A remark on affine complete rings

The notion of affine complete universal algebras has been studied by a number of authors. One of the problems investigated is the following: When is a direct product of affine complete universal algebras affine complete?

Grätzer [3] for example shows the affine completeness of Boolean algebras, Iskander [4] determines the affine complete subdirect products of finite prime-fields. Nöbauer [7] shows that the finite direct product of rings with unity is affine complete, if and only if all factors are affine complete. All affine complete abelian groups have also been determined (see [5],[6],[7]). Dorninger and Nöbauer [2] and Dorninger and Eigenthaler [1] have obtained a number of results on affine complete lattices. Further examples of affine complete algebras can be found e.g. in the papers of Werner [8],[9].

In this note we will characterize the affine completeness of an infinite direct product of commutative rings with unity.

By I we denote an arbitrary index set. Further let $\mathcal{R}_i$ for each $i \in I$ be a commutative ring with unity, and denote by $\mathcal{R}$ the direct product $\mathcal{R} = \prod_{i \in I} \mathcal{R}_i$. The theorem we are going to prove is stated as follows:

Theorem 1 *Let $k \in \mathbb{N}$. The direct product $\mathcal{R}$ of commutative rings with unity $(\mathcal{R}_i)_{i \in I}$ is k-affine complete, if and only if the following two conditions are satisfied:*

1. *All rings $\mathcal{R}_i$ are k-affine complete,*
2. *There exists a number $M_1 \in \mathbb{N}$, such that every unary polynomial function on the ring $\mathcal{R}_i$ for all but finitely many indices $i \in I$, can be realized by a polynomial of degree at most M_1.*

The first of the two following lemmata generalizes condition *2* to functions of more than one variable. We will use this generalization in the proof of the above theorem.

Lemma 1 *Assume condition 2 holds for the family $(\mathcal{R}_i)_{i \in I}$ of commutative rings with unity. Then a stronger condition is also true as follows:*

For each $k \in \mathbb{N}$ and family $(p_i)_{i \in I}$ where p_i denotes a k-place polynomial of $\mathcal{R}_i$, we can find some $M_k \in \mathbb{N}$ and another family of polynomials $(q_i)_{i \in I}$, such that for each $i \in I$ the degree of q_i is at most M_k and the function induced by q_i coincides with that induced by p_i.

We note that this number M_k also depends on the family $(p_i)_{i\in I}$ we start with.

Proof : To prove this statement we use induction on k. For $k = 1$ our assertion is an immediate consequence of *2*. So let $k \in \mathbb{N}, k > 1$ and suppose our assertion is true for $k-1$. We start with a family of k-place polynomials $(p_i)_{i\in I}$. Let

$$p_i(x_1,\ldots,x_k) = a_{i,n}(x_1,\ldots,x_{k-1})x_k^n + \ldots + a_{i,0}(x_1,\ldots,x_{k-1}).$$

In this representation n depends on i. From condition *2* we get a cofinite set $I' \subseteq I$ and a number M_1 such that we find polynomials $h_{i,n}$ for each $i \in I'$ and $n \in \mathbb{N}$ of degree at most M_1 that represent as functions the monomials x^n, that is $h_{i,n}(x) = x^n$ for each $x \in \mathcal{R}_i$. So we may write (as functions)

$$p_i(x_1,\ldots,x_k) = a_{i,n}(x_1,\ldots,x_{k-1})h_{i,n}(x_k) + \ldots + a_{i,0}(x_1,\ldots,x_{k-1}) =$$

$$= b_{i,M_1}(x_1,\ldots,x_{k-1})x_k^{M_1} + \ldots + b_{i,0}(x_1,\ldots,x_{k-1})$$

for each $i \in I'$. As a consequence of the inductive hypothesis we find polynomials $c_{i,l}(x_1,\ldots,x_{k-1})$ for each $l \in \{0,\ldots,M_1\}$ and $i \in I$ which induce the same functions as $b_{i,l}(x_1,\ldots,x_{k-1})$ and have bounded degree, i.e. there are numbers $m_l \in \mathbb{N}$ such that

$$\deg\left(c_{i,l}(x_1,\ldots,x_{k-1})\right) \leq m_l \text{ for } i \in I', l = 0,\ldots,M_1$$

holds. So we get

$$p_i(x_1,\ldots,x_k) = c_{i,n}(x_1,\ldots,x_{k-1})x_k^{M_1} + \ldots + c_{i,0}(x_1,\ldots,x_{k-1}) \qquad (1)$$

as functions for $i \in I'$. The degree of the polynomial q_i on the right hand side of equation *(1)* is at most $\max(m_1,\ldots,m_{M_1}) + M_1$. For $i \notin I'$ we take $q_i = p_i$. Obviously the family $(q_i)_{i\in I}$ satisfies the desired properties.

□

Lemma 2 *For each $i \in I$ let f_i be a compatible k-place function of $\mathcal{R}_i$. Then the product function f defined on $\mathcal{R}$ by*

$$f((a_i^1)_{i\in I},\ldots,(a_i^k)_{i\in I}) = (f(a_i^1,\ldots,a_i^k)_i)_{i\in I}$$

is again compatible.

Proof : It is sufficient to show that f is compatible with every principle congruence. This follows from the fact that in a direct product of rings with unity each principle ideal is a product ideal.

□

Now we are in position to prove Theorem 1.

Proof : To prove the 'only if' part of the theorem, suppose $\mathcal{R}$ is k-affine complete. Let f be any compatible k-place function on $\mathcal{R}_{i_0}$ for some $i_0 \in I$ and consider the function $\hat{f}$ on $\mathcal{R}$ defined as $\hat{f}\left((a_i^1)_{i\in I}, \ldots, (a_i^k)_{i\in I}\right) = (b_i)_{i\in I}$ with

$$b_i = \begin{cases} f(a_{i_0}^1, \ldots, a_{i_0}^k) & \text{for } i = i_0, \\ 0 & \text{for } i \neq i_0. \end{cases}$$

Due to Lemma 2 $\hat{f}$ is compatible with every ideal of $\mathcal{R}$. We find then a k-place polynomial $\hat{p}$ of $\mathcal{R}$, which realizes $\hat{f}$. Taking the i_0-th projections of each coefficient of $\hat{p}$ we obtain a polynomial p_{i_0} of $\mathcal{R}_{i_0}$ which clearly realizes f. That shows that $\mathcal{R}_{i_0}$ is k-affine complete.

Now suppose on the contrary, that condition *2* fails to be true. We can then find a sequence $i_l \in I$ ($l \in \mathbb{N}$) of distinct indices, and polynomial functions f_{i_l} of $\mathcal{R}_{i_l}$ which cannot be written as polynomials with degree at most l. We consider the function $\hat{f}$ of $\mathcal{R}$ which is defined as $\hat{f}\left((a_i)_{i\in I}\right) = (b_i)_{i\in I}$ where

$$b_i = \begin{cases} f_{i_l}(a_{i_l}) & \text{if } i = i_l \text{ and } l \in \mathbb{N}, \\ 0 & \text{else.} \end{cases}$$

This function again is compatible on $\mathcal{R}$, and therefore must be represented by some polynomial $\hat{p}$ of $\mathcal{R}$. If the degree of $\hat{p}$ equals n, every function f_{i_l} is realized as a polynomial of $\mathcal{R}_{i_l}$ of degree at most n, again by taking the i_l-th projection of each coefficient. This is a contradiction to our choice of f_{i_l} for $l > n$.

It remains to show the sufficiency of our conditions *1* and *2*. Let f be a compatible k-place function on $\mathcal{R}$, then f decomposes into a direct product of functions f_i which are compatible in $\mathcal{R}_i$. This means

$$f\left((a_i^1)_{i\in I}, \ldots, (a_i^k)_{i\in I}\right) = (b_i)_{i\in I} \text{ , with}$$

$$b_i = f_i(a_i^1, \ldots, a_i^k).$$

From assumption *1* we find a polynomial p_i for each $i \in I$ representing the function f_i. By Lemma 1 we obtain other polynomials q_i, which induce the same functions as p_i, but have bounded degree. We can therefore define a polynomial q as 'product polynomial', i.e. we just take the families of corresponding coefficients of the polynomials q_i as coefficients of q. The function f is clearly realized by q.

□

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