{((g - zf) + ([f, c_1 - \overline{z}c_0] + [g, d_1 - \overline{z}d_0])h_0 + [f, h_0](c_1 + zd_1)-$$

$$-[g, h_0](c_0 + zd_0); f + ([f, c_0] + [g, d_0])h_0 - [f, h_0]d_1 + [g, h_0]d_0)|(f; g) \in A_P\}.$$

Let $z \in \varrho(A_P) \cap M$ and assume that $(u; \tilde{u}) \in R(z)$. Then there exists an element $(f; g) \in A_P$, such that

$$u = (g - zf) + ([f, c_1 - \overline{z}c_0] + [g, d_1 - \overline{z}d_0])h_0 + [f, h_0](c_1 + zd_1) - [g, h_0](c_0 + zd_0) \quad (5.4)$$

and

$$\tilde{u} = f + ([f, c_0] + [g, d_0])h_0 - [f, h_0]d_1 + [g, h_0]d_0. \quad (5.5)$$

We will consider the components of (5.4) with respect to the decomposition

$$\mathcal{P} = \mathcal{H}_n[\dot{+}](\langle h_0 \rangle \dot{+} \langle h_1 \rangle)[\dot{+}]\mathcal{P}_c^\perp. \quad (5.6)$$

Let $f = f_n + \varphi_0 h_0 + \varphi_1 h_1 + f_r$ and $g = g_n + \gamma_0 h_0 + \gamma_1 h_1 + g_r$ with respect to (5.6), then we have, as $u \in \mathcal{H}$

$$0 = \gamma_1 - z\varphi_1 \text{ , i.e. } \gamma_1 = z\varphi_1. \quad (5.7)$$

Relation (5.4) thus takes the form

$$u = (g - zf) + ([f_n, c_1 - \overline{z}c_0] + [g_n, d_1 - \overline{z}d_0])h_0 + \varphi_1 a(z) \quad (5.8)$$

and shows that

$$u_n = (g_n - zf_n) + \varphi_1 a(z). \quad (5.9)$$

By applying $R_P(z)$ to (5.9) we get

$$f_n = R_P(z)u_n - \varphi_1 R_P(z)a(z) \quad (5.10)$$

and, substituting into (5.9)

$$g_n = u_n + zR_P(z)u_n - \varphi_1(zR_P(z) + I)a(z). \quad (5.11)$$

Now apply $(A_P - z)^{-1}$ to u: (5.8) implies that

$$(A_P - z)^{-1}u = f + ([f_n, c_1 - \overline{z}c_0] + [g_n, d_1 - \overline{z}d_0])R'_P(z)h_0 + \varphi_1 R_P(z)a(z),$$

on the other hand $(A_P - z)^{-1}u = R_P(z)u_n + \mu_0 R'_P(z)h_0$. Multiplying by h_0 we obtain

$$\mu_0[R'_P(z)h_0, h_0] = \varphi_1 + ([f_n, c_1 - \overline{z}c_0] + [g_n, d_1 - \overline{z}d_0])[R'_P(z)h_0, h_0]. \quad (5.12)$$

Using (5.10) and (5.11) we find

$$[f_n, c_1 - \overline{z}c_0] + [g_n, d_1 - \overline{z}d_0] = [R_P(z)u_n, c_1 - \overline{z}c_0] -$$

$$-\varphi_1[R_P(z)a(z), c_1 - \overline{z}c_0] + [u_n, d_1 - \overline{z}d_0] + z[R_P(z)u_n, d_1 - \overline{z}d_0] -$$

$$-\varphi_1[(zR_P(z) + I)a(z), d_1 - \overline{z}d_0] =$$

$$= [R_P(z)u_n, a(\overline{z})] + [u_n, d_1 - \overline{z}d_0] - \varphi_1([R_P(z)a(z), a(\overline{z})] + [a(z), d_1 - \overline{z}d_0]) =$$

$$= [u_n, b(\overline{z})] - \varphi_1[a(z), b(\overline{z})].$$

Thus φ_1 computes from (5.12) as

$$\varphi_1 = -\frac{\mu_0 - [u_n, b(\overline{z})]}{-\frac{1}{\tau(z)} + [a(z), b(\overline{z})]} = -\frac{[u, \chi(\overline{z})]}{-\frac{1}{\tau(z)} + q(z)}, \tag{5.13}$$

where $\tau(z) = [R_P'(z)h_0, h_0]$.

Consider in particular the element $u = 0$. Due to (5.13) we have $\varphi_1 = 0$, and (5.8) together with (5.10) and (5.11) shows that $g = zf$. Since $z \in \varrho(A_P)$ and $(f; g) = (f; zf) \in A_P$ we find $f = g = 0$. Hence $\tilde{u} = 0$, i.e. $R(z)$ is an operator. This implies that $z \in \varrho(A)$ and we find

$$\varrho(A) \supseteq \varrho(A_P) \cap M,$$

in particular $\varrho(A) \neq \emptyset$. Since $R(z)$ is an operator we may write $\tilde{u} = R(z)u$.

We obtain from (5.5) and (5.10)

$$[R(z)u - R_P(z)u_n, u] = [f - f_n, u] - \varphi_1[(d_1 - zd_0), u] - \varphi_1[R_P(z)a(z), u] =$$

$$= \varphi_1(\overline{\mu_0} - [b(z), u]),$$

thus, using (5.13)

$$[R(z)u, u] = [R_P(z)u_n, u_n] - [\chi(z), u]\frac{1}{-\frac{1}{\tau(z)} + q(z)}[u, \chi(\overline{z})].$$

Consider the case that $\varrho(A_P) \neq \emptyset$ but assume that $\tau(z)$ coincides with $\frac{1}{q(z)}$. For $z \in \varrho(A_P)$ let $(f'; g') \in A_P'$ be such that $g' - zf' = h_0$ and put

$$f_n = -[f', h_0]R_P(z)a(z),$$

$$g_n = -[f', h_0](zR_P(z) + I)a(z).$$

It is checked by a straightforward calculation using the above formulas and the fact that $[f', h_0] = \tau(z) \neq 0$, that $f' + f_n \in \ker(A - z)$ Therefore $\varrho(A) = \emptyset$.

It remains to study the case that $\varrho(A_P) = \emptyset$, i.e. $\ker(A_P - z) = \ker(A_P' - z) \neq \{0\}$. As $\mathcal{H}_n^\perp$ is a π_1-space, we have that $\ker(A_P' - z)$ is constant on $\mathbb{C} \setminus \mathbb{R}$ and has dimension 1 (see [3]). If $\ker(A_P' - z) = \langle h_0 \rangle$, i.e. $(h_0; zh_0) \in A_P'$, we have $h_0 \in \ker(A - z)$ and therefore $\varrho(A) = \emptyset$.

If $\ker(A_P' - z) \neq \langle h_0 \rangle$ write $\ker(A_P' - z) = \langle h \rangle$. Then h is neutral and $h \in A_P'(0)$, thus $h \in \mathcal{R}(A_P' - z)^\circ$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. As $\mathcal{H}_n^\perp$ is a π_1-space $\mathcal{R}(A_P' - z)$ is positive semidefinite and we can assume that h and h_0 are skewly linked. This shows that $h_0 \notin \mathcal{R}(A'_P - z)$. We will prove that $\varrho(A) \neq \emptyset$, i.e. $\ker(A - z) = \{0\}$. Assume on the contrary that $(f; g) \in A_P$ such that $\Psi(f; g) \in \ker(A - z)$. Let $f = f_n + f'$ and $g = g_n + g'$ with respect to $\mathcal{P} = \mathcal{H}_n[\dot{+}]\mathcal{H}_n^{\perp}$. The fact that $\Psi(f; g) \in \ker(A - z)$ implies

$$(g' - zf') + ([f_n, c_1 - \overline{z}c_0] + [g_n, d_1 - \overline{z}d_0])h_0 = 0. \tag{5.14}$$

As $h_0 \notin \mathcal{R}(A_P - z)$ we find $g' - zf' = 0$ which shows that $f' \in \ker(A_P - z) = \langle h \rangle$, i.e. $f' = \lambda h$. Furthermore

$$0 = (g_n - zf_n) + [f, h_0](c_1 + zd_1) - [g, h_0](c_0 + zd_0) = (g_n - zf_n) + \lambda a(z),$$

i.e. $g_n - zf_n = -\lambda a(z)$. Thus we have $f_n = -\lambda R_P(z)a(z)$ and $g_n = -\lambda(zR_P(z) + I)a(z)$. Now (5.14) shows that

$$0 = [f_n, c_1 - \overline{z}c_0] + [g_n, d_1 - \overline{z}d_0] = [g_n - zf_n, b(\overline{z})] = -\lambda q(z).$$

If $q(z) \neq 0$ we find $\lambda = 0$, thus $f = 0$ and $g = 0$, a contradiction.

Let $h = \alpha_0 h_0 + \alpha_1 h_1 + h_r$. If $q(z) = 0$ for some $z \in \mathbb{C} \setminus \mathbb{R}$ consider the element

$$(\alpha_1 R_P(z)a(z) - h; \alpha_1(zR_P(z) + I)a(z) - zh) \in A_P.$$

By substituting this element into the relations (5.4) and (5.5), it follows by a straightforward computation that

$$\ker(A - z) = R(z)(0) \neq \{0\}.$$

In order to compute $[(A - z)^{-1}u, u]$ consider (5.8). It shows that

$$\mu_0 h_0 = (g' - zf') + ([f_n, c_1 - \overline{z}c_0] + [g_n, d_1 - \overline{z}d_0])h_0.$$

Again, as $h_0 \notin \mathcal{R}(A'_P - z)$, we have $g' - zf' = 0$, $f' = \lambda h$ and

$$\mu_0 - ([f_n, c_1 - \overline{z}c_0] + [g_n, d_1 - \overline{z}d_0]) = 0.$$

Substituting (5.10) and (5.11) we obtain

$$\mu_0 - ([u_n, b(\overline{z})] - \lambda q(z)) = 0$$

as $\lambda = \varphi_1$ and therefore

$$\lambda = -\frac{[u, \chi(\overline{z})]}{q(z)}.$$

From (5.5) and (5.10) we find

$$[R(z)u, u] = [R_P(z)u_n, u_n] - [u, \chi(\overline{z})]\frac{1}{q(z)}[\chi(z), u].$$

This corresponds to the parameter $\tau(z) = \infty$.

$\square$

From the proof of Theorem 2 we have the following

Corollary 1 *Let $A \supseteq S$ and $A_P \supseteq S_P$ be selfadjoint relations, $A = \Psi(A_P)$. If $\varrho(A_P) \neq \emptyset$ then $\varrho(A) \neq \emptyset$ if and only if $[(A_P - z)^{-1}h_0, h_0] \neq \frac{1}{q(z)}$, in fact*

$$(\varrho(A) \cap \varrho(A_P)) \setminus \mathbb{R} = \{z \in \varrho(A_P) \setminus \mathbb{R} | \tau(z) = [(A_P - z)^{-1}h_0, h_0] \neq \frac{1}{q(z)}\}.$$

If $\varrho(A_P) = \emptyset$ then $\varrho(A) \neq \emptyset$ if and only if for one and hence for all $z \in \mathbb{C} \setminus \mathbb{R}$

$$\ker (A_P - z) \neq \langle h_0 \rangle.$$

In this case

$$\varrho(A) \setminus \mathbb{R} = \{z \in \mathbb{C} \setminus \mathbb{R} | q(z) \neq 0\}.$$

In the following we give an interpretation of the expressions $\chi(z)$ and $q(z)$ as the defect elements and Q-function, respectively, of a certain symmetric relation S_1 and a selfadjoint extension A_1 of S_1. In fact

$$S_1 = \Psi(S_P \dot{+} (0 \times \langle h_1 \rangle)),$$

and

$$A_1 = \Psi(S_P \dot{+} (0 \times \langle h_0, h_1 \rangle)).$$

By Corollary 1 we have $\varrho(A_1) \neq \emptyset$.

Proposition 4 *We have $\langle \chi(z) \rangle = \mathcal{R}\,(S_1 - \overline{z})^{\perp}$ and*

$$\chi(z) = (I + (z - w)(A_1 - z)^{-1})\chi(w).$$

The function $q(z)$ is the Q-function of S_1 and A_1, i.e.

$$\frac{q(z) - \overline{q(w)}}{z - \overline{w}} = [\chi(z), \chi(w)]. \tag{5.15}$$

Proof : We first prove that $\chi(z) \perp \mathcal{R}\,(S_1 - \overline{z})$. Note

$$S_1 = S + \langle (d_0; h_1 - c_0) \rangle,$$

thus

$$\mathcal{R}\,(S_1 - \overline{z}) = \mathcal{R}\,(S - \overline{z}) + \langle h_1 - (c_0 + \overline{z}d_0) \rangle. \tag{5.16}$$

We have

$$[\chi(z), h_1 - (c_0 + \overline{z}d_0)] = [b(z), c_0 + \overline{z}d_0] + [b(z), -(c_0 + \overline{z}d_0)] = 0,$$

which shows that $\chi(z)$ is orthogonal to the second summand on the right hand side of (5.16). Furthermore

$$\mathcal{R}\,(S - \overline{z}) = \{(g - \overline{z}f) + ([f, c_1 - zc_0] + [g, d_1 - zd_0])h_0 | (f; g) \in S_P\},$$

and

$$[\chi(z), (g - \overline{z}f) + ([f, c_1 - zc_0] + [g, d_1 - zd_0])h_0] = ([c_1 - zc_0, f] + [d_1 - zd_0, g]) -$$

$$-[b(z), g - \overline{z}f] = ([c_1 - zc_0, f] + [d_1 - zd_0, g]) - [a(z), f] -$$

$$-[d_1 - zd_0, g - \overline{z}f] = 0$$

shows that also $\chi(z) \perp \mathcal{R}(S - \overline{z})$. Thus the first assertion is proved.

Denote by A_P the selfadjoint relation

$$A_P = S_P \dot{+} (0 \times \langle h_0, h_1 \rangle)),$$

and by $R(z)$ and $R_P(z)$ the resolvents

$$R(z) = (A_1 - z)^{-1} \text{ and } R_P(z) = (A_P - z)^{-1},$$

respectively. The next step in the proof of Proposition 4 is to prove the relation

$$(I + (z - w)R(z))\chi(w) = \chi(z). \tag{5.17}$$

We reconsider the proof of Theorem 2 in the case that $A = A_1$ and A_P defined as above ($A = \Psi(A_P)$), but with an element

$$u = u_n + \mu_0 h_0 + \mu_1 h_1,$$

which is not any more contained in $\mathcal{H}$. As for $(f; g) \in A_P$ we have $\varphi_0 = \varphi_1 = 0$ (5.7) reads as

$$\gamma_1 = \mu_1,$$

and therefore the relations (5.4) and (5.5) become

$$u = (g - zf_n) + ([f_n, c_1 - \overline{z}c_0] + [g_n, d_1 - \overline{z}d_0])h_0 - \mu_1(c_0 + zd_0) \tag{5.18}$$

and

$$R(z)u = f_n + ([f_n, c_0] + [g_n, d_0])h_0 + \mu_1 d_0. \tag{5.19}$$

Applying $R_P(z)$ to both sides of (5.18) yields (note that $R_P(z)h_0 = R_P(z)h_1 = 0$)

$$R_P(z)u = f_n - \mu_1 R_P(z)(c_0 + zd_0), \tag{5.20}$$

which gives together with (5.19)

$$R(z)u = R_P(z)u_n + \mu_1(R_P(z)(c_0 + zd_0) + d_0) + ([f_n, c_0] + [g_n, d_0])h_0. \tag{5.21}$$

From (5.20) we get

$$[f_n, c_0] = [R_P(z)u_n, c_0] + \mu_1[R_P(z)(c_0 + zd_0), c_0]$$

and

$$[f_n, d_0] = [R_P(z)u_n, d_0] + \mu_1[R_P(z)(c_0 + zd_0), d_0],$$

which implies together with (5.18)

$$[g_n, d_0] = z[R_P(z)u_n, d_0] + z\mu_1[R_P(z)(c_0 + zd_0), d_0] + \mu_1[c_0 + zd_0, d_0] + [u_n, d_0].$$

Substituting into (5.21) this shows that

$$R(z)u = R_P(z)u_n + \mu_1(R_P(z)(c_0 + zd_0) + d_0) +$$

$$+[u_n+\mu_1(c_0+zd_0),d_0+R_P(\overline{z})(c_0+\overline{z}d_0)]h_0. \tag{5.22}$$

The element $\chi(w)$ is explicitly given as

$$\chi(w)=h_1-R_P(w)((c_1-wc_0)+w(d_1-wd_0))-(d_1-wd_0)-$$
$$-[R_P(w)((c_1-wc_0)+w(d_1-wd_0))+(d_1-wd_0),c_0+\overline{w}d_0]h_0.$$

Using (5.22) we obtain

$$(I+(z-w)R(z))\chi(w)=(I+(z-w)R_P(z))\chi(w)+(z-w)(R_P(z)(c_0+zd_0)+d_0+$$
$$+[(c_0+zd_0)-R_P(w)((c_1-wc_0)+w(d_1-wd_0))-(d_1-wd_0),d_0+R_P(\overline{z})(c_0+\overline{z}d_0)]h_0)=$$
$$=h_1+R_P(z)(-(c_1-wc_0)-w(d_1-wd_0)-(z-w)(d_1-wd_0)+(z-w)c_0+z(z-w)d_0)-$$
$$-(d_1-wd_0)+(z-w)d_0+[R_P(w)((c_1-wc_0)+w(d_1-wd_0))+(d_1-wd_0),$$
$$,-c_0-\overline{w}d_0+(\overline{w}-\overline{z})d_0+(\overline{w}-\overline{z})R_P(\overline{z})(c_0+\overline{z}d_0)]h_0+$$
$$+(z-w)[c_0+zd_0,d_0+R_P(\overline{z})(c_0+\overline{z}d_0)]h_0=$$
$$=h_1-R_P(z)((c_1-zc_0)+z(d_1-zd_0))-(d_1-zd_0)-[R_P(z)((c_1-wc_0)+$$
$$+w(d_1-wd_0))+(I+(z-w)R_P(z))(d_1-wd_0)-(z-w)d_0-R_P(z)((z-w)(c_0+zd_0)),$$
$$,c_0+\overline{z}d_0]h_0=\chi(z).$$

To obtain the last but one equality sign we have used the relation

$$[c_0+zd_0,d_0]=[d_0,c_0+\overline{z}d_0],$$

which holds as $(c_0;d_0)\in S_P=S_P^+$, and by the resolvent identity

$$(z-w)R_P(z)R_P(w)=R_P(z)-R_P(w).$$

Thus (5.17) is proved, and therefore $\chi(z)$ parametrizes the defect spaces of S_1 appropriately.

In order to show that $q(z)$ is the Q-function of S_1 and A_1 it thus suffices to prove the relation (5.15). As a straightforward computation shows the function $q(z)$ is real, i.e.

$$q(z)=\overline{q(\overline{z})}=-[\chi(z),a(\overline{z})].$$

We compute

$$\frac{q(z)-q(\overline{w})}{z-\overline{w}}=-\frac{[\chi(z),a(\overline{z})]-[\chi(\overline{w}),a(w)]}{z-\overline{w}}=$$
$$=-\frac{[(I+(z-\overline{w})R(z))\chi(\overline{w}),a(\overline{z})]-[\chi(\overline{w}),a(w)]}{z-\overline{w}}=-[\chi(\overline{w}),R(\overline{z})a(\overline{z})]-$$
$$-[\chi(\overline{w}),\frac{a(\overline{z})-a(w)}{\overline{z}-w}]\overset{(5.22)}{=}$$
$$\overset{(5.22)}{=}-[\chi(\overline{w}),R_P(\overline{z})a(\overline{z})+[a(\overline{z}),d_0+R_P(z)(c_0+zd_0)]h_0]-[\chi(\overline{w}),-c_0+d_1-(w+\overline{z})d_0]=$$
$$\overset{S_P=S_P^+}{=}-[c_0+zd_0,d_1-\overline{z}d_0]-[c_0+zd_0,R_P(\overline{z})a(\overline{z})]-[b(\overline{w}),c_0+wd_0]+[b(\overline{w}),b(\overline{z})]=$$
$$=[\chi(\overline{w}),\chi(\overline{z})]=[\chi(z),\chi(w)].$$

Thus all assertions of Proposition 4 are proved.

□

Remark 4 Due to Proposition 4 we have $q \in \mathcal{N}_{\kappa'}$ with $\kappa' \leq \kappa+1$, if κ denotes the negative index of $\mathcal{H}$. Using the characterization (5.2) of the set of parameters, we find that the exception of the parameter $\tau(z) = \frac{1}{q(z)}$ in Theorem 2 occurs if and only if q is a rational function of degree κ' or q is rational and of degree $\kappa'+1$ and $0 \leq \lim_{\eta\to\infty} \frac{i\eta}{q(i\eta)} < \infty$.

Proposition 4 shows in particular that q is not identically zero. As

$$\mathcal{R}\,(S_1 - z) = \mathcal{R}\,(S - z) + \langle h_1 - (c_0 + zd_0)\rangle$$

and $u \in \mathcal{H}$ we have $u \in \mathcal{R}\,(S_1 - z)$ if and only if $u \in \mathcal{R}\,(S - z)$. Since we have assumed that u is a module element for S, u is also a module element for S_1, in particular $[u, \chi(\overline{z})] \neq 0$ for one and hence for all $z \in \mathbb{C}^+$ $(\mathbb{C}^-)$ with possible exception of an isolated set (compare [11]).

Theorem 2 leads to a parametrization of the u-resolvents of S, i.e. of the functions of the form $(u \in \mathcal{H})$

$$r_u(z) = [(A - z)^{-1}u, u].$$

With a similar proof as in Section 3 of [11] we find

Proposition 5 *The u-resolvents of S are parametrized by*

$$r_u(z) = \frac{w_{11}(z)(-\frac{1}{\tau(z)}) + w_{12}(z)}{w_{21}(z)(-\frac{1}{\tau(z)}) + w_{22}(z)},\ \tau(z) \in \mathcal{T}, \tag{5.23}$$

where

$$w_{11}(z) = \frac{[R_P(z)u_n, u_n]}{[u, \chi(\overline{z})]},\ w_{12}(z) = \frac{[R_P(z)u_n, u_n]q(z)}{[u, \chi(\overline{z})]} - [\chi(z), u],$$

$$w_{21}(z) = \frac{1}{[u, \chi(\overline{z})]},\ w_{22}(z) = \frac{q(z)}{[u, \chi(\overline{z})]}.$$

The matrix

$$W(z) = \begin{pmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{pmatrix}$$

is the u-resolvent matrix of S_1. It satisfies the equation

$$\frac{W(z)JW(w)^* - J}{z - \overline{w}} = \begin{pmatrix} Q(z) \\ -P(z) \end{pmatrix} (Q(w)^* \quad -P(w)^*)$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *and* $(f = f_n + \varphi_0 h_0 + \varphi_1 h_1)$

$$P(z)f = \frac{[f, \chi(\overline{z})]}{[u, \chi(\overline{z})]}.$$

$$Q(z)f = [R_P(z)f_n, u_n] - \frac{[f, \chi(\overline{z})]}{[u, \chi(\overline{z})]}[R_P(z)u_n, u_n] + \varphi_1[R_P(z)(f_0 + zg_0) + g_0, u_n].$$

6 Analytic characterization of the parameters

In the following two sections we prove that the class $\mathcal{T}$ of parameters is given by (5.2) and investigate some properties of these parameter functions.

Theorem 3 *The function τ is an element of*

$$\mathcal{T}_1 = \mathcal{N}_0 \cup \{\tau \in \mathcal{N}_1 | 0 \leq \lim_{\eta\to\infty} i\eta\tau(i\eta) < \infty\},$$

if and only if τ admits a representation

$$\tau(z) = [(A-z)^{-1}v, v], \tag{6.1}$$

where A is a selfadjoint relation in a π_1-space $\mathcal{P}$ with $\varrho(A) \neq \emptyset$ and $v \in \mathcal{P}$ is neutral.

I.e. we have $\mathcal{T}_1 = \mathcal{T}$, when $\mathcal{T}$ is defined as in Section 5. The proof of Theorem 3 is split up into several lemmata.

Assume that τ admits a representation (6.1).

Lemma 6 *Let $\tau(z) = [(A-z)^{-1}v, v]$ with A, v as in Theorem 3, then $\tau \in \mathcal{T}_1$. If $\tau \in \mathcal{N}_1$ then $A(0)$ is positive definite or A is an operator.*

Proof : Note that

$$N_\tau(z,w) = [(A-z)^{-1}v, (A-w)^{-1}v]. \tag{6.2}$$

As the negative index of $\mathcal{P}$ is 1 we have $\tau \in \mathcal{N}_0 \cup \mathcal{N}_1$.

Assume that $f \in \mathcal{N}_1$. Then (6.2) and $\{(A-z)^{-1}v | z \in \varrho(A)\} \subseteq \overline{\mathcal{D}(A)}$ shows that $\overline{\mathcal{D}(A)}$ contains a negative element. Then $\overline{\mathcal{D}(A)}$ is nondegenerated, as otherwise it would contain a two dimensional nonpositive subspace. Therefore $A(0)$ $(= \overline{\mathcal{D}(A)}^\perp)$ is positive definite or trivial. Furthermore (see [3])

$$\mathcal{P} = \overline{\mathcal{D}(A)}[\dot{+}]A(0), \tag{6.3}$$

and we may decompose A as

$$A = A_s[\dot{+}]A_\infty$$

where A_s is a selfadjoint operator in $\overline{\mathcal{D}(A)}$ and $A_\infty = \{0\} \times A(0)$. Hence the resolvent $(A-z)^{-1}$ can be written as an operator matrix

$$(A-z)^{-1} = \begin{pmatrix} (A_s - z)^{-1} & 0 \\ 0 & 0 \end{pmatrix} : \begin{matrix} \overline{\mathcal{D}(A)} \\ [\dot{+}] \\ A(0) \end{matrix} \to \begin{matrix} \overline{\mathcal{D}(A)} \\ [\dot{+}] \\ A(0) \end{matrix}.$$

Let $v = v_s + v_\infty$ be the decomposition of v with respect to (6.3), then

$$\tau(z) = [(A-z)^{-1}v, v] = [(A_s - z)^{-1}v_s, v_s].$$

Since $0 = [v,v] = [v_s, v_s] + [v_\infty, v_\infty]$ and $[v_\infty, v_\infty] \geq 0$ we have $[v_s, v_s] \leq 0$. As $\lim_{\eta\to\infty} i\eta(A_s - i\eta)^{-1}v_s = -v_s$ (see [4], Theorem 2.4) we have $\lim_{\eta\to\infty} i\eta\tau(i\eta) \geq 0$.

□

In order to prove the converse implication of the theorem we consider first the case that $\tau \in \mathcal{N}_1 \cap \mathcal{T}_1$.

Lemma 7 *Let $\tau \in \mathcal{N}_1 \cap \mathcal{T}_1$, then τ has a representation (6.1).*

Proof : As $0 \leq \lim_{\eta\to\infty} i\eta\tau(i\eta) < \infty$ implies that $\lim_{\eta\to\infty} \tau(i\eta) = 0$ and that $\lim_{\eta\to\infty} y|\operatorname{Im}\tau(i\eta)|$ exists, a result of [10] shows that there is a π_1-space $\mathcal{P}_s$, a selfadjoint operator A_s and an element $u \in \mathcal{P}_s$, such that

$$\tau(z) = [(A_s - z)^{-1}u, u].$$

Due to $0 \leq \lim_{\eta\to\infty} i\eta\tau(i\eta) < \infty$ and [4], Theorem 2.4 we have

$$[u, u] = -\lim_{\eta\to\infty} i\eta\tau(i\eta) \leq 0.$$

If u is neutral we are done, otherwise define

$$\mathcal{P} = \mathcal{P}_s[\dot{+}]\langle h\rangle$$

where $[h, h] = 1$, and

$$A = A_s[\dot{+}](0 \times \langle h\rangle).$$

Then A is a selfadjoint relation in the π_1-space $\mathcal{P}$ and we have $\ker((A-z)^{-1}) = \langle h\rangle$ and $\mathcal{R}((A-z)^{-1}) \subseteq \mathcal{P}_s$. Let $v = u + h\sqrt{-[u,u]}$, then

$$\tau(z) = [(A_s - z)^{-1}u, u] = [(A - z)^{-1}v, v],$$

and $[v, v] = 0$.

□

In the following let $\tau \in \mathcal{N}_0$. Denote by $\mathcal{L}_\tau$ the inner product space

$$\mathcal{L}_\tau = \{\sum_{z\in\mathbb{C}\setminus\mathbb{R}} \xi_z e_z | \xi_z \in \mathbb{C}, \xi_z \neq 0 \text{ only for finitely many } z \in \mathbb{C}\setminus\mathbb{R}\}$$

with the inner product defined by

$$[e_z, e_w] = N_\tau(z, w).$$

Remark 5 It is well known that τ admits the representation

$$\tau(z) = \overline{\tau(z_0)} + (z - \overline{z_0})[(I + (z - z_0)(A_\tau - z)^{-1}e_{z_0}, e_{z_0}]$$

with a selfadjoint relation A_τ with $z_0 \in \varrho(A_\tau)$ acting in a certain π_0-space $\mathcal{P}_\tau$. The space $\mathcal{P}_\tau$ is obtained as the completion

$$\mathcal{P}_\tau = \widehat{\mathcal{L}_\tau/\mathcal{L}_\tau^\circ}.$$

For details on this construction see [9] and [10].

Definition 4 *Let $\mathcal{L}$ be the inner product space*

$$\mathcal{L} = \mathcal{L}_\tau \dot{+} \langle h_0, h_1\rangle$$

endowed with the inner product given by

$$[f,g]_{\mathcal{L}} = [f,g]_{\mathcal{L}_\tau} \textit{ for } f,g \in \mathcal{L}_\tau,$$
$$[e_z,h_0]_{\mathcal{L}} = \overline{[h_0,e_z]_{\mathcal{L}}} = \tau(z) \textit{ and}$$
$$[e_z,h_1]_{\mathcal{L}} = [h_1,e_z]_{\mathcal{L}} = 0 \textit{ for } z \in \mathbb{C}\setminus\mathbb{R}$$

and where h_0 and h_1 are skewly linked, i.e.

$$[h_0,h_1]_{\mathcal{L}} = [h_1,h_0]_{\mathcal{L}} = 1 \textit{ and } [h_0,h_0]_{\mathcal{L}} = [h_1,h_1]_{\mathcal{L}} = 0.$$

If no confusion can occur we will drop the index at the inner product. Note that the elements h_0 and h_1 are by definition linearly independent of each other and of $\mathcal{L}_\tau$.

Lemma 8 *The inner product space $\mathcal{L}$ is a π_1-lineal and*

$$\mathcal{L}^\circ = \{f - [f,h_0]h_1 | f \in \mathcal{L}_\tau^\circ\}.$$

Proof : Obviously the element $h_0 - h_1$ is negative. Assume that $\mathcal{G} \subseteq \mathcal{L}$ is a two dimensional negative subspace. Then, by a dimension argument,

$$\mathcal{G} \cap (\mathcal{L}_\tau[\dot{+}]\langle h_1\rangle) \neq \{0\}$$

which yields a contradiction as $\tau \in \mathcal{N}_0$ implies that $\mathcal{L}_\tau[\dot{+}]\langle h_1\rangle$ is positive semidefinite.

To prove the second assertion let $f' = f + \varphi_0 h_0 + \varphi_1 h_1 \in \mathcal{L}^\circ$ where $f \in \mathcal{L}_\tau$, then

$$\varphi_0 = [f + \varphi_0 h_0 + \varphi_1 h_1, h_1] = 0$$

and

$$0 = [f + \varphi_0 h_0 + \varphi_1 h_1, h_0] = \varphi_1 + [f,h_0],$$

thus $f' = f - [f,h_0]h_1$. The converse inclusion follows similarly.

□

Definition 5 *Let S be the relation*

$$\langle (0;h_1),(e_z;h_0 + z e_z) | z \in \mathbb{C}\setminus\mathbb{R}\rangle \subseteq \mathcal{L}^2.$$

Lemma 9 *The relation S is symmetric.*

Proof : Recall that τ is real, i.e. $\tau(\overline{z}) = \overline{\tau(z)}$. Let $z \neq \overline{w}$, then

$$[e_z, h_0 + w e_w] - [h_0 + z e_z, e_w] = [e_z,h_0] - [h_0,e_w] -$$
$$-(z-\overline{w})[e_z,e_w] = \tau(z) - \tau(\overline{w}) - (z-\overline{w})\frac{\tau(z)-\tau(\overline{w})}{z-\overline{w}} = 0.$$

If $z = \overline{w}$ we have

$$[e_z, h_0 + \overline{z} e_{\overline{z}}] - [h_0 + z e_z, e_{\overline{z}}] = \tau(z) - \overline{\tau(\overline{z})} = 0.$$

As $h_1 \perp e_z$ for each $z \in \mathbb{C}\setminus\mathbb{R}$ the lemma is proved.

□

Let $\mathcal{P}$ be the π_1-space constructed from $\mathcal{L}$ (see [9]), and denote by A the relation $A = \overline{S/(\mathcal{L}^\circ)^2} \subseteq \mathcal{P}^2$.

Remark 6 Note that, even if $\mathcal{L}_\tau$ is nondegenerated, the closure of $\mathcal{L}_\tau$ in $\mathcal{P}$ need not be isomorphic to $\mathcal{P}_\tau$ (compare Corollary 2).

Lemma 10 *The relation A is selfadjoint and has a nonempty resolvent set. In fact $\mathbb{C}\setminus\mathbb{R}\subseteq \varrho(A)$.*

Proof : Clearly A is a closed symmetric relation. Let $z, w \in \mathbb{C}\setminus\mathbb{R}$, then

$$(S-z)^{-1}h_0 = e_z, \tag{6.4}$$

$$(S-z)^{-1}h_1 = 0 \text{ and}$$

$$(S-z)^{-1}e_w = \frac{e_w - e_z}{w-z} \text{ for } w \neq z.$$

For $z \in \mathbb{C}\setminus\mathbb{R}$ let

$$C_z(S) = \{(g - zf; g - \overline{z}f) | (f; g) \in S\}$$

be the Cayley transform of S. Since S is symmetric C_z is an isometric operator. As τ is continuous at z we have $e_w \to e_z$ in the inner product topology of $\mathcal{L}$ if $z \to w$ (see [9]). This implies together with

$$\mathcal{D}\left(C_z(S)\right) = \mathcal{R}\left(S-z\right) \text{ and } \mathcal{R}\left(C_z(S)\right) = \mathcal{R}\left(S-\overline{z}\right)$$

that $\mathcal{D}\left(C_z(S)\right)$ and $\mathcal{R}\left(C_z(S)\right)$ are dense in $\mathcal{L}$. Thus

$$C_z(S)(\mathcal{L}^\circ) \subseteq \mathcal{L}^\circ,$$

and therefore $C_z(S)/(\mathcal{L}^\circ)^2$ is an isometric densely defined operator in $\mathcal{P}$. Its closure V_z is an everywhere defined continuous isometry and the relation

$$C_z(S/(\mathcal{L}^\circ)^2) = C_z(S)/(\mathcal{L}^\circ)^2$$

shows that V_z is the Cayley transform of A. As

$$(A-z)^{-1} = \frac{1}{z-\overline{z}}(V_z - I)$$

we have $z \in \varrho(A)$ and A is selfadjoint.

□

From the definition of the inner product on $\mathcal{P}$ and from (6.4) it is obvious that τ admits a representation of the form (6.1):

$$\tau(z) = [(A-z)^{-1}h_0, h_0].$$

The preceeding lemmata imply Theorem 3.

7 Some properties of the functions $\tau \in \mathcal{T}$

If we restrict ourselves in Theorem 3 to minimal relations, the connection between selfadjoint relations and parameter functions becomes (up to unitary equivalence) a one to one correspondence.

Definition 6 *Let A be a selfadjoint relation in a Pontrjagin space $\mathcal{P}$ with $\varrho(A) \neq \emptyset$ and let $v \in \mathcal{P}$. Then A is called v-minimal (or, equivalently, v is called a generating element for A) if*

$$\overline{\langle v, (A-z)^{-1}v | z \in \varrho(A) \rangle} = \mathcal{P}.$$

Proposition 6 *Let $A_1, \mathcal{P}_1, v_1$ $(A_2, \mathcal{P}_2, v_2)$ be such that A_1 (A_2) is v_1 (v_2) -minimal and v_1 (v_2) is neutral. Then*

$$[(A_1 - z)^{-1}v_1, v_1] = [(A_2 - z)^{-1}v_2, v_2],\ z \in \varrho(A_1) \cap \varrho(A_2)$$

implies that A_1 and A_2 are unitary equivalent.

For any $\tau \in \mathcal{T}$ $(\tau \neq 0)$ there exists a π_1-space $\mathcal{P}$, a neutral element $v \in \mathcal{P}$ and a v-minimal selfadjoint relation A, such that τ is represented as u-resolvent.

Proof : The uniqueness part of the assertion is proved as in [10]. Also it is clear that τ can be represented by a minimal relation, simply if one restricts A to the Pontrjagin space constructed from the lineal $\langle v, (A-z)^{-1}v | z \in \varrho(A) \rangle$. Thus it remains to prove that $\langle v, (A-z)^{-1}v | z \in \varrho(A) \rangle$ actually is a π_1-lineal. Assume on the contrary that it is positive semidefinite. As v is neutral v is in fact isotropic, in particular

$$v \perp (A-z)^{-1}v,$$

i.e. $\tau = 0$, a contradiction.

□

In the following we investigate the relation between $\tau \in \mathcal{T}$ and a representing relation A more closely.

Proposition 7 *Let $\tau \in \mathcal{T}$ and let*

$$\tau(z) = [(A-z)^{-1}v, v]$$

with a selfadjoint relation A in a π_1-space $\mathcal{P}$ and a neutral element v. Assume that A is v-minimal. Then

(i) *$\tau \in \mathcal{N}_1$ if and only if $A(0) = \{0\}$ or $A(0)$ is positive definite.*

(ii) *$\tau \in \mathcal{N}_0$ and*

$$\lim_{\eta \to \infty} \eta |\tau(i\eta)| < \infty \qquad (7.1)$$

if and only if $A(0)$ is negative definite.

(iii) *$\tau \in \mathcal{N}_0$ and the limit (7.1) does not exist if and only if $A(0)$ is neutral.*

Proof : If $\tau \in \mathcal{N}_1$, then Lemma 6 can be applied and shows that $A(0) = \{0\}$ or $A(0)$ is positive definite.

Assume that $A(0) = \{0\}$, then

$$\lim_{\eta \to \infty} i\eta(A - i\eta)^{-1}v = -v.$$

Thus

$$\mathcal{P} = \overline{\langle (A - z)^{-1}v | z \in \varrho(A) \rangle},$$

which shows that $\tau \in \mathcal{N}_1$. Assume that $A(0)$ ($\neq \{0\}$) is positive definite, then $\mathcal{P} = \overline{\mathcal{D}(A)}[\dot{+}]A(0)$ and therefore $\overline{\mathcal{D}(A)}$ is a π_1-space. As

$$\overline{\langle (A - z)^{-1}v | z \in \varrho(A) \rangle} \subseteq \overline{\mathcal{D}(A)} \tag{7.2}$$

and the left hand side of (7.2) has codimension at most 1 in $\mathcal{P}$, we have equality in (7.2) and we find $\tau \in \mathcal{N}_1$.

Let $\tau \in \mathcal{N}_0$ satisfy (7.1). Then there exists a Hilbert space $\mathcal{P}_0$, a selfadjoint operator A_0 and a generating element $u \in \mathcal{P}_0$ exists, such that

$$\tau(z) = [(A_0 - z)^{-1}u, u].$$

Let $\mathcal{P}_1 = \mathcal{P}_0[\dot{+}]\langle h \rangle$ where $[h, h] = -1$, and set

$$A_1 = A_0 \dot{+} (0 \times \langle h \rangle). \tag{7.3}$$

Then $\tau(z) = [(A_1 - z)^{-1}v, v]$ where $v = u + h\sqrt{[u, u]}$. As A_0 is u-minimal A_1 is v-minimal. Obviously $A_1(0)$ $(= \langle h \rangle)$ is negative definite. Proposition 6 implies that $A(0)$ is also negative definite.

If $A(0)$ is negative definite, then $\mathcal{P} = \overline{\mathcal{D}(A)}[\dot{+}]A(0)$ and $\overline{\mathcal{D}(A)}$ is positive definite. Thus (7.2) shows that $\tau \in \mathcal{N}_0$. Corresponding to (6.3) A can be decomposed as

$$A = A_0[\dot{+}]A_\infty$$

and $\tau(z) = [(A_0 - z)^{-1}u, u]$ where $v = u + v_\infty$ with $u \in \overline{\mathcal{D}(A)}$, $v_\infty \in A(0)$ and where A_0 is a selfadjoint operator in the Hilbert space $\overline{\mathcal{D}(A)}$. Now [7] shows that (7.1) holds.

Due to the minimality assumption we have in any case $\dim A(0) \leq 1$. Therefore $A(0)$ is either positive or negative definite or neutral. Hence *(iii)* is a consequence of *(i)* and *(ii)*.

□

The (up to unitary equivalence) unique v-minimal relation representing a function $\tau \in \mathcal{T}$ can be determined explicitly.

Remark 7 In case *(i)* of Proposition 7 the operator A_s in Lemma 7 can be chosen minimal. Then A is v-minimal.

Proposition 8 *In case (iii) of Proposition 7 the relation A constructed in Section 6 (see Definition 5) is h_0-minimal. If case (ii) of Proposition 7 occurs the relation A is not h_0-minimal. Then the subspace*

$$\mathcal{P}_1 = \overline{\mathcal{L}_\tau} \dot{+} \langle h_0 \rangle,$$

which does not contain h_1, is a π_1-space and reduces A to a h_0-minimal relation.

Proof : Consider the construction given in Section 6. If $\mathcal{L}^\circ \neq \{0\}$ we identify $f \in \mathcal{L}$ with its canonical image in $\mathcal{L}/\mathcal{L}^\circ$. We have

$$\mathcal{L}_\tau = \langle (A-z)^{-1} h_0 | z \in \mathbb{C} \setminus \mathbb{R} \rangle.$$

Note that, as $h_1 \perp \mathcal{L}_\tau$ but $[h_1, h_0] = 1$, the element h_0 is not in $\overline{\mathcal{L}_\tau}$. Hence $\overline{\mathcal{L}_\tau}$ itself has codimension 1 or 2. We consider the alternative $h_1 \in \overline{\mathcal{L}_\tau}$, or $h_1 \notin \overline{\mathcal{L}_\tau}$. This means

$$h_1 \in \overline{\mathcal{L}_\tau} \dot{+} \langle h_0 \rangle = \overline{\mathcal{L}_\tau \dot{+} \langle h_0 \rangle},$$

or $h_1 \notin \overline{\mathcal{L}_\tau \dot{+} \langle h_0 \rangle}$, i.e. whether A is h_0-minimal or not.

If $h_1 \in \overline{\mathcal{L}_\tau}$, the subspace $\overline{\mathcal{L}_\tau}$ has codimension 1 and thus

$$A(0) = \overline{\mathcal{D}(A)}^\perp = \overline{\mathcal{L}_\tau}^\perp = \langle h_1 \rangle.$$

Since then A is h_0-minimal, Proposition 7 shows that case *(iii)* occurs. Note that in this case $\overline{\mathcal{L}_\tau}$ is positive semidefinite and degenerated.

If $h_1 \notin \overline{\mathcal{L}_\tau}$, then $\overline{\mathcal{L}_\tau}$ has codimension 2 and is positive definite as $\mathcal{P}$ is a π_1-space. Thus

$$A(0) = \overline{\mathcal{L}_\tau}^\perp = \langle h_1, f + h_0 \rangle$$

for some $f \in \overline{\mathcal{L}_\tau}$, $f \neq 0$. We have

$$\mathcal{P}_1 = \overline{\mathcal{L}_\tau} \dot{+} \langle h_0 \rangle = \overline{\mathcal{L}_\tau} [\dot{+}] \langle f + h_0 \rangle, \tag{7.4}$$

and

$$0 = [f, f + h_0] = [f, f] + [f, h_0],$$

which shows that

$$[f + h_0, f + h_0] = [f + h_0, f] + [f, h_0] = -[f, f] < 0.$$

Therefore the right hand side of (7.4) is a fundamental decomposition, which shows that $\mathcal{P}_1$ is a π_1-space, in particular nondegenerated. Since $\mathcal{P}_1$ is invariant for each resolvent of A, the restriction of A to $\mathcal{P}_1$ is again selfadjoint and clearly $\langle h_0 \rangle$-minimal. As

$$(A \cap \mathcal{P}_1^2)(0) = \langle f + h_0 \rangle$$

Proposition 7 shows that case *(ii)* occurs.

□

In case *(ii)* the relation $A \cap \mathcal{P}_1^2$ is unitary equivalent to the relation (7.3).

Corollary 2 *The closure of* $\overline{\mathcal{L}_\tau}^{\mathcal{P}}$ *coincides with* $\mathcal{P}_\tau$ *(compare Remark 6 and Remark 5) if and only if case (ii) of Proposition 7 occurs.*

Proof : The trace topology on $\mathcal{L}_\tau$ induced by $\mathcal{P}$ coincides with the topology induced by the inner product of $\mathcal{L}_\tau$ if and only if $\overline{\mathcal{L}_\tau}^{\mathcal{P}}$ is nondegenerated.

□

Due to Satz 1.5 of [10] (see also [7]) case *(ii)* of Proposition 7 occurs if and only if τ admits a representation of the form

$$\tau(z) = [(A_0 - z)^{-1}u, u]$$

with a selfadjoint operator A_0 in a π_0-space $\mathcal{P}_0$ and an element $u \in \mathcal{P}_0$. There A_0 and $\mathcal{P}_0$ can be chosen as in Remark 5, $A_0 = A_\tau$, $\mathcal{P}_0 = \mathcal{P}_\tau$ and $u = (A_\tau - z_0)e_{z_0}$.

Denote by Φ the functional $\Phi : \mathcal{L}_\tau \to \mathbb{C}$ defined as

$$\Phi(e_z) = \tau(z).$$

Proposition 9 *Case (ii) of Proposition 7 occurs if and only if Φ induces a continuous (well defined) functional on $\mathcal{P}_\tau$. In this case u is the unique element representing Φ as*

$$\Phi(f) = [f, u]_{\mathcal{P}_\tau}.$$

Proof : For the definition of $\mathcal{L}_\tau$, $\mathcal{P}_\tau$ and A_τ recall Remark 5. Assume first that case *(ii)* occurs. Then Φ induces a well defined functional on $\mathcal{L}_\tau/\mathcal{L}_\tau^\circ$: Let $\sum\limits_{i=1}^{n} \lambda_i e_{z_i} \in \mathcal{L}_\tau^\circ$, then it is shown in [14] that

$$\tau(z) = \frac{\sum\limits_{i=1}^{n} \lambda_i \tau(z_i) \prod_{j\neq i}(z - z_j)}{\sum\limits_{i=1}^{n} \lambda_i \prod_{j\neq i}(z - z_j)}.$$

As in particular $\lim_{\eta\to\infty} \tau(i\eta) = 0$ we must have

$$\sum_{i=1}^{n} \lambda_i \tau(z_i) = 0,$$

i.e. $\Phi(\sum\limits_{i=1}^{n} \lambda_i e_{z_i}) = 0$. It is shown in [10] that $e_{z_0} \in \mathcal{D}(A_\tau - z_0)$ and that $\tau(z) = [(A_\tau - z)^{-1}u, u]$, where $u = (A_\tau - z_0)e_{z_0}$. As

$$(A_\tau - z)^{-1}u = (A_\tau - z)^{-1}(A_\tau - z_0)e_{z_0} = e_z$$

we have

$$\Phi(e_z) = \tau(z) = [(A_\tau - z)^{-1}u, u] = [e_z, u],$$

in particular Φ is continuous.

Conversely, let Φ induce a continuous functional on $\mathcal{P}_\tau$, then there exists an element $u \in \mathcal{P}_\tau$, such that $\Phi(e_z) = \tau(z) = [e_z, u]$. Due to [10] it is enough to show that

$$(e_{z_0}; u) \in A_\tau - z_0 \text{ , i.e. } (e_{z_0}; u + z_0 e_{z_0}) \in A_\tau = A_\tau^+.$$

Indeed, we have for $(e_z - e_{z_0}; ze_z - z_0 e_{z_0}) \in A_\tau$:

$$[e_z - e_{z_0}, u + z_0 e_{z_0}] - [ze_z - z_0 e_{z_0}, e_{z_0}] = [e_z, u] - [e_{z_0}, u] +$$
$$+(\overline{z_0} - z_0)[e_z e_{z_0}] + (z_0 - \overline{z_0})[e_{z_0}, e_{z_0}] = \tau(z) - \tau(z_0) +$$
$$+(\overline{\tau(z_0)} - \tau(z)) + (\tau(z_0) - \overline{\tau(z_0)}) = 0.$$

□

8 An example

In this section we apply the preceeding results to a situation which arises from a certain extrapolation problem. We shall briefly recall some definitions and results, for an exact treatment see [6], [10] and [12].

Let $0 < a \leq \infty$ and $\kappa \in \mathbb{N}_0$, then $\mathcal{P}_{\kappa;a}$ denotes the set of all continuous complex valued functions $F : (-2a, 2a) \to \mathbb{C}$ such that

$$F(-t) = \overline{F(t)} \text{ for } 0 \leq t \leq 2a$$

holds and that the kernel

$$\mathcal{F}_F(s,t) = F(t-s)$$

has κ negative squares for $0 \leq t, s \leq 2a$.

A function $F \in \mathcal{P}_{\kappa;a}$ generates a π_κ-space: The vector space $\mathcal{L}_a(F)$ consisting of all arbitrarily often differentiable functions

$$f : (-a, a) \to \mathbb{C}$$

which have compact support, endowed with the inner product

$$[f,g] = \int_{-a}^{a}\int_{-a}^{a} F(t-s)f(t)\overline{g(s)}\,dtds$$

is a π_κ-lineal. Thus its completion $\mathcal{P}_a(F)$ is a π_κ-space.

Let A_a be the closure of the symmetric operator defined by

$$f \to -if' \text{ for } f \in \mathcal{L}_a(F).$$

Then the relation

$$-i\int_0^\infty \hat{F}(t)e^{-itz}\,dt = [(A-z)^{-1}u, u], \tag{8.1}$$

where u is a certain element of $\mathcal{P}_a(F)$, establishes a bijective correspondence between continuations of $F \in \mathcal{P}_{\kappa;a}$ to $\mathcal{P}_{\kappa;\infty}$and minimal selfadjoint extensions A of A_a acting in a π_κ-space.

Let $0 < a < \infty$ and consider the function

$$F(t) = 1 - |t| \text{ for } -2a \leq t \leq 2a.$$

This example has been considered by H.Langer and Z.Sasvari. If $a < 1$ ($a > 1$) we have $F \in \mathcal{P}_{0;a}$ ($F \in \mathcal{P}_{1;a}$), if $a = 1$ the space $\mathcal{L}_a(F)$ is positive semidefinite and degenerated. In the case $a < 1$ ($a > 1$) F admits infinitely many extensions to $\mathcal{P}_{0;\infty}$ ($\mathcal{P}_{1;\infty}$), if $a = 1$ there exists exactly one extension to $\mathcal{P}_{0;\infty}$. The extensions (in case $a \neq 1$) are connected by (8.1) to the u-resolvents of A_a. In fact the classical theory of resolvent matrices shows (see [11]) that the relation

$$-i\int_0^\infty \hat{F}(t)e^{-itz}\,dt = \frac{w_{11}(z)(-\frac{1}{\tau(z)}) + w_{12}(z)}{w_{21}(z)(-\frac{1}{\tau(z)}) + w_{22}(z)},$$

where

$$W(z) = \begin{pmatrix} \frac{\sin az - z\cos az}{(a-1)z} & \frac{(1-(a-1)z^2)\sin az - az\cos az}{z^2} \\ \frac{z\cos az}{a-1} & (a-1)z\sin az + \cos az \end{pmatrix},$$

establishes a one to one correspondence between the extensions of $F|_{(-2a,2a)}$ in $\mathcal{P}_{0;\infty}$ if $a < 1$ ($\mathcal{P}_{1;\infty}$ if $a > 1$) and the set $\mathcal{N}_0$ of parameters.

We will determine the u-resolvents of A_a in the case $a = 1$, i.e. when A_a is an operator in a degenerated space. This yields a parametrization of the extensions of $F|_{(-2,2)}$ in $\mathcal{P}_{1;\infty}$.

In order to do the necessary computations we introduce a - unitarily equivalent - model. Denote by $\mathcal{H}$ the vector space

$$\mathcal{H} = L^2[-1,1]\dot{+}\mathbb{C}$$

endowed with the inner product

$$\left[\begin{pmatrix} f \\ \varphi \end{pmatrix}, \begin{pmatrix} g \\ \gamma \end{pmatrix}\right] = 2(f,g) + \varphi\overline{\gamma} - (f, 1_{[-1,1]})\overline{\gamma} - \varphi(1_{[-1,1]}, g),$$

where $(.,.)$ denotes the usual inner product on $L^2[-1,1]$. Here $1_{[\alpha,\beta]}$ denotes the function

$$1_{[\alpha,\beta]}(t) = \begin{cases} 1 \text{ for } t \in [\alpha,\beta] \\ 0 \text{ for } t \notin [\alpha,\beta] \end{cases}.$$

A straightforward computation shows that

$$\mathcal{H}^\circ = \langle h_0 \rangle \text{ with } h_0 = \begin{pmatrix} \frac{1}{2}1_{[-1,1]} \\ 1 \end{pmatrix},$$

and we put

$$\mathcal{H}_n = L^2[-1,1]$$

to obtain a decomposition

$$\mathcal{H} = \mathcal{H}_n[\dot{+}]\langle h_0 \rangle$$

as in Section 2, (2.1). Further denote by S the symmetric operator in $\mathcal{H}$ defined by

$$S\begin{pmatrix} f \\ \varphi \end{pmatrix} = \begin{pmatrix} -if' \\ 0 \end{pmatrix} \text{ for } \begin{pmatrix} f \\ \varphi \end{pmatrix} \in \mathcal{D}(S)$$

where

$$\mathcal{D}(S) = \{\begin{pmatrix} f \\ f(1) \end{pmatrix} \in \mathcal{H} | f \text{ abs.cont.}, f' \in L^2[-1,1], f(-1) = 0\}.$$

Lemma 11 *The operator S satisfies the conditions (2.2) and (2.3) and has defect numbers $(1,1)$. Moreover we have $\sigma(S_P) \subseteq \mathbb{R}$.*

Proof : As S is an operator ($S(0) = 0$) (2.2) is clearly satisfied. In order to show that S satisfies (2.3) it suffices, due to Lemma 1, to note that

$$h_0 = \frac{1}{2}\begin{pmatrix} 1_{[-1,1]} \\ 2 \end{pmatrix} \notin \mathcal{D}(S).$$

We have for $z \in \mathbb{C} \setminus \mathbb{R}$

$$\mathcal{R}(S - z) = \{\begin{pmatrix} -if' - zf \\ -zf(1) \end{pmatrix} | f \text{ abs.cont.}, f' \in L^2[-1,1], f(-1) = 0\}.$$

As $-if'-zf=g,\ f(-1)=0$ is uniquely solvable

$$\operatorname{codim}\mathcal{R}\left(S-z\right)=1,$$

hence S has defect $(1,1)$.

To prove $\sigma(S_P)\subseteq\mathbb{R}$ it suffices to show that $h_0\notin\mathcal{R}\left(S-z\right)$ whenever $z\notin\mathbb{R}$. Assume that

$$2h_0=\left(\begin{array}{c}1|_{[-1,1]}\\ 2\end{array}\right)=(S-z)\left(\begin{array}{c}f\\ f(1)\end{array}\right)=\left(\begin{array}{c}-if'-zf\\ -zf(1)\end{array}\right).$$

The equation

$$-if'-zf=1,\ f(-1)=0$$

has the solution

$$f(t)=\frac{1}{z}(e^{iz(t+1)}-1).$$

We compute $f(1)=\frac{1}{z}(e^{2iz}-1)$, hence $-zf(1)=2$ implies that $-e^{2iz}=1$, and we find

$$z=\frac{2k+1}{2}\pi\in\mathbb{R}.$$

□

The projection P of $\mathcal{H}$ onto $\mathcal{H}_n$ with kernel $\mathcal{H}^\circ$ is given by

$$P\left(\begin{array}{c}f\\ \varphi\end{array}\right)=f-\frac{\varphi}{2}1_{[-1,1]}.$$

Lemma 12 *The operator S_P is given by*

$$S_Pf=-if' \text{ for } f\in\mathcal{D}\left(S_P\right),$$

where

$$\mathcal{D}\left(S_P\right)=\{f\in L^2[-1,1]|f \text{ abs.cont.}, f'\in L^2[-1,1], f(-1)=-f(1)\}.$$

Proof : We have

$$S_P=\{(\left(\begin{array}{c}f-\frac{f(1)}{2}1_{[-1,1]}\\ 0\end{array}\right);\left(\begin{array}{c}-if'\\ 0\end{array}\right))|\left(\begin{array}{c}f\\ f(1)\end{array}\right)\in\mathcal{D}\left(S\right)\}=$$

$$=\{(f;-if')|f \text{ abs.cont.}, f'\in L^2[-1,1], f(-1)=-f(1)\}.$$

□

Also the resolvent of S_P can be computed explicitly.

Lemma 13 *Let $f\in L^2[-1,1]$, then*

$$(S_P-z)^{-1}f=\frac{ie^{izt}}{2\cos z}(e^{-iz}\int_{-1}^{t}f(s)e^{-izs}\,ds-e^{iz}\int_{t}^{1}f(s)e^{-izs}\,ds). \tag{8.2}$$

Proof : We are looking for an element $g \in \mathcal{D}(S_P)$, such that

$$(S_P - z)g = -ig' - zg = f.$$

A solution of this equation is of the form

$$g(t) = ie^{izt} \int_{-1}^{t} f(s)e^{-izs}\, ds + ce^{izt}$$

with $c \in \mathbb{C}$. From the condition $g \in \mathcal{D}(S_P)$ we find

$$ce^{-iz} = g(-1) = -g(1) = -ie^{iz} \int_{-1}^{1} f(s)e^{-izs}\, ds - ce^{iz},$$

which implies

$$c = \frac{-ie^{iz}}{2\cos z} \int_{-1}^{1} f(s)e^{-izs}\, ds.$$

Thus

$$\begin{aligned} g &= \frac{ie^{izt}}{2\cos z} \left(2\cos z \int_{-1}^{t} f(s)e^{-izs}\, ds - e^{iz} \int_{-1}^{1} f(s)e^{-izs}\, ds\right) = \\ &= \frac{ie^{izt}}{2\cos z} \left(e^{-iz} \int_{-1}^{t} f(s)e^{-izs}\, ds - e^{iz} \int_{t}^{1} f(s)e^{-izs}\, ds\right). \end{aligned}$$

□

We proceed determining elements $(c_0; d_0)$ and $(c_1; d_1)$ in Proposition 3.

Lemma 14 *With the notation of Proposition 4 we have*

$$c_0(t) = \frac{1}{2\cosh 1} \sinh t,\;\; d_0(t) = \frac{-i}{2\cosh 1} \cosh t, \tag{8.3}$$

$$c_1(t) = 0,\;\; d_1(t) = 0.$$

Proof : As $\mathcal{R}(S) \subseteq \mathcal{H}_n$ we have $(c_1; d_1) = 0$. Furthermore $(c_0; d_0) \in S_P$, i.e. $d_0(t) = -ic_0(t)'$. In order to prove the remaining assertion it suffices to show that with c_0 and d_0 as in (8.3) the relation $\Psi(S_P) = S$ holds. We have

$$\Psi(S_P) = \{(\begin{pmatrix} f \\ 0 \end{pmatrix} + ([f, c_0] + [-if', -id_0]) \begin{pmatrix} \frac{1}{2}1_{[-1,1]} \\ 1 \end{pmatrix}; \begin{pmatrix} -if' \\ 0 \end{pmatrix})| f \in \mathcal{D}(S_P)\},$$

and

$$\begin{aligned} &[f, \frac{1}{2\cosh 1} \sinh t] + [-if', \frac{-i}{2\cosh 1} \cosh t] = \\ &= \frac{1}{2\cosh 1} \left(2\int_{-1}^{1} f(t)\sinh t\, dt + 2\int_{-1}^{1} f(t)'\cosh t\, dt\right) = \frac{1}{\cosh 1} [f(t)\cosh t]|_{t=-1}^{1} = \\ &= \frac{1}{\cosh 1} (f(1)\cosh 1 - f(-1)\cosh(-1)) = 2f(1). \end{aligned}$$

Thus

$$\Psi(S_P) = \{(\begin{pmatrix} f + f(1)1_{[-1,1]} \\ 2f(1) \end{pmatrix}; \begin{pmatrix} -if' \\ 0 \end{pmatrix})| f \in \mathcal{D}(S_P)\} = S.$$

□

Now we put

$$u = \begin{pmatrix} 1_{[0,1]} \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} u_n(t) \\ 0 \end{pmatrix}}_{\in \mathcal{H}_n} + \underbrace{\begin{pmatrix} \frac{1}{2}1_{[-1,1]} \\ 1 \end{pmatrix}}_{\in \langle h_0 \rangle},$$

where

$$u_n(t) = \begin{cases} -\frac{1}{2} \text{ for } -1 \leq t < 0 \\ \frac{1}{2} \text{ for } 0 \leq t \leq 1 \end{cases}.$$

Lemma 15 *With the notation of Section 5 we have*

$$q(z) = z^2 \tan z,$$

$$\chi(z) = h_1 - \frac{iz}{2\cos z} e^{izt} + z \tan z \cdot h_0$$

and

$$[(S_P - z)^{-1} u_n, u_n] = \frac{\tan z}{z^2} - \frac{1}{z}.$$

Proof : We first compute $a(z)$:

$$a(z) = (c_1 - zc_0) + z(d_1 - zd_0) = \frac{z}{2\cosh 1}(iz \cosh t - \sinh t).$$

Using (8.2) and the relations

$$\int \sinh s e^{-izs}\, ds = -\frac{1}{2}\left(\frac{e^{-(iz-1)s}}{iz-1} - \frac{e^{-(iz+1)s}}{iz+1}\right) \tag{8.4}$$

$$\int \cosh s e^{-izs}\, ds = -\frac{1}{2}\left(\frac{e^{-(iz-1)s}}{iz-1} + \frac{e^{-(iz+1)s}}{iz+1}\right) \tag{8.5}$$

a straightforward computation shows that

$$(S_P - z)^{-1} a(z) = \frac{iz}{2\cos z} e^{izt} - \frac{iz}{2\cosh 1}\cosh t.$$

Therefore we have

$$b(z) = \frac{iz}{2\cos z} e^{izt}.$$

Using again (8.4) and (8.5) we find

$$q(z) = [a(z), b(\overline{z})] = 2\int_{-1}^{1} \frac{z}{2\cosh 1}(iz\cosh t - \sinh t)\frac{-iz}{2\cos z} e^{-izt}\, dt =$$

$$= \frac{-iz^2}{2\cosh 1 \cos z}\int_{-1}^{1}(iz\cosh t - \sinh t)e^{-izt}\, dt = z^2 \tan z.$$

As $c_1 = d_1 = 0$

$$[b(z), c_0 + \overline{z}d_0] = -\frac{1}{z}[b(z), a(\overline{z})] = -\frac{1}{z}q(z) = z \tan z$$

we find

$$\chi(z) = h_1 - \frac{iz}{2\cos z}e^{izt} + z\tan z \cdot h_0.$$

We proceed computing $[(S_P - z)^{-1}u_n, u_n]$. For $t \geq 0$ we find from (8.2) by an elementary computation

$$(S_P - z)^{-1}u_n(t) = \frac{e^{iz(t-1)}}{2z\cos z} - \frac{1}{2z},$$

for $t < 0$

$$(S_P - z)^{-1}u_n(t) = -\frac{e^{iz(t+1)}}{2z\cos z} + \frac{1}{2z}.$$

Therefore

$$[(S_P - z)^{-1}u_n, u_n] = 2\int_{-1}^{1}(S_P - z)^{-1}u_n(s)\overline{u_n(s)}\,ds =$$

$$= 2\int_0^1 (\frac{e^{iz(t-1)}}{2z\cos z} - \frac{1}{2z})\frac{1}{2}\,ds + 2\int_{-1}^{0}(-\frac{e^{iz(t+1)}}{2z\cos z} + \frac{1}{2z})(-\frac{1}{2})\,ds =$$

$$= -\frac{1}{z} + \frac{\tan z}{z^2}.$$

□

Theorem 2 and Proposition 5 now imply the following result:

Proposition 10 *The formula*

$$-i\int_0^\infty \hat{F}(t)e^{-itz}\,dt = \frac{w_{11}(z)(-\frac{1}{\tau(z)}) + w_{12}(z)}{w_{21}(z)(-\frac{1}{\tau(z)}) + w_{22}(z)},$$

where

$$W(z) = \begin{pmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{pmatrix} = \begin{pmatrix} \frac{\sin z}{z^2} - \frac{\cos z}{z} & -\cos z - z\sin z \\ \cos z & z^2 \sin z \end{pmatrix}, \tag{8.6}$$

establishes a bijective correspondence between extensions of $F|_{(-2,2)}$ in $\mathcal{P}_{0;\infty} \cup \mathcal{P}_{1;\infty}$ and the set $\mathcal{T} \cup \{\infty\}$ of parameters. The unique extension of $F|_{(-2,2)}$ in $\mathcal{P}_{0;\infty}$ corresponds to the parameter $\tau(z) = 0$.

Proof : Substituting the result of Lemma 15 into the formulas of Proposition 5 yields (8.6). As $q(z)$ is not a rational function Remark 4 shows that the exclusion of $\frac{1}{q(z)}$ from the set of parameters does not occur.

□

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The first author was supported by "Fonds zur Förderung der wissenschaftlichen Forschung" of Austria, Project P 09832-MAT.

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AMS Classification Numbers: 47B50, 47B25, 42A82