Directing functionals and de Branges space completions in almost Pontryagin spaces

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The following theorem holds: Let $L$ be a – not necessarily nondegenerated or complete – positive semidefinite inner product space carrying an antilinear isometric involution, and let $S$ be a symmetric operator in $L$. If $S$ possesses a universal directing functional $\Phi : L \times \mathbb{C} \to \mathbb{C}$ which is real w.r.t. the given involution, and the closure of $S$ in the completion of $L$ has defect index $(1, 1)$, then there exists a de Branges (Hilbert-) space $B$ such that $x \mapsto \Phi(x, \cdot)$ maps $L$ isometrically onto a dense subspace of $B$ and the multiplication operator in $B$ is the closure of the image of $S$ under this map.

In this paper we consider a version of universal directing functionals defined on an open set $\Omega \subseteq \mathbb{C}$ instead of the whole plane, and inner product spaces $L$ having finite negative index. We seek for representations of $S$ in a class of reproducing kernel almost Pontryagin spaces of functions on $\Omega$ having de Branges-type properties. Our main result is a version of the above stated theorem, which gives conditions making sure that $\Phi$ establishes such a representation. This result is accompanied by a converse statement and some supplements.

As a corollary, we obtain that if a de Branges-type inner product space of analytic functions on $\Omega$ has a reproducing kernel almost Pontryagin space completion, then this completion is a de Branges-type almost Pontryagin space. This is an important fact in applications. The corresponding result in the case that $\Omega = \mathbb{C}$ and $L$ is positive semidefinite is well-known, often used, and goes back (at least) to work of M.Riesz in the 1920’s.