Inductive topologies in universal algebras

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1 Introduction

Treating topological algebra it seems natural to ask whether on a given algebraic structure there exists a topology such that all fundamental operations are continuous or not. Some results about this problem can be found in [4], [5], [7], [8], [9], [10], [11] or [12]. In many places topologies which are uniformizable, possess a countable basis of neighborhoods and are such that all fundamental operations are uniformly continuous are of importance. As examples I-adic topologies on rings, where I is an ideal of a ring may serve, see e.g. [3] or [6]. Also in the theory of topological groups one often deals with such topologies.

In this paper we consider a universal algebra $\langle A, F\rangle$. Denote with $\mathcal{U}$ the set of all uniformities on A, which satisfy the first axiom of countability and are such that the family of translations of A is uniformly equicontinuous. Our aim is to describe $\mathcal{U}$ in an 'algebraic' way.

J.O.Kiltinen defined in [8] so called inductive ring topologies to investigate the existence of non trivial topologies of the above kind on commutative rings with unity. We will adapt the idea of this inductive process to the case of a universal algebra. This provides a method to investigate the above mentioned question in a general setting.

In §2 we construct certain uniformities on the algebra of terms $\mathcal{T}_F(X)$. This construction could be used in a more general topological context. We make use of evaluation homomorphisms to obtain uniformities in $\mathcal{U}$. All uniformities in $\mathcal{U}$ can be constructed in this way, as is shown in §3. Finally, in §4 we modify our method to describe all so called equivalence uniform structures on A holding the above properties.

From the viewpoint of general topology our results can be interpreted as a description of all metrizable uniformities with a prescribed uniformly equicontinuous family of functions.

We use the notation of [1] and [2] for topological and algebraic notions, respectively. Throughout the paper we assume that all constant functions are fundamental operations, which is no loss of generality.

2 Construction of uniformities on the algebra of terms

Let $\langle A, F\rangle$ be a universal algebra on the set A of type F. Assume that all constant functions are fundamental operations. For $f \in F$ let $\sigma(f)$ be the arity of f. We denote with $\mathcal{T}_F(X)$ the algebra of terms of type F over the set $X = \{X_n | n = 1, 2, 3, \ldots\}$ of indeterminates.

Let $f \in F, 1 \leq i \leq \sigma(f)$ and let $b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_{\sigma(f)}$ be elements of A. Then the function

$$t(x) = f(b_1, \dots, b_{i-1}, x, b_{i+1}, \dots, b_{\sigma(f)})$$

is called a translation. We denote the set of all translations with $T(A)$.

A translation can be understood either as function on $\mathcal{T}_F(X)$ or as function on A itself. It will be clear from the context on which domain a considered translation operates.

Definition 1 (First inductive process) *Let $(M_k)_{k \in \mathbf{N}}$ be a sequence of relations on $\mathcal{T}_F(X)$ or A. Define for $k, n \in \mathbb{N}$, $k \leq n$ sets W_k^n as*

$$\begin{aligned} &1.\ W_k^n = M_k \text{ if } k = n, \text{ and} \\ &2.\ W_k^n = \left\{ (t(e), t(e')) \in \mathcal{T}_F(X)^2 \,\middle|\, (e, e') \in W_{k+1}^n, t \in T(A) \right\} \cup W_k^{n-1} \text{ if } k < n. \end{aligned} \tag{1}$$

Further denote with W_k the union $W_k = \bigcup\limits_{n \geq k} W_k^n$.

In the following denote with $\circ$ the relational product.

Definition 2 (Second inductive process) *Let $(M_k)_{k \in \mathbf{N}}$ be a sequence of relations on $\mathcal{T}_F(X)$ or A. Define for $k, n \in \mathbb{N}$, $k \leq n$ sets V_k^n as*

$$\begin{aligned} &1.\ V_k^n = M_k \text{ if } k = n, \text{ and} \\ &2.\ V_k^n = (V_{k+1}^n \circ V_{k+1}^n) \cup V_k^{n-1} \text{ if } k < n. \end{aligned} \tag{2}$$

Further denote with V_k the union $V_k = \bigcup\limits_{n \geq k} V_k^n$.

We have defined relations on $\mathcal{T}_F(X)$ (or on A) by an inductive procedure, using the following arrangement of the sets W_k^n (V_k^n, respectively).

$$\begin{array}{cccccccccc}
W_1^1 & & W_1^2 & & W_1^3 & & W_1^4 & & & \\
& \ddots & \uparrow & \ddots & \uparrow & \ddots & \uparrow & & & \cdots\cdots \\
& & W_2^2 & & W_2^3 & & W_2^4 & & & \\
& & & \ddots & \uparrow & \ddots & \uparrow & \ddots & \uparrow & \\
& & & & W_3^3 & & W_3^4 & & W_3^5 & \\
& & & & & \ddots & \uparrow & \ddots & \uparrow & \cdot\cdot \\
& & & & & & W_4^4 & & W_4^5 & \\
& & & & & & & \ddots & \uparrow & \\
& & & & & & & & W_5^5 & \\
& & & & & & & & & \ddots
\end{array} \tag{3}$$

To prove assertions on such inductively defined sets we will often use 'induction on the diagonals' of scheme (3). This means that we let $n = k + l$ $(l \geq 0)$ and use induction on l.

Now we investigate some properties of these inductive constructions.

Lemma 1 *Let $(M_k)_{k\in\mathbf{N}}$ be a sequence of relations on $\mathcal{T}_F(X)$ or A. Suppose $M_{k+1} \subseteq M_k$ holds for each $k \in \mathbb{N}$, and consider the relations W_k^n and V_k^n given by (1) and (2), respectively. Then $W_{k+1}^{n+1} \subseteq W_k^n$ and $V_{k+1}^{n+1} \subseteq V_k^n$ holds for $k, n \in \mathbb{N}$, $k \leq n$. Thus we also have $W_{k+1} \subseteq W_k$ and $V_{k+1} \subseteq V_k$ for each $k \in \mathbb{N}$.*

Proof : Let $n = k + l$ and use induction on l. The assumption of the lemma settles the case $l = 0$. For $l > 0$ we have from the inductive hypothesis

$$W_{(k+1)+1}^{(k+1)+(l-1)+1} \subseteq W_{(k+1)}^{(k+1)+(l-1)} \subseteq W_{(k+1)-1}^{(k+1)+(l-1)-1}.$$

The assertion now follows from the definition (1) of W_k^n. For the relations V_k^n the proof is analogous.

□

Let Δ always be the identity relation on the considered set. Denote with $\mathcal{R}$ the set of all sequences of relations $(M_k)_{k\in\mathbf{N}}$ on $\mathcal{T}_F(X)$, which satisfy

$$\Delta \subseteq M_k,\ M_k^{-1} = M_k \text{ and } M_{k+1} \subseteq M_k \text{ for } k \in \mathbb{N}. \tag{4}$$

Lemma 2 *Let $(M_k)_{k\in\mathbf{N}} \in \mathcal{R}$. Then the sequence $(W_k)_{k\in\mathbf{N}}$ given by Definition 1 is an element of $\mathcal{R}$, and additionally satisfies*

$$t(W_{k+1}) \subseteq W_k \textit{ for } t \in T(A) \textit{ and } k \in \mathbb{N}.$$

Proof : The proof follows immediately from our definitions and Lemma 1.

□

Lemma 3 *Let $(M_k)_{k\in\mathbf{N}} \in \mathcal{R}$ satisfy*

$$t(M_{k+1}) \subseteq M_k \textit{ for } t \in T(A) \textit{ and } k \in \mathbb{N}. \tag{5}$$

Then the sequence $(V_k)_{k\in\mathbf{N}}$ given by Definition 2 is element of $\mathcal{R}$, satisfies (5) and

$$V_{k+1} \circ V_{k+1} \subseteq V_k \textit{ for } k \in \mathbb{N}. \tag{6}$$

Proof : It is obvious that $(V_k)_{k\in\mathbf{N}}$ is element of $\mathcal{R}$, whereas (6) is a consequence of our definitions and the fact that the rows of the corresponding scheme of form (3) are ascending chains. To proof (5) first note that it is enough to show

$$t\left(V_{k+1}^n\right) \subseteq V_k^n \text{ for } t \in T(A) \text{ and } k \leq n.$$

Now we use induction on the diagonals of (3). So let $(p,q) \in V^n_{k+1}$, then either $(p,q) \in V^{n-1}_{k+1}$ or $(p,q) = (p,r) \circ (r,q)$ with $(p,r),(r,q) \in V^n_{k+2}$. Applying the inductive hypotheses we find in both cases $(t(p),t(q)) \in V^n_k$.

□

Let us recall the notion of uniform equicontinuity: If $\mathcal{N}$ is a uniform structure on a set A and T is a family of functions on A, T is called uniformly equicontinuous, if for each neighbourhood $N \in \mathcal{N}$ there is a common neighbourhood $M \in \mathcal{N}$, such that $t(M) \subseteq N$ holds for each $t \in T$.

In the sequel we denote with $\mathcal{U}_t$ the set of all uniformities on $\mathcal{T}_F(X)$, which satisfy the first axiom of countability, and are such that the family $T(A)$ is uniformly equicontinuous.

Combining the above results we obtain the following theorem.

Theorem 1 *Let $(M_k)_{k\in\mathbf{N}} \in \mathcal{R}$. Then the sequence $(V_k)_{k\in\mathbf{N}}$ constructed by applying the first inductive process to $(M_k)_{k\in\mathbf{N}}$, and applying the second inductive process to the resulting sequence $(W_k)_{k\in\mathbf{N}}$, forms a basis of a uniformity $\mathcal{N}^{ind}((M_k)_{k\in\mathbf{N}})$ on $\mathcal{T}_F(X)$. Furthermore $\mathcal{N}^{ind}((M_k)_{k\in\mathbf{N}})$ is element of $\mathcal{U}_t$.*

Remark 1 From $T(A)$ being uniformly equicontinuous, we know that all fundamental operations are uniformly continuous. If the type F is bounded, i.e. for some $N \in \mathbb{N}$ there is no $f \in F$ with $\sigma(f) > N$, the family of fundamental operations is uniformly equicontinuous. If we furthermore assume that we have only finitely many fundamental operations of type $\sigma(f) \geq 1$, then $T(A)$ being uniformly equicontinuous is even equivalent to all fundamental operations being uniformly continuous.

Remark 2 If for each $k \in \mathbb{N}$ we have some $n(k) \in \mathbb{N}$ with $M_{n(k)} \circ M_{n(k)} \subseteq M_k$, then the sequence $(M_k)_{k\in\mathbf{N}}$ forms the basis of some uniformity $\mathcal{N}$ on $\mathcal{T}_F(X)$. In this case $\mathcal{N}^{ind}((M_k)_{k\in\mathbf{N}})$ is coarser than $\mathcal{N}$.

In the next section we will need one more result concerning the first and second inductive process.

Lemma 4 *Let $(M_k)_{k\in\mathbf{N}}$ and $(N_k)_{k\in\mathbf{N}}$ be sequences of relations on $\mathcal{T}_F(X)$ or A with $M_k \subseteq N_k$ for $k \in \mathbb{N}$. Further let $(W_k)_{k\in\mathbf{N}}$ ($(V_k)_{k\in\mathbf{N}}$, respectively) be the sequence obtained by applying the first (second, respectively) inductive process to $(M_k)_{k\in\mathbf{N}}$. If $(N_k)_{k\in\mathbf{N}}$ satisfies*

$$t(N_{k+1}) \subseteq N_k \text{ for } t \in T(A) \text{ and } k \in \mathbb{N},$$

then $W_k \subseteq N_k$ for $k \in \mathbb{N}$. If $(N_k)_{k\in\mathbf{N}}$ satisfies

$$N_{k+1} \circ N_{k+1} \subseteq N_k \text{ for } k \in \mathbb{N},$$

then $V_k \subseteq N_k$ for $k \in \mathbb{N}$.

Proof : Using induction on the diagonals the proof is straightforward.

□

3 Inductive uniformities on A

We are going to use so called evaluation homomorphisms to transport uniformities of $\mathcal{T}_F(X)$ to A.

Let us recall: For a sequence $(a_i)_{i\in\mathbf{N}}$ of elements of A the mapping

$$\Phi_{(a_i)} : \mathcal{T}_F(X) \to A,$$

which maps X_i to a_i for each $i \in \mathbb{N}$ and acts identically on A is called an evaluation homomorphism.

Let $\kappa : A^{\mathbb{N}} \to \mathcal{R}$ be a mapping, assigning to each sequence $(a_i)_{i\in\mathbf{N}}$ of elements of A a sequence $\left(M_k^{(a_i)}\right)_{k\in\mathbb{N}}$, holding the properties (4). From each sequence $\left(M_k^{(a_i)}\right)_{k\in\mathbb{N}}$ we obtain a basis $\left(V_k^{(a_i)}\right)_{k\in\mathbb{N}}$ of $\mathcal{N}^{ind}\left(\left(M_k^{(a_i)}\right)_{k\in\mathbb{N}}\right)$ as in Theorem 1.

Definition 3 *Let κ be a mapping, $\kappa : A^{\mathbb{N}} \to \mathcal{R}$. For $(a_i)_{i\in\mathbf{N}} \in A^{\mathbb{N}}$ and $k \in \mathbb{N}$ let*

$$V_k{'}^{(a_i)} = \Phi_{(a_i)}\left(V_k^{(a_i)}\right),$$

and denote with $V_k{'}^{\kappa}$ the union

$$V_k{'}^{\kappa} = \bigcup_{(a_i)\in A^{\mathbb{N}}} V_k{'}^{(a_i)}.$$

Applying the second inductive process to the sequence $(V_k{'}^{\kappa})_{k\in\mathbb{N}}$ we obtain a sequence $(U_k^{\kappa})_{k\in\mathbb{N}}$ of relations on A.

The sequence $(U_k^{\kappa})_{k\in\mathbb{N}}$ will give a uniformity on A holding the desired properties.

Lemma 5 *The sequence $(V_k{'}^{\kappa})_{k\in\mathbb{N}}$ possesses the properties (4), and*

$$t\left(V'^{\kappa}_{k+1}\right) \subseteq V_k{'}^{\kappa} \text{ for } t \in T(A) \text{ and } k \in \mathbb{N}. \tag{7}$$

Proof : Recall that all constant functions are fundamental operations, and note that in our assertions only unary functions occur. Thus it is enough to show the appropriate properties for the sets $V_k{'}^{(a_i)}$. For these sets (4) is obvious. As $\Phi_{(a_i)}$ is a homomorphism (7) holds.

□

Let now $\mathcal{N} \in \mathcal{U}$, and let $(N_k)_{k\in\mathbf{N}}$ be a basis of $\mathcal{N}$ holding the properties (4), $N_{k+1} \circ N_{k+1} \subseteq N_k$ and $t(N_{k+1}) \subseteq N_k$ for $t \in T(A)$ and $k \in \mathbb{N}$.

Definition 4 *For a sequence $(a_i)_{i\in\mathbf{N}}$ of elements of A and $k \in \mathbb{N}$ denote with $M_k^{(a_i)}$ the relation*

$$M_k^{(a_i)} = \left\{(p,q) \in \mathcal{T}_F(X)^2 \middle| \left(\Phi_{(a_i)}(p), \Phi_{(a_i)}(q)\right) \in N_k\right\} \tag{8}$$

on $\mathcal{T}_F(X)$.

Obviously $(M_k^{(a_i)})_{k\in\mathbb{N}} \in \mathcal{R}$. Thus we may define $\kappa_{\mathcal{N}} : A^{\mathbb{N}} \to \mathcal{R}$ as $\kappa_{\mathcal{N}}\left((a_i)_{i\in\mathbf{N}}\right) = (M_k^{(a_i)})_{k\in\mathbb{N}}$.

Lemma 6 *For each $k \in \mathbb{N}$ we have $N_k \subseteq {V_k'}^{\kappa_{\mathcal{N}}}$*

Proof : For any pair $(a,b) \in N_k$ consider the sequence $(a_i)_{i\in\mathbf{N}} = (a,b,b,b,\ldots)$. We have $(X_1,X_2) \in M_k^{(a_i)}$, and therefore $(X_1,X_2) \in V_k^{(a_i)}$. Thus $(a,b) = \Phi_{(a_i)}((X_1,X_2)) \in {V_k'}^{(a_i)} \subseteq {V_k'}^{\kappa_{\mathcal{N}}}$.

□

Theorem 2 *Let κ be a mapping, $\kappa : A^{\mathbb{N}} \to \mathcal{R}$. Then the sequence $(U_k^{\kappa})_{k\in\mathbb{N}}$ given in Definition 3 forms a basis of a uniformity $\mathcal{N}_{ind}(\kappa)$ on A. We have $\mathcal{N}_{ind}(\kappa) \in \mathcal{U}$. Conversly let $\mathcal{N} \in \mathcal{U}$, then $\mathcal{N} = \mathcal{N}_{ind}(\kappa_{\mathcal{N}})$.*

Proof : From Lemma 2 and Lemma 3 we get that $(U_k^{\kappa})_{k\in\mathbb{N}}$ is a basis of some uniformity $\mathcal{N}_{ind}(\kappa)$ on A. That $T(A)$ is uniformly equicontinuous with respect to $\mathcal{N}_{ind}(\kappa)$ is proved as (5) of Lemma 3.

To prove the converse statement note $N_k \subseteq {V_k'}^{\kappa_{\mathcal{N}}} \subseteq U_k^{\kappa_{\mathcal{N}}}$. To show the other inclusion, consider the relations $V_k^{(a_i)}$. From Lemma 4 we get $V_k^{(a_i)} \subseteq M_k^{(a_i)}$, and (8) shows ${V_k'}^{(a_i)} = \Phi_{(a_i)}\left(V_k^{(a_i)}\right) \subseteq N_k$. Thus ${V_k'}^{\kappa} \subseteq N_k$, and applying Lemma 4 finishes the proof.

□

We will call $\mathcal{N}_{ind}(\kappa)$ the inductive uniformity on A induced by κ.

Remark 3 Note that we could also use a partial mapping κ', defined on some subset $D(\kappa') \subseteq A^{\mathbb{N}}$, for the construction of the inductive uniformity. But $\mathcal{N}_{ind}(\kappa')$ coincides with the inductive uniformity induced by the mapping κ, which acts as κ' on $\mathrm{D}(\kappa')$, and assigns $(\Delta)_{k\in\mathbb{N}}$ to each sequence $(a_i)_{i\in\mathbf{N}} \notin D(\kappa')$.

Remark 4 For a given mapping κ we have uniformities $\mathcal{N}^{ind}\left(\left(M_k^{(a_i)}\right)_{k\in\mathbb{N}}\right)$ and $\mathcal{N}_{ind}(\kappa)$. The evaluation homomorphisms

$$\Phi_{(a_i)} : \left\langle \mathcal{T}_F(X), \mathcal{N}^{ind}\left(\left(M_k^{(a_i)}\right)_{k\in\mathbb{N}}\right)\right\rangle \to \langle A, \mathcal{N}_{ind}(\kappa)\rangle$$

are uniformly continuous. Thus $\mathcal{N}_{ind}(\kappa)$ is coarser than the final uniformity on A with respect to the family $\{\Phi_{(a_i)} | (a_i)_{i\in\mathbf{N}} \in A^{\mathbb{N}}\}$.

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4 A modified construction for the description of equivalence uniform structures

Definition 5 *A uniform structure $\mathcal{N}$ on a set A is called an equivalence uniform structure if it possesses a basis of equivalence relations.*

A sequence $(M_k)_{k\in\mathbb{N}}$ of equivalence relations on $\mathcal{T}_F(X)$ satisfying $M_{k+1} \subseteq M_k$ for each $k \in \mathbb{N}$ obviously holds (4), that is $(M_k)_{k\in\mathbb{N}} \in \mathcal{R}$. Applying the first inductive process we obtain relations W_k for $k \in \mathbb{N}$. In general these will not be equivalence relations, but, according to Lemma 2 they are at least reflexive and symmetric. For $k \in \mathbb{N}$ let V_k be the transitive cover of W_k:

$$V_k = \bigcup_{n\in\mathbb{N}} (W_k)^n .$$

Then V_k is an equivalence relation.

Lemma 7 *If $t\,(W_{k+1}) \subseteq W_k$ for $t \in T(A)$ and $k \in \mathbb{N}$ holds, the transtive covers V_k also satisfy*

$$t\,(V_{k+1}) \subseteq V_k \text{ for } t \in T(A) \text{ and } k \in \mathbb{N}.$$

Proof : It is enough to show $t\,((W_{k+1})^n) \subseteq V_k$ for $t \in T(A)$ and $k \in \mathbb{N}$. To prove this let $(p,q) \in (W_{k+1})^n$, i.e. let

$$(p,q) = (p,r_1) \circ (r_1,r_2) \circ \ldots \circ (r_{n-1},q),$$

where each pair is an element of W_{k+1}. Then we have

$$(t(p),t(q)) = (t(p),t(r_1)) \circ (t(r_1),t(r_2)) \circ \ldots \circ (t(r_{n-1}),t(q))\,,$$

where each pair is an element of W_k. Thus $(t(p),t(q)) \in (W_k)^n \subseteq V_k$.

□

Lemma 7 implies that the sequence $(V_k)_{k\in\mathbb{N}}$ forms the basis of an equivalence uniform structure $\mathcal{N}^{eq}((M_k)_{k\in\mathbb{N}}) \in \mathcal{U}_t$. Let $\mathcal{R}^{eq}$ be the set of all sequences $(M_k)_{k\in\mathbb{N}} \in \mathcal{R}$, such that each M_k is an equivalence relation. If κ is a mapping $\kappa : A^{\mathbb{N}} \to \mathcal{R}^{eq}$ we construct the relations $V_k^{(a_i)}$ from $\kappa((a_i)_{i\in\mathbb{N}})$ as above. Again define

$$V_k'^{\kappa} = \bigcup_{(a_i)\in A^{\mathbb{N}}} \Phi_{(a_i)}\left(V_k'^{(a_i)}\right).$$

Finally let U_k^{κ} be the transitiv cover of $V_k'^{\kappa}$.

The sequence $(U_k^{\kappa})_{k\in\mathbb{N}}$ again will give a uniformity on A holding the desired properties. To establish the analogous result to Theorem 2 let $\mathcal{N} \in \mathcal{U}$ be an equivalence uniform structure, and let $(N_k)_{k\in\mathbb{N}}$ be a basis of $\mathcal{N}$ holding the properties (4), $N_k \circ N_k \subseteq N_k$ and $t(N_{k+1}) \subseteq N_k$ for $t \in T(A)$ and $k \in \mathbb{N}$.

As in (8) let for any sequence $(a_i)_{i\in\mathbb{N}}$ of elements of A

$$M_k^{(a_i)} = \left\{(p,q) \in \mathcal{T}_F(X)^2 \middle| \left(\Phi_{(a_i)}(p), \Phi_{(a_i)}(q)\right) \in N_k\right\}.$$

Obviously $M_k^{(a_i)}$ is an equivalence relation. Thus the mapping $\kappa_{\mathcal{N}}$ with $\kappa_{\mathcal{N}}\,((a_i)_{i\in\mathbb{N}}) = M_k^{(a_i)}$ maps $A^{\mathbb{N}}$ to $\mathcal{R}^{eq}$. Similar to Lemma 6 and Lemma 4, we find $B_k \subseteq U_k^{\kappa_{\mathcal{N}}}$ for each $k \in \mathbb{N}$, and

Lemma 8 *Let $(M_k)_{k\in\mathbb{N}}$ and $(N_k)_{k\in\mathbb{N}}$ be sequences of relations on $\mathcal{T}_F(X)$, which satisfy $M_k \subseteq N_k$ for each $k \in \mathbb{N}$. Further let each N_k be an equivalence relation and let $N_{k+1} \subseteq N_k$ for $k \in \mathbb{N}$. Then then transitive covers V_k of $(M_k)_{k\in\mathbb{N}}$ also satisfy $V_k \subseteq N_k$ for $k \in \mathbb{N}$.*

Theorem 3 *Let κ be a mapping, $\kappa : A^{\mathbb{N}} \to \mathcal{R}^{eq}$. Then the sequence $(U_k^\kappa)_{k\in\mathbb{N}}$ constructed above forms a basis of an equivalence uniformity $\mathcal{N}_{eq}(\kappa)$ on A. We have $\mathcal{N}_{eq}(\kappa) \in \mathcal{U}$. Conversly let $\mathcal{N} \in \mathcal{U}$ be an equivalence uniformity, then $\mathcal{N} = \mathcal{N}_{ind}(\kappa_{\mathcal{N}})$.*

Proof : We show that $T(A)$ is uniformly equicontinuous. Since we have $\Delta \subseteq \Phi_{(a_i)}\left(V_k^{(a_i)}\right) = {V_k}'^{(a_i)}$, and $\Phi_{(a_i)}$ is a homomorphism, Lemma 7 implies that all translations satisfy

$$f(b_1, \ldots, b_{i-1}, {V_{k+1}}'^{(a_i)}, b_{i+1}, \ldots, b_\sigma) \subseteq {V_k}'^{(a_i)}.$$

Therefore the same holds for the relations ${V_k}'^{\kappa}$ and U_k^κ. The remaining assertions prove similar to Theorem 2.

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