MULTIPLE POINT INTERPOLATION IN NEVANLINNA CLASSES

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A Nevanlinna-Pick type interpolation problem for generalized Nevanlinna functions is considered. We prescribe the values of the function and its derivatives up to a certain order at finitely many points of the upper half plane. An operator theoretic approach is used to parametrize the solutions of this interpolation problem by means of selfadjoint extensions of a certain symmetry.

1 Introduction

A function f is said to be of Nevanlinna class $\mathcal{N}_\nu^\pi$ if f is meromorphic in the open upper half plane $\mathbb{C}^+$ and the so called Nevanlinna kernel

$$N_f(z,w) = \frac{f(z) - \overline{f(w)}}{z - \overline{w}} \text{ for } z, w \in \varrho(f)$$

has π positive and ν negative squares. Here $\varrho(f)$ denotes the domain of analyticity of f in $\mathbb{C}^+$ and π and ν are nonnegative integers or ∞. More explicitly this means that for each number $n \in \mathbb{N}$ and points $z_1, \dots, z_n \in \varrho(f)$ the quadratic form

$$\sum_{i,j=1}^{n} N_f(z_i, z_j)\xi_i\overline{\xi_j} \tag{1}$$

has at most π positive (ν negative) squares, and that for some choice of n and $z_1, \dots, z_n$ this upper bound is attained. If there is no upper bound for the number of positive (negative) squares of the forms (1) put $\pi = \infty$ ($\nu = \infty$). We consider only such classes $\mathcal{N}_\nu^\pi$ where at least one index is finite. Denote by $\mathcal{N}_\nu$ the union $\mathcal{N}_\nu = \bigcup_{\pi=0}^{\infty} \mathcal{N}_\nu^\pi$ where ν is finite.

A multiple point interpolation problem is a problem where not only values for the function itself, but also for its derivatives up to a certain order are prescribed. The interpolation data thus consist of numbers $n \in \mathbb{N}$, $k_1, \dots, k_n \in \mathbb{N}_0$ and points $z_1, \dots, z_n \in \mathbb{C}^+$ and $w_{ij} \in \mathbb{C}$ for $j = 0, \dots, k_i$ and $i = 1, \dots, n$. A solution is a function f satisfying

$$f^{(j)}(z_i) = w_{ij} \text{ for } j = 0, \dots, k_i \text{ and } i = 1, \dots, n. \tag{2}$$

In this note we develop an operator theoretic method to describe the solutions of the interpolation problem (2) which are contained in some Nevanlinna class. This generalizes the well known operator method of Krein and Langer concerning simple point interpolation which has been developed in [17] and [20].

In the classical case the so called Pick matrix plays an important role. We introduce in Section 2 a generalized Pick matrix for multiple point interpolation problems. Furthermore we recall the operator representation of an $\mathcal{N}_\nu$-function f and compute the derivatives of f by means of this representation. This motivates the structure of the generalized Pick matrix. In Section 3 a model space, i.e. an inner product space $\mathcal{H}$ and a symmetric operator S, is associated to the given data and investigated. After these considerations we are in position to prove the main results (Theorem 1, Proposition 2 and Proposition 3) of this note which is done in Section 4. They establish a bijective correspondence between the solutions of the multiple point interpolation problem and the selfadjoint extensions of the above mentioned symmetric operator. More precisely, our result can be formulated as follows (here e_{10} is a certain element of $\mathcal{H}$):

The formula

$$f_{\tilde{S}}(z) = \overline{w_{10}} + (z - \overline{z_1})[(I + (z - z_1)(\tilde{S} - z)^{-1})e_{10}, e_{10}]. \tag{3}$$

establishes a bijective correspondence between the solutions of the interpolation problem (2) within $\mathcal{N}_\nu^\pi$ and the selfadjoint extensions $\tilde{S}$ of S acting in a Pontryagin space $\mathcal{P} \supseteq \mathcal{H}$ with positive (negative) index π (ν), which contain $z_1, \ldots, z_n$ in their resolvent set and are e_{10}-minimal.

In the proof some results of [16], [17] and [22] are used.

Similar results for the simple point case and some generalizations to interpolation problems by matrix valued functions can be found e.g. in [6], [8], [13], [14], [20] or [22]. In the definite case also some generalizations of Nevanlinna-Pick type problems to the case of so-called Nevanlinna-pairs are given in [1], [2] and [11].

In the classical case one can give a parametrization of the solutions of (2) by means of a fractional linear transformation involving a parameter function, e.g. by starting from a formula like (3) and using the theory of so-called resolvent matrices (see e.g. [18]), i.e. apply Krein's formula for the generalized resolvents of a symmetry. Since in our case the space $\mathcal{H}$ is in general degenerated this theory cannot be applied. For some partial results in this direction (concerning different types of interpolation problems) see [3], [11], [12] and [19].

We would like to remark that interpolation by rational functions is also covered by the theory developed here, as the set of (real) rational functions equals the union of all sets $\mathcal{N}_\nu^\pi$ with both indices finite (see e.g. [21]). The generalized Pick matrix is in this context sometimes replaced by the so called generalized Löwner matrix which has a quite similar structure. Of course rational interpolation problems are well studied, see e.g. [4], [5] or [9]. For further approaches to rational interpolation problems and an extensive coverage of the literature we refer to [7].

We will use some results concerning indefinite inner product spaces and their linear operators which can be found e.g. in [10] or [15].

2 Derivatives of $\mathcal{N}_\nu$-functions

Let $f \in \mathcal{N}_\nu$, then it is well known (see e.g. [17]) that f has a representation by means of a selfadjoint relation $\tilde{S}$ in a Pontryagin space $\mathcal{P}$ with negative index ν. This representation can be obtained as follows: Denote by $\varrho(f)$ the domain of analyticity of f in $\mathbb{C}^0$ $(= \mathbb{C}^+ \cup \mathbb{C}^-)$ and let $\mathcal{H}_f$ be the inner product space consisting of all formal sums $\sum_{z\in\varrho(f)} x_z e_z$ where all but finitely many coefficients vanish, endowed with the inner product given by

$$[e_z, e_w] = \frac{f(z) - \overline{f(w)}}{z - \overline{w}} \text{ for } z, w \in \varrho(f), z \neq \overline{w},$$

$$[e_z, e_{\overline{z}}] = f'(z) \text{ for } z \in \varrho(f).$$

Then $\mathcal{P}$ is the Pontryagin space obtained from $\mathcal{H}_f$ by completion of the factor space $\mathcal{H}_f/\mathcal{H}_f{}^\circ$, where $\mathcal{H}_f{}^\circ$ denotes the isotropic part of $\mathcal{H}_f$. Let $\tilde{A}$ be the operator in $\mathcal{H}_f$ with domain $\mathcal{D}\left(\tilde{A}\right) = \{\sum_{z\in\varrho(f)} x_z e_z \in \mathcal{H}_f | \sum_{z\in\varrho(f)} x_z = 0\}$ which acts as $\tilde{A}(\sum_{z\in\varrho(f)} x_z e_z) = \sum_{z\in\varrho(f)} z x_z e_z$. Then $\tilde{S}$ is obtained from $\tilde{A}$ by factorization and closure, and the relation

$$f(z) = \overline{f(z_0)} + (z - \overline{z_0})[(I + (z - z_0)(\tilde{S} - z)^{-1})e_{z_0}, e_{z_0}] \tag{4}$$

holds where z_0 is a fixed point of $\varrho(f)$. Here the resolvent set of $\tilde{S}$ coincides with the domain of analyticity of f. Note that the function f (and thus also its derivatives) is real, i.e. $f(\overline{z}) = \overline{f(z)}$. This can be proved by a straightforward computation.

Denote by $s(z)$ the expression $s(z) = (I + (z - z_0)(\tilde{S} - z)^{-1})e_{z_0}$. From (4) we can compute the derivatives of f by means of derivatives of $s(z)$:

Lemma 1 *Let $f \in \mathcal{N}_\nu$, and let $\tilde{S}$ be such that the representation (4) holds. Then for $k \geq 1$ we have*

$$f^{(k)}(z) = (z - \overline{z_0})[s^{(k)}(z), e_{z_0}] + k[s^{(k-1)}(z), e_{z_0}].$$

The proof of Lemma 1 is immediate by an easy computation.

Suppose now we are given finitely many points $z_1, \ldots, z_n \in \mathbb{C}^+ \cap \varrho(f)$ and numbers $k_1, \ldots, k_n \in \mathbb{N}_0$, and we are interested in the values of $f^{(k)}(z_i)$ for $k = 0, \ldots, k_i$ and $i = 1, \ldots, n$. We may assume that $z_0 = z_1$. As then $s(z_1) = e_{z_1} = e_{z_0}$, Lemma 1 shows that the information about the values $f^{(k)}(z_i)$ is contained in the subspace

$$\mathcal{G} = \langle s^{(k)}(z_i) | k = 0, \ldots, k_i, i = 1, \ldots, n\rangle$$

of $\mathcal{P}$. To determine the inner product $[.,.]$ on $\mathcal{G}$ explicitly we will use the following relation.

Lemma 2 *The Nevanlinna kernel of f can be computed via $s(z)$:*

$$\frac{f(z) - \overline{f(w)}}{z - \overline{w}} = [s(z), s(w)].$$

Proof : We will repeatedly use the formula

$$\frac{g(z)h(z) - g(\overline{w})h(\overline{w})}{z - \overline{w}} = \frac{g(z) - g(\overline{w})}{z - \overline{w}} h(z) + g(\overline{w}) \frac{h(z) - h(\overline{w})}{z - \overline{w}}.$$

Note that the function f defined by (4) is real, i.e. $\overline{f(z)} = f(\overline{z})$. Thus

$$\frac{f(z) - \overline{f(w)}}{z - \overline{w}} = \frac{f(z) - f(\overline{w})}{z - \overline{w}} = \frac{(z - \overline{z_0})[s(z), e_{z_0}] - (\overline{w} - \overline{z_0})[s(\overline{w}), e_{z_0}]}{z - \overline{w}} =$$

$$= [s(z), e_{z_0}] + (\overline{w} - \overline{z_0})[\frac{s(z) - s(\overline{w})}{z - \overline{w}}, e_{z_0}].$$

As

$$[\frac{s(z) - s(\overline{w})}{z - \overline{w}}, e_{z_0}] = \frac{(\overline{w} - z_0)(\overline{w} - z)^{-1}e_{z_0} - (z - z_0)(z - z)^{-1}e_{z_0}}{\overline{w} - z} =$$

$$= (\overline{w} - z)^{-1}e_{z_0} + (z - z_0)\frac{(\overline{w} - z)^{-1}e_{z_0} - (z - z)^{-1}e_{z_0}}{\overline{w} - z} =$$

$$= (\overline{w} - z)^{-1}e_{z_0} + (z - z_0)(\overline{w} - z)^{-1}(z - z)^{-1}e_{z_0},$$

we find

$$\frac{f(z) - f(\overline{w})}{z - \overline{w}} = [e_{z_0}, e_{z_0}] + [(z - z_0)(z - z)^{-1}e_{z_0}, e_{z_0}] + (\overline{w} - \overline{z_0})[(\overline{w} - z)^{-1}e_{z_0}, e_{z_0}] +$$

$$+ (\overline{w} - \overline{z_0})[(z - z_0)(\overline{w} - z)^{-1}(z - z)^{-1}e_{z_0}, e_{z_0}]. \tag{5}$$

On the other hand we have

$$[s(z), s(\overline{w})] = [e_{z_0} + (z - z_0)(z - z)^{-1}e_{z_0}, e_{z_0} + (w - z_0)(w - z)^{-1}e_{z_0}] =$$

$$= [e_{z_0}, e_{z_0}] + [(z - z_0)(z - z)^{-1}e_{z_0}, e_{z_0}] + [e_{z_0}, (w - z_0)(w - z)^{-1}e_{z_0}] +$$

$$+ [(z - z_0)(z - z)^{-1}e_{z_0}, (w - z_0)(w - z)^{-1}e_{z_0}]. \tag{6}$$

Obviously the right hand sides of (5) and (6) coincide.

□

From the above lemma we find that

$$[s^{(k)}(z_i), s^{(l)}(z_j)] = \frac{\partial^k}{\partial z^k}\frac{\partial^l}{\partial \overline{w}^l}\left(\frac{f(z) - \overline{f(w)}}{z - \overline{w}}\right)\Bigg|_{\substack{z=z_i\\ w=z_j}} \tag{7}$$

holds. Thus the information about the values of f and its derivatives at the points z_i is contained in the generalized Pick matrix associated with the points $z_1, \ldots, z_n$

$$\mathbb{P} = (P^{ij})_{j,i=1}^{n} \tag{8}$$

where P^{ij} is a block of size $(k_j + 1) \times (k_i + 1)$ with entries

$$p_{lk}^{ij} = \frac{\partial^k}{\partial z^k}\frac{\partial^l}{\partial \overline{w}^l}\left(\frac{f(z) - \overline{f(w)}}{z - \overline{w}}\right)\Bigg|_{\substack{z=z_i\\ w=z_j}} \quad \text{for } l = 0, \ldots, k_j, k = 0, \ldots, k_i. \tag{9}$$

In the case $k_1 = \ldots = k_n = 0$ we obtain the classical Pick matrix.

The following lemma gives some information about the action of $\tilde{S}$ on $\mathcal{G}$.

Lemma 3 *We have*

(i) $\sum_{i=1}^{n} x_i s(z_i) \in \mathcal{D}\left(\tilde{S}\right)$ *whenever* $\sum_{i=1}^{n} x_i = 0$, *and*

$$(\sum_{i=1}^{n} x_i s(z_i), \sum_{i=1}^{n} x_i z_i s(z_i)) \in \tilde{S}.$$

(ii) $s^{(k)}(z_i) \in \mathcal{D}\left(\tilde{S}\right)$ *whenever* $k \geq 1$, *and*

$$(s^{(k)}(z_i), z_i s^{(k)}(z_i) + k s^{(k-1)}(z_i)) \in \tilde{S}.$$

Proof : Note that

$$\sum_{i=1}^{n} x_i s(z_i) = \underbrace{(\sum_{i=1}^{n} x_i)}_{=0} e_{10} + \sum_{i=1}^{n} x_i (z_i - z_1)(\tilde{S} - z_i)^{-1} e_{10},$$

and that

$$((\tilde{S} - z_i)^{-1} e_{10}, e_{10} + z_i (\tilde{S} - z_i)^{-1} e_{10}) \in \tilde{S}.$$

Thus

$$(\sum_{i=1}^{n} x_i (z_i - z_1)(\tilde{S} - z_i)^{-1} e_{10}, \sum_{i=1}^{n} x_i (z_i - z_1) e_{10} + \sum_{i=1}^{n} x_i (z_i - z_1) z_i (\tilde{S} - z_i)^{-1} e_{10}) \in \tilde{S}$$

which proves (i), as $\sum_{i=1}^{n} x_i z_1 e_{10} = 0$ and thus the right hand side equals $\sum_{i=1}^{n} x_i z_i s(z_i)$.

As

$$s^{(k)}(z_i) = (z_i - z_1) k! (\tilde{S} - z_i)^{-(k+1)} e_{10} + k! (\tilde{S} - z_i)^{-k} e_{10}$$

we find $s^{(k)} \in \mathcal{D}\left(\tilde{S}\right)$ and, as

$$k s^{(k-1)}(z_i) = (z_i - z_1) k! (\tilde{S} - z_i)^{-k} e_{10} + k! (\tilde{S} - z_i)^{-(k-1)} e_{10})$$

holds:

$$(k s^{(k-1)}(z_i), s^{(k)}(z_i)) \in (\tilde{S} - z_i)^{-1}$$

which yields the assertion.

□

3 An inner product space connected to the interpolation data

Consider now a multiple point interpolation problem, i.e. let $n \in \mathbb{N}$, $k_1, \ldots, k_n \in \mathbb{N}_0$, $z_1, \ldots, z_n \in \mathbb{C}^k$ and $w_{ik} \in \mathbb{C}$ for $k = 0, \ldots, k_i$ and $i = 1, \ldots, n$ be given. From this data we can buildt the generalized Pick matrix $\mathbb{P}$ associated with the points $z_1, \ldots, z_n$.

Lemma 4 *The entries p_{lk}^{ij} $(l=0,\dots,k_j, k=0,\dots,k_i; i,j=1,\dots,n)$ of the generalized Pick matrix associated with the points $z_1,\dots,z_n$ are given explicitly as*

$$p_{lk}^{ij}=\sum_{h=0}^{k}\binom{k}{h}w_{ih}\frac{(-1)^{k-h}(k+l-h)!}{(z_i-\overline{z_j})^{k+l+1-h}}+\sum_{h=0}^{l}\binom{l}{h}\overline{w_{jh}}\frac{(-1)^{l-h}(k+l-h)!}{(\overline{z_j}-z_i)^{k+l+1-h}}. \tag{10}$$

Proof : As the entries p_{lk}^{ij} equal $\frac{\partial^k}{\partial z^k}\frac{\partial^l}{\partial\overline{w}^l}\left(\frac{f(z)-\overline{f(w)}}{z-\overline{w}}\right)\Big|_{\substack{z=z_i\\ w=z_j}}$ it suffices to proof the formula

$$\begin{aligned}\frac{\partial^k}{\partial z^k}\frac{\partial^l}{\partial\overline{w}^l}\left(\frac{f(z)-\overline{f(w)}}{z-\overline{w}}\right)=\sum_{h=0}^{k}\binom{k}{h}f(z)^{(h)}\frac{(-1)^{k-h}(k+l-h)!}{(z-\overline{w})^{k+l+1-h}}+\\ +\sum_{h=0}^{l}\binom{l}{h}\overline{f(w)^{(h)}}\frac{(-1)^{l-h}(k+l-h)!}{(\overline{w}-z)^{k+l+1-h}}.\end{aligned} \tag{11}$$

In order to prove (11) use induction on l and k. For $l=k=0$ the assertion is clearly true. As (11) is symmetric with respect to l and k we only have to do the inductive step for one of l and k, say, $k\mapsto k+1$, i.e. we have to differentiate the right hand side of (10) with respect to z. We then obtain

$$\begin{aligned}&\sum_{h=0}^{k}\binom{k}{h}(-1)^{k-h}(k+l-h)![f(z)^{(h+1)}\frac{1}{(z-\overline{w})^{k+l+1-h}}+f(z)^{(i)}\frac{-(k+l+1+h)}{(z-\overline{w})^{k+l+2-h}}]+\\ &+\sum_{h=0}^{k}\binom{l}{h}f(\overline{w})^{(h)}(-1)^{l-h}(k+l-h)!(k+l+1-h)\frac{1}{(\overline{w}-z)^{k+l+2-h}}=\\ &=\sum_{h=1}^{k}[\underbrace{\binom{k}{h-1}+\binom{k}{h-1}}_{=\binom{k+1}{h}}](-1)^{(k+1)-h}((k+1)+l-h)!f(z)^{(h)}\frac{1}{(z-\overline{w})^{(k+1)+l+1-h}}+\\ &+\underbrace{\binom{k}{k}}_{=\binom{k+1}{k+1}}l!f(z)^{(k+1)}\frac{1}{(z-\overline{w})^{l+1}}+\underbrace{\binom{k}{0}}_{\binom{k+1}{0}}(-1)^{k+1}((k+1)+l)!f(z)\frac{1}{(z-\overline{w})^{(k+1)+l+1-h}}+\\ &+\sum_{h=0}^{k}\binom{l}{h}f(\overline{w})^{(h)}(-1)^{l-h}((k+1)+l-h)!\frac{1}{(\overline{w}-z)^{(k+1)+l+1-h}}\end{aligned}$$

which proves the assertion.

□

Taking the subspace $\mathcal{G}$ as a model we define an inner product space associated with the interpolation data.

Definition 1 *Let $\mathcal{H}$ be the inner product space of all formal sums*

$$\mathcal{H}=\{\sum_{\substack{k=0,\dots,k_i\\ i=1,\dots,n}}x_{ik}e_{ik}|x_{ik}\in\mathbb{C}\}$$

endowed with the inner product defined by

$$[e_{ik},e_{jl}]=p_{lk}^{ij}.$$

Remark 1 If we identify the element $\sum_{\substack{k=0,\ldots,k_i\\ i=1,\ldots,n}} x_{ik}e_{ik}$ with the column vector

$$(x_{10},\ldots,x_{1k_1};x_{20},\ldots,\ldots,\ldots,x_{nk_n})^T, \tag{12}$$

then the Gram matrix of $[.,.]$ on $\mathcal{H}$ is the generalized Pick matrix associated with $z_1,\ldots,z_n$.

Note that, as the elements e_{ik} are per definitionem linearly independent we have $\dim\mathcal{H} = \sum_{i=1}^{n}(k_i+1)$. In our model $\mathcal{G}$ in general only the inequality $\dim\mathcal{G} \leq \sum_{i=1}^{n}(k_i+1)$ holds.

We define an operator S acting in $\mathcal{H}$ (compare Lemma 3).

Definition 2 *Let S be the operator with domain*

$$\mathcal{D}(S) = \{\sum_{\substack{k=0,\ldots,k_i\\ i=1,\ldots,n}} x_{ik}e_{ik} | \sum_{i=1}^{n} x_{i0} = 0\} =$$

$$= \langle \sum_{j=1}^{n} x_{j0}e_{j0}, e_{ik} | \sum_{j=1}^{n} x_{j0} = 0, k=1,\ldots,k_i, i=1,\ldots,n\rangle,$$

which acts as

$$S(\sum_{i=1}^{n} x_{i0}e_{i0}) = \sum_{i=1}^{n} z_i x_{i0}e_{i0} \tag{13}$$

and, for $k=1,\ldots,k_i, i=1,\ldots,n$

$$Se_{ik} = z_i e_{ik} + k e_{i,k-1}. \tag{14}$$

Remark 2 If we again identify the element $\sum_{\substack{k=0,\ldots,k_i\\ i=1,\ldots,n}} x_{ik}e_{ik}$ with the column vector (12), S admits a representation as a block matrix

$$S \subseteq (S_{ij})_{i,j=1}^{n} \tag{15}$$

where the blocks S_{ij} are of size $(k_j+1)\times(k_i+1)$, the off-diagonal blocks contain only zeros and the diagonal blocks are of the form

$$S_{ii} = \begin{pmatrix} z_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & k_i \\ 0 & & & z_i \end{pmatrix} \text{ for } i=1,\ldots,n.$$

Before we investigate some properties of S we give another lemma.

Lemma 5 *The entries of the generalized Pick matrix satisfy the following relations ($i,j \in \{1,\ldots,n\}$):*

(i) $(z_i - \overline{z_j})p_{lk}^{ij} = -kp_{l,k-1}^{ij} + lp_{l-1,k}^{ij}$ *for $l,k \geq 1$,*

(ii) $(z_i - \overline{z_j})p_{l0}^{ij} = -\overline{w_{jl}} + lp_{l-1,0}^{ij}$ *for $l \geq 1$ and similarly* $(z_i - \overline{z_j})p_{0k}^{ij} = -kp_{0,k-1}^{ij} + w_{ik}$ *for $k \geq 1$,*

(iii) $(z_i - \overline{z_j})p^{ij}_{00} = w_{i0} - \overline{w_{j0}}$.

Proof : We use the explicit formula for p^{ij}_{lk} given in Lemma 4. Then we find

$$(z_i - \overline{z_j})p^{ij}_{lk} = \sum_{h=0}^{k} \binom{k}{h} w_{ih} \frac{(-1)^{k-h}(k+l-h)!}{(z_i - \overline{z_j})^{k+l-h}} - \sum_{h=0}^{l} \binom{l}{h} \overline{w_{jh}} \frac{(-1)^{l-h}(k+l-h)!}{(\overline{z_j} - z_i)^{k+l-h}},$$

$$lp^{ij}_{l-1,k} = \sum_{h=0}^{k} l \binom{k}{h} w_{ih} \frac{(-1)^{k-h}(k+l-1-h)!}{(z_i - \overline{z_j})^{k+l-h}} - \sum_{h=0}^{l-1} l \binom{l}{h} \overline{w_{jh}} \frac{(-1)^{l-h}(k+l-1-h)!}{(\overline{z_j} - z_i)^{k+l-h}}$$

and

$$kp^{ij}_{l,k-1} = \sum_{h=0}^{k-1} k \binom{k}{h} w_{ih} \frac{(-1)^{k-h}(k+l-1-h)!}{(z_i - \overline{z_j})^{k+l-h}} - \sum_{h=0}^{l} \binom{l}{h} \overline{w_{jh}} \frac{(-1)^{l-h}(k+l-1-h)!}{(\overline{z_j} - z_i)^{k+l-h}}.$$

Comparing the coefficients of the term

$$w_{ih} \frac{(-1)^{k-h}}{(z_i - \overline{z_j})^{k+l-h}}$$

for $h = 0, \ldots, k-1$ in the first sums of the above right hand sides we find that we have to prove

$$(k+l-1-h)!(l\binom{k}{h} + k\binom{k-1}{h}) = (k+l-h)!\binom{k}{h}.$$

This relation is proved by direct computation. The coefficients of

$$\overline{w_{jh}} \frac{(-1)^{l-h}}{(\overline{z_j} - z_i)^{k+l-h}}$$

of the second sums on the right hand sides give (for $h = 0, \ldots, l-1$)

$$(k+l-h)!\binom{l}{h} = (k+l-1-h)!(l\binom{l-1}{h} + k\binom{l}{h})$$

which is also proved by direct computation. For $h = k$ the coefficients equal obviously.

To prove the second assertion of the lemma we compute (see Lemma 4)

$$(z_i - \overline{z_j})p^{ij}_{l0} = l!\frac{w_{i0}}{(z_i - \overline{z_j})^l} - \sum_{h=0}^{l} \binom{l}{h} \overline{w_{jh}} \frac{(-1)^{l-h}(l-h)!}{(\overline{z_j} - z_i)^{l-h}}$$

and

$$lp^{ij}_{l-1,0} = l(l-1)!\frac{w_{i0}}{(z_i - \overline{z_j})^l} - l\sum_{h=0}^{l-1} \binom{l-1}{h} \overline{w_{jh}} \frac{(-1)^{l-h}(l-1-h)!}{(\overline{z_j} - z_i)^{l-h}}.$$

Putting these expression together finishes the proof.

The last assertion is an immediate consequence of the definition of p^{ij}_{00}.

$\square$

Proposition 1 *The operator S is symmetric and has no eigenvectors.*

Proof : The matrix (15) representing S (or, more exactly, the extension of S given by (15)) obviously has the eigenvalues z_i with corresponding eigenvectors e_{i_0} $(i = 1, \ldots, n)$. But these are exactly the elements excluded by the choice of $\mathcal{D}(S)$. Thus S has no nonzero eigenvectors.

To prove that S is symmetric we have to show that

$$[Sx, y] = [x, Sy] \tag{16}$$

for $x, y \in \mathcal{D}(S)$. First consider the case $x = e_{ik}$ and $y = e_{jl}$ where $k, l \geq 1$. Then

$$[Sx, y] = [z_i e_{ik} + k e_{i,k-1}, e_{jl}] = z_i [e_{ik}, e_{jl}] +$$

$$+ k [e_{i,k-1}, e_{jl}] = z_1 p_{lk}^{ij} + k p_{l,k-1}^{ij},$$

and similarly

$$[x, Sy] = \overline{z_j} p_{lk}^{ij} + l p_{l-1,k}^{ij}.$$

Lemma 5 thus implies (16). If $x = \sum\limits_{i=1}^{n} x_{i0} e_{i0}$ where $\sum\limits_{i=1}^{n} x_{i0} = 0$ and $y = e_{jl}$ with $l \geq 1$ we find

$$[Sx, y] = \sum_{i=1}^{n} x_{i0} [z_i e_{i0}, e_{jl}] = \sum_{i=1}^{n} x_{i0} e_{i0}$$

and

$$[x, Sy] = \sum_{i=1}^{n} x_{i0} [e_{i0}, z_j e_{jl} + l e_{j,l-1}] =$$

$$= \sum_{i=1}^{n} x_{i0} \overline{z_j} p_{l0}^{ij} + l \sum_{i=1}^{n} x_{i0} p_{l-1,0}^{ij}.$$

Thus we find, again using Lemma 5

$$[Sx, y] - [x, Sy] = \sum_{i=1}^{n} x_{i0} \underbrace{\left((z_i - \overline{z_j}) p_{l0}^{ij} + l p_{l-1,0}^{ij} \right)}_{= -f(\overline{z_j})^{(l)}}. \tag{17}$$

The right hand side of (17) equals 0, as $\sum\limits_{i=1}^{n} x_{i0} = 0$. Finally consider the case that $x = \sum\limits_{i=1}^{n} x_{i0} e_{i0}$ and $y = \sum\limits_{j=1}^{n} y_{j0} e_{j0}$ where $\sum\limits_{i=1}^{n} x_{i0} = \sum\limits_{j=1}^{n} y_{i0} = 0$. We have

$$[Sx, y] - [x, Sy] = \sum_{i,j=1}^{n} x_{i0} \overline{y_{j0}} (z_i - \overline{z_j}) p_{00}^{ij} =$$

$$= \sum_{i=1}^{n} x_{i0} f(z_i) \overline{\sum_{j=1}^{n} y_{j0}} - \sum_{j=1}^{n} \overline{y_{j0}} f(\overline{z_j}) \sum_{i=1}^{n} x_{i0} = 0.$$

$\square$

Lemma 6 *For each $i \in \{1, \ldots, n\}$ and $k \in \{0, \ldots, k_i\}$ we have $e_{10} \in \mathcal{R}\left((S - z_i)^{-(k+1)}\right)$. In fact for $i \neq 1$*

$$(S - z_i)^{-(k+1)} e_{10} = \sum_{h=0}^{k} \frac{(-1)^{k-h}}{h!(z_i - z_1)^{k-h+1}} e_{ih} + \frac{(-1)^{k+1}}{(z_i - z_1)^{k+1}} e_{10} \tag{18}$$

holds, and for $i = 1$ we have

$$(S - z_i)^{-(k+1)} e_{10} = \frac{1}{k!} e_{1k}.$$

Proof : To prove the first assertion use induction on k. If $k = 0$ (18) follows immediately from the definition (13). Then we compute

$$(\tilde{S} - z_i)^{-1} (\tilde{S} - z_i)^{-(k+1)} e_{10} = \sum_{h=0}^{k} \frac{(-1)^{k-h}}{h!(z_i - z_1)^{k-h+1}} \frac{1}{h+1} e_{ih} + \frac{(-1)^{k+1}}{(z_i - z_1)^{k+1}} \frac{e_{i0} - e_{10}}{z_i - z_1} =$$

$$= \sum_{h=0}^{k+1} \frac{(-1)^{(k+1)-h}}{h!(z_i - z_1)^{(k+1)-h+1}} e_{ih} + \frac{(-1)^{(k+1)+1}}{(z_i - z_1)^{(k+1)+1}} e_{10} = (\tilde{S} - z_i)^{-(k+2)} e_{10}$$

where we used (13) and (14).

The second assertion is clear from (14).

□

4 Correspondence of solutions and extensions

In this section we will prove the main results of the present note which establish a connection between the solutions of the interpolation problem (2) and the selfadjoint extensions of S acting in some Pontryagin space extending $\mathcal{H}$. These results follow similar as in the classical case (see [17], [20], [22]).

But first we have to recall some notions. We call an extension $\tilde{S}$ of S acting in a Pontryaginspace $\mathcal{P}$ $\langle e_{10} \rangle$-minimal if

$$\overline{\langle (\tilde{S} - z)^{-1} e_{10} | z \in \varrho(\tilde{S}) \rangle} = \mathcal{P}.$$

We call two extensions $\tilde{S}_1$ and $\tilde{S}_2$ acting in Pontryaginspaces $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively $\langle e_{10} \rangle$-unitary equivalent if there is a unitary operator $U : \mathcal{P}_1 \to \mathcal{P}_2$ which satisfies $U e_{10} = e_{10}$ (remember that both, $\mathcal{P}_1$ and $\mathcal{P}_2$, are extensions of $\mathcal{H}$) and $\tilde{S}_1 = U^{-1} \tilde{S}_2 U$.

Theorem 1 *Let $n \in \mathbb{N}$, $z_1, \ldots, z_n \in \mathbb{C}^+$, $k_1, \ldots, k_n \in \mathbb{N}_0$ and $w_{ik} \in \mathbb{C}$ for $k = 0, \ldots, k_i, i = 1, \ldots, n$ be given. Assume that the generalized Pick matrix is regular. Then the solutions of the multiple point interpolation problem*

$$f^{(k)}(z_i) = w_{ik} \text{ for } k = 0, \ldots, k_i \text{ and } i = 1, \ldots, n$$

within a Nevanlinna class correspond to the selfadjoint (relational) extensions $\tilde{S}$ of the symmetric operator S which operate in a Pontryagin space $\mathcal{P}$ extending $\mathcal{H}$ and contain the points $z_1, \ldots, z_n$ in their resolvent set. This connection is established by the formula

$$\tilde{S} \mapsto f_{\tilde{S}}(z) = \overline{w_{10}} + (z - \overline{z_1})[(I + (z - z_1)(\tilde{S} - z)^{-1}) e_{10}, e_{10}].$$

Proof : Assume that f is a solution of the considered problem, then the results of Section 2 show that there exist $\tilde{S}$ and $\mathcal{P}$ extending S and $\mathcal{H}$, respectively and satisfy the required conditions. The fact that $z_1, \ldots, z_n \in \varrho(\tilde{S})$ follows similar as in [22].

Suppose conversely that $\tilde{S}$ extends S and that $z_1, \ldots, z_n \in \varrho(\tilde{S})$. In order to compute $f^{(k)}(z_i)$ we have to compute (use similar notation as in Section 2) $s^{(k)}(z_i)$ for $k = 0, \ldots, k_i$. In case $k = 0$ we find from Lemma 6 ($i \neq 1$)

$$(I + (z_i - z_1)(\tilde{S} - z)^{-1})e_{10} = e_{10} + (z_i - z_1)\frac{e_{i0} - e_{10}}{z_1 - z_1} = e_{i0}$$

(the case $i = 1$ is obvious) whereas for $k \geq 1$ a straightforward computation again using Lemma 6 implies

$$\frac{\partial^k}{\partial z^k}(I + (z - z_1)(\tilde{S} - z)^{-1})|_{z=z_i} e_{10} = e_{ik}.$$

Thus (iii) of Lemma 5 shows

$$f(z_i) = \overline{w_{10}} + (z_i - \overline{z_1})\underbrace{[e_{i0}, e_{10}]}_{=p^{i1}_{00}} = w_{i0},$$

and (ii) of Lemma 5 together with Lemma 1 shows ($k = 1, \ldots, k_i$)

$$f^{(k)}(z_i) = (z_i - \overline{z_1})\underbrace{[s^{(k)}(z_i), e_{10}]}_{=p^{i1}_{0k}} + k\underbrace{[s^{(k-1)}(z_i), e_{10}]}_{=p^{i0}_{0,k-1}} = w_{ik}.$$

□

Proposition 2 *The correspondence $\tilde{S} \mapsto f_{\tilde{S}}$ between extensions and solutions becomes bijective if we consider only $\langle e_{10}\rangle$-minimal extensions of S and do not distinguish between $\langle e_{10}\rangle$-unitary equivalent extensions.*

Proof : It is proved in [22] that two minimal extensions induce the same function if and only if they are $\langle e_{10}\rangle$-unitary equivalent. As we can restrict our attention to $\langle e_{10}\rangle$-minimal extensions in any case, the assertion is clear.

□

There is also a connection between the indices of the space $\mathcal{P}$ and the indices of the Nevanlinna class $f_{\tilde{S}}$ belongs to. Although the following result is well known (see e.g. [17] or [22]) and does not use the fact that $\mathcal{H} \subset \mathcal{P}$ we include it, especially with view to Remark 3.

Proposition 3 *If $\tilde{S}$ acts in a Pontryagin space with index (π, ν) then the corresponding solution $f_{\tilde{S}}$ is in a class $\mathcal{N}^{\pi'}_{\nu'}$ with $\pi' \leq \pi$ and $\nu' \leq \nu$. If we assume that $\tilde{S}$ is minimal the full number of positive and negative squares is attained, i.e. $\pi' = \pi$ and $\nu' = \nu$.*

Remark 3 Assume that the generalized Pick matrix has π_0 positive and ν_0 negative eigenvalues and is regular. Then there exist solutions of (2) in those classes $\mathcal{N}^{\pi}_{\nu}$ where $\pi \geq \pi_0$ and $\nu \geq \nu_0$.

On the other hand it is obvious that in classes $\mathcal{N}^{\pi}_{\nu}$ where $\pi < \pi'$ or $\nu < \nu'$ no solutions can exist.

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