On the existence of congruence-uniform structures on universal algebras

1 Introduction

In many algebraic structures like groups, rings or lattices there is made use of topologies which are naturally given by the algebraic structure. As an example may serve a m-adic topology on a ring (see e.g. [5]) or subgroup topologies on (abelian) groups (see e.g. [10] or [13]).

In a similar way a universal algebra may be equipped with a topological structure. A fact in common with both examples mentioned above is that not only a topology but a uniform structure is given, and that this uniformity is determined by a fundamental system of entourages consisting of congruence relations (ideals or normal subgroups, respectively). Thus the following definition, which can be found in [2] or [6] gives a generalization of the situation above:

Definition 1 *Let $\mathcal{A} = < A, F >$ be a universal algebra of type F with underlying set A. A uniform structure $\mathcal{N}$ on A is called a congruence-uniform structure (*cus*) if there exists a fundamental system of entourages consisting of congruence relations.*

As a matter of fact all fundamental operations of $\mathcal{A}$ are uniformly continuous with respect to a cus. This is even more true as each family of n-ary polynomial functions ($n \in \mathbb{N}$) forms an equicontinuous family of uniformly continuous mappings.

A question which arises naturally is whether or not for a given universal algebra there exists a cus. Obviously the set $\{\Delta\}$ consisting of the equality relation only and the set $\{\nabla\}$ where ∇ denotes the complete relation both form a filterbase on A^2. Thus at least the discrete and indiscrete uniform structures on A are cus. To exclude such cases we will restrict our attention to proper cus (pcus), which are cus such that the induced topology is Hausdorff and not discrete.

In [8] the rings which do not allow any pcus are determined. In the present article the existence of pcus will be related to the atomic structure of the lattice $\mathsf{Con}\,\mathcal{A}$ of all congruences of $\mathcal{A}$. This leads to a characterization of those universal algebras which allow a pcus.

In §2 the connection mentioned above is established and a characterization is given for the case of congruence distributive universal algebras. §3 deals with the general case. In §4 we study the existence of pcus on a universal algebra which is isomorphic to some direct product. In paragraphs §5 and §6 we apply the general results to multioperatorgroups and lattices, giving characterizations for the existence of pcus in either case.

The notation used in this article is similar to [4] and [7] concerning algebraic notions and [1] for used topological notions. The last two paragraphs make use of some theorems that can be found in [7] and [14].

2 General results

To start with note the following quite obvious fact.

Lemma 1 *Let $\mathcal{A}$ be a universal algebra . There exists a* pcus *on $\mathcal{A}$ if and only if there exists a set $M \subseteq \mathsf{Con}\,\mathcal{A}$ of congruence relations with the following properties:*

1. *M is a filterbase*
2. *$\Delta \notin M$*
3. *$\bigcap_{\theta \in M} \theta = \Delta$*

We now give a necessary condition for the existence of a pcus, which will be used in proving our first main theorem. This result partly generalizes Theorem 1 of [8].

Proposition 1 *Let $\mathcal{A}$ be a universal algebra . If there exists a mapping $\kappa : \mathsf{Con}\,\mathcal{A} \to \mathsf{Con}\,\mathcal{A}$ with the properties*

1. *$|\{\theta \in \mathsf{Con}\,\mathcal{A}\ |\kappa(\theta) \neq \Delta\}| < \infty$*
2. *$\forall \theta \in \mathsf{Con}\,\mathcal{A}\ : \mathsf{height}[\Delta, \kappa(\theta)] < \infty$*
3. *$\forall (a, b) \notin \Delta : \theta(a, b) \cap [\bigcup_{\theta \in \mathsf{Con}\,\mathcal{A}} \kappa(\theta)] \neq \Delta$*

then there does not exist any pcus *on $\mathcal{A}$.*

Proof: Let $\emptyset \neq \Sigma \subseteq \mathsf{Con}\,\mathcal{A}$ be a filterbase, then without limitation of generality we may assume

$$\forall \psi_1, \psi_2 \in \Sigma \ : \ \psi_1 \wedge \psi_2 \in \Sigma \tag{1}$$

$$\Delta \notin \Sigma \tag{2}$$

We will then prove $\bigwedge_{\theta \in \Sigma} \theta \neq \Delta$.
For this purpose let $\theta \in \mathsf{Con}\,\mathcal{A}$ and consider the set $\Omega_\theta = \{\kappa(\theta) \wedge \psi | \psi \in \Sigma\}$ which is partially ordered by inclusion.

Step 1 : Ω_θ has a smallest element.
First note that Ω_θ has minimal elements because Ω_θ is not empty and starting with an arbitrary element of Ω_θ we construct a descending chain in Ω_θ. Any such chain can only have finite lenght not exceeding $\mathsf{height}[\Delta, \kappa(\theta)] + 1$ because of $\Omega_\theta \subseteq [\Delta, \kappa(\theta)]$ and assumption *2*. Thus we may construct a minimal element in at most $\mathsf{height}[\Delta, \kappa(\theta)] + 1$ steps. Because Ω_θ obviously is closed with respect to finite intersection, there can be at most one minimal element which is therefore the smallest element.

Denote in the following by $\psi(\theta)$ an element of Σ such that $\kappa(\theta) \wedge \psi(\theta)$ is the smallest element of Ω_θ. According to assumption *1* there is $n \in \mathbb{N}$ such that we may denote by $\{\theta_1, \ldots, \theta_n\}$ the set of all congruences θ with $\kappa(\theta) \neq \Delta$.

Let $\psi = \psi(\theta_1) \wedge \ldots \wedge \psi(\theta_n)$, then $\psi \in \Sigma$ and $\psi \wedge \kappa(\theta_i) \subseteq \psi(\theta_i) \wedge \kappa(\theta_i)$ for $i = 1, \ldots n$ and therefore $\psi \wedge \kappa(\theta_i)$ is the smallest element of $\Omega(\theta_i)$ for each i.

Step 2 : There is a number $i \in \{1, \ldots, n\}$ such that $\psi \wedge \kappa(\theta_i) \neq \Delta$.
Suppose $\forall i \in \{1, \ldots, n\} \;:\; \psi \wedge \kappa(\theta_i) = \Delta$. Then $\Delta = \psi \cap (\bigcup_{i=1}^{n} \kappa(\theta_i)) = \psi \cap (\bigcup_{\theta \in \mathsf{Con}\,\mathcal{A}} \kappa(\theta))$. As $\psi \in \Sigma$ we have $\psi \neq \Delta$. Choose any pair $(a, b) \in \psi \setminus \Delta$ then $\theta(a, b) \cap (\bigcup_{\theta \in \mathsf{Con}\,\mathcal{A}} \kappa(\theta)) = \Delta$. This contradicts assumption *3*.
Let e.g. $\psi \wedge \kappa(\theta_1) \neq \Delta$. Then $\forall \phi \in \Sigma \;:\; \psi \wedge \kappa(\theta_1) \subseteq \phi \wedge \kappa(\theta_1) \subseteq \phi$ which implies the fact we aimed for.

□

Now we are in position to prove our first main theorem, which relates the existence of pcus to the atomic structure of $\mathsf{Con}\,\mathcal{A}$. Following the notation of [7] we call a lattice atomic if every element lies above an atom. Note that this condition is weaker than the requirement that every element is the join of atoms.

Theorem 1 *Let $\mathcal{A}$ be a universal algebra . Then the following implications hold:*

1. *If $\mathsf{Con}\,\mathcal{A}$ is not atomic then there exists a* pcus*.*
2. *If $\mathsf{Con}\,\mathcal{A}$ is atomic and has only finitely many atoms then there exists no* pcus*.*

Proof: ad *1* : Let θ be a congruence which does not contain an atom and denote by Ω the set of all descending chains in $\mathsf{Con}\,\mathcal{A}$ which start with θ and do not contain Δ. Ω is partially ordered by inclusion and is not empty since the one-element chain $\{\theta\}$ is element of Ω. If there is an ascending chain $(\alpha_i)_{i \in I}$ of elements of Ω then the set-theoretic union $\bigcup_{i \in I} \alpha_i$ is also a chain which starts with θ and does not contain Δ. Thus $\bigcup_{i \in I} \alpha_i$ is an upper bound for each α_i in Ω. According to Zorn's Lemma there is a maximal element of Ω which we will denote by α_M. Obviously α_M satisfies the properties (1) and (2) and therefore defines a cus. Let $\psi = \bigwedge_{\phi \in \alpha_M} \phi$ then $\psi = \Delta$. For if $\psi \neq \Delta$, there would be a congruence $\psi_1 \neq \Delta, \psi_1 \subset \psi$ because ψ cannot be an atom. Thus the chain $\alpha_M \vee \{\psi_1\}$ would properly contain α_M which contradicts the choice of α_M. The cus given by the chain α_M therefore is proper.
ad *2* : Let $\kappa : \mathsf{Con}\,\mathcal{A} \to \mathsf{Con}\,\mathcal{A}$ be the mapping defined by

$$\kappa(\theta) = \begin{cases} \Delta & , \quad \theta \text{ is not an atom} \\ \theta & , \quad \theta \text{ is an atom} \end{cases}$$

Then the following statements are true:

1. $|\{\theta \in \mathsf{Con}\,\mathcal{A} \mid \kappa(\theta) \neq \Delta\}| < \infty$ because there are only finitely many atoms.
2. $\mathsf{height}[\Delta, \kappa(\theta)] = 1, 0$ whether or not θ is an atom.
3. For each pair $(a, b) \notin \Delta$ the set $\theta(a, b) \cap [\bigcup_{\psi \in \mathsf{Con}\,\mathcal{A}} \kappa(\psi)]$ is not equal to Δ because there is an atom which is contained in $\theta(a, b)$.

Applying Proposition 1 we obtain our result.

□

According to the above theorem only the case of $\mathsf{Con}\,\mathcal{A}$ being atomic and having infinitely many atoms requires further attention.

If $\mathsf{Con}\,\mathcal{A}$ is distributive, there always exists pcus in this case. There is however not only one pcus, but a rich structure of pcus. To make this idea more precise we give the following definition:

Definition 2 *A family of $(\mathcal{N}_{ij})_{i,j\in\mathbb{N}}$ of* pcus *is called a fan family if*

1. *$\forall i \in \mathbb{N}, j_1 < j_2 \in \mathbb{N}\ :\ \mathcal{N}_{ij_2}$ is finer than $\mathcal{N}_{ij_1}$*
2. *$\forall i_1 \neq i_2, j_1, j_2 \in \mathbb{N}\ :\ \mathcal{N}_{i_1 j_1}$ and $\mathcal{N}_{i_2 j_2}$ are incomparable.*

Call two fan families $(\mathcal{N}_{ij})$ and $(\mathcal{N}'_{ij})$ elementwise incomparable if any two elements $\mathcal{N}_{ij}$ and $\mathcal{N}_{kl}$ are incomparable.

If I is an index set then denote with $\lambda(I)$ the cardinality of the set of all limit numbers of I where I is wellordered such that it has a maximal element.

Note that if I is infinite we have $\lambda(I) = |I|$.

A sufficient condition for the existence of pcus is given by the following result, which leads to our characterization theorem for congruence distributive universal algebras.

Proposition 2 *Let $(\theta_i)_{i\in I}$ be an infinite family of distinct atoms of the atomic lattice $\mathsf{Con}\,\mathcal{A}$ with the property*

$$\forall J \subseteq I, \theta \text{ atom of } \mathsf{Con}\,\mathcal{A}\ :\ \theta \subseteq \bigvee_{j\in J} \theta_j \iff \theta = \theta_i \text{ for some } i \in J \qquad (3)$$

then there exists at least $|I|$ elementwise incomparable fan families of pcus *on $\mathcal{A}$.*

Proof: <u>Step 1 :</u> Let $N = \{i_k | k \in \mathbb{N}\}$ be a subset of I which is order isomorphic to $\mathbb{N}$. Then the family $\Theta_n = \bigvee_{k\geq n} \theta_{i_k}$ for $n \in \mathbb{N}$ is a descending chain in $\mathsf{Con}\,\mathcal{A}$. Obviously no congruence Θ_n is equal to Δ. We now show that the cus given by this chain is proper: Suppose $\bigcap_{n\in\mathbb{N}} \Theta_n \neq \Delta$. Then there would be an atom θ which is contained in this intersection and thus in each congruence Θ_n. Now $\theta \subseteq \Theta_1 = \bigvee_{k\geq 1} \theta_{i_k}$ implies $\theta = \theta_{i_k}$ for some number k. But then $\theta = \theta_{i_k} \subseteq \Theta_{i_{k+1}}$ leads to a contradiction.

Given a set N as above we may consider the sets $N_p^l = \{i_k \in N | k = p^{2^{l}\cdot h}, h \in \mathbb{N}\}$ for any prime number p. The pcus $\mathcal{N}_{pl}$ constructed as above from these sets form a fan family of pcus. This is an immediate consequence of the criterion for the comparability of two pcus constructed in the above way shown in the next step of the proof.

<u>Step 2 :</u> Let $N_1 = \{i_k | k \in \mathbb{N}\}$ and $N_2 = \{j_k | k \in \mathbb{N}\}$ be two subsets of I; we obtain two pcus $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively by step 1. Then $\mathcal{N}_1$ is finer than $\mathcal{N}_2$ if and only if there exists a number $n_0 \in \mathbb{N}$ such that $\{i_k \in N_1 | k \geq n_0\} \subseteq N_2$.

To prove this let first $\{i_k \in N_1 | k \geq n_0\} \subseteq N_2$ and let Θ_n^2 be a set of the filterbase obtained in step 1 for $\mathcal{N}_2$. Obviously there is a number $m \geq n_0$ such that $i_m \in N_2$ and $i_m \geq j_n$. Then $\Theta_m^1 = \bigvee_{k\geq m} \theta_{i_k} \subseteq \bigvee_{l\geq n} \theta_{j_l} = \Theta_n^2$ and thus $\mathcal{N}_1$ is finer than $\mathcal{N}_2$.

For the converse suppose that $\mathcal{N}_1$ is finer than $\mathcal{N}_2$. Then there is a number m such that $\Theta^1_m \subseteq \Theta^2_1$. Now for $k \geq m \ : \ \theta_{i_k} \subseteq \Theta^1_m \subseteq \Theta^2_1 = \bigvee_{l\geq 1} \theta_{j_l} \Rightarrow \theta_{i_k} = \theta_j$ for some $j \in N_2$. Thus $\{i_k | k \geq m\} \subseteq N_2$.

Step 3 : Let i_0 be a limit number in the wellorderd set I and let j_1 be any element smaller than i_0. Then by defining elements $j_n = j_{n-1}+1$ inductively for $n \in \mathbb{N}$ we obtain a set $\{j_n | n \in \mathbb{N}\}$ which is order isomorphic to $\mathbb{N}$. Now we write this set as a disjoint union of countably many countable sets, which all are order isomorphic to $\mathbb{N}$. From each of these countable sets we obtain a fan family of pcus as shown in step 1. Refering again to step 2 these fan families are elementwise incomparable. Thus there are at least $\aleph_0 \cdot \lambda(I) = \aleph_0 \cdot |I| = |I|$ incomparable fan families.

□

Now we get as an immediate consequence the following theorem.

Theorem 2 *Let $\mathcal{A}$ be a universal algebra with distributive lattice of congruences. Then there exists a* pcus *on $\mathcal{A}$ if and only if either $\mathcal{A}$ is not atomic or $\mathcal{A}$ has infinitely many atoms. If the set* $\mathsf{At}(\mathcal{A})$ *of all atoms of $\mathcal{A}$ is infinite then there exists at least* $|\mathsf{At}(\mathcal{A})|$ *elementwise incomparable fan families of* pcus.

Proof: In view of Theorem 1 and Proposition 2 it remains to show that condition (3) holds in a congruence distributive universal algebra when $\{\theta_i | i \in I\}$ is the set of all atoms of $\mathsf{Con}\,\mathcal{A}$. To show this let θ be an atom of $\mathsf{Con}\,\mathcal{A}$ and let $\theta \subseteq \bigvee_{j\in J} \theta_j$ for $J \subseteq I$. This is true if and only if $\theta = \theta \wedge (\bigvee_{j\in J} \theta_j) = \theta \wedge (\bigcup_{E\subseteq J, |E|<\infty} \bigvee_{j\in E} \theta_j) = \bigcup_{E\subseteq J,|E|<\infty} (\theta \wedge \bigvee_{j\in E} \theta_j) = \bigcup_{E\subseteq J,|E|<\infty} \bigvee_{j\in E} (\theta \wedge \theta_j)$ Because θ and θ_j are atoms either $\theta \wedge \theta_j = \Delta$ if $\theta \neq \theta_j$ or $\theta \wedge \theta_j = \theta$ if $\theta = \theta_j$. Thus the above equation is true if and only if $\theta = \theta_j$ for some $j \in J$, which shows (3).

□

3 Congruence atomic universal algebras

We will now discuss in greater detail the remaining case where $\mathsf{Con}\,\mathcal{A}$ is atomic and has infinitely many atoms, without making further assumptions on $\mathcal{A}$. For the rest of this paragraph let all universal algebras have these properties and denote by $(\psi_i)_{i\in I_A}$ the family of all atoms of $\mathsf{Con}\,\mathcal{A}$ (where I_A is infinite). Also denote by $\alpha : \mathsf{Con}\,\mathcal{A} \to \mathcal{P}(I_A)$ the mapping defined by

$$\alpha(\phi) = \{i \in I_A | \psi_i \subseteq \phi\}$$

Lemma 2 *Some basic properties of the mapping α are*

1. *α is monotonic, i.e. $\phi_1 \subseteq \phi_2 \Rightarrow \alpha(\phi_1) \subseteq \alpha(\phi_2)$*
2. *$\alpha(\phi_1 \wedge \phi_2) \subseteq \alpha(\phi_1) \cap \alpha(\phi_2)$ and $\alpha(\phi_1 \vee \phi_2) \supseteq \alpha(\phi_1) \cup \alpha(\phi_2)$*
3. *The* cus *induced by a filterbase Σ of congruence relations is proper if and only if $\bigcap_{\phi\in\Sigma} \alpha(\phi) = \emptyset$*

Proof:

1. $i \in \alpha(\phi_1) \Rightarrow \psi_i \subseteq \phi_1 \subseteq \phi_2 \Rightarrow i \in \alpha(\phi_2)$
2. This assertion is an immediate consequence of the fact α being monotonic.
3. Assume the topology induced by Σ is not proper. Then $\bigcap_{\phi\in\Sigma}\phi \neq \Delta \Rightarrow \exists i_0 \in I_A \ : \ \psi_{i_0} \subseteq \bigcap_{\phi\in\Sigma}\phi \Rightarrow \forall\phi \in \Sigma \ : \ i_0 \in \alpha(\phi) \Rightarrow i_0 \in \bigcap_{\phi\in\Sigma}\alpha(\phi)$.

   For the converse suppose $i_0 \in \bigcap_{\phi\in\Sigma}\phi$. Then by the same conclusions as above one obtains that Σ cannot be proper.

□

In group theory one defines a normal subgroup which is called the socle of the group. In a similar manner we give the following definition.

Definition 3 *Let $\mathcal{A}$ be a universal algebra . We will call the set*

$$\sigma(\mathcal{A}\ ) = \{\bigvee_{i\in J}\psi_i | J \subseteq I_A\}$$

the socle of $\mathcal{A}$.

Concerning the question of the existence of a pcus we may restrict our attention to $\sigma(\mathcal{A}\ )$.

Proposition 3 *There exists a* pcus *on $\mathcal{A}$ if and only if there exists a* pcus *which has a filterbase of neighbourhoods consisting of sets of the form $\bigvee_{i\in J}\psi_i$ where $J \subseteq I_A$.*

Proof: Let Σ be the base of a pcus. Then we obtain a base of a pcus of the required form by

$$\Sigma' = \{\bigvee_{i\in\alpha(\phi)}\psi_i | \phi \in \Sigma\}$$

Because of $\mathcal{A}$ being atomic $\Delta \notin \Sigma'$ holds. Also $\bigcap_{\theta\in\Sigma'}\theta \subseteq \bigcap_{\phi\in\Sigma}\phi = \Delta$ which means Σ' being proper. The property of Σ' being a filterbase immediately follows from Lemma 2.

The converse is obvious.

□

For our next theorem we need another notation.

Definition 4 *The mapping $^- : \mathcal{P}(I_A) \to \mathcal{P}(I_A)$ which is defined as $\overline{J} = \{i \in I_A | \psi_i \subseteq \bigvee_{j\in J}\psi_j\}$ will be called the atomic closure operator on I_A.*

Note that $^-$ can be obtained as the closure operator corresponding to a certain relation: Let $R \subseteq I_A \times \mathsf{Con}\,\mathcal{A}$ be the relation defined by

$$i\,R\,\theta \iff \psi_i \leq \theta.$$

The so called polarities[1] of R compute as $(J \subseteq I_A)$

$$J^{\rightarrow} = \{\theta \in \mathsf{Con}\,\mathcal{A}\ | jR\theta\ \forall j \in J\} = [\bigvee_{j\in J} \psi_j, \nabla]$$

and

$$(J^{\rightarrow})^{\leftarrow} = \{i \in I_A | iR\theta\ \forall \theta \in J^{\rightarrow}\} = \{i \in I_A | \psi_i \subseteq \bigvee_{j\in J} \psi_j\} = \bar{J}.$$

Some properties of $^-$ are collected in the following Lemma.

Lemma 3 *Let J, J_1, J_2 be subsets of I_A, then*

1. $\overline{\emptyset} = \emptyset$ *and* $J \subseteq \overline{J}$
2. $J_1 \subseteq J_2 \Rightarrow \overline{J_1} \subseteq \overline{J_2}$
3. $\overline{(J_1 \cap J_2)} \subseteq \overline{J_1} \cap \overline{J_2}$ *and* $\overline{(J_1 \cup J_2)} \supseteq \overline{J_1} \cup \overline{J_2}$
4. $\overline{\overline{J}} = \overline{J}$
5. $\alpha(\phi) = \overline{\alpha(\phi)}$

Proof: The statements 1, 2 and 4 are elementary properties of a closure operator (see Theorem 2.21 of [11]). Furthermore 3 follows immediately from 2. To prove 5 we only have to note that

$$i \in \overline{\alpha(\phi)} \Rightarrow \psi_i \subseteq \bigvee_{j\in\alpha(\phi)} \psi_j \subseteq \phi \Rightarrow i \in \alpha(\phi)$$

holds.

□

Unfortunately $\overline{J_1 \cup J_2} \subseteq \overline{J_1} \cup \overline{J_2}$ fails to be true in general and so we cannot talk about $^-$ as a closure operator in the sense of general topology. Nevertheless we will call a set $J \subseteq I_A$ with $J = \overline{J}$ a $^-$-closed set.

Theorem 3 *Let $\mathcal{A}$ be a universal algebra with atomic lattice of congruences. Then $\mathcal{A}$ allows a* pcus *if and only if there exists a filterbase $(J_i)_{i\in I}$ of non-empty $^-$-closed sets on I_A with*

$$\bigcap_{i\in I} J_i = \emptyset \tag{4}$$

Proof: If Σ is base for a pcus then by using the mapping α one obtains a family of subsets $\{\alpha(\phi)|\phi \in \Sigma\}$ of I_A. The properties we require were proved in the above Lemmas 2 and 3.

[1] see [11] p.51.

If a filterbase $\{J_i | i \in I\}$ of $^-$-closed subsets of I_A with property (4) is given, then one verifies easily (again referring to the Lemmas 2 and 3) that the system

$$\Sigma = \{ \bigvee_{j \in J_i} \psi_j | i \in I \}$$

is a filterbase with $\Delta \notin \Sigma$ and $\bigcap_{\phi \in \Sigma} \phi = \Delta$. Thus Σ determines a pcus on $\mathcal{A}$.

□

Note the formal similarity of condition (4) with the condition of compactness for a topological space given by a closure operator $^-$.

We will now formulate the above theorem in terms of the lattice

$$\mathcal{L} = < \{J \subseteq I_A | \overline{J} = J\}, \subseteq >$$

Proposition 4 [2] *The operator $^-$ is an algebraic closure operator and thus the lattice $\mathcal{L}$ is algebraic.*

Proof: Using Lemma 3 we get $\overline{\bigcap_{i \in I} J_i} \subseteq \bigcap_{i \in I} \overline{J_i} = \bigcap_{i \in I} J_i$ for each family $(J_i)_{i \in I}$ of $^-$-closed sets. Let now $(J_i)_{i \in I}$ be an ascending chain of $^-$-closed sets and let $l \in \overline{\bigcup_{i \in I} J_i}$. We have to show that $l \in \bigcup_{i \in I} J_i$. Now $\psi_l \subseteq \bigvee_{j \in \bigcup_{i \in I} J_i} \psi_j = \bigcup_{E \subseteq \bigcup_{i \in I} J_i} (\bigvee_{j \in E} \psi_j)$. ψ_l being an atom implies $\exists E_0 \ : \ \psi_l \subseteq \bigvee_{j \in E_0} \psi_j \subseteq \bigvee_{j \in J_i} \psi_j$ for some J_i. Because J_i is closed we obtain $l \in J_i \subseteq \bigcup_{i \in I} J_i$. Thus $^-$ is actually algebraic. Theorem 2.16 of [11] now implies that $\mathcal{L}$ is an algebraic lattice.

□

Let $\tilde{\mathcal{L}}$ denote the dual lattice of $\mathcal{L}$. Then the above Theorem 3 is formulated in terms of the lattice $\tilde{\mathcal{L}}$ as follows:

Theorem 4 *There exists no* pcus *on the universal algebra $\mathcal{A}$ if and only if the 1-element of the lattice $\tilde{\mathcal{L}}$ is compact.*

Proof: The 1-element of $\tilde{\mathcal{L}}$ being compact means that if $1 = \bigvee_{i \in I} L_i$ for some family of elements L_i of $\tilde{\mathcal{L}}$ there is a finite subset I' of I such that $1 = \bigvee_{i \in I'} L_i$. Returning to the lattice $\mathcal{L}$ we get our condition being equivalent to the following: If the empty set is infimum of an arbitrary family of elements of $\mathcal{L}$ then it is already infimum of a finite subfamily. Thus 1 being not compact in $\tilde{\mathcal{L}}$ is equivalent to the existence of a filterbase of elements of $\mathcal{L}$ with property (4).

□

Combining our previous Theorems 1 and 4 we obtain our characterization result for a universal algebra without making further assumptions.

Theorem 5 *Let $\mathcal{A}$ be a universal algebra . There exists a* pcus *on $\mathcal{A}$ if and only if either* Con $\mathcal{A}$ *is not atomic or* Con $\mathcal{A}$ *is atomic but the 1-element in the lattice $\tilde{\mathcal{L}}$ is not compact.*

[2] For the notions used in this Proposition see e.g. [4] or [9].

4 Direct products

Let the universal algebra $\mathcal{A}$ be isomorphic to a direct product $\prod_{i\in I}\mathcal{A}_i$ of a family of universal algebras $(\mathcal{A}_i)_{i\in I}$. Then we relate the property of the existence of pcuson $\mathcal{A}$ to the same property for the direct factors.

Because of the next proposition we may focus our attention to the case of finite direct products.

Proposition 5 [3] *Let $\mathcal{A} = \prod_{i\in I}\mathcal{A}_i$ where I is infinite. There exists then at least $|I|$ elementwise incomparable fan families of* pcus.

Proof: Let $\{i_k | k \in \mathbb{N}\} \subseteq I$ be a subset of I which is order isomorphic to $\mathbb{N}$. For a subset J of I let

$$\chi_J = \prod_{i\in I}\delta_i(J) \quad \text{where } \delta_i(J) = \begin{cases} \Delta\ , i \notin J \\ \nabla\ , i \in J \end{cases}$$

For $n \in \mathbb{N}$ let $\Theta_n = \chi_{\{i_k | k \geq n, k\in\mathbb{N}\}}$ then the family $(\Theta_n)_{n\in\mathbb{N}}$ is a descending chain of congruences of $\mathcal{A}$ which does not contain Δ. But $\bigcap_{n\in\mathbb{N}}\Theta_n = \Delta$. To show this let $(a,b) = ((a_i)_{i\in I},(b_i)_{i\in I})$ be an element of $\bigcap_{n\in\mathbb{N}}\Theta_n$. If $i \in I \setminus \{i_k | k \in \mathbb{N}\}$ then a_i obviously equal b_i. If $i = i_l \in \{i_k | k \in \mathbb{N}\}$ then $(a,b) \in \Theta_{l+1}$ also leads to $a_i = b_i$. Thus $a = b$.
Proceeding as in the proof of Proposition 2 we obtain the assertion of this proposition.

□

So we have to consider the case $|I| < \infty$. But first let us introduce one more notation.

Definition 5 *Let $\mathcal{A}$ be a universal algebra . Then denote by* $\mathsf{Cu}\mathcal{A}$ *the set of all* cus *on $\mathcal{A}$. If for two filterbases $\Sigma, \Sigma' \subseteq \mathsf{Con}\,\mathcal{A}$ the relation $\sim$ is defined by*

$$\Sigma \sim \Sigma' \iff \textit{the induced } \mathsf{cus} \textit{ coincide}$$

then there is a bijective correspondence between $\mathsf{Cu}\mathcal{A}$ *and the set*

$$\{\Sigma | \Sigma \subseteq \mathsf{Con}\,\mathcal{A} \ \textit{ is a filterbase}\}/_\sim$$

Proposition 6 *Let $\mathcal{A} = \prod_{i\in I}\mathcal{A}_i$ be the direct product of the family $(\mathcal{A}_i)_{i\in I}$. Then there is a mapping*

$$\beta : \prod_{i\in I}\mathsf{Cu}\mathcal{A}_i \longrightarrow \mathsf{Cu}\mathcal{A} \tag{5}$$

which is one-to-one.

Proof: For a family $(\Sigma_i)_{i\in I}$ of filterbases in $\mathsf{Con}\,\mathcal{A}_i$ define $\beta((\Sigma_i)_{i\in I}) = \{\prod_{i\in I}\phi_i\}$ where ϕ_i is equal to ∇ for all but finitely many $i \in I$ and for any other I let $\phi_i \in \Sigma_i$. Then $\beta((\Sigma_i)_{i\in I})$ is a subset of $\mathsf{Con}\,\mathcal{A}$ and obviously is a filterbase. Thus it induces a cus on $\mathcal{A}$. We have to show that β is well defined i.e. that two families

[3] We may remind of the notation given in Definition 2.

$(\Sigma_i)_{i\in I}$ and $(\Sigma'_i)_{i\in I}$ where for each $i \in I$ the same cus is induced by Σ_i and Σ'_i have the same image under β. Then we may regard β as a mapping $\beta : \prod_{i\in I}\mathsf{Cu}\mathcal{A}_i \rightarrow \mathsf{Cu}\mathcal{A}$. But this fact is immediate because $\beta((\Sigma_i)_{i\in I})$ induces the product uniformity of the uniformities induced by each Σ_i. With the same conclusion it also follows that β is one-to-one.

□

Corrollary 1 *If $\mathcal{A} = \prod_{i\in I}\mathcal{A}_i$ and one universal algebra $\mathcal{A}_i$ allows a* pcus *then $\mathcal{A}$ allows a* pcus*.*

To characterize the existence of a pcus on $\mathcal{A}$ in terms of pcus on direct factors we have to make further assumptions and will use the notion of [3]. So let us recall some notation:

Definition 6 *Let $\mathcal{A} = \prod_{i\in I}\mathcal{A}_i$ be the direct product of a family $(\mathcal{A}_i)_{i\in I}$. A congruence $\theta \in \mathsf{Con}\,\mathcal{A}$ is said to have property (P) if there exists a family $(\theta_i)_{i\in I}$ of congruences $\theta_i \in \mathsf{Con}\,\mathcal{A}_i$ such that $\theta = \prod_{i\in I}\theta_i$.*

The direct product $\prod_{i\in I}\mathcal{A}_i$ is said to have property (P) if each congruence of $\mathcal{A}$ satisfies (P). Similar a class K of universal algebra (especially an equational class) is said to hold (P) if each direct product of members of K holds (P).

Proposition 7 *Let the direct product $\mathcal{A} = \prod_{i\in I}\mathcal{A}_i$ have property (P). Then the mapping β is onto.*

Proof: First of all note that if $\mathcal{A}$ holds (P) the index set I must be finite [4](see Theorem 6 of [3]). Therefore β is given by $\beta((\Sigma_i)_{i\in I}) = \{\prod_{i\in I}\phi_i|\phi_i \in \Sigma_i\}$.
Suppose now Σ is a filterbase in $\mathsf{Con}\,\mathcal{A}$ which induces a cus on $\mathcal{A}$. Then for each member $\phi \in \Sigma$ there exists congruences $\phi_i \in \mathsf{Con}\,\mathcal{A}_i$ such that $\phi = \prod_{i\in I}\phi_i$, namely $\phi_i = \pi_i(\phi)$ where π_i denotes the projection mapping onto the i-th component. The sets $\Sigma_i = \{\phi_i|\phi_i = \pi_i(\phi), \phi \in \Sigma\}$ are filterbases in $\mathsf{Con}\,\mathcal{A}_i$ (which is proved straightforward). Thus we may consider for this family $\beta((\Sigma_i)_{i\in I})$. This uniformity is finer than the uniformity we started with. But all projection mappings [5]$\pi_i : (\mathcal{A}, \Sigma) \rightarrow (\mathcal{A}_i, \Sigma_i)$ are uniformly continous and therefore the uniformity induced by Σ must be finer than the product uniformity of the family $(\Sigma_i)_{i\in I}$ which coincides with $\beta((\Sigma_i)_{i\in I})$. Thus $\beta((\Sigma_i)_{i\in I})$ equals the uniformity induced by Σ and β is onto.

□

In [3] one can find conditions when (P) holds. Necessary for (P) is that the index set I is finite - which is the case we are interested in. Combining the above results we obtain a characterization whether or not $\mathcal{A}$ allows a pcus.

Theorem 6 *Let $\mathcal{A} = \prod_{i\in I}\mathcal{A}_i$ be a direct product of the family $(\mathcal{A}_i)_{i\in I}$. If the set I is infinite then $\mathcal{A}$ allows a* pcus*. The condition that one $\mathcal{A}_i$ allows a* pcus*is sufficient for $\mathcal{A}$ to allow a* pcus*. If $\mathcal{A}$ satisfies (P) this condition is also necessary.*

[4]We exclude the case of some direct factors being one-element algebras which does not make any difference for our purposes.

[5]$(\mathcal{A}, \Sigma)$ denotes the uniform space on $\mathcal{A}$ with Σ as a fundamental system of entourages.

5 Multioperatorgroups

Our first application of the theorems given in the previous paragraphs will essentially concern multioperatorgroups. We are considering a little more extensive class of universal algebras given by the following definition.

Definition 7 *A universal algebra* $\mathcal{A} \; =< A, F >$ *is said to admit a multioperatorgroup structure if there is a multioperatorgroup G(*$\mathcal{A}$ *) with underlying set A which is polynomially equivalent with* $\mathcal{A}$ *.*

For the rest of this paragraph denote by σ(G($\mathcal{A}$)) the socle [6] of the multioperatorgroup G($\mathcal{A}$). As a result of [14] [7] the socle of a multioperatorgroup is a direct product of a family of minimal normal subgroups.

To prepare the main theorem of this paragraph where we will make further assumptions on $\mathcal{A}$ we give a proposition which makes use of Theorem 1 and holds in general.

Proposition 8 *Let G be a multioperatorgroup. Then the following implications hold (e denotes the identity element of G):*

1. *If there exists a normal subgroup $N \trianglelefteq G$ of G with $N \cap \sigma(G) = \{e\}$ then G allows a* pcus*.*
2. *If $\sigma(G) = \prod_{i \in I} N_i$ where N_i are minimal normal subgroups of G and I is infinite then there are at least $|I|$ elementwise incomparable fan families of* pcus*.*

Proof: To prove the first statement note that under the assumptions of *1* $\mathsf{Con}\, G$ is not atomic. Thus Theorem 1 implies the existence of a pcus.
Using the same method as in the proof of Proposition 5 we obtain the required number of filterbases consisting of normal subgroups of $\sigma(G)$. Because every automorphism of G operates componentwise on $\sigma(G)$ and because of the special structure of the filterbases constructed in Proposition 5 we get that the elements of the above filterbases are even normal subgroups of the group G. Thus we have the required number of pcus on G.

□

Now we state our theorem which characterizes the existence of pcus on some special classes of multioperatorgroups. An example for a variety for which this theorem is applicable is the variety of commutative rings with unity.

Theorem 7 *Let $\mathcal{A}$ be a universal algebra which admits a multioperatorgroup structure. If $\mathcal{A}$ is in a variety satisfying the conditions of Fraser and Horn [8] then 1 and 2 are equivalent:*

1. *$\mathcal{A}$ allows a* pcus*.*

[6] see [14] p. 272f.
[7] Theorem 27 on p. 272.
[8] see [3] Theorem 5.

2. *Either there exists a normal subgroup* $N \trianglelefteq G(\mathcal{A})$ *with* $N \cap \sigma(G(\mathcal{A})) = \{e\}$ *or* $\sigma(G(\mathcal{A})) = \prod_{i\in I} N_i$ *where* I *is infinite.*

Proof: The implication '*2*⇒ *1*' is stated in the above proposition, so assume *2* does not hold. Then we show that $\mathsf{Con}\, G(\mathcal{A})$ is atomic and has only finitely many atoms. This implies by using $\mathsf{Con}\,\mathcal{A} \cong \mathsf{Con}\, G(\mathcal{A})$ and Theorem 1 that *1* is false.
Step 1 : Let $\theta \in \mathsf{Con}\,\mathcal{A}$ be represented by the normal subgroup N. Then $N \cap \sigma(G(\mathcal{A})) \neq \{e\}$. Because $G(\mathcal{A})$ is polynomially equivalent with $\mathcal{A}$ it also satisfies the conditions of Fraser and Horn. Writing $\sigma(G(\mathcal{A})) = \prod_{i\in I} N_i$ (I is of course finite by our assumption) we may write $N = \prod_{i\in I} N'_i$ where N'_i is a normal subset of N_i. Again using the argument that every automorphism of $G(\mathcal{A})$ operates componentwise on $\sigma(G(\mathcal{A})) = \prod_{i\in I} N_i$ we get that the subgroups N'_i are even normal subgroups of $G(\mathcal{A})$. Because of $N \cap \sigma(G(\mathcal{A})) \neq \{e\}$ we get that at least one normal subgroup N'_i must equal N_i which implies $N_i \subseteq N$. The congruence induced by N_i is an atom contained in θ.
Step 2 : Now let N be the normal subgroup induced by an atom of $\mathsf{Con}\, G(\mathcal{A})$. The previous step tells us that there exists a normal subgroup N_i contained in N. Thus $N = N_i$ for some $i \in I$. Because I is finite $\mathsf{Con}\, G(\mathcal{A})$ can have only finitely many atoms.

□

6 Lattices

To start with note that the lattice of congruences of a lattice $\mathcal{V}$ is always distributiv. Thus we may apply Theorem 2. But $\mathsf{Con}\,\mathcal{V}$ has some more special properties e.g. $\mathsf{Con}\,\mathcal{V}$ is pseudo boolean [9]. We may recapitulate in this place the definition of a pseudo boolean lattice.

Definition 8 *Let* $\mathcal{V}$ *be a lattice;* $a, b, c \in \mathcal{V}$. *Then* c *is called a pseudo complement of* a *with respect to* b *if*

$$\forall x \in \mathcal{V} \ : \ x \preceq c \iff x \wedge a \preceq b$$

It is easy to see that at most one pseudo complement may exist.

If $\mathcal{V}$ *is a lattice with zero-element and if for each pair of elements* a, b *the pseudo complement exists,* $\mathcal{V}$ *is called a pseudo boolean lattice. In such a lattice we will denote the pseudo complement* c *of a pair of elements* a, b *by* $c = a \hookleftarrow b$.

Let again be $\{\psi_i | i \in I_A\}$ be the set of all atoms of the lattice $\mathsf{Con}\,\mathcal{V}$ and appropriate to Definition 3 let $\sigma(\mathcal{V}) = \{\bigvee_{i\in J} \psi_i | J \subseteq I_A\}$. The socle $\sigma(\mathcal{V})$ has a greatest element as subset of $\mathsf{Con}\,\mathcal{V}$ namely $\bigvee_{i\in I_A} \psi_i = \bigvee_{\psi\in\sigma(\mathcal{V})} \psi$.

Now we come to our characterization theorem on the existence of pcus in lattices.

[9]See Theorem 31.8 of [7].

Theorem 8 *Let $\mathcal{V}$ be a lattice. $\mathcal{V}$ does not allow a proper* cus *if and only if the following two conditions are satisfied.*

$$((\bigvee_{i \in I_A} \psi_i) \hookleftarrow \Delta) = \Delta \tag{6}$$

$$|\sigma(\mathcal{V})| < \infty \tag{7}$$

Proof: Step 1 : Obviously $\mathsf{Con}\,\mathcal{V}$ has only finitely many atoms if and only if $|\sigma(\mathcal{V})| < \infty$.

Step 2 : We show that the validity of (6) is equivalent to $\mathsf{Con}\,\mathcal{V}$ being atomic.
First let $\mathsf{Con}\,\mathcal{V}$ be atomic and assume $((\bigvee_{i \in I_A} \psi_i) \hookleftarrow \Delta) \neq \Delta$. Then we find an atom $\phi \subseteq ((\bigvee_{i \in I_A} \psi_i) \hookleftarrow \Delta)$. But this implies $\phi = \phi \wedge \bigvee_{i \in I_A} \psi_i \subseteq \Delta$ which is a contradiction.
For the converse let ϕ be any congruence of $\mathcal{V}$, $\phi \neq \Delta$. Then $\phi \wedge \bigvee_{i \in I_A} \psi_i \neq \Delta$ as $\phi \wedge \bigvee_{i \in I_A} \psi_i = \Delta$ would imply $\phi \subseteq ((\bigvee_{i \in I_A} \psi_i) \hookleftarrow \Delta)$ which is a contradiction as by our assumption the right hand side equals Δ. Thus we have (see [7] Theorem 25.4) $\Delta \neq \phi \wedge \bigvee_{i \in I_A} \psi_i = \bigvee_{i \in I_A} (\phi \wedge \psi_i)$ which is only possible if there is an atom ψ_0 with $\psi_0 \subseteq \phi$. So we found that $\mathsf{Con}\,\mathcal{V}$ is atomic.
Applying Theorem 2 we obtain our assertion.

□

Now we want to proceed with a proposition which is of purely algebraic nature and characterizes the existence of pcus in a subclass of lattices. To prove it we make use of theorems on congruence lattices of lattices which can be found in [7]. Again we may recapitulate some notation.

Definition 9 [10] *A lattice $\mathcal{V}$ is called relatively complemented if for each pair of elements $a \preceq b$ and $x \in [a, b]$ there exists a relative complement which is an element y such that $x \vee y = b$ and $x \wedge y = a$.*

Proposition 9 *Let $\mathcal{V}$ be a lattice satisfying the following two properties:*

1. *For two elements $a \preceq b$ one may find a finite maximal chain from a to b.*
2. *$\mathcal{V}$ is modular or $\mathcal{V}$ is relatively complemented.*

Then there exists a pcus *on $\mathcal{V}$ if and only if $|\mathsf{Con}\,\mathcal{V}| \geq \aleph_0$.*

Proof: Our conditions imply that $\mathsf{Con}\,\mathcal{V}$ is a boolean algebra (see Theorem 31.9 of [7]). Because $\mathsf{Con}\,\mathcal{V}$ is also a complete lattice we have that $\mathsf{Con}\,\mathcal{V}$ is atomic if and only if it is isomorphic to a boolean algebra $\mathcal{P}(M)$ where M is some set. Such an algebra has only finitely many atoms if and only if it is finite. This establishes our assertion.

□

Finally we want to relate to a given lattice $\mathcal{V}$ a quasicompact T_0- space and characterize the existence of pcus on $\mathcal{V}$ in terms of this topological object.

[10] See [7] p. 48f.

Definition 10 *In the sequel we will call a boolean algebra $\mathcal{T}$ with an additional operation $^-$ a topological boolean algebra if $^-$ is a closure operator in the sense of general topology, i.e. satisfies the axioms*

1. $x \subseteq y \Rightarrow \overline{x} \subseteq \overline{y}$
2. $x \subseteq \overline{x}, \overline{x} = \overline{\overline{x}}$
3. $\overline{x \cup y} = \overline{x} \cup \overline{y}, \overline{0} = 0$

An element $x \in T$ is called open if $\overline{x'} = x'$. The set of all open elements of T will be denoted by $\dot{T}$.

Note that a topological boolean algebra in our sense need not be a topoligical boolean algebra in the sense that all fundamental operations are continous.

Proposition 10 *Let $\mathcal{V}$ be a lattice. We may assign to $\mathcal{V}$ a quasicompact T_0- space which is constructed by the following procedure:*

1. *The lattice $\mathsf{Con}\,\mathcal{V}$ is a pseudo boolean lattice, thus isomorphic to $\dot{T}$ where T is a topological boolean algebra.*
2. *T is as a boolean algebra isomorphic to the set of clopen sets of some compact totally disconnected topological space $< Y, \tau' >$.*
3. *On Y we define another topology τ'' which has as basis all clopen sets of $< Y, \tau' >$ that correspond to open elements of the boolean algebra T via the isomorphism of step 2.*
4. *Now we factorize the space $< Y, \tau'' >$ with respect to the relation* [11]

$$x, y \in Y : \; x \sim y \iff \mathcal{U}(x) = \mathcal{U}(y)$$

We obtain the associated T_0-space $< X, \tau >$ which is quasicompact. The cannonical mapping $\pi :< Y, \tau'' > \rightarrow < X, \tau >$ is continuous, open and closed.

Proof: The first step taken is exactly the statement of Theorem 25.7 of [7]. Using the theorem of Stone we obtain the representation of T as asserted in step *2*. For step *3* we have to show that the set

$$B = \{O \subseteq Y | O \text{ is clopen}, \overline{O'} = O'\}$$

is a basis for some topology which is a straightforward conclusion of the axioms for $^-$. The last step is the well known construction of the associated T_0-space (see e.g. [12] p. 120f). It remains to show that $< X, \tau >$ is quasicompact. This follows from the fact that $< Y, \tau'' >$ is coarser than $< Y, \tau' >$ and that $< X, \tau >$ is a continuous and open image of $< Y, \tau'' >$.

□

Some properties of this assignment we need are given by the following lemmata.

[11] Denote by $\mathcal{U}(x)$ the system of all neighbourhoods of the point x.

Lemma 4 *A congruence of $\mathcal{V}$ is an atom in the lattice $\mathsf{Con}\,\mathcal{V}$ if and only if it corresponds to an open subset O of Y with the property*

$$\forall x \in O \ : \ \bigcap_{U \in \mathcal{U}(x)} U = O \tag{8}$$

When constructing the associated T_0-space such open sets exactly are the inverse images of the isolated points of $< X, \tau >$.

Proof: Assume θ is an atom in $\mathsf{Con}\,\mathcal{V}$. Then the corresponding set O is an open set in $< Y, \tau'' >$. Thus for $x \in O$ we have $\bigcap_{U \in \mathcal{U}(x)} U \subseteq O$. Suppose there exist $x, y \in O$ with $y \notin \bigcap_{U \in \mathcal{U}(x)} U$. Then there would be an open set $U_0 \in \mathcal{U}(x)$ with $y \notin U_0$. But then for the corresponding congruences Δ, ϕ, θ of $\emptyset, U_0 \cap O, O$ we would have $\Delta \subset \phi \subset \theta$ which contradicts θ being an atom.
Now assume property (8). If there is a congruence ϕ with $\Delta \neq \phi \subseteq \theta$ where θ is the congruence corresponding to O we have for the corresponding set O' of ϕ $\emptyset \neq O' \subseteq O$. So for any $x \in O'$ we have $O' \supseteq \bigcap_{U \in \mathcal{U}(x)} U = O$. Thus $O' = O$ which implies $\phi = \theta$. Therefore θ is an atom in $\mathsf{Con}\,\mathcal{V}$.
If O is an open set satisfying (8) we obviously get $|\pi(O)| = 1$ and thus $\pi(O)$ is an isolated point. Conversely let $\tilde{x}$ be an isolated point of $< X, \tau >$. Then $\pi^{-1}(\tilde{x})$ is open which gives

$$\forall x \in \pi^{-1}(\tilde{x}) \ : \ \pi^{-1}(\tilde{x}) \supseteq \bigcap_{U \in \mathcal{U}(x)} U$$

As for all $x \in \pi^{-1}(\tilde{x})$ the neighbourhood filters coincide, we have $\pi^{-1}(\tilde{x}) \subseteq \bigcap_{U \in \mathcal{U}(X)} U$ which establishes (8).

□

Lemma 5 *The lattice $\mathsf{Con}\,\mathcal{V}$ is atomic if and only if the set*

$$\mathsf{Iso}\,(< X, \tau >) = \{x \in X | x \textit{ is isolated point}\}$$

is dense in $< X, \tau >$.

Proof: First let $\mathsf{Con}\,\mathcal{V}$ be atomic and let $\tilde{O}$ be a non-empty open set in $< X, \tau >$. Then $O = \pi^{-1}(\tilde{O})$ is open in $< Y, \tau'' >$ and therefore there must be a set $\emptyset \neq O' \subseteq O$ of the base of the topology τ''. This set corresponds to a congruence relation $\theta \neq \Delta$. Thus there is an atom in $\mathsf{Con}\,\mathcal{V}$ which is contained in θ. The corresponding isolated point of X then is contained in $\tilde{O}$. Reversing this process we obtain the converse.

□

Now we are ready to formulate our criterion for the existence of a pcus in terms of the - via Proposition 10 - corresponding topological space $< X, \tau >$.

Theorem 9 *The lattice $\mathcal{V}$ does not allow a* pcus *if and only if we find a finite dense subset consisting of isolated points of the corresponding space $< X, \tau >$.*

Proof: Using the fact that every dense subset of a topological space must contain the set Iso this is an immediate consequence of the above lemmata and Theorem 2. □

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