Admissible majorants for de Branges spaces of entire functions

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Abstract

For a given de Branges space $\mathcal{H}(E)$ we investigate de Branges subspaces defined in terms of majorants on the real axis: if ω is a nonnegative function on $\mathbb{R}$, we consider the subspace

$$\mathcal{R}_\omega(E) = \mathrm{Clos}_{\mathcal{H}(E)} \left\{ F \in \mathcal{H}(E) : \; \exists C > 0 : |E^{-1}F| \leq C\omega \text{ on } \mathbb{R} \right\}.$$

We show that $\mathcal{R}_\omega(E)$ is a de Branges subspace, describe all subspaces of this form, and study the majorants ω such that $\mathcal{R}_\omega(E) = \mathcal{H}(E)$. We give a criterion for the existence of positive minimal majorants and characterize finite-dimensional subspaces of the form $\mathcal{R}_\omega(E)$ in terms of minimal majorants.

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1 Introduction and preliminaries

The theory of Hilbert spaces of entire functions introduced by L. de Branges is an important branch of modern analysis. It is an intriguing example for a fruitful interplay of function theory and operator theory, which has deep applications in mathematical physics, namely in differential operators and scattering theory.

One of the striking features of a de Branges space is the structure of its de Branges subspaces (that is, subspaces which are themselves de Branges spaces) revealed by de Branges' Ordering Theorem. This theorem states, roughly speaking, that, for a given space, the set of all its de Branges subspaces 'with the same real zeros' is totally ordered with respect to set-theoretic inclusion. However, given an individual de Branges space, there is no explicit way to determine the chain of its de Branges subspaces.

In a recent series of papers V. Havin and J. Mashreghi introduced the notion of admissible majorants for shift-coinvariant subspaces of the Hardy space. Since de Branges spaces are, essentially, particular shift-coinvariant subspaces of the Hardy space, this notion is applicable. Of course, due to the rich structure of de Branges spaces, much more specific results than in the general setting can be expected.

It is the aim of our present work to show that admissible majorants give rise to de Branges subspaces and to study the structure of these subspaces. Our main results are a description of all subspaces which are induced by admissible majorants, a criterion for the existence of minimal majorants which are separated from zero, and a description of finite-dimensional subspaces in terms of minimal majorants.

As already indicated in the above abstract, an admissible majorant defines a de Branges subspace by means of a restriction on the growth along the real axis. It is an interesting observation that this concept is complementary to imposing growth conditions off the real axis. In the recent paper [KW1] de Branges subspaces were defined by means of restriction on mean type. We will see that the subspaces defined by majorants cannot be described by mean type conditions. Hence these two methods can, in conjunction, lead to a description of the whole chain of subspaces of a de Branges space. An elaboration of this idea will be subject of future work.

Let us describe the organization and content of the present paper in more detail. After this paragraph we proceed with the preliminaries-part of this introductory section. There we set up our notation and recall some basic facts concerning de Branges spaces and admissible majorants. Also the exact definitions of all the used terms will be given. Section 2 is devoted to the study of subspaces induced by majorants by means of Definition 2.1 and Proposition 2.2. The main result in this context is the characterization of those subspaces which can be realized in this way, given in Theorem 2.6 and Proposition 2.11. As corollaries we obtain a couple of conditions for density of 'small' functions in a given de Branges space. Moreover, we give some, rather general, examples to illustrate these results. In Section 3 we turn to a thorough investigation of minimal majorants. Our main result is Theorem 3.2 where we relate minimal majorants to one-dimensional subspaces. In combination with Theorem 2.6 this leads to a characterization of existence of minimal majorants separated from zero, cf. Theorem 3.8. Finally, we present a description of finite-dimensional subspaces induced by majorants by means of minimal majorants, cf. Theorem 3.12 and Proposition 3.15.

a. de Branges spaces

An entire function E is said to belong to the *Hermite-Biehler class* $\mathcal{HB}$, if it has no zeros in the open upper half plane $\mathbb{C}^+$ and satisfies

$$|E(\bar{z})| < |E(z)|, \; z \in \mathbb{C}^+ .$$

In what follows we denote by $H^2(\mathbb{C}^+)$ the Hardy class in the upper half plane, see e.g. [RR]. Moreover, throughout this paper we will, for any function F, denote by $F^\#$ the function $F^\#(z) := \overline{F(\bar{z})}$.

1.1 Definition. If $E \in \mathcal{H}B$, the *de Branges space* $\mathcal{H}(E)$ is defined as the set of all entire functions F which have the property that

$$\frac{F}{E}, \frac{F^\#}{E} \in H^2(\mathbb{C}^+) .$$

Moreover, $\mathcal{H}(E)$ will be endowed with the norm

$$\|F\|_E := \Big(\int_{\mathbb{R}} \left| \frac{F(t)}{E(t)} \right|^2 dt \Big)^{1/2}, \; F \in \mathcal{H}(E) .$$

It is shown in [dB, Theorem 21] that $\mathcal{H}(E)$ is a Hilbert space with respect to the norm $\|.\|_E$.

1.2 Remark. The definition of $\mathcal{H}(E)$ given above can be reformulated. In fact, an entire function F belongs to $\mathcal{H}(E)$ if and only if

$$\frac{F}{E}, \frac{F^\#}{E} \in N(\mathbb{C}^+), \; \operatorname{mt} \frac{F}{E}, \operatorname{mt} \frac{F^\#}{E} \leq 0 ,$$

and

$$\frac{F}{E}\Big|_{\mathbb{R}} \in L^2(\mathbb{R}) .$$

Here $N(\mathbb{C}^+)$ denotes the set of all functions of bounded type in $\mathbb{C}^+$, and $\operatorname{mt} f$ denotes the mean type of a function $f \in N(\mathbb{C}^+)$, i.e.

$$\operatorname{mt} f := \limsup_{y \to +\infty} \frac{1}{y} \log |f(iy)| ,$$

see e.g. [RR]. In fact, this is the original definition given in [dB].

It is an important feature that de Branges spaces can be characterized axiomatically, cf. [dB, Problem 50, Theorem 23]. Let $\mathcal{H}$ be a nonzero Hilbert space whose elements are entire functions. Then $\mathcal{H}$ is equal to a space $\mathcal{H}(E)$ including equality of norms if and only if $\mathcal{H}$ satisfies:

- **(dB1)** for every $v \in \mathbb{C}$ the point evaluation functional $\chi_v : F \mapsto F(v)$ is continuous on $\mathcal{H}$;

(dB2) if $F \in \mathcal{H}$, then also $F^\# \in \mathcal{H}$, and we have

$$\|F^\#\| = \|F\|, \ F \in \mathcal{H};$$

(dB3) if $F \in \mathcal{H}$ and $z_0 \in \mathbb{C} \setminus \mathbb{R}$ with $F(z_0) = 0$, then also

$$\frac{z - \overline{z_0}}{z - z_0} F(z) \in \mathcal{H}, \text{ and } \left\| \frac{z - \overline{z_0}}{z - z_0} F(z) \right\| = \|F\| .$$

1.3 Remark. If a Hilbert space $\mathcal{H}$ which satsifies (dB1)-(dB3) is given, the function $E \in \mathcal{HB}$ which realizes $\mathcal{H}$ as $\mathcal{H}(E)$ is not unique. In fact, if $E_1, E_2 \in \mathcal{HB}$, then $\mathcal{H}(E_1) = \mathcal{H}(E_2)$ including equality of norms, if and only if

$$(A_2, B_2) = (A_1, B_1)U$$

where $A_k = \frac{1}{2}(E_k + E_k^\#)$, $B_k = \frac{i}{2}(E_k - E_k^\#)$, $k = 1, 2$, and where U is a 2×2-matrix with real entries and determinant 1.

By (dB1), a de Branges space $\mathcal{H}$ is a reproducing kernel Hilbert space of entire functions. This means that there exists a (unique) function $K(v, z)$, entire in z and in $\bar{v}$, such that for every fixed $v \in \mathbb{C}$ we have

$$K(v, \cdot) \in \mathcal{H} \text{ and } (F, K(v, \cdot)) = F(v), \ F \in \mathcal{H} .$$

If $\mathcal{H}$ is realized as $\mathcal{H}(E)$ with some $E \in \mathcal{HB}$, the reproducing kernel of $\mathcal{H}$ can be written explicitly in terms of E. In fact, we have

$$K(v, z) = \frac{E(z)\overline{E(v)} - E^\#(z)E(\bar{v})}{2\pi i(\bar{v} - z)}.$$

For an entire function G let $\mathfrak{d}(G) : \mathbb{C} \to \mathbb{N}$ be the map which assigns to a point v its multiplicity as a zero of G. For a de Branges space $\mathcal{H}$ we put

$$\mathfrak{d}(\mathcal{H})(v) := \min_{F \in \mathcal{H}} \mathfrak{d}(F)(v) .$$

Then for any $E \in \mathcal{HB}$ with $\mathcal{H} = \mathcal{H}(E)$ we have

$$\mathfrak{d}(\mathcal{H}(E))(t) = \mathfrak{d}(E)(t), \ t \in \mathbb{R} ,$$

cf. [dB, Problem 45]. Note that, by (dB3), we always have $\mathfrak{d}(\mathcal{H})|_{\mathbb{C} \setminus \mathbb{R}} = 0$.

The following observation is often useful, cf. [dB, Problem 44].

1.4 Remark. If S is a real entire function (by *real* we mean that $S = S^\#$) which has no zeros off the real axis, then $SE \in \mathcal{HB}$ if and only if $E \in \mathcal{HB}$. Moreover, in this situation, the map $F(z) \mapsto S(z)F(z)$ is an isometry of $\mathcal{H}(E)$ onto $\mathcal{H}(SE)$. This shows that one can often restrict considerations to the case where $E \in \mathcal{HB}$ has no real zeros, or, equivalently, that $\mathcal{H}$ satisfies the additional axiom

(Z) $\mathfrak{d}(\mathcal{H}) \equiv 0$.

For a more general viewpoint on this subject see [KW1, Lemma 2.4].

Let $v \in \mathbb{C}$ and $F \in \mathcal{H}(E)$ with $F(v) = 0$ be given. If either $v \in \mathbb{C} \setminus \mathbb{R}$ or $v \in \mathbb{R}$ and $\mathfrak{d}(F)(v) > \mathfrak{d}(E)(v)$, then also

$$\frac{F(z)}{z - v} \in \mathcal{H}(E)\,. \tag{1.1}$$

b. dB-subspaces

A most prominent role in the theory of de Branges spaces is played by their subspaces.

1.5 Definition. A subset L of a de Branges space $\mathcal{H}$ is called a *dB-subspace*, if it is itself, with the norm inherited from $\mathcal{H}$, a de Branges space. We shall denote the set of all dB-subspaces of a given space $\mathcal{H}$ by $\operatorname{Sub}(\mathcal{H})$.

In view of the above axiomatic characterization of de Branges spaces, a subset L of $\mathcal{H}$ is a dB-subspace if and only if

(Sub1) L is a closed linear subspace of $\mathcal{H}$;

(Sub2) if $F \in L$, then also $F^{\#} \in L$;

(Sub3) if $F \in L$ and $z_0 \in \mathbb{C} \setminus \mathbb{R}$ with $F(z_0) = 0$, then also $\frac{z-\overline{z}_0}{z-z_0} F(z) \in L$.

Those dB-subspaces L of a given de Branges space $\mathcal{H}$ which additionally satisfy

(SubZ) $\mathfrak{d}(L) = \mathfrak{d}(\mathcal{H})$

are of particular importance. The set of all such dB-subspaces will be denoted by $\operatorname{Sub}^s(\mathcal{H})$. Note that, if $\mathcal{H}$ and L are written as $\mathcal{H}(E)$ and $\mathcal{H}(E_1)$, respectively, with some $E, E_1 \in \mathcal{H}B$, then the validity of (SubZ) just means that $\mathfrak{d}(E_1)|_{\mathbb{R}} = \mathfrak{d}(E)|_{\mathbb{R}}$.

1.6 Remark. In the situation of Remark 1.4, we have $L \in \operatorname{Sub}(\mathcal{H}(E))$ if and only if $S \cdot L \in \operatorname{Sub}(\mathcal{H}(SE))$, and also $L \in \operatorname{Sub}^s(\mathcal{H}(E))$ if and only if $S \cdot L \in \operatorname{Sub}^s(\mathcal{H}(SE))$.

One of the most fundamental and deep results in the theory of de Branges spaces is the so-called Ordering Theorem of de Branges, cf. [dB, Theorem 35] (we state only a somewhat weaker version which suffices for our needs):

1.7. de Branges' Ordering Theorem: *Let $\mathcal{H}$ be a de Branges space and let $\mathfrak{d}:\mathbb{R}\to\mathbb{N}\cup\{0\}$ be given. Then the set*

$$\big\{L\in\operatorname{Sub}(\mathcal{H}):\ \mathfrak{d}(L)=\mathfrak{d}\big\}$$

is totally ordered with respect to set-theoretic inclusion.

1.8 Example. An important example of a de Branges space is the classical *Paley-Wiener space* $\mathcal{P}W_a$, $a>0$. It can be defined as the space of all entire functions of exponential type at most a, whose restrictions to the real axis belong to $L^2(\mathbb{R})$. The norm in the space $\mathcal{P}W_a$ is given by the usual L^2-norm,

$$\|F\|:=\Big(\int_{\mathbb{R}}|F(t)|^2\,dt\Big)^{1/2},\ F\in\mathcal{P}W_a\,.$$

It is a consequence of a theorem of M.G. Krein, cf. [RR, Examples/Addenda 2, p. 134], that $\mathcal{P}W_a=\mathcal{H}(e^{-iaz})$. The chain $\operatorname{Sub}^s(\mathcal{P}W_a)$ is given as

$$\operatorname{Sub}^s(\mathcal{P}W_a)=\big\{\mathcal{P}W_b:\ 0<b\leq a\big\}\,.$$

Recall that, by the Theorem of Paley-Wiener, the space $\mathcal{P}W_a$ coincides with the Fourier image of the space of square summable functions supported on the interval $[-a,a]$.

1.9 Example. More general examples of de Branges spaces occur in the theory of canonical (or Hamiltonian) systems of differential equations, cf. e.g. [dB, Theorems 37,38], [GK], [HSW]. Let H be a 2×2-matrix valued function defined for $t\in[0,l]$, such that $H(t)$ is real and nonnegative, the entries of $H(t)$ belong to $L^1([0,l])$ and $H(t)$ does not vanish on any nonempty interval. We call an interval $(\alpha,\beta)\subseteq[0,l]$ H-indivisible, if for some $\varphi\in\mathbb{R}$ and some scalar function $h(t)$ we have

$$H(t)=h(t)\begin{pmatrix}\cos^2\varphi & \cos\varphi\sin\varphi\\ \cos\varphi\sin\varphi & \sin^2\varphi\end{pmatrix},\quad \text{a.e. } t\in(\alpha,\beta).$$

Let $W(t,z)$ be the (unique) solution of the initial value problem

$$\frac{\partial}{\partial t}W(t,z)J=zW(t,z)H(t),\ t\in[0,l]\,,$$
$$W(0,z)=I$$

where J denotes the matrix

$$J:=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

and $z\in\mathbb{C}$. The function H is called the Hamiltonian of this system.

Put $(A_t(z),B_t(z)):=(1,0)W(t,z)$, $t\in[0,l]$, and $E_t(z):=A_t(z)-iB_t(z)$. Then (*i*) $E_t \in \mathcal{H}B$, $t \in (0, l]$, and $E_0 = 1$.

(*ii*) If $0 < s \leq t \leq l$, then $\mathcal{H}(E_s) \subseteq \mathcal{H}(E_t)$ and the set-theoretic inclusion map is contractive. If s is not an inner point of an H-indivisible interval, it is actually isometric.

(*iii*) We have

$$\operatorname{Sub}^s(\mathcal{H}(E_l)) = \{\mathcal{H}(E_t) : t \text{ not inner point of } H\text{-indivisible interval}\}.$$

Paley-Wiener spaces can be realized in this way. In fact, if $H(t) = I$, $t \in [0, l]$, then $E_t(z) = e^{-itz}$.

c. Admissible majorants

Let us recall the notion of an admissible majorant of a de Branges space $\mathcal{H}(E)$, cf. [HM1, HM2].

1.10 Definition. Let $E \in \mathcal{H}B$. A nonnegative function ω on the real axis $\mathbb{R}$ is said to be an *admissible majorant* for the space $\mathcal{H}(E)$, if there exists a nonzero function $F \in \mathcal{H}(E)$ such that $|E(x)^{-1}F(x)| \leq \omega(x)$, $x \in \mathbb{R}$. The set of all admissible majorants for $\mathcal{H}(E)$ is denoted by $\operatorname{Adm}(E)$.

1.11 Remark. If $E_1, E_2 \in \mathcal{H}B$ generate the same space, i.e. $\mathcal{H}(E_1) = \mathcal{H}(E_2)$ including equality of norms, then $\operatorname{Adm}(E_1) = \operatorname{Adm}(E_2)$. This follows from an elementary estimate using Remark 1.3.

1.12 Remark. One may consider a slightly more general definition of admissible majorants, if one assumes that the estimate $|E^{-1}F| \leq \omega$ holds almost everywhere on $\mathbb{R}$ with respect to Lebesgue measure (as in [HM1, HM2]). However, since $E^{-1}F$ is continuous on $\mathbb{R}$ for any $F \in \mathcal{H}(E)$, the requirement that $|E^{-1}F| \leq \omega$ everywhere on $\mathbb{R}$ does not lead to any substantial loss of generality. In fact, if we restrict our attention to majorants ω which are semicontinuous from below, then these notions are equivalent.

A natural necessary condition for a function ω to be an admissible majorant is the convergence of the logarithmic integral

$$\int_{\mathbb{R}} \frac{\log^- \omega(x)}{1 + x^2}\, dx < \infty. \tag{1.2}$$

Here $\log^- t := \max(-\log t, 0)$. Indeed, since $F/E \in H^2(\mathbb{C}^+)$, we have

$$\int_{\mathbb{R}} \frac{\log^- \omega(x)}{1 + x^2}\, dx \leq \int_{\mathbb{R}} \frac{\log^- |F(x)/E(x)|}{1 + x^2}\, dx < \infty,$$

see, e.g., [HJ], p. 32–36.

The description of admissible majorants for the Paley-Wiener spaces is a classical problem of harmonic analysis. By what we just said, any admissible majorant for a space $\mathcal{P}W_a$ must satisfy (1.2). The fact that this, obvious, necessary condition is in many cases also sufficient is the content of the famous Beurling–Malliavin Multiplier Theorem, cf. [BM]:

1.13. ***Beurling-Malliavin Multiplier Theorem:*** *Let ω be a positive function on $\mathbb{R}$ satisfying (1.2), and assume that the function $\log\omega$ is Lipschitz on $\mathbb{R}$. Then ω is an admissible majorant for every space $\mathcal{P}W_a$, $a > 0$.*

This is one of the deepest results of harmonic analysis and several different proofs of it are known (see [HJ, HMN, K]). A typical example of a majorant admissible for all spaces $\mathcal{P}W_a$ is the function $\omega(x) = \exp(-|x|^\beta)$ where $\beta \in (0, 1)$. This theorem is referred to as *Multiplier Theorem* since it means that for any $a > 0$ there exists a nonzero multiplier $f \in \mathcal{P}W_a$ such that $f\omega^{-1} \in L^\infty(\mathbb{R})$.

Admissible majorants for general de Branges spaces (and even in a more general setting of the so-called star-invariant subspaces of the Hardy class) were studied for the first time by V.P. Havin and J. Mashreghi in [HM1, HM2], where a complete parametrization of the class $\operatorname{Adm}(E)$ is found and a number of conditions sufficient for admissibility are obtained. Further applications of this approach may be found in [BH, BBH] and in [HMN] where a new and essentially simpler proof of the Beurling-Malliavin theorem is given.

A certain subclass of admissible majorants is of particular interest.

1.14 Definition. Let $E \in \mathcal{HB}$. We say that an admissible majorant ω for $\mathcal{H}(E)$ is *separated from zero*, if each point $x \in \mathbb{R}$ has a neighbourhood $U(x) \subseteq \mathbb{R}$ such that

$$\inf\big\{\omega(t) : t \in U(x)\big\} > 0 .$$

The set of all admissible majorants for $\mathcal{H}(E)$ which are separated from zero will be denoted by $\operatorname{Adm}^+(E)$.

Note that, clearly, the condition for ω to be separated from zero is equivalent to the following: for every interval $[a, b] \subseteq \mathbb{R}$ we have $\inf\{\omega(t) : t \in [a, b]\} > 0$.

1.15 Example. Examples of admissible majorants can be obtained from elements of $\mathcal{H}(E)$. For $F \in \mathcal{H}(E) \setminus \{0\}$, consider the function

$$\omega_F(x) := \left|\frac{F(x)}{E(x)}\right|, \ x \in \mathbb{R} .$$

Then, by definition, $|E(x)^{-1}F(x)| \leq \omega_F(x)$, and hence ω_F is an admissible majorant for $\mathcal{H}(E)$. Clearly, in this situation, we have $\omega_F \in \operatorname{Adm}^+(E)$ if and only if $\mathfrak{d}(F)|_{\mathbb{R}} = \mathfrak{d}(E)|_{\mathbb{R}}$.

2 Subspaces in terms of admissible majorants

Throughout this paper we will use the following notation: we write $f \lesssim g$ if there exists a positive constant C such that $f \leq Cg$ for all admissible values of variables. Moreover, we write $f \asymp g$ if $f \lesssim g$ and $g \lesssim f$.

Since the relation $\lesssim$ is reflexive and transitive, it induces an order on equivalence classes of functions modulo the equivalence relation $\asymp$. In particular, given $E \in \mathcal{H}B$, we obtain an order on the set $\operatorname{Adm}(E)/_{\asymp}$ as well as on $\operatorname{Adm}^+(E)/_{\asymp}$. Clearly, $\operatorname{Adm}(E)$ and $\operatorname{Adm}^+(E)$ are saturated with respect to $\asymp$, i.e. if $\omega \in \operatorname{Adm}(E)$, or $\omega \in \operatorname{Adm}^+(E)$, and $\omega_1 \asymp \omega$, then also $\omega_1 \in \operatorname{Adm}(E)$, or $\omega_1 \in \operatorname{Adm}^+(E)$, respectively.

Admissible majorants give rise to dB-subspaces of $\mathcal{H}(E)$.

2.1 Definition. For $E \in \mathcal{H}B$ and $\omega \in \operatorname{Adm}(E)$ define

$$\mathcal{R}_\omega(E) := \operatorname{Clos}_{\mathcal{H}(E)} \left\{ F \in \mathcal{H}(E) : |E(x)^{-1}F(x)| \lesssim w(x), x \in \mathbb{R} \right\}.$$

2.2 Proposition. *Let $E \in \mathcal{H}B$ and let $\omega \in \operatorname{Adm}(E)$. Then the space $\mathcal{R}_\omega(E)$ is a dB-subspace of $\mathcal{H}(E)$. The assignment $\omega \mapsto \mathcal{R}_\omega(E)$ defines a monotone map of $\operatorname{Adm}(E)/_{\asymp}$ into $\operatorname{Sub}(\mathcal{H}(E))$. Moreover, $\omega \in \operatorname{Adm}^+(E)$ if and only if $\mathcal{R}_\omega(E) \in \operatorname{Sub}^s(\mathcal{H}(E))$.*

Proof. Since $\mathcal{R}_\omega(E)$ is by definition the closure of the linear space

$$R_\omega(E) := \left\{ F \in \mathcal{H}(E) : |E(x)^{-1}F(x)| \lesssim \omega(x), x \in \mathbb{R} \right\},$$

it is a closed linear subspace of $\mathcal{H}(E)$. We need to show that $\mathcal{R}_\omega(E)$ has the properties (Sub2) and (Sub3).

Clearly, $R_\omega(E)$ is invariant under the map $F \mapsto F^\#$. Since this map is continuous with respect to the norm of $\mathcal{H}(E)$, also $\mathcal{R}_\omega(E) = \operatorname{Clos}_{\mathcal{H}} R_\omega(E)$ is invariant under $F \mapsto F^\#$, i.e. (Sub2) holds.

Let $F \in R_\omega(E)$ and $v \in \mathbb{C} \setminus \mathbb{R}$ with $F(v) = 0$ be given. Then also

$$\frac{z - \bar{v}}{z - v} F(z) \in R_\omega(E),$$

i.e. $R_\omega(E) \cap \ker \chi_v$, where χ_v is the point evaluation functional at v, is mapped into $R_\omega(E)$ by the map $\Phi : F(z) \mapsto \frac{z-\bar{v}}{z-v} F(z)$. Note that, in particular, one can always find an element $G \in R_\omega(E)$ with $G(v) = 1$.

Since Φ maps $\ker\chi_v$ isometrically and, thus, continuously into $\mathcal{H}(E)$, it follows that

$$\Phi\big(\operatorname{Clos}_{\mathcal{H}(E)}(R_\omega(E)\cap\ker\chi_v)\big)\subseteq\operatorname{Clos}_{\mathcal{H}(E)}R_\omega(E)=\mathcal{R}_\omega(E)\,.$$

We shall establish (Sub3) by showing that

$$\operatorname{Clos}_{\mathcal{H}(E)}(R_\omega(E)\cap\ker\chi_v)=\mathcal{R}_\omega(E)\cap\ker\chi_v\,.$$

To see this let $F\in\mathcal{R}_\omega(E)\cap\ker\chi_v$ be given and choose $F_n\in R_\omega(E)$ such that $F_n\to F$. Moreover, choose $G\in R_\omega(E)$ with $G(v)=1$. Since $F_n(v)\to F(v)=0$, we have $F_n-F_n(v)G\to F$ and hence $F\in\operatorname{Clos}_{\mathcal{H}(E)}(R_\omega(E)\cap\ker\chi_v)$. The converse inclusion is trivial.

If $\omega_1,\omega_2\in\operatorname{Adm}(E)$, $\omega_1\lesssim\omega_2$, then, clearly, $R_{\omega_1}(E)\subseteq R_{\omega_2}(E)$ and, therefore, $\mathcal{R}_{\omega_1}(E)\subseteq\mathcal{R}_{\omega_2}(E)$. It follows that $\mathcal{R}_\omega(E)$ depends only on the equivalence class $\omega/_{\asymp}$ and is monotone.

We come to the proof of the last assertion. Let $\omega\in\operatorname{Adm}(E)$. Assume first that $\mathcal{R}_\omega(E)\in\operatorname{Sub}^s(\mathcal{H}(E))$ and let $t\in\mathbb{R}$ be given. Choose $F\in\mathcal{R}_\omega(E)$ with $\mathfrak{d}(F)(t)=\mathfrak{d}(E)(t)$. Then, by continuity, there exists $\delta>0$ and a compact neighbourhood $U(t)$ of t such that $|F(x)/E(x)|\geq\delta$, $x\in U(t)$. Choose a sequence $G_n\in R_\omega(E)$ such that $G_n\to F$ in the norm of $\mathcal{H}(E)$. Then G_n also converges to F locally uniformly. Since $\mathfrak{d}(G_n)(x)\geq\mathfrak{d}(E)(x)$ for all $n\in\mathbb{N}$ and $x\in\mathbb{R}$, by the Maximium Modulus Principle, $G_n/E\to F/E$ locally uniformly on $\mathbb{C}\setminus\{v\in\mathbb{C}^-:\ E(v)=0\}$. Hence there exists $n\in\mathbb{N}$ such that $|G_n(x)/E(x)|\geq\delta/2$, $x\in U(t)$. Let $C>0$ be such that $|E(x)^{-1}G_n(x)|\leq C\omega(x)$, $x\in\mathbb{R}$. It follows that

$$\inf_{x\in U(t)}\omega(x)\geq\frac{\delta}{2C}>0\,,$$

and we see that $\omega\in\operatorname{Adm}^+(E)$.

Conversely, assume that $\omega\in\operatorname{Adm}^+(E)$. Let $t\in\mathbb{R}$ be given and choose $F\in R_\omega(E)\setminus\{0\}$. Put $n:=\mathfrak{d}(F)(t)-\mathfrak{d}(E)(t)$, then $n\in\mathbb{N}\cup\{0\}$ and the function $\frac{F(z)}{(z-t)^n}$ belongs to $\mathcal{H}(E)$. Let $U(t)$ be a compact neighbourhood of t such that $\inf_{x\in U(t)}\omega(x)>0$. Then, by continuity of $(x-t)^nF(x)/E(x)$,

$$\left|\frac{F(x)}{(x-t)^nE(x)}\right|\lesssim\omega(x),\ x\in U(t)\,.$$

Since $\frac{1}{|x-t|}$ is bounded for $x\notin U(t)$, it is clear that

$$\left|\frac{F(x)}{(x-t)^nE(x)}\right|\lesssim\left|\frac{F(x)}{E(x)}\right|\lesssim\omega(x),\ x\notin U(t)\,.$$

We see that $\frac{F(z)}{(z-t)^n} \in R_\omega(E)$. Thus, $\mathcal{R}_\omega(E)$ contains an element G with $\mathfrak{d}(G)(t) = \mathfrak{d}(E)(t)$.

❑

2.3 Remark. Let us note that taking the closure $\mathrm{Clos}_{\mathcal{H}(E)}$ in the definition of $\mathcal{R}_\omega(E)$ is actually necessary in order to obtain de Branges subspaces. Although the linear space $R_\omega(E)$ always satisfies (Sub2) and (Sub3), it will not, in general, be closed. In fact, if one assumes that $\omega \in L^2(\mathbb{R})$, then the linear space $R_\omega(E)$ is not closed unless it is finite-dimensional. This is seen by an application of a theorem of Grothendieck with the probability measure

$$d\mu(x) := \frac{\omega^2(x)}{\int_{\mathbb{R}} \omega^2(t)\, dt}\, dx\,,$$

cf. [R, Theorem 5.2]. Note that the assumption $\omega \in L^2(\mathbb{R})$ is not too restrictive; for example, it is met by every admissible majorant of the form ω_F, $F \in \mathcal{H}(E) \setminus \{0\}$, cf. Example 1.15.

In connection with Remarks 1.4 and 1.6 the following observation is often useful.

2.4 Remark. Let $E \in \mathcal{H}B$ and let S be a real entire function which has no zeros off the real axis. Then $\mathrm{Adm}(E) = \mathrm{Adm}(SE)$ and $\mathrm{Adm}^+(E) = \mathrm{Adm}^+(SE)$. Moreover, for any $\omega \in \mathrm{Adm}(E)$, we have $\mathcal{R}_\omega(SE) = S \cdot \mathcal{R}_\omega(E)$.

Indeed, by Remark 1.4, the map $F \mapsto SF$ is an isometry of $\mathcal{H}(E)$ onto $\mathcal{H}(SE)$. Since, by analyticity, $\left|\frac{(SF)(x)}{(SE)(x)}\right| = \left|\frac{F(x)}{E(x)}\right|$, $x \in \mathbb{R}$, for all $F \in \mathcal{H}(E)$, we have $|E^{-1}F| \lesssim \omega$ if and only if $|(SE)^{-1}(SF)| \lesssim \omega$. Thus $\mathrm{Adm}(E) = \mathrm{Adm}(SE)$ and $R_\omega(SE) = S \cdot R_\omega(E)$. By the bicontinuity of the map $F \mapsto SF$, we also have $\mathcal{R}_\omega(SE) = S \cdot \mathcal{R}_\omega(E)$.

2.5 Definition. Let $E \in \mathcal{H}B$. Denote by $\mathfrak{R}$ the map

$$\mathfrak{R} : \begin{cases} \mathrm{Adm}(E) & \to & \mathrm{Sub}(\mathcal{H}(E)) \\ \omega & \mapsto & \mathcal{R}_\omega(E) \end{cases}$$

In the next theorem we characterize the dB-subspaces of a given space $\mathcal{H}(E)$ which are of the form $\mathcal{R}_\omega(E)$. This is the first main result of this paper.

2.6 Theorem. *Let $E, E_1 \in \mathcal{H}B$ be given, such that $\mathcal{H}(E_1) \in \mathrm{Sub}(\mathcal{H}(E))$. Then $\mathcal{H}(E_1) \in \mathfrak{R}(\mathrm{Adm}(E))$ if and only if $\mathrm{mt}\, \frac{E_1}{E} = 0$.*

2.7 Remark. The mean type condition in this theorem does not depend on the choice of E and E_1. In fact, by Remark 1.3, if $\mathcal{H}(E_1) = \mathcal{H}(E_2)$ with equality of norms, then $\mathrm{mt}\, \frac{E_1}{E_2} = \mathrm{mt}\, \frac{E_2}{E_1} = 0$.

In the proof of Theorem 2.6 we will use a class of dB-subspaces defined by a growth condition, cf. [KW1]: If $\mathcal{H}(E)$ is a de Branges space and $\beta_+, \beta_- \leq 0$, denote by $\mathcal{H}(E)_{(\beta_+,\beta_-)}$ the linear subspace

$$\mathcal{H}(E)_{(\beta_+,\beta_-)} := \left\{ F \in \mathcal{H}(E) : \operatorname{mt} \frac{F}{E} \leq \beta_+, \operatorname{mt} \frac{F^\#}{E} \leq \beta_- \right\}.$$

Then the space $\mathcal{H}(E)_{(\beta_+,\beta_-)}$ is closed. Moreover, if $\beta_+ = \beta_-$, it actually belongs to $\operatorname{Sub}^s(\mathcal{H}(E)) \cup \{0\}$, cf. [KW1, Lemma 2.6, Corollary 5.2].

2.8 Lemma. *Let $\mathcal{H}(E)$ be a de Branges space, $\beta < 0$, and assume that $\mathcal{H}(E)_{(\beta,\beta)} \neq \{0\}$. Then, for all $\beta' \in (\beta, 0]$, we have*

$$\dim \left(\mathcal{H}(E)_{(\beta',\beta')} / \mathcal{H}(E)_{(\beta,\beta)} \right) = \infty .$$

Proof. It is enough to show that for all β with $\mathcal{H}(E)_{(\beta,\beta)} \neq \{0\}$ and $\beta' \in (\beta, 0]$ we have $\mathcal{H}(E)_{(\beta,\beta)} \neq \mathcal{H}(E)_{(\beta',\beta')}$. To see this, choose $F \in \mathcal{H}(E)_{(\beta,\beta)} \setminus \{0\}$ and put $\alpha := \operatorname{mt} \frac{F}{E}$. Then the function $G(z) := e^{i(\alpha-\beta')z}F(z)$ belongs to $\mathcal{H}(E)$, cf. [KW1, Lemma 2.6], and satisfies $\operatorname{mt} \frac{G}{E} = \beta'$. Moreover, since $\alpha \leq \beta \leq \beta'$, we have $\operatorname{mt} \frac{G^\#}{E} = \alpha - \beta' + \frac{F^\#}{E} \leq \alpha - \beta' + \beta \leq \beta'$. Hence $G \in \mathcal{H}_{(\beta',\beta')} \setminus \mathcal{H}_{(\beta,\beta)}$. ❑

2.9 Lemma. *Let $E, E_1 \in \mathcal{H}B$, $\mathcal{H}(E_1) \in \operatorname{Sub}(\mathcal{H}(E))$, and $\beta < 0$ be given. Then $\mathcal{H}(E_1) \subseteq \mathcal{H}(E)_{(\beta,\beta)}$ if and only if $\operatorname{mt} \frac{E_1}{E} \leq \beta$.*

Proof. Assume that $\mathcal{H}(E)_{(\beta,\beta)} \neq \{0\}$. Then [KW1, Lemma 5.5] implies that $\mathcal{H}(E)_{(\beta,\beta)} = \mathcal{H}(E_\beta)$ with $E_\beta \in \mathcal{H}B$ and $\operatorname{mt} \frac{E_\beta}{E} = \beta$. Hence, if $\mathcal{H}(E_1) \subseteq \mathcal{H}(E)_{(\beta,\beta)}$, we get

$$\operatorname{mt} \frac{E_1}{E} = \underbrace{\operatorname{mt} \frac{E_1}{E_\beta}}_{\leq 0} + \operatorname{mt} \frac{E_\beta}{E} \leq \beta .$$

Conversely, if $\operatorname{mt} \frac{E_1}{E} \leq \beta$, we obtain for every $F \in \mathcal{H}(E_1) \setminus \{0\}$

$$\operatorname{mt} \frac{F}{E} = \operatorname{mt} \left(\frac{F}{E_1} \cdot \frac{E_1}{E} \right) = \operatorname{mt} \frac{F}{E_1} + \operatorname{mt} \frac{E_1}{E} \leq \beta .$$

Hence $F \in \mathcal{H}(E)_{(\beta,0)}$. Since with F also $F^\#$ belongs to $\mathcal{H}(E_1)$, the same argument will show that $F \in \mathcal{H}(E)_{(0,\beta)}$ and, therefore, $F \in \mathcal{H}(E)_{(\beta,\beta)}$. ❑

Proof (of Theorem 2.6). Let $E, E_1 \in \mathcal{H}B$, $\mathcal{H}(E_1) \in \operatorname{Sub}(\mathcal{H}(E))$, be given.

Sufficiency: Assume that $\operatorname{mt} \frac{E_1}{E} = 0$.
Since $\mathcal{H}(E_1) \in \operatorname{Sub}(\mathcal{H}(E))$, we have $\mathfrak{d}(E_1)|_{\mathbb{R}} \geq \mathfrak{d}(E)|_{\mathbb{R}}$. Define ω as

$$\omega(x) := \frac{|E_1(x)|}{(1+|x|)|E(x)|}, \ x \in \mathbb{R}, \tag{2.1}$$

then ω is a continuous and nonnegative function on $\mathbb{R}$. Let $v \in \mathbb{C} \setminus \mathbb{R}$ and consider the reproducing kernel

$$K_1(v,z) = \frac{E_1(z)\overline{E_1(v)} - E_1^{\#}(z)E_1(\bar{v})}{2\pi i(\bar{v}-z)}$$

of $\mathcal{H}(E_1)$. Then we have for $x \in \mathbb{R}$,

$$\begin{aligned} |K_1(v,x)| &= \frac{1}{2\pi}\left|\frac{E_1(x)\overline{E_1(v)} - E_1^{\#}(x)E_1(\bar{v})}{\bar{v}-x}\right| \leq \\ &\leq \frac{1}{\pi}\max\{|E_1(v)|,|E_1(\bar{v})|\} \cdot \max_{t\in\mathbb{R}} \frac{1+|t|}{|t-\bar{v}|} \cdot \frac{|E_1(x)|}{1+|x|} = C\omega(x)|E(x)| \end{aligned} \tag{2.2}$$

where $C := \frac{1}{\pi}\max\{|E_1(v)|,|E_1(\bar{v})|\}\max_{t\in\mathbb{R}} \frac{1+|t|}{|t-\bar{v}|}$. Hence $|E(x)^{-1}K_1(v,x)| \lesssim \omega(x)$, $E(x) \neq 0$, and by continuity this inequality holds for all $x \in \mathbb{R}$. Hence $\omega \in \operatorname{Adm}(E)$ and $K_1(v,\cdot) \in R_\omega(E)$. Since the linear span of the reproducing kernels $K_1(v,\cdot)$, $v \in \mathbb{C}\setminus\mathbb{R}$, is dense in $\mathcal{H}(E_1)$, we conclude that $\mathcal{H}(E_1) \subseteq \mathfrak{R}(\omega)$.

Conversely, let $F \in R_\omega(E)$. Then $F \in \mathcal{H}(E)$ and, since by [KW1, §2] always $E_1 \in \mathcal{H}(E) + z\mathcal{H}(E)$,

$$\frac{F}{E_1} = \frac{F}{E} \cdot \frac{E}{E_1} \in N(\mathbb{C}^+).$$

Moreover, by our assumption that $\operatorname{mt} \frac{E_1}{E} = 0$, we have

$$\operatorname{mt} \frac{F}{E_1} = \operatorname{mt} \frac{F}{E} + \operatorname{mt} \frac{E}{E_1} = \operatorname{mt} \frac{F}{E} \leq 0.$$

Since $F^{\#}$ also belongs to $\mathcal{H}(E)$ whenever F does, this argument also applies to $F^{\#}$ and we obtain

$$\frac{F^{\#}}{E_1} \in N(\mathbb{C}^+), \ \operatorname{mt} \frac{F^{\#}}{E_1} \leq 0.$$

Since $F \in R_\omega(E)$, i.e. $|F(x)| \lesssim \omega(x)|E(x)|$, $x \in \mathbb{R}$, we have

$$\left|\frac{F(x)}{E_1(x)}\right| \lesssim \frac{1}{1+|x|} \in L^2(\mathbb{R}).$$

It follows that $F \in \mathcal{H}(E_1)$ for any $F \in R_\omega(E)$. Thus, also $\mathfrak{R}(\omega) \subseteq \mathcal{H}(E_1)$. Altogether, we conclude that $\mathcal{H}(E_1) = \mathfrak{R}(\omega)$.

Necessity: Assume that $\mathcal{H}(E_1) = \mathfrak{R}(\omega)$ *for some* $\omega \in \operatorname{Adm}(E)$.

Let us assume on the contrary that $\operatorname{mt} \frac{E_1}{E} = \beta < 0$. Then, by Lemma 2.9, $\mathcal{H}(E_1) \subseteq \mathcal{H}_{(\beta,\beta)}$.

Consider the map

$$\Phi : \begin{cases} \mathcal{H}_{(\beta,\beta)} & \to & \mathcal{H}(E) \\ F(z) & \mapsto & e^{i\beta z} F(z) \end{cases}$$

cf. [KW1, Lemma 2.6]. This map is isometric and, therefore, continuous. Since $R_\omega(E) \subseteq \mathcal{H}_{(\beta,\beta)}$ it follows that $\Phi(R_\omega(E)) \subseteq \mathcal{H}(E)$. Clearly, we have $|\Phi(F)(x)| = |F(x)|$, $x \in \mathbb{R}$. Thus, $\Phi(R_\omega(E)) \subseteq R_\omega(E)$ and, consequently, $\Phi(\mathcal{R}_\omega(E)) \subseteq \mathcal{R}_\omega(E)$. Hence, if $F \in \mathcal{R}_\omega(E)$, then for every $n \in \mathbb{N}$ we have

$$\Phi^n(F) \in \mathcal{R}_\omega(E) \subseteq \mathcal{H}(E)\,.$$

However,

$$\operatorname{mt} \frac{\Phi^n(F)}{E} = \operatorname{mt} \frac{e^{in\beta z} F(z)}{E(z)} = -n\beta + \operatorname{mt} \frac{F}{E}\,.$$

If $F \neq 0$ and n is chosen sufficiently large, we have a contradiction since, due to the inclusion $\Phi^n(F) \in \mathcal{H}(E)$, always $\operatorname{mt} \frac{\Phi^n(F)}{E} \leq 0$ must hold.

❑

As a byproduct of the proof of Theorem 2.6 we obtain the following result which will be of importance in our further investigation of the structure of $\operatorname{Adm}(E)$.

2.10 Corollary. *Let* $E \in \mathcal{H}B$ *and* $\omega \in \operatorname{Adm}(E)$. *Then there exists* $F \in \mathcal{H}(E) \setminus \{0\}$ *such that* $\mathfrak{R}(\omega) = \mathfrak{R}(\omega_F)$.

Proof. Choose $E_1 \in \mathcal{H}B$ such that $\mathfrak{R}(\omega) = \mathcal{H}(E_1)$. Then $\operatorname{mt} \frac{E_1}{E} = 0$, and, as we have seen in the proof of sufficiency of Theorem 2.6,

$$\mathcal{H}(E_1) = \mathfrak{R}\Big(\frac{|E_1(x)|}{(1+|x|)|E(x)|} \Big)\,.$$

Let K_1 be the reproducing kernel of $\mathcal{H}(E_1)$ and fix $v \in \mathbb{C}^+$. Then, cf. (2.2),

$$|K_1(v,x)| \lesssim \frac{|E_1(x)|}{1+|x|}\,.$$

On the other hand,

$$|K_1(v,x)| \geq \frac{1}{2\pi|\bar{v}-x|} \big| |E_1(x)\overline{E_1(v)}| - |\overline{E_1(x)}E_1(\bar{v})| \big| \geq$$

$$\geq \frac{1}{2\pi} \min_{t\in\mathbb{R}} \frac{1+|t|}{|t-\bar{v}|} \frac{|E_1(x)|}{1+|x|} \big| |\overline{E_1(v)}| - |E_1(\bar{v})| \big| .$$

Since $v \in \mathbb{C}^+$, we have $|E_1(v)| > |E_1(\bar{v})|$. It follows that

$$|K_1(v,x)| \asymp \frac{|E_1(x)|}{1+|x|} ,$$

and hence

$$\mathfrak{R}\Big(\frac{|E_1(x)|}{(1+|x|)|E(x)|}\Big) = \mathfrak{R}(\omega_{K_1(v,\cdot)}) .$$

❑

The dB-subspaces of highest interest are those which satisfy (SubZ), i.e. the elements of $\operatorname{Sub}^s(\mathcal{H}(E))$. Correspondingly, the admissible majorants of highest interest are the elements of $\operatorname{Adm}^+(E)$. From Theorem 2.6 we deduce a characterization of $\mathfrak{R}(\operatorname{Adm}^+(E))$.

2.11 Proposition. *Let $E, E_1 \in \mathcal{H}B$ be such that $\mathcal{H}(E_1) \in \operatorname{Sub}(\mathcal{H}(E))$. Then the following are equivalent:*

- (i) $\mathcal{H}(E_1) \in \mathfrak{R}(\operatorname{Adm}^+(E))$;
- (ii) $\mathcal{H}(E_1) \in \operatorname{Sub}^s(\mathcal{H}(E))$ *and* $\operatorname{mt} \frac{E_1}{E} = 0$;
- (iii) $\mathcal{H}(E_1) \in \operatorname{Sub}^s(\mathcal{H}(E))$ *and* $\mathcal{H}(E_1) \supseteq \bigcup_{\beta<0} \mathcal{H}(E)_{(\beta,\beta)}$.

Proof. Combining Theorem 2.6 and Proposition 2.2 we immediately see that (i) is equivalent to (ii).

Assume that (ii) holds. Since $\operatorname{Sub}^s(\mathcal{H}(E)) \cup \{0\}$ is totally ordered with respect to set theoretic inclusion, we either have

(1) $\exists\, \beta < 0 \,:\, \mathcal{H}(E_1) \subseteq \mathcal{H}(E)_{(\beta,\beta)}$;

or

(2) $\forall\, \beta < 0 \,:\, \mathcal{H}(E_1) \supseteq \mathcal{H}(E)_{(\beta,\beta)}$.

In the second case $\mathcal{H}(E_1) \supseteq \bigcup_{\beta<0} \mathcal{H}(E)_{(\beta,\beta)}$, i.e. (iii) is valid. In the first case we obtain from Lemma 2.9 that $\operatorname{mt} \frac{E_1}{E} \leq \beta$ for some $\beta < 0$ which contradicts (ii). Hence the case (1) cannot take place and we are done.

Conversely, assume that (iii) holds. If we had $\beta_0 := \operatorname{mt} \frac{E_1}{E} < 0$, then Lemma 2.9 would imply $\mathcal{H}(E_1) \subseteq \mathcal{H}(E)_{(\beta_0,\beta_0)}$. By (iii), $\bigcup_{\beta<0} \mathcal{H}(E)_{(\beta,\beta)} \subseteq \mathcal{H}(E)_{(\beta_0,\beta_0)}$. This contradicts Lemma 2.8, since $\mathcal{H}(E_1) \neq \{0\}$.

❑

From this result we obtain a criterion for density of a set $R_\omega(E)$ in $\mathcal{H}(E)$. Results of this type are of interest since density of $R_\omega(E)$ means that all elements of $\mathcal{H}(E)$ can be approximated by functions F satisfying $|E^{-1}F| \lesssim \omega$ on the real axis, i.e. by, in a certain sense, 'small' functions.

2.12 Corollary. *Let $E \in \mathcal{H}B$.*

(*i*) *If the linear space $\mathcal{L}_0 := \bigcup_{\beta<0} \mathcal{H}(E)_{(\beta,\beta)}$ is dense in $\mathcal{H}(E)$, then for every $\omega \in \operatorname{Adm}^+(E)$ the linear space $R_\omega(E)$ is dense in $\mathcal{H}(E)$. Unless $\dim \mathcal{H}(E) = 1$, also the converse holds.*

(*ii*) *Assume that $\operatorname{Clos}_{\mathcal{H}(E)} \mathcal{L}_0 = \mathcal{H}(E)$ and let $F_0 \in \mathcal{H}(E)$, $\mathfrak{d}(F_0)|_{\mathbb{R}} = \mathfrak{d}(E)|_{\mathbb{R}}$. Then the set*

$$\big\{F \in \mathcal{H}(E) : |F(x)| \lesssim |F_0(x)|, x \in \mathbb{R}\big\}$$

is dense in $\mathcal{H}(E)$.

Proof. The asserted implication in (i) follows immediately from Proposition 2.11, $(i) \Rightarrow (iii)$. To prove the converse, let $\dim \mathcal{H}(E) > 1$ and assume that $\mathcal{L}_0$ is not dense in $\mathcal{H}(E)$. If $\mathcal{L}_0 = \{0\}$, let L be any element of $\operatorname{Sub}^s(\mathcal{H}(E)) \setminus \{\mathcal{H}(E)\}$. Note that this set is nonempty since $\dim \mathcal{H}(E) > 1$. If $\mathcal{L}_0 \neq \{0\}$, put $L := \operatorname{Clos}_{\mathcal{H}(E)} \mathcal{L}_0$. Then, also in this case, $L \in \operatorname{Sub}^s(\mathcal{H}(E)) \setminus \{\mathcal{H}(E)\}$. By Proposition 2.11, we have $L = \mathfrak{R}(\omega_0)$ for some $\omega_0 \in \operatorname{Adm}^+(E)$. We see that $R_{\omega_0}(E)$ is not dense in $\mathcal{H}(E)$.

To establish the assertion (ii), apply (i) with the majorant ω_{F_0}.

❑

We would like to illustrate the above statements by some examples. First let us make explicit two extreme cases.

2.13 Example. Let $E \in \mathcal{H}B$.

(*i*) Assume that $\tau_E := \operatorname{mt} \frac{E^\#}{E} < 0$. Then $\mathcal{R}_\omega(E) = \mathcal{H}(E)$ for all $\omega \in \operatorname{Adm}^+(E)$. Indeed, in the present situation we have, by [KW1, Theorem 2.7, (ii)],

$$\operatorname{Clos}_{\mathcal{H}(E)} \bigcup_{\beta<0} \mathcal{H}(E)_{(\beta,\beta)} = \mathcal{H}(E)\,.$$

(*ii*) Assume that E is of zero exponential type. Then every element $L \in \operatorname{Sub}^s(E)$ can be written as $L = \mathfrak{R}(\omega)$ for some $\omega \in \operatorname{Adm}^+(E)$. In particular, assuming $\dim \mathcal{H}(E) > 1$, there exist admissible majorants in $\mathrm{Adm}^+(E)$ such that $R_\omega(E)$ is not dense.

To see this, note that, by [KW1, Lemma 5.6], we have $\mathcal{H}(E)_{(\beta,\beta)} = \{0\}$ for all $\beta < 0$. An application of Proposition 2.11 yields the present assertion.

Next we look at the classical setting of Paley-Wiener spaces.

2.14 Example. The Paley-Wiener space $\mathcal{P}W_a = \mathcal{H}(e^{-iaz})$, $a > 0$, satisfies the condition of Example 2.13, (i). We conclude that, whenever $\omega \in \mathrm{Adm}^+(e^{-iaz})$, the set

$$\big\{F \in \mathcal{P}W_a :\, |F(x)| \lesssim \omega(x), x \in \mathbb{R}\big\} \tag{2.3}$$

is dense in $\mathcal{P}W_a$.

In particular, this can be applied to $\omega(x) = |F_0(x)|$ whenever F_0 is an entire function of exponential type at most a which belongs to $L^2(\mathbb{R})$ and has no real zeros. By the Beurling-Malliavin Theorem, the set (2.3) is dense also if $\omega : \mathbb{R} \to (0,\infty)$ satisfies (1.2) and $\log w$ is Lipschitz on $\mathbb{R}$.

Finally, let us give an example where some, but not all, dB-subspaces can be realized as $\mathfrak{R}(\omega)$. This example also shows that the concepts of dB-subspaces defined by majorants on the one hand and by mean type conditions on the other, are in a way complementary.

2.15 Example. Consider a canonical system on $[0,l]$ with Hamiltonian H, cf. Example 1.9. Then we have $1 \in \mathcal{H}(E_t) + z\mathcal{H}(E_t)$, $t \in (0,l]$. The function E_t belongs to $N(\mathbb{C}^+)$ and

$$\tau(t) := \mathrm{mt}\, E_t = \int_0^t \sqrt{\det H(s)}\, ds\,.$$

Note that τ is a continuous and nondecreasing function on $[0,l]$. We obtain from Proposition 2.11 that a space $\mathcal{H}(E_t)$, where $t \in (0,l]$ is not an inner point of an indivisible interval, belongs to $\mathfrak{R}(\mathrm{Adm}^+(E_l))$ if and only if $\tau(t) = \tau(l)$. On the other hand, by Lemma 2.9, we have for $\beta \leq 0$

$$\mathcal{H}(E_l)_{(\beta,\beta)} = \begin{cases} \mathcal{H}(E_{s(\beta)}), & s(\beta) > 0 \\ \{0\}, & \text{otherwise} \end{cases}$$

where

$$s(\beta) := \sup\big\{t \in [0,l] :\, \tau(t) = \tau(l) + \beta\big\}\,.$$

We close this section with a result which shows that subspaces $\mathfrak{R}(\omega)$ where $\omega \in \mathrm{Adm}(E)$ has a certain monotonicity property, are either finite dimensional or, in a certain sense, quite big, cf. Proposition 2.17 below. Its proof relies on the following lemma.

2.16 Lemma. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space whose elements are entire functions, let $\alpha \in \mathbb{C}$, $|\alpha| > 1$, and assume that $F(\alpha z) \in \mathcal{H}$ whenever $F(z) \in \mathcal{H}$. Then $\dim \mathcal{H} < \infty$ and $\mathcal{H}$ consists entirely of polynomials.*

Proof. Consider the linear operator

$$\Phi : \begin{cases} \mathcal{H} & \to \ \mathcal{H} \\ F(z) & \mapsto \ F(\alpha z) \end{cases}$$

Since point evaluation is continuous in $\mathcal{H}$, its graph is closed. By the Closed Graph Theorem, therefore, Φ is bounded.

Put $\Phi_1 := \frac{1}{\|\Phi\|}\Phi$. Then, for every $F \in \mathcal{H}$ and $n \in \mathbb{N}$, we have

$$\Phi_1^n(F) = \frac{F(\alpha^n z)}{\|\Phi\|^n} \in \mathcal{H}, \ \|\Phi_1^n(F)\| \leq \|F\| \,.$$

Let $M := \sup_{|z|=1} \|\chi_z\|$ where χ_z denotes the point evaluation functional at z. Then, by the Principle of Uniform Boundedness, $M < \infty$ and we have

$$\left| \frac{F(\alpha^n z)}{\|\Phi\|^n} \right| \leq M \|\Phi_1^n(F)\| \leq M\|F\|, \ |z| = 1 \,. \tag{2.4}$$

Let F have the power series expansion $F(z) = \sum_{k=0}^{\infty} a_k z^k$. Then for any $r > 0$

$$a_k = \frac{1}{2\pi i} \oint_{|\zeta|=r} \frac{F(\zeta)}{\zeta^{k+1}} \, d\zeta$$

and, thus,

$$|a_k| \leq \frac{1}{r^k} \max_{|\zeta|=r} |F(\zeta)| \,.$$

If we put $r = \alpha^n$ and use (2.4), it follows that

$$|a_k| \leq \frac{1}{|\alpha|^{nk}} M\|F\| \cdot \|\Phi\|^n = M\|F\| \left(\frac{\|\Phi\|}{|\alpha|^k} \right)^n \,.$$

Since $n \in \mathbb{N}$ was arbitrary, this implies that for $k > \frac{\ln\|\Phi\|}{\ln|\alpha|}$ we must have $a_k = 0$. Hence every element of $\mathcal{H}$ is a polynomial of degree at most $\frac{\ln\|\Phi\|}{\ln|\alpha|}$. In particular, $\dim \mathcal{H} < \infty$.

❑

2.17 Proposition. *Let $E \in \mathcal{HB}$ and let $\omega \in \operatorname{Adm}(E)$ be continuous. Then* (*at least*) *one of the following holds true:*

(*i*) $\dim \mathcal{R}_\omega(E) < \infty$ *and* $\mathcal{R}_\omega(E)$ *consists entirely of polynomials;*

(*ii*) *whenever $k > 1$ is such that $\omega(kx)|E(kx)| \lesssim \omega(x)|E(x)|$, there exists $F(z) \in \mathcal{R}_\omega(E)$ with $F(kz) \notin \mathcal{H}(E)$.*

Proof. Assume that there exists $k > 1$ such that $\omega(kx)|E(kx)| \lesssim \omega(x)|E(x)|$ and that $F(kz) \in \mathcal{H}(E)$ whenever $F \in \mathcal{R}_\omega(E)$. By the preceding lemma, it suffices to show that, actually, $F(kz) \in \mathcal{R}_\omega(E)$ whenever $F \in \mathcal{R}_\omega(E)$.

To this end, consider the map

$$\Phi : \begin{cases} \mathcal{R}_\omega(E) & \to & \mathcal{H}(E) \\ F(z) & \mapsto & F(kz) \end{cases}$$

By the Closed Graph Theorem, the present hypothesis implies that Φ is continuous. However,

$$|F(kz)| \lesssim \omega(kx)|E(kx)| \lesssim \omega(x)|E(x)|,\ F \in R_\omega(E)\,,$$

and, thus, $\Phi(R_\omega(E)) \subseteq R_\omega(E)$. By continuity, also $\Phi(\mathcal{R}_\omega(E)) \subseteq \mathcal{R}_\omega(E)$.

❑

2.18 Remark. Note that, if $E_1 \in \mathcal{HB}$ has no real zeros and $d := \dim \mathcal{H}(E_1) < \infty$, then

$$\mathcal{H}(E_1) = \operatorname{span}\left\{1, z, \dots, z^{d-1}\right\}.$$

Hence in this case, $F(kz) \in \mathcal{H}(E_1)$ for all $F \in \mathcal{H}(E_1)$. This shows that if in Proposition 2.17 we have $\omega \in \operatorname{Adm}^+(E)$, the cases (*i*) and (*ii*) exclude each other.

3 Minimal majorants

In this section we will have a closer look at the order structure of $\operatorname{Adm}(E)/_\asymp$ and $\operatorname{Adm}^+(E)/_\asymp$, respectively.

It is, for example, trivial that if $\omega \in \operatorname{Adm}(E)$ and $\omega_1 : \mathbb{R} \to [0, \infty)$ is such that $\omega \lesssim \omega_1$, then also $\omega_1 \in \operatorname{Adm}(E)$. Consequently, every finite subset of $\operatorname{Adm}(E)$ (or of $\operatorname{Adm}^+(E)$, respectively) has an upper bound. Also, it is trivial, that $\operatorname{Adm}(E)$ cannot contain maximal elements. More intriguing is the question of existence of lower bounds or minimal elements.

3.1 Definition. An admissible majorant ω is said to be *minimal* if its equivalence class $\omega/_\asymp$ is a minimal element of $\mathrm{Adm}(E)/_\asymp$. This means that for every admissible majorant $\tilde{\omega}$ with $\tilde{\omega} \lesssim \omega$, we must have $\tilde{w} \asymp w$.

Our investigation is based on the following result, which shows that minimal admissible majorants correspond to one-dimensional dB-subspaces of $\mathcal{H}(E)$.

3.2 Theorem. *Let $E \in \mathcal{H}B$. If $\omega \in \mathrm{Adm}(E)$ is minimal in $\mathrm{Adm}(E)/_\asymp$, then $\dim \mathcal{R}_\omega(E) = 1$. Conversely, if $\omega \in \mathrm{Adm}(E)$ and $\dim \mathcal{R}_\omega(E) = 1$, then there exists $\omega_0 \in \mathrm{Adm}(E)$, which is minimal in $\mathrm{Adm}(E)/_\asymp$, such that*

$$\mathcal{R}_\omega(E) = \mathcal{R}_{\omega_0}(E) .$$

Proof.

Step 1: Let $\omega \in \mathrm{Adm}(E)$ and assume that $\dim \mathcal{R}_\omega(E) > 1$. We show that ω is not minimal in $\mathrm{Adm}(E)/_\asymp$.

Since $\dim \mathcal{R}_\omega(E) > 1$, we also have $\dim R_\omega(E) > 1$. Choose linearly independent elements F_1, F_2 of $R_\omega(E)$. Fix $v \in \mathbb{C} \setminus \mathbb{R}$ and choose $\alpha_1, \alpha_2 \in \mathbb{C}$, not both zero, such that $\alpha_1 F_1(v) + \alpha_2 F_2(v) = 0$. Put

$$F(z) := \frac{\alpha_1 F_1(z) + \alpha_2 F_2(z)}{z - v} .$$

Then $F \in R_\omega(E)$ and does not vanish identically. Hence $\omega_F \lesssim \omega$. However, we have

$$\Big|\frac{F(x)}{E(x)}\Big| \lesssim \frac{\omega(x)}{1+|x|},\ x \in \mathbb{R} ,$$

and hence $\inf_{x\in\mathbb{R}} \frac{\omega_F(x)}{\omega(x)} = 0$. Thus, $\omega \not\lesssim \omega_F$. It follows that ω is not minimal in $\mathrm{Adm}(E)/_\asymp$.

Step 2: Let $F \in \mathcal{H}(E) \setminus \{0\}$ and $\dim \mathcal{R}_{\omega_F}(E) = 1$. Then ω_F is minimal in $\mathrm{Adm}(E)/_\asymp$.

Let $\omega \in \mathrm{Adm}(E)$ be given such that $\omega \lesssim \omega_F$, and choose $G \in R_\omega(E) \setminus \{0\}$. Then G also belongs to $R_{\omega_F}(E)$. Our assumption that $\dim \mathcal{R}_{\omega_F}(E) = 1$ implies $F = \lambda G$ for some $\lambda \in \mathbb{C}$. It follows that, for some appropriate constant $C > 0$,

$$\omega_F(x) = \left|\frac{F(x)}{E(x)}\right| = |\lambda| \left|\frac{G(x)}{E(x)}\right| \leq C|\lambda|\omega(x) .$$

Hence $\omega_F \lesssim \omega$, and we see that ω_F is minimal in $\mathrm{Adm}(E)/_\asymp$.

Step 3: Let $\omega \in \operatorname{Adm}(E)$ and $\dim \mathcal{R}_\omega(E) = 1$. Then for every $F \in \mathcal{R}_\omega(E) \setminus \{0\}$ we have $\omega_F \lesssim \omega$ and $\mathcal{R}_\omega(E) = \mathcal{R}_{\omega_F}(E)$.

Fix $F \in \mathcal{R}_\omega(E) \setminus \{0\}$, and consider ω_F. Since $\mathcal{R}_\omega(E)$ is finite-dimensional, we have $\mathcal{R}_\omega(E) = R_\omega(E)$. Thus, $\omega_F \lesssim \omega$ and so $\mathcal{R}_{\omega_F}(E) \subseteq \mathcal{R}_\omega(E)$. Since $\dim \mathcal{R}_\omega(E) = 1$, this implies that $\dim \mathcal{R}_{\omega_F}(E) = 1$, and thus, clearly, also $\mathcal{R}_{\omega_F}(E) = \mathcal{R}_\omega(E)$.

Step 4: The proof of the theorem is now easily completed. Assume that ω is minimal, then by Step 1 we must have $\dim \mathcal{R}_\omega(E) = 1$. Assume conversely that $\dim \mathcal{R}_\omega(E) = 1$. Choose $F \in \mathcal{R}_\omega(E) \setminus \{0\}$, then, by Step 3, $\mathcal{R}_\omega(E) = \mathcal{R}_{\omega_F}(E)$ and, by Step 2, ω_F is minimal.

❑

3.3 Corollary. *Let $\omega \in \operatorname{Adm}(E)$. Then ω is minimal in $\operatorname{Adm}(E)/_{\asymp}$ if and only if $\dim \mathcal{R}_\omega(E) = 1$ and $\omega \asymp \omega_F$ for some $F \in \mathcal{H}(E) \setminus \{0\}$. In this case $\omega \asymp \omega_F$ for any $F \in \mathcal{R}_\omega(E) \setminus \{0\}$.*

Proof. Assume that $\omega \in \operatorname{Adm}(E)$ is minimal in $\operatorname{Adm}(E)/_{\asymp}$. By the above theorem we have $\dim \mathcal{R}_\omega(E) = 1$. By Step 3 of its proof, for $F \in R_\omega(E) \setminus \{0\}$, the majorant ω_F satisfies $\omega_F \lesssim \omega$. By minimality of ω, this implies $\omega_F \asymp \omega$. The converse is just Step 2 of the above proof.

❑

3.4 Remark. It should be emphasized that, if $\omega \in \operatorname{Adm}(E)$ has the property that $\dim \mathcal{R}_\omega(E) = 1$, it does not necessarily follow that ω itself is minimal.

For example, let $E(z) := (z + i)(z + 2i)$. Then $\mathcal{H}(E) = \operatorname{span}\{1, z\}$ and we see that $\omega(x) := |E(x)|^{-1}$ and $\omega_1(x) := |E(x)|^{-1}\sqrt{|x| + 1}$ belong to $\operatorname{Adm}^+(E)$ and that

$$\mathcal{R}_\omega(E) = \mathcal{R}_{\omega_1}(E) = \operatorname{span}\{1\}\,.$$

However, clearly, ω is essentially smaller than ω_1.

Next let us note that minimal admissible majorants always exist.

3.5 Corollary. *Let $E \in \mathcal{H}B$ and assume that $\dim \mathcal{H}(E) > 1$. Then the set $\operatorname{Adm}(E)/_{\asymp}$ contains uncountably many minimal elements.*

Proof. Let $x_0 \in \mathbb{R}$ and consider the function $S_\alpha(z) := e^{i\alpha}E(z) - e^{-i\alpha}E^\#(z)$, $\alpha \in [0, \pi)$. Then there exists at most one number $\alpha \in [0, \pi)$ such that $S_\alpha \in \mathcal{H}(E)$. Let $t \in \mathbb{R}$ be given and assume that t is not a zero of a function

S_α which belongs to $\mathcal{H}(E)$. Then, by [dB, Theorem 22], the space $\mathcal{R}_{\omega_{K(t,\cdot)}}(E)$ is one-dimensional; in fact

$$\mathcal{R}_{\omega_{K(t,\cdot)}}(E) = \operatorname{span}\{K(t,\cdot)\}\,.$$

By Corollary 3.3, $\omega_{K(t,\cdot)}$ is minimal in $\operatorname{Adm}(E)/_{\asymp}$.

Since $\dim \mathcal{H}(E) > 1$, no two of the elements $K(t,\cdot)$, $t \in \mathbb{R}$, are linearly dependent. Hence no two of the spaces $\mathcal{R}_{\omega_{K(t,\cdot)}}(E)$ coincide and, therefore, no two of the majorants $\omega_{K(t,\cdot)}$ define the same equivalence class in $\operatorname{Adm}(E)/_{\asymp}$.

❑

For admissible majorants separated from zero the situation is significantly different. Below we will show that the set $\operatorname{Adm}^+(E)/_{\asymp}$ need not necessarily contain minimal elements and give a criterion for the existence of minimal elements in $\operatorname{Adm}^+(E)/_{\asymp}$.

We start with a simple observation which shows that for a majorant $\omega \in \operatorname{Adm}^+(E)$ minimality in $\operatorname{Adm}^+(E)/_{\asymp}$ is the same as minimality in $\operatorname{Adm}(E)/_{\asymp}$.

3.6 Lemma. *Let $\omega \in \operatorname{Adm}^+(E)$ be given. Then $\omega/_{\asymp}$ is a minimal element in $\operatorname{Adm}^+(E)/_{\asymp}$ if and only if it is minimal in $\operatorname{Adm}(E)/_{\asymp}$.*

Proof. Let $\omega/_{\asymp}$ be a minimal element of $\operatorname{Adm}^+(E)/_{\asymp}$ and assume that $\omega_1 \in \operatorname{Adm}(E)$ is such that $\omega_1 \leq \omega$ and $\inf_{\mathbb{R}} \omega_1/\omega = 0$. It is elementary to see that, since $\omega \in \operatorname{Adm}^+(E)$, there exists a function ω_2 separated from zero and such that $\omega_1 \leq \omega_2 \leq \omega$, $\inf_{\mathbb{R}} \omega_2/\omega = 0$. Thus, $\omega_2 \in \operatorname{Adm}^+(E)$, $\omega_2 \lesssim \omega$, but $\omega_2 \not\asymp \omega$, which contradicts the minimality of $\omega/_{\asymp}$ in $\operatorname{Adm}^+(E)/_{\asymp}$.

❑

Moreover, let us make the following observation.

3.7 Lemma. *Let $E \in \mathcal{H}B$. Then the space $\mathcal{H}(E)$ contains a real function S with*

$$\mathfrak{d}(S)|_{\mathbb{R}} = \mathfrak{d}(E)|_{\mathbb{R}} \text{ and } \mathfrak{d}(S)|_{\mathbb{C}\setminus\mathbb{R}} = 0\,, \tag{3.1}$$

if and only if there exists $L \in \operatorname{Sub}^s(\mathcal{H}(E))$ such that $\dim L = 1$. In this case there exists, up to constant real multiples, exactly one real function $S \in \mathcal{H}(E)$ which satisfies (3.1).

Proof. If $S = S^\#$ and (3.1) holds, then clearly $L := \operatorname{span}\{S\}$ satisfies (Sub1)-(Sub3) and (SubZ). Conversely, assume that $L \in \operatorname{Sub}^s(\mathcal{H}(E))$ is one-dimensional. By (Sub2) there exists $S = S^\# \in L \setminus \{0\}$. Since, for a zero

v of S, the functions $S(z)$ and $\frac{S(z)}{z-v}$ are linearly independent, it follows from (Sub3) and (1.1) that S must satisfy (3.1).

If S_1, S_2 are real elements of $\mathcal{H}(E)$ which both satisfy (3.1), then $\operatorname{span}\{S_1\}$ and $\operatorname{span}\{S_2\}$ are one-dimensional elements of $\operatorname{Sub}^s(\mathcal{H}(E))$. Hence, by the Ordering Theorem, $\operatorname{span}\{S_1\} = \operatorname{span}\{S_2\}$.

❑

Combining Theorem 3.2 with Theorem 2.6 now leads to

3.8 Theorem. *Let $E \in \mathcal{H}B$. Then there exists a minimal element in $\operatorname{Adm}^+(E)/_\asymp$ if and only if the following hold:*

(*i*) *there exists $L \in \operatorname{Sub}^s(\mathcal{H}(E))$ with $\dim L = 1$;*

(*ii*) *for all $\beta < 0$ we have $\mathcal{H}(E)_{(\beta,\beta)} = \{0\}$.*

In this case there exists exactly one minimal element in $\operatorname{Adm}^+(E)/_\asymp$, namely $\omega_S/_\asymp$ where S is the (up to scalar multiples unique) real element of $\mathcal{H}(E)$ with $\mathfrak{d}(S)|_\mathbb{R} = \mathfrak{d}(E)|_\mathbb{R}$, $\mathfrak{d}(S)|_{\mathbb{C}\setminus\mathbb{R}} = 0$.

Proof. Assume that the conditions (*i*) and (*ii*) hold. Let L be the one-dimensional element of $\operatorname{Sub}^s(\mathcal{H}(E))$, and let S be as in Lemma 3.7. By Proposition 2.11 there exists $\omega \in \operatorname{Adm}^+(E)$ such that $L = \mathfrak{R}(\omega)$. By Step 3 of the proof of Theorem 3.2, we have $\mathcal{R}_{\omega_S}(E) = L$, and ω_S is minimal by Step 2. Since S satisfies (3.1), we have $\omega_S \in \operatorname{Adm}^+(E)$.

Assume that ω is a minimal element of $\operatorname{Adm}^+(E)/_\asymp$. By Lemma 3.6 and Theorem 3.2, we have $\dim \mathcal{R}_\omega(E) = 1$. Hence (*i*) holds. Moreover, by Theorem 2.6, we must have $\mathcal{R}_\omega(E) \supseteq \mathcal{H}_{(\beta,\beta)}$ for all $\beta < 0$. Thus, $\dim \mathcal{H}_{(\beta,\beta)} \in \{0,1\}$ for all $\beta < 0$. If for some $\beta < 0$ we have $\dim \mathcal{H}_{(\beta,\beta)} = 1$, we would have $\mathcal{H}_{(\beta,\beta)} = \mathcal{H}_{(\beta',\beta')}$ for all $\beta' \in (\beta, 0)$, which contradicts Lemma 2.8.

Let ω_1, ω_2 be minimal elements of $\operatorname{Adm}^+(E)/_\asymp$. By Lemma 3.6 and Theorem 3.2 we have $\dim \mathcal{R}_{\omega_j}(E) = 1$, $j = 1, 2$. Since $\mathcal{R}_{\omega_j}(E) \in \operatorname{Sub}^s(\mathcal{H}(E))$, it follows that $\mathcal{R}_{\omega_1}(E) = \mathcal{R}_{\omega_2}(E) = \operatorname{span}\{S\}$ where S is as in Lemma 3.7. By Corollary 3.3 we have $\omega_j \asymp \omega_S$, $j = 1, 2$.

❑

3.9 Remark. The present Theorem 3.8 is a (slight) generalization of a result of V.P. Havin and J. Mashreghi, cf. [HM1] (see also [B, BH]), which states the following:

> *Assume that $E \in \mathcal{H}B$ is of zero exponential type. Then there exists a positive and continuous minimal majorant in* $\operatorname{Adm}(E)$

if and only if $1 \in \mathcal{H}(E)$. *Moreover, in case of existence, this majorant is given by* $\omega = |E|^{-1}$, *and any other continuous positive minimal majorant* $\omega_1 \in \operatorname{Adm}(E)$ *satisfies* $\omega_1 \asymp |E|^{-1}$.

Note that, in the present setting, the inclusion $1 \in \mathcal{H}(E)$ is equivalent to $|E|^{-1} \in L^2(\mathbb{R})$. A number of conditions sufficient for the inclusion $1 \in \mathcal{H}(E)$ may be found in [B, KW2].

3.10 Example. Let $\mathcal{H}(E) = \mathcal{P}W_a$. Then $\operatorname{Adm}^+(E)/_{\asymp}$ does not contain minimal elements, since $\operatorname{Sub}^s(\mathcal{P}W_a)$ does not contain a one-dimensional element. A related example may be found in [HM1].

3.11 Example. Consider a canonical system defined on $[0,l]$ with Hamiltonian H. Then $\operatorname{Adm}^+(E_l)$ contains a minimal element if and only if for some $\epsilon > 0$ the interval $(0,\epsilon)$ is indivisible and $\det H(t) = 0$, a.e. $t \in [0,l]$. In this case the minimal majorant is given by $|E_l(x)|^{-1}$.

Our final aim is to characterize finite dimensional spaces induced by majorants in terms of minimal majorants, cf. Proposition 3.15. This result will be deduced from Theorem 3.2 and the following statement, which is also of independent interest. For an admissible majorant $\omega \in \operatorname{Adm}(E)$ and $k \in \mathbb{N} \cup \{0\}$ we put $\omega^{[k]}(x) := (1+|x|)^k \omega(x)$.

3.12 Theorem. *Let* $E \in \mathcal{H}B$ *and* $\omega \in \operatorname{Adm}(E)$. *Then for every* $k \in \mathbb{N} \cup \{0\}$ *we have*

$$\operatorname{codim}_{\mathcal{R}_{\omega^{[k+1]}}(E)} \mathcal{R}_{\omega^{[k]}}(E) \leq 1, \ \mathfrak{d}(\mathcal{R}_{\omega^{[k+1]}}(E)) = \mathfrak{d}(\mathcal{R}_{\omega^{[k]}}(E)) .$$

Assume that additionally $\dim \mathcal{R}_\omega(E) < \infty$, *and put*

$$N := -\dim \mathcal{R}_\omega(E) +$$

$$+ \sup \left\{ \dim L : \ L \in \operatorname{Sub}(\mathcal{H}(E)), \dim L < \infty, \, \mathfrak{d}(L)|_{\mathbb{R}} = \mathfrak{d}(\mathcal{R}_\omega(E))|_{\mathbb{R}} \right\} .$$

Then

$$\mathcal{R}_\omega(E) \subsetneq \mathcal{R}_{\omega^{[1]}}(E) \subsetneq \ldots \subsetneq \mathcal{R}_{\omega^{[N]}}(E) = \mathcal{R}_{\omega^{[N+1]}}(E) = \ldots$$

Proof. The proof proceeds in several steps. Before we continue, let us note that

$$\omega^{[k]}(x) \leq \omega^{[k+1]}(x), \ x \in \mathbb{R}, k \in \mathbb{N} \cup \{0\},$$

and hence $\mathcal{R}_{\omega^{[k]}}(E) \subseteq \mathcal{R}_{\omega^{[k+1]}}(E)$.

To shorten notation we will throughout the proof write $R = R_\omega(E)$, $\mathcal{R} = \mathcal{R}_\omega(E)$, $R_k = R_{\omega^{[k]}}(E)$ and $\mathcal{R}_k = \mathcal{R}_{\omega^{[k]}}(E)$. For a dB-subspace L of $\mathcal{H}(E)$ we denote by $\mathcal{S}_L$ the operator of multiplication by z in L and by $\operatorname{dom} \mathcal{S}_L$ the domain of $\mathcal{S}_L$, that is, the set of all functions F in L such that

$zF(z) \in L$.

Step 1: For every $k \in \mathbb{N} \cup \{0\}$ we have $\operatorname{codim}_{\mathcal{R}_{k+1}} \mathcal{R}_k \leq 1$ and $\mathfrak{d}(\mathcal{R}_{k+1}) = \mathfrak{d}(\mathcal{R}_k)$.

Let $G \in R_{k+1}$. We fix $v \in \mathbb{C} \setminus \mathbb{R}$ such that $G(v) \neq 0$, and consider the difference quotient operator

$$\Phi_G^v : \begin{cases} \mathcal{R}_{k+1} & \to & \mathcal{R}_{k+1} \\ F(z) & \mapsto & \frac{F(z)G(v)-G(z)F(v)}{z-v} \end{cases}$$

Then Φ_G^v is a bounded linear operator. The fact that $G \in R_{k+1}$ implies that for every $F \in R_{k+1}$ we have

$$\begin{aligned} \Big|\frac{(\Phi_G^v F)(x)}{E(x)}\Big| = \frac{1}{|x-v|}\Big|\frac{F(z)G(v)-G(z)F(v)}{E(x)}\Big| \lesssim \\ \lesssim \frac{1}{|x-v|}\omega^{[k+1]}(x) \lesssim \omega^{[k]}(x)\,, \end{aligned} \tag{3.2}$$

i.e. $\Phi_G^v(R_{k+1}) \subseteq R_k$. By continuity, also

$$\Phi_G^v\big(\mathcal{R}_{k+1}\big) \subseteq \mathcal{R}_k\,.$$

Let $\mathcal{S}_{k+1} := \mathcal{S}_{\mathcal{R}_{k+1}}$. It is easy to see that $\operatorname{dom} \mathcal{S}_{k+1} \subseteq \Phi_G^v(\mathcal{R}_{k+1})$. However, by [dB, Theorem 29],

$$\operatorname{codim}_{\mathcal{R}_{k+1}} \Big(\operatorname{Clos}_{\mathcal{R}_{k+1}} \operatorname{dom} \mathcal{S}_{k+1} \Big) \leq 1\,,$$

and, thus, also $\operatorname{codim}_{\mathcal{R}_{k+1}} \mathcal{R}_k \leq 1$. Since $\mathfrak{d}(\operatorname{Clos}_{\mathcal{R}_{k+1}} \operatorname{dom} \mathcal{S}_{k+1}) = \mathfrak{d}(\mathcal{R}_{k+1})$ and, as we have just seen, $\operatorname{Clos}_{\mathcal{R}_{k+1}} \operatorname{dom} \mathcal{S}_{k+1} \subseteq \mathcal{R}_k$, we find that $\mathfrak{d}(\mathcal{R}_k) = \mathfrak{d}(\mathcal{R}_{k+1})$.

Step 2: Assume that $\dim \mathcal{R} < \infty$, and that there exists $L \in \operatorname{Sub}(\mathcal{H}(E))$ with $\operatorname{codim}_L \mathcal{R} = 1$ and $\mathfrak{d}(L) = \mathfrak{d}(\mathcal{R})$. Then $L = \mathcal{R}_1$.

Since $\dim L < \infty$, $\operatorname{dom} \mathcal{S}_L$ is a dB-subspace of L with $\operatorname{codim}_L \operatorname{dom} \mathcal{S}_L = 1$ and $\mathfrak{d}(\operatorname{dom} \mathcal{S}_L)|_{\mathbb{R}} = \mathfrak{d}(L)|_{\mathbb{R}} = \mathfrak{d}(\mathcal{R})|_{\mathbb{R}}$. By the Ordering Theorem, $\mathcal{R} = R = \operatorname{dom} \mathcal{S}_L$. Choose $G \in \mathcal{R}$ and $v \in \mathbb{C}$ with $G(v) \neq 0$. Such a choice is possible since $\mathcal{R} \neq \{0\}$. If $F \in L$, then $\Phi_G^v F \in \operatorname{dom} \mathcal{S}_L = R$. Hence

$$F(z) = \frac{1}{G(v)}\big[(z-v)(\Phi_G^v F)(z) + G(z)F(v)\big] \in R_1\,.$$

Thus, $L \subseteq \mathcal{R}_1$ and, by Step 1, $L = \mathcal{R}_1$.

Step 3: End of the proof.

The desired assertion follows by an inductive argument. Since $(\omega^{[k]})^{[1]} = \omega^{[k+1]}$, we obtain from Step 2 that $\mathcal{R}_{\omega^{[k]}} \subsetneq \mathcal{R}_{\omega^{[k+1]}}$ as long as $k < N$, since the existence of a finite-dimensional dB-subspace with $L \supsetneq \mathcal{R}_{\omega^{[k]}}$ and $\mathfrak{d}(L) = \mathfrak{d}(\mathcal{R})$ implies that there exists a dB-subspace $\tilde{L}$ with $\mathfrak{d}(\tilde{L}) = \mathfrak{d}(\mathcal{R})$ and $\operatorname{codim}_{\tilde{L}} \mathcal{R}_{\omega^{[k]}} = 1$, cf. [dB, Problem 101, Problem 110].

Finally, assume that $\mathcal{R}_{\omega^{[N]}} \subsetneq \mathcal{R}_{\omega^{[k]}}$ for some $k > N$. Then $L := \mathcal{R}_{\omega^{[k]}}$ is a dB-subspace with $\mathfrak{d}(L) = \mathfrak{d}(\mathcal{R})$ and, by what we have just proved,

$$\operatorname{codim}_L \mathcal{R} > \operatorname{codim}_{\mathcal{R}_{\omega^{[N]}}} \mathcal{R} = N .$$

This contradicts the definition of N.

❑

The following example shows that the assumption $\dim \mathcal{R}_\omega(E) < \infty$ in the second part of Theorem 3.12 cannot be dropped.

3.13 Example. We shall construct $E \in \mathcal{H}B$ and $\omega \in \operatorname{Adm}^+(E)$ such that there exists $L \in \operatorname{Sub}^s(\mathcal{H}(E))$ with $\operatorname{codim}_L \mathcal{R}_\omega(E) = 1$, but $\mathcal{R}_{\omega^{[k]}}(E) = \mathcal{R}_\omega(E)$ for all $k \in \mathbb{N} \cup \{0\}$.

Consider the canonical system on $[0, 2]$ with Hamiltonian

$$H(t) := \begin{cases} I & , \ t \in [0,1) \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & t \in [0,2] \end{cases}$$

and put $E := E_2$. Then $E \in \mathcal{H}B$, $\mathfrak{d}(E) \equiv 0$, and $E := A - iB$ is explicitly given as

$$(A, B) := (\cos z, \sin z + z \cos z) = (\cos z, \sin z) \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} .$$

Hence

$$\operatorname{Sub}^s(\mathcal{H}(E)) = \big\{\mathcal{P}W_a : 0 < a \leq 1\big\} \cup \{\mathcal{H}(E)\} .$$

Moreover, $\operatorname{codim}_{\mathcal{H}(E)} \mathcal{P}W_1 = 1$ and in fact

$$\mathcal{H}(E) = \mathcal{P}W_1 \dot{+} \operatorname{span}\{\cos z\} .$$

By Proposition 2.11, for every $\omega \in \operatorname{Adm}^+(E)$ we have $\mathcal{R}_\omega(E) \supseteq \mathcal{P}W_1$.

The function E is explicitly given as $E(z) = \cos z - i(\sin z + z \cos z)$. We will give some estimates of E. Let $x \in \mathbb{R}$, $|x| \geq 1$, and assume that $|\cos x| \leq \frac{1}{2|x|}$. Then $|\sin x| \geq \sqrt{3}/2$ and

$$|\sin x + x \cos x| \geq \big||\sin x| - |x \cos x|\big| \geq \frac{\sqrt{3} - 1}{2} .$$

It follows that

$$|E(x)| \geq \min\Big\{ \min_{t\in[-1,1]} |E(t)|, \frac{\sqrt{3}-1}{2}, \frac{1}{2|x|}\Big\} \gtrsim \frac{1}{1+|x|},\ x\in\mathbb{R}\,. \tag{3.3}$$

Trivially, we have the following estimate from above:

$$|E(x)| \leq 1+|x|,\ x\in\mathbb{R}\,. \tag{3.4}$$

We show that for all $\beta \in (0,1)$ the function $\omega_\beta(x) := e^{-|x|^\beta}$ belongs to $\operatorname{Adm}^+(E)$. Choose $\beta' \in (\beta,1)$. By the Beurling-Malliavin Theorem, there exists $F \in \mathcal{P}W_1 \setminus \{0\}$ such that

$$|F(x)| \lesssim e^{-|x|^{\beta'}},\ x\in\mathbb{R}\,.$$

It follows that $F \in \mathcal{H}(E)$ and, by (3.3),

$$\Big|\frac{F(x)}{E(x)}\Big| \lesssim |F(x)|(1+|x|) \lesssim e^{-|x|^{\beta'}}(1+|x|) \lesssim e^{-|x|^\beta},\ x\in\mathbb{R}\,.$$

We show that $\mathcal{R}_{\omega_\beta}(E) = \mathcal{P}W_1$. Assume that $R_{\omega_\beta}(E) \not\subseteq \mathcal{P}W_1$. Then there exist $\lambda\in\mathbb{C}$, $\lambda\neq 0$, and $F_0 \in \mathcal{P}W_1$, such that $F_0 + \lambda\cos z \in R_{\omega_\beta}(E)$, i.e.

$$\Big|\frac{F_0(x)+\lambda\cos x}{E(x)}\Big| \lesssim \omega_\beta(x)\,.$$

By (3.4),

$$|\cos x| \lesssim (1+|x|)\,\omega_\beta(x) + |F_0(x)|,\ x\in\mathbb{R}\,.$$

We have a contradiction, since both F_0 and $(1+|x|)\omega_\beta(x)$ are in $L^2(\mathbb{R})$.

For all $\beta \in (0,1)$ and $k \in \mathbb{N}\cup\{0\}$ we have $\mathcal{R}_{\omega^{[k]}}(E) = \mathcal{P}W_1$. Choose $\beta' \in (0,\beta)$, then $(1+|x|)^k e^{-|x|^\beta} \lesssim e^{-|x|^{\beta'}}$, i.e. $\omega_\beta^{[k]} \lesssim \omega_{\beta'}$. Hence $\mathcal{R}_{\omega_\beta^{[k]}}(E) \subseteq \mathcal{R}_{\omega_{\beta'}}(E) = \mathcal{P}W_1$.

3.14 Example. The phenomenon considered in the previous example depends upon the fact that ω decreases rapidly. To see this, let E be as in Example 3.13, and consider the function

$$\omega(x) := \Big|\frac{\sin(x+i)}{(1+|x|)E(x)}\Big|\,.$$

Since the reproducing kernel $K_1(i,\cdot)$ of the Paley-Wiener space $\mathcal{P}W_1$ is equal to

$$K_1(i,z) = \frac{\sin(z+i)}{\pi(z+i)}\,,$$

we conclude from the proof of Corollary 2.10 that $\mathcal{R}_\omega(E)=\mathcal{P}W_1$. However, since $|\sin(x+i)|\geq e+e^{-1}$, we have

$$\Big|\frac{\cos x}{E(x)}\Big|\lesssim(1+|x|)\omega(x)=\omega^{[1]}(x)\,,$$

and hence $\mathcal{R}_{\omega^{[1]}}(E)=\mathcal{H}(E)$.

We come to our characterization of finite-dimensional spaces $\mathfrak{R}(\omega)$.

3.15 Proposition. *Let $E\in\mathcal{H}B$ and $\omega\in\operatorname{Adm}(E)$. Then $\dim\mathcal{R}_\omega(E)<\infty$ if and only if there exists ω_0, minimal in $\operatorname{Adm}(E)/_\asymp$, and $n\in\mathbb{N}\cup\{0\}$ such that*

$$\mathcal{R}_\omega(E)=\mathcal{R}_{\omega_0^{[n]}}(E)\,. \tag{3.5}$$

In this case the minimal number n such that $\mathcal{R}_\omega(E)$ can be represented as in (3.5) is $\dim\mathcal{R}_\omega(E)-1$.

Proof. Assume that $\mathcal{R}_\omega(E)$ can be represented as in (3.5) with some minimal majorant ω_0 and some $n\in\mathbb{N}\cup\{0\}$. Then, by Theorem 3.2 and the first assertion of Theorem 3.12, we have

$$\dim\mathcal{R}_\omega(E)=\dim\mathcal{R}_{\omega_0^{[n]}}(E)\leq n+1\,.$$

In particular, it follows that the number n in any representation (3.5) is at least $\dim\mathcal{R}_\omega(E)-1$.

To complete the proof of the present proposition, it is enough to construct for a given finite-dimensional space $\mathcal{R}_\omega(E)$ a minimal majorant ω_0 such that $\mathcal{R}_\omega(E)=\mathcal{R}_{\omega_0^{[d]}}(E)$ with $d=\dim\mathcal{R}_\omega(E)-1$.

Write $\mathcal{R}_\omega(E)=\mathcal{H}(E_1)$ with some $E_1\in\mathcal{H}B$, then $\operatorname{mt}\frac{E_1}{E}=0$. Since $\dim\mathcal{R}_\omega(E)<\infty$, there exists a dB-subspace L of $\mathcal{H}(E)$ with $\dim L=1$ and $\mathfrak{d}(L)=\mathfrak{d}(\mathcal{R}_\omega(E))$. Write $L=\mathcal{H}(E_2)$ with some $E_2\in\mathcal{H}B$. Then there exists a 2×2-matrix polynomial P with $\det P=1$, such that

$$(A_1,B_1)=(A_2,B_2)P$$

where $A_k=\frac{1}{2}(E_k+E_k^\#)$, $B_k=\frac{i}{2}(E_k-E_k^\#)$, $k=1,2$, cf. [dB, Theorem 33, Problem 110]. It follows that $\operatorname{mt}\frac{E_2}{E}=\operatorname{mt}\frac{E_1}{E}=0$, and hence there exists a minimal majorant ω_0 with $\mathcal{R}_{\omega_0}(E)=L$. Put $d:=\dim\mathcal{R}_\omega(E)-1$. Then, by the second part of Theorem 3.12, we have $\dim\mathcal{R}_{\omega_0^{[d]}}(E)=\dim\mathcal{R}_\omega(E)$. Moreover, $\mathfrak{d}(\mathcal{R}_{\omega_0^{[d]}}(E))=\mathfrak{d}(L)=\mathfrak{d}(\mathcal{R}_\omega(E))$. By the Ordering Theorem, $\mathcal{R}_{\omega_0^{[d]}}(E)=\mathcal{R}_\omega(E)$.

❑

We close the present paper with a discussion of the family $\omega^{[k]}(x) := (1+|x|)^k\omega(x)$, $k\in\mathbb{Z}$, of possible majorants. For $\omega\in\operatorname{Adm}(E)$ put

$$N_-(\omega) := \inf\left\{k\in\mathbb{Z} :\, \omega^{[k]}\in\operatorname{Adm}(E)\right\}.$$

Then $-\infty \leq N_-(\omega) \leq 0$, and, if $k\in\mathbb{Z}$ is such that $k\geq N_-(\omega)$, then $\omega^{[k]}\in\operatorname{Adm}(E)$.

3.16 Example. Let us show by examples that all cases for N_- can occur.

(*i*) Consider the space $\mathcal{P}W_a = \mathcal{H}(e^{-iaz})$. Then, by the Beurling-Malliavin Theorem, all functions $(1+|x|)^k e^{-|x|^\beta}$, $k\in\mathbb{Z}$, $\beta\in(0,1)$ belong to $\operatorname{Adm}(e^{-iaz})$. Hence, if we put $\omega(x) := e^{-|x|^\beta}$, we have $N_-(\omega) = -\infty$.

(*ii*) Let $\mathcal{P}$ denote the linear space of all polynomials. Consider a space $\mathcal{H}(E)$ which has the property that $\mathcal{P}\subseteq\mathcal{H}(E)$ and $\operatorname{Clos}_{\mathcal{H}(E)}\mathcal{P} = \mathcal{H}(E)$ (see [B, KW2]). Then $\omega_0\equiv 1$ is a minimal majorant in $\operatorname{Adm}^+(E)$. For a given number $N\in\mathbb{Z}$, $N\leq 0$, put $\omega := \omega_0^{[-N]}$. Then, clearly, $\omega^{[k]} = \omega_0^{[k-N]}$. Hence, for $k\geq N$, we have $\omega^{[k]}\in\operatorname{Adm}^+(E)$. If $k<N$, we have

$$\inf_{x\in\mathbb{R}}\frac{\omega^{[k]}(x)}{\omega_0(x)} = \inf_{x\in\mathbb{R}}\frac{\omega_0^{[k-N]}(x)}{\omega_0(x)} = 0\,,$$

and hence, by the minimality of ω_0, $\omega^{[k]}\notin\operatorname{Adm}(E)$. We conclude that $N_-(\omega) = N$.

Let us next introduce the two numbers

$$n_-(\omega) := \sup\left\{k\in\mathbb{Z}, k\geq N_-(\omega) :\, \mathfrak{R}(\omega^{[k]}) = \mathfrak{R}(\omega^{[l]}), \forall l\in\mathbb{Z}, N_-\leq l\leq k\right\},$$

$$n_+(\omega) := \inf\left\{k\in\mathbb{Z}, k\geq N_-(\omega) :\, \mathfrak{R}(\omega^{[k]}) = \mathfrak{R}(\omega^{[l]}), \forall l\in\mathbb{Z}, l\geq k\right\}.$$

Then $n_-(\omega), n_+(\omega)\in[N_-(\omega),\infty]$.

To show that these numbers can have quite arbitrary behaviour, let us provide one more example.

3.17 Example. Let $d\in\mathbb{N}$ be given, and consider a canonical system defined on $[0,l]$, $l := d+2$, with a Hamiltonian H such that

(*i*) $H(t) = I$, $t\in[0,1]$;

(*ii*) each of the intervals $(k,k+1)$, $k=1,\dots,d$, is H-indivisible, but no interval which contains an integer is H-indivisible;

(*iii*) no interval $(d+1,d+1+\epsilon)$, $\epsilon>0$, is H-indivisible, and $\det H(t) = 0$, $t\in(d+1,d+2]$.

Note that $\mathcal{H}(E_1) = \mathcal{P}W_1$. Consider the function

$$\omega(x) := \Big|\frac{\sin(x+i)}{(1+|x|)E_l(x)}\Big| .$$

As in the proof of Corollary 2.10, we see that $\omega \in \mathrm{Adm}^+(E_l)$, and $\mathfrak{R}(\omega) = \mathcal{H}(E_1)$. Since the function $K_1(i,z) := \frac{\sin(z+i)}{z+i}$ belongs to $\mathcal{H}(E_l)$ and has infinitely many zeros, we conclude that $N_-(\omega) = -\infty$. Consider the space $\mathcal{H}(E_{d+1})$. Then $\operatorname{codim}_{\mathcal{H}(E_{d+1})} \mathcal{H}(E_1) = d$.

Consider the function $A_{d+1} := \frac{1}{2}(E_{d+1} + E_{d+1}^{\#})$. Since $d+1$ is not a left endpoint of an H-indivisible interval, $A_{d+1} \notin \mathcal{H}(E_{d+1})$. Since $\dim \mathcal{H}(E_{d+1}) = \infty$, this function has infinitely many zeros, say $a_1, a_2, \ldots$. We have $\frac{A_{d+1}(z)}{z-a_1} \in \mathcal{H}(E_{d+1})$, and hence

$$\frac{A_{d+1}(z)}{(z-a_1)(z-a_2)} \in \operatorname{dom} \mathcal{S}_{\mathcal{H}(E_{d+1})} .$$

However, $\mathcal{H}(E_d) = \operatorname{Clos}_{\mathcal{H}(E_{d+1})} \operatorname{dom} \mathcal{S}_{\mathcal{H}(E_{d+1})}$, and hence $\frac{A_{d+1}(z)}{(z-a_1)(z-a_2)} \in \mathcal{H}(E_d)$. Proceeding inductively, we obtain

$$\frac{A_{d+1}(z)}{(z-a_1)\cdots(z-a_k)} \in \mathcal{H}(E_{d+2-k}),\ k = 1, \ldots, d+1 .$$

There exists a 2×2-matrix polynomial P of degree d such that

$$(A_{d+1}(z), B_{d+1}(z)) = (\cos z, \sin z) P(z) .$$

Hence

$$\Big|\frac{A_{d+1}(z)}{(z-a_1)E_l(x)}\Big| \lesssim (1+|x|)^d \omega(x) = \omega^{[d]}(x) ,$$

and so $\frac{A_{d+1}(z)}{z-a_1} \in R_{\omega^{[d]}}(E_l)$. Since $\frac{A_{d+1}(z)}{z-a_1} \notin \mathcal{H}(E_d)$, we have $R_{\omega^{[d]}}(E_l) \not\subseteq \mathcal{H}(E_d)$. It follows that $\mathcal{R}_{\omega^{[d]}}(E_l) \supseteq \mathcal{H}(E_{d+1})$. Since $\operatorname{codim}_{\mathcal{R}_{\omega^{[d]}}(E_l)} \mathcal{H}(E_1) \leq d$, we conclude that $\mathcal{R}_{\omega^{[d]}}(E_l) = \mathcal{H}(E_{d+1})$.

For every majorant $\hat{\omega} \in \mathrm{Adm}^+(E_l)$ we must have $\mathcal{H}(E_1) \subseteq \mathfrak{R}(\hat{\omega})$, and so $\mathfrak{R}(\omega^{[k]}) = \mathcal{H}(E_1)$, $k \leq 0$. On the other hand, if $k > d$ and $\mathcal{R}_{\omega^{[k]}}(E_l) \supsetneq \mathcal{R}_{\omega^{[d]}}(E_l)$, then, by Theorem 3.12,

$$1 \leq \operatorname{codim}_{\mathcal{R}_{\omega^{[k]}}(E_l)} \mathcal{H}(E_{d+1}) = \operatorname{codim}_{\mathcal{R}_{\omega^{[k]}}(E_l)} \mathcal{R}_{\omega^{[d]}}(E_l) \leq k - d < \infty .$$

This contradicts our construction of E_l, cf. the requirement (iii) on H. Thus

$$\ldots = \mathfrak{R}(\omega^{[-1]}) = \mathfrak{R}(\omega^{[0]}) \subsetneq \mathfrak{R}(\omega^{[1]}) \subsetneq \ldots \subsetneq \mathfrak{R}(\omega^{[d]}) = \mathfrak{R}(\omega^{[d+1]}) = \ldots$$

where

$$\mathfrak{R}(\omega^{[k]}) = \mathcal{H}(E_{k+1}),\ k = 0, \ldots, d .$$

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