Definitizable normal linear operators on Krein spaces

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# Preliminaries
- Spectral Theorem of normal operators on Hilbert Spaces
- Selfadjoint definitizable operators

# Dragging operators
- Embedding of Hilbert space in Krein space
- Embedding induced by definitizable operator

# Zero-dimensional Ideals
- Structure of zero-dimensional Ideals
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# Functional Calculus for normal, definitizable operators
- Functional Calculus
- Spectrum of $N$
Preliminaries

Dragging operators

Zero-dimensional Ideals

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- reobtain $E$ by $E(\Delta) = 1_\Delta(N)$. 

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- few hope even for general selfadjoint operators in Krein spaces;
- in the early 1980’s Heinz Langer gave some sort of Spectral Theorem for definitizable selfadjoint operators:
Selfadjoint definitizable operators

Theorem (Heinz Langer)

Let $A \in B(K)$ be selfadjoint and definitizable, i.e. $A = A^*$ and $[p(A)x, x] \geq 0$ for all $x \in K$ for some $0 \neq p \in \mathbb{R}[z]$. Then:
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- Riesz-projection $E(\{z\})$ corresponding to $z \in p^{-1}\{0\} \setminus \mathbb{R}$ satisfies $E(\{z\})^* = E(\{\bar{z}\})$ and $\text{ran\ } E(\{z\})$ is neutral, i.e. $[x, x] = 0$, $x \in \text{ran\ } E(\{z\})$. 

Main idea of the proof: $E(\Delta)$ were obtained with contour integrals around $\Delta$. 

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**Definition**

A normal $N \in B(\mathcal{K})$ is called definitizable, if the selfadjoint operators $A := \frac{N+N^*}{2}$ and $B := \frac{N-N^*}{2i}$ are definitizable in the sense that there exist $p, q \in \mathbb{R}[z] \setminus \{0\}$ such that

$$[p(A)x, x] \geq 0 \quad \text{and} \quad [q(B)x, x] \geq 0 \quad \text{for all} \quad x \in \mathcal{K}.$$
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With this straightforward definition it was possible to derive a functional calculus.
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There are unsatisfactory phenomena with this concept of definitizability. For example, it could be that a bijective $N = A + iB$ is definitizable in the above sense, but $N^{-1}$ is not.
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**Definition**

For a normal $N \in B(K)$ we call $p(x, y) \in \mathbb{R}[x, y]$ a definitizing polynomial for $N$, if

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and $N$ definitizable if there exist non-zero definitizing polynomials.
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and $N$ definitizable if there exist non-zero definitizing polynomials. For definitizable $N$ let $I$ be the ideal generated by all definitizing polynomials in $\mathbb{C}[x, y]$.

Rest of the talk is devoted to a Spectral Theorem for definitizable normal operators with a zero-dimensional $I$, i.e. $\dim \mathbb{C}[x, y]/I < \infty$. 
Method of Dragging Operators
Embedding of Hilbert space in Krein space

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{K}$ be a Krein space, and let $T : \mathcal{H} \to \mathcal{K}$ be bounded linear embedding (injective).
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$$
\mathcal{H} \overset{T}{\longrightarrow} \mathcal{K}
$$

**Definition**
For $C \in B(\mathcal{K})$ define

$$
\Theta(C) := T^{-1}CT = (T \times T)^{-1}(C),
$$

where is $C$ identified with its graph viewed as a subspace of $\mathcal{K} \times \mathcal{K}$, i.e. as a linear relation.
Embedding of Hilbert space in Krein space

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But if $C(TT^[*]) = (TT^[*])C$, then $\Theta(C') \in B(\mathcal{H})$.

**Theorem (R.Pruckner,K.; M.Dritschel, J.Rovnyak)**

$\Theta : C \mapsto T^{-1}CT$ constitutes $\ast$-Algebra Homomorphism

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\Theta : \{TT^[*]\}' \subseteq B(\mathcal{K}) \rightarrow \{T^*[T]\}' \subseteq B(\mathcal{H})
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satisfying $\Theta(I_\mathcal{K}) = I_\mathcal{H}$. 
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In particular: $\Theta(N)$ is normal if $N \in \left\{ TT^* \right\}'$ is normal.
Embedding of Hilbert space in Krein space

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Embedding induced by definitizable operator

- Given a definitizable normal $N = A + iB \in B(K)$ we denoted by $I \trianglelefteq \mathbb{C}[z]$ the ideal generated by all definitizing polynomials ($\in \mathbb{R}[z]$) for $N$. 
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$$\langle x, y \rangle := \left[\left(\sum_{k} p_k(A, B)\right)x, y\right], \quad x, y \in \mathcal{K};$$
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- \( T \) is injective due to
\[
\ker T = \iota(\mathcal{K})[\perp] = \mathcal{K}/\{x \in \mathcal{K} : \langle x, x \rangle = 0\}[\perp] = \{0\}.
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Embedding induced by definitizable operator

\( \mathcal{H} \) and \( T \) constructed from \([\left( \sum_k p_k(A, B) \right), ., .]\)
Embedding induced by definitizable operator

Construct $\mathcal{H}_k$ and $T_k$ in the same way from $[p_k(A, B)\ldots]$
Embedding induced by definitizable operator

Exist contractions $R_k : \mathcal{H}_k \rightarrow \mathcal{H}$ with $\sum_k R_k R_k^* = I_\mathcal{H}$
Embedding induced by definitizable operator

Proposition

\[ TT^\ast = \sum_k p_k(A, B) \ (\in B(K)) \text{ and } N = A + iB \in \{TT^\ast\}' \]

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Therefore:

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Structure of zero-dimensional Ideals
Primary decomposition

Applying the Noether-Lasker Theorem from ring theory to the special situation of a zero-dimensional ideal $I$ in $\mathbb{C}[x, y]$ we obtain minimal primary decomposition

$$I = Q_1 \cap \cdots \cap Q_m$$
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- $Q_j \not\supseteq \bigcap_{i \neq j} Q_i$ for all $j = 1, \ldots, m$.
- $P_j \neq P_i$ for $i \neq j$, where $P_j := \sqrt{Q_j}$ denotes the radical $\left\{ f \in \mathbb{C}[x, y] : f^k \in Q_j \text{ for some } k \in \mathbb{N} \right\}$. 
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- zero-dimensionality implies uniqueness of the above decomposition, and maximality of the $P_j$’s.
Primary decomposition

- Thus, \( P_j = \{ p \in \mathbb{C}[x, y] : p(a_j) = 0 \} \) for unique and pairwise distinct \( a_j \in \mathbb{C}^2 \).
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• For the variety $V(I) = \{ a \in \mathbb{C}^2 : f(a) = 0 \text{ for all } f \in I \}$ induced by $I$, we have $V(I) = \{ a_1, \ldots, a_m \}$. 
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• We write $Q(a) := Q_j$, $P(a) := P_j$, if $a = a_j \in V(I)$. Hence,

$$I = \bigcap_{a \in V(I)} Q(a)$$
Primary decomposition, real Ideal

Our $I$, which is the ideal generated in $\mathbb{C}[x, y]$ by all definitizing polynomials $p \in \mathbb{R}[x, y]$, satisfies $I = I^\#$, where $p^\#(x, y) = p(\bar{x}, \bar{y})$. 
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- $Q(a)^# = Q(a^#)$, $P(a)^# = P(a^#)$.
- We consider $V_\mathbb{R}(I) := \{a_j : a_j \in \mathbb{R}^2\}$ as a subset of $\mathbb{C}$ and $V(I) \setminus \mathbb{R}^2$ as a subset of $\mathbb{C}^2$; $V(I) \setminus \mathbb{R}^2$ invariant under $(\xi, \eta) \mapsto (\bar{\xi}, \bar{\eta})$. 
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- For $w \in V_{\mathbb{R}}(I)$ algebra $A(w) := \mathbb{C}[x, y]/(P(w) \cdot Q(w))$ is finite dimensional, because with $Q(w)$ also $P(w) \cdot Q(w)$ is primary and zero-dimensional.
Primary decomposition, real Ideal

Our \( I \), which is the ideal generated in \( \mathbb{C}[x, y] \) by all definitizing polynomials \( p \in \mathbb{R}[x, y] \), satisfies \( I = I^\# \), where \( p^\#(x, y) = p(\bar{x}, \bar{y}) \). \( a \in V(I) \Rightarrow a^\# \in V(I) \), where \( a^\# \in \mathbb{C}^2 \) is entrywise conjugation.

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- For \( \zeta \in V(I) \setminus \mathbb{R}^2 \) consider the finite-dimensional algebra \( B(\zeta) := \mathbb{C}[x, y]/Q(\zeta) \).
Function class $\mathcal{M}_N$

Define Function class $\mathcal{M}_N$ consisting of functions

$$\phi : \left( \sigma(\Theta(N)) \cup V_{\mathbb{R}}(I) \right) \cup V(I) \setminus \mathbb{R}^2 \rightarrow$$

$$\subseteq \mathbb{C} \quad \subseteq \mathbb{C}^2$$

$$\mathbb{C} \cup \bigcup_{w \in V_{\mathbb{R}}(I)} A(w) \cup \bigcup_{\zeta \in V(I) \setminus \mathbb{R}^2} B(\zeta)$$
Function class $\mathcal{M}_N$

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$$\phi : \left( \sigma(\Theta(N)) \cup V_{R}(I) \right) \cup V(I) \setminus \mathbb{R}^2 \rightarrow \mathbb{C} \cup \bigcup_{w \in V_{R}(I)} A(w) \cup \bigcup_{\zeta \in V(I) \setminus \mathbb{R}^2} B(\zeta)$$

such that

$$\phi(z) \in \begin{cases} B(z) & \text{for } z \in V(I) \setminus \mathbb{R}^2, \\ A(z) & \text{for } z \in V_{R}(I), \\ \mathbb{C} & \text{otherwise} \end{cases}$$
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Define Function class $\mathcal{M}_N$ consisting of functions

$$\phi : (\sigma(\Theta(N)) \cup V_\mathbb{R}(I)) \cup V(I) \setminus \mathbb{R}^2 \to \mathbb{C} \cup \bigcup_{w \in V_\mathbb{R}(I)} A(w) \cup \bigcup_{\zeta \in V(I) \setminus \mathbb{R}^2} B(\zeta)$$

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$\mathcal{M}_N$ becomes $*$-algebra when provided with pointwise with multiplication, addition and with $\phi \mapsto \phi^\#$, where

$$\phi^\#(z) := \phi(z^\#)^\#.$$
Function class $\mathcal{F}_N$

$I$ generated by finitely many definitizable $p_1, \ldots, p_m \in \mathbb{R}[z]$. 
Function class $\mathcal{F}_N$

$I$ generated by finitely many definitizable $p_1, \ldots, p_m \in \mathbb{R}[z]$. For all $w \in V_\mathbb{R}(I)$, which are non-isolated in $\sigma(\Theta(N)) \cup V_\mathbb{R}(I)$, we define a function $\chi_w$ on a sufficiently small neighbourhood of $w$ by

$$\chi_w(x + iy) := \max_{k=1, \ldots, m} |p_k(x, y)|.$$
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'Order of growth' towards $w$ is independent of the chosen generators $p_1, \ldots, p_m$. 
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'Order of growth' towards $w$ is independent of the chosen generators $p_1, \ldots, p_m$.

Definition
We denote by $\mathcal{F}_N$ the set of all elements $\phi \in \mathcal{M}_N$ such that
Function class $F_N$ is measurable and bounded on $\sigma(\Theta(N)) \setminus \mathbb{R}(I)$.

For each $w \in \mathbb{R}(I)$ which is non-isolated in $\sigma(\Theta(N)) \cup \mathbb{R}(I)$, $|\phi(z) - \phi(w)|$ is $0(\chi_w(z))$ as $z \in \sigma(\Theta(N)) \setminus \mathbb{R}(I)$.

Here $\phi(w)$ stands for $r(Re z, Im z)$, where $r \in C[x,y]$ is a representant of the coset $\phi(w) \in A(w) := C[x,y] / (P(w) \cdot Q(w))$. $F_N$ constitutes sub-$\ast$-algebra of $M_N$. 
Function class $\mathcal{F}_N$

- $z \mapsto \phi(z)$ is measurable and bounded on $\sigma(\Theta(N)) \setminus V_\mathbb{R}(I)$;
- for each $w \in V_\mathbb{R}(I)$ which is non-isolated in $\sigma(\Theta(N)) \cup V_\mathbb{R}(I)$

\[ \phi(z) - \phi(w)\big|_{x=\text{Re} z, y=\text{Im} z} = O(\chi_w(z)) \text{ as } \sigma(\Theta(N)) \setminus V_\mathbb{R}(I) \ni z \to w. \]
Function class $\mathcal{F}_N$

- $z \mapsto \phi(z)$ is measurable and bounded on $\sigma(\Theta(N)) \setminus V_\mathbb{R}(I)$;
- for each $w \in V_\mathbb{R}(I)$ which is non-isolated in $\sigma(\Theta(N)) \cup V_\mathbb{R}(I)$

$$\phi(z) - \phi(w)|_{x=\text{Re } z, y=\text{Im } z} = O(\chi_w(z)) \quad \text{as} \quad \sigma(\Theta(N)) \setminus V_\mathbb{R}(I) \ni z \to w.$$ 

Here $\phi(w)|_{x=\text{Re } z, y=\text{Im } z}$ stands for $r(\text{Re } z, \text{Im } z)$, where $r \in \mathbb{C}[x, y]$ is a representant of the coset $\phi(w) \in A(w) := \mathbb{C}[x, y]/(P(w) \cdot Q(w))$. 
Function class $\mathcal{F}_N$

• $z \mapsto \phi(z)$ is measurable and bounded on $\sigma(\Theta(N)) \setminus V_R(I)$;
• for each $w \in V_R(I)$ which is non-isolated in $\sigma(\Theta(N)) \cup V_R(I)$

$$\phi(z) - \phi(w)|_{x=\text{Re} z, y=\text{Im} z} = O(\chi_w(z)) \quad \text{as} \quad \sigma(\Theta(N)) \setminus V_R(I) \ni z \to w.$$  

Here $\phi(w)|_{x=\text{Re} z, y=\text{Im} z}$ stands for $r(\text{Re} z, \text{Im} z)$, where $r \in \mathbb{C}[x, y]$ is a representant of the coset $\phi(w) \in \mathcal{A}(w) := \mathbb{C}[x, y]/(P(w) \cdot Q(w))$.

$\mathcal{F}_N$ constitutes sub-$\ast$-algebra of $\mathcal{M}_N$. 
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Function class $\mathcal{F}_N$

Lemma

$\mathcal{F}_N$ 'contains' $\mathbb{C}[x,y]$ in the following sense:

For $s \in \mathbb{C}[x,y]$ the function $s_N \in \mathcal{M}_N$ defined by

$$s_N(z) = \begin{cases} 
  s(\text{Re } z, \text{Im } z), & z \in \sigma(\Theta(N)) \setminus V_R(I), \\
  s + (P(z) \cdot Q(z)) \in \mathcal{A}(z), & z \in V_R(I), \\
  s + Q(z) \in \mathcal{B}(z), & z \in V(I) \setminus \mathbb{R}^2.
\end{cases}$$

belongs to $\mathcal{F}_N$. 
Function class $\mathcal{F}_N$

Lemma

For each $\phi \in \mathcal{F}_N$ there exists a $p(x, y) \in \mathbb{C}[x, y]$ and $f_1, \ldots, f_m \in \mathcal{B}(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I))$ with $f_k(w) = 0$ for $w \in V_{\mathbb{R}}(I)$ such that

$$\phi(z) = p_N(z) + \sum_{k} f_k(z) \cdot (p_k)_N(z)$$

for all $z \in \sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)$, and that $\phi(\zeta) = p_N(\zeta)$ for all $\zeta \in V(I) \setminus \mathbb{R}^2$.
FUNCTIONAL CALCULUS FOR NORMAL, DEFINITIZABLE OPERATORS
Definition
For \( \phi \in \mathcal{F}_N \) let \( p(x, y) \in \mathbb{C}[x, y] \) and \( f_1, \ldots, f_m \in \mathcal{B}(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)) \) be as in the previous lemma.
Functional Calculus

Definition

For $\phi \in \mathcal{F}_N$ let $p(x, y) \in \mathbb{C}[x, y]$ and $f_1, \ldots, f_m \in \mathcal{B}(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I))$ be as in the previous lemma. Then we define

$$\phi(N) := p(A, B) + \sum_{k=1}^{m} T_k \circ \left( \int_{\sigma(\Theta_k(N))} f_k \, dE_k \right) \circ T_k^{[*]}.$$
Theorem (Functional Calculus for normal operators)

- $\phi(N)$ independent of particular choice of $p \in \mathbb{C}[x, y]$ and $f_1, \ldots, f_m$ as long as the assertion of the previous lemma is satisfied.
Functional Calculus

Theorem (Functional Calculus for normal operators)

- \( \phi(N) \) independent of particular choice of \( p \in \mathbb{C}[x, y] \) and \( f_1, \ldots, f_m \) as long as the assertion of the previous lemma is satisfied.

- The mapping \( \phi \mapsto \phi(N) \) constitutes \(*\)-homomorphism from \( \mathcal{F}_N \) into \( \{N, N^*\}'' \ (\subseteq B(K)) \);
Theorem (Functional Calculus for normal operators)

- \( \phi(N) \) independent of particular choice of \( p \in \mathbb{C}[x,y] \) and \( f_1, \ldots, f_m \) as long as the assertion of the previous lemma is satisfied.
- The mapping \( \phi \mapsto \phi(N) \) constitutes \(*\)-homomorphism from \( \mathcal{F}_N \) into \( \{N, N^*\}'' \ (\subseteq B(K)) \);
- For \( s \in \mathbb{C}[x,y] \) and \( \phi = s_N \in \mathcal{F}_N \) we have \( \phi(N) = s(A, B) \);
**Spectrum of** $N$

**Proposition**

\[
\sigma(N) = \sigma(\Theta(N)) \cup (V_{\mathbb{R}}(I) \cap \sigma(N)) \cup \\
\{ \alpha + i\beta : (\alpha, \beta) \in V(I) \setminus \mathbb{R}^2, \alpha + i\beta, \bar{\alpha} + i\bar{\beta} \in \sigma(N) \}
\]
Spectrum of $N$

Proposition

$$\sigma(N) = \sigma(\Theta(N)) \cup (V_R(I) \cap \sigma(N)) \cup \{\alpha + i\beta : (\alpha, \beta) \in V(I) \setminus \mathbb{R}^2, \alpha + i\beta, \bar{\alpha} + i\bar{\beta} \in \sigma(N)\}$$

Accordingly, Functional Calculus is supported on

$$\sigma(\Theta(N)) \cup (V_R(I) \cap \sigma(N)) \cup \{(\alpha, \beta) \in V(I) \setminus \mathbb{R}^2 : \alpha + i\beta, \bar{\alpha} + i\bar{\beta} \in \sigma(N)\}$$