The order problem for canonical systems

FWF Project Proposal, Joint Project (RFBR-FWF)

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Abstract

Canonical systems are differential equations of a specific form which frequently appear in natural sciences. For example in Hamiltonian mechanics, where they model the motion of a particle under the influence of a time-dependent potential, or as generalizations of Sturm-Liouville problems, e.g. in the study of a vibrating string with non-homogeneous mass distribution. A canonical system is given by a locally integrable function taking positive semidefinite real matrices as values, its Hamiltonian.

A chain of Hilbert spaces of entire functions is closely connected with a canonical system, known as the associated chain of de Branges spaces. The theory of such spaces plays a prominent role in the investigation of the spectrum of canonical systems. For example it is the basis for the Inverse Spectral Theorem which states that a Hamiltonian is essentially uniquely determined by its spectral function.

The elements of the de Branges spaces associated with a canonical system are entire functions, and hence it is natural to pose the question how the growth (in particular order and type) of these functions is related to the Hamiltonian. With exception of some particular cases, the only known general result in this direction is that all functions are of order at most one and finite exponential type. In fact, the exponential type can be computed from the Hamiltonian by means of a simple formula.

This project aims at establishing general quantitative relations between the Hamiltonian on the one hand, and growth referring to small exponential orders or to proximate orders (growth functions) on the other. Moreover, we will investigate the consequences of such relations for the spectral theory of canonical systems.

The basic problem breaks into three tasks: Estimates for growth in terms of the Hamiltonian (direct problem), characterisation of those Hamiltonians whose associated de Branges spaces possess or do not exceed a prescribed growth (inverse problem), estimating or determining the growth in terms of the spectral measure associated with the canonical system. Answers to these basic questions will lead, in particular, to statements about the asymptotic distribution of eigenvalues of a system in case its spectrum is discrete. Several classes of canonical systems are of particular interest. For example, Jacobi matrices and Schrödinger operators can be considered as canonical systems. The first are closely related to the power moment problem, the second are basic objects in quantum mechanics. The general results shall be applied to these situations.

Methods of various areas of analysis will be employed to tackle the problem. These include: the theory of differential operators (Volterra integral equations, variational techniques, Levinson-type theorems), complex analysis (growth and zero-distribution, singular integrals, subharmonic functions), and functional analysis (reproducing kernel Hilbert spaces, Krein’s theory of entire operators, resolvent matrices).
Classification

This project belongs to the scientific area of mathematical analysis. It is related to mathematical physics, in particular to quantum mechanics, and to probability theory. In the classification categories of Statistik Austria the topics that correspond to the project are: Analysis (1103, 70%), Mathematical physics (1225, 20%), and Probability theory (1118, 10%).
1 Introduction

The theory of canonical systems was founded in works of Stieltjes, Weyl, Markov, Krein, and de Branges. Modern studies in the area proceed in several directions of which we would like to mention direct and inverse spectral relations, systems with inner singularities, or matrix problems. Contemporary interest in canonical systems is related to physical problems (oscillations of loaded strings, propagation of shallow water waves, inverse problems in reflection seismology, etc.), and problems in analysis and probability (inverse spectral problems for differential operators, moment problems, birth-death processes, etc.). As one concrete instance, let us mention the work of Remling [Rem02] where the traditional approach to the inverse spectral problem for the Schrödinger equation based on Marchenko transformation operators is understood in terms of canonical systems.

The key result in the theory is the one-to-one correspondence between regular (singular) canonical systems and their monodromy matrices (their Weyl-coefficients, respectively). This result was discovered in the 1960’s by de Branges, cf. [Bra68]. In the setting of strings with a non-homogeneous mass distribution, it goes back to work of Krein in the early 1950’s. De Branges’ proof is based on his theory of Hilbert spaces of entire functions, and on associating with a canonical system a naturally parameterised chain of such spaces.

A long known formula computes the exponential type of a space (i.e., the maximal exponential type of one of its elements) of the associated chain from the canonical system. It states that the exponential type of the space corresponding to the parameter \( t \) is given as

\[
\int_0^t \sqrt{\det H(x)} \, dx
\]

where \( H \) is the Hamiltonian of the system, cf. [Bra61, Theorem X]. A typical application of this formula is to provide spectral asymptotics for the Dirac system on a finite interval.

However, for many important special classes of canonical systems (e.g., for those arising from Schrödinger equations, strings, or Jacobi matrices and moment problems), all spaces of the corresponding chain are known to be of minimal exponential type, and the Hamiltonians are known to have determinant zero. In such cases dealing with exponential type does not give any information, and it is natural to seek for formulas determining growth with respect to a finer scale: One may ask to compute the order of the spaces of the chain, which may be - and often is - strictly smaller than 1. Even more refined, one may study growth with respect to a growth function (i.e., a proximate order in the sense of Valiron), and ask for formulas computing the respective type (if it is finite).

During the past decades a wealth of examples of spaces with small growth, sometimes even of zero order, were exhibited. In contrast, very few general results were established. This makes a detailed study of growth properties on a general and abstract level a necessity. It is the objective of the presently proposed project to start and carry out such a program.
2 The main players: Hamiltonians, de Branges spaces, Weyl-coefficients

A Hamiltonian $H$ is a function defined on some (finite or infinite) interval $[0, L)$, which takes real and nonnegative $2 \times 2$-matrices as values, is locally integrable in $[0, L)$, and does not vanish on any set of positive measure. The canonical system with Hamiltonian $H$ is the differential equation

$$y'(x) = zJH(x)y(x), \quad x \in [0, L), \quad (1)$$

where $z$ is a complex parameter (the eigenvalue parameter) and $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

For a canonical system with Hamiltonian $H$ we denote by $W_H(z, x) := W_H(z)^T$, $x \in [0, L)$, the (transposed of the) fundamental solution of (1), i.e., the unique solution of the initial value problem

$$\frac{d}{dx}W(x, z)J = zW(x, z)H(x), \quad x \in [0, L), \quad W(0, z) = I.$$ 

In the spectral theory of the equation (1) two fundamentally different cases occur, depending on the growth of $H$ towards $L$: We say that $H$ is regular (or in Weyl’s limit circle case), if $\int_0^L \text{tr} H(x) \, dx < \infty$. Otherwise, we say that $H$ is singular (in Weyl’s limit point case).

Case $H$ regular: If $H$ is regular, the system (1) is solvable up to $L$, and we refer to $W_H(z) := W_H(L, z)$ as the monodromy matrix of $H$. This matrix function is entire and iJ-inner (meaning that the reproducing kernel $\frac{W(z)JW(w)^* - J}{z-w}$ is positive semidefinite). The operator $S$ associated with (1) by prescribing the boundary condition $y(0) = 0$ ($y = (y_1, y_2)^T$) is a closed simple and entire symmetry with defect index $(1, 1)$. The monodromy matrix $W_H$ can be used to describe all spectral functions of this operator by means of Krein’s resolvent formula.

Case $H$ singular: If $H$ is singular, the limit ($W_H(z, x) = (W_H(x, z))_{ij}$) exists for $z \in \mathbb{C} \setminus \mathbb{R}$ independently of the parameter $\tau \in \mathbb{R} \cup \{\infty\}$. This function is called the Weyl-coefficient of $H$. It is a Herglotz-function (meaning that it is analytic, satisfies $q_H(z) = \overline{q_H(z)}$, and has nonnegative imaginary part throughout the upper half-plane). We refer to the measure $\mu_H$ in the Herglotz-integral representation of $q_H$ as the Clark-measure of $H$. The above described operator $S$ is selfadjoint, and its spectral measure is $\mu_H$.

The key result in the theory is the Inverse Spectral Theorem of de Branges. It states that

1. The assignment “$H \mapsto W_H$” is a bijection of the set of all regular Hamiltonians (up to changes of scale) and the set of all entire $iJ$-inner matrix functions.

2. The assignment “$H \mapsto q_H$” is a bijection of the set of all singular Hamiltonians (up to changes of scale) and the set of all Herglotz-functions.
The proof of this result is based on de Branges’ theory of Hilbert spaces of entire functions.

A de Branges space is a Hilbert space generated by a reproducing kernel constructed from a Hermite-Biehler function $E$ (by this we mean that $E$ is entire and satisfies $|E(z)| \leq |E(\tau)|$ throughout the upper half-plane). These spaces can also be introduced axiomatically. The subclass of de Branges spaces which appear in the context of canonical systems consists of those which are invariant with respect to forming difference quotients.

Now the connection between canonical systems and de Branges spaces is made as follows. The direct part: Given $H$, for each $x \in [0, L)$ the fundamental solution $W_H(x, z)$ induces a de Branges space (invariant with respect to difference quotients). This family of spaces is naturally parameterized by $x$, and forms a chain where each space (with some exceptions related to so-called $H$-indivisible intervals) is isometrically contained in all spaces with larger parameter. If $H$ is regular, the largest space is the one induced by the monodromy matrix. If $H$ is singular, these spaces are isometrically contained in the space $L^2(\mu_H)$ and their union is dense in this space. Conversely, the inverse part: Given a de Branges space which is invariant under difference quotients, there exists a (essentially unique) regular Hamiltonian $H$, such that the given space is induced by the monodromy matrix of $H$. Given a Herglotz-function $q$ (with measure $\mu$), then there exists a (again essentially unique) singular Hamiltonian, such that its chain of de Branges spaces is contained isometrically in $L^2(\mu)$ and its union is dense.

Canonical systems appear in a variety of contexts. The following instances may be regarded as classical and commonly known. Despite this, explicit accounts appeared only rather recently in the literature.

**Canonical systems related with Hamburger moment problems**

A sequence $(a_n)_{n=0}^{\infty}$ of real numbers is called a (Hamburger-) moment sequence, if there exists a positive Borel measure $\mu$ on the real line, such that

$$a_n = \int_{\mathbb{R}} x^n \, d\mu(x).$$

By simple rescaling one may always restrict all considerations without loss of generality to probability measures.

Given a moment sequence $(a_n)_{n=0}^{\infty}$ one can construct a Hamiltonian. It consists of a sequence of indivisible intervals whose lengths and types are computed from the numbers $a_n$. Conversely, if $H$ is a Hamiltonian which consists of a sequence of indivisible intervals, then there exists a unique moment sequence, such that $H$ arises in the above way.

All classical notions from moment problem theory (e.g., determinacy/indeterminacy, orthogonal polynomials of the first and second kinds, three term recurrence relation, N-extremal solution) have their counterparts in the language of canonical systems (referring to the above mentioned list: singular/regular, fundamental solution, the canonical differential equation, spectral measure of canonical selfadjoint extension).
Details, in particular explicit formulae, can be found in [Kac99]. Due to this intimate connection, following Kac, we call a Hamiltonian of Hamburger type if it consists of a sequence of indivisible intervals.

**Canonical systems related with Krein strings**

A string $S[L, m]$ (in the sense of Krein) is given by its length $L \in (0, \infty]$ and its mass distribution function $m : [0, L] \rightarrow [0, \infty]$ (a nondecreasing function). The Krein-Feller operator $-D_{m}Dx$ associated with the string occurs when applying Fourier’s method to the partial differential equation which describes the vibration of the inhomogenous string with mass distribution $m$.

With a string $S[L, m]$ one can associate a canonical system, namely by specifying its Hamiltonian as

$$H_{S[L, m]}(x) := \begin{pmatrix} 1 & -m(x) \\ -m(x) & m(x)^2 \end{pmatrix}, \quad x \in [0, L].$$

The spectral function of the Krein-Feller operator (in other words, the principal Titchmarsh-Weyl coefficient of the string) is closely related with the Weyl-coefficient of this system (choosing a particular boundary condition at $L$ if $H_{S[L, m]}$ is regular).

Details about this connection, and other canonical systems related with strings, can be found in [KWW07].

**Canonical systems related with Schrödinger equations**

Let an integrable potential on an interval $[0, L]$ be given. Then we denote by $y_{1}$ and $y_{2}$ the solutions of the homogenous equation $-\frac{d^2}{dx^2} + V = 0$ with initial values

$$y_{1}(0) = 1, \quad y'_{1}(0) = 0, \quad y_{1}(0) = 0, \quad y'_{1}(0) = 1,$$

and assign to $V$ the Hamiltonian

$$H_{V}(x) := \begin{pmatrix} y_{1}(x)^2 & y_{1}(x)y_{2}(x) \\ y_{1}(x)y_{2}(x) & y_{2}(x)^2 \end{pmatrix}, \quad x \in [0, L].$$

For sufficiently smooth Hamiltonians this construction can be reversed.

The canonical system with Hamiltonian $H_{V}$ is closely related to the Schrödinger equation with potential $V$. In fact, if a function $y(x, z)$ solves the equation $-\frac{d^2}{dx^2}y(x, z) + V(x)y(x, z) = zy(x, z)$, then the function

$$u(x, z) := \left( \begin{array}{c} y_{1}(x) \\ y'_{1}(x) \\ y_{2}(x) \\ y'_{2}(x) \end{array} \right)^{-1} \left( \begin{array}{c} y(x, z) \\ y'(x, z) \end{array} \right)$$

solves the canonical system.
Up to this simple transformation, the spaces of the chain corresponding to $H_V$ are generated by the function $E(x, z) := y(x, z) + iy'(x, z)$ where $y(x, z)$ and the solution of $-\frac{d^2}{dx^2} y(x, z) + V(x)y(x, z) = zy(x, z)$ with $y(0, z) = 1, y'(0, z) = 0$. As a set, the space generated by $E(x, z)$ is given as the space of all cosine transforms with parameter $\sqrt{z}$ of square integrable functions on $[0, L]$. Its inner product can be computed via a certain integral operator.

Details about this connection, and some interesting features, can be found in [Rem02].

3 Proposed research and preliminary work performed

3.1 Presently known results and examples

In this section we collect some results and examples from the literature which are related to growth properties. These facts serve as source, motivation, and model, for our conjectures and the expected results of this project. The provided selection is by far not exhaustive, but hopefully illustrative.

To start with, the classical result of Krein and de Branges that exponential growth of spaces can be computed via the determinant of the Hamiltonian.

**Theorem 1** ([Kre51], [Bra61]). Let $H$ be a regular Hamiltonian defined on the interval $[0, L]$, and let $W_H$ be its monodromy matrix. Then

$$et W_H = \int_0^L \sqrt{\det H(t)} \, dt.$$  \hspace{1cm} (2)

Consider a Hamiltonian whose determinant vanishes identically. Then this result tells us that all spaces of the corresponding chain are of minimal exponential type, but does not give any further - finer - information on the growth of the spaces. It is a central observation that growth measurement on a more refined scale than “exponential type” is a complex matter, and that virtually any behaviour may occur. Subsequently we focus on situations where smaller growth occurs.

We mainly deal with Hamiltonians whose determinant vanishes identically. For such, let us introduce one notation: If $\det H = 0$, we can write $H(x) = \text{tr} H(x)\xi_{\phi_H(x)}^T\xi_{\phi_H(x)}$ where $\xi_\phi := (\cos \phi, \sin \phi)^T$. We call the function $\phi_H$ (which is determined up to integer multiples of $\pi$) the rotation angle of $H$.

The regular case.

First we give some theorems dealing with classes of systems, then more concrete examples where different growth behaviours are exhibited.

A (infinite) source of intriguing examples is found in the theory of power moment problems. For instance, passing from a Stieltjes- to the symmetrized Hamburger moment problem in a theorem due to Borichev and Sodin, we obtain the following fact.
Theorem 2 ([BS98]). Let $\lambda$ be a growth function with $\lambda(r) = o(r)$. Then there exists a regular Hamiltonian $H$ of Hamburger type, such that its monodromy matrix $W_H$ has finite and positive $\lambda$-type.

Classes of examples for which a natural bound of growth prevails arise in connection with second-order differential operators. We mention two results in this direction. The first one is related to Krein-Feller operators and the string equation, and goes back to Krein [Kre52], see also [KK68]. The below is a slightly stronger formulation taken from a recent work of Winkler and Woracek.

Theorem 3 ([WW12]). Let $H$ be a regular Hamiltonian whose determinant vanishes identically, and assume that its rotation angle $\varphi_H$ can be chose piecewise monotone and bounded. Then the monodromy matrix $W_H$ is of growth at most order $\frac{1}{2}$ finite type.

Let us remark that the upper bound given in this result need not be attained.

The second statement deals with canonical systems arising in connection with Schrödinger operators. It follows from work of Remling.

Theorem 4 ([Rem02]). Let $L \in (0, \infty)$ and let $V \in L^1(0, L)$. Then the monodromy matrix $W_{H_V}$ is of order $\frac{1}{2}$ finite and positive type.

Now we turn to concrete examples. These not only make even more evident that different growth behaviours indeed appear “in nature”, but also that it is necessary to investigate growth functions rather than just using the usual scale of order and type.

First, we exhibit instances where the monodromy matrix grows very slow. This is due to some work of Christiansen, Berg, and Pedersen.

Example 5 ([Chr04], [BP07]). Using some general results, e.g., the Riesz criterion or the Krein condition, it is often possible to conclude that a concrete moment sequence is indeterminate. Contrasting this, there are rather few examples known of indeterminate moment sequences for which the corresponding monodromy matrix can be computed explicitly. One class of such sequences are indeterminate moment problems within the $q$-Askey scheme. These include a variety of situations featured by classical orthogonal polynomials, e.g., $q$-Laguerre or Stieltjes-Wigert polynomials.

For indeterminate moment problems within the $q$-Askey scheme the corresponding monodromy matrices can be given explicitly in terms of special functions (mostly hypergeometric functions). It turns out that these monodromy matrices are of finite and positive $\lambda$-type with respect to the growth function $\lambda(r) := (\log r)^\alpha$, where the value of $\alpha$ may depend on the situation under consideration (but mostly is equal to 2). In particular, these functions are of zero order.

Next, we mention one example from probability theory where rational (positive) orders appear. This is taken from work of Berg and Valent, and of Gilewicz, Leopold, and Valent.
Example 6 ([BV94], [GLV05]). Birth-and-death processes are a particular kind of stationary Markov processes whose state space is the nonnegative integers. They model the time evolution of some population. The transition probabilities are a solution of the forward Kolmogoroff equation, and this yields a connection to the theory of orthogonal polynomials and in turn to canonical systems (for details see, e.g., [KM57]).

For several cases order and type of the corresponding monodromy matrices was computed. It depends on the asymptotic behaviour on a small time scale of the one-step transition probabilities. It turns out that for quartic processes the monodromy matrix is of order $\frac{1}{4}$ and for cubic processes of order $\frac{1}{3}$. The type with respect to the respective order is finite and positive, and can be calculated (in fact, as the value of some elliptic integral).

Fractal strings are studied in the literature, especially in connection with Markov processes. The following example, where irrational orders appear, is taken from work of Freiberg.

Example 7 ([Fre00], [Fre05]). Let $C$ be the classical Cantor set, and let $S_1$ and $S_2$ denote the functions $S_1(x) := \frac{1}{3}x$ and $S_2(x) := \frac{1}{3}x + \frac{2}{3}$ defined on the unit interval $[0, 1]$. Moreover, for $\rho \in (0, 1)$, let $\mu_\rho$ be the unique probability measure on $[0, 1]$ with

$$\mu_\rho(A) = \rho \mu_\rho(S_1^{-1}(A)) + (1 - \rho) \mu_\rho(S_2^{-1}(A))$$

for each Borel subset $A$ of $[0, 1]$. Then $\text{supp } \mu_\rho = C$.

The distribution function $m_\rho(x) := \mu_\rho([0, x])$, $x \in [0, 1]$, is the mass function of a regular string. Provided that $\log(\frac{\rho}{3})/\log(\frac{1 - \rho}{3})$ is irrational, order and type of the corresponding monodromy matrix can be computed: Denote by $n(r)$ the counting function of the spectrum of the corresponding Krein-Feller operator (i.e., the number of spectral points in the interval $[-r, r]$). Then the limit $\lim_{r \to \infty} \frac{n(r)}{r}$ exists and is finite and positive, where $\gamma \in (0, \frac{1}{2})$ is the unique solution of the equation

$$\left(\frac{\rho}{3}\right)^\gamma + \left(\frac{1 - \rho}{3}\right)^\gamma = 1.$$

Finally, an example where again growth different from classical order and type can be observed. This is taken from Kaltenbäck and Woracek [KW05].

Example 8 ([KW05]). Let $\xi$ denote the Riemann $\xi$-function, and set

$$E(z) := \xi\left(\frac{1}{2} + i\sqrt{z}\right).$$

Due to the functional equation $\xi(1 - s) = \xi(s)$, this formula defines an entire function. It is of Hermite-Biehler class, and hence generates a de Branges space. The spectrum of the operator in this space whose resolvents are given as difference quotients with the function $E$ encodes the zeros of the function $\zeta(z)$.

The de Branges space generated by $E$ contains the constant function 1, in particular gives rise to a canonical system. It turns out that the Hamiltonian of this system is of Hamburger type, and that its monodromy matrix is of finite and positive $\lambda$-type with $\lambda(r) := \sqrt{r} \log r$. 
The singular case.

The case of a singular Hamiltonian $H$ is much more complicated than the case of a regular one: Instead of the monodromy matrix $W_H$, the Weyl-coefficient $q_H$ and its Clark-measure $\mu_H$ has to be considered. This immediately brings up two problems.

1. If $q_H$ is meromorphic in the whole plane, it is natural to use the counting function of the poles of $q_H$ as a measure for growth (since, in the regular case, this corresponds to counting the poles of the quotient of two entries of the monodromy matrix, i.e., the zeros of one entry). However, generically, the Clark-measure $\mu_H$ will not be discrete.

2. By the very definition of $q_H$ a limiting procedure has to be undertaken in order to relate the chain of spaces with the Weyl-coefficient.

Concerning (1), at present, there are no results available which indicate how to quantitatively measure the “growth” of $q_H$ when the Clark-measure is not discrete. Moreover, it seems to be highly involved to characterize those Hamiltonians with a discrete Clark-measure (in an operator theoretic language: those Hamiltonians whose selfadjoint realizations have compact resolvents). There are only some sufficient conditions and some necessary conditions known, which are due to Kac [Kac95]. For canonical systems generated by strings, compactness of resolvents can be characterized in a neat way. This is a result of Kac and Krein [KK58].

Concerning (2), it is clear from the existing literature that serious difficulties are encountered. Passing to a limit already in the regular case may completely alter growth: For a regular system arising from a moment problem, each space of the chain (with exception of the space generated by the monodormy matrix itself) is finite-dimensional and contains only polynomials. Contrasting this, as we mentioned previously, the growth of the monodromy matrix may be arbitrary (the only restriction is that it must be of minimal exponential type).

Let us have a closer look at the discrete case. Following the philosophy that the distribution of the sequence of poles of $q_H$ is a natural measure for growth, it is an obvious task to study the convergence exponent of this sequence. In an operator theoretic language, this means to ask whether selfadjoint realizations belong to a Neumann-von Schatten class. The, up to our knowledge, only result in this direction which does not require any extra conditions on the nature of the system, is the following characterization of the Hilbert-Schmidt property.

**Theorem 9** ([KW07]). Let $H$ be a singular Hamiltonian on the interval $I = [0, L)$. Then the sequence $(x_n)_{n \in \mathbb{N}}$ of the poles of $q_H$ satisfies

$$\sum_{n=1}^{\infty} \frac{1}{x_n^2} < \infty,$$

Concerning (2), it is clear from the existing literature that serious difficulties are encountered. Passing to a limit already in the regular case may completely alter growth: For a regular system arising from a moment problem, each space of the chain (with exception of the space generated by the monodormy matrix itself) is finite-dimensional and contains only polynomials. Contrasting this, as we mentioned previously, the growth of the monodromy matrix may be arbitrary (the only restriction is that it must be of minimal exponential type).

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if and only if there exists an angle \( \phi(H) \in [0, \pi) \) with 
\[
(G(x) := \int_0^x H(t) \, dt)
\]
\[
\int_0^L \xi_{\phi(H)}^T H(t) \xi_{\phi(H)} \, dt < \infty, \quad (3)
\]
\[
\int_0^L \xi_{\phi(H)}^T G(t) \xi_{\phi(H)} + \frac{\xi_{\phi(H)}^T H(t) \xi_{\phi(H)}}{2} \, dt < \infty. \quad (4)
\]

In the setting of strings also other symmetrically normed ideals have been investigated. This is work of Kac (for the case of trace-class going back to Krein). The following two results deal with quite different behaviour. First, singular strings with discrete but rather dense spectrum. Second, strings with sparse spectrum. For simplicity we provide not the most general formulations.

**Theorem 10** ([Kac62], [KK68]). Let \( S[L, m] \) be a singular string with \( L = \infty \), \( m(\infty) < \infty \). Assume that \( \lim_{x \to \infty} x(m(\infty) - m(x)) = 0 \), so that the Clark-measure of the corresponding Hamiltonian is discrete.

Denote by \( (a_n)_{n \in \mathbb{N}} \) the sequence of poles of the Weyl-coefficient \( q_{H_m} \), and let \( k \in \mathbb{N} \). Then, in order that \( \sum_{n=1}^{\infty} \frac{1}{a_n} < \infty \), it is necessary and sufficient that

\[
\int_Q U(x_1, x_2)U(x_2, x_3) \cdots U(x_{n-1}, x_n)U(x_n, x_1) \, dx_1 \cdots dx_n < \infty,
\]

where

\[
U(x, s) := \begin{cases} m(\infty) - m(s), & x \leq s \\ m(\infty) - m(x), & x > s \end{cases}
\]

and

\[
Q := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq \cdots \leq x_n\}.
\]

**Theorem 11** ([Kac86]). Let \( S[L, m] \) be a string with \( m(L) < \infty \). Assume that \( \int_0^L x \, dm(x) \) < \( \infty \), so that the resolvents of the Krein-Feller operator are trace-class operators. Moreover, let \( \chi : [0, \infty) \to [0, \infty) \) be a concave and nondecreasing function, which is continuous at the point \( 0 \) and \( \chi(0) = 0 \).

Denote by \( (a_n)_{n \in \mathbb{N}} \) the spectrum of the Krein-Feller operator. Then, in order that \( \sum_{n=1}^{\infty} \chi\left(\frac{1}{a_n}\right) < \infty \), it is necessary and sufficient that (for one and hence for all \( l \in (0, \infty) \))

\[
\int_0^L \int_0^{s_x(t)} \chi^+(u_x(t)) \, dt \, dm(x) < \infty,
\]

where \( \chi^+ \) is the right-derivative of \( \chi \), and

\[
u_x(s) := s(m(x+s) - m(x-s)), \quad s \in [0, \min\{x, L-x\}]
\]

\[
s_x(t) := \sup \left\{ s \in [0, \min\{x, L-x\}] : u_x(s) \leq t \right\}.
\]
Finally, let us mention one example where growth of order $\frac{1}{2}$ maximal type occurs. The following is due to Lagarias.

**Example 12 ([Lag09])**. The Morse potential $V_k(x) := \frac{1}{4} e^{2x} + k e^x$ with $k \in \mathbb{R}$ is known from quantum mechanics as an approximation for the radial potential of diatomic molecules. In this physical context the corresponding Schrödinger operator is considered on the whole line or on a left infinite half-line. Its spectrum is known, the half-axis $[0, \infty)$ is covered by absolutely continuous spectrum and on the left half-axis at most finitely many spectral points may occur.

In a number theoretic context, the Schrödinger operator with Morse potential is considered on a right infinite half-line. Then it is in limit point case at $\infty$, its spectrum is discrete. The corresponding eigenfunctions are given essentially as Whittaker functions $W_{k, \mu}(x)$ where $\mu$ is the eigenvalue parameter. It is known that the zeros of Whittaker functions considered as functions of the parameter $\mu$ mimic in some respects the zeros of the Riemann Zeta-function. This goes back to some work of Polya, who dealt with $K$-Bessel functions. In some sense this means that the Schrödinger operator with Morse potential can be viewed as a toy model for a Hilbert-Polya operator. In the mentioned work of Lagarias this connection and its significance is explained in some more detail. Among others, the asymptotics of the Morse spectrum are determined (and hence knowledge about the zeros of Whittaker functions was obtained). It turns out that $n(r)$ denotes the counting function of the spectrum

$$n(r) = c_1 \sqrt{r} \log r + c_2 \sqrt{r} + O(1), \quad r \to \infty,$$

with some $c_1, c_2 > 0$.

**Perturbation results.**

We mention three theorems of perturbative nature. The first one says that under certain small perturbations of the Clark-measure the spaces of the chain of the corresponding Hamiltonian remain the same as sets (of course, inner products will change). As an immediate consequence, under such perturbations all growth properties are stable. The following result was shown by Winkler. Its proof is in flavour similar to the Gelfand-Levitan approach to inverse problems (and the result actually contains the classical theorem of Gelfand and Levitan). In the setting of strings a similar result was given in [DK78].

**Theorem 13** ([Win00]). Let $H_i, i = 1, 2$, be singular Hamiltonians. Assume that the corresponding Clark-measures $\mu_{H_i}$ satisfy

(i) $\mu_{H_2} \ll \mu_{H_1}$,

(ii) There exists $N \in \mathbb{N}$ and $C > 0$, such that $(M := \mathbb{R} \setminus [-N, N])$

$$\mu_{H_1}|_M \ll \mu_{H_2}|_M, \quad \frac{d|\mu_{H_1} - \mu_{H_2}|}{d\mu_{H_2}}(x) \leq \frac{C}{1 + x^2}, \quad x \in M.$$
Then the chains corresponding to $H_1$ and $H_2$ contain the same spaces as sets (not including inner products).

The second theorem is a recent result of Borichev and Sodin, which deals with exponential growth. We do not state the original theorem, but rather provide a (somewhat less detailed) formulation adopted to the present context.

It is noteworthy that the given conditions are essentially sharp. In the proof methods are employed which were originally developed for solving the Bernstein approximation problem.

**Theorem 14** ([BS11]). Let $H_i$, $i = 1, 2$, be singular Hamiltonians defined on respective intervals $[0, L_i)$. Assume that there exist $\delta > 0$, $c > 0$, $n \in \mathbb{N}$, such that the corresponding Clark-measures $\mu_{H_i}$ satisfy

$$I_{x, \delta} := [x - e^{-\delta|x|}, x + e^{-\delta|x|}], I_{x, \delta}^2 := [x - 2e^{-\delta|x|}, x + 2e^{-\delta|x|}]$$

$$\mu_{H_2}(I_{x, \delta}) \leq C(1 + |x|)^n (\mu_{H_1}(I_{x, \delta}^2 + e^{-2\delta|x|}), \quad x \in \mathbb{R}.$$ 

Then

$$\int_0^{L_2} \sqrt{\det H_2(t)} \, dt \leq \int_0^{L_1} \sqrt{\det H_1(t)} \, dt.$$ 

The third result is a local uniqueness theorem. It says that exponentially small perturbations of the Weyl-coefficient do not influence a beginning section of the Hamiltonian an vice versa. In particular, such perturbations do not influence growth of the spaces of this beginning section. This result goes back to Krein and Langer. Similar local uniqueness theorems were shown in the setting of Schrödinger equations by Simon [Sim99], and in the setting of canonical systems with inner singularities by Langer and Woracek [LW11].

**Theorem 15** ([KL]). Let $H_i$, $i = 1, 2$, be a singular Hamiltonian on $[0, L_i)$. Let $a > 0$, and set

$$l_i := \sup \{ x \in [0, L_i) : \int_0^x \sqrt{\det H_i(t)} \, dt < a \}.$$ 

Then the following are equivalent.

(i) $H_1|_{[0, l_1]} \sim H_2|_{[0, l_2]}$.

(ii) There exists $\theta \in (0, \pi)$ such that for every $\varepsilon > 0$

$$q_{H_1}(re^{i\theta}) - q_{H_2}(re^{i\theta}) = O\left(e^{(-2a+\varepsilon)r\sin \theta}\right), \quad r \to \infty.$$ 

(iii) The estimate

$$q_{H_1}(z) - q_{H_2}(z) = O\left((\text{Im} \, z)^3 e^{-2a \text{Im} \, z}\right), \quad |z| \to \infty,$$

prevails uniformly in each Stolz-angle $\Gamma_{\alpha} := \{ z \in \mathbb{C} : \alpha \leq \arg z \leq \pi - \alpha \}, \alpha \in (0, \pi)$. 

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3.2 Conjectures and expected results

The planned research revolves around several intuitive ideas which are motivated from the readily known results and examples. We formulate these vague principles as four “conjectures”.

The first pair of conjectures claims direct relations between properties of a Hamiltonian and properties of the corresponding monodromy matrix or Weyl-coefficient. Since exponential growth is well-understood by means of the formula (2), the main task is to understand Hamiltonians whose determinant vanishes identically and growth functions which are growing slower than \( r \).

**Conjecture 1.** Let \( H(x) \) be a regular Hamiltonian with \( \det H(x) \equiv 0 \). Denote by \( \varphi_H(x) \) its rotation angle, and by \( W_H(z) \) its monodromy matrix. Moreover, let \( \lambda \) be a growth function with \( \lambda(r) = o(r) \). Then: The growth of the entire function \( W_H(z) \) w.r.t. \( \lambda \) is correlated with

- the speed of rotation (i.e., the growth of \( \varphi_H(x) \)),
- the regularity of rotation (e.g., piecewise monotonicity),
- the size of the set of values covered by \( \varphi_H(x) \) (e.g., in terms of capacity, taking into account multiply covered areas).

All this understands of course measured appropriately w.r.t. \( \lambda \).

**Conjecture 2.** Let \( H(x) \) be a singular Hamiltonian with \( \det H(x) \equiv 0 \). Denote by \( q_H(z) \) its Weyl-coefficient (with corresponding Clark-measure \( \mu_H \)), and by \( W_H(x, z) \) its fundamental solution. Moreover, let \( \lambda \) be a growth function with \( \lambda(r) = o(r) \). Then: Existence of uniform bounds w.r.t. \( x \) for the growth of the entire functions \( W_H(x, z) \) w.r.t. \( \lambda \) is correlated with density of the measure \( \mu_H \) measured appropriately w.r.t. \( \lambda \). If \( q_H \) is meromorphic in the whole plane, existence of such bounds is related to the distribution of the poles of \( q_H \).

The second pair of conjectures takes up a perturbative viewpoint.

**Conjecture 3.** Let \( H(x) \) be a (regular or singular) Hamiltonian, and denote by \( W_H(x, z) \) its fundamental solution. Then: Small perturbations of \( H(x) \) do not change the growth of \( W_H(x, z) \) (including uniformity of bounds w.r.t. \( x \)).

**Conjecture 4.** Let \( H(x) \) be a singular Hamiltonian, let \( q_H(z) \) be its Weyl-coefficient (with corresponding Clark-measure \( \mu_H \)) and \( W_H(x, z) \) its fundamental solution. Then: Small perturbations of \( \mu_H \) do not change the growth of \( W_H(x, z) \) (including uniformity of bounds w.r.t. \( x \)).

Our aim is to give precise and quantitative meaning to this intuition. In the optimal case, for non-perturbative results, direct and inverse results in the form of “if and only if”–theorems shall be established. For many questions in this circle of ideas, this is a probably challenging but also
promising task; direct and inverse results relating properties of a Hamiltonian with properties of its monodormy matrix or Wely-coefficient, are often involved and powerful. As experience tells, for some questions, it may be unrealistic to expect “if and only if”-theorems; in such cases necessary conditions and sufficient conditions shall be established which fall apart as few as possible. For results of perturbative nature, optimally, theorems should be complemented by examples which show that the provided conditions are sharp.

We expect to obtain/solve the following results/tasks.

(1) Necessary conditions and sufficient conditions in the terms mentioned in Conjecture 1 for a regular Hamiltonian \( H \) in order that its monodromy matrix \( W_H \) has finite and positive \( \lambda \)-type (\( \lambda \) a growth function). These should be “if and only if”-theorems (at least) under some additional assumptions (e.g., smoothness), as well as for classes of Hamiltonians appearing in applications (e.g., moment problems or strings).

(2) Similar as in the previous item for a singular Hamiltonian \( H \) with discrete Clark-measure in order that the sequence of poles of \( q_H \) has a finite and positive \( \lambda \)-density. Extensions of the known Neumann-von Schatten class conditions (in the setting of strings) to arbitrary Hamiltonians and to other symmetrically normed ideals.

(3) To find out what “measured appropriately w.r.t. \( \lambda \)” in Conjecture 2 means, and to prove corresponding relations between \( H \) and the “density” of \( \mu_H \).

(4) Stability theorems for growth characteristics of spaces of entire functions under perturbations of the corresponding Clark-measure of the Hamiltonian. In particular, conditions for smallness of a perturbation in the flavour of Theorem 13 and Theorem 14, which ensure stability of \( \lambda \)-growth (rather than stability of the actual spaces as sets). Examples which show that the obtained results are sharp (or indicate how far the gap to sharpness is).

(5) Extend the obtained results to canonical systems with inner singularities which appear in the context of Pontryagin space theory. The crucial task in this respect is to understand the regular case.

(6) Apply the previous results to provide information on the spectrum (in particular, spectral asymptotics) for concrete differential- or difference- operators of mathematical physics (e.g., Schrödinger or Dirac operators) or probability theory (e.g., birth-death processes). Extend this analysis to operators with singular coefficients.

We expect that the most innovative aspects are related to items (1), (3), and (4): It seems a major challenge to find out how to quantitatively measure growth/regularity/capacity of the rotation angle function w.r.t. a growth function \( \lambda \), how density of a measure w.r.t. \( \lambda \) can be defined, and how large a perturbation may be to ensure stability of growth.
We expect that the most striking of the applications in (6) are those related to operators from mathematical physics with singular coefficients (a nowadays very active field).

3.3 Potential additional aspects

A problem related to positive definite functions.

A continuous function \( f : \mathbb{R} \to \mathbb{C} \) with \( f(-t) = \overline{f(t)} \) is called positive definite, if for each choice of \( n \in \mathbb{N} \) and \( t_1, \ldots, t_n \in \mathbb{R} \) the quadratic form \( \sum_{i,j=1}^{n} f(t_i - t_j) \xi_i \xi_j \) is positive semidefinite.

By Bochner’s theorem, a function \( f \) is positive definite if and only if it is the Fourier transform of a finite positive Borel measure on \( \mathbb{R} \). This fact can be used to relate positive definite functions with canonical systems: The measure \( \sigma_f \) corresponding to \( f \) in Bochner’s theorem is the Clark-measure of some Hamiltonian \( H_f \). Not every canonical system can be produced in this way, since in general a Clark-measures only needs to satisfy \( \int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty \). However, this is only a minor restriction: the general case can be reduced to this particular one with a simple construction (see, e.g., the work of Krein and Langer [KL] or the work [LW11] where the argument is carried out explicitly). Hence, the theory of positive definite functions can be regarded in a way as equivalent to the theory of canonical systems.

Let \( a \geq 0 \) be given, and consider the restriction \( f|_{[-a,a]} \). This function has at least one positive definite extension to the whole line (namely \( f \)). However, there may also be other positive functions than \( f \) which extend \( f|_{[-a,a]} \), and, if so, then there already exist infinitely many such. The reason for this dichotomy can be well-understood from an operator point of view: To the function \( f|_{[-a,a]} \) there corresponds a symmetric operator which may or may not be selfadjoint, and its spectral functions give rise to extensions of \( f|_{[-a,a]} \). If \( f|_{[-a,a]} \) is uniquely extendable, we call the point a determining for \( f \).

A treatise of positive definite functions from an operator theoretic point of view is given in [GG97], for an approach taking the viewpoint of harmonic analysis (which is probably the more common one) we refer to [Sas94].

Let \( f \) be a positive definite function. A part of the chain of de Branges corresponding to \( H_f \) can be determined explicitly in terms of \( f \): Let \( a \geq 0 \) be not determining. Then the completion of the linear space \( \mathcal{H}_a := \text{span}\{e^{itz} : t \in (-\frac{a}{2}, \frac{a}{2})\} \) with respect to the inner product defined by (sesquilinearity and) the relation \( (e^{ixz}, e^{iyz}) := f(x - y) \) is a de Branges space and belongs to the chain of \( H_f \).

The part of the chain which is obtained in the above way may be a very small part. For example, if \( f \) is real-analytic in some neighbourhood of 0, each value \( a > 0 \) is determining for \( f \) and hence only the (trivial) space \( \text{span}\{1\} \) corresponding to “\( a = 0 \)” can be obtained. On the other hand, it may also exhaust the whole chain. This happens, e.g., for the function \( f(t) := e^{-|t|} \). In general, those spaces of the chain can not be obtained from \( f \) which belong to an interval where the determinant of \( H_f \) vanishes: Set \( A_f := \sup\{a \geq 0 : a \text{ not determining for } f\} \).
and let \( a \in (0, A_f) \). Then the gap
\[
H_a \neq \bigcap_{b \in (a,A_f)} H_b
\] (5)
corresponds to existence of an interval with \( \det H_f = 0 \). The following question appears naturally: Let \( a \in (0, A_f) \). Can (5) be characterized in terms of local properties of \( f \) at \( a \)? If yes, which; if no, which other properties of \( f \) come into play? And it seems to be a deep problem to answer this question.

The connection with the topics of the project is made as follows: Appearance of a gap (5) means that there exist intervals in the chain where the spaces can be distinguished only with finer growth measures than exponential type.

**De Branges spaces and functional models.**

One important aspect of the theory of de Branges spaces is its role in functional models for some classes of linear operators. To construct a functional model for a linear operator means to find a unitary equivalence with some linear operator acting in a function space (often consisting of analytic functions). The, probably, most notable instance of functional models is the Nagy–Foias model for contractions. Often de Branges spaces serve as the model spaces in functional models. For instance let us mention the recent work of Silva and Toloza [ST10] and Martin [Mar11], where self-adjoint extensions of entire operators (in the sense of M.G.Krein) are studied, or the work of Gubreev and Tarasenko [GT10], who construct a model for non-dissipative operators with a two-dimensional imaginary part.

Let \( A \) be the operator of multiplication by the independent variable \( x \) in a space \( L^2(\mu) \) with a pure point measure \( \mu = \sum_n \mu_n \delta_{t_n} \) satisfying \( t_n \neq 0 \) and \( |t_n| \to \infty \) as \( |n| \to \infty \). Then \( A \) is a self-adjoint operator with simple and discrete spectrum, and \( 0 \notin \rho(A) \). Let \( a, b \) be two functions such that \( a, b \notin L^2(\mu) \) but \( \frac{a(x)}{1+|x|}, \frac{b(x)}{1+|x|} \in L^2(\mu) \), and let \( \kappa \in \mathbb{C} \). We associate with each data \( (a, b, \kappa) \) of this kind a linear operator \( L \) as follows:

\[
\text{dom} L := \left\{ y \in L^2(\mu) : \exists y_0 \in \text{dom} A, c \in \mathbb{C} \text{ s.t. } y = y_0 + c \cdot A^{-1} a, \kappa c + (y_0, b)_{L^2(\mu)} = 0 \right\},
\]
\[
Ly := Ay_0, \quad y \in \text{dom} L.
\]

The decomposition \( y = y_0 + c \cdot A^{-1} a \) in the above formula is unique, and hence \( L \) is well-defined. Moreover, \( \text{dom} L \) is dense.

Each operator \( L \) constructed in the above manner, is called a singular rank one perturbation of \( A \), and a functional model was constructed in recent work of Baranov and Yakubovich, cf. [BY12]. Thereby, the model operator acts in an appropriate de Branges space. Operators of this type appear naturally in analysis of the Scrödinger equation with nonlocal interaction. The following problems are considered to be deep and interesting questions:

- When does \( L \) have a complete system of eigenvectors?
– When does completeness of eigenvectors of $L$ imply the completeness of eigenvectors of $L^*$?

– When does $L$ admit the spectral synthesis (i.e., when is any $L$-invariant subspace generated by the eigenvectors it contains)?

– For which operators $A$ does there exist a singular rank one perturbation $L$ whose spectrum is concentrated at infinity (i.e., such that $L$ is the inverse of a Volterra operator)?

Using the mentioned de Branges space model, these problems can be related to problems about sets of uniqueness and interpolation in de Branges spaces. These, in turn, are known to be related to asymptotic behaviour and zero distribution, and hence to growth properties.

The above circle of ideas is thus loosely related to the core problems of the project; some influences in either direction are expected and shall be explored.

3.4 Methods and approaches

Solving the problems in the project will require to apply and combine methods from various fields. Among them: spectral theory, differential operators, entire functions, operator models, perturbation theory, subharmonic functions, or indefinite inner product spaces. There will appear a remarkable combination of classical methods (growth theory for entire functions, the Carleman method for estimates of subharmonic functions, the Nevanlinna parameterization in the moment problem, etc.) and modern methods (functional models, reproducing kernel Hilbert spaces, singular integrals, etc.).

Among the techniques of the theory of differential operators that are going to be used, we emphasize Volterra integral equations, the Gelfand-Levitan method, and variational techniques. Among the techniques studying the spaces of entire functions arising in the problems we emphasize the majorization theory in spaces of entire functions developed in works of Havin-Mashregh and Baranov-Woracek, and the theory of canonical systems with inner singularities developed by Kaltenbäck-Woracek. It is also expected that modern techniques for singular integrals (developed during the last decade by Makarov and Poltoratski) will play an important role.

We also would like to emphasize the two basically different approaches we intend to use: The non-perturbative approach related to Conjectures 1 and 2, and the perturbative one related to Conjectures 3 and 4. Of course, major new results may also occur when combining these.

4 Project planning

The number of researchers involved in the present project allows to a certain extent to attack different expected results in parallel. Of course, there are some logical constraints on the order
in which problems can be solved: The items containing the most innovative aspects have to come first, considering singular Hamiltonians makes sense only after the regular case is (at least partially) settled, extensions to problems with singular coefficients can be undertaken only once the classical cases are understood. Concerning applications, our aim is to develop them from an as early as possible stage. Dealing with concrete situations is expected to make the theory more interesting and accessible for a broader community, and to stimulate ongoing abstract research.

A rough time plan can schematically be presented as follows (numbers refer to the list of expected results, a “⋆” to potential additional aspects):

The research of the Austrian and Russian teams, and of the potential Ph.D.-candidate, is highly intertwined and expected to happen largely in close collaboration. Of course, each involved researcher has a somewhat different expertise. We expect that major contributions to the tasks under consideration will be distributed roughly as indicated in the following scheme (again numbers refer to the list of expected results, and a “⋆” to potential additional aspects):

5 Dissemination strategy and use of instrumentation

We plan to regularly publish papers in established international journals, as an outcome of the research within this project. Expected are several papers per year focussing on the topics of the project. All publications will be available online on arXiv and on my personal webpage and will clearly mention the project. We will promote the research by giving talks on international conferences and in seminars in other related research institutions whenever there is a possibility
for it. In addition, we plan to establish a project specific website which explains topics, open
problems, and connections with other areas, and reports about the progress. It shall also
contain an extensive (as exhaustive as possible) collection of literature. The aim is to promote
this direction of research and to attract other colleagues to participate. The Ph.D.-candidate is
expected to finish his thesis and to recieve his Ph.D. degree.

No particular use of equipment is planned, apart from laptop and basic computer equip-
ment. The infrastructure at the Institute for Analysis and Scientific Computing at the Vienna
University of Technology is perfectly sufficient for the research within the proposed project.

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Appendix A. Presentation of the project partners

A.1. The Austrian team

The Austrian team consists of Harald Woracek (Vienna University of Technology) who acts as project applicant (at the FWF) and principal investigator, Michael Kaltenbäck (Vienna University of Technology), and Peter Yuditskii (Johannes Kepler University Linz).

Harald Woracek:
In my scientific work I deal with
- De Branges spaces of entire functions;
- Spectral theory for Hamiltonian systems, strings, and Sturm-Liouville equations;
- Indefinite inner product spaces and their operators;
- Interpolation- and continuation problems of classical analysis, moment problems.

I started my career as a researcher with my Ph.D.-studies under the supervision of H.Langer, and became familiar with the deep work of M.G.Krein on operator theory at this early stage. In the time of my Ph.D. I mainly dealt with indefinite versions of classical interpolation problems (Nevanlinna-Pick problem, continuation of positive definite functions). I developed and employed operator theoretic methods to solve such problems.

In a second period I became familiar with de Branges’ theory of Hilbert spaces of entire functions, and carried out the program to generalize this theory and the theory of canonical systems to the indefinite (Pontryagin space) setting. This happened in close collaboration with my colleague and friend Michael Kaltenbäck. The task to properly define canonical systems with inner singularities, develop an operator model, construct a Weyl-coefficient, and prove an Inverse Spectral Theorem, was a long standing open problem. It was posed and approached by M.G.Krein more than 40 years ago. He succeeded in studying some particular classes of systems, mainly in collaboration with my Ph.D.-supervisor H.Langer. A full solution was obtained only during the past 15 years in the series “Pontryagin spaces of entire functions I–VI” by Michael Kaltenbäck and myself.

In my recent research I focus on:

(1) Questions from spectral theory, mainly motivated by mathematical physics. For instance, spectral theory of equations with singular coefficients, inverse problems, quantum graphs.

(2) Classical topics revolving around de Branges’ spaces of entire functions. For instance, moment problems, approximation problems, growth and majorization.

Concerning (1) I often collaborate with M.Langer (University of Strathclyde, Glasgow), concerning (2) with A.Baranov (St. Petersburg State University, St. Petersburg), who is also the
principal investigator of the Russian team of the presently proposed project. The topics of the project are mainly related with (2). However, especially for applications to concrete problems of recent interest, (1) will be important.

Michael Kaltenbäck:
Michael Kaltenbäck is a specialist in operator theory in indefinite inner product spaces and in de Branges spaces; both topics being integral parts of the present project. He also has extensive knowledge in harmonic analysis, manifolds, and other fields of contemporary analysis, and is capable to carry out deep research involving notions from very different areas of analysis. Since the present project lies on the border of functional analysis, complex analysis, and Fourier analysis, this ability is important and promises significant contributions.

In the past years he had to cope with a heavy teaching load and was intensively engaged in administrative matters, e.g., building up new study plans at our university. Hence his time for research was rather limited. However, these matters are now settled, and he will be able to have a focus on project related research.

Peter Yuditskii:
Peter Yuditskii is a specialist in complex analysis and spectral theory of functions and operators with applications to mathematical physics. He has worked on topics like extremal problems, functional models of operators and analytic matrix functions, moment problems and orthogonal polynomials, function theory in multiply connected domains, or inverse scattering problems. His expertise in the theory of de Branges spaces is without any doubt outstanding. Just to cite one colleague: “Peter is the one who understands de Branges’ theory the deepest”.

Most recently, he contributed significantly to a better understanding of the Kotani-Last conjecture, which originally claimed a relation between (almost) periodicity of the potential of a Schrödinger- or Jacobi operator and the presence of absolutely continuous spectrum. Interestingly, in this context canonical systems appear naturally together with a measure of growth relative to a certain conformal mapping between multiply connected domains.

For the present project, especially his experience with moment problems, $L^1$-extremal problems, and spectral theory of Schrödinger operators, will be an important skill.

A.2. The Russian team

The Russian team consists of Anton Baranov (St. Petersburg State University) who acts as project applicant (at the RFBR) and as principal investigator, Yurii Belov (Chebyshev Laboratory), and Roman Romanov (V.Fock Institute for Physics and St. Petersburg State University).

Anton Baranov:
The main scientific interests of Anton Baranov are related with shift-coinvariant subspaces of Hardy spaces (often called model subspaces, due to their role in the Nagy-Foiaš model for
contractions in Hilbert space) and with operators acting in these spaces. De Branges spaces are a particular class of model subspaces, namely those that correspond to inner functions which are meromorphic in the whole plane. Coming from the school of the outstanding mathematicians V.Havin and N.Nikolski, Anton Baranov became a leading expert in the field.

More specifically, he deals in his research with

- Weighted Bernstein inequalities for model spaces and their applications;

- Geometric properties of families of reproducing kernels (completeness, Riesz bases and their stability);

- Admissible majorants for model subspaces (in particular for de Branges spaces) and Beurling-Malliavin type theorems.

A Bernstein inequality estimates a weighted norm of the derivative of a function in a model subspace in terms of its natural $L^p$-norm. Anton Baranov proposed a new approach to inequalities of this kind, and used weighted Bernstein inequalities to obtain Carleson-type embedding theorems (with compact embedding operator). The description problem of such embeddings is a long-standing open question posed by Cohn in 1982. This problem remains open, however, the results obtained by Anton Baranov are the most general currently known results and contain previous results by Cohn, Volberg and Treil, Cima and Matheson.

The geometric properties of families of reproducing kernels in a reproducing kernel Hilbert space model spaces are of great importance for understanding its structure. E.g., the completeness property for systems of reproducing kernels is equivalent to the description of uniqueness sets, while being a Riesz basis for such system means that any data from some weighted $l^2$ space may be interpolated by elements of the space. Specializing from the setting of de Branges spaces, completeness and Riesz bases problems contain the classical (and very difficult) problems about exponential systems.

Stability of complete systems or Riesz bases under small perturbations is important for applications. Classical results about stability of exponential systems are due to Paley-Wiener, Redheffer and Kadets (the so-called 1/4-theorem). Anton Baranov obtained a number of new stability results.

Recently, he solved (jointly with Yurii Belov, who is also a member of the Russian team) a deep problem posed by N.Nikolski. Namely, whether the system biorthogonal to a complete and minimal system of reproducing kernels in a model subspace is complete (a property which is known to hold for systems of exponentials by a famous theorem of Young). Interestingly it turns out that this remains true for a class of de Branges spaces, but in general the biorthogonal system may fail to be complete.

The theory of majorization is a classical area. In the setting of Paley-Wiener spaces, a wide class of admissible majorants is described by the famous Beurling-Malliavin multiplier theorem.
Admissible majorants for general model spaces were for the first time considered by V.Havin and J.Mashreghi in 2003. This approach was further developed by Anton Baranov in a series of papers jointly with V.Havin and A.Borichev. In a series of joint papers with H.Woracek (principal investigator of the Austrian team) he studied the interplay of admissible majorants and the structure of subspaces of the de Branges spaces. This work revolves around the question which spaces of the chain of de Branges subspaces of a given de Branges space can be generated by majorization on some subset of the complex plane. Naturally, this topic is closely related to questions about growth, illustrative examples can be constructed using the connections between growth and zero-distribution in the general setting of proximate orders.

In the context of the present project, his expertise in de Branges spaces, theory of admissible majorants, and Riesz-basis properties and their stability, will be the most important features.

**Yurii Belov:**
The research of Yurii Belov revolves around de Branges spaces, reproducing kernels, and functional analysis in Hardy-spaces (shift-invariant subspaces, Riesz-basis properties, Beurling-Malliavin type theorems, etc.).

He is a specialist in complex analysis, and has readily proved to be a powerful scientist. For example, he has solved the Carlsson-Sundberg completeness problem, and (jointly with Anton Baranov) the already mentioned problem posed by N.Nikolski asking for completeness of the biorthogonal system. It should be mentioned that the latter topic turns out to be closely related to growth questions (absence of growth on an exponential scale).

In the present context his experience with de Branges space, model operators, Beurling-Malliavin majorants, and complex analysis in general will be of prime importance.

**Roman Romanov:**
The main research interests of Roman Romanov are spectral theory of operators, especially analysis of operators by methods of functional models, and application to operators of mathematical physics. Recently, his research included the theory of de Branges spaces and inverse problems for strings and canonical systems. For instance, he investigated neutron transport operators, asymptotics of their solutions, and the question of complete non-selfadjointness. A problem which, for certain perturbations, turned out to be related with the Beurling gap problem (a classical question in analysis).

His solid background in mathematical physics as well as in operator theory and differential equations makes him an important member of the team. It is expected that he contributes significant results to both: abstract theory (when it comes to analyzing and estimating solutions of equations), and applications (first, for coming up with examples meaningful from a physical point of view, and second, when it comes to interpreting the abstract results). Also his experience with perturbation theory will prove to be very helpful in the present context.
A.3. The Ph.D.-candidate (N.N.)

The Ph.D.-student (who is at present not known) is expected to contribute significantly to the following tasks.

- Finding upper estimates of growth in terms of (regular) Hamiltonians.
- Carrying out limiting procedures in the singular but discrete case.
- Getting involved into results of perturbative nature. Providing examples for sharpness of (to be obtained) theorems.
- Investigating concrete operators from analysis, mathematical physics and probability theory. For instance, moment problems related with birth-death processes or classical orthogonal polynomials, Schrödinger- or Dirac- operators, the Camassa-Holm equation, or Markov processes related with fractal strings.

By carrying out this research, she/he will prove her/his abilities as a researcher. The expected outcome will certainly be sufficient to complete a thesis and qualify for the Ph.D.-degree.

Her/his workplace is situated at the Department for Analysis and Scientific Computing of the Vienna University of Technology, and his thesis will be supervised mainly by the project applicant H.Woracek. However, she/he is expected to visit St. Petersburg on a regular basis. On the one hand, this shall give her/him the opportunity to closely collaborate with the members of the Russian team and receive advanced training from them. On the other hand, it shall provide experience at foreign research institutions (in particular, the outstanding Chebyshev Laboratory led by the Fields medallist S.Smirnov), and enable her/him to establish and cultivate close contacts with a larger circle of St. Petersburg mathematicians. Moreover, especially at a later stage of the project, she/he is expected to actively participate in international conferences. This will help to establish scientific contacts with others mathematicians of the worldwide community.

We believe that her/his career, as well as her/his scientific and personal development, will greatly benefit from such experiences and contacts.

A.4. Benefit of a close cooperation

Each participant of the teams is in his way a specialist in the central topics of the project: canonical systems and de Branges spaces. However, each of the participants had approached this area with a somewhat different background: Anton Baranov and Yurii Belov from the side of complex analysis and model subspaces, Harald Woracek and Michael Kaltenbäck from the side of operator theory and indefinite inner product spaces, Roman Romanov from mathematical physics and spectral theory, and Peter Yuditskii from spectral theory and complex analysis.

The following scheme is a very rough illustration of the areas of expertise of the teams.
Major areas of expertise:  

A . . . Austrian team  

R . . . Russian team  

Expected areas of development of the Ph.D.-candidate:  

P  

Bringing together these expertises promises a fruitful collaboration and significant new results. On the one hand, the participants are close enough to find interest in the same questions and speak the same “mathematical language”. On the other hand, their experiences are sufficiently far apart to imply mutual stimulation and broadening horizons. This will give rise to research perspectives which could not be obtained without a close cooperation among the members of the teams.

The Austrian team will benefit from the strong background of the Russian team in classical complex analysis and mathematical physics, and the Russian team will benefit from the strong background of the Austrian team in operator theory and spectral theory.
Appendix B. Requested budget

The requested budget is composed of:

**Personnel costs:** One Ph.D.-position (75%) for three years. According to the FWF salary scale: €34700 per year.

**Travel costs:** The present project shall create the financial basis for an environment of continuous teamwork. Therefore we put emphasize on travelling. In particular, the Ph.D.-candidate is expected to travel on a regular basis.

We expect travel costs of €12180 per year. This amount is comprised of

- One yearly visit of two weeks for each member of the Austrian team.
- Three yearly visits of two weeks for the Ph.D.-candidate.

The costs for one visit of two full working weeks (travelling Sunday–Saturday) are estimated as €2030. This amount is computed as follows:

Travel (flight): €500.

Accommodation (mid-range hotel), 13 nights: \[13 \times €90 = €1170.\]

Daily charges (according to the RGV rates for each full 24 hours, using the mean of scales 2a and 2b): \[12 \times €30 = €360.\]

**Material costs:** One Laptop and one tablet computer/thinkpad to be used mainly by the Ph.D.-candidate. Approximate total amount: €2000 for the whole duration of the project.

Since the Ph.D.-candidate is expected to travel frequently, we regard this equipment as necessary and appropriate.

*Subtitle (excluding general costs): €142640*

**General costs (obligatory):** According to the rules of the FWF, the general costs for this project amount to €7132. This part of the funding shall be used primarily for attendance at conferences. A part shall be used for creating and maintaining a specific website related to the project topic.

*Total: €149772*
C. Curricula Vitae / Publication lists

HARALD WORACEK
Project applicant (FWF-side). Principal investigator of the Austrian team.

Personal information

Name Harald Woracek
Academic degree Ph.D.
Present position ao.Univ.Prof., Vienna University of Technology,
Institute for Analysis and Scientific Computing,
Research Group for Functional Analysis
Date of birth 10 July 1969
Place of birth Vienna, Austria
Nationality Austria
Marital status Married
Office address Institute for Analysis and Scientific Computing,
Vienna University of Technology,
Wiedner Hauptstrasse 8-10/101, 1040 Vienna, Austria
Phone: +43(0)1 58801 10112, Fax: +43(0)1 58801 10199
Home address Rosensteingasse 16/1/16, 1170 Vienna, Austria
Electronic address harald.woracek@tuwien.ac.at
http://asc.tuwien.ac.at/index.php?id=woracek

Education

1999 Habilitation at the Vienna University of Technology.
1992 - 1994 Ph.D. (Dr.techn.), Vienna University of Technology.
Thesis: “Das verallgemeinerte Nevanlinna-Pick Problem im entarteten
Fall”. Supervisor: Prof.Dr.Heinz Langer.
1987 - 1992 Master study “Technische Mathematik” (Dipl.-Ing.), Vienna University
of Technology. Thesis: “Polynomsfunktionen und Interpolation in kom-
mutativen Ringen mit Einselement”. Supervisor: Prof.Dr.Hans Kaiser.
Affiliations

Since 1999 ao.Univ.Prof. at the Institute for Analysis and Scientific Computation of the Vienna University of Technology.

1992 - 1999 Assistant at the Institute for Analysis and Scientific Computation (formerly Institut für Analyis und Technische Mathematik), Vienna University of Technology.


Research interests

De Branges spaces of entire functions, Spectral theory for Hamiltonian systems and Sturm-Liouville equations (especially with singular potentials), Indefinite inner product spaces and their operators (especially symmetric operators and their selfadjoint extension), Interpolation-and extrapolation- problems, Moment problems, Functional analysis and complex analysis in general.

Services


– Associate editor of Complex Analysis and Operator Theory.


– Reviews for major journals (JMAA, CAOT, LAA, etc.). Reviews of research projects for the NWO and DFG.

– Host for the Erasmus Mundus PostDoc Dr.Sergey Simonov during the academic year 2011/12.


Memberships

ÖMG (Österreichische Mathematische Gesellschaft),

SCM (Societat Catalana de Matemàtiques)
International cooperation partners

- Anton Baranov: St. Petersburg State University, St. Petersburg, Russia.
- Matthias Langer: University of Strathclyde, Glasgow, UK.
- Sergey Simonov: Dublin Institute of Technology, Dublin, Ireland.
- Henrik Winkler: Ilmenau University of Technology, Ilmenau, Germany.
- Vyacheslav Pivovarchik: South-Ukrainian state Pedagogical University, Odessa, Ukraine.

Publications (since 2009)

All publications are available on my website: http://asc.tuwien.ac.at/index.php?id=woracek


32
Recent manuscripts (submitted for publication)

All manuscripts are available for download from my website, and preprints are published in the Preprint Series of my department: http://asc.tuwien.ac.at/index.php?id=132


10 most important publication


ANTON BARANOV
Project applicant (RFBR-side). Principal investigator of the Russian team.

Personal information

Name Anton Baranov
Academic degree Ph.D.
Present position Professor, St. Petersburg State University,
Department of Mathematics and Mechanics,
Chair of Mathematical Analysis
and
Senior Researcher, Chebyshev Laboratory,
St. Petersburg State University
Date of birth 11 April 1977
Place of birth Leningrad, Russia
Nationality Russia
Marital status not married
Office address Department of Mathematics and Mechanics,
St. Petersburg State University,
28, Universitetskii prosp., Petrodvorets, 198504, Russia
Phone: +7 812 4284211
Electronic address a.baranov@ev13934.spb.edu
http://www.math.spbu.ru/analysis/engl/staff.html
http://en.chebyshev.spb.ru/staff/baranov

Education

2011 Doctor of Science (Habilitation), St. Petersburg Branch of Steklov Mathematical Institute of Russian Academy of Sciences.
2005 Ph.D. in Mathematics, University of Bordeaux, Thesis Adviser Prof.N.K.Nikolski.
2003 Candidate in Mathematics and Physics (Ph.D.), St. Petersburg Branch of Steklov Mathematical Institute of Russian Academy of Sciences, Thesis Adviser Prof.V.P.Havin.
1999 Graduated (with Honours Diploma) from St. Petersburg State University.
Affiliations

Since 2012  Professor at the Department of Mathematics and Mechanics of the St. Petersburg State University.

2004 - 2012  Associate professor at the Department of Mathematics and Mechanics of the St. Petersburg State University.

2005 - 2006  Postdoctoral position at the Institute of Mathematics of the Royal Institute of Technology (KTH), Stockholm.

2002 - 2003  Assistant professor at the Department of Mathematics and Mechanics of the St. Petersburg State University.

1999 - 2002  PhD candidate at the Department of Mathematics and Mechanics of the St. Petersburg State University.

Research interests

Complex Analysis, Harmonic Analysis, Operator Theory

Awards

- Abramov Prize of St. Petersburg Mathematical Society (2008)

Grants and Fellowships

- Göran Gustafsson postdoctoral fellowship, Royal Institute of Technology (KTH), Stockholm (2005-2006).
- French Government Fellowship for participation in Collaboration program ”Thèse en cotutelle” between St. Petersburg State University and University Bordeaux 1, France (2003–2005).
Services

- Managing editor of *Algebra i Analiz* (translated as *Saint Petersburg Mathematical Journal*).

- Co-editor of *Linear and Complex Analysis* (a collection of papers dedicated to V.P.Havin on the occasion of His 75th Birthday), American Mathematical Society, 2009.

- Co-editor of the Section *De Branges spaces* of the Springer Reference on Operator Theory.


International cooperation partners

- *Harald Woracek*: Vienna University of Technology, Vienna, Austria.

- *Alexander Borichev*: Université de Provence, Marseille, France.


- *Emmanuel Fricain*: Université de Lille, Lille, France.

- *Donald Sarason*: University of California, Berkeley, USA.

- *Dmitry Yakubovich*: Universidad Autónoma de Madrid, Madrid, Spain.

- *Javad Mashreghi*: Université Laval, Quebec, Canada.

- *Eugueny Abakoumov*: Université Paris-Est Marne-la-Vallée, France.

Publications (since 2009)

Most publications are available for download from arXiv: [http://arxiv.org](http://arxiv.org)


Recent manuscripts (submitted for publication)

All manuscripts are available for download from http://arxiv.org.


10 most important publication


Yuri Belov  
Russian team.  

Personal information  

Name Yurii Belov  
Academic degree Ph.D.  
Present position Researcher, Chebyshev Laboratory, St. Petersburg State University  
Date of birth 25 November 1981  
Place of birth Leningrad, Russia  
Nationality Russia  
Marital status Married, 1 child  
Office address Chebyshev Laboratory, St. Petersburg State University, 14th Line 29B, Vasilyevsky Island, St. Petersburg 199178, Russia  
Electronic address j_b_juri_belov@mail.ru  

http://en.chebyshev.spb.ru/staff/belov  

Education  

2003 Graduated from St. Petersburg State University, Thesis: “On Hilbert transform of R being Lipschitz”.  

Affiliations

Since 2011  PostDoc at the Chebyshev Laboratory of the St. Petersburg State University.

2009 - 2011  PostDoc at the Faculty of Information Technology, Mathematics and Electrical Engineering of the Norwegian University of Science and Technology.

2007 - 2009  Faculty Associate at the Department of Mathematical Sciences II of the St. Petersburg Electrotechnical University.

2008  PreDoc at the Department of Mathematics and Mechanics of the St. Petersburg State University.

2005 - 2007  PreDoc at the Faculty of Information Technology, Mathematics and Electrical Engineering of the Norwegian University of Science and Technology.

2003 - 2005  Teaching Fellow at the Department of Mathematical Sciences II of the St. Petersburg Electrotechnical University.

Research interests

De Branges spaces, systems of reproducing kernels, shift-invariant subspaces, spaces of entire functions.

Publications (since 2009)


Michael Kaltenbäck
Austrian team.

Personal information

- **Name**: Michael Kaltenbäck
- **Academic degree**: Ph.D.
- **Present position**: ao.Univ.Prof., Vienna University of Technology, Institute for Analysis and Scientific Computing, Research Group for Functional Analysis
- **Date of birth**: 10 September 1971
- **Place of birth**: Laakirchen, Austria
- **Nationality**: Austria
- **Marital status**: not married, 1 child
- **Office address**: Institute for Analysis and Scientific Computing, Vienna University of Technology, Wiedner Hauptstrasse 8-10/101, 1040 Vienna, Austria
- **Phone**: +43(0)1 58801 10122, Fax: +43(0)1 58801 10199
- **Home address**: Leonard Bernstein Straße 8/2/7.12, 1220 Vienna, Austria
- **Electronic address**: michael.kaltenbaeck@tuwien.ac.at
  
  http://asc.tuwien.ac.at/index.php?id=419

Education

- **1999**: Habilitation at the Vienna University of Technology.
- **1996**: Ph.D. (Dr.techn.), Vienna University of Technology. Thesis: “Some questions related to symmetric operators in Hilbert spaces”. Supervisor: Prof. Dr. Heinz Langer.
Affiliations

Since 2000  ao.Univ.Prof. at the Institute for Analysis and Scientific Computation of the Vienna University of Technology.
1999 - 2000  Erwin Schrödinger fellow at the Weizmann Institute of Science, Israel.
1996 - 1999  Assistant at the Institute for Analysis and Scientific Computation (formerly Institut für Analyis und Technische Mathematik), Vienna University of Technology.

Research interests

Indefinite inner product spaces and their operators, De Branges spaces of entire functions, Dirichlet spaces, harmonic analysis and complex analysis.

Publications (since 2009)


**Roman Romanov**  
Russian team.

### Personal information

<table>
<thead>
<tr>
<th>Name</th>
<th>Roman V. Romanov</th>
</tr>
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<tbody>
<tr>
<td>Academic degree</td>
<td>Ph.D.</td>
</tr>
<tr>
<td>Present position</td>
<td>Associate Professor, St. Petersburg State University, Faculty of Physics and Senior Researcher, Laboratory of Quantum Networks, V.Fock Institute for Physics at the Faculty of Physics, St. Petersburg State University</td>
</tr>
<tr>
<td>Date of birth</td>
<td>4 July 1973</td>
</tr>
<tr>
<td>Place of birth</td>
<td>Leningrad, Russia</td>
</tr>
<tr>
<td>Nationality</td>
<td>Russia</td>
</tr>
<tr>
<td>Marital status</td>
<td>Married, 1 child</td>
</tr>
<tr>
<td>Office address</td>
<td>Laboratory of Quantum Networks, Institute for Physics, St. Petersburg State University, 28, Universitetskii prosp., Petrodvorets, 198504, Russia</td>
</tr>
<tr>
<td>Electronic address</td>
<td><a href="mailto:morovom@gmail.com">morovom@gmail.com</a></td>
</tr>
</tbody>
</table>

### Education

<table>
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<th>Year range</th>
<th>Details</th>
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<tr>
<td>1996 - 1999</td>
<td>Ph.D., St. Petersburg State University, Faculty of Physics, Department of Mathematical Physics.</td>
</tr>
</tbody>
</table>
Affiliations

Since 2005
ass.Prof. at the Department of Mathematical Physics of the St. Petersburg State University, and Senior Researcher at the Laboratory of Quantum Networks of the V.Fock Institute of Physics, St. Petersburg State University.

2003 - 2004
Junior Researcher at the Laboratory of Quantum Networks of the V.Fock Institute of Physics, St. Petersburg State University.

2002 - 2004
Research Fellow at the Department of Computer Science of the Cardiff University.

2001
Visiting Assistant Professor at the Department of Mathematics of the University of Alabama at Birmingham.

2000
Research Fellow at the School of Mathematical Sciences of the University of Sussex.

Research interests

Spectral theory of differential operators, non-selfadjoint operators, scattering theory

Services


- Reviewer for Mathematical Reviews.


- Supervision of Master candidates.

Publications (since 2009)


Peter Yuditskii
Austrian team.

**Personal information**

Name: Peter Yuditskii  
Academic degree: Ph.D.  
Present position: Senior Researcher, FWF Projects Leader, J. Kepler University of Linz, Institute for Analysis, Group for Dynamical Systems and Approximation Theory  
Date of birth: 12 November 1955  
Place of birth: Kharkov  
Nationality: Austrian  
Marital status: Married, 2 children  
Office address: Institute for Analysis, J. Kepler University of Linz, Altenberger Str. 69, 4040 Linz, Austria  
Phone: +43/732/2468-9184, Fax: +43/732/2468-1561  
Electronic address: Petro.Yudytskiy@jku.at  
http://www.dynamics-approx.jku.at/yuditskii/

**Education**

2011  
Habilitation at the J. Kepler University of Linz.

1987  
Ph.D. in Mathematics, Kharkov State University, Kharkov, USSR.  
Thesis: “Functional models and generalized interpolation”.

1978  
Diploma in Mathematics, Kharkov State University, Kharkov, USSR.
Affiliations

Since 2003  Head of group “Dynamical systems and approximation theory” at the J. Kepler University of Linz, Linz, Austria.

2005 - 2006  Associate Professor, Department of Mathematics, Bar Ilan University, Israel (left for family reasons in 2006).

2000 - 2003, 2008  Associate Professor, Department of Mathematics, Michigan State University, East Lansing, USA.

1999  Visiting Professor at the J. Kepler University of Linz, Linz, Austria.

1997  Visiting Professor at the University Paris 7, Paris, France.

1996  Visiting Professor at the Weizmann Institute of Science, Rehovot, Israel.

Research interests

Spectral theory of functions and operators and mathematical physics, functional models of operators, analytic matrix functions, extremal problems of Chebyshev type, asymptotics of orthonormal polynomials, spectral function theory in multiply connected domains, Ruelle operators on Julia sets, almost-periodic Schrödinger operators with Cantor–type spectrum, inverse scattering problem for operators with disconnected spectrum

Awards


Featured reviews


Grants and Fellowships


Services

- Member of Scientific and Organizing Committees of CAOTA 2011 (“Complex Analysis, Operator Theory, and Approximation”, International conference dedicated to the memory of Franz Peherstorfer), Linz.
- Supervision of Ph.D. students: Stanislav Kupin, Ph.D. Bordeaux, 2000 (joint with Prof. N. Nikolski), Ionela Moale, Ph.D. Linz, 2011 (joint with Prof. J. Cooper).
- Reviews for major journals.

Publications (since 2009)


