Stability of order and type under perturbation of the spectral measure

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Abstract: It is known that the type of a measure is stable under perturbations consisting of exponentially small redistribution of mass and exponentially small additive summands. This fact can be seen as stability of de Branges chains in the corresponding L^2-spaces.

We investigate stability of de Branges chains in L^2-spaces under perturbations having the same form, but allow other magnitudes for the error. The admissible size of a perturbation is connected with the maximal growth of functions in the chain and is measured by means of a growth function λ. The main result is a Fast Growth Theorem. It states that an alternative takes place when passing to a perturbed measure: either the original de Branges chain remains dense or its closures must contain functions with faster growth than λ. For the growth function $\lambda(r) = r$, i.e. exponentially small perturbations, the afore mentioned known fact is reobtained.

We propose a notion of order of a measure and show stability and monotonicity properties of this notion. The cases of exponential type (order 1) and very slow growth (logarithmic order ≤ 2) turn out to be particular.

AMS MSC 2010: 46E22, 30D15, 41A30, 34B20
Keywords: de Branges space, order and type of a measure, perturbation of measures, growth function, weighted approximation

1 Introduction

Let μ be a finite positive Borel measure on the real line. The supremum in $[0, \infty]$ of all numbers $a \geq 0$ such that the linear span of the exponentials $\{z \mapsto e^{itz} : |t| \leq a\}$ is not dense in $L^2(\mu)$ is called the *type of* μ; we denote it by $T[\mu]$. It is a famous problem in harmonic analysis – the *type problem* – to determine $T[\mu]$. The origins of this problem lie in work of Kolmogoroff and Wiener about stationary Gaussian processes, and there are intimate connections with various topics of analysis. A vast literature on the type problem exists which culminates in the recent work [Pol13] of A.Poltoratski.

Let us consider one aspect of type which occurs in connection with the spectral theory of differential operators.

`The Schrödinger operator:` Let V be an integrable potential on a finite interval $[0, L]$, and let μ be the spectral measure of the corresponding Schrödinger operator $-\frac{d^2}{dx^2} + V$. Then μ is discrete and $\operatorname{supp} \mu \subseteq [0, \infty)$, say, $\operatorname{supp} \mu = \{x_n : n \in \mathbb{N}\}$ with $0 \leq x_1 < x_2 < \ldots$. The type of the symmetrised measure (here $\delta_{\{x\}}$ denotes the unit point mass at x)

$$\hat{\mu} := \sum_{n \in \mathbb{N}} \mu(\{x_n\}) \big(\delta_{\{\sqrt{x_n}\}} + \delta_{\{-\sqrt{x_n}\}} \big)$$

‡This work was supported by a joint project of the Austrian Science Fund (FWF, I 1536–N25) and the Russian Foundation for Basic Research (RFBR, 13-01-91002-ANF). A.Baranov was supported by Dmitry Zimin's Dynasty Foundation.

governs the asymptotic distribution of the spectrum and is easily computed:

$$\lim_{n\to\infty}\frac{n}{\sqrt{x_n}}=\frac{1}{2}T[\hat{\mu}]=\frac{L}{2\pi}.$$

In particular, $T[\hat{\mu}]$ is always finite and positive.

`The Krein-Feller operator:` The situation changes when considering a Krein-Feller operator $-D_mD_x$ associated to a string with mass distribution m being defined on a finite interval $[0,L]$ and having finite total mass. The spectral measure μ is again discrete and supported on the positive half-axis. The type of the symmetrised measure $\hat{\mu}$ is again finite, governs the spectral asymptotics, and can be computed:

$$\lim_{n\to\infty}\frac{n}{\sqrt{x_n}}=\frac{1}{2}T[\hat{\mu}]=\int_0^L\sqrt{m'(x)}\,dx.$$

Apparently it may happen that $T[\hat{\mu}]=0$. In fact, there is a variety of examples where

$$0<\lim_{n\to\infty}\frac{n^{\frac{1}{\rho}}}{\sqrt{x_n}}<\infty \tag{1.1}$$

for some $\rho\in(0,1)$, see, e.g., [Fre05].

`The Jacobi operator:` Much more drastically, for the spectral measure of a Jacobi operator associated with an indeterminate Stieltjes moment sequence always $T[\hat{\mu}]=0$, and eigenvalue asymptotics like

$$\#\{n\in\mathbb{N}:|x_n|\leq r\}\sim\alpha(\log r)^\beta \tag{1.2}$$

are no rarity, see, e.g., [BP07].

Analysing spectral asymptotics as in (1.1) or (1.2) requires to deal with orders less than 1 or even with growth measured on a logarithmic scale, rather than with exponential type.

> *Does there exist an underlying concept of "order of a measure μ" for μ in the class of positive finite measures on the real line ?*

In order to suggest a – natural and potentially meaningful – candidate for such a concept, recall that $T[\mu]$ can be characterised in several different ways. Most ways are very much fitted to the study of exponential type, however, one of them allows immediate passing to a finer scale of growth properties and smaller orders.

> *Let $(\mathcal{H}_t)_{t\in I}$ be the unique maximal chain of de Branges spaces $\mathcal{H}_t$ which are isometrically contained in $L^2(\mu)$ and invariant under difference quotients. Then $T[\mu]$ is the supremum of exponential types of functions in*
>
> $$\mathcal{L}:=\bigcup_{t\in I}\mathcal{H}_t. \tag{1.3}$$

Now the following definition comes naturally.

1.1 Definition.

- The *order of* μ is the supremum of all orders of functions in $\mathcal{L}$.
- Provided μ has finite order, the *type of* μ *w.r.t. its order* is the supremum of all types w.r.t. the order of μ of functions in $\mathcal{L}$.

We denote the order of μ by $\rho[\mu]$, and its type (w.r.t. $\rho[\mu]$) by $\tau[\mu]$. $\Diamond$

Since the spaces $\mathcal{H}_t$ are invariant under forming difference quotients, certainly $\rho[\mu] \in [0,1]$. The type $\tau[\mu]$ may take any value in $[0,\infty]$. One can easily construct examples which show that $\rho[\mu]$ may assume any value in $[0,1]$, and that, prescribing $\rho \in [0,1)$, all values in $[0,\infty]$ may appear as $\tau[\mu]$ for some measure μ with $\rho[\mu] = \rho$.

1.2 Remark. We should say it immediately that the notion of order and type of a measure proposed above is designed as an analogue of the type of a measure in the limit point situation. There are other (possible and meaningful) notions which arise as analogues of the limit circle situation. For some more explanations about this, see Section 3 of the conference presentation [Wor16]. $\Diamond$

A brief account of our present results:

Roughly speaking, the present paper is a generalisation to orders different from 1 of the work [BS11a] of A.Borichev and M.Sodin[1]. There exponential type was investigated, monotonicity of type was shown when μ is majorised by $\tilde{\mu}$ up to an exponentially small error, and stability of type under exponentially small perturbations followed.

Our results will be established in the context of growth classes defined by growth functions (i.e., proximate orders) rather than usual order and type. Also, as in [BS11a], we work with power bounded measures rather than finite ones (but this is only a minor point).

In the following items `(1)`–`(3)` we give a summary of our work.

`(1)` We introduce a majorisation relation "$\mu \preceq \tilde{\mu}$ w.r.t. (λ_1, λ_2)" which depends on a pair (λ_1, λ_2) of growth functions, cf. Definition 2.16. This relation expresses majorisation of μ by a perturbation of $\tilde{\mu}$, where the perturbation is composed of a "shifting/redistribution of mass"–part inside intervals of length $e^{-\lambda_1}$, and an "additive"–part limited by $e^{-\lambda_2}$.

One is tempted to think of majorisation w.r.t. smaller functions λ_1 as majorisation up to a larger error. Having in mind the most simple cases (like small shifts of point masses located at a well-separated sequence) this is a good intuition, in general it is misleading: relations $\preceq$ for different λ_1 are incomparable.

Majorisation w.r.t. $\lambda_1(r) = \mathrm{o}(r)$ or $r = \mathrm{O}(\lambda_1(r))$ are equally interesting cases, and show very different behaviour. The reason for this is the automatic presence of exponential type and completely regular growth w.r.t. order 1, occuring because functions from the de Branges chains are of Cartwright class.

The case of exponentially small perturbations treated in [BS11a] is reobtained using $\lambda_1(r) = \delta r$, $\lambda_2(r) = 2\delta r$, cf. Remark 2.17.

[1] This paper has appeared as [BS11b], however, after publication some gaps were found. These are fixed in the cited arXiv version, and therefore we shall always refer to the "pre"–print.

(2) Our main result is Theorem 3.1 which reveals an alternative. Let $\mathcal{L} = \bigcup_{t\in I}\mathcal{H}_t$ and $\tilde{\mathcal{L}} = \bigcup_{t\in\tilde{I}}\tilde{\mathcal{H}}_t$ be built as in (1.3) for μ and $\tilde{\mu}$, respectively. If $\mu \preceq \tilde{\mu}$ w.r.t. (λ_1, λ_2) where λ_1 is a growth function, and one has the a priori knowledge that all functions in $\tilde{\mathcal{L}}$ grow correspondingly slow, then the space $\tilde{\mathcal{L}}$ is contained in $L^2(\mu)$ and either is dense or its $L^2(\mu)$-closure is a de Branges space which contains functions of comparatively fast growth. We refer to results of that type as *Fast Growth Theorems*.

If $\lambda_1(r) = \mathrm{o}(r)$, density may indeed be lost. The Fast Growth Theorem then says that this loss must be balanced with occurance of fast growing functions. Contrasting this, if $r = \mathrm{O}(\lambda_1(r))$, density must be preserved (since balancing is not possible), cf. Corollary 3.2.

An interesting case occurs when $\mathcal{L}$ is the space of all polynomials which we discuss in Theorem 3.9. It turns out that, in order to have the conclusion of a Fast Growth Theorem, redistribution of mass in tremendously larger intervals than in Theorem 3.1 can be allowed: the function λ_1 might even tend to $-\infty$ at a very moderate speed.

(3) The Fast Growth Theorem gives rise to stability results. Assume the a priori knowledge that all function in $\mathcal{L} \cup \tilde{\mathcal{L}}$ grow slow. Then majorisation of measures implies a quasi-monotonicity property of type, cf. Theorem 3.6. The fact that for smaller growth than exponential type only quasi-monotonicity is present is an intrinsic phenomenon. It is due to the possible occurance of irregular zero distribution and corresponding irregular growth behaviour. Interestingly, this phenomenon disappears for functions of very slow growth, cf. Theorem 3.8.

When μ and $\tilde{\mu}$ mutually majorise each other, a much stronger property holds. Namely, the de Branges chains of μ and $\tilde{\mu}$ coincide (of course non-isometrically), cf. Theorem 3.3. In particular, if $\mu \preceq \tilde{\mu}$ and $\tilde{\mu} \preceq \mu$, then order and type of μ and $\tilde{\mu}$ coincide.

Structuring of the manuscript and detailed description:

The paper is structured according to the following table.

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`Section 2:` This section is inteded to explain the necessary notions for formulating our main results in a logically consistent, but as as brief and focussed as possible, manner. The main players are: (1) de Branges spaces and chains of such spaces isometrically included in a space $L^2(\mu)$, (2) algebraic de Branges spaces and their closure in spaces $L^2(\mu)$, (3) classes of entire functions defined by restrictions on growth, (4) a majorisation relation of measures quantified by a pair of fairly regular behaving functions.

`Section 3:` This section is devoted to formulate our main results. First, the Fast Growth Theorem (Theorem 3.1) and the Coincidence Theorem (Theorem 3.3) which deal with the chains of de Branges spaces themselves. Second, we turn to stabilty and monotonicity of order and type. In this context it is seen that a Quasi-Monotonicity Theorem holds, cf. Theorem 3.6. We show in Corollary 3.7 that for fast growth (at least as exponential type) true monotonicity holds. From this [BS11a, Theorem 1.3 and Corollary 1.4] are reobtained. The argument in the proof of Corollary 3.7 explains in a neat way why the case of exponential type is much different (and much simpler) than the case of growth with respect to an order less than 1. The bottom line being that bounded type implies regularity of growth and dominant growth along the imaginary axis. Interestingly also on the other end of the growth band, for very slow growing function, true monotonicity holds. This is shown in Theorem 3.8. Finally, we discuss a condition known from previous work about density of polynomials, and show that this condition is preserved under very large perturbations, cf. Theorem 3.9.

`Section 4:` We study weight functions and weighted C_0-spaces in a rather general setting. The main achievement is Theorem 4.12, where we establish a way to pass from L^2-spaces to weighted C_0-spaces. The – probably – first time a result of this type appeared, is in work of A.Bakan dealing with density of polynomials, cf. [Bak98, Theorem 4.1], [Bak08, Theorem 1.3]. Later on, this passage from $L^2(\mu)$ to $C_0(W)$ was applied in the already cited work of A.Borichev and M.Sodin, cf. [BS11a, Theorem 2.8]. It should be noted that the proofs in the mentioned literature rely on analyticity (used in the form of a theorem often attributed to M.Riesz and S.N.Mergelyan; a general version is [Pit83, Proposition 2.4]). However, this does not reflect the actual situation: Bakan-type theorems are of purely topological nature.

`Section 5:` De Branges' theorem on weigthed polynomial approximation says that non-density of polynomials in a weighted C_0-space is equivalent to existence of entire functions with certain properties. We discuss a version of this result for algebraic de Branges spaces instead of the space of polynomials. This version was deduced along the lines of de Branges' original argument in [BW13]. Independently M.Sodin and P.Yuditskii proposed a different approach based on Chebyshev alternance following their earlier work [SY92], [SY97], and [BS11a].

The Sodin-Yuditskii approach yields finer knowledge about those entire functions whose existence is claimed in de Branges' theorem, and this is a key ingredient in our arguments.

`Section 6:` This is a collection of some further necessary preliminaries. First we discuss some properties of growth functions and of those functions quantifying the size of perturbation in the majorisation of measures. Second, we make – on a general level – the connection between the notions of infinite index of determinacy of a measure (cf. Definition 2.5 which is a general version of [BD95] where the space of polynomials is considered) and stable density (as known from [BS11a] for the space $\mathcal{E}(a)$, see also Definition 4.9).

`Section 7:` An inclusion result is established: under a growth assumption, square-integrability is inherited when passing to a majorised measure (Theorem 7.1). The proof of this fact is fairly elementary; we give the necessary estimates in an explicit way.

`Section 8:` First, a smoothening operation with weight functions is introduced. This construction corresponds on the level of weight functions to passing from a measure to a majorised one. Second, we establish an estimate for canonical products when shifting zeroes.

`Section 9:` We complete the proof of the Fast Growth Theorem. The line of the argument follows [BS11a], and the theory we built up in the previous sections enables us to successfully proceed that way.

`Section 10:` We complete the proof of the other assertions stated in Section 3.

`Appendix A:` The only available source for the Sodin-Yuditskii approach to the general de Branges' theorem on weigthed approximation is the conference presentation [SY12] and personal communication. Otherwise, their proof remained unpublished. With the kind permission of M.Sodin and P.Yuditskii we elaborate it in this appendix.

`Appendix B:` We give the proof of existence and uniqueness of a distinguished chain of de Branges spaces in $L^2(\mu)$ where μ is a measure having at most power growth.

2 An introduction to the main players

In this section we introduce the main objects occuring in our present investigation, and recall some basic properties. Namely, we discuss de Branges spaces and chains, power bounded measures, closures of algebraic de Branges spaces, growth classes of entire functions, and a growth dependent majorisation relation between measures.

2.1 De Branges spaces and distinguished chains

Recall the definition of a de Branges space.

2.1 Definition. Let $\mathcal{H}$ be a linear space whose elements are complex-valued functions[2], $\mathcal{H} \neq \{0\}$, and let $(.,.)_{\mathcal{H}}$ be a positive definite inner product on $\mathcal{H}$. We call $\langle \mathcal{H}, (.,.)_{\mathcal{H}} \rangle$ a *de Branges space*, if the following axioms are fullfilled ($\|.\|_{\mathcal{H}}$ denotes the norm induced by $(.,.)_{\mathcal{H}}$).

(dB1) The elements of $\mathcal{H}$ are entire functions, and $\langle \mathcal{H}, (.,.)_{\mathcal{H}} \rangle$ is a reproducing kernel Hilbert space.

(dB2) If $F \in \mathcal{H}$ and $w \in \mathbb{C} \setminus \mathbb{R}$ with $F(w) = 0$, then also the function $\frac{F(z)}{z-w}$ belongs to $\mathcal{H}$, and

$$\Big\| \frac{z - \overline{w}}{z - w} F(z) \Big\|_{\mathcal{H}} = \|F\|_{\mathcal{H}}.$$

(dB3) If $F \in \mathcal{H}$, then also the function $F^{\#}(z) := \overline{F(\overline{z})}$ belongs to $\mathcal{H}$, and

$$\|F^{\#}\|_{\mathcal{H}} = \|F\|_{\mathcal{H}}.$$

$\Diamond$

A more concrete approach to de Branges spaces proceeds via a certain class of entire functions and reproducing kernels, cf. [Bra68, §19]: We call an entire function a *Hermite-Biehler function*, if it satisfies

$$|E(\overline{z})| < |E(z)|, \quad z \in \mathbb{C}^+.$$

Given a Hermite-Biehler function E, consider the function K_E defined as

$$K_E(w, z) = \frac{i}{2\pi} \frac{E(z)E^{\#}(\overline{w}) - E^{\#}(z)E(\overline{w})}{z - \overline{w}}, \quad z, w \in \mathbb{C},$$

where the formula has to be interpreted as a derivative if $z = \overline{w}$. Then K_E is a positive semidefinite kernel, and the reproducing kernel Hilbert space $\mathcal{H}(E)$ generated by K_E is a de Branges space. Conversely, for each de Branges space $\langle \mathcal{H}, (.,.)_{\mathcal{H}} \rangle$ there exist Hermite-Biehler functions E, such that the reproducing kernel of $\langle \mathcal{H}, (.,.)_{\mathcal{H}} \rangle$ coincides with K_E. Given a Hermite-Biehler function E, we denote the de Branges space it generates as $\mathcal{H}(E)$.

The structure theory for the set of all de Branges spaces which are contained isometrically in a space $L^2(\mu)$ plays a particularly important role. To once state it very precisely: by saying that some set of entire functions is *contained in* $L^2(\mu)$ we mean that the operator mapping an entire function to the equivalence class μ-a.e. of its restriction to the real line maps it into $L^2(\mu)$, and by saying that an inner product space is *contained isometrically in* $L^2(\mu)$ we mean that this operator maps it isometrically into $L^2(\mu)$.

A substantial portion of de Branges' theory can be summarised as follows; the standard reference, where all listed facts are found, is [Bra68]. Let μ be a positive Borel measure on the real line[3], $\mu \neq 0$. Then the following facts hold true.

[2]Here, and always, we tacitly assume that linear operations are defined by pointwise addition and scalar multiplication.

[3]We include into the term *Borel measure* that compact sets have finite measure.

❋ The set of all de Branges spaces which are isometrically contained in $L^2(\mu)$ is nonempty. Each two maximal chains in this set (subsets which are totally order w.r.t. set-theoretic inclusion, and maximal with this property) are either equal or disjoint, and the whole set is the disjoint union of all its maximal chains.

❋ Let $\mathcal{C}$ be a maximal chain. Then

$$\operatorname{Clos}_{L^2(\mu)} \bigcup_{\mathcal{L}\in\mathcal{C}} \mathcal{L} = L^2(\mu),$$

$$\Big(\operatorname{Clos} \bigcup_{\mathcal{L}\in\mathcal{C},\mathcal{L}\subsetneq\mathcal{H}} \mathcal{L}\Big) \in \mathcal{C} \quad\text{and}\quad \dim\Big(\mathcal{H}\Big/\operatorname{Clos} \bigcup_{\mathcal{L}\in\mathcal{C},\mathcal{L}\subsetneq\mathcal{H}} \mathcal{L}\Big) \leq 1,$$

$$\text{if } \mathcal{H}\in\mathcal{C} \text{ and } \mathcal{H} \text{ is not minimal element of } \mathcal{C},$$

and

$$\dim \bigcap_{\mathcal{L}\in\mathcal{C}} \mathcal{L} \leq 1,$$

$$\bigcap_{\mathcal{L}\in\mathcal{C},\mathcal{L}\supsetneq\mathcal{H}} \mathcal{L} \in \mathcal{C} \quad\text{and}\quad \dim\Big(\bigcap_{\mathcal{L}\in\mathcal{C},\mathcal{L}\supsetneq\mathcal{H}} \mathcal{L}\Big/\mathcal{H}\Big) \leq 1,$$

$$\text{if } \mathcal{H}\in\mathcal{C} \text{ and } \mathcal{H} \text{ is not maximal element of } \mathcal{C}.$$

❋ Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be maximal chains. Then $\mathcal{C}_1=\mathcal{C}_2$ if and only if the following two conditions are satisfied:

- (i) There exist $\mathcal{H}_i\in\mathcal{C}_i$, $i=1,2$, and $F_i\in\mathcal{H}_i\setminus\{0\}$, $i=1,2$, such that $\frac{F_1}{F_2}$ is a meromorphic function of bounded characteristic in $\mathbb{C}^+$.
- (ii) There exist $\mathcal{H}_i\in\mathcal{C}_i$, $i=1,2$, such that for each $x\in\mathbb{R}$ the minimal multiplicity of x as a zero of functions $F_1\in\mathcal{H}_1\setminus\{0\}$ is the same as the minimal multiplicity of x as a zero of functions $F_2\in\mathcal{H}_2\setminus\{0\}$.

❋ If μ is Poisson integrable, i.e., $\int_{\mathbb{R}}\frac{1}{1+x^2}\,d\mu(x)<\infty$, then there exists a unique maximal chain $\mathcal{C}$ in $L^2(\mu)$, such that each space $\mathcal{H}\in\mathcal{C}$ is invariant under forming difference quotients, i.e.,

$$\frac{F(z)-F(w)}{z-w}\in\mathcal{H},\quad F\in\mathcal{H}, w\in\mathbb{C}.$$

2.2 Power bounded measures

Sticking to the case of Poisson integrable (or even finite) measures is not natural. The appropriate class to deal with are measures whose tails do not grow faster than some power.

2.2 Definition. We say that a positive Borel measures on $\mathbb{R}$ has *at most power growth*, if there exists $n\in\mathbb{N}$ such that $\int_{\mathbb{R}}\frac{d\mu(x)}{(1+x^2)^n}<\infty$. The set of all nonzero positive Borel measures having at most power growth is denoted by $\mathbb{M}$. $\Diamond$

In the present context it is vital to be able to single out one particular maximal chain in $L^2(\mu)$ whenever $\mu\in\mathbb{M}$ (analogous as above for the Poisson integrable case). The fact that this is possible may be viewed as a sign that the class $\mathbb{M}$ is indeed “appropriate”.

The starting point for the next result is the observation that a de Branges space $\mathcal{H}$ is invariant under forming difference quotients if and only if $1\in\mathcal{H}+z\mathcal{H}$.

2.3 Proposition. *Let $\mu \in \mathbb{M}$. Then there exists a unique maximal chain $\mathcal{C}$ in $L^2(\mu)$, such that*

$$\forall \mathcal{H} \in \mathcal{C}: \quad \exists n \in \mathbb{N}_0 : 1 \in \mathcal{H} + z\mathcal{H} + \ldots + z^n\mathcal{H}. \tag{2.1}$$

The minimum of all numbers $n \in \mathbb{N}_0$ such that $(1+x^2)^{-n}$ is integrable w.r.t. μ is equal to the minimal number $n \in \mathbb{N}_0$ such that (2.1) holds (in particular it is independent of $\mathcal{H} \in \mathcal{C}$).

If $\mathcal{H} \in \mathcal{C}$ and $F \in \mathcal{H}$, then F is of bounded type in $\mathbb{C}^+$ and $\mathbb{C}^-$, i.e., of Cartwright class.

The proof of Proposition 2.3 is carried out by reduction to the Poisson integrable case.

2.4 Definition. Let $\mu \in \mathbb{M}$. Then we denote the distinguished maximal chain exhibited in Proposition 2.3 as $\mathcal{C}[\mu]$. ♢

The following property of a measure $\mu \in \mathbb{M}$ is a strengthening of the property that $L^2(\mu)$ itself is not a de Branges space. In [BS11a] it is called "stable density", however, we prefer the term established in the literature about moment problems.

2.5 Definition. Let $\mu \in \mathbb{M}$. We say that μ has *infinite index of determinacy*, if for every finitely supported positive measure ν on $\mathbb{R}$ the space $\bigcup_{\mathcal{H}\in\mathcal{C}[\mu]} \mathcal{H}$ is dense in $L^2(\mu+\nu)$. ♢

A systematic and general treatment of this notion can be found in [Wor, §4].

2.6 Remark. Observe that a measure with infinite index of determinacy necessarily has infinite support. For if μ is finitely supported, then $\dim \bigcup_{\mathcal{H}\in\mathcal{C}[\mu]} \mathcal{H} = \dim L^2(\mu) < \infty$, and hence $\bigcup_{\mathcal{H}\in\mathcal{C}[\mu]} \mathcal{H}$ is closed and not dense in any space $L^2(\mu+\nu)$ where $\operatorname{supp}\nu \not\subseteq \operatorname{supp}\mu$.

On the other hand, if μ is not discrete then μ has infinite index of determinacy, cf. [Wor, Corollary 4.7]. ♢

2.3 Closures of algebraic de Branges spaces

Let us introduce linear spaces which satisfy the "algebraic part" of the axioms for a de Branges space. In order to save on notation, we shall include in this definition the assumption that the space is invariant under division of *real* zeroes.

2.7 Definition. Let $\mathcal{L}$ be a linear space, $\mathcal{L} \neq \{0\}$. We call $\mathcal{L}$ an *algebraic de Branges space*, if

- **(a-dB1)** the elements of $\mathcal{L}$ are entire functions;
- **(a-dB2)** if $F \in \mathcal{L}$ and $w \in \mathbb{C}$ with $F(w) = 0$, then also the function $\frac{F(z)}{z-w}$ belongs to $\mathcal{L}$;
- **(a-dB3)** if $F \in \mathcal{L}$, then also the function $F^{\#}(z) := \overline{F(\overline{z})}$ belongs to $\mathcal{L}$.

♢

Note the following property of algebraic de Branges spaces: If $\mathcal{D}$ is a set of algebraic de Branges spaces which is totally ordered w.r.t. set-theoretic inclusion, then $\bigcup_{\mathcal{H}\in\mathcal{D}}\mathcal{H}$ is an algebraic de Branges space.

Simple examples of algebraic de Branges spaces are obtained from spaces of polynomials.

2.8 Example. Let S be a zerofree entire function with $S = S^{\#}$. Then each of the spaces

$$\mathcal{P}_d[S] := \operatorname{span}\big\{S(z), \dots, z^{d-1}S(z)\big\},\ d \in \mathbb{N}, \qquad \mathcal{P}_\infty[S] := S \cdot \mathbb{C}[z],$$

is an algebraic de Branges space.

Recall in this place that every finite dimensional algebraic de Branges space is of the form $\mathcal{P}_d[S]$ with some real and zerofree function S, cf. [Bra68, Problem 88]. $\Diamond$

Another standard example of an algebraic de Branges space is obtained from a chain of de Branges spaces.

2.9 Example. Let μ be a positive Borel measure on $\mathbb{R}$, $\mu \neq 0$, let $\mathcal{C}$ be a maximal chain in $L^2(\mu)$, and consider the union $\mathcal{L} := \bigcup_{\mathcal{H}\in\mathcal{C}}\mathcal{H}$. Provided that for each $x \in \mathbb{R}$ this union contains some function which does not vanish at x, $\mathcal{L}$ is an algebraic de Branges space.

For a measure $\mu \in \mathbb{M}$ and the distiguished chain $\mathcal{C}[\mu]$ this requirement is fulfilled. $\Diamond$

Next, we recall a fact which was proved (in a more general context) in [Pit83]. For particular cases of measures and algebraic de Branges spaces, e.g., for the space of polynomials, this is a classical result.

❋ Let μ be a positive Borel measure on the real line, $\mu \neq 0$, and let $\mathcal{L}$ be an algebraic de Branges space which is contained in $L^2(\mu)$. Then either $\operatorname{Clos}_{L^2(\mu)} \mathcal{L} = L^2(\mu)$ or this closure is a de Branges space contained isometrically in $L^2(\mu)$. To once state it very precise: by this we mean that there exists a de Branges space $\mathcal{H}$ which contains $\mathcal{L}$ and such the operator mapping an entire function to the equivalence class μ-a.e. of its restriction to the real line maps $\mathcal{H}$ isometrically onto the closure in $L^2(\mu)$ of the image of $\mathcal{L}$.

An explicit deduction from the results of [Pit83] is given in [BW13, Theorem 2.3]. Note here the following slight subtelty: We have assumed that $\mathcal{L}$ is invariant w.r.t. division of real zeroes. Hence, if $\mathcal{L}$ contains a function which does not vanish identically but whose restriction to the real line is equal to 0 μ-a.e., then $\mathcal{L}$ is already dense in $L^2(\mu)$.

❋ The two possiblities in the previous item do not exclude each other. In fact, $L^2(\mu)$ itself is a de Branges space $\mathcal{H}$ (invariant under division of real zeroes), if and only if $\mathcal{H}$ is maximal element of the maximal chain having $\mathcal{H}$ as an element.

2.4 Growth Classes

The notions of order and type of an entire function have been extended to the more refined scale of proximate orders by G.Valiron in the 1920's. On this scale the growth of a function is compared to functions of the form $e^{\tau\lambda(r)}$ rather than $e^{\tau r^\rho}$, where λ is growing sufficiently regularly.

2.10 Definition. A function $\lambda : [0, \infty) \to \mathbb{R}^+$ is called a *growth function* if it satisfies the following axioms.

(gf1) The function λ is differentiable, strictly increasing, $\log r = \mathrm{o}(\lambda(r))$, and $\lambda(0) = 1$.[4]

(gf2) The limit $\rho_\lambda := \lim_{r\to\infty} \frac{\log \lambda(r)}{\log r}$ exists and is finite and non-negative.

(gf3) $\lim_{r\to\infty} \dfrac{r\lambda'(r)}{\lambda(r)} = \rho_\lambda$.[5]

The logarithm of a growth function is called a *proximate order*. ◊

Typical examples of growth functions are functions λ which are, for large enough r, given as

$$\lambda(r) = r^a \cdot \big(\log r\big)^{b_1} \cdot \big(\log\log r\big)^{b_2} \cdot \ldots \cdot \big(\underbrace{\log \cdots \log r}_{m\text{-times iterate}}\big)^{b_m}, \tag{2.2}$$

where $a \geq 0$ and $b_1, \ldots, b_m \in \mathbb{R}$ such that $\log r = \mathrm{o}(\lambda(r))$. Comparison of the growth of the maximum modulus of an entire function with functions of the particular form (2.2) goes back to E.Lindelöf in the first years of the 20th century.

Most theorems known for order and type have their analogues in the context of proximate orders, cf. [Lev56, Section I.12] or [LG86, Section I.6]. In particular, the growth of an entire function with respect to some growth function is related to the density of its zeroes (w.r.t. the same growth function), cf. [Rub96, Theorems 13.5.2–4].

2.11 Definition. Let λ be a growth function. The λ*-type* of an entire function F is

$$\tau_\lambda[F] := \limsup_{|z|\to\infty} \frac{\log |F(z)|}{\lambda(|z|)},$$

and we denote the set of all entire functions of finite λ-type as $\mathcal{G}(\lambda, \infty)$.

The *indicator w.r.t.* λ of a function $F \in \mathcal{G}(\lambda, \infty)$ is

$$h_\lambda[F](\phi) := \limsup_{r\to\infty} \frac{\log |F(re^{i\phi})|}{\lambda(r)}, \quad \phi \in [0, 2\pi),$$

and the *lower indicator w.r.t.* λ of F is

$$\underline{h}_\lambda[F](\phi) := \liminf_{r\to\infty} \frac{\log |F(re^{i\phi})|}{\lambda(r)}, \quad \phi \in [0, 2\pi).$$

More specific growth classes are defined as

$$\mathcal{G}(\lambda, c) := \Big\{F \in \mathcal{G}(\lambda, \infty) : h_\lambda[F](0), h_\lambda[F](\pi) \leq c\Big\}, \quad c \in \mathbb{R}.$$

◊

[4] The essence of a growth function is its behaviour at infinity. It would be enough, but would not lead to a significantly more general notion, to assume differentiability and monotonicity only for all sufficiently large r and drop the normalisation at 0.

[5] Instead of this often the condition $\lim_{r\to\infty} \Big(r\frac{\lambda'(r)}{\lambda(r)} \Big/ \frac{\log \lambda(r)}{\log r}\Big) = 1$ is required. If $\rho_\lambda > 0$, then this is equivalent to (gf3). However, if $\rho_\lambda = 0$, then (gf3) is weaker. For example consider λ as in (2.2) with $a = 0$ and $b_1 > 0$. In this context we should also draw the readers attention to [Lev80, p.32,№ 1].

Note that the condition in the definition of $\mathcal{G}(\lambda, c)$ is a limitation of growth along the *real axis*.

2.12 Remark. If λ is a growth function with $\lambda(r) = \mathrm{o}(r)$, then $\mathcal{G}(\lambda, c) = \{0\}$ whenever $c < 0$. This follows since a function of minimal exponential type cannot tend to zero along any line unless it vanishes identically. $\Diamond$

Following to the idea to consider order and type of a measure as in Definition 1.1, we give the following general definition.

2.13 Definition. Let $\mu \in \mathbb{M}$. Then we denote

$$\begin{aligned} h_\lambda[\mu](\phi) &:= \sup\Big\{h_\lambda[F](\phi) : F \in \bigcup_{\mathcal{H}\in\mathcal{C}[\mu]} \mathcal{H}\Big\}, \\ \underline{h}_\lambda[\mu](\phi) &:= \sup\Big\{\underline{h}_\lambda[F](\phi) : F \in \bigcup_{\mathcal{H}\in\mathcal{C}[\mu]} \mathcal{H}\Big\}, \\ \tau_\lambda[\mu] &:= \sup\Big\{\tau_\lambda[F] : F \in \bigcup_{\mathcal{H}\in\mathcal{C}[\mu]} \mathcal{H}\Big\}. \end{aligned}$$

$\Diamond$

2.14 Remark. If $r = \mathrm{o}(\lambda(r))$, then always $h_\lambda[\mu] = 0$. This follows since all functions $F \in \bigcup_{\mathcal{H}\in\mathcal{C}[\mu]} \mathcal{H}$ are of Cartwright class and hence of finite exponential type, cf. Proposition 2.3. $\Diamond$

2.5 Majorisation of measures

In our version of majorisation of measures two properties play a role which a function $\lambda : [0, \infty) \to \mathbb{R}$ may or may not have. Namely:

(P1) The function λ is differentiable. Either $\lambda(0) = 1$ and λ is strictly increasing, or $\lambda(0) = -1$ and λ is strictly decreasing.

(P2) $\lim_{r\to\infty} \lambda'(r)e^{-\lambda(r)} = 0$.

The main objective is the case that λ is a growth function. Indeed, if λ is a growth function, then (P1) holds by definition and (P2) holds since

$$\lambda'(r)e^{-\lambda(r)} = \frac{1}{r} \cdot \underbrace{r\frac{\lambda'(r)}{\lambda(r)}}_{\to\rho_\lambda} \cdot \underbrace{\lambda(r)e^{-\lambda(r)}}_{\leq 1}.$$

We include the decreasing case because of occurrance of large perturbations in Theorem 3.9. Examples for negative and decreasing functions λ satisfying (P1) and (P2) are functions with very moderate decay.

2.15 Example. Consider a function λ subject to (P1) which is, for sufficiently large r, of the form

$$\lambda(r) = a \log r + \psi(r),$$

where $a \in [-1, 0)$, $\psi \geq 0$, $\psi = \mathrm{o}(\log r)$, $\psi'(r) = \mathrm{o}(\frac{1}{r})$, and $\lim_{r\to\infty} \psi(r) = \infty$ if $a = -1$. Then λ satisfies (P2), since

$$\lambda'(r)e^{-\lambda(r)} = ar^{-1-a}e^{-\psi(r)} + r^{-a}\psi'(r)e^{-\psi(r)}.$$

Suitable functions ψ being, e.g.,

$$\psi(r) := (\log\log r)^{b_2} + \ldots + \big(\underbrace{\log\cdots\log r}_{m\text{-times iterate}}\big)^{b_m},$$

where $b_2, \ldots, b_m \in \mathbb{R}$ with at least one $b_i > 0$ if $a = -1$. $\Diamond$

2.16 Definition. Let $\sigma, \nu \in \mathbb{M}$, let $\lambda_1, \lambda_2 : [0, \infty) \to \mathbb{R}$, and assume that λ_1 satisfies (P1) and (P2), and that λ_2 satisfies (P1) with $\lambda_2' > 0$. If

$$\begin{aligned}
&\exists\, c_0, c_1, c_2 \text{ with } c_1 \geq 1, c_0, c_2 \geq 0 \quad \forall\, x \in \mathbb{R} : \\
&\sigma\big((x - e^{-\lambda_1(|x|)}, x + e^{-\lambda_1(|x|)})\big) \leq \qquad\qquad (2.3)\\
&\qquad\qquad c_0 \nu\big((x - c_1 e^{-\lambda_1(|x|)}, x + c_1 e^{-\lambda_1(|x|)})\big) + c_2 e^{-\lambda_2(|x|)},
\end{aligned}$$

we say that σ *is majorised by* ν *w.r.t.* (λ_1, λ_2) and write "$\sigma \preceq \nu$ *w.r.t.* (λ_1, λ_2)".

Writing "$\sigma \preceq \nu$ w.r.t. (λ_1, λ_2)" always implicitly includes that the parameters λ_1 and λ_2 are subject to the stated requirements. $\Diamond$

2.17 Remark. The relation $\preccurlyeq$ used in [BS11a] is obtained as

$$\sigma \preccurlyeq \nu \qquad \Longleftrightarrow \qquad \exists\, \delta > 0 : \sigma \preceq \nu \text{ w.r.t. } (\delta r, 2\delta r)$$

$\Diamond$

The relation "$\sigma \preceq \nu$ w.r.t. (λ_1, λ_2)" should be viewed as a majorisation of σ by a perturbation of ν, where this perturbation is composed of a "shifting/redistribution of mass"–part whose extent is limited by $e^{-\lambda_1}$, and an "additive"–part which does not exceed $e^{-\lambda_2}$.

Concerning this intuition one important word of caution is in order.

☡: *The relation "$\preceq$ w.r.t. (λ_1, λ_2)" gets stronger when the parameter λ_2 is increased pointwise, but increasing the parameter λ_1 pointwise leads to incomparable relations.*

In particular, majorisation w.r.t. a parameter $\lambda_1(r) := r^\rho$ where $\rho > 1$ is incomparable with the majorisation used in [BS11a].

3 Statement of the main results

The below Theorem 3.1 is our main result. In its statement two parameters are involved which should be fitted to each other:

— A pair (λ, c) where λ is a growth function and $c \in [0, \infty)$. This parameter quantifies the a priori knowledge on $\mathcal{C}[\tilde{\mu}]$ as well as the strength of the conclusion.

— A pair of functions (λ_1, λ_2). This parameter quantifies the permitted size of the perturbation.

3.1 Theorem (Fast Growth Theorem)**.** *Let $\mu, \tilde{\mu} \in \mathbb{M}$ where $\tilde{\mu}$ has infinite index of determinacy, let λ be a growth function and let $c \in [0, \infty)$. Assume the a priori knowledge that*

`[Chain]` $\tilde{\mathcal{L}} := \bigcup_{\tilde{\mathcal{H}} \in \mathcal{C}[\tilde{\mu}]} \tilde{\mathcal{H}} \subseteq \mathcal{G}(\lambda, c).$

Assume that $\mu \preceq \tilde{\mu}$ *w.r.t.* (λ_1, λ_2)*, where*

`[A]` $\lambda_1 \geq 2c^+\lambda$ *with some* $c^+ > c$*, and* $\lambda_2 \geq 2\lambda_1$*.*

Then $\tilde{\mathcal{L}}$ *is contained in* $L^2(\mu)$*, and*

$$\textit{either} \quad \operatorname{Clos}_{L^2(\mu)} \tilde{\mathcal{L}} = L^2(\mu) \quad \textit{or} \quad \operatorname{Clos}_{L^2(\mu)} \tilde{\mathcal{L}} \not\subseteq \mathcal{G}(\lambda, c). \tag{3.1}$$

The case that λ_1 grows at least linearly deserves particular attention. It is special due to the automatic presence of bounded type.

3.2 Corollary. *Let* $\mu, \tilde{\mu} \in \mathbb{M}$ *where* $\tilde{\mu}$ *has infinite index of determinacy, let* λ_1 *be a growth function with* $r = \mathrm{O}(\lambda_1(r))$*. If* $\mu \preceq \tilde{\mu}$ *w.r.t.* $(\lambda_1, 2\lambda_1)$*, then* $\tilde{\mathcal{L}}$ *is a dense subspace of* $L^2(\mu)$*.*

Concerning the conclusion (3.1) of Theorem 3.1, a word of caution is in order.

☡: *If the first case in the alternative* (3.1) *takes place, we still do not claim that* $\mathcal{C}[\mu] \subseteq \mathcal{C}[\tilde{\mu}]$*. If the second case in the alternative* (3.1) *takes place, we still do not claim that* $\mathcal{C}[\mu] \supseteq \mathcal{C}[\tilde{\mu}]$*.*

Contrasting this notice, if μ and $\tilde{\mu}$ mutually majorise each other, a conclusion about the chains themselves can be drawn.

3.3 Theorem (Coincidence Theorem)**.** *Let* $\mu, \tilde{\mu} \in \mathbb{M}$ *with infinite index of determinacy, let* λ *be a growth function and let* $c \in [0, \infty)$*. Assume the a priori knowledge that*

`[2Chain]` $\displaystyle\bigcup_{\mathcal{H} \in \mathcal{C}[\mu]} \mathcal{H} \cup \bigcup_{\tilde{\mathcal{H}} \in \mathcal{C}[\tilde{\mu}]} \tilde{\mathcal{H}} \subseteq \mathcal{G}(\lambda, c).$

Assume that $\mu \preceq \tilde{\mu}$ *w.r.t.* (λ_1, λ_2) *and* $\tilde{\mu} \preceq \mu$ *w.r.t.* $(\tilde{\lambda}_1, \tilde{\lambda}_2)$*, where*

`[2A]` $\lambda_1, \tilde{\lambda}_1 \geq 2c^+\lambda$ *with some* $c^+ > c$*, and* $\tilde{\lambda}_2 \geq 2\tilde{\lambda}_1$*,* $\lambda_2 \geq 2\lambda_1$*.*

Then $\mathcal{C}[\mu] = \mathcal{C}[\tilde{\mu}]$*.*

Let us again explicitly state the case of fast growing λ_1.

3.4 Corollary. *Let* $\mu, \tilde{\mu} \in \mathbb{M}$ *with infinite index of determinacy, let* $\lambda_1, \tilde{\lambda}_1$ *be growth functions with* $r = \mathrm{O}(\lambda_1(r))$*,* $r = \mathrm{O}(\tilde{\lambda}_1(r))$*. If* $\mu \preceq \tilde{\mu}$ *w.r.t.* $(\lambda_1, 2\lambda_1)$ *and* $\tilde{\mu} \preceq \mu$ *w.r.t.* $(\tilde{\lambda}_1, 2\tilde{\lambda}_1)$*, then* $\mathcal{C}[\mu] = \mathcal{C}[\tilde{\mu}]$*.*

Next, we turn to stability and monotonicity of order and type. It is a consequence of the Coincidence Theorem that two measure who mutually majorise each other have the same order and type.

3.5 Corollary. *Let* $\mu, \tilde{\mu} \in \mathbb{M}$ *with infinite index of determinacy, let* $\rho \geq \max\{\rho[\mu], \rho[\tilde{\mu}]\}$*, assume that* $\tau_{r^\rho}[\mu], \tau_{r^\rho}[\tilde{\mu}] < \infty$*, and set*

$$c := \max\Big\{h_{r^\rho}[\mu](0),\, h_{r^\rho}[\tilde{\mu}](0),\, h_{r^\rho}[\mu](\tfrac{\pi}{2}),\, h_{r^\rho}[\tilde{\mu}](\tfrac{\pi}{2})\Big\}.$$

Moreover, let $\delta > 0$*. If* $\mu \preceq \tilde{\mu}$ *and* $\tilde{\mu} \preceq \mu$ *w.r.t.* $((2c+\delta)r^\rho, (4c+2\delta)r^\rho)$*, then*

$$\rho[\mu] = \rho[\tilde{\mu}] \quad \textit{and} \quad \tau[\mu] = \tau[\tilde{\mu}].$$

If $\rho \geq 1$*, we always have* $\tau_{r^\rho}[\mu], \tau_{r^\rho}[\tilde{\mu}] < \infty$ *and* $c = 0$*.*

If one has only majorisation in one direction, and not mutual majorisation, type satisfies only a quasi-monotonicity property.

3.6 Theorem (Quasi-Monotonicity Theorem)**.** *Let $\mu, \tilde{\mu} \in \mathbb{M}$ and assume that $\mathcal{C}[\mu]$ has no maximal element. Let λ be a growth function, $c \in [0, \infty)$, and set*

$$\theta := \begin{cases} \frac{\pi}{2} & , \quad \rho_\lambda \leq \frac{1}{2} \\ \frac{\pi}{2}(\frac{1}{\rho_\lambda} - 1) & , \quad \frac{1}{2} < \rho_\lambda \leq 1 \end{cases}.$$

Assume that `[2Chain]` *holds and that $\mu \preceq \tilde{\mu}$ w.r.t. (λ_1, λ_2) as in* `[A]`*. Then*

$$\underline{h}_\lambda[\mu](\phi) \leq h_\lambda[\tilde{\mu}](\phi), \quad |\phi - \frac{\pi}{2}| < \theta.$$

If $\rho_\lambda \in (\frac{1}{2}, 1), \lambda(r) = \mathrm{o}(r^{\rho_\lambda})$, or $\rho_\lambda = 1$, this inequality holds also for $\phi = \frac{\pi}{2} \pm \theta$.

Observe here that only the lower indicator $\underline{h}_\lambda[\mu]$ is estimated from above.

As a consequence of the Quasi-Monotonicity Theorem, for exponential type true monotonicity follows. This includes [BS11a, Theorem 1.3], remember here Remark 2.17.

3.7 Corollary. *Let $\mu, \tilde{\mu} \in \mathbb{M}$ and assume that $\mathcal{C}[\mu]$ has no maximal element. Let λ_1 be a growth function with $r = \mathrm{O}(\lambda_1(r))$. If $\mu \preceq \tilde{\mu}$ w.r.t. $(\lambda_1, 2\lambda_1)$, then $\tau_r[\mu] \leq \tau_r[\tilde{\mu}]$. In fact, $h_r[\mu](\phi) \leq h_r[\tilde{\mu}](\phi)$, $\phi \in [0, 2\pi)$.*

Also on the other end of the growth scale, for very slow growing functions, true monotonicity is present

3.8 Theorem. *Let $\mu, \tilde{\mu} \in \mathbb{M}$ and assume that $\mathcal{C}[\mu]$ has no maximal element. Let λ be a growth function with $\lambda(r) = \mathrm{O}([\log r]^2)$, and let $c \in [0, \infty)$. Assume that* `[2Chain]` *holds and that $\mu \preceq \tilde{\mu}$ w.r.t. (λ_1, λ_2) as in* `[A]`*. Then*

$$\tau_\lambda[\mu] \leq \tau_\lambda[\tilde{\mu}].$$

Observe that in the situation of this theorem certainly $\rho_\lambda = 0$ and hence $h_\lambda[\mu]$ is constant equal to $\tau_\lambda[\mu]$, cf. footnote №15 on page 38.

Finally, we discuss density of polynomials. From [Wor, Corollary 6.13] we know the following result (which is a theorem of "fast-growth type"):

If $\mu \in \mathbb{M}$ and

`[Meas]` $$\int_{-\infty}^{\infty} e^{2c^+ \lambda(|x|)}\, d\mu(x) < \infty \text{ with some } c^+ > c,$$

then either $\mathbb{C}[z]$ is dense in $L^2(\mu)$ or $\mathrm{Clos}_{L^2(\mu)} \mathbb{C}[z] \not\subseteq \mathcal{G}(\lambda, c)$.

In view of this fact, it is noteworthy that `[Meas]` is stable under very large perturbations.

3.9 Theorem. *Let λ be a growth function and $c \in [0, \infty)$. Let $\mu, \tilde{\mu} \in \mathbb{M}$, assume that $\tilde{\mu}$ satisfies* `[Meas]`*, and let $c^+ > c$ be as in this condition. If $\mu \preceq \tilde{\mu}$ w.r.t. (λ_1, λ_2), where*

`[B]` $\lambda_1 \geq (\rho_\lambda^+ - 1) \log r$ *with some* $\rho_\lambda^+ > \rho_\lambda$, *and* $\lambda_2 \geq 2c^+ \lambda$,

then μ satisfies `[Meas]`.

Observe that the condition `[B]` allows λ_1 to decrease to $-\infty$ when $\rho_\lambda < 1$.

PART I:
The Toolbox

In this part of the paper we provide some necessary tools and preliminaries.

— Weight functions and a Bakan-type theorem relating (stable) density in spaces $L^2(\mu)$ and $C_0(W)$; these results are of topological nature.

— A previously unpublished approach to de Branges' theorem on weighted polynomial approximation due to M.Sodin and P.Yuditskii.

— Some properties of growth functions and functions of the kind appearing in the majorisation relation of measures; these considerations are mainly elementary.

— The connection between infinite index of determinacy and stable density.

4 Weighted C_0-spaces

Throughout this section let Ω be a locally compact and σ-compact metrisable topological space. For example, one could think of Ω as being the real line, or an open or closed subset of the euclidean space $\mathbb{R}^n$. Moreover, we denote by $C(\Omega)$ the space of complex valued continuous functions on Ω, and by $C_{00}(\Omega)$ its subspace of all compactly supported functions.

If h is a complex valued function on Ω and $\alpha \in \mathbb{C}$, we write $\lim_{x\to\infty} h(x) = \alpha$ if

$$\forall \varepsilon > 0\ \exists K \subseteq \Omega, K \text{ compact } \forall x \in \Omega \setminus K : \quad |h(x) - \alpha| < \varepsilon \tag{4.1}$$

Let $\overline{\Omega} = \Omega \cup \{\infty\}$ be the one-point compactification of Ω. Then $\lim_{x\to\infty} h(x) = \alpha$ in the sense of (4.1), if and only if $\lim_{x\to\infty} h(x) = \alpha$ with respect to the topology of $\overline{\Omega}$.

4.1 Weight functions

4.1 Definition. Let $W : \Omega \to (0, \infty]$ be a function. If W is lower semicontinuous and not identically equal to ∞, we call W a *weight function on* Ω.

For a weight function W on Ω we set

$$\Omega_W := \{x \in \Omega : W(x) \neq \infty\}.$$

◊

With each weight function we associate a space of continuous functions.

4.2 Definition. Let W be a weight function on Ω. Then we denote[6]

$$C_0(W) := \Big\{ f \in C(\Omega) : \lim_{x\to\infty} \frac{f(x)}{W(x)} = 0 \Big\},$$

$$\|f\|_W := \sup_{x\in\Omega} \Big| \frac{f(x)}{W(x)} \Big|, \quad f \in C_0(W).$$

◊

[6]We set $\frac{a}{\infty} := 0$, $a \in \mathbb{C}$.

For example, if $W(x) = 1$, $x \in \Omega$, the space $C_0(W)$ is just the usual Banach space $C_0(\Omega)$ of all continuous functions f on Ω which vanish at infinity.

For each weight function the space $C_0(W)$ is a linear space, and $\|.\|_W$ is a seminorm on $C_0(W)$. Observe that $\|.\|_W$ is a norm if and only if the set Ω_W is dense in Ω. Unless specified differently, all topological notions applied within $C_0(W)$ refer to the locally convex (but not necessarily Hausdorff) topology induced by the seminorm $\|.\|_W$.

Since a weight function is lower semicontinuous, it is bounded away from zero on every compact set. This shows that for each weight function W on Ω it holds that $C_{00}(\Omega) \subseteq C_0(W)$.

4.3 Remark. Let W_1 and W_2 be weight functions on Ω. Assume that $\Omega_{W_1} = \Omega_{W_2}$ and that the quotient $\frac{W_2}{W_1}|_{\Omega_{W_1}}$ has a continuous extension to a function $L : \Omega \to (0, \infty)$. Then the map $\lambda : f \mapsto L \cdot f$ is an isometric isomorphism of $C_0(W_1)$ onto $C_0(W_2)$. $\Diamond$

Sometimes it is practical to pass to continuous weight functions which do not assume the value infinity. The following statement provides a tool to do this.

4.4 Lemma. *Let W be a weight function on Ω. Then there exists a continuous function ω which takes finite and positive values such that $\omega(x) \leq W(x)$, $x \in \Omega$.*

Proof. Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of open and relatively compact subsets of Ω such that

$$\overline{\Omega_n} \subseteq \Omega_{n+1},\ n \in \mathbb{N}, \qquad \bigcup_{n\in\mathbb{N}} \Omega_n = \Omega, \tag{4.2}$$

and set $\Omega_{-1} = \Omega_0 := \emptyset$. The family

$$\mathcal{M} := \big\{\Omega_{2n} \setminus \overline{\Omega_{2n-3}} :\ n \in \mathbb{N}\big\}$$

is a locally finite open cover of Ω. Let $(\chi_n)_{n\in\mathbb{N}}$ be a continuous partition of unity subordinate to $\mathcal{M}$. Observe that the covering $\mathcal{M}$ is not only locally finite, but that a point x can belong to at most two elements of $\mathcal{M}$. Namely, setting $n(x) := \min\{k \in \mathbb{N} :\ x \in \Omega_k\}$, $x \in \Omega$, we obtain

$$x \in \Omega_{2n} \setminus \Omega_{2n-3} \quad \Leftrightarrow \quad \begin{cases} n \in \{\frac{n(x)+1}{2}\} & ,\quad n(x) \text{ odd}, \\ n \in \{\frac{n(x)}{2}, \frac{n(x)}{2} + 1\}, & \quad n(x) \text{ even}. \end{cases} \tag{4.3}$$

Set

$$\sigma_n := \inf\big\{W(x) :\ x \in \Omega_{2n}\big\}, \quad n \in \mathbb{N}.$$

Since W is lower semicontinuous and Ω_{2n} is relatively compact, we have $\sigma_n > 0$, $n \in \mathbb{N}$. Moreover, $\sigma_n \geq \sigma_{n+1}$, $n \in \mathbb{N}$, in particular $\sup\{\sigma_n : n \in \mathbb{N}\} = \sigma_1 < \infty$.

Now we consider the function

$$\omega(x) := \sum_{n=1}^{\infty} \sigma_n \chi_n(x), \quad x \in \Omega.$$

This is a continuous function, and assumes positive and finite values. Due to (4.3) we have for each $x \in \Omega$

$$\omega(x) = \left\{ \begin{array}{ll} \sigma_{\frac{n(x)+1}{2}}(x)\chi_{\frac{n(x)+1}{2}}(x) & ,\ n(x) \text{ odd} \\ \sigma_{\frac{n(x)}{2}}(x)\chi_{\frac{n(x)}{2}}(x) + \sigma_{\frac{n(x)}{2}+1}(x)\chi_{\frac{n(x)}{2}+1}(x)\,, & n(x) \text{ even} \end{array} \right\} \leq W(x).$$

❑

4.5 Lemma. *Let W and $\tilde{W}$ be a weight functions, and assume that there exists $C \in (0,\infty)$ with*[7]

$$\tilde{W}(x) \le CW(x), \quad x \in \Omega.$$

Then $C_0(\tilde{W})$ is contained in $C_0(W)$. The set-theoretic inclusion map $\iota : C_0(\tilde{W}) \to C_0(W)$ is continuous. Each subset $D \subseteq C_0(\tilde{W})$ which is dense in $C_0(\tilde{W})$ w.r.t. $\|.\|_{\tilde{W}}$ is also a dense in $C_0(W)$ w.r.t. $\|.\|_W$.

Proof. For each $f \in C_0(\tilde{W})$ and $x \in \Omega$ it holds that

$$\Big|\frac{f(x)}{W(x)}\Big| \le C\Big|\frac{f(x)}{\tilde{W}(x)}\Big|.$$

This shows that $C_0(\tilde{W}) \subseteq C_0(W)$ and that the set-theoretic inclusion map is bounded. Since $C_{00}(\Omega) \subseteq C_0(\tilde{W})$, the space $C_0(\tilde{W})$ is dense in $C_0(W)$ w.r.t. $\|.\|_W$. Since the set-theoretic inclusion map is continuous and has dense range, it maps dense subsets of $C_0(\tilde{W})$ to dense subsets of $C_0(W)$. ❑

4.6 Lemma. *Let W be a weight function. There exists a countable subset of $C_{00}(\Omega)$ which is dense in $C_0(W)$.*

Proof. We consider first the case that W is continuous and $\Omega_W = \Omega$. Then the map

$$\lambda : f \mapsto \frac{f}{W}$$

is an isometric isomorphism of $C_0(W)$ onto $C_0(\Omega)$. Moreover, it carries $C_{00}(\Omega)$ onto itself. Hence, it is enough to prove the desired assertion for $C_0(\Omega)$. The one-point compactification $\overline{\Omega}$ of Ω is second countable and Hausdorff[8] and hence the space $C(\overline{\Omega})$ is separable. Fix a countable dense subset $\{f_m : m \in \mathbb{N}\}$ of $C(\overline{\Omega})$. Moreover, for each $n \in \mathbb{N}$, choose a partition of unity $\chi_n, \hat{\chi}_n$ subordinate to the open cover $\{\Omega_{n+1}, \Omega\backslash\overline{\Omega_n}\}$, where Ω_n are as in (4.2). Consider the function $g_{n,m} := \chi_n \cdot f_m$. Then $\operatorname{supp} g_{n,m} \subseteq \Omega_{n+1} \subseteq \overline{\Omega_{n+1}}$, and hence is $\operatorname{supp} g_{n,m}$ is compact. We are going to show that

$$D := \{g_{n,m} : n, m \in \mathbb{N}\} \subseteq C_{00}(\Omega)$$

is dense in $C_0(\Omega)$.

Let $f \in C_0(\Omega)$ and $\varepsilon > 0$ be given. Choose a compact set K with $|f(x)| \le \varepsilon$, $x \in \Omega \setminus K$, choose $m \in \mathbb{N}$ such that $\|f - f_m\|_\infty \le \varepsilon$ (where we regard $C_0(\Omega)$ as a subspace of $C(\overline{\Omega})$), and choose $n \in \mathbb{N}$ such that $\Omega_n \supseteq K$. Then

$$\|f - g_{n,m}\|_\infty = \|f - \chi_n f_m\|_\infty \le \|f - \chi_n f\|_\infty + \|\chi_n(f - f_m)\|_\infty \le 2\varepsilon.$$

This shows that indeed D is dense in $C_0(\Omega)$, and hence the set

$$D_W := \{f \cdot W : f \in D\}$$

[7] We set $a \cdot \infty := \infty$, $a \in (0,\infty)$.

[8] Let Ω_n be as in (4.2). Since Ω is σ-compact, it is Lindelöf. Together with metrisability, this gives that Ω is second countable. Let $\mathcal{B}$ be a countable base for the topology of Ω. Consider $x \in O \subseteq \overline{\Omega}$ with O open in $\overline{\Omega}$. If $x \ne \infty$, then we find a set $U \in \mathcal{B}$ with $x \in U \subseteq O \cap \Omega \subseteq O$. If $x = \infty$, then O^c is a compact subset of Ω. Hence, there exists Ω_n with $\Omega_n \supseteq O^c$. This gives $\infty \in \overline{\Omega} \setminus \overline{\Omega_n} \subseteq O$. However, $\overline{\Omega} \setminus \overline{\Omega_n}$ is open in $\overline{\Omega}$. We have shown that

$$\mathcal{B} \cup \{\overline{\Omega} \setminus \overline{\Omega_n} : n \in \mathbb{N}\}$$

is a base for the topology of $\overline{\Omega_n}$. Since Ω is Hausdorff, also $\overline{\Omega}$ has this property.

is dense in $C_0(W)$.

Now consider an arbitrary weight function W. Choose a continuous and finite weight ω with $\omega \le W$ according to Lemma 4.4. By what we just showed the correspondingly defined set D_ω is dense in $C_0(\omega)$. By Lemma 4.5 this set is dense in $C_0(W)$. ❑

4.2 The topological dual of $C_0(W)$

Knowledge about duals of weighted C_0-spaces (actually, in a much more general setting than the present) was obtained in the 1960's by W.H.Summers following the work of L.Nachbin, cf. [Sum69], [Sum70], [Nac65].

We denote by $\mathbb{M}_+(\Omega)$ the set of all positive Borel measure on Ω, and by $\mathbb{M}_b(\Omega)$ the space of all complex (bounded) Borel measures on Ω endowed with the norm $\|\mu\| := |\mu|(\Omega)$, where $|\mu|$ denotes the total variation of the complex measure μ.

Consider the map T which assigns to each measure $\mu \in \mathbb{M}_b(\Omega)$ the linear functional $T\mu$ defined as

$$(T\mu)f := \int_\Omega \frac{1}{W} f \, d\mu, \quad f \in C_0(W). \tag{4.4}$$

Obviously, T is well-defined and maps $\mathbb{M}_b(\Omega)$ into $C_0(W)'$, in fact

$$\|T\mu\| \le \|\mu\|, \quad \mu \in \mathbb{M}_b(\Omega).$$

The following statement is a consequence of [Sum70, Theorems 3.1 and 4.5]. First one settles the case that Ω_W is dense in Ω, which is the situation considered in [Sum70]. This requires just some standard approximation arguments, e.g., Lusin's Theorem [Rud87, p.55, 2.24]. Then one passes to the general case using isometry of the restriction operator $f \mapsto f|_{\overline{\Omega_W}}$ and Tietze's extension theorem. We will not go into details.

4.7 Theorem (Summers)**.** *The map T defined by* (4.4) *maps $\mathbb{M}_b(\Omega)$ surjectively onto $C_0(W)'$. For each $\mu \in \mathbb{M}_b(\Omega)$, the following statements hold*[9]*.*

- (*i*) *We have $T\mu = T(\mathbb{1}_{\Omega_W}\mu)$ and $\|T\mu\| = \|\mathbb{1}_{\Omega_W}\mu\|$.*
- (*ii*) *The functional $T\mu$ is real (i.e. $\forall f \in C_0(\Omega), f \ge 0 : \ (T\mu)f \in \mathbb{R}$), if and only if $\mathbb{1}_{\Omega_W}\mu$ is a real-valued measure.*

4.3 A Bakan type theorem

In this subsection we relate spaces of the types $C_0(W)$ and $L^2(\mu)$, where W is a weight function on Ω and μ is a positive Borel measure on Ω. We use a unifying notation in order to cover density as well as what is known as “stable density” from [BS11a, Definitions 1.2 and 2.7] or as “infinite index of determinacy” from the theory of power moment problems, cf. [BD95].

4.8 Definition. A *scale* $(\phi_n)_{n=1}^\infty$ *on* Ω is a sequence of functions $\phi_n \in C(\Omega)$, $n \in \mathbb{N}$, with

$$1 \le \phi_1 \le \phi_2 \le \dots .$$

[9]We write $h\mu$ for the measure which is absolutely continuous with respect to μ and has Radon–Nikodym derivative h.

We always set $\phi_0 := 1$.

Let $\mu \in \mathbb{M}_+(\Omega)$, let W be a weight on Ω, and let $\mathcal{X}_N$, $N \in \mathbb{N}_0$, be either the sequence of spaces $L^2(\phi_N^2 d\mu)$, $N \in \mathbb{N}_0$, or $C_0(\phi_N^{-1} W)$, $N \in \mathbb{N}_0$. For $A \subseteq \mathbb{C}^\Omega$ set

$$A_N[\mathcal{X}_0] := \big\{ f \in \mathcal{X}_N : \phi_0 f, \ldots, \phi_N f \in A \big\}, \quad N \in \mathbb{N}_0.$$

We say that A is *dense in $\mathcal{X}_0$ w.r.t. the scale $(\phi_n)_{n=1}^\infty$*, if

$$\forall\, N \in \mathbb{N}_0 : \quad \operatorname{Clos}_{\mathcal{X}_N} A_N[\mathcal{X}_0] = \mathcal{X}_N.$$

◊

Observe that $A_N[\mathcal{X}_0]$ can also be written as

$$A_N[\mathcal{X}_0] := \big\{ f \in \mathbb{C}^\Omega : \phi_0 f, \ldots, \phi_{N-1} f \in A, \phi_N f \in A \cap \mathcal{X}_0 \big\}, \quad N \in \mathbb{N}_0.$$

Density w.r.t. a scale is in two ways stronger than just density. Namely, the set $A_N[\mathcal{X}_0]$ is a (generally proper) subset of $A_{N-1}[\mathcal{X}_0]$, and the norm in the space $\mathcal{X}_N$ is (generally) stronger than the norm of $\mathcal{X}_{N-1}$. To be precise, $C_{00}(\Omega)$ is a dense subset of $\mathcal{X}_N$ for all $N \in \mathbb{N}_0$, and hence density of $A_N[\mathcal{X}_0]$ in $\mathcal{X}_N$ implies density of $A_{N-1}[\mathcal{X}_0]$ in $\mathcal{X}_{N-1}$.

The first example is the trivial one: $\phi_n := 1$, $n \in \mathbb{N}$. Then a set A is dense in $\mathcal{X}_0$ w.r.t. the scale $(\phi_n)_{n\in\mathbb{N}}$ if and only if $\operatorname{Clos}_{\mathcal{X}_0}[A \cap \mathcal{X}_0] = \mathcal{X}_0$. Second, stable density in the sense of [BS11a] is covered as follows.

4.9 Definition. Let $A \subseteq \mathbb{C}^\mathbb{R}$.

(*i*) If $\mu \in \mathbb{M}_+(\mathbb{R})$, we say that A is *stably dense in $L^2(\mu)$*, if A is dense in $L^2(\mu)$ w.r.t. the scale $((1+|x|)^n)_{n=1}^\infty$.

(*ii*) If W is a weight function on $\mathbb{R}$, we say that A is *stably dense in $C_0(W)$*, if A is dense in $C_0(W)$ w.r.t. the scale $((1+|x|)^n)_{n=1}^\infty$.

◊

It is a simple but basic observation that for sufficiently small weights density transfers from $C_0(W)$ to $L^2(\mu)$.

4.10 Lemma. *Let W be a weight function on Ω, let $\mu \in \mathbb{M}_+(\Omega)$, and assume that $W \in L^2(\mu)$. Then $C_0(W)$ is contained in $L^2(\mu)$. The set-theoretic inclusion map $\iota : C_0(W) \to L^2(\mu)$ is continuous and has dense range. In particular, it maps dense subsets to dense subsets.*

Proof. The set $\Omega \setminus \Omega_W$ is a μ-zero set, and for $f \in C_0(W)$ we can estimate

$$\|f\|_\mu^2 = \int_\Omega |f|^2 \, d\mu = \int_{\Omega_W} \Big|\frac{f}{W}\Big|^2 W^2 \, d\mu \le \|f\|_W^2 \|W\|_\mu^2. \tag{4.5}$$

Thus $f \in L^2(\mu)$ and the inclusion map is bounded with norm not exceeding $\|W\|_\mu$. To see that $C_0(W)$ is dense in $L^2(\mu)$, remember that $C_0(W)$ contains $C_{00}(\Omega)$, and that $C_{00}(\Omega)$ is dense in $L^2(\mu)$. ❑

The relation (4.5) implies in particular that a function with $\|f\|_W = 0$ is equal to zero μ-a.e.

4.11 Corollary. *Let W be a weight function on Ω, let $\mu \in \mathbb{M}_+(\Omega)$, and assume that $W \in L^2(\mu)$. Let $A \subseteq \mathbb{C}^\Omega$ and let $(\phi_n)_{n=1}^\infty$ be a scale on Ω. If A is dense in $C_0(W)$ w.r.t. the scale $(\phi_n)_{n=1}^\infty$, then A is dense in $L^2(\mu)$ w.r.t. $(\phi_n)_{n=1}^\infty$.*

Proof. Let $N \in \mathbb{N}_0$. Since $W \in L^2(\mu)$ also $\phi_N^{-1} W \in L^2(\phi_N^2 d\mu)$, and hence $C_0(\phi_N^{-1} W) \subseteq L^2(\phi_N^2 \mu)$. It follows that $A_N[C_0(W)] \subseteq A_N[L^2(\mu)]$. Moreover, since $A_N[C_0(W)]$ is dense in $C_0(\phi_N^{-1} W)$, it is also dense in $L^2(\phi_N^2 \mu)$. ❑

In the next theorem we establish a partial converse of this fact; this is a Bakan-type result.

4.12 Theorem. *Let $\mu \in \mathbb{M}_+(\Omega)$, $\mu \neq 0$, let $(\phi_n)_{n=1}^\infty$ be a scale on Ω, and let $A \subseteq C(\Omega)$. Assume that A is dense in $L^2(\mu)$ w.r.t. the scale $(\phi_n)_{n=1}^\infty$. Then there exists a weight W on Ω, such that $W \in L^2(\mu)$ and A is dense in $C_0(W)$ w.r.t. the scale $(\phi_n)_{n=1}^\infty$.*

Given a continuous and positive function $\omega \in L^2(\mu)$, the weight W can be chosen such that $W \geq \omega$.

Proof. Assume that a continuous and positive function ω which belongs to $L^2(\mu)$ is given.

Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of open and relatively compact subsets of Ω such that

$$\overline{\Omega_n} \subseteq \Omega_{n+1},\ n \in \mathbb{N}, \qquad \bigcup_{n\in\mathbb{N}} \Omega_n = \Omega,$$

and set $\Omega_{-1} = \Omega_0 := \emptyset$. The family

$$\mathcal{M} := \big\{ \Omega_{2n} \setminus \overline{\Omega_{2n-3}} :\ n \in \mathbb{N} \big\}$$

is a locally finite open cover of Ω. Let $(\chi_n)_{n\in\mathbb{N}}$ be a continuous partition of unity subordinate to $\mathcal{M}$. Observe that the covering $\mathcal{M}$ is not only locally finite, but that a point x can belong to at most two elements of $\mathcal{M}$.

For each $n \in \mathbb{N}$ we can choose a countable subset of $C_{00}(\Omega)$ which is dense in $C_0(\phi_n^{-1}\omega)$. Arranging the union of these sets in one sequence, we obtain $\{d_l :\ l \in \mathbb{N}\} \subseteq C_{00}(\Omega)$ which is dense in $C_0(\phi_n^{-1}\omega)$ for all $n \in \mathbb{N}$.

Let $k, l \in \mathbb{N}$. Since A is dense in $L^2(\mu)$ w.r.t. the scale $(\phi_n)_{n=1}^\infty$, we find a function $h_{k,l} \in A_k[L^2(\mu)]$ such that

$$\|h_{k,l} - d_l\|_{L^2(\phi_k^2 d\mu)} \leq \frac{1}{3^{k+l}}.$$

Then we have

$$\frac{1}{9^{k+l}} \geq \int_\Omega |h_{k,l} - d_l|^2 \phi_k^2\, d\mu = \sum_{n=1}^\infty \int_\Omega |h_{k,l} - d_l|^2 \phi_k^2 \chi_n\, d\mu.$$

Hence we can choose a nondecreasing sequence $(b_{k,l;n})_{n\in\mathbb{N}}$ of real numbers with $b_{k,l;1} \geq 1$ and $\lim_{n\to\infty} b_{k,l;n} = \infty$, such that

$$\sum_{n=1}^\infty b_{k,l;n} \int_\Omega |h_{k,l} - d_l|^2 \phi_k^2 \chi_n\, d\mu \leq \frac{1}{4^{k+l}}.$$

Now we define a function W on Ω as

$$W(x) := \Big[\omega(x)^2 + \sum_{k,l,n=1}^{\infty} 2^{k+l} b_{k,l;n} |h_{k,l}(x) - d_l(x)|^2 \phi_k^2 \chi_n(x)\Big]^{\frac{1}{2}}, \quad x \in \Omega.$$

As the supremum of continuous functions W is lower semicontinuous. Moreover, $W(x) \geq \omega(x)$, $x \in \Omega$, in particular W takes values in $(0, \infty]$. The estimate

$$\int_\Omega W^2\, d\mu = \int_\Omega \omega^2\, d\mu + \sum_{k,l=1}^{\infty} 2^{k+l} \sum_{n=1}^{\infty} b_{k,l;n} \int_\Omega |h_{k,l} - d_l|^2 \phi_k^2 \chi_n\, d\mu \leq$$
$$\leq \|\omega\|_\mu^2 + \sum_{k,l=1}^{\infty} \frac{1}{2^{k+l}} = \|\omega\|_\mu^2 + 1$$

shows that $W \in L^2(\mu)$. In particular, since $\mu \neq 0$, the function W cannot be identically equal to ∞. Thus, W is a weight function on Ω.

Now we fix $N \in \mathbb{N}$. We show that $h_{k,l} \in C_0(\phi_N^{-1} W)$, $k \geq N$, $l \in \mathbb{N}$. The fact that

$$W(x)^2 \geq \sum_{n=1}^{\infty} 2^{k+l} b_{k,l;n} |h_{k,l}(x) - d_l(x)|^2 \phi_k^2 \chi_n(x), \quad x \in \Omega,$$

yields the basic estimate

$$\Big|\frac{h_{k,l}(x) - d_l(x)}{W(x)}\Big|^2 \leq \Big[2^{k+l} \phi_k^2 \sum_{n=1}^{\infty} b_{k,l;n} \chi_n(x)\Big]^{-1}, \quad x \in \Omega. \tag{4.6}$$

If $m \in \mathbb{N}$ and $x \notin \Omega_{2m-2}$, then $\sum_{n=1}^{\infty} b_{k,l;n} \chi_n(x) \geq b_{k,l;m}$. Hence,

$$\Big|\frac{h_{k,l}(x)}{\phi_N^{-1} W(x)}\Big|^2 \leq \frac{|\phi_N(x)|^2}{2^{k+l} |\phi_k(x)|^2 b_{k,l;m}} \leq \frac{1}{b_{k,l;m}},$$
$$x \notin \Omega_{2m-2} \cup \operatorname{supp} d_l,\ k \geq N\ , m \in \mathbb{N},\ l \in \mathbb{N}.$$

Since $\lim_{m\to\infty} b_{k,l;m} = \infty$, this shows that indeed $h_{k,l} \in C_0(\phi_N^{-1} W)$, $k \geq N$, $l \in \mathbb{N}$. It follows that $h_{k,l} \in A_N[C_0(W)]$, $k \geq N$, $l \in \mathbb{N}$.

We use (4.6) once again to show $\lim_{k\to\infty} \|h_{k,l} - d_l\|_{C_0(\phi_N^{-1} W)} = 0$. Since $\sum_{n=1}^{\infty} b_{k,l;n} \chi_n(x) \geq 1$, $x \in \Omega$, (4.6) indeed gives

$$\Big|\frac{h_{k,l}(x) - d_l(x)}{|\phi_N(x)|^{-1} W(x)}\Big|^2 \leq \frac{\phi_N^2}{2^{k+l} \phi_k^2} \leq \frac{1}{2^k}, \quad x \in \Omega,\ k \geq N,\ l \in \mathbb{N}.$$

It follows that $\{d_l : l \in \mathbb{N}\} \subseteq \operatorname{Clos}_{C_0(\phi_N^{-1} W)} A_N[C_0(W)]$.

Finally, since $\phi_N^{-1} W \geq \phi_N^{-1} \omega$, we may apply Lemma 4.5 which yields that $\{d_l : l \in \mathbb{N}\}$ is dense in $C_0(\phi_N^{-1} W)$, and hence that

$$\operatorname{Clos}_{C_0(\phi_N^{-1} W)} A_N[C_0(W)] = C_0(\phi_N^{-1} W).$$

❏

5 De Branges' theorem on weighted approximation

In [Bra59] L.de Branges showed that non-density of polynomials in $C_0(W)$ is equivalent to existence of entire functions with certain properties, so-called *Krein class* functions. This result is fundamentally different from other characterisations of non-density given by S.N.Mergelyan, N.I.Achieser or others (see, e.g., [Koo98, Chapter VI]) which characterise non-density in terms of the function (sometimes called the *Hall–majorant*)

$$\mathfrak{m}(z) := \sup\big\{|p(z)| : p \in \mathbb{C}[z], \|p\|_W \le 1\big\}, \quad z \in \mathbb{C}.$$

Apparently, the value $\mathfrak{m}(z)$ is nothing but the norm of the point-evaluation functional at z.

In this section we disscuss a variant of de Branges' Theorem, which is valid for algebraic de Branges spaces instead of the space of polynomials, and is adapted to deal with stable density.

A suitable replacement for Krein class functions is needed.

5.1 Definition. Let $\mathcal{L}$ be an algebraic de Branges space, let W be a weight function on $\mathbb{R}$, and let $m, n \in \mathbb{Z}$. Then we define the *weighted Krein class* $\mathcal{K}_{m,n}(\mathcal{L}, W)$ as the set of all entire function B which have the following properties.

(K1) The function B satisfies $B = B^\#$ and all its zeros are real and simple. It does not vanish identically and has at least one zero.

(K2) For each $F \in \mathcal{L}$, the function $\frac{F}{B}$ is of bounded type in $\mathbb{C}^+$.

(K3) For each $F \in \mathcal{L}$ we have

$$|y|^{1-m} \cdot |F(iy)| = \mathrm{o}\big(|B(iy)|\big), \quad y \to \infty.$$

(K4) If $x \in \mathbb{R}$ with $B(x) = 0$ then $W(x) < \infty$, and

$$\sum_{x:\, B(x)=0} \frac{1}{(1+|x|)^n} \frac{W(x)}{|B'(x)|} < \infty.$$

◊

Note the following obvious properties of weighted Krein classes.

5.2 Remark. Let $\mathcal{L}$ be an algebraic de Branges space and let W be a weight function on $\mathbb{R}$.

(*i*) If $m \le m'$ and $n \le n'$, then $\mathcal{K}_{m,n}(\mathcal{L}, W) \subseteq \mathcal{K}_{m',n'}(\mathcal{L}, W)$.

(*ii*) Let $B \in \mathcal{K}_{m,n}(\mathcal{L}, W)$. If $x_0 \in \Omega_W$ with $B(x_0) \ne 0$, then

$$(z - x_0)B(z) \in \mathcal{K}_{m-1,n-1}(\mathcal{L}, W).$$

If B has at least two zeroes and $x_0 \in \mathbb{R}$ with $B(x_0) = 0$, then

$$\frac{B(z)}{(z - x_0)} \in \mathcal{K}_{m+1,n+1}(\mathcal{L}, W).$$

(*iii*) For $l \in \mathbb{Z}$, set $W_l(x) := (1+|x|)^{-l} W(x)$. Then

$$\mathcal{K}_{m,n}(\mathcal{L}, W_{l+k}) = \mathcal{K}_{m,n+k}(\mathcal{L}, W_l), \quad m,n,l,k \in \mathbb{Z}.$$

$\Diamond$

For an algebraic de Branges space $\mathcal{L}$ and a number $l \in \mathbb{Z}$ set

$$\mathcal{L}_l := \begin{cases} \{F \in \mathcal{L} : z^l F(z) \in \mathcal{L}\}, & l \geq 0, \\ \mathcal{L} + z\mathcal{L} + \ldots + z^{|l|}\mathcal{L} \quad , & l < 0. \end{cases} \tag{5.1}$$

Note that $(\mathcal{L}_l)_{l \in \mathbb{Z}}$ is a decreasing chain (w.r.t. inclusion) of algebraic de Branges spaces. Moreover, recall that

$$\dim \mathcal{L} = \inf \big\{ l \in \mathbb{N} : \mathcal{L}_l = \{0\} \big\},$$

where the infimum of the empty set is understood as ∞, cf. [Wor11, Lemma 2.11].

The next property of Krein classes relies on the structure of $\mathcal{L}$ as algebraic de Branges space.

5.3 Lemma. *Let $\mathcal{L}$ be an algebraic de Branges space, let $l, k \in \mathbb{Z}$, and assume that $\mathcal{L}_l, \mathcal{L}_{l+k} \neq \{0\}$. Moreover, let W be a weight function. Then*

$$\mathcal{K}_{m,n}(\mathcal{L}_{l+k}, W) = \mathcal{K}_{m+k,n}(\mathcal{L}_l, W), \quad m,n \in \mathbb{Z}.$$

Proof. It is enough to show that for all $m,n,l \in \mathbb{Z}$ with $\mathcal{L}_l \neq \{0\}$, it holds that $\mathcal{K}_{m,n}(\mathcal{L}_l, W) = \mathcal{K}_{m+1,n}(\mathcal{L}_{l-1}, W)$. The inclusion "$\supseteq$" is easy to see. From $z\mathcal{L}_l \subseteq \mathcal{L}_{l-1}$ we get that for $F \in \mathcal{L}_l$ and $B \in \mathcal{K}_{m+1,n}(\mathcal{L}_{l-1}, W)$

$$|y|^{1-m}|F(iy)| = |y|^{1-(m+1)}|(iy)F(iy)| = \mathrm{o}(B(iy)).$$

The reverse inclusion uses that $\mathcal{L}$ is an algebraic de Branges space in form of the property

$$F, G \in \mathcal{L}_l, w \in \mathbb{C} \quad \Rightarrow \quad \frac{F(z)G(w) - G(w)F(z)}{z - w} \in \mathcal{L}_{l+1}.$$

Fix $G \in \mathcal{L}_l$ and $w \in \mathbb{C}$ with $G(w) \neq 0$. Moreover, let $B \in \mathcal{K}_{m,n}(\mathcal{L}_l, W)$ and $F \in \mathcal{L}_{l-1}$. Set

$$H(z) := \frac{F(z) - \frac{G(z)}{G(w)} F(w)}{z - w},$$

then $H \in \mathcal{L}_l$ and hence $|y|^{1-m}|H(iy)| = \mathrm{o}(B(iy))$. Since also $|y|^{1-m}|G(iy)| = \mathrm{o}(B(iy))$, it follows that

$$|y|^{1-(m+1)}|F(iy)| = \mathrm{o}(B(iy)).$$

Hence, $B \in \mathcal{K}_{m+1,n}(\mathcal{L}_{l-1}, W)$. ❑

Next, let us observe that functions belonging to a weighted Krein class usually will have infinitely many zeroes.

5.4 Lemma. *Let $\mathcal{L}$ be an algebraic de Branges space, let W be a weight function on $\mathbb{R}$, and let $m, n \in \mathbb{N}_0$, $N \in \mathbb{N}$. If $\mathcal{K}_{m,n}(\mathcal{L}, W)$ contains a function having at most N zeroes then $\dim \mathcal{L} \leq m + N$.*

Proof. Assume $B \in \mathcal{K}_{m,n}(\mathcal{L}, W)$ and that B has exactly N zeroes, say $x_1, \ldots, x_N$. Set $S(z) := \big[\prod_{i=1}^{N}(z - x_i)\big]^{-1} B(z)$, then S is real and zerofree. The function $\frac{F}{S}$ is entire, of bounded type in $\mathbb{C}^+$ and $\mathbb{C}^-$, and satisfies [10]

$$\Big|\frac{F(iy)}{S(iy)}\Big| \lesssim |y|^{m-1+N}, \quad y \to \pm\infty.$$

Hence, $\frac{F}{S}$ is a polynomial of degree at most $m-1+N$. Thus $\dim \mathcal{L} \leq m+N$. ❑

The following result is a general version of de Branges' Theorem. It relates non-density of an algebraic de Branges space $\mathcal{L}$ with existence of functions in a weighted Krein class. A proof along the lines of de Branges' original argument was given in [BW13].

5.5 Theorem. *Let $\mathcal{L}$ be an algebraic de Branges space, let W be a weight function on $\mathbb{R}$, and assume that $\mathcal{L} \subseteq C_0(W)$. Then the following are equivalent.*

(*i*) *$\mathcal{L}$ is dense in $C_0(W)$.*

(*ii*) *$\mathcal{K}_{0,0}(\mathcal{L}, W) = \emptyset$.*

Let us deduce a version for stable density.

5.6 Proposition. *Let $\mathcal{L}$ be an algebraic de Branges space, let W be a weight function on $\mathbb{R}$, and assume that $\mathcal{L} \subseteq C_0(W)$. Then the following are equivalent.*

(*i*) *$\mathcal{L}$ is stably dense in $C_0(W)$.*

(*ii*) *$\displaystyle\bigcup_{m,n\in\mathbb{N}_0} \mathcal{K}_{m,n}(\mathcal{L}, W) = \emptyset$.*

Proof. First we settle the case that $\mathcal{L}$ is finite-dimensional. Set $d := \dim \mathcal{L}$, then $\mathcal{L} = \operatorname{span}\{S(z), \ldots, z^{d-1}S(z)\}$ with some real and zerofree function S, cf. Example 2.8. Choose $x_0 \in \mathbb{R}$ with $W(x_0) \neq 0$, then $(z-x_0)S(z) \in \mathcal{K}_{d-1,0}(\mathcal{L}, W)$ Hence, the union in (*ii*) is nonempty. Since the set $\{(z+i)^k F(z) : k = 0, \ldots, N\}$ is linearly independent unless F vanishes identically, we have $\mathcal{L}_N[C_0(W)] = \{0\}$ when $N \geq \dim \mathcal{L}$. However, $C_0((1 + |x|)^n W(x)) \neq \{0\}$, and hence $\mathcal{L}$ is not stably dense in $C_0(W)$.

From now on assume that $\dim \mathcal{L} = \infty$. The space $\mathcal{L}$ is contained in $C_0(W)$ and invariant under division of zeroes. Hence (using the notation from Remark 5.2, (*iii*), and (5.1)),

$$\mathcal{L}_N[C_0(W)] = \big\{f \in C_0(W_N) : f(z), \ldots, (z+i)^N f(z) \in \mathcal{L}\big\} = \mathcal{L}_N, \quad N \in \mathbb{N}_0.$$

[10] Let us once and forall fix a commonly used notation. We write

$$f(x) \lesssim g(x),\ x \in M \quad :\Longleftrightarrow \quad \exists\, C > 0\, \forall\, x \in M : f(x) \leq Cg(x).$$

Notice that this relation sees the sign of f and g, e.g., we have $-n \lesssim 1, n \in \mathbb{N}$. Moreover, we write

$$f(x) \asymp g(x),\ x \in M \quad :\Longleftrightarrow \quad f(x) \lesssim g(x),\ x \in M \text{ and } g(x) \lesssim f(x),\ x \in M$$

and $f \gtrsim g$ means that $g \lesssim f$.

We conclude that $\mathcal{L}$ is stably dense in $C_0(W)$ if and only if

$$\bigcup_{N\in\mathbb{N}_0} \mathcal{K}_{0,0}(\mathcal{L}_N, W_N) = \emptyset.$$

However, $\mathcal{K}_{0,0}(\mathcal{L}_N, W_N) = \mathcal{K}_{N,N}(\mathcal{L}, W)$, and hence

$$\bigcup_{N\in\mathbb{N}_0} \mathcal{K}_{0,0}(\mathcal{L}_N, W_N) = \bigcup_{N\in\mathbb{N}_0} \mathcal{K}_{N,N}(\mathcal{L}, W) = \bigcup_{m,n\in\mathbb{N}_0} \mathcal{K}_{m,n}(\mathcal{L}, W).$$

❑

5.7 Corollary. *Let $\mathcal{L}$ be an algebraic de Branges space, let W be a weight function on $\mathbb{R}$, and assume that $\mathcal{L} \subseteq C_0(W)$.*

(*i*) *If $\dim \mathcal{L} < \infty$, then $\mathcal{L}$ is not stably dense in $C_0(W)$.*

(*ii*) *If Ω_W is not discrete and $\mathcal{L}$ is dense in $C_0(W)$, then $\mathcal{L}$ is stably dense in $C_0(W)$.*

Proof. Item (i) was shown in the first paragraph of the proof of Proposition 5.6, and item (ii) follows from Remark 5.2, (ii). ❑

Another consequence of Proposition 5.6 is that stable density is inherited when the weight is changed inside a finite interval.

5.8 Lemma. *Let $\mathcal{L}$ be an algebraic de Branges space, let W_1, W_2 be weight functions on $\mathbb{R}$, and assume that $\mathcal{L} \subseteq C_0(W_1) \cap C_0(W_2)$. If there exist $T > 0$ and $N \in \mathbb{N}_0$ such that*

$$\frac{W_1(x)}{(1+|x|)^N} \lesssim W_2(x), \quad |x| > T,$$

and $\mathcal{L}$ is stably dense in $C_0(W_1)$, then $\mathcal{L}$ is also stably dense in $C_0(W_2)$.

Proof. Assume on the contrary that $\mathcal{L}$ is not stably dense in $C_0(W_2)$. Then we can choose $m, n \in \mathbb{N}_0$ and $B_2 \in \mathcal{K}_{m,n}(\mathcal{L}, W_2)$. Since $\mathcal{L}$ is stably dense in $C_0(W_1)$, it has infinite dimension. Lemma 5.4 ensures that B_2 has infinitely many zeroes. Let $x_1, \ldots, x_k$ be the zeroes of B_2 in $[-T, T]$, and set

$$B_1(z) := \Big[\prod_{i=1}^{k}(z - x_i)\Big]^{-1} B_2(z).$$

Then $B_1 \in \mathcal{K}_{m+k,n+k}(\mathcal{L}, W_2)$, and (with some appropriate constant $c > 0$)

$$\sum_{x:B_1(x)=0} \frac{1}{(1+|x|)^{n+k+N}} \frac{W_1(x)}{|B_1'(x)|} \le c \sum_{x:B_1(x)=0} \frac{1}{(1+|x|)^{n+k}} \frac{W_2(x)}{|B_1'(x)|} < \infty.$$

Thus, $B_1 \in \mathcal{K}_{m+k,n+k+N}(\mathcal{L}, W_1)$, and we have reached a contradiction. ❑

Independently, and in parallel to our work [BW13], M.Sodin and P.Yuditskii proposed a different approach to Theorem 5.5 based on Chebyshev alternance. It follows ideas of their earlier work [SY92], [SY97] and the method from [BS11a] [11].

[11] The proof given in this paper is very much adapted to the case of exponential type.

This approach was presented in a conference talk, cf. [SY12], but remained unpublished otherwise. The Sodin-Yuditskii method yields, under the assumption of non-density, some additional information on the constructed Krein class function.

5.9 Theorem. *Let $\mathcal{L}$ be an infinite dimensional algebraic de Branges space, and let W be a weight function on $\mathbb{R}$. Assume that $\mathcal{L} \subseteq C_0(W)$ but that $\mathcal{L}$ is not dense in $C_0(W)$. Denote by $\mathfrak{m}_\mathcal{L}$ the corresponding Hall-majorant*

$$\mathfrak{m}_\mathcal{L}(z) := \sup\big\{|F(z)| : F \in \mathcal{L}, \|F\|_W \leq 1\big\}, \quad z \in \mathbb{C}.$$

Then there exists a function $B \in \mathcal{K}_{3,2}(\mathcal{L}, W)$ [12] which has infinitely many zeroes and satisfies

$$|B(z)| \lesssim (1 + |z|) \cdot \mathfrak{m}_\mathcal{L}(z), \quad z \in \Gamma_\vartheta, \tag{5.2}$$

uniformly in each Stolz angle Γ_ϑ, $\vartheta \in (0, \frac{\pi}{2})$.

The knowledge about $\mathcal{K}_{3,2}(\mathcal{L}, W)$ expressed by (5.2) is a key ingredient to our investigations, and the fact that it is unpublished left us in a somewhat unsatisfactory situation. Thus, we decided – with the kind permission of M.Sodin and P.Yuditskii – to make the details of their proof available in an appendix to this paper, cf. Appendix A.

We are going to use the Sodin-Yuditskii Theorem in form of the following

5.10 Proposition. *Let λ be a growth function and $c \in [0, \infty]$, let $\mu \in \mathbb{M}$, and let W be a weight function on $\mathbb{R}$. Assume that $W \in L^2(\mu)$. If $\mathcal{H} \in \mathcal{C}[\mu]$ with*

$$\mathcal{H} \neq L^2(\mu), \quad \mathcal{H} \subseteq \mathcal{G}(\lambda, c), \quad \dim\big(\mathcal{H} \cap C_0(W)\big) = \infty,$$

then

$$\mathcal{K}_{3,2}\big(\mathcal{H} \cap C_0(W), W\big) \cap \mathcal{G}(\lambda, c) \neq \emptyset.$$

Proof. The assumptions of Theorem 5.9 are satisfied with $\mathcal{L} := \mathcal{H} \cap C_0(W)$: $\mathcal{L}$ clearly is an algebraic de Branges space, by assumption $\dim \mathcal{L} = \infty$, and $\mathcal{L}$ is not dense in $C_0(W)$ by Corollary 4.11. Thus we find a function $B \in \mathcal{K}_{3,2}(\mathcal{L}, W)$ which is bounded by the Hall-majorant $\mathfrak{m}_\mathcal{L}$ in the sense of (5.2).

We have $C_0(W) \subseteq L^2(\mu)$ and the corresponding inclusion map is bounded. Since, for each $z \in \mathbb{C}$, the point evaluation functional satisfies

$$\begin{array}{ccc} \langle \mathcal{H} \cap C_0(W), \|.\|_W \rangle & \xrightarrow{\ \subseteq\ } & \langle \mathcal{H}, \|.\|_\mu \rangle \\ & {\scriptstyle f \mapsto f(z)} \searrow & \downarrow {\scriptstyle f \mapsto f(z)} \\ & & \mathbb{C} \end{array}$$

we have

$$\mathfrak{m}_\mathcal{L}(z) \lesssim \sup\big\{|F(z)| : F \in \mathcal{H}, \|F\|_\mu \leq 1\big\} =: \nabla_{\langle \mathcal{H}, \|.\|_\mu \rangle}(z), \quad z \in \mathbb{C}.$$

However, since $\mathcal{H} \subseteq \mathcal{G}(\lambda, c)$, it holds that

$$\limsup_{|z| \to \infty} \frac{\log \nabla_{\langle \mathcal{H}, \|.\|_\mu \rangle}(z)}{\lambda(|z|)} < \infty \quad \text{and} \quad \limsup_{x \to \pm\infty} \frac{\log \nabla_{\langle \mathcal{H}, \|.\|_\mu \rangle}(x)}{\lambda(|x|)} \leq c,$$

see, e.g., [Wor, Lemma 5.2]. This shows that $B \in \mathcal{G}(\lambda, c)$. ❑

[12] We believe, but do not know, that one can obtain a similar result for $\mathcal{K}_{0,0}(\mathcal{L}, W)$.

6 Various preliminaries

6.1 Growth functions

In this short subsection we list some elementary properties of growth functions which are frequently used. Let λ be a growth function.

❋ The limit relation

$$\lim_{r\to\infty}\frac{\lambda(Cr)}{\lambda(r)}=C^{\rho_\lambda}$$

holds uniformly in C on compact subsets of $(0,\infty)$, cf. [Lev80, I.12.Lemma 5] or [LG86, Theorem 1.18]. In particular,

$$x_n,y_n>0,\ \lim_{n\to\infty}\frac{x_n}{y_n}=1 \quad\Rightarrow\quad \lim_{n\to\infty}\frac{\lambda(x_n)}{\lambda(y_n)}=1 \tag{6.1}$$

❋ Let $\sigma>0$. Then, for all sufficiently large r, the function $\frac{\lambda(r)}{r^\sigma}$ is increasing to ∞ if $\sigma<\rho_\lambda$ and decreasing to 0 if $\sigma>\rho_\lambda$. This follows from the relation

$$\Big[\frac{\lambda(r)}{r^\sigma}\Big]'=\frac{\lambda(r)}{r^{\sigma+1}}\Big(-\sigma+\frac{r\lambda'(r)}{\lambda(r)}\Big)$$

and the fact that $\lim_{r\to\infty}\frac{r\lambda'(r)}{\lambda(r)}=\rho_\lambda$ by (gf3).

6.2 Functions with (P1),(P2)

To start with, let us observe the following simple fact.

6.1 Lemma. *Let $\lambda:[0,\infty)\to\mathbb{R}$ be subject to* (P1) *and* (P2)*. Then for every $C>0$ there exists $r_0>0$ such that*

$$\lambda(r)\geq-\log r+C,\quad r\geq r_0.$$

In particular, $\lim_{r\to\infty}\frac{1}{r}e^{-\lambda(r)}=0$.

Proof. This is trivial when λ is positive. If λ is negative, hence decreasing, it follows by integrating the inequality $|\lambda'(r)e^{-\lambda(r)}|\leq\epsilon$ which holds for arbitrary $\epsilon>0$ when r is sufficiently large. ❑

With a function λ we associate a sequence of real points and intervals.

6.2 Definition. Let $\lambda:[0,\infty)\to\mathbb{R}$ be subject to (P1) and (P2), set

$$a(r):=r+e^{-\lambda(r)},\quad r\in[0,\infty),$$

and let $a^{[n]}$ be the n-times iterate of a. Define points x_n and intervals $I_{n,c}$ as

$$\begin{aligned}&x_n:=a^{[n]}(0),\ n\in\mathbb{N},\qquad x_n:=0,\ n\in\mathbb{Z},n\leq0,\\&I_{n,c}:=\big(x_n-ce^{-\lambda(x_n)},x_n+ce^{-\lambda(x_n)}\big)\cap[0,\infty),\ n\in\mathbb{Z},c\geq1.\end{aligned}\tag{6.2}$$

◊

Since λ is continuous, passing to the limit in the recursive definition of x_n yields that $\lim_{n\to\infty} x_n = \infty$. However, the sequence $(x_n)_{n\in\mathbb{N}}$ cannot grow too fast: from Lemma 6.1 we obtain that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1. \tag{6.3}$$

6.3 Remark. The interval $I_{n,1}$ contains $[x_n, x_{n+1})$. Hence, for each $m, m' \in \mathbb{N}_0$, $m \le m'$, the family $\{I_{n,1} : m \le n \le m'\}$ covers the interval $[x_m - e^{-\lambda(x_m)}, x_{m'+1})$. In other words, we have [13]

$$\sum_{n=m}^{m'} \mathbb{1}_{I_{n,1}} \ge \mathbb{1}_{(x_m - e^{-\lambda(x_m)}, x_{m'+1})} \ge \mathbb{1}_{[x_m, x_{m'+1})}.$$

$\Diamond$

In the next lemma we collect some still elementary but slightly more involved facts.

6.4 Lemma. *Let* $\lambda : [0,\infty) \to \mathbb{R}$ *be subject to* (P1) *and* (P2). *Then the following statements hold.*

(*i*) $\forall\, c \ge 1\ \exists\, k_+(c), k_-(c) \in \mathbb{N} : I_{n,c} \subseteq \big(x_{n-k_-(c)}, x_{n+k_+(c)}\big), n \in \mathbb{N}_0.$

(*ii*) $\forall\, c \ge 1 : \gamma(\lambda, c) := \sup_{n\in\mathbb{N}_0} \dfrac{\max\{e^{-\lambda(t)} : t \in [x_{n-k_-(c)}, x_{n+k_+(c)}]\}}{\min\{e^{-\lambda(t)} : t \in [x_{n-k_-(c)}, x_{n+k_+(c)}]\}} < \infty.$

(*iii*) *Set* $c' := \gamma(\lambda, c)(c + k_+(c) + k_-(c))$, *then*

$$\forall c \ge 1, n \in \mathbb{N}_0, x \in [x_{n-k_-(c)}, x_{n+k_+(c)}] : I_{n,c} \subseteq \big[x - c' e^{-\lambda(x)}, x + c' e^{-\lambda(x)}\big].$$

Proof. Set $\alpha(r) := e^{-\lambda(r)}$. The crucial fact for the proof of (*i*) and (*ii*) is that

$$a^{[l]}(x) = x + l\alpha(x) + R_l(x) \text{ with } R_l = o(\alpha(x)), \quad l \in \mathbb{N}, \tag{6.4}$$

which we shall establish by induction on l.

For $l = 1$ this relation trivially holds with $R_1 := 0$. Assume that (6.4) holds for some $l \in \mathbb{N}$. We have $a^{[l]}(x) > x$, and the mean value theorem provides us with $\xi_x \in [x, a^{[l]}(x)]$ such that

$$a^{[l+1]}(x) = a\big(a^{[l]}(x)\big) = a(x) + \big(a^{[l]}(x) - x\big)a'(\xi_x) = a(x) + \big(l\alpha(x) + R_l(x)\big)a'(\xi_x).$$

We continue rewriting the right hand side as

$$\begin{aligned} a(x) + \big(l\alpha(x) + R_l(x)\big)a'(\xi_x) &= (x + \alpha(x)) + \big(l\alpha(x) + R_l(x)\big)(1 + \alpha'(\xi_x)) = \\ &= x + (l+1)\alpha(x) + \underbrace{\big[R_l(x) + l\alpha(x)\alpha'(\xi_x) + R_l(x)\alpha'(\xi_x)\big]}_{:=R_{l+1}(x)}. \end{aligned}$$

Since $R_l = o(\alpha)$ and $\lim_{x\to\infty} \alpha'(\xi_x) = \lim_{x\to\infty} \alpha'(x) = 0$, also $R_{l+1} = o(\alpha)$.

[13] We denote by $\mathbb{1}_Y$ the characteristic function of the set Y.

For the proof of (i), let $c \geq 1$ be given. Let k be the smallest integer greater than c, and choose $n_0 \in \mathbb{N}$ such that

$$\frac{|R_k(x)|}{\alpha(x)} \leq k - c,\ x \geq x_{n_0}, \quad \frac{R_{k+1}(x) - R_k(x)}{\alpha(x)} \leq \frac{k}{c} - 1,\ x \geq x_{n_0-k}.$$

Giving an upper bound for $I_{n,c}$ is now easy. Namely, by (6.4), for each $n \geq n_0$

$$\sup I_{n,c} = x_n + c\alpha(x_n) \leq x_n + \Big(k - \frac{|R_k(x_n)|}{\alpha(x_n)}\Big)\alpha(x_n) \leq a^{[k]}(x_n) = x_{n+k}.$$

We turn to the lower bound. If λ is positive and hence increasing, the function α is decreasing and we have

$$c\alpha(x_n) \leq k\alpha(x_n) \leq \alpha(x_{n-1}) + \ldots + \alpha(x_{n-k}).$$

Thus

$$\inf I_{n,c} = x_n - c\alpha(x_n) \geq x_n - \alpha(x_{n-1}) - \ldots - \alpha(x_{n-k}) = x_{n-k}.$$

Assume that λ is negative and hence that α is increasing. From (6.4) we have for each $n \geq n_0$

$$\begin{aligned}\alpha(x_n) = x_{n+1} - x_n &= a^{[k+1]}(x_{n-k}) - a^{[k]}(x_{n-k}) \\ &= \alpha(x_{n-k}) + \big(R_{k+1}(x_{n-k}) - R_k(x_{n-k})\big),\end{aligned}$$

and hence

$$\begin{aligned}\inf I_{n,c} = x_n - c\alpha(x_n) &= x_n - c\Big(1 + \frac{R_{k+1}(x_{n-k}) - R_k(x_{n-k})}{\alpha(x_{n-k})}\Big)\alpha(x_{n-k}) \\ &\geq x_n - k\alpha(x_{n-k}) \geq x_n - \alpha(x_{n-1}) - \ldots - \alpha(x_{n-k}) = x_{n-k}.\end{aligned}$$

Thus $I_{n,c} \subseteq (x_{n-k}, x_{n+k})$ for all $n \geq n_0$. Set $k_-(c) := k + n_0$ and choose $k_+(c)$ such that $k_+(c) \geq k$ and $x_{k_+(c)} \geq \sup \bigcup_{n=1}^{n_0} I_{n,c}$. Then, certainly, $I_{n,c} \subseteq (x_{n-k_-(c)}, x_{n+k_+(c)})$, $n = 0, \ldots, n_0$. Since $k_-(c), k_+(c) \geq k$, this inclusion also holds for all larger values of n.

For the proof of (ii) we use (6.4) to rewrite

$$\begin{aligned}\frac{\alpha(x_{n+k_+(c)})}{\alpha(x_{n-k_-(c)})} &= \frac{x_{n+k_+(c)+1} - x_{n+k_+(c)}}{\alpha(x_{n-k_-(c)})} \\ &= \frac{a^{[k_-(c)+k_+(c)+1]}(x_{n-k_-(c)}) - a^{[k_-(c)+k_+(c)]}(x_{n-k_-(c)})}{\alpha(x_{n-k_-(c)})} \\ &= \frac{\alpha(x_{n-k_-(c)}) + R_{k_-(c)+k_+(c)+1}(x_{n-k_-(c)}) - R_{k_-(c)+k_+(c)}(x_{n-k_-(c)})}{\alpha(x_{n-k_-(c)})}.\end{aligned}$$

This shows that $\lim_{n\to\infty} \frac{\alpha(x_{n+k_+(c)})}{\alpha(x_{n-k_-(c)})} = 1$. In particular, this quotient is bounded above and away from zero. Using monotonicity of α yields (ii).

To show (iii), first observe that

$$\begin{aligned}x_{n+k_+(c)} - x_{n-k_-(c)} &= \sum_{i=n-k_-(c)}^{n+k_+(c)-1} e^{-\lambda(x_i)} \\ &\leq (k_+(c) + k_-(c)) \max\big\{e^{-\lambda(t)} : t \in [x_{n-k_-(c)}, x_{n+k_+(c)}]\big\} \\ &\leq (k_+(c) + k_-(c))\gamma(\lambda, c) \min\big\{e^{-\lambda(t)} : t \in [x_{n-k_-(c)}, x_{n+k_+(c)}]\big\}.\end{aligned}$$

Now we estimate, for given $x \in [x_{n-k_-(c)}, x_{n+k_+(c)}]$,

$$\begin{aligned}\sup I_{n,c} &= x_n + ce^{-\lambda(x_n)} \le x_{n+k_+(c)} + c\max\big\{e^{-\lambda(t)} : t \in [x_{n-k_-(c)}, x_{n+k_+(c)}]\big\} \\ &\le x_{n-k_-(c)} + \Big[(k_+(c)+k_-(c))\gamma(\lambda,c) + c\gamma(\lambda,c)\Big]\cdot \\ &\qquad\qquad \cdot \min\big\{e^{-\lambda(t)} : t \in [x_{n-k_-(c)}, x_{n+k_+(c)}]\big\} \\ &\le x + c'e^{-\lambda(x)}.\end{aligned}$$

On the other hand

$$\begin{aligned}\inf I_{n,c} &= x_n - ce^{-\lambda(x_n)} \ge x_{n-k_-(c)} - c\max\big\{e^{-\lambda(t)} : t \in [x_{n-k_-(c)}, x_{n+k_+(c)}]\big\} \\ &\ge x_{n+k_+(c)} - \Big[(k_+(c)+k_-(c))\gamma(\lambda,c) + c\gamma(\lambda,c)\Big]\cdot \\ &\qquad\qquad \cdot \min\big\{e^{-\lambda(t)} : t \in [x_{n-k_-(c)}, x_{n+k_+(c)}]\big\} \\ &\ge x - c'e^{-\lambda(x)}.\end{aligned}$$

❑

For later reference, let us explicitly state some consequences of the properties listed in Lemma 6.4.

6.5 Corollary. *Let $\lambda : [0,\infty) \to \mathbb{R}$ be subject to* (P1) *and* (P2)*.*

(*i*) $\displaystyle\sum_{n=0}^{\infty} \mathbb{1}_{I_{n,c}} \le k_+(c) + k_-(c) =: d(\lambda, c)$.

(*ii*) *For every growth function σ we have* $\lim_{n\to\infty} \frac{\sigma(x_{n+k_+(c)})}{\sigma(x_{n-k_-(c)})} = 1$.

(*iii*) $\displaystyle\int_{x_n}^{x_n+e^{-\lambda(x_n)}} e^{\lambda(t)}\,dt \ge \min\{e^{\lambda(t)} : t \in [x_n, x_n + e^{-\lambda(x_n)}]\}e^{-\lambda(x_n)} \ge \frac{1}{\gamma(\lambda,1)}$.

Proof. Item (*i*) follows from Lemma 6.4, (*i*), since any given point x can be contained in at most $k_+(c) + k_-(c)$ intervals of the form $(x_{n-k_-(c)}, x_{n+k_+(c)})$. For (*ii*) it is enough to remember (6.1) and (6.3). Item (*iii*) is immediate from Lemma 6.4, (*ii*). ❑

6.3 Stable density vs. infinite index of determinacy

In this subsection we discuss stable density in L^2-spaces. This is done by reducing to the $C_0(W)$-situation with help of our Bakan-type theorem.

First, the analogue of Corollary 5.7.

6.6 Lemma. *Let $\mathcal{L}$ be an algebraic de Branges space, let $\mu \in \mathbb{M}_+(\mathbb{R})$, $\mu \ne 0$, and assume that $\mathcal{L} \subseteq L^2(\mu)$.*

(*i*) *If $\dim \mathcal{L} < \infty$, then $\mathcal{L}$ is not stably dense in $L^2(\mu)$.*

(*ii*) *If μ is not discrete and $\mathcal{L}$ is dense in $L^2(\mu)$, then $\mathcal{L}$ is stably dense in $L^2(\mu)$.*

Proof. Item (i) is again clear by linear independence of $\{F(z),\dots,z^nF(z)\}$ for F not identically zero. To show (ii) apply Theorem 4.12 to obtain a weight W, such that $W\in L^2(\mu)$ and that $\mathcal{L}\cap C_0(W)$ is dense in $C_0(W)$. Since μ is not discrete, also Ω_W cannot be discrete. By Corollary 5.7, (ii), the space $\mathcal{L}\cap C_0(W)$ is stably dense in $C_0(W)$. Now Corollary 4.11 applies and yields that even $\mathcal{L}\cap C_0(W)$ is stably dense in $L^2(\mu)$. ❑

The following connection is known for the space of polynomials from [BD95] and for the space $\mathcal{E}(a)$ of Fourier transforms of fast decaying functions from [BS11a, Appendix B].

6.7 Proposition. *Let $\mathcal{L}$ be an algebraic de Branges space, let $\mu\in\mathbb{M}_+(\mathbb{R})$, $\mu\neq 0$, and assume that $\mathcal{L}\subseteq L^2(\mu)$. Then the following are equivalent.*

- (i) *$\mathcal{L}$ is stably dense in $L^2(\mu)$.*
- (ii) *For every compactly supported measure $\nu\in\mathbb{M}_+(\mathbb{R})$, the space $\mathcal{L}$ is stably dense in $L^2(\mu+\nu)$.*
- (iii) *For every finitely supported measure $\nu\in\mathbb{M}_+(\mathbb{R})$, the space $\mathcal{L}$ is dense in $L^2(\mu+\nu)$.*

Proof. Obviously (ii) implies the other two properties. We are going to show "$(i)\Rightarrow(ii)$" and "$(iii)\Rightarrow(i)$".

Assume $\mathcal{L}$ is stably dense in $L^2(\mu)$ and let a compactly supported measure ν be given. By Theorem 4.12 we can choose a weight $W\in L^2(\mu)$, such that $\mathcal{L}$ is stably dense in $C_0(W)$. Choose $T>0$ such that $\operatorname{supp}\nu\subseteq(-T,T)$, and choose a weight function $\tilde W$ which is finite and continous in $[-T,T]$ and coincides with W outside of this interval. By Lemma 5.8, $\mathcal{L}$ is stably dense in $C_0(\tilde W)$. However, $\tilde W\in L^2(\mu+\nu)$ and Corollary 4.11 yields that $\mathcal{L}$ is stably dense in $L^2(\mu+\nu)$.

Now assume that (iii) holds. Then, in particular, $\mathcal{L}$ is dense in $L^2(\mu)$. If μ is not discrete, $\mathcal{L}$ is stably dense by Lemma 6.6, (ii). Hence, assume that μ is discrete. Let $N\in\mathbb{N}$, choose N different points $x_1,\dots,x_N\in\mathbb{R}\setminus\operatorname{supp}\mu$, and let ν be the finitely supported measure having unit point mass at each of the points x_i, $i=1,\dots,N$. For $f\in L^2((1+|x|)^{2N}d\mu)$, the function $g(x):=f(x)\prod_{i=1}^N(x-x_i)$ belongs to $L^2(\mu+\nu)$. Hence, we find functions $G_n\in\mathcal{L}$, $n\in\mathbb{N}$, with $\lim_{n\to\infty}G_n=g$ in $L^2(\mu+\nu)$. In particular, it holds that $\lim_{n\to\infty}G_n(x_i)=0$, $i=1,\dots,N$.

We have $\dim L^2(\mu+\nu)\geq N+1$, and hence $\mathcal{L}_{N-1}\neq\{0\}$. Thus we find functions $H_i\in\mathcal{L}$, $i=1,\dots,N$, with $H_i(x_j)=\delta_{ij}$, $i,j=1,\dots,N$. The functions

$$\tilde G_n(z):=G_n(z)-\sum_{i=1}^N G_n(x_i)H_i(z)$$

belong to $\mathcal{L}$, vanish at the points $x_1,\dots,x_N$, and converge to g in $L^2(\mu)$. The functions

$$F_n(z):=\Big[\prod_{i=1}^N(z-x_i)\Big]^{-1}\tilde G_n(z)$$

belong to $\mathcal{L}_N[L^2(\mu)]$, and

$$\lim_{n\to\infty}\int_{\mathbb{R}}\big|F_n(x)-f(x)\big|\cdot\Big|\prod_{i=1}^N(x-x_i)\Big|\,d\mu=\lim_{n\to\infty}\int_{\mathbb{R}}\big|\tilde G_n(x)-g(x)\big|\,d\mu=0.$$

Since $x_i \notin \operatorname{supp}\mu$, this implies that $\lim_{n\to\infty} F_n = f$ in $L^2((1+|x|)^{2N}d\mu)$. ❑

Proposition 6.7 immediately gives the following

6.8 Corollary. *Let $\mu \in \mathbb{M}$. Then μ has infinite index of determinacy, if and only if $\bigcup_{\mathcal{H}\in\mathcal{C}[\mu]} \mathcal{H}$ is stably dense in $L^2(\mu)$.*

Next, we show a result which is needed in the later proofs but is also of independent interest. It deals with density of the domain of the multiplication operator.

6.9 Definition. Let $\mathcal{L}$ be an algebraic de Branges space. Then we denote

$$\mathbb{D}(\mathcal{L}) := \big\{F \in \mathcal{L} : zF(z) \in \mathcal{L}\big\}.$$

◊

Note that $\mathbb{D}(\mathcal{L})$ is again an algebraic de Branges space provided that $\dim \mathcal{L} > 1$ (to ensure $\mathbb{D}(\mathcal{L}) \neq \{0\}$).

6.10 Proposition. *Let $\mu \in \mathbb{M}$ and assume that $\mathcal{C}[\mu]$ has no maximal element. Let $\mathcal{L}$ be an algebraic de Branges space with $\mathcal{L} \subseteq L^2(\mu)$ and such that there exists a function in $\mathcal{L}\setminus\{0\}$ which is of bounded type in $\mathbb{C}^+$ and $\mathbb{C}^-$. Then $\mathcal{L}$ is dense in $L^2(\mu)$ if and only if $\mathbb{D}(\mathcal{L})$ is dense in $L^2(\mu)$.*

Proof. Starting from μ we define two measures μ', μ''.

— μ not discrete: Choose $a<b$ such that $\mathbb{1}_{\mathbb{R}\setminus[a,b]}\mu$ is still not discrete, and set

$$\mu' := \mathbb{1}_{\mathbb{R}\setminus[a,b]}\mu, \quad \mu'' := \mu + \delta,$$

where δ denotes the unit point mass at $\frac{1}{2}(a+b)$.

— μ discrete: Choose $a<b$ with $\mu([a,b]) = 0$, and set

$$\mu' := \mu, \quad \mu'' := \mu + \delta.$$

Now we consider a space $\mathcal{L}$ as in the statement of the proposition and show that

$$\mathcal{L} \text{ dense in } L^2(\mu) \ \Leftrightarrow\ \mathcal{L} \text{ dense in } L^2(\mu') \ \Leftrightarrow\ \mathcal{L} \text{ dense in } L^2(\mu'') \tag{6.5}$$

Since $\mu \geq \mu'$ and $\mu'' \geq \mu'$ the implications "⇒" on the left and "⇐" on the right are clear.

Assume first that μ is not discrete. Then also μ' and μ'' are not discrete and hence all three measures have infinite index of determinacy. By [Wor, Theorem 6.7] we have $\mathcal{C}[\mu] = \mathcal{C}[\mu'] = \mathcal{C}[\mu'']$. If $\mathcal{L}$ is not dense in $L^2(\mu)$, then $\operatorname{Clos}_{L^2(\mu)} \mathcal{L} \in \mathcal{C}[\mu] = \mathcal{C}[\mu']$. By assumption $\mathcal{C}[\mu]$ has no maximal element. Thus $\mathcal{L}$ is not dense in $L^2(\mu')$. If $\mathcal{L}$ is not dense in $L^2(\mu'')$ argue in the same way. Second, consider the case that μ is discrete. Again, $\mathcal{L}$ being dense in $L^2(\mu'')$ implies that $\mathcal{L}$ is dense in $L^2(\mu)$, since $\mu \leq \mu''$. Since $\mathcal{C}[\mu]$ has no maximal element, [Wor, Theorem 4.10] leaves us with two cases:

(*i*) $\mathcal{C}[\mu''] = \mathcal{C}[\mu] \dot\cup \{\mathcal{H}_0\}$ with $\bigcup_{\mathcal{H}\in\mathcal{C}[\mu]} \mathcal{H}$ dense in $\mathcal{H}_0$;

(*ii*) $\mathcal{C}[\mu''] = \mathcal{C}[\mu]$.

If $\mathcal{L}$ is not dense in $L^2(\mu'')$, we have $\mathcal{H} := \operatorname{Clos}_{L^2(\mu'')} \mathcal{L} \in \mathcal{C}[\mu'']$ and is not equal to $L^2(\mu'')$. Apparently, therefore, $\mathcal{H} \in \mathcal{C}[\mu]$. It follows that $\mathcal{L}$ is not dense in $L^2(\mu)$. This finishes the proof of (6.5).

Let $\mathcal{L}$ be a space as in the statement of the proposition, and assume that $\mathcal{L}$ is dense in $L^2(\mu)$ but $\mathbb{D}(\mathcal{L})$ is not. Denote $\mathcal{H} := \operatorname{Clos}_{L^2(\mu)} \mathbb{D}(\mathcal{L})$. Since $\mathcal{C}[\mu]$ has no maximal element, we can choose $\tilde{\mathcal{H}} \in \mathcal{C}[\mu]$ with $\dim \tilde{\mathcal{H}}/\mathcal{H} \geq 2$. Then $\mathbb{D}(\tilde{\mathcal{H}}) \not\subseteq \mathcal{H}$, since $\dim \tilde{\mathcal{H}}/\operatorname{Clos}_{\tilde{\mathcal{H}}} \mathbb{D}(\tilde{\mathcal{H}}) \leq 1$. Choose $F \in \mathbb{D}(\tilde{\mathcal{H}}) \setminus \mathcal{H}$, and set $x_0 := \frac{1}{2}(a+b)$. Then $(z-x_0)F(z) \in \tilde{\mathcal{H}} \subseteq L^2(\mu'')$, and we find a sequence $L_n \in \mathcal{L}$ with $L_n \to (z-x_0)F(z)$ in $L^2(\mu'')$. In particular, $L_n(x_0) \to 0$, whence we may assume from the start that $L_n(x_0) = 0$, $n \in \mathbb{N}$. Then

$$G_n(z) := \frac{L_n(z)}{z-x_0} \in \mathbb{D}(\mathcal{L}) \subseteq \mathcal{H}$$

and $G_n \to F$ in $L^2(\mu')$. This implies that $F \in \mathcal{H}$: if μ is not discrete, $\mathcal{H} \in \mathcal{C}[\mu] = \mathcal{C}[\mu']$, and if μ is discrete, $\mu' = \mu$. We reached a contradiction. ❑

PART II:

Proof of the Fast Growth Theorem and its consequences

The proof of Theorem 3.1 occupies the first three sections of this part.

— We show that growth restrictions imply integrability properties (this is needed to establish $\tilde{\mathcal{L}} \subseteq L^2(\mu)$).

— A smoothening operation with weight functions and some estimates for canonical products (technical but essential).

— Carrying out the proof by passing to C_0-spaces by virtue of Bakan's Theorem and appealing to de Branges' Theorem (one important point is to have stability of Krein classes when passing to smoothened weights).

The fourth and last section of this part contains the proofs of the other results stated in Section 3.

7 An inclusion result

In this section we show that square-integrability can be deduced from growth properties.

7.1 Theorem (Inclusion Theorem)**.** *Let λ be a growth function and $c \in [0, \infty)$. Let $\mu, \tilde{\mu} \in \mathbb{M}$, and assume that $\mu \preceq \tilde{\mu}$ w.r.t. (λ_1, λ_2). Assume that there exists $b > c$ such that*

$$\int\limits_{-\infty}^{\infty} e^{2b\lambda(|x|)} \cdot e^{-\lambda_1(|x|)}\, d\tilde{\mu}(x) < \infty, \qquad \int\limits_{0}^{\infty} e^{2b\lambda(r)} \cdot e^{\lambda_1(r)-\lambda_2(r)}\, dr < \infty. \tag{7.1}$$

Then $\mathcal{G}(\lambda, c) \cap L^2(\tilde{\mu}) \subseteq L^2(\mu)$.

The following statement contains the core estimate. We formulate it in a very explicit way that allows to keep track of various constants appearing in the estimate.

7.2 Lemma. *Let $\mu, \tilde{\mu} \in \mathbb{M}$, assume that $\mu \preceq \tilde{\mu}$ w.r.t. (λ_1, λ_2), and let c_0, c_1, c_2 be constants such that* (2.3) *holds. Let λ be a growth function, $\beta_1, \beta_2 \in \mathbb{R}$, and assume that*

$$C_1 := \int\limits_0^\infty e^{\beta_1\lambda(x)} e^{-\lambda_1(x)} d\tilde{\mu}(x) < \infty, \quad C_2 := \int\limits_0^\infty e^{\beta_2\lambda(r)} e^{\lambda_1(r)-\lambda_2(r)} dr < \infty. \tag{7.2}$$

Let $R \geq 0$ and $\varepsilon > 0$, and set (points x_n, intervals I_{n,c_1}, and constants $k_\pm(c_1)$, constructed from λ_1 as in Definition 6.2 *and Lemma* 6.4, (i)*)*

$$\begin{aligned} n_0 &:= \min\Big\{n \in \mathbb{N}_0 : x_{n-k_-(c_1)} \geq R \text{ and } \frac{\lambda(x_{m+k_+(c_1)})}{\lambda(x_{m-k_-(c_1)})} \leq 1 + \varepsilon, m \geq n\Big\}, \\ R_0 &:= \inf \bigcup_{n \geq n_0} I_{n,1}, \quad R_1 := \inf \bigcup_{n \geq n_0} I_{n,c_1}. \end{aligned}$$

Let $f \in C^1([0,\infty))$ with

$$h := \sup_{x \geq R} \frac{\log|f(x)|}{\lambda(x)} < \infty, \quad h' := \sup_{x \geq R} \frac{\log|f'(x)|}{\lambda(x)} < \infty, \tag{7.3}$$

and[14]

$$\beta_1 \geq h(1+\varepsilon)^{\operatorname{sgn} h} + h'(1+\varepsilon)^{\operatorname{sgn} h'}, \quad \beta_2 \geq 2h(1+\varepsilon)^{\operatorname{sgn} h}. \tag{7.4}$$

Then (numbers $\gamma(\lambda_1, c_1)$ and $d(\lambda_1, c_1)$ as in Lemma 6.4*, (ii), and Corollary* 6.5*, (i), respectively)*

$$\int_{R_0}^{\infty} |f(t)|^2 \, d\mu(t) \leq c_0 d(\lambda_1, c_1) \int_{R_1}^{\infty} |f(t)|^2 \, d\tilde{\mu}(t) + 2c_0(c_1+1)\gamma(\lambda_1, c_1) d(\lambda_1, c_1) C_1 + c_2 \gamma(\lambda_1, 1) C_2.$$

Proof.
Step 1; The basic estimate: Using the estimate from below in Remark 6.3 and the mean value theorem of integration, we get with an appropriate choice of $\eta_n \in I_{n,1}$

$$\int_{R_0}^{\infty} |f|^2 \, d\mu \leq \int_{R_0}^{\infty} \sum_{n=n_0}^{\infty} \mathbb{1}_{I_{n,1}} \cdot |f|^2 \, d\mu = \sum_{n=n_0}^{\infty} \int_{I_{n,1}} |f|^2 \, d\mu = \sum_{n=n_0}^{\infty} |f(\eta_n)|^2 \mu\big(I_{n,1}\big)$$
$$\leq c_0 \sum_{n=n_0}^{\infty} |f(\eta_n)|^2 \tilde{\mu}\big(I_{n,c_1}\big) + c_2 \sum_{n=n_0}^{\infty} |f(\eta_n)|^2 e^{-\lambda_2(x_n)}. \tag{7.5}$$

The first of these sums can be related to the $L^2(\tilde{\mu})$-norm of f. Namely, using the estimate from above in Corollary 6.5, (i), we get with an appropriate choice of $\xi_n \in I_{n,c_1}$

$$\int_{R_1}^{\infty} |f|^2 \, d\tilde{\mu} \geq \frac{1}{d(\lambda_1, c_1)} \sum_{n=n_0}^{\infty} \int_{I_{n,c_1}} |f|^2 \, d\tilde{\mu} = \frac{1}{d(\lambda_1, c_1)} \sum_{n=n_0}^{\infty} |f(\xi_n)|^2 \tilde{\mu}\big(I_{n,c_1}\big). \tag{7.6}$$

Continuity of f' provides us with $\zeta_n \in \overline{I_{n,c_1}}$ such that $\max_{t \in \overline{I_{n,c_1}}} |f'(t)| = |f'(\zeta_n)|$, and we can estimate

$$\begin{aligned} \big| |f(\eta_n)|^2 - |f(\xi_n)|^2 \big| &= \big| |f(\eta_n)| - |f(\xi_n)| \big| \cdot \big(|f(\eta_n)| + |f(\xi_n)| \big) \\ &\leq \big| f(\eta_n) - f(\xi_n) \big| \cdot \big(|f(\eta_n)| + |f(\xi_n)| \big) \\ &\leq \max_{t \in \overline{I_{n,c_1}}} |f'(t)| \cdot |\eta_n - \xi_n| \cdot \big(|f(\eta_n)| + |f(\xi_n)| \big) \\ &\leq |f'(\zeta_n)| \cdot (c_1+1) e^{-\lambda_1(x_n)} \cdot \big(|f(\eta_n)| + |f(\xi_n)| \big). \end{aligned} \tag{7.7}$$

[14] For convenience, we set $\operatorname{sgn} 0 := +1$.

Putting together (7.5), (7.6), and (7.7), we obtain

$$\int_{R_0}^{\infty} |f|^2 \, d\mu \le c_0 d(\lambda_1, c_1) \int_{R_1}^{\infty} |f|^2 \, d\tilde{\mu}$$

$$+ c_0(c_1+1) \sum_{n=n_0}^{\infty} |f'(\zeta_n)| \big(|f(\eta_n)| + |f(\xi_n)|\big) \cdot e^{-\lambda_1(x_n)} \tilde{\mu}(I_{n,c_1}) \tag{7.8}$$

$$+ c_2 \sum_{n=n_0}^{\infty} |f(\eta_n)|^2 \cdot e^{-\lambda_2(x_n)}. \tag{7.9}$$

Step 2; A bound for (7.8)*:* We consider the summands $|f'(\zeta_n)| \cdot |f(\eta_n)| \cdot e^{-\lambda_1(x_n)} \tilde{\mu}(I_{n,c_1})$ in (7.8). Note that

$$h = \sup_{x \ge R} \frac{\log^+ |f(x)|}{\lambda(x)} \text{ if } h \ge 0, \quad h' = \sup_{x \ge R} \frac{\log^+ |f'(x)|}{\lambda(x)} \text{ if } h' \ge 0.$$

We obtain

$$|f'(\zeta_n)| \begin{cases} \le \exp\Big[\underbrace{\frac{\log^+ |f'(\zeta_n)|}{\lambda(\zeta_n)}}_{0 \le \downarrow \le h'} \cdot \underbrace{\frac{\lambda(\zeta_n)}{\lambda(x_{n-k_-(c_1)})}}_{\le 1+\varepsilon} \cdot \lambda(x_{n-k_-(c_1)}) \Big], & h' \ge 0 \\ = \exp\Big[\underbrace{\frac{\log |f'(\zeta_n)|}{\lambda(\zeta_n)}}_{\le h' < 0} \cdot \underbrace{\frac{\lambda(\zeta_n)}{\lambda(x_{n+k_+(c_1)})}}_{\ge (1+\varepsilon)^{-1}} \cdot \lambda(x_{n+k_+(c_1)}) \Big], & h' < 0 \end{cases}$$

$$|f(\eta_n)| \begin{cases} \le \exp\Big[\underbrace{\frac{\log^+ |f(\eta_n)|}{\lambda(\eta_n)}}_{0 \le \downarrow \le h} \cdot \underbrace{\frac{\lambda(\eta_n)}{\lambda(x_{n-k_-(c_1)})}}_{\le 1+\varepsilon} \cdot \lambda(x_{n-k_-(c_1)}) \Big], & h \ge 0 \\ = \exp\Big[\underbrace{\frac{\log |f(\eta_n)|}{\lambda(\eta_n)}}_{\le h < 0} \cdot \underbrace{\frac{\lambda(\eta_n)}{\lambda(x_{n+k_+(c_1)})}}_{\ge (1+\varepsilon)^{-1}} \cdot \lambda(x_{n+k_+(c_1)}) \Big], & h < 0 \end{cases}$$

$$e^{-\lambda_1(x_n)} = \begin{cases} \underbrace{\frac{e^{-\lambda_1(x_n)}}{e^{-\lambda_1(x_{n+k_+(c_1)})}}}_{\le \gamma(\lambda_1, c_1)} \cdot e^{-\lambda_1(x_{n+k_+(c_1)})}, & \lambda_1 \text{ nondecreasing} \\ \underbrace{\frac{e^{-\lambda_1(x_n)}}{e^{-\lambda_1(x_{n-k_-(c_1)})}}}_{\le \gamma(\lambda_1, c_1)} \cdot e^{-\lambda_1(x_{n-k_-(c_1)})}, & \lambda_1 \text{ nonincreasing} \end{cases}$$

Together, using that λ is increasing,

$$|f'(\zeta_n)| \cdot |f(\eta_n)| \cdot e^{-\lambda_1(x_n)} \tilde{\mu}\big(I_{n,c_1}\big)$$
$$\le \gamma(\lambda_1, c_1) \int_{I_{n,c_1}} \exp\Big[\Big(h'(1+\varepsilon)^{\operatorname{sgn} h'} + h(1+\varepsilon)^{\operatorname{sgn} h}\Big)\lambda(x)\Big] \cdot e^{-\lambda_1(x)} \, d\tilde{\mu}(x).$$

The summands $|f'(\zeta_n)| \cdot |f(\xi_n)| \cdot e^{-\lambda_1(x_n)}\tilde{\mu}(I_{n,c_1})$ in (7.8) are estimated in the same way. Summing over n and using the bound Corollary 6.5, (i), yields

$$
\begin{aligned}
(7.8) \;\; &\leq 2c_0(c_1+1)\gamma(\lambda_1,c_1)d(\lambda_1,c_1) \\
&\cdot \int_{R_1}^{\infty} \exp\Big[\Big(h'(1+\varepsilon)^{\operatorname{sgn} h'} + h(1+\varepsilon)^{\operatorname{sgn} h}\Big)\lambda(x)\Big] \cdot e^{-\lambda_1(x)}\, d\tilde{\mu}(x) \\
&\leq 2c_0(c_1+1)\gamma(\lambda_1,c_1)d(\lambda_1,c_1) \cdot \int_0^{\infty} e^{\beta_1\lambda(x)} \cdot e^{-\lambda_1(x)}\, d\tilde{\mu}(x).
\end{aligned}
$$

Step 3; A bound for (7.9)*:* Since λ_2 is nondecreasing, we can estimate

$$
|f(\eta_n)| \begin{cases} \leq \exp\Big[\underbrace{\dfrac{\log^+|f(\eta_n)|}{\lambda(\eta_n)}}_{0\leq \downarrow \leq h} \cdot \underbrace{\dfrac{\lambda(\eta_n)}{\lambda(x_{n-1})}}_{\leq 1+\varepsilon} \cdot \lambda(x_{n-1})\Big], & h \geq 0 \\[2ex] = \exp\Big[\underbrace{\dfrac{\log|f(\eta_n)|}{\lambda(\eta_n)}}_{\leq h<0} \cdot \underbrace{\dfrac{\lambda(\eta_n)}{\lambda(x_n)}}_{\geq (1+\varepsilon)^{-1}} \cdot \lambda(x_n)\Big] , & h < 0 \end{cases}
$$

Using Corollary 6.5, (iii), and that λ is increasing, we obtain

$$
|f(\eta_n)|^2 e^{-\lambda_2(x_n)} \leq \gamma(\lambda_1,1)\int_{x_{n-1}}^{x_n} \exp\Big[2h(1+\varepsilon)^{\operatorname{sgn} h}\lambda(r)\Big] \cdot e^{-\lambda_2(r)+\lambda_1(r)}\, dr.
$$

Summing over n yields

$$
\begin{aligned}
(7.9) \;\; &\leq c_2\gamma(\lambda_1,1) \cdot \int_{x_{n_0-1}}^{\infty} \exp\Big[2h(1+\varepsilon)^{\operatorname{sgn} h}\lambda(r)\Big] \cdot e^{-\lambda_2(r)+\lambda_1(r)}\, dr \\
&\leq c_2\gamma(\lambda_1,1) \cdot \int_0^{\infty} e^{\beta_2\lambda(r)} \cdot e^{\lambda_1(r)-\lambda_2(r)}\, dr.
\end{aligned}
$$

❑

Using Lemma 7.2, it is not difficult to deduce Theorem 7.1.

Proof of Theorem 7.1. Let $F \in \mathcal{G}(\lambda,c)$ be given, then also $F' \in \mathcal{G}(\lambda,c)$ [15]. Set

$$
a := \max\Big\{\limsup_{x\to\infty}\frac{\log|F(x)|}{\lambda(x)}, \limsup_{x\to\infty}\frac{\log|F'(x)|}{\lambda(x)}\Big\},
$$

then $a \leq c$. By the assumption (7.1), the integrals (7.2) are finite for $\beta_1 = \beta_2 := 2b$. Since $b > a$, we can choose $R \geq 0$ sufficiently large and $\varepsilon > 0$ sufficiently small, such that the relations (7.4) hold.

[15] This is a classical fact. For growth functions λ with $\rho_\lambda > 0$, it is well-known and can be deduced from results in standard textbooks, e.g., from [Lev80, Ch.1,Theorems 27,28]. The case that $\rho_\lambda = 0$ is probably less widely known. It follows using that the indicator function w.r.t. λ actually is constant. This result goes back to [Gol62]. A more recent reference, which contains a nice proof due to W.Hayman, is [BP07, Appendix].

Now assume that $F|_{\mathbb{R}} \in L^2(\tilde{\mu})$. Then Lemma 7.2 implies that for some $R_0 > 0$ the integral $\int_{R_0}^{\infty} |F(t)|^2 \, d\mu(t)$ is finite. Using the same argument with the function $F(-z)$, we obtain that, for some $R_0' > 0$, $\int_{R_0'}^{\infty} |F(t)|^2 \, d\mu(t) < \infty$. Since F is continuous, the integral over the interval $[-R_0', R_0]$ is certainly finite, and together thus $F \in L^2(\mu)$. ❑

Combining the Inclusion Theorem with [Wor, Theorem 3.5], we obtain an ordering-type theorem for certain growth defined beginning sections of chains. This result also shows what might happen when we drop the condition `[2Chain]` in the Coincidence Theorem.

7.3 Proposition (Section Ordering)**.** *Let $\mu, \tilde{\mu} \in \mathbb{M}$, let λ be a growth function, and let $c \in [0, \infty)$. Assume that $\mu \preceq \tilde{\mu}$ w.r.t. (λ_1, λ_2) and $\tilde{\mu} \preceq \mu$ w.r.t. $(\tilde{\lambda}_1, \tilde{\lambda}_2)$, where* `[2A]` *holds. Then*

$$\big\{\mathcal{H} \in \mathcal{C}[\mu] : \mathcal{H} \subseteq \mathcal{G}(\lambda, c)\big\} \left\langle \begin{matrix} \subseteq \\ \textit{or} \\ \supseteq \end{matrix} \right\rangle \big\{\mathcal{H} \in \mathcal{C}[\tilde{\mu}] : \mathcal{H} \subseteq \mathcal{G}(\lambda, c)\big\} \tag{7.10}$$

Proof. Since $\tilde{\mu} \preceq \mu$ and $\mu \preceq \tilde{\mu}$, we get from Theorem 7.1 that

$$\mathcal{H} \in \mathcal{C}[\mu], \mathcal{H} \subseteq \mathcal{G}(\lambda, c) \Rightarrow \mathcal{H} \subseteq L^2(\tilde{\mu}), \quad \mathcal{H} \in \mathcal{C}[\tilde{\mu}], \mathcal{H} \subseteq \mathcal{G}(\lambda, c) \Rightarrow \mathcal{H} \subseteq L^2(\mu)$$

However, we know from [Wor, Theorem 3.5(1)] that

$$\big\{\mathcal{H} \in \mathcal{C}[\mu] : \mathcal{H} \subseteq L^2(\tilde{\mu})\big\} \left\langle \begin{matrix} \subseteq \\ \texttt{or} \\ \supseteq \end{matrix} \right\rangle \big\{\mathcal{H} \in \mathcal{C}[\tilde{\mu}] : \mathcal{H} \subseteq L^2(\mu)\big\}$$

Depending on which of "$\subseteq$" and "$\supseteq$" holds in this relation, in (7.10) the same inclusion takes place. ❑

Observe that, in the situation of the Coincidence Theorem, the conclusion of Proposition 7.3 is much weaker than the conclusion of Theorem 3.3.

8 Preparation

8.1 A construction of weight functions

We study the following construction carried out with weight functions.

8.1 Definition. Let $\sigma_1, \sigma_2 : [0, \infty) \to \mathbb{R}$ be continuous functions and let $\gamma > 0$. Then we define, for each weight function W on $\mathbb{R}$,

$$\mathcal{S}[W](x) := \min\Big\{ \inf\big\{W(t) : |t - x| \le \gamma e^{-\sigma_1(|x|)}\big\}, e^{\sigma_2(|x|)}\Big\}, \quad x \in \mathbb{R}. \tag{8.1}$$

◊

We suppress explicit notation of the parameters $\mathcal{S}$ on $\sigma_1, \sigma_2, \gamma$.

8.2 Lemma. *The function $\mathcal{S}[W]$ defined by* (8.1) *is an everywhere finite weight function.*

Proof. The infimum in the first argument of the minimum (8.1) is attained since W is lower semicontinuous. In particular, it is for all $x \in \mathbb{R}$ positive. The function W being lower semicontinuous, the infimum in the first argument of (8.1) being taken over a closed interval, and the function σ_1 being continuous, implies that this first argument is a lower semicontinuous function of x. The term in the second argument is continuous, positive and finite. ❑

8.3 Remark. For later reference, let us state the following obvious properties.

(*i*) We have $\mathcal{S}[W](x) \leq W(x)$, $x \in \mathbb{R}$. In particular, thus

$$C_0(\mathcal{S}[W]) \subseteq C_0(W), \qquad \|f\|_{\mathcal{S}[W]} \geq \|f\|_W, \ f \in C_0(\mathcal{S}[W]).$$

(*ii*) Assume that $\sigma_1 \leq \sigma_1'$, $\sigma_2 \leq \sigma_2'$, $\gamma \geq \gamma'$, and let $\mathcal{S}$ and $\mathcal{S}'$ be the corresponding operators defined as in (8.1). Then, for every weight W, we have $\mathcal{S}[W](x) \leq \mathcal{S}'[W](x)$, $x \in \mathbb{R}$. In particular, thus

$$C_0(\mathcal{S}[W]) \subseteq C_0(\mathcal{S}'[W]), \qquad \|f\|_{\mathcal{S}[W]} \geq \|f\|_{\mathcal{S}'[W]}, \ f \in C_0(\mathcal{S}[W]).$$

♢

In the next two results, we exhibit a dichotomic situation. When σ_1 and σ_2 grow sufficiently fast, passing from W to $\mathcal{S}[W]$ is compatible with growth classes (Lemma 8.4), and when σ_1 and σ_2 grow sufficiently slow, passing from W to $\mathcal{S}[W]$ is compatible with an integrability property w.r.t. pairs of majorised measures (Lemma 8.5).

8.4 Lemma. *Let $\sigma_1, \sigma_2 : [0,\infty) \to \mathbb{R}$ be continuous functions and let $\gamma > 0$. Let λ be a growth function and let $c \in [0,\infty)$. Let W be a weight function on $\mathbb{R}$, and $\omega : \mathbb{R} \to (0,\infty)$ be a function with $\omega(x) \leq W(x)$, $x \in \mathbb{R}$. Assume that $\sigma_1, \sigma_2, \omega, \gamma$ satisfy*[16]

$$\exists\, r_1 > 0 : \ \sigma_1(|x|) + \log\omega(x) > 0, \sigma_2(|x|) > 0 \textit{ for } |x| \geq r_1, \tag{8.2}$$

$$\lim_{r\to\infty} \frac{1}{r} e^{-\sigma_1(r)} = 0, \tag{8.3}$$

$$\frac{\omega(t)}{\omega(x)} \asymp 1, \ x,t \in \mathbb{R}, |t-x| \leq \gamma e^{-\sigma_1(|x|)}, \tag{8.4}$$

$$\limsup_{r\to\infty} \frac{\lambda(r)}{\sigma_2(r)} < \frac{1}{c}, \tag{8.5}$$

$$\limsup_{|x|\to\infty} \frac{\lambda(|x|)}{\sigma_1(|x|) + \log\omega(x)} < \frac{1}{c}. \tag{8.6}$$

Then

$$C_0\big(\mathcal{S}[W]\big) \cap \mathcal{G}(\lambda, c) = C_0(W) \cap \mathcal{G}(\lambda, c).$$

Proof. By Remark 8.3, (*i*), it is enough to show the inclusion $C_0(W) \cap \mathcal{G}(\lambda,c) \subseteq C_0(\mathcal{S}[W])$. Let $f \in C_0(W) \cap \mathcal{G}(\lambda,c)$ be given. We need to estimate the quotients of f by each of the two arguments in the minimum (8.1). In the following denote

$$h_0 := \limsup_{x\to\infty} \frac{\log|f(x)|}{\lambda(x)}, \quad h_0' := \limsup_{x\to\infty} \frac{\log|f'(x)|}{\lambda(x)}.$$

[16] In (8.5) and (8.6) we understand $\frac{1}{0} := \infty$.

The quotient of f by the second argument is easy to treat. For each $\epsilon > 0$ we have for all sufficiently large points $x > 0$ that

$$\Big|\frac{f(x)}{e^{\sigma_2(x)}}\Big| \leq \exp\Big[\big(h_0+\epsilon\big)\lambda(x) - \sigma_2(x)\Big] = \exp\Big[\sigma_2(x)\Big((h_0+\epsilon)\frac{\lambda(x)}{\sigma_2(x)} - 1\Big)\Big].$$

Our assumption (8.5) ensures that for sufficiently small values of $\epsilon > 0$

$$\lim_{r\to\infty} \sigma_2(r) = +\infty, \quad \limsup_{r\to\infty} \Big((h_0+\epsilon)\frac{\lambda(r)}{\sigma_2(r)} - 1\Big) < 0.$$

Thus $\lim_{x\to+\infty} \big|\frac{f(x)}{e^{\sigma_2(x)}}\big| = 0$. The limit along the negative real axis is evaluated in the same way.

We turn to the quotient of f by the first argument in (8.1). For each $x > 0$ choose a point $\xi_x \in [x - \gamma e^{-\sigma_1(x)}, x + \gamma e^{-\sigma_1(x)}]$ such that

$$W(\xi_x) = \inf\big\{W(t) : |t - x| \leq \gamma e^{-\sigma_1(x)}\big\}.$$

For an appropriate point η_x on the line segment $\operatorname{co}\{x, \xi_x\}$ connecting x with ξ_x we have

$$\Big|\frac{f(x)}{W(\xi_x)}\Big| \leq \Big|\frac{f(\xi_x)}{W(\xi_x)}\Big| + \frac{|\xi_x - x|\cdot|f'(\eta_x)|}{W(\xi_x)}. \tag{8.7}$$

If x tends to $+\infty$, also ξ_x and η_x tend to $+\infty$ by (8.3). Since $f \in C_0(W)$, this yields that

$$\lim_{x\to\infty} \Big|\frac{f(\xi_x)}{W(\xi_x)}\Big| = 0.$$

For each $\epsilon > 0$ we have for all sufficiently large points $x > 0$ that

$$\begin{aligned}\frac{|\xi_x - x|\cdot|f'(\eta_x)|}{W(\xi_x)} &\leq \gamma e^{-\sigma_1(x)} \cdot e^{(h_0'+\epsilon)\lambda(\eta_x)} \cdot \frac{1}{\omega(\xi_x)} \\ &\overset{(6.1),(8.3),(8.4)}{\lesssim} \exp\Big[(h_0'+\epsilon)(1+\epsilon)\lambda(x) - \sigma_1(x) - \log\omega(x)\Big].\end{aligned}$$

Our assumption (8.6) ensures that for sufficiently small values of $\epsilon > 0$

$$\begin{aligned}&\lim_{|x|\to\infty} \big[\sigma_1(|x|) + \log\omega(|x|)\big] = +\infty, \\ &\limsup_{|x|\to\infty} \Big((h_0'+\epsilon)(1+\epsilon)\frac{\lambda(|x|)}{\sigma_1(|x|) + \log\omega(|x|)} - 1\Big) < 0.\end{aligned}$$

Hence also the second summand on the right side of (8.7) tends to 0 when x approaches $+\infty$. The limit along the negative real axis is evaluated in the same way. ❑

8.5 Lemma. *Let $\mu, \tilde\mu \in \mathbb{M}$, assume that $\mu \preceq \tilde\mu$ w.r.t. (λ_1, λ_2), and let c_0, c_1, c_2 be constants such that* (2.3) *holds. Let $\sigma_1, \sigma_2 : [0,\infty) \to \mathbb{R}$ be continuous functions and let $\gamma > 0$. Assume that σ_2 positive and nondecreasing, and that $\sigma_1, \sigma_2, \gamma$ satisfy (constants $\gamma(\lambda_1, c_1)$ and $k_\pm(c_1)$ as in Lemma* 6.4*)*

$$\sigma_1 \leq \lambda_1, \quad \gamma \geq \gamma(\lambda_1, c_1)(c_1 + k_+(c_1) + k_-(c_1)), \tag{8.8}$$

$$\lim_{\frac{t}{s}\to 1,\, t,s>0} \frac{\sigma_2(t)}{\sigma_2(s)} = 1, \tag{8.9}$$

$$\int_0^\infty e^{\beta\sigma_2(r)} \cdot e^{\lambda_1(r)-\lambda_2(r)}\, dr < \infty \text{ for some } \beta > 2. \tag{8.10}$$

Then, for each weight function W,

$$W \in L^2(\tilde{\mu}) \;\Rightarrow\; \mathcal{S}[W] \in L^2(\mu).$$

Proof. Let points x_n and intervals $I_{n,c}$ be defined as in Definition 6.2 using the function λ_1. Then we have the estimate

$$\int\limits_0^\infty \mathcal{S}[W](x)^2 d\mu(x) = \sum_{n=0}^\infty \int\limits_{x_n}^{x_{n+1}} \mathcal{S}[W](x)^2 d\mu(x) \le \sum_{n=0}^\infty \Big[\sup_{x\in[x_n,x_{n+1})} \mathcal{S}[W](x)\Big]^2 \mu(I_{n,1})$$

$$\le c_0 \sum_{n=0}^\infty \Big[\sup_{x\in[x_n,x_{n+1})} \Big(\inf\{W(t) : |t-x| \le \gamma e^{-\sigma_1(x)}\}\Big)\Big]^2 \tilde{\mu}(I_{n,c_1}) \tag{8.11}$$

$$+ c_2 \sum_{n=1}^\infty \Big[\sup_{x\in[x_n,x_{n+1})} e^{\sigma_2(x)}\Big]^2 e^{-\lambda_2(x_n)}. \tag{8.12}$$

Our aim is to show that each of the sums in (8.11) and (8.12) is finite.

Using Lemma 6.4, (iii), and our present assumption (8.8), we obtain that

$$I_{n,c_1} \subseteq [x - \gamma e^{-\sigma_1(x)}, x + \gamma e^{-\sigma_1(x)}], \quad x \in [x_n, x_{n+1}),$$

and hence

$$\sup_{x\in[x_n,x_{n+1})} \Big(\inf\{W(t) : |t-x| \le \gamma e^{-\sigma_1(x)}\}\Big)$$
$$\le \sup_{x\in[x_n,x_{n-1})} \Big(\inf\{W(t) : t \in I_{n,c_1}\}\Big) = \inf\{W(t) : t \in I_{n,c_1}\}.$$

In turn, it follows that (with the constant $d(\lambda_1, c_1)$ as in Corollary 6.5, (i))

$$(8.11) \;\le c_0 \sum_{n=0}^\infty \Big[\inf_{t\in I_{n,c_1}} W(t)\Big]^2 \tilde{\mu}(I_{n,c_1}) \le c_0 \sum_{n=0}^\infty \int_{I_{n,c_1}} W(x)^2 \, d\tilde{\mu}(x)$$
$$\le c_0 \int_{\mathbb{R}} \sum_{n=0}^\infty \mathbb{1}_{I_{n,c_1}}(x) W(x)^2 \, d\tilde{\mu}(x) \le c_0 d(\lambda_1, c_1) \int_{\mathbb{R}} W(x)^2 \, d\tilde{\mu}(x) < \infty.$$

Since $\lim_{n\to\infty} \frac{x_{n+1}}{x_{n-1}} = 1$, our assumption (8.9) implies $\lim_{n\to\infty} \frac{\sigma_2(x_{n+1})}{\sigma_2(x_{n-1})} = 1$. Using that σ_2 is nondecreasing, we find for each $\varepsilon > 0$ a number $n_0 \in \mathbb{N}$ such that

$$\sup_{x\in[x_{n-1},x_{n+1})} \sigma_2(x) \le (1+\varepsilon) \cdot \inf_{x\in[x_{n-1},x_{n+1})} \sigma_2(x), \quad n \ge n_0.$$

Using Corollary 6.5, (iii), we see that for each $n \ge n_0$

$$\Big[\sup_{x\in[x_{n-1},x_{n+1})} e^{\sigma_2(x)}\Big]^2 \cdot e^{-\lambda_2(x_n)}$$
$$\le \Big[\sup_{x\in[x_{n-1},x_{n+1})} e^{\sigma_2(x)}\Big]^2 \cdot \gamma(\lambda_1, 1) \int\limits_{x_{n-1}}^{x_n} e^{\lambda_1(t)} dt \cdot e^{-\lambda_2(x_n)}$$
$$\le \gamma(\lambda_1, 1) \int\limits_{x_{n-1}}^{x_n} e^{2(1+\varepsilon)\sigma_2(t)} \cdot e^{\lambda_1(t)} \cdot e^{-\lambda_2(t)} dt$$

Making a sufficiently small choice for $\varepsilon > 0$ and using (8.10), it follows that

$$(8.12) \ \leq c_2 \gamma(\lambda_1, 1) \int_0^\infty e^{2(1+\varepsilon)\sigma_2(t)} \cdot e^{\lambda_1(t)-\lambda_2(t)}\, dt < \infty.$$

Finiteness of the integral $\int_{-\infty}^0 W(x)^2\, d\mu(x)$ is shown in the same way. ❑

8.2 Shifting of zeroes in canonical products

Let $\vec{\alpha} := (\alpha_n)_{n\in\mathbb{N}}$ be a sequence of real numbers with the following properties:

(S1) The sequence $\vec{\alpha}$ consists of pairwise distinct non-zero points and has no finite accumulation point.

(S2) Denote by $\vec{\alpha}^+$ and $\vec{\alpha}^-$ the subsequences of $\vec{\alpha}$ consisting of all positive or negative, respectively, elements of ξ arranged according to increasing modulus [17]. Then the following limits exist in $[0, \infty)$ and are equal [18]

$$\beta := \lim_{n\to\infty} \frac{n}{\vec{\alpha}_n^+} = \lim_{n\to\infty} \frac{n}{|\vec{\alpha}_n^-|}.$$

(S3) The limit

$$\lim_{r\to\infty} \sum_{|\alpha_n|\leq r} \frac{1}{\alpha_n}$$

exists in $\mathbb{R}$.

Then we denote

$$P_{\vec{\alpha}}(z) := \lim_{r\to\infty} \prod_{|\alpha_n|\leq r} \Big(1 - \frac{z}{\alpha_n}\Big), \quad z \in \mathbb{C}.$$

The function $P_{\vec{\alpha}}$ is an entire function of finite exponential type with

$$\lim_{r\to\infty} \frac{\log |P_{\vec{\alpha}}(re^{i\vartheta})|}{r} = \pi\beta |\sin\vartheta|, \quad \vartheta \in (0, \pi) \cup (\pi, 2\pi).$$

The fact that $P_{\vec{\alpha}}(z)$ is of exponential type is a consequence of Lindelöf's Theorem (see, e.g., [Boa54, §2.10.3]), and [Boa54, §8.3.1] yields the above formula for the exponential indicator of $P_{\vec{\alpha}}$.

In the following pair of lemmata we provide some simple facts about what happens off the real axis when perturbing the zeroes α_n.

8.6 Lemma. *Let $\vec{\alpha} := (\alpha_n)_{n\in\mathbb{N}}$ and $\vec{\beta} := (\beta_n)_{n\in\mathbb{N}}$ be two sequences subject to* (S1). *Assume that*

$$\sum_{n=1}^\infty \frac{|\beta_n - \alpha_n|}{|\alpha_n|} < \infty. \tag{8.13}$$

[17] Both sequences are supposed to have an empty or finite or infinite index set of the form $\{n \in \mathbb{N} : n < N\}$ for some $N \in \mathbb{N}_0 \cup \{\infty\}$.

[18] Here we tacitly understand the limit of a finite or empty sequence as being equal to 0.

Then, for each $\vartheta \in (0, \frac{\pi}{2}]$*, the product*

$$Q(z) := \prod_{n=1}^{\infty} \frac{1 - \frac{z}{\beta_n}}{1 - \frac{z}{\alpha_n}}$$

converges uniformly in the Stolz angle $\Gamma_\vartheta := \{z \in \mathbb{C} : \vartheta \leq \arg z \leq \pi - \theta\}$*, and*

$$Q(z) \asymp 1, \quad z \in \Gamma_\vartheta.$$

Proof. Choose a sequence τ_n of positive numbers such that

$$\lim_{n\to\infty} \tau_n = 0, \quad \sum_{n=1}^{\infty} \frac{|\beta_n - \alpha_n|}{\tau_n|\alpha_n|} < \infty.$$

Let U_n be the open disk centred at α_n with radius $\tau_n|\alpha_n|$. For each point z on the boundary of U_n it holds that

$$\begin{aligned}\left|\frac{1 - \frac{z}{\beta_n}}{1 - \frac{z}{\alpha_n}} - 1\right| &= \left|\frac{z}{\beta_n}\right| \cdot \frac{|\beta_n - \alpha_n|}{|z - \alpha_n|} \leq \frac{1 + \frac{|z-\alpha_n|}{|\alpha_n|}}{1 - \frac{|\beta_n-\alpha_n|}{|\alpha_n|}} \cdot \frac{|\beta_n - \alpha_n|}{|z - \alpha_n|} \\ &= \underbrace{\frac{1 + \tau_n}{1 - \frac{|\beta_n-\alpha_n|}{|\alpha_n|}}}_{\to 1} \cdot \frac{|\beta_n - \alpha_n|}{\tau_n|\alpha_n|} \lesssim \frac{|\beta_n - \alpha_n|}{\tau_n|\alpha_n|}, \ n \in \mathbb{N}.\end{aligned} \tag{8.14}$$

The function $\frac{1-\frac{z}{\beta_n}}{1-\frac{z}{\alpha_n}} - 1$ is analytic on a region containing $\mathbb{C} \setminus U_n$ and has a removable singularity at ∞. By the maximum principle the bound (8.14) prevails throughout $\mathbb{C} \setminus U_n$. It follows that the product

$$Q(z) := \prod_{n=1}^{\infty} \frac{1 - \frac{z}{\alpha_n}}{1 - \frac{z}{\beta_n}}$$

converges uniformly in $\mathbb{C} \setminus \bigcup_{n=1}^{\infty} U_n$ and

$$|Q(z)| \asymp 1, \ z \in \mathbb{C} \setminus \bigcup_{n=1}^{\infty} U_n.$$

Since the ratio between radius and absolute value of the centre of the disk U_n tends to zero, each Stolz angle Γ_ϑ intersects at most finitely many of the disks U_n. This yields the required assertion. ❑

8.7 Lemma. *Let* $\vec{\alpha} := (\alpha_n)_{n\in\mathbb{N}}$ *and* $\vec{\beta} := (\beta_n)_{n\in\mathbb{N}}$ *be two sequences subject to* (S1) *for which* (8.13) *holds. Assume moreover that* $\vec{\alpha}$ *satisfies* (S2) *and* (S3)*, and that* $\alpha_n > 0$ *if and only if* $\beta_n > 0$*. Then also* β_n *satisfies* (S2) *and* (S3)*, and the following statements hold.*

(*i*) *For each* $\vartheta \in (0, \frac{\pi}{2}]$ *we have*

$$|P_{\vec{\beta}}(z)| \asymp |P_{\vec{\alpha}}(z)|, \quad z \in \Gamma_\vartheta.$$

(*ii*) *Let λ be a growth function. Then $P_{\vec\alpha}$ is of finite λ-type if and only if $P_{\vec\beta}$ has this property. If $P_{\vec\alpha}$ is of finite λ-type, then*

$$\limsup_{r\to\infty}\frac{\log|P_{\vec\alpha}(re^{i\vartheta})|}{\lambda(r)}=\limsup_{r\to\infty}\frac{\log|P_{\vec\beta}(re^{i\vartheta})|}{\lambda(r)},\quad \vartheta\in[0,2\pi).$$

Proof. The summability (8.13) gives in particular that $\lim_{n\to\infty}\frac{\beta_n}{\alpha_n}=1$. Writing $\vec\alpha^+=(\alpha_{n(l)})$ and $\vec\beta^+=(\beta_{m(l)})$, it holds that $n(l)=m(l)$ due to our assumption on signs. Hence,

$$\lim\frac{\beta_n^+}{\alpha_n^+}=1.$$

The same applies to $\vec\alpha^-$ and $\vec\beta^-$, and we conclude that $\vec\beta$ satisfies (S2). Writing

$$\Big|\frac{1}{\beta_n}-\frac{1}{\alpha_n}\Big|=\Big|\frac{\beta_n}{\alpha_n}-1\Big|\frac{|\alpha_n|}{|\beta_n|}\frac{1}{|\alpha_n|}\lesssim\Big|\frac{\beta_n}{\alpha_n}-1\Big|,$$

shows that the series $\sum_{n=1}^\infty\big|\frac{1}{\beta_n}-\frac{1}{\alpha_n}\big|$ converges. Hence, $\vec\beta$ satisfies (S3).

Clearly, $P_{\vec\beta}=Q\cdot P_{\vec\alpha}$. Thus, Lemma 8.6 immediately implies (*i*). For (*ii*) note that the functions $P_{\vec\alpha}$ and $P_{\vec\beta}$ are of order at most 1. Hence, the Phragmen-Lindelöf principle applies in each angle with opening less than π. Since $|Q(z)|\asymp 1$ on each ray not lying on the real axis, it follows that $P_{\vec\alpha}$ is of finite λ-type if and only if $P_{\vec\beta}$ has this property. Moreover, clearly,

$$\limsup_{r\to\infty}\frac{\log|P_{\vec\alpha}(re^{i\vartheta})|}{\lambda(r)}=\limsup_{r\to\infty}\frac{\log|P_{\vec\beta}(re^{i\vartheta})|}{\lambda(r)},\quad \vartheta\in(0,\pi)\cup(\pi,2\pi).$$

Since the indicator is a continuous function of ϑ, this equality holds for all values of ϑ. ❑

The next proposition plays a crucial role in the proof of Theorem 3.1. It compares the values of the derivatives of canonical products $P_{\vec\alpha}(z)$ and $P_{\vec\beta}(z)$ at their respective zeroes α_n and β_n. Its proof is a modification of an argument due to A. Borichev and M. Sodin [BS11b].

8.8 Proposition. *Let $\vec\alpha:=(\alpha_n)_{n\in\mathbb N}$ be a sequence subject to* (S1)–(S3). *Assume that $P_{\vec\alpha}\in\mathcal G(\lambda,c)$ and*

$$|P_{\vec\alpha}'(\alpha_n)|\ge(1+|x|)^{-\tilde N}$$

for some $\tilde N\in\mathbb N$. Furthermore, let $c_+>c$ and assume that $\vec\beta:=(\beta_n)_{n\in\mathbb N}$ is a sequence satisfying (S1) *such that*

$$|\beta_n-\alpha_n|\le\gamma e^{-c_+\lambda(|\alpha_n|)}.$$

Then the sequence $\vec\beta$ satisfies (S2) *and* (S3), *and*

$$|P_{\vec\beta}'(\beta_n)|\asymp|P_{\vec\alpha}'(\alpha_n)|,\quad n\in\mathbb N.$$

In the proof we use an estimate for the separation of a sequence $\vec\alpha=(\alpha_n)_{n\in\mathbb N}$. We denote the separation by $s_{\vec\alpha}$, i.e.,

$$s_{\vec\alpha}(n):=\min\big\{|\alpha_k-\alpha_n|:k\ne n\big\}.$$

8.9 Lemma. *Let λ be a growth function, $d \in [0,\infty)$, and let $B \in \mathcal{G}(\lambda, d)$ be a function with $B(0) = 1$ and $B(\overline{z}) = \overline{B(z)}$, $z \in \mathbb{C}$. Denote by $\vec{\alpha} := (\alpha_n)_{n=1,2,\dots}$ the (finite or infinite) sequence of real zeroes of B, and let $\omega : \mathbb{R} \to (0,\infty)$ be a function with $\omega(\alpha_n) \lesssim |B'(\alpha_n)|$, $n = 1,2,\dots$. Then, for each $d^+ > d$,*

$$s_{\vec{\alpha}}(n) \gtrsim \min\big\{|\alpha_n|, \omega(\alpha_n)e^{-d^+\lambda(|\alpha_n|)}\big\},\ n = 1,2,\dots. \tag{8.15}$$

Proof. If there are only finitely many real zeroes, this is of course trivial. Hence, assume that B has infinitely many real zeroes.

For each $n \in \mathbb{N}$ choose $m(n) \in \mathbb{N}$ with $s_{\vec{\alpha}}(n) = |\alpha_n - \alpha_{m(n)}|$. We claim that

$$|B'(\alpha_n)| + |B'(\alpha_{m(n)})| = |B'(\alpha_n) - B'(\alpha_{m(n)})|.$$

If one of α_n and $\alpha_{m(n)}$ is a multiple zero of B, this is trivial. If both are simple, the signs of $B'(\alpha_n)$ and $B'(\alpha_{m(n)})$ are different, and again the claim holds.

The function B'' is of finite λ-type and satisfies

$$\limsup_{|x|\to\infty} \frac{\log|B''(x)|}{\lambda(x)} \leq d.$$

Let $d^+ > d$ be given and choose $d^* \in (d, d^+)$. Then we can estimate

$$\begin{aligned}
\omega(\alpha_n) &\lesssim |B'(\alpha_n)| \leq |B'(\alpha_n)| + |B'(\alpha_{m(n)})| = |B'(\alpha_n) - B'(\alpha_{m(n)})| \\
&= \Big|\int_{\alpha_n}^{\alpha_{m(n)}} B''(x)\,dx\Big| \leq \Big|\int_{\alpha_n}^{\alpha_{m(n)}} |B''(x)|\,dx\Big| \\
&\lesssim \underbrace{|\alpha_n - \alpha_{m(n)}|}_{=s_{\vec{\alpha}}(n)} \cdot e^{d^*\lambda(\max\{|\alpha_n|,|\alpha_{m(n)}|\})}, \quad n \in \mathbb{N},
\end{aligned}$$

and obtain

$$s_{\vec{\alpha}}(n) \gtrsim \omega(\alpha_n)e^{-d^*\lambda(\max\{|\alpha_n|,|\alpha_{m(n)}|\})}, \quad n \in \mathbb{N}.$$

Choose $\varepsilon > 0$ such that $d^*(1+\varepsilon) \leq d^+$, and choose $\delta > 0$ and $r_0 > 0$ such that

$$\frac{\lambda(r(1+\delta))}{\lambda(r)} \leq 1 + \varepsilon, \quad r \geq r_0.$$

Consider the sets

$$\begin{aligned}
M_1 &:= \big\{n \in \mathbb{N} : s_{\vec{\alpha}}(n) \geq \delta|\alpha_n|\big\}, \\
M_2 &:= \big\{n \in \mathbb{N} : s_{\vec{\alpha}}(n) < \delta|\alpha_n|,\ |\alpha_n|, |\alpha_{m(n)}| \geq r_0\big\}.
\end{aligned}$$

The asserted inequality (8.15) trivially holds when n ranges over M_1. Let $n \in M_2$. Then

$$\begin{aligned}
d^*\lambda(\max\{|\alpha_n|,|\alpha_{m(n)}|\}) &\leq d^*\lambda(|\alpha_n| + s_{\vec{\alpha}}(n)) \leq d^*\lambda\big(|\alpha_n|(1+\delta)\big) \\
&\leq d^*(1+\varepsilon)\lambda(|\alpha_n|) \leq d^+(|\alpha_n|).
\end{aligned}$$

We see that

$$s_{\vec{\alpha}}(n) \gtrsim \omega(\alpha_n)e^{-d^+\lambda(|\alpha_n|)}, \quad n \in M_2.$$

Since $\mathbb{N} \setminus (M_1 \cup M_2)$ is finite, the assertion (8.15) follows. ❑

Proof of Proposition 8.8. Lemma 8.7 together with the fact that $\log r = o(\lambda(r))$ imply that the sequence $\vec{\beta}$ satisfies (S2) and (S3).

Step 1. Let $0 < \varepsilon < c_+ - c$. By Lemma 8.9, there exists $A > 0$ such that the discs D_m with the centers α_m and the radius $Ae^{-(c+\varepsilon)\lambda(|\alpha_m|)}$ are disjoint. Then there exists a constant $A_3 > 0$ such that for any m and for $z \in \partial D_m$, we have

$$\left|\left(1-\frac{z}{\beta_m}\right)\left(1-\frac{z}{\alpha_m}\right)^{-1}-1\right| = \frac{|\beta_m-\alpha_m|\cdot|z|}{|\alpha_m-z|\cdot|\beta_m|} \le A_3 e^{-(c_+-c-\varepsilon)\lambda(|\alpha_m|)}.$$

By the maximum principle, the same estimate is true for $z \notin D_m$. Hence, for $n \ne m$, we have

$$\left|\left(1-\frac{\alpha_n}{\beta_m}\right)\left(1-\frac{\alpha_n}{\alpha_m}\right)^{-1}-1\right| \le A_3 e^{-(c_+-c-\varepsilon)\lambda(|\alpha_m|)}.$$

Since $\log r = o(\lambda(r))$ and $\sum_m \alpha_m^2 < \infty$, we conclude that

$$\sum_m e^{-(c_+-c-\varepsilon)\lambda(|\alpha_m|)} < \infty,$$

and so there exist constants $A_4, A_5 > 0$ such that for all n,

$$A_4 \le \prod_{m\ne n}\left|\frac{1-\alpha_n/\beta_m}{1-\alpha_n/\alpha_m}\right| \le A_5. \tag{8.16}$$

We have

$$\begin{aligned}\left|\frac{P'_{\vec{\beta}}(\beta_n)}{P'_{\vec{\alpha}}(\alpha_n)}\right| = \frac{|\alpha_n|}{|\beta_n|}\prod_{m\ne n}\left|\frac{1-\beta_n/\beta_m}{1-\alpha_n/\alpha_m}\right| &= \prod_{m\ne n}\left|\frac{1-\alpha_n/\beta_m}{1-\alpha_n/\alpha_m}\right| \\ &+ \frac{\prod_{m\ne n}|1-\beta_n/\beta_m| - \prod_{m\ne n}|1-\alpha_n/\beta_m|}{\prod_{m\ne n}|1-\alpha_n/\alpha_m|}.\end{aligned} \tag{8.17}$$

The first term in (8.17) is both bounded and bounded away from zero by (8.16).

Step 2. To estimate the last term in (8.17), put $f_n(z) = P_{\vec{\beta}}(z)/(1-z/\beta_n)$. We show that for any $\varepsilon_1 \in (0, c_+ - c)$ there exists $A_6 > 0$ such that for all n

$$|f'_n(x)| \le A_6 \exp((c+\varepsilon_1)\lambda(|\alpha_n|)), \qquad |x-\alpha_n| \le \gamma e^{-c_+\lambda(|\alpha_n|)}. \tag{8.18}$$

By Lemma 8.8, $P_{\vec{\beta}} \in \mathcal{G}(\lambda, c)$. Hence, for any $\varepsilon_1 > 0$ there exists a constant A_7 such that

$$|P_{\vec{\beta}}(z)| + |P'_{\vec{\beta}}(z)| \le A_6 e^{(c+\varepsilon_1)\lambda(|z|)}, \qquad z \in \mathbb{C},\ |\operatorname{Im} z| \le 1.$$

Then, for $|z-\beta_n| = \min\left(1, \frac{|\beta_n|}{\lambda(|\beta_n|)}\right)$,

$$|f'_n(z)| \le \left|\frac{\beta_n \tilde{B}'(z)}{z-\beta_n}\right| + \left|\frac{\beta_n \tilde{B}(z)}{(z-\beta_n)^2}\right| \le A_7|\beta_n|\Big(2+\frac{\lambda(|\beta_n|)}{|\beta_n|}+\frac{\lambda^2(|\beta_n|)}{|\beta_n|^2}\Big)e^{(c+\varepsilon_1)\lambda(|z|)}.$$

Taking a slightly larger ε_1 and a larger constant we can get rid of the factor in front of the exponent and obtain that $|f_n'(z)| \le A_8 e^{(c+\varepsilon_1)\lambda(|z|)}$. Since $\big||z| - |\alpha_n|\big| = O\big(\frac{|\alpha_n|}{\lambda(\alpha_n)}\big)$, we use again the fact that $\lambda'(r) = O(r^{-1}\lambda(r))$ to conclude that $\lambda(|z|) - \lambda(|\alpha_n|) = O(1)$. By the maximum principle,

$$|f_n'(z)| \le A_6 \exp((c+\varepsilon_1)\lambda(|\alpha_n|)), \qquad |z - \beta_n| \le \min\left(1, \frac{|\beta_n|}{\lambda(|\beta_n|)}\right)$$

for some constant A_6 independent on n. In particular (8.18) holds.

Step 3. To complete the estimate of the last term in (8.17), note that

$$\begin{aligned}\Big| \prod_{m\neq n} (1-\beta_n/\beta_m) - \prod_{m\neq n}(1-\alpha_n/\beta_m)\Big| &= |f_n(\beta_n) - f_n(\alpha_n)| \\ &\le \gamma e^{-c_+\lambda(|\alpha_n|)} \cdot A_6 e^{(c+\varepsilon_1)\lambda(|\alpha_n|)} \le \gamma A_6 e^{-(c_+-c-\varepsilon_1)\lambda(|\alpha_n|)},\end{aligned}$$

where $c + \varepsilon_1 < c_+$. Note that for the denominator we have

$$\prod_{m\neq n} |1-\alpha_n/\alpha_m| = |\alpha_n P_{\vec\alpha}'(\alpha_n)| \ge (1+|\alpha_n|)^{-\tilde N+1},$$

Thus, the last term in (8.17) tends to zero as $n \to \infty$, and so $|P_{\vec\beta}'(\beta_n)| \asymp |P_{\vec\alpha}'(\alpha_n)|$. ❑

9 Carrying out the argument

From now on we assume that we are in the situation of the Fast Growth Theorem. That means:

Let λ be a growth function and $c \in [0,\infty)$. Let $\mu, \tilde\mu \in \mathbb{M}$ with $\tilde\mu$ having infinite index of determinacy, and assume that $\mu \preceq \tilde\mu$ w.r.t. (λ_1,λ_2). Further assume that

`[Chain]` $\qquad \tilde{\mathcal{L}} := \bigcup_{\tilde{\mathcal{H}}\in\mathcal{C}[\tilde\mu]} \tilde{\mathcal{H}} \subseteq \mathcal{G}(\lambda, c)\ ,$

and that

$$\big(\exists\, c^+ > c : \lambda_1 \ge 2c^+\lambda\big) \quad \textit{and} \quad \lambda_2 = 2\lambda_1.$$

Assuming equality in the conditions for λ_2 is no loss in generality; remember what we said in the paragraph after Remark 2.17.

Let us collect the given data in a table:

Given constants and functions:
— λ growth function
— c, c^+ with $0 \le c < c^+ < \infty$
— $\lambda_1(r) \ge 2c^+\lambda(r)$ and $\lambda_2 = 2\lambda_1$
— c_0, c_1, c_2 as in Definition 2.16 for $\mu \preceq \tilde\mu$ w.r.t. (λ_1, λ_2)

In the subsequent arguments several functions and constants will be chosen appropriately. Again we collect them in a table:

Chosen constants and functions:

- b with $c < b < c^+$
- d_1, d_2 with $c < d_2 < d_1 < c^+$
- $\tilde{N} \in \mathbb{N}$ with $\int_{-\infty}^{\infty} (1+x^2)^{-2\tilde{N}}\, d\tilde{\mu}(x) < \infty$
- $\gamma(\lambda_1, c_1)$ and $k_\pm(c_1)$ as in Lemma 6.4
- $\omega(x) := (1+x^2)^{-\tilde{N}}$
- $\sigma_1 := \lambda_1$, $\sigma_2 := d_1\lambda$, and $\gamma := \gamma(\lambda_1, c_1)\big(c_1 + k_+(c_1) + k_-(c_1)\big)$

The argument proceeds in five steps.

⇛ `Step 1:` `We show that` $\tilde{\mathcal{L}} \subseteq L^2(\mu)$.

We check finiteness of the integrals in (7.1): Since $\log r = \mathrm{o}(\lambda(r))$ and $\tilde{\mu}$ has at most power growth, we have

$$\int_{-\infty}^{\infty} e^{2b\lambda(|x|)} e^{-\lambda_1(|x|)}\, d\tilde{\mu}(x) \le \int_{-\infty}^{\infty} \exp\big[\underbrace{2(b-c^+)}_{<0}\lambda(|x|)\big]\, d\tilde{\mu}(x) < \infty,$$

$$\int_0^{\infty} e^{2b\lambda(r)} e^{\lambda_1(r)-\lambda_2(r)}\, dr \le \int_0^{\infty} \exp\big[\underbrace{2(b-c^+)}_{<0}\lambda(r)\big]\, dr < \infty.$$

An application of Theorem 7.1 yields

$$\tilde{\mathcal{L}} \subseteq \mathcal{G}(\lambda, c) \cap L^2(\tilde{\mu}) \subseteq L^2(\mu).$$

⇛ `Step 2:` `We construct weight functions` $W, \tilde{W}$ `on` $\mathbb{R}$ `such that`

$$\begin{aligned} &\tilde{W} \in L^2(\tilde{\mu}), \quad \tilde{\mathcal{L}} \texttt{ is stably dense in } C_0(\tilde{W}),\\ &W \in L^2(\mu), \quad C_0(W) \cap \mathcal{G}(\lambda, c) = C_0(\tilde{W}) \cap \mathcal{G}(\lambda, c),\\ &\omega(x) \lesssim \min\{W(x), \tilde{W}(x)\}, \quad x \in \mathbb{R}. \end{aligned} \tag{9.1}$$

Note that the function $\log \omega|_{[0,\infty)}$ is decreasing and negative, and that $\omega \in L^2(\tilde{\mu})$.

Since $\tilde{\mu}$ has infinite index of determinacy, the space $\tilde{\mathcal{L}}$ is stably dense in $L^2(\tilde{\mu})$, cf. Proposition 6.7. Now Theorem 4.12 provides us with a weight function $\tilde{W}$ such that

$$\tilde{W} \in L^2(\tilde{\mu}), \quad \tilde{\mathcal{L}} \text{ stably dense in } C_0(\tilde{W}), \quad \omega(x) \le \tilde{W}(x),\ x \in \mathbb{R}.$$

We apply the smoothening operator $\mathcal{S}$ from Definition 8.1 with the parameters $\sigma_1, \sigma_2, \omega, \gamma$: denote $W := \mathcal{S}[\tilde{W}]$.

The properties (9.1) are established by applying Lemma 8.4 and Lemma 8.5. We need to check the required hypothesis.

We have $|\log\omega(r)| = \mathrm{o}(\sigma_1(r))$. The conditions (8.2) and (8.3) are now obvious. For (8.4) note that ω is rational and $\sigma_1 \geq 0$. For (8.5) observe that

$$\frac{\lambda(r)}{\sigma_2(r)} = \frac{1}{d_1} < \frac{1}{c},$$

and for (8.6) compute

$$\limsup_{|x|\to\infty} \frac{\lambda(|x|)}{\sigma_1(|x|)+\log\omega(x)} \leq \limsup_{|x|\to\infty} \frac{\lambda(|x|)}{2c^+\lambda(|x|)(1+\mathrm{o}(1))} = \frac{1}{2c^+} < \frac{1}{c}.$$

Lemma 8.4 applies and yields

$$C_0(W)\cap\mathcal{G}(\lambda,c) = C_0(\tilde{W})\cap\mathcal{G}(\lambda,c).$$

The condition (8.8) holds by definition (even with equality), and (8.9) holds since σ_2 is a growth function, cf. (6.1). For (8.10) chose β with $2 < \beta < 2\frac{c^+}{d_1}$, and compute

$$\beta\sigma_2(r) + \lambda_1(r) - \lambda_2(r) \leq (\beta d_1 - 2c^+)\lambda(r)$$

Lemma 8.5 applies and yields $W \in L^2(\mu)$.

It remains to show that $\omega \lesssim W$. Since ω satisfies (8.4), we have

$$\omega(x) \asymp \inf\big\{\omega(t) : |t-x| \leq \gamma e^{-\sigma_1(|x|)}\big\} \leq \inf\big\{\tilde{W}(t) : |t-x| \leq \gamma e^{-\sigma_1(|x|)}\big\},\ x\in\mathbb{R}.$$

By the definitions of $\log\omega$ and σ_2 we have $\lim_{|x|\to\infty}(\log\omega(x) - \sigma_2(x)) = -\infty$, and hence $\omega(r) = \mathrm{o}(e^{\sigma_2(r)})$. Together, thus,

$$\omega(x) \lesssim W(x),\ x\in\mathbb{R}.$$

We continue using proof by contradiction: Assume from now on that

$$\operatorname{Clos}_{L^2(\mu)}\tilde{\mathcal{L}} \neq L^2(\mu) \text{ and } \operatorname{Clos}_{L^2(\mu)}\tilde{\mathcal{L}} \subseteq \mathcal{G}(\lambda,c).$$

`⇛ Step 3:` `The function` B `and estimates for the separation of its zeroes.`

Set $\mathcal{H} := \operatorname{Clos}_{L^2(\mu)}\tilde{\mathcal{L}}$, then $\mathcal{H}$ belongs to the chain $\mathcal{C}[\mu]$. Since $\tilde{W}$ is everywhere positive, $\dim C_0(\tilde{W}) = \infty$ and it follows that $\dim[\tilde{\mathcal{L}}\cap C_0(\tilde{W})] = \infty$ by density. However, $\tilde{\mathcal{L}}\cap C_0(\tilde{W}) = \tilde{\mathcal{L}}\cap C_0(W) \subseteq \mathcal{H}\cap C_0(W)$ since $\tilde{\mathcal{L}}\subseteq\mathcal{G}(\lambda,c)$. Thus also $\dim\big[\mathcal{H}\cap C_0(W)\big] = \infty$. Proposition 5.10 tells us that

$$\mathcal{K}_{3,2}(\mathcal{H}\cap C_0(W), W)\cap\mathcal{G}(\lambda,c) \neq \emptyset.$$

Since W is everywhere positive, we can use Remark 5.2, (*ii*), and Lemma 5.4 to add or remove finitely many zeroes in Krein classes with weight W. Clearly, doing this does not lead out of $\mathcal{G}(\lambda,c)$. Hence, we can choose a function

$$B \in \mathcal{K}_{0,0}(\mathcal{H}\cap C_0(W), W)\cap\mathcal{G}(\lambda,c), \quad B(0) = 1.$$

Denote the sequence of zeroes of B as $\vec{\alpha} = (\alpha_n)_{n\in\mathbb{N}}$. Observe that B is of Cartwright class, and hence $\vec{\alpha}$ satisfies (S1)–(S3) and $B = P_{\vec{\alpha}}$ (see, e.g., [Lev80, V.4, Theorem 11]). Moreover, note that the convergence exponent $\rho_{\vec{\alpha}}$ of the sequence $\vec{\alpha}$ does not exceed the order of B which, in turn, does not exceed ρ_λ.

Next we estimate the separation of $\vec{\alpha}$ from below. Convergence of the series $\sum_{n=1}^{\infty} \frac{W(\alpha_n)}{|B'(\alpha_n)|}$ implies that

$$\omega(\alpha_n) \lesssim W(\alpha_n) \lesssim |B'(\alpha_n)|,\ n \in \mathbb{N}.$$

Now Lemma 8.9 yields

$$s_{\vec{\alpha}}(n) \gtrsim \min\big\{|\alpha_n|, \omega(\alpha_n)e^{-d_2\lambda(|\alpha_n|)}\big\}. \tag{9.2}$$

This is used to establish

$$e^{-\sigma_1(|\alpha_n|)} = \mathrm{o}(s_{\vec{\alpha}}(n)),\ n \to \infty. \tag{9.3}$$

We have to estimate the quotients by each argument of the minimum in (9.2). Since σ_1 satisfies (P1) and (P2), it holds that $e^{-\sigma_1(|\alpha_n|)} = \mathrm{o}(|\alpha_n|)$. Next, we have

$$e^{-\sigma_1(x)} \cdot \frac{e^{d_2\lambda(|x|)}}{\omega(x)} \leq (1+x^2)^{-\tilde{N}} \exp\big[(d_2 - 2c^+)\lambda(|x|)\big]$$

Apparently the expression on the right side tends to 0 for $|x| \to \infty$.

$\Rightarrow$ `Step 4:` `We construct a perturbation` $\vec{\beta} = (\beta_n)_{n=1}^{\infty}$ `of the sequence` $\vec{\alpha}$`, and show that`

$$|P'_{\vec{\beta}}(\beta_n)| \asymp |P'_{\vec{\alpha}}(\alpha_n)|,\ n \in \mathbb{N}, \quad |P_{\vec{\beta}}(iy)| \asymp |P_{\vec{\alpha}}(iy)|,\ y \in \mathbb{R}. \tag{9.4}$$

The relation (9.3) implies that there exists $n_0 \in \mathbb{N}$ such that the intervals

$$\big[\alpha_n - \gamma e^{-\sigma_1(|\alpha_n|)}, \alpha_n + \gamma e^{-\sigma_1(|\alpha_n|)}\big], \quad n \geq n_0,$$

are pairwise disjoint, and that each of these intervals contains neither the point 0 nor a point α_m different from α_n.

Convergence of the series $\sum_{n=1}^{\infty} \frac{W(\alpha_n)}{|B'(\alpha_n)|}$ and the fact that $B' \in \mathcal{G}(\lambda, c)$ implies that

$$W(\alpha_n) \lesssim |B'(\alpha_n)| \lesssim e^{d_2\lambda(|\alpha_n|)} = \mathrm{o}\big(e^{\sigma_2(|\alpha_n|)}\big).$$

Hence, there exists $n_1 \geq n_0$ such that

$$W(\alpha_n) = \inf\big\{\tilde{W}(t) : |t - \alpha_n| \leq \gamma e^{-\sigma_1(|\alpha_n|)}\big\},\ n \geq n_1.$$

For $n \geq n_1$ choose $\beta_n \in [\alpha_n - \gamma e^{-\sigma_1(|\alpha_n|)}, \alpha_n + \gamma e^{-\sigma_1(|\alpha_n|)}]$ such that

$$\tilde{W}(\beta_n) = W(\alpha_n), \quad n \geq n_1. \tag{9.5}$$

For all other n set $\beta_n := \alpha_n$.

We have

$$\Big|\frac{\beta_n}{\alpha_n} - 1\Big| \lesssim \frac{e^{-\sigma_1(|\alpha_n|)}}{|\alpha_n|} = \frac{1}{|\alpha_n|} e^{-2c^+\lambda(|\alpha_n|)}$$

and hence $\sum_{n\in\mathbb{N}}\left|\frac{\beta_n}{\alpha_n}-1\right|<\infty$. Lemma 8.7, (i), applies and yields

$$|P_{\vec{\beta}}(iy)|\asymp|P_{\vec{\alpha}}(iy)|,\quad y\in\mathbb{R}.$$

Since $c_+\lambda<\lambda_1=\sigma_1$, Proposition 8.8 applies and yields

$$|P'_{\vec{\beta}}(\beta_n)|\asymp|P'_{\vec{\alpha}}(\alpha_n)|,\ n\in\mathbb{N}.$$

`⇛ Step 5:` `We finish the proof by deducing`

$$\mathcal{K}_{n_1-1,n_1-1}(\tilde{\mathcal{L}}\cap C_0(\tilde{W}),\tilde{W})\neq\emptyset,$$

`which contradicts that` $\tilde{\mathcal{L}}$ `is stably dense in` $C_0(\tilde{W})$.

Set

$$\tilde{B}(z):=\Big[\prod_{n=1}^{n_1-1}\Big(1-\frac{z}{\beta_n}\Big)\Big]^{-1}P_{\vec{\beta}}(z).$$

Using (9.4), (9.5), and that $\frac{\beta_n}{\alpha_n}\to 1$, it follows that (with some appropriate constants $C,C',C''>0$)

$$\sum_{x:\tilde{B}(x)=0}\frac{1}{(1+|x|)^{n_1-1}}\frac{\tilde{W}(x)}{|\tilde{B}'(x)|}\leq C\sum_{n=n_1}^{\infty}\frac{\tilde{W}(\beta_n)}{|P'_{\vec{\beta}}(\beta_n)|}$$
$$\leq C'\sum_{n=n_1}^{\infty}\frac{W(\alpha_n)}{|P'_{\vec{\alpha}}(\alpha_n)|}\leq C'\sum_{x:B(x)=0}\frac{W(x)}{|B'(x)|}<\infty,$$

$$\lim_{y\to\infty}\frac{y^{2-n_1}\cdot|F(iy)|}{|\tilde{B}(iy)|}=\lim_{y\to\infty}\frac{y\cdot|F(iy)|}{|P_{\vec{\beta}}(iy)|}\leq C''\lim_{y\to\infty}\frac{y\cdot|F(iy)|}{|P_{\vec{\alpha}}(iy)|}=0,$$
$$F\in\tilde{\mathcal{L}}\cap C_0(W).$$

By convergence of the series we have

$$|P'_{\vec{\beta}}(\beta_n)|\gtrsim\tilde{W}(\beta_n)\gtrsim\omega(\beta_n)\asymp\frac{1}{\beta_n^{2\tilde{N}}}$$

and hence

$$\sum_{n=1}^{\infty}\frac{1}{\beta_n^{2\tilde{N}+2}|P'_{\vec{\beta}}(\beta_n)|}\leq C\sum_{n=1}^{\infty}\frac{1}{\beta_n^2}<\infty.$$

It follows that $P_{\vec{\beta}}$ is of bounded type in $\mathbb{C}^+$ and $\mathbb{C}^-$, and hence that $\tilde{B}^{-1}F$ is of bounded type for all $F\in\tilde{\mathcal{L}}$. Alltogether, thus,

$$\tilde{B}\in\mathcal{K}_{n_1-1,n_1-1}(\tilde{\mathcal{L}}\cap C_0(W),\tilde{W}).$$

However, by (9.1) and $\tilde{\mathcal{L}}\subseteq\mathcal{G}(\lambda,c)$, we have $\tilde{\mathcal{L}}\cap C_0(W)=\tilde{\mathcal{L}}\cap C_0(\tilde{W})$.

❑ `The proof of the Fast Growth Theorems is complete` ❑

10 Consequences of Theorem 3.1

Corollary 3.2 follows due to presence of bounded type.

Proof of Corollary 3.2. Our aim is to apply Theorem 3.1 with $\lambda(r) := r$ and $c := 0$. All functions in $\tilde{\mathcal{L}}$ are of bounded type in the upper and lower half-planes, and hence belong to $\mathcal{G}(r,0)$. Thus `[Chain]` holds. Since $r = \mathrm{O}(\lambda_1(r))$, we can choose $c^+ > 0$ so small that $r \leq \frac{1}{2c^+}\lambda_1(r)$. Then `[A]` is satisfied.

Theorem 3.1 yields the alternative (3.1). Assume $\tilde{\mathcal{L}}$ were not dense in $L^2(\mu)$. Then $\mathrm{Clos}_{L^2(\mu)} \tilde{\mathcal{L}}$ is a member of the chain $\mathcal{C}[\mu]$, whence contained in $\mathcal{G}(r,0)$ and this is a contadiction. ❑

The Coincidence Theorem follows by combining the Fast Growth Theorem with a result from [Wor].

Proof of Theorem 3.3. The assumptions of Theorem 3.1 are fullfilled for μ and $\tilde{\mu}$, as well as when the roles of μ and $\tilde{\mu}$ are exchanged. We conclude first of all that

$$\mathcal{L} \subseteq L^2(\tilde{\mu}) \quad \text{and} \quad \tilde{\mathcal{L}} \subseteq L^2(\mu).$$

By [Wor, Theorem 3.5(i)] either $\mathcal{C}[\tilde{\mu}]$ is a beginning section of $\mathcal{C}[\mu]$ or vice versa. For definiteness, assume the first case takes place.

If $\mathcal{C}[\tilde{\mu}] \neq \mathcal{C}[\mu]$ then $\mathcal{C}[\tilde{\mu}] \subsetneq \mathcal{C}[\mu]$, and for any space $\mathcal{H} \in \mathcal{C}[\mu] \setminus \mathcal{C}[\tilde{\mu}]$ it holds that $\tilde{\mathcal{L}} \subseteq \mathcal{H}$. Therefore

$$\mathrm{Clos}_{L^2(\mu)} \tilde{\mathcal{L}} \subseteq \mathcal{H} \subseteq \mathcal{G}(\lambda, c),$$

whence the second alternative in (3.1) is ruled out. However, $\tilde{\mathcal{L}}$ also cannot be dense in $L^2(\mu)$ since μ has infinite index of determinacy and in particular $\mathcal{H} \neq L^2(\mu)$. We reached a contradiction. ❑

Proof of Corollary 3.4. For the same reasons as in the proof of Corollary 3.2 we can apply the Coincidence Theorem with $\lambda(r) := r$ and $c := 0$. ❑

Proof of Corollary 3.5. Apply the Coincidence Theorem with the parameters

$$\lambda(r) := r^\rho,\ c,\ \lambda_1(r) := (2c+\delta)r^\rho,\ \lambda_2(r) := 2\lambda_1(r).$$

Observe here that always $c \geq 0$. ❑

The proofs of the (quasi-) monotonicity Theorems 3.6 and 3.8 is based on the argument used in [Bra68, Theorem 26]. Let us analyse this argument.

10.1. De Branges Theorem 26: There is given a positive measure μ, a de Branges space $\mathcal{H}(E)$ which is contained isometrically in $L^2(\mu)$, and an entire function S. The following properties are assumed:

- `(a)` $S \in L^2\big(\frac{d\mu(x)}{1+x^2}\big)$ and $\frac{S}{E}, \frac{S^\#}{E}$ have no real poles,
- `(b)` $\frac{S}{E}, \frac{S^\#}{E}$ are of bounded type in $\mathbb{C}^+$,
- `(b`$_{26}$`)` $\frac{S(\pm iy)}{E(iy)} = \mathrm{o}(1)$, $y \to +\infty$,
- `(c)` if $Q \in \mathcal{H}(E) + z\mathcal{H}(E)$ and $Q = 0$ μ-a.e., then $Q = 0$ identically.

It is concluded that $S \in \mathcal{H}(E) + z\mathcal{H}(E)$.

The proof consists of three steps.

Step 1: For $h \in L^2(\mu) \ominus \mathcal{H}(E)$ consider

$$L_h(z) := \frac{S(z)}{G(z)} \int_{-\infty}^{\infty} \frac{1}{x-z} G(x)\overline{h(x)} d\mu(x) - \int_{-\infty}^{\infty} \frac{1}{x-z} S(x)\overline{h(x)} d\mu(x) \quad (10.1)$$

where $G \in \mathcal{H}(E) + z\mathcal{H}(E)$.

It is shown that this expression does not depend on G and defines an entire function.

This follows using the algebraic identity

$$G(w)\frac{\tilde{G}(z)S(w) - S(z)\tilde{G}(w)}{z-w} - \tilde{G}(w)\frac{G(z)S(w) - S(z)G(w)}{z-w} = \\ = S(w)\frac{\tilde{G}(z)G(w) - G(z)\tilde{G}(w)}{z-w}$$

and property `(a)`.

Step 2: It is shown that $L_h = 0$ for all $h \in L^2(\mu) \ominus \mathcal{H}(E)$.

This follows using the properties `(b)` and `(b`$_{26}$`)` by an application of the Phragmen-Lindelöf principle.

Step 3: It is shown that indeed $S \in \mathcal{H}(E) + z\mathcal{H}(E)$.

This is seen as follows (the argument in [Bra68] is formulated slightly different): For each $w \in \mathbb{C}^+$ choose $G_w \in \mathcal{H}(E)$ with

$$G_w(x) = \frac{S(x)E(w) - E(x)S(w)}{z-w} \ \mu\text{-a.e.},$$

and consider

$$T_w(z) := \frac{1}{E(w)}\big[E(z)S(w) + (z-w)G_w(z)\big] \in \mathcal{H}(E) + z\mathcal{H}(E).$$

Then $T_w(x) = S(x)$ μ-a.e., and $T_w(w) = S(w)$.

For $w, w_0 \in \mathbb{C}^+$ we have $T_w - T_{w_0} \in \mathcal{H}(E) + z\mathcal{H}(E)$ and $(T_w - T_{w_0})(x) = 0$ μ-a.e. Property `(c)` implies that $T_w = T_{w_0}$ identically. This yields $S(w) = T_{w_0}(w)$, $w \in \mathbb{C}^+$, and hence $S = T_{w_0} \in \mathcal{H}(E) + z\mathcal{H}(E)$.

$\Diamond$

Proof of Theorems 3.6 *and* 3.8. As usual denote $\tilde{\mathcal{L}} := \bigcup_{\mathcal{H} \in \mathcal{C}[\tilde{\mu}]} \mathcal{H}$. We assume towards a contradiction that the conclusion of Theorem 3.6 or Theorem 3.8, respectively, is false, and are going to establish

$$\exists\, \mathcal{H} \in \mathcal{C}[\mu]: \quad \tilde{\mathcal{L}} \subseteq \mathcal{H} + z\mathcal{H}. \quad (10.2)$$

Using this claim it is easy to derive a contradiction: (10.2) implies that $\mathbb{D}(\tilde{\mathcal{L}}) \subseteq \mathcal{H}$. Since $\mathcal{C}[\mu]$ has no maximal element, $\mathbb{D}(\tilde{\mathcal{L}})$ is not dense in $L^2(\mu)$. By Proposition 6.10 also $\tilde{\mathcal{L}}$ is not dense in $L^2(\mu)$ and hence $\mathrm{Clos}_{L^2(\mu)} \tilde{\mathcal{L}} \in \mathcal{C}[\mu]$.

However, by the Fast Growth Theorem (note that the assumptions of Theorem 3.1 are certainly fulfilled)

$$\operatorname{Clos}_{L^2(\mu)} \tilde{\mathcal{L}} \not\subseteq \mathcal{G}(\lambda, c),$$

which contradicts `[2Chain]`.

The proof of the claim (10.2) consists of two tasks. Namely, to find $\mathcal{H}$ and to prove $S \in \mathcal{H} + z\mathcal{H}$ for all $S \in \tilde{\mathcal{L}}$ (where, clearly, it is enough to consider S with $S = S^{\#}$).

The first of these tasks is easily completed using our indirect hypothesis.

`- for Theorem` 3.6: Let ϕ be in the described range of angles with $\underline{h}_\lambda[\mu](\phi) > h_\lambda[\tilde{\mu}](\phi)$, and fix τ_1, τ_2 with

$$h_\lambda[\tilde{\mu}](\phi) < \tau_1 < \tau_2 < \underline{h}_\lambda[\mu](\phi).$$

Choose $F_0 \in \bigcup_{\mathcal{H} \in \mathcal{C}[\mu]} \mathcal{H}$ with $\underline{h}_\lambda[F_0](\phi) > \tau_2$, and let $\mathcal{H} \in \mathcal{C}[\mu]$ be such that $F_0 \in \mathcal{H}$.

`- for Theorem` 3.8: Fix τ_1, τ_2 with

$$\tau_\lambda[\tilde{\mu}] < \tau_1 < \tau_2 < \tau_\lambda[\mu].$$

Choose $F_0 \in \bigcup_{\mathcal{H} \in \mathcal{C}[\mu]} \mathcal{H}$ with $\tau_\lambda[F_0] > \tau_2$, and let $\mathcal{H} \in \mathcal{C}[\mu]$ be such that $F_0 \in \mathcal{H}$.

The second task will be completed along the lines of the argument explained in 10.1 using the assumptions of our theorems instead of $(\mathtt{b}_{26})$. We start with a preliminary observation.

By the Fast Growth Theorem we have $\tilde{\mathcal{L}} \subseteq L^2(\mu)$. Since $\tilde{\mathcal{L}}$ consists of functions from spaces in $\mathcal{C}[\tilde{\mu}]$, every function in $\tilde{\mathcal{L}}$ is of bounded type in $\mathbb{C}^{\pm}$. Write $\mathcal{H} = \mathcal{H}(E)$ with some Hermite-Biehler function E. Since $\mathcal{H} \in \mathcal{C}[\mu]$, the function E is of bounded type in $\mathbb{C}^{\pm}$ and has no real zeroes. This shows that `(a)` and `(b)` of 10.1 hold. Since $\mathcal{C}[\mu]$ has no maximal element, $\mathcal{H} \neq L^2(\mu)$, and [Bra68, Problem 69] implies that also `(c)` holds.

Now let $S \in \tilde{\mathcal{L}}$, $S = S^{\#}$, be given. Then Step 1 of 10.1 can be applied, and Step 3 will apply once Step 2 is completed. We are thus left with the task to show "$L_h = 0$ for all $h \in L^2(\mu) \ominus \mathcal{H}$" (where again it is enough to consider h with $h = \overline{h}$).

`- for Theorem` 3.6: The relation

$$|F_0(re^{i\phi})| \leq \Big(\frac{|E(re^{i\phi})|^2 - |E(re^{-i\phi})|^2}{4\pi r \sin\phi}\Big)^{\frac{1}{2}} \|F_0\|_{\mathcal{H}} \lesssim |E(re^{i\phi})| \tag{10.3}$$

yields $\underline{h}_\lambda[F_0](\phi) \leq \underline{h}_\lambda[E](\phi)$ and in turn $e^{\tau_2 \lambda(r)} \lesssim |E(re^{i\phi})|$. The relation

$$\limsup_{y \to +\infty} \frac{\log|S(iy)|}{\lambda(y)} \leq h_\lambda[\tilde{\mu}] < \tau_1$$

yields $|S(re^{i\phi})| \lesssim e^{\tau_1 \lambda(r)}$. Together,

$$\frac{|S(re^{i\phi})|}{|E(re^{i\phi})|} \lesssim e^{(\tau_1 - \tau_2)\lambda(r)} = \mathrm{o}(1). \tag{10.4}$$

If $\phi = \frac{\pi}{2}$, this is ($\mathtt{b}_{26}$) and Step 2 of 10.1 directly applies. Assume that $\phi \neq \frac{\pi}{2}$. Then, certainly, $\rho_\lambda < 1$.

The first step is to show that L_h has order at most ρ_λ with minimal type if $\lambda(r) = \mathrm{o}(r^{\rho_\lambda})$. We know that L_h is of finite exponential type and, using e.g. the function $A := \frac{1}{2}(E + E^\#)$ for G in (10.1), that $L_h^\# = L_h$. Hence, it suffices to consider the growth of L_h on rays $\mathcal{R} := \{re^{i\psi} : r > 0\}$ with $\psi \in (0, \pi)$.

Now we use the function E for G in (10.1). Both integrals in (10.1) tend to 0 when z tends to infinity along the ray $\mathcal{R}$, and the function S satisfies

$$\log |S(re^{i\psi})| \leq C\lambda(r), \quad r > 0,$$

with some constant $C > 0$. In order to estimate $\log |E(re^{i\psi})|$ we use [Boa54, Theorem 3.7.1]. Fix $\delta \in [0, 1-\rho_\lambda)$ where $\delta = 0$ is only permitted if $\lambda(r) = \mathrm{o}(r^{\rho_\lambda})$. Set $\eta := \frac{1}{4} \sin \psi$ and let $\epsilon > 0$. Then [Boa54, Theorem 3.7.1] provides us with $R_0 > 0$ and, for each $R > R_0$, with a finite set of disks $\Omega_i(R)$, $i \in I(R)$, such that the sum of their radii is at most ηR and such that

$$\log |E(z)| \geq -\epsilon R^{\rho_\lambda + \delta}, \quad |z| \leq R, \ z \notin \Omega(R) := \bigcup_{i \in I(R)} \Omega_i(R).$$

However, keep in mind that R_0 as well as the disks $\Omega_i(R)$ may depend on δ and ϵ. These exceptional disks arise from an application of the Boutroux-Cartan Lemma [Boa54, Lemma 3.4.1], hence each of them contains at least one zero of E. The zeroes of E are all located in the lower half plane, and hence Ω is contained in the half-plane $\{z \in \mathbb{C} : \operatorname{Im} z \leq 2\eta R\}$.

Let $r > \frac{1}{2}R_0$, and choose $R > R_0$ such that $\frac{1}{2}R < r \leq R$. Then $\operatorname{Im} re^{i\psi} = r \sin \psi > R \frac{\sin \psi}{2} = 2\eta R$. Thus

$$\log |E(re^{i\psi})| \geq -\epsilon R^{\rho_\lambda + \delta} \geq -\epsilon 2^{\rho_\lambda + \delta} r^{\rho_\lambda + \delta}.$$

It follows that

$$\log \Big| \frac{S(re^{i\psi})}{E(re^{i\psi})} \Big| \leq C\lambda(r) + \epsilon 2^{\rho_\lambda + \delta} r^{\rho_\lambda + \delta} \leq \epsilon r^{\rho_\lambda + \delta} \Big(2^{\rho_\lambda + \delta} + C \frac{\lambda(r)}{r^{\rho_\lambda + \delta}} \Big), \ r \geq \frac{1}{2} R_0.$$

We conclude that

$$\limsup_{r \to \infty} \frac{1}{r^{\rho_\lambda + \delta}} \log |L_h(re^{i\psi})| \leq \epsilon C', \quad \psi \in (0, \pi),$$

where

$$C' := \sup_{r \geq 1} \Big(2^{\rho_\lambda + \delta} + C \frac{\lambda(r)}{r^{\rho_\lambda + \delta}} \Big).$$

The Phragmen-Lindelöf principle yields that the order of L_h does not exceed $\rho_\lambda + \delta$. Choosing $\delta > 0$ arbitrarily small implies that it is at most ρ_λ. Assume now that $\lambda(r) = \mathrm{o}(r^{\rho_\lambda})$. Then we use $\delta = 0$, remember that $\epsilon > 0$ was arbitrary, and note that C' is independent of ϵ. It follows that L_h is of minimal type w.r.t. the order ρ_λ.

By (10.4) and symmetry the function L_h tends to 0 along the rays with angle ϕ and $-\phi$. The allowed range of ϕ is defined in such a way that the Phragmen-Lindelöf principle applies to both sectors bounded by these rays. We conclude that indeed $L_h = 0$.

- `for Theorem` 3.8: Again referring to the reproducing kernel estimate (10.3) we obtain $\tau_\lambda[E] > \tau_2$. Choose a zero z_0 of E and set $\hat{E}(z) := \frac{E(z)}{z-z_0}$. Then also $\tau_\lambda[\hat{E}] > \tau_2$. Moreover, denote

$$\hat{M}(r) := \max_{|z|=r} |\hat{E}(z)|, \qquad \hat{n}(r) := \#\{z : |z| \le r, \hat{E}(z) = 0\},$$
$$\hat{Q}(r) := r \int_r^\infty \frac{1}{x^2} \hat{n}(x)\, dx.$$

Choose a function Δ which increases to $+\infty$, such that

$$\lim_{r\to\infty} \Delta(r) \cdot \frac{\log r}{\lambda(r)} = 0.$$

Then [Boa54, Lemma 3.5.10] provides us with $R_0 > 0$ and, for each $R \ge R_0$, disks $\Omega_{R,i}$, $i \in I_R$, such that

(i) the sum of the radii of $\Omega_{R,i}$ does not exceed $\frac{1}{4}R$,

(ii) $\log|\hat{E}(z)| \ge \log \hat{M}(2R) - \hat{Q}(2R)\Delta(R)$ for $|z| \le R$ with $z \notin \bigcup_{i\in I_R} \Omega_{R,i}$.

The disks $\Omega_{R,i}$ arise from an application of the Boutroux-Cartan Lemma, hence each of them contains at least one zero of $\hat{E}$. Remembering the proof of [Boa54, Lemma 3.5.8], thus

$$\#I_R \le \hat{n}(2R) \lesssim \log R.$$

Choose a sequence $R_n \nearrow +\infty$ such that $\log \hat{M}(R_n/2) \ge \tau_2 \lambda(R_n/2)$. Let $T_{n,i}$, $n \in \mathbb{N}, i \in I_{R_n}$, be the torus $T_{n,i} := \bigcup_{\theta\in\mathbb{R}} e^{i\theta}\Omega_{R_n,i}$. Then the set

$$\{z \in \mathbb{C} : R_n/2 \le |z| \le R_n\} \setminus \bigcup_{i\in I_{R_n}} T_{n,i} \tag{10.5}$$

is the disjoint union of at most $\#I_{R_n} + 1 \lesssim \log R_n$ tori. Since the sum of the widths of the tori $T_{n,i}$ does not exceed $R_n/4$, there exists a torus $T(n)$ in the representation of (10.5) whose width is at least equal to 2 (in fact, there must be much larger ones).

Let $\epsilon > 0$ be so small that $\frac{\tau_2}{(1+\epsilon)^2} \ge \tau_1$. From $\lim_{r\to\infty} \frac{\lambda(r)}{\lambda(r/2)} = 1$ and our choice of Δ we obtain that, for all sufficiently large n,

$$\log \hat{M}(2R_n) \ge \log \hat{M}(R_n/2) \ge \tau_2 \lambda(R_n/2) \ge \frac{\tau_2}{1+\epsilon} \lambda(R_n/2),$$
$$\frac{\hat{Q}(2R_n)\Delta(R_n)}{\log \hat{M}(2R_n)} \le 1 - \frac{1}{1+\epsilon}.$$

For such n thus

$$\log|\hat{E}(z)| \ge \frac{\tau_2}{(1+\epsilon)^2} \lambda(R_n) \ge \frac{\tau_2}{(1+\epsilon)^2} \lambda(|z|), \quad z \in T(n).$$

For n sufficiently large $|S(z)| \le \tau_1 \lambda(|z|)$, $z \in T(n)$, and hence

$$\frac{|S(z)|}{|\hat{E}(z)|} \le 1, \quad z \in T(n).$$

In order to estimate L_h, use $\hat{E}$ for G in (10.1). Off the real axis the integrals are bounded by $\frac{C}{|\operatorname{Im} z|}$ where

$$C := \max\Big\{ \int_{-\infty}^{\infty} \hat{E}(x)h(x)\,d\mu(x), \int_{-\infty}^{\infty} S(x)h(x)\,d\mu(x) \Big\}.$$

This gives

$$|L_h(z)| \leq \frac{2C}{|\operatorname{Im} z|}, \quad z \in T(n), z \notin \mathbb{R}. \tag{10.6}$$

The function $\log|L_h(z)|$ is subharmonic, whence

$$\log|L_h(z)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|L_h(z+e^{i\theta})|\,d\theta.$$

Let r_n be such that the circle $\{z \in \mathbb{C} : |z| = r_n\}$ is exactly in the middle of the torus $T(n)$. Then each disk $\{\zeta : |\zeta - z| \leq 1\}$ where $|z| = r_n$ is entirely contained in $T(n)$, and we obtain

$$\log|L_h(z)| \leq \log(2C) + C', \quad |z| = r_n,$$

where

$$C' := \sup_{z\in\mathbb{C}\setminus\mathbb{R}} \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|\operatorname{Im}(z+e^{i\theta})|}\,d\theta.$$

It follows that L_h is constant. However, (10.6) gives a sequence tending to $i\infty$ along which L_h tends to 0. Hence, L_h vanishes identically.

❑

Proof of Corollary 3.7. Let $\mathcal{H} \in \mathcal{C}[\mu]$ and write $\mathcal{H} = \mathcal{H}(E)$. The function E is of bounded type and has no zeroes in $\mathbb{C}^+$. Hence,

$$h_r[E](\phi) = \underline{h}_r[E](\phi) = \lim_{r\to\infty} \frac{1}{r}\log\big|E(re^{i\phi})\big| = \tau_r[E]\cdot\sin\phi, \quad \phi \in (0,\pi).$$

Now an application of the Quasi-Monotonicity Theorem yields the desired assertion. ❑

Proof of Theorem 3.9. Let $c^+ > c$ be as provided by `[Meas]` for $\tilde{\mu}$, and fix b, b^* with $c < b^* < b < c^+$. We check the condition (7.1) for b. Since $\log r = \mathrm{o}(\lambda(r))$ and $\tilde{\mu}$ satisfies the condition `[Meas]`, we have

$$\int\limits_{\mathbb{R}\setminus(-1,1)} e^{2b\lambda(|x|)}e^{-\lambda_1(|x|)}\,d\tilde{\mu}(x)$$
$$\leq \int\limits_{\mathbb{R}\setminus(-1,1)} \exp\big[\underbrace{2(b-c^+)}_{<0}\lambda(|x|)\big]\cdot|x|^{1-\rho_\lambda^+}\cdot e^{2c^+\lambda(|x|)}\,d\tilde{\mu}(x) < \infty,$$

$$\int\limits_1^\infty e^{2b\lambda(r)}e^{\lambda_1(r)-\lambda_2(r)}\,dr \leq \int\limits_1^\infty \exp\big[\underbrace{2(b-c^+)}_{<0}\lambda(r)\big]r^{\rho_\lambda^+-1}\,dr < \infty.$$

The integrals over the intervals $(-1,1)$ and $[0,1)$, respectively, are clearly finite.

Our aim is to apply Lemma 7.2 with $\beta_1 = \beta_2 := 2b$, and the function $f(x) := e^{b^*\lambda(x)}$. Clearly, for arbitrary $R > 0$, we have $h = b^*$. Next, $\lim_{r\to\infty} r\frac{\lambda'(r)}{\lambda(r)} = \rho_\lambda$ implies that for all large enough x

$$f'(x) = b^*\lambda'(x)e^{b^*\lambda(x)} \leq b^*(\rho_\lambda + 1)\frac{\lambda(x)}{x} \cdot e^{b^*\lambda(x)} \leq \lambda(x)e^{b^*\lambda(x)}.$$

Thus

$$\limsup_{x\to\infty} \frac{\log|f'(x)|}{\lambda(x)} \leq \limsup_{x\to\infty} \Big(\frac{\log\lambda(x)}{\lambda(x)} + b^*\Big) = b^*.$$

By choosing R large enough and $\varepsilon > 0$ small enough the condition (7.4) can be satisfied, and it follows that

$$\int_{R_0}^{\infty} e^{2b^*\lambda(x)}\,d\mu < \infty.$$

The integral along a half-axis unbounded to the left is estimated in the same way. Thus `[Meas]` holds for μ with the number $b^* > c$. ❑

Appendix A. The Sodin-Yuditskii approach to de Branges' theorem

This appendix is devoted to the proof of Theorem 5.9. From now on assume that $\mathcal{L}$ and W are given subject to the assumptions of this theorem.

It is a classical fact, going back at least to M.Riesz, that the Hall-majorant is an everywhere finite and continuous function. This property is actually established by L.Pitt in a very general context, cf. [Pit83]. Note here that $\mathcal{L}$ being not dense implies that $\mathcal{L}$ is contained injectively in $C_0(W)$.

A.1 Remark. We have the estimate

$$|F(z)| \leq \mathfrak{m}(z)\|F\|_W, \quad F \in \mathcal{L}.$$

Since $\mathfrak{m}$ is (in particular) locally bounded, this implies that each $\|.\|_W$-bounded subset of $\mathcal{L}$ is relatively compact in $\mathbb{H}(\mathbb{C})$ with respect to locally uniform convergence. ♢

The following fact can be extracted from [Pit83, Propositions 2.4 and 2.3].

A.2 Lemma. *Let $w \in \mathbb{C}^+$. Then there exists a function $H_0 \in \mathcal{L}$ with $\frac{H_0(x)}{x-w} \notin \operatorname{Clos}_{C_0(W)}\mathcal{L}$.*

Proof. Assume that $\frac{F(z)}{z-w} \in \overline{\mathcal{L}}$, $F \in \mathcal{L}$. The operator $R_w : f \mapsto \frac{f(x)}{x-w}$ maps $C_0(W)$ boundedly into $C_0(W)$, and it follows that $R_w(\overline{\mathcal{L}}) \subseteq \overline{\mathcal{L}}$. Thus also

$$\frac{f(x)}{(x-w)^n} \in \overline{\mathcal{L}}, \quad n \in \mathbb{N}_0. \tag{A.1}$$

We consider the annihilator of $\overline{\mathcal{L}}$. To this end remember the W.Summer's description of the dual space, cf. Theorem 4.7.

Let μ be a complex measure with $\int_{\mathbb{R}} f(x)\frac{d\mu(x)}{W(x)} = 0$, $f \in \overline{\mathcal{L}}$. Let $x_0 \in \mathbb{R}$ and choose $F \in \mathcal{L}$ with $F(x_0) \neq 0$. Consider the Cauchy-transform

$$C(z) := \int_{\mathbb{R}} \frac{1}{x-z} \frac{F(x)d\mu(x)}{W(x)}.$$

Then

$$C^{(n)}(z) = (-1)^n n! \int_{\mathbb{R}} \frac{1}{(x-z)^{n+1}} \frac{F(x)d\mu(x)}{W(x)}, \quad n \in \mathbb{N}_0,$$

and (A.1) implies that $C^{(n)}(w) = 0$, $n \in \mathbb{N}_0$. Thus $C(z) = 0$, $z \in \mathbb{C}^+$.

Let J be the isometric involution $f \mapsto \overline{f}$ on $C_0(W)$. From $J(\mathcal{L}) = \mathcal{L}$ we conclude that also $J(\overline{\mathcal{L}}) = \overline{\mathcal{L}}$, and hence that $\int_{\mathbb{R}} f(x)\frac{\overline{d\mu(x)}}{W(x)} = 0$, $f \in \overline{\mathcal{L}}$. Thus

$$\overline{C(\overline{z})} = \int_{\mathbb{R}} \frac{1}{x-z} \frac{\overline{F(x)}\,\overline{d\mu(x)}}{W(x)}, \quad z \in \mathbb{C}^+.$$

The Cauchy-transform being identically zero on $\mathbb{C}\setminus\mathbb{R}$ now implies that $\frac{F(x)d\mu(x)}{W(x)}$ is the zero measure. Thus the point x_0 has an open neighbourhood which is a $\frac{d\mu(x)}{W(x)}$-zero set. It follows that $\frac{d\mu(x)}{W(x)} = 0$, whence the functional induced by μ on $C_0(W)$ is zero. Thus $\overline{\mathcal{L}} = C_0(W)$. ❑

The next lemma is a stripped-down version of [Pit83, Proposition 3.4] for $C_0(W)$.

A.3 Lemma. *Let $\mathcal{F}$ be a $\|.\|_W$-bounded subset of $\mathcal{L}$, and let $F, G \in \operatorname{Clos}_{\mathbb{H}(\mathbb{C})} \mathcal{F}$ with G not identically zero. Then $\frac{F}{G}$ is a meromorphic function of bounded type in $\mathbb{C}^+$ and in $\mathbb{C}^-$.*

Proof. Fix $w \in \mathbb{C}^+$. The space $\mathcal{L}$ is not dense in $C_0(W)$, invariant under $.^{\#}$, and contains for each $x \in \mathbb{R}$ a function which does not vanish at x. Hence Lemma *A.*2 applies and shows that there exists a function $H_0 \in \mathcal{L}$ with $\frac{H_0(x)}{x-w} \notin \operatorname{Clos}_{C_0(W)} \mathcal{L}$. Choose a complex Borel measure μ on $\mathbb{R}$ with

$$\int_{\mathbb{R}} \frac{H(x)}{W(x)}\, d\mu(x) = 0,\ H \in \mathcal{L}, \qquad \int_{\mathbb{R}} \frac{H_0(x)}{(x-w)W(x)}\, d\mu(x) \neq 0.$$

Now let $F \in \operatorname{Clos}_{\mathbb{H}(\mathbb{C})} \mathcal{F}$ be given, and choose $F_n \in \mathcal{F}$, $n \in \mathbb{N}$, with $\lim_{n\to\infty} F_n = F$ locally uniformly. Then, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \frac{F_n(x)H_0(z) - F_n(z)H_0(x)}{x-z} \frac{1}{W(x)}\, d\mu(x) \\ &= H_0(z) \int_{\mathbb{R}} \frac{1}{x-z} \frac{F_n(x)}{W(x)}\, d\mu(x) - F_n(z) \int_{\mathbb{R}} \frac{1}{x-z} \frac{H_0(x)}{W(x)}\, d\mu(x). \end{aligned}$$

Since $\sup_{n\in\mathbb{N}} \|F_n\|_W < \infty$ the bounded convergence theorem applies, and we conclude that

$$\frac{F(z)}{H_0(z)} = \Big(\int_{\mathbb{R}} \frac{1}{x-z} \frac{F(x)}{W(x)} d\mu(x) \Big) \cdot \Big(\int_{\mathbb{R}} \frac{1}{x-z} \frac{H_0(x)}{W(x)} d\mu(x) \Big)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Cauchy integrals of complex measures are functions of bounded type, and the assertion follows. ❑

Since $\dim \mathcal{L} > 1$ (actually infinite), and $\mathcal{L}$ is invariant under $.^{\#}$, we can choose $w \in \mathbb{C}^+$ and $w' \in \mathbb{C} \setminus \mathbb{R}$, such that

$$M := \big\{F \in \mathcal{L} : F = F^{\#}, F(w) = w'\big\} \neq \emptyset.$$

Set $r_M := \inf\{\|F\|_W : F \in M\}$, then

$$r_M \geq \frac{|w'|}{\mathfrak{m}(w)} > 0.$$

Since the set $\{F \in M : \|F\|_W \leq r_M + 1\}$ is relatively compact, we find functions $F_n \in M$, $n \in \mathbb{N}$, such that

$$r_M = \lim_{n\to\infty} \|F_n\|_W, \qquad A := \lim_{n\to\infty} F_n \text{ exists in } \mathbb{H}(\mathbb{C}).$$

Clearly, $A = A^{\#}$ and $A(w) = w'$. In particular, A is not constant. Moreover, we have

$$|A(z)| = \lim_{n\to\infty} |F_n(z)| \leq \liminf_{n\to\infty} \big[\mathfrak{m}(z)\|F_n\|_W\big] = r_M \mathfrak{m}(z), \quad z \in \mathbb{C}, \qquad \text{(A.2)}$$

and for each $F \in \mathcal{L}$ the quotient $\frac{F}{A}$ is of bounded type in $\mathbb{C}^+$ and $\mathbb{C}^-$.

A function A constructed in the above way has several important properties, which are proved using the method of *Markov corrections*.

A.4 Proposition. *Set*

$$\mathcal{M}_A := \big\{G \in \mathcal{L} : \ G = G^{\#}, \|G\|_W < r_M, \ A - G \text{ not constant}\big\},$$
$$\Lambda_A := \Big\{x \in \mathbb{R} : \frac{|A(x)|}{W(x)} = r_M\Big\}.$$

Then for each $G \in \mathcal{M}_A$ the following statements hold.

(*i*) *$A - G$ has only real zeroes.*

(*ii*) *Every zero of $A - G$ is simple.*

(*iii*) *Let $x_1 < x_2$ be two consecutive zeroes of $A - G$. Then $(x_1, x_2) \cap \Lambda_A \neq \emptyset$.*

Note that, since A is not constant, we have $0 \in \mathcal{M}_A$. Moreover, for each $F \in \mathcal{L}$, $F = F^{\#}$, there exists $t > 0$ with $tF \in \mathcal{M}_A$. In particular, therefore, $\mathcal{L} = \operatorname{span} \mathcal{M}_A$.

In the proof of the above proposition, we frequently use the following *point-wise convexity* property of the weighted supremum norm $\|.\|_W$.

A.5 Lemma. *Let $\Omega \subseteq \mathbb{R}$, $t : \mathbb{R} \to [0, 1]$, and set $t_0 := \inf\{t(x) : x \in \Omega\}$. Then*

$$\big\|\mathbb{1}_\Omega\big(F(1-t) + Gt\big)\big\|_W \leq \|\mathbb{1}_\Omega F\|_W (1 - t_0) + \|\mathbb{1}_\Omega G\|_W t_0,$$
$$F, G \in \mathbb{H}(\mathbb{C}), \|\mathbb{1}_\Omega F\|_W \geq \|\mathbb{1}_\Omega G\|_W.$$

Proof. We have, for each $x \in \Omega$,

$$\begin{aligned}
&\frac{1}{W(x)}\big|F(x)(1-t(x))+G(x)t(x)\big| \\
&\qquad\qquad \le \frac{1}{W(x)} \cdot \begin{cases} |G(x)| &, \quad |G(x)| \ge |F(x)| \\ |F(x)|(1-t_0)+|G(x)|t_0\,, & |G(x)|<|F(x)| \end{cases} \\
&\qquad\qquad \le \begin{cases} \|\mathbb{1}_\Omega G\|_W &, \quad |G(x)| \ge |F(x)| \\ \|\mathbb{1}_\Omega F\|_W(1-t_0)+\|\mathbb{1}_\Omega G\|_W t_0\,, & |G(x)|<|F(x)| \end{cases} \\
&\qquad\qquad \le \|\mathbb{1}_\Omega F\|_W(1-t_0)+\|\mathbb{1}_\Omega G\|_W t_0.
\end{aligned}$$

❏

Proof of Proposition A.4.
Item (i): Showing that $A-G$ has only real zeroes is the technically simplest task. Assume on the contrary that $w_0 \in \mathbb{C}\setminus\mathbb{R}$ and $(A-G)(w_0)=0$. Since $A-G$ is not constant and $\lim_{n\to\infty}(F_n-G)=A-G$ locally uniformly, we find $n_0\in\mathbb{N}$ and a sequence $(w_n)_{n=n_0}^\infty$ such that

$$\lim_{n\to\infty} w_n = w_0, \quad (F_n-G)(w_n)=0, \quad |w_n-w_0| \le \frac{1}{2}|\operatorname{Im} w_0|.$$

Set

$$T_n := \frac{(z-w)(z-\overline{w})}{(z-w_n)(z-\overline{w_n})},$$

and consider the Markov corrections

$$H_{n,\delta} := F_n - \delta T_n(F_n-G), \quad n \ge n_0, \tag{A.3}$$

where $\delta > 0$ (and a specific choice for δ will be made later). Since $\mathcal{L}$ is an algebraic de Branges space, we have $H_{n,\delta}\in\mathcal{L}$. Moreover, $H_{n,\delta}=H_{n,\delta}^{\#}$ and $H_{n,\delta}(w)=F_n(w)=w'$. In total, thus, $H_{n,\delta}\in M$.

The functions T_n satisfy

$$\inf\big\{T_n(x): x\in\mathbb{R}, n\ge n_0\big\}>0, \quad \sup\big\{T_n(x): x\in\mathbb{R}, n\ge n_0\big\}<\infty.$$

Now we choose $\delta := \sup\{T_n(x): x\in\mathbb{R}, n\ge n_0\}^{-1}$. Setting $t_n(x):=\delta T_n(x)$, $x\in\mathbb{R}$, the function $H_{n,\delta}$ can be written as the pointwise convex combination

$$H_{n,\delta}(x) = F_n(x)(1-t_n(x)) + G(x)t_n(x), \quad x\in\mathbb{R}, n\ge n_0, \tag{A.4}$$

and Lemma $A.5$ implies

$$\|H_{n,\delta}\|_W \le \|F_n\|_W(1-t_{n,0}) + \|G\|_W t_{n,0}, \quad n \ge n_0,$$

where $t_{n,0} := \inf\{t_n(x): x\in\mathbb{R}\}$. Clearly, $\inf\{t_{n,0}: n\ge n_0\}>0$, and hence $\|H_{n,\delta}\|_W < r_M$ for sufficiently large n. We have reached a contradiction.

Item (ii): For showing that $A-G$ has no multiple real zeroes the argument essentially repeats, but in a slightly more complicated fashion. Assume on the contrary that $w_0 \in \mathbb{R}$ and $(A-G)(w_0) = (A-G)'(w_0) = 0$. First, choose $\epsilon > 0$ such that

$$\|G\|_W + 3\epsilon < r_M.$$

Second, since $\frac{|A|}{W}$ is upper semicontinuous and $A(w_0) = G(w_0)$, we can choose $r > 0$ such that

$$\frac{|A(x)|}{W(x)} \leq \|G\|_W + \epsilon, \ x \in I := [w_0 - 2r, w_0 + 2r].$$

Choose $n_0 \in \mathbb{N}$ and two sequences $(w_{1,n})_{n=n_0}^{\infty}$, $(w_{2,n})_{n=n_0}^{\infty}$, such that

$$\lim_{n\to\infty} w_{i,n} = w_0, \quad \frac{F_n(z) - G(z)}{(z - w_{1,n})(x - w_{2,n})} \text{ entire}, \quad |w_{i,n} - w_0| \leq r,$$
$$w_{1,n}, w_{2,n} \in \mathbb{R} \text{ or } w_{1,n} = \overline{w_{2,n}}.$$

Since W is lower semicontinuous and $\lim_{n\to\infty} F_n = A$ locally uniformly, we can make the choice of n_0 such that in addition

$$\frac{|F_n(x)|}{W(x)} \leq \|G\|_W + 2\epsilon, \quad x \in I. \tag{A.5}$$

Set

$$T_n := \frac{(z-w)(z-\overline{w})}{(z-w_{1,n})(z-w_{2,n})}, \tag{A.6}$$

and let $H_{n,\delta}$ be the function (A.3). Then again $H_{n,\delta} \in M$.

The functions T_n satisfy

$$\begin{aligned} &\inf\big\{T_n(x) : x \in \mathbb{R} \setminus I, n \geq n_0\big\} > 0, \\ &\sup\big\{T_n(x) : x \in \mathbb{R} \setminus I, n \geq n_0\big\} < \infty. \end{aligned} \tag{A.7}$$

Moreover, since $\lim_{n\to\infty} T_n(z)(F_n(z) - G(z)) = \frac{(z-w)(z-\overline{w})}{(z-w_0)^2}(A(z) - G(z))$ locally uniformly, we have in particular

$$c := \sup\Big\{\frac{|T_n(x)(F_n(x) - G(x))|}{W(x)} : x \in I, n \geq n_0\Big\} < \infty.$$

Now we choose $\delta := \min\{\sup\{T_n(x) : x \in \mathbb{R}, n \geq n_0\}^{-1}, \frac{\epsilon}{c}\}$. Set

$$t_n(x) := \mathbb{1}_{\mathbb{R}\setminus I} \cdot \delta T_n(x), \ x \in \mathbb{R}, \quad t_{n,0} := \inf\{t_n(x) : x \in \mathbb{R} \setminus I\}, \tag{A.8}$$

and consider the representation (A.4) of $H_{n,\delta}$ on $\mathbb{R} \setminus I$. Using (A.5) we find

$$\|\mathbb{1}_{\mathbb{R}\setminus I} F_n\|_W = \|F_n\|_W > \|G\|_W \geq \|\mathbb{1}_{\mathbb{R}\setminus I} G\|_W,$$

and pointwise convexity implies

$$\begin{aligned} \|\mathbb{1}_{\mathbb{R}\setminus I} H_{n,\delta}\|_W &\leq \|\mathbb{1}_{\mathbb{R}\setminus I} F_n\|_W (1 - t_{n,0}) + \|\mathbb{1}_{\mathbb{R}\setminus I} G\|_W t_{n,0} \\ &\leq \|F_n\|_W (1 - t_{n,0}) + \|G\|_W t_{n,0}, \quad n \geq n_0. \end{aligned}$$

Again $\inf\{t_{n,0} : n \geq n_0\} > 0$, and hence $\|\mathbb{1}_{\mathbb{R}\setminus I} H_{n,\delta}\|_W < r_M$ for all sufficiently large n. For $x \in I$ we have

$$\frac{|H_{n,\delta}(x)|}{W(x)} \leq \frac{|F_n(x)|}{W(x)} + \delta \frac{|T_n(x)(F_n(x) - G(x))|}{W(x)} \leq \|G\|_W + 3\epsilon,$$

and hence $\|\mathbb{1}_I H_{n,\delta}\|_W < r_M$. In total $\|H_{n,\delta}\|_W < r_M$ for large n, and we reached a contradiction.

Item (iii): Let $x_1 < x_2$ be two consecutive zeroes of $A - G$, and assume on the contrary that $(x_1, x_2) \cap \Lambda_A = \emptyset$. Since $\frac{|A(x_i)|}{W(x_i)} = \frac{|G(x_i)|}{W(x_i)} \le \|G\|_W < r_M$, we then have

$$\max\Big\{\frac{|A(x)|}{W(x)} : x \in [x_1, x_2]\Big\} < r_M,$$

and can choose $r > 0$ such that ($I := [x_1 - 2r, x_2 + 2r]$)

$$\max\Big\{\frac{|A(x)|}{W(x)} : x \in I\Big\} < r_M.$$

Choose $n_0 \in \mathbb{N}$ and sequences $(w_{1,n})_{n=n_0}^{\infty}$, $(w_{2,n})_{n=n_0}^{\infty}$, such that

$$\lim_{n\to\infty} w_{i,n} = x_i, \quad (F_n - G)(w_{i,n}) = 0, \quad w_{i,n} \in [x_1 - r, x_2 + r], \quad w_{1,n} \neq w_{2,n},$$
$$c := \sup\Big\{\frac{|F_n(x)|}{W(x)} : x \in I, n \ge n_0\Big\} < r_M.$$

The functions (A.6) are again subject to (A.7), and $T_n(F_n - G)$ converges locally uniformly.

Now we choose δ sufficiently small so that

$$\delta \cdot \sup\big\{T_n(x) : x \in \mathbb{R} \setminus I, n \ge n_0\big\} \le 1,$$
$$\delta \cdot \sup\Big\{\frac{|T_n(x)(F_n(x) - G(x))|}{W(x)} : x \in I, n \ge n_0\Big\} < \frac{r_M - c}{2}.$$

Notation being as in (A.8), pointwise convexity implies that $\|\mathbb{1}_{\mathbb{R}\setminus I} H_{n,\delta}\|_W < r_M$ for large n. On the interval I we estimate

$$\|\mathbb{1}_I H_{n,\delta}\|_W \le \max_{x\in I} \frac{|F_n(x)|}{W(x)} + \delta \max_{x\in I} \frac{|T_n(x)(F_n(x) - G(x))|}{W(x)} \le \frac{c + r_M}{2} < r_M.$$

Again, we reached the contradiction that $\|H_{n,\delta}\|_W < r_M$ for large n. ❑

Proposition $A.4$ is supplemented by the following simple facts.

A.6 Lemma. *Let $x_1 < x_2$ be two consecutive points of Λ_A, and let $G \in \mathcal{M}_A$.*

(i) If $\operatorname{sgn} A(x_1) \neq \operatorname{sgn} A(x_2)$, then $A - G$ has a zero in (x_1, x_2).

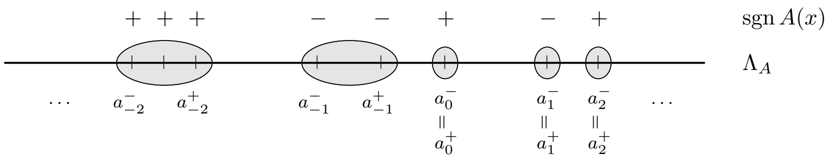
(ii) If $\operatorname{sgn} A(x_1) = \operatorname{sgn} A(x_2)$, then $A - G$ has no zero in (x_1, x_2).

Proof. If $x \in \Lambda_A$, then $\frac{|A(x)|}{W(x)} = r_M > \|G\|_W$, and hence

$$\operatorname{sgn} A(x) = \operatorname{sgn}(A - G)(x).$$

The first statement follows immediately. Assume now that $\operatorname{sgn} A(x_1) = \operatorname{sgn} A(x_2)$ and $A - G$ has a zero y in (x_1, x_2). By Proposition $A.4$, (i), all zeroes of $A - G$ are simple, thus $A - G$ changes sign at y. Hence there must exist a second zero of $A - G$ in (x_1, x_2). Proposition $A.4$, (iii), implies that $(x_1, x_2) \cap \Lambda_A \neq \emptyset$. This contradicts the fact that x_1 and x_2 are consecutive points of Λ_A. ❑

Consider the discrete set Λ_A and group it in blocks where $A(x)$ has the same sign:



(A.9)

Formally, we obtain (possibly one-sided or two-sided infinite) sequences $(a_n^-)_{n_-<n<n_+}$, $(a_n^+)_{n_-<n<n_+}$, such that

$$\cdots \leq a_{n-1}^+ < a_n^- \leq a_n^+ < a_{n+1}^- \leq \cdots$$

$$a_n^\pm \in \Lambda_A, \quad \Lambda_A \subseteq \bigcup_{n_-<n<n_+} [a_n^-, a_n^+], \quad \operatorname{sgn} A(x) = (-1)^n,\ x \in [a_n^+, a_{n+1}^-] \cap \Lambda_A.$$

A.7 Corollary. *Let the sequences $(a_n^\pm)_{n_-<n<n_+}$ be as above, and set*

$$J_n := (a_n^+, a_{n+1}^-), \quad J_{-\infty} := (-\infty, \inf \Lambda_A),\ J_\infty := (\sup \Lambda_A, \infty).$$

For each $G \in \mathcal{M}_A$ the function $A - G$ has

- (i) *exactly one zero in each interval J_n, $n_- < n < n_+$,*
- (ii) *at most one zero in each of $J_{-\infty}$ and J_∞,*
- (iii) *no zeroes in $\mathbb{R} \setminus \big[J_{-\infty} \cup J_\infty \cup \bigcup_{n_-<n<n_+} J_n\big]$.*

Proof. By Lemma A.6, (i), the function $A - G$ has a zero in each J_n, $n_- < n < n_+$. By Proposition A.4, (iii), it has at most one zero in each such interval as well as in $J_{-\infty}$ and J_∞. The third item holds by Lemma A.6, (ii). ❑

A.8 Lemma. *The function A has infinitely many zeroes. In particular, Λ_A is infinite and consists of infinitely many blocks as in* (A.9).

Proof. Assume that A has only finitely many zeroes, say N many. Then the set Λ_A must consist of at most $N+1$ blocks, and hence for each $G \in \mathcal{M}_A$ the function $A - G$ has at most $N + 2$ zeroes.

For $G \in \mathcal{M}_A$ let p_G be the monic polynomial with the same zeroes as $A-G$. Then the function

$$H_G := \frac{A-G}{A} \cdot \frac{p_0}{p_G}$$

is real and entire, zerofree, and of bounded type in $\mathbb{C}^+$ and $\mathbb{C}^-$. Using, e.g., the product representation [Lev80, Theorem V.11], it follows that H_G is constant. This yields

$$G(z) = \frac{A(z)}{p_0(z)}\big[p_0(z) - H_G p_G(z)\big] \in \frac{A(z)}{p_0(z)} \cdot \Big\{p \in \mathbb{C}[z] : \deg p \leq N+2\Big\}.$$

Since $\mathcal{L} = \operatorname{span} \mathcal{M}_A$, we find $\dim \mathcal{L} \leq N + 3$, a contradiction. ❑

We have reached the stage to actually construct the required function B. From each of the blocks of Λ_A in (A.9) choose one point. Then, by the interlacing property in Corollary $A.7$ and the fact that there are infinitely many such blocks, there exists a real and meromorphic function q_0 with

$$\operatorname{Im} q_0(z) \geq 0, \quad z \in \mathbb{C}^+,$$

which has zeroes precisely at the zeroes of A and poles precisely at the chosen points of Λ_A (see, e.g., [Lev80, Theorem VII.1]). Now define B to be the real entire function

$$B := \frac{1}{q_0} A.$$

A.9 Lemma. *For each $G \in \mathcal{M}_A$, the function* $\operatorname{Im} \frac{A-G}{B}$ *does not change sign in $\mathbb{C}^+$.*

Proof. For the same reason as above we find a real and meromorphic function q_G with nonnegative imaginary part in $\mathbb{C}^+$ which has zeroes precisely at the zeroes of $A - G$ and poles precisely at chosen points of Λ_A, i.e., at the zeroes of B. The function $\frac{A-G}{B}\frac{1}{q_G}$ is real, entire, and zerofree. It can be written in the form

$$\frac{A-G}{B}\frac{1}{q_G} = \Big(1 - \frac{G}{A}\Big) q_0 \frac{1}{q_G},$$

and hence is of bounded type in $\mathbb{C}^+$. Thus, it is identically equal to some nonzero real constant, say γ_G. This shows that

$$\frac{A-G}{B} = \gamma_G \cdot q_G,$$

and hence its imaginary part has no sign changes in $\mathbb{C}^+$. ❑

It is now easy to establish the required properties of B.

Proof of Theorem 5.9. The properties (K1) hold by construction of B. For each $G \in \mathcal{M}_A$ we have the representation

$$\frac{G}{B} = q_0 - \gamma_G \cdot q_G.$$

This shows that $\frac{G}{B}$ is of bounded type in $\mathbb{C}^+$ and $\mathbb{C}^-$, and that

$$\lim_{|y|\to\infty} \frac{1}{iy}\frac{G(iy)}{B(iy)} \text{ exists in } \mathbb{R}.$$

Since $\mathcal{L} = \operatorname{span} \mathcal{M}_A$, both of these properties also hold for every function of $\mathcal{L}$. This proves (K2) and (K3'). The summability condition (K4') holds by

$$\sum_{x:B(x)=0} \frac{1}{1+x^2}\frac{W(x)}{B'(x)} = \frac{1}{r_M} \sum_{x:B(x)=0} \frac{1}{1+x^2}\Big|\frac{A(x)}{B'(x)}\Big|$$
$$= \frac{1}{r_M} \sum_{x:B(x)=0} \frac{1}{1+x^2} \operatorname{Res}(q_0; x) < \infty,$$

remember here [Lev80, Theorem VII.2]. Finally, for the additional assertion, it suffices to remember (A.2) and that a function with nonnegative imaginary part in $\mathbb{C}^+$ is uniformly $\lesssim |z|$ in each Stolz angle. ❑

Appendix B. Existence of a distinguished chain

In this appendix we provide a direct proof of Proposition 2.3. This is done by reducing to the Poisson integrable case.

To start with, we would like to point out the following facts. Assume that $\mathcal{H}$ is a de Branges space, and that $1 \in \mathcal{H} + \ldots + z^n\mathcal{H}$ for some $n \in \mathbb{N}$. That means we can write

$$1 = f_0 + zf_1 + \ldots + z^n f_n \quad \text{with some } f_0, \ldots, f_n \in \mathcal{H}.$$

From this we see that for every point $x \in \mathbb{R}$ there must exist a function in $\mathcal{H}$ which does not vanish at x. Moreover, the function f_0 cannot vanish identically (it cannot vanish at the point 0), and dividing by f_0 leads to $\frac{1}{f_0} = 1 + z\frac{f_1}{f_0} + \ldots + z^n\frac{f_n}{f_0}$. This shows that f_0 is of bounded type in $\mathbb{C}^+$ and $\mathbb{C}^-$. Thus all functions of $\mathcal{H}$ have this property.

The uniqueness part of the assertion follows now immediately from de Branges' theory, cf. [Bra68, Theorem 35]. Moreover, if (2.1) holds for some $n \in \mathbb{N}$, then clearly

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^n}\, d\mu(x) \leq \sum_{k=0}^{n} \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n-k}} \cdot |f_k|^2 d\mu(x) < \infty.$$

We turn to the existence part. Let $\mu \in \mathbb{M}$ be given. If μ is Poisson integrable (or finite), we know from [Bra68] that there exists a chain $\mathcal{C}$ in $L^2(\mu)$ with $1 \in \mathcal{H} + z\mathcal{H}$ (or $1 \in \mathcal{H}$, respectively) for all $\mathcal{H} \in \mathcal{C}$. Hence, let us assume that $n \geq 2$ and

$$\int_{-\infty}^{\infty} \frac{d\mu(x)}{(1+x^2)^n} < \infty, \quad \int_{-\infty}^{\infty} \frac{d\mu(x)}{(1+x^2)^{n-1}} = \infty.$$

Set $d\nu(x) := (1+x^2)^{-n} d\mu(x)$. Clearly, ν is finite but not finitely supported (and hence $\dim L^2(\nu) = \infty$). Hence, we can choose a de Branges space $\mathring{\mathcal{K}}$ which is contained isometrically in $L^2(\nu)$, contains the constant function 1, has dimension not less than $n+1$, and is not equal to $L^2(\nu)$. Set

$$D := \big\{F \in \mathring{\mathcal{K}} : z^n F(z) \in \mathring{\mathcal{K}}\big\},$$

then D is an algebraic de Branges space, $D \neq \{0\}$, and $D \subseteq L^2(\mu)$. Let us show that D is not dense in $L^2(\mu)$. Assume the contrary, and let $f \in L^2(\nu)$. Then the function $g(x) := (1+x^2)^{-n} f(x)$ belongs to $L^2(\mu)$, and we find $F_k \in D$ with $F_k \to g$ in $L^2(\mu)$. This yields

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} \big|(x+i)^n F_k(x) - f(x)\big|^2 d\nu(x) = \lim_{k\to\infty} \int_{-\infty}^{\infty} \big|F_k(x) - g(x)\big|^2 d\mu(x) = 0.$$

However, $(z+i)^n F_k \in \mathring{\mathcal{K}}$, and hence $f\mathring{\mathcal{K}}$. Since f was arbitrary, this contradicts the fact that $\mathring{\mathcal{K}} \neq L^2(\nu)$.

Since D is not dense in $L^2(\mu)$, its closure $\mathring{\mathcal{H}} := \operatorname{Clos}_{L^2(\mu)} D$ is a de Branges space. Now de Branges' theory furnishes us with a chain $\mathcal{C}$ having $\mathring{\mathcal{H}}$ as an element. The functions in D are of bounded type in both half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$, and hence all functions in $\bigcup_{\mathcal{H}\in\mathcal{C}} \mathcal{H}$ share this property. The space D is invariant w.r.t. division of zeroes (also of real zeroes), and hence for each $x \in \mathbb{R}$ the space $\bigcup_{\mathcal{H}\in\mathcal{C}} \mathcal{H}$ contains a function which does not vanish at x.

Next we show that $\mathcal{C}$ does not contain any finite-dimensional spaces. Assume the contrary, then $\mathcal{C}$ also contains a 1-dimensional space $\mathcal{H} = \operatorname{span}\{S\}$ where S can be chosen to be real. By the above properties, S is zerofree and of bounded type in $\mathbb{C}^{\pm}$, hence constant. This contradicts the fact that μ is not finite.

Finally, we shall prove that $1 \in \mathcal{H} + \ldots + z^n\mathcal{H}$ for every space $\mathcal{H}$ in the chain. Let $\mathcal{H} \in \mathcal{C}$ be given. Since $\dim \mathcal{H} = \infty$, we can choose $F \in \mathcal{H}$, $F = F^{\#}$, such that F is not a polynomial of degree less than $n-1$. Thus the function $F(z) - i$ has at least $n-1$ zeroes counted according to their multiplicities (if F is transcendental, observe that it cannot simultaneously omit the values $+i$ and $-i$). Clearly, all of its zeroes are nonreal. Choose $n-1$ of them, say, $w_1, \ldots, w_{n-1}$ (multiple zeroes listed according to their multiplicity), and consider

$$G(z) := \frac{F(z) - i}{\prod_{j=1}^{n-1}(z - w_j)}.$$

This function is of bounded type in $\mathbb{C}^+$ and $\mathbb{C}^-$, and

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} |G(x)|^2 \, d\mu(x) < \infty.$$

Write $\mathcal{H} = \mathcal{H}(E)$ and set $S_{\vec{\alpha}} := \frac{1}{2}\big(e^{i\vec{\alpha}}E + e^{-i\vec{\alpha}}E^{\#}\big)$, $\vec{\alpha} \in [0, \pi)$. Since E is a Hermite-Biehler function (and $S_{\vec{\alpha}} = S_{\vec{\alpha}}^{\#}$, respectively) and of bounded type, the functions E and $S_{\vec{\alpha}}$ are of Polya class. In particular,

$$\frac{1}{E(iy)} = \mathrm{O}(1), \ \frac{1}{S_{\vec{\alpha}}(iy)} = \mathrm{O}(1), \quad y \in [1, \infty).$$

By the reproducing kernel estimate in $\mathcal{H}(E)$ we have

$$\sqrt{y}\frac{|F(iy)|}{|E(iy)|} = \mathrm{O}(1), \quad y \in [1, \infty).$$

The function $S_{\vec{\alpha}+\frac{\pi}{2}}^{-1} S_{\vec{\alpha}}$ has nonnegative imaginary part throughout the upper half-plane, whence $|S_{\vec{\alpha}}(iy)| \gtrsim \frac{1}{y}|S_{\vec{\alpha}+\frac{\pi}{2}}(iy)|$. Thus $\frac{1}{y}\frac{|E(iy)|}{|S_{\vec{\alpha}}(iy)|} = \mathrm{O}(1)$, and in turn $\frac{1}{\sqrt{y}}\frac{|F(iy)|}{|S_{\vec{\alpha}}(iy)|} = \mathrm{O}(1)$. Alltogether, therefore,

$$\lim_{y \to \infty} \frac{|G(iy)|}{|E(iy)|} = \lim_{y \to \infty} \frac{|G(iy)|}{|S_{\vec{\alpha}}(iy)|} = 0.$$

The same argument shows that also the limit for $y \to -\infty$ is 0. Now [Bra68, Theorem 26,Problem 70] implies that $G \in \mathcal{H} + z\mathcal{H}$, and in turn that $1 \in \mathcal{H} + \ldots + z^n\mathcal{H}$.

The proof of Proposition 2.3 is finished.

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